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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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DYNAMICS OF HELICAL VORTEX FILAMENTS IN NONVISCOUS INCOMPRESSIBLE FLOWS

by Martin Donati, Christophe Lacave & Evelyne Miot

Abstract. — In this paper, we study concentrated solutions of the three-dimensional Euler equations in helical symmetry without swirl. We prove that any helical vorticity solution initially concentrated around helices of pairwise distinct radii remains concentrated close to filaments. As suggested by the vortex filament conjecture, we prove that those filaments are translating and rotating helices. Similarly to what is obtained in other frameworks, the localization is weak in the direction of the movement but strong in its normal direction and holds on an arbitrary long time interval in the naturally rescaled time scale. In order to prove this result, we derive a new explicit formula for the singular part of the Biot–Savart kernel in a two-dimensional reformulation of the problem. This allows us to obtain an appropriate decomposition of the velocity field to reproduce recent methods used to describe the dynamics of vortex rings or point-vortices for the lake equation.

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Martin Donati, CNRS, Université de Poitiers, LMA, Poitiers, France • *E-mail :* `Martin.Donati@math.univ-poitiers.fr`

Christophe Lacave, Univ. Savoie Mont Blanc, CNRS, LAMA, ISTerre, 73000 Chambéry, France • *E-mail :* `Christophe.Lacave@univ-smb.fr`

Evelyne Miot, Univ. Grenoble Alpes, CNRS, IF, 38000 Grenoble, France • *E-mail :* `Evelyne.Miot@univ-grenoble-alpes.fr`

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Résumé (*Dynamique des filaments de tourbillon hélicoïdaux dans les écoulements incompressibles non visqueux*). — Dans cet article, nous étudions les solutions concentrées des équations d'Euler tridimensionnelles avec symétrie hélicoïdale sans swirl. Nous prouvons que toute solution dont la vorticité est initialement concentrée autour d'hélices de rayons deux à deux distincts reste concentrée à proximité de filaments. Nous établissons que ces filaments sont des hélices en translation et en rotation, en accord avec la conjecture des filaments de tourbillon. La localisation est faible dans la direction du mouvement mais forte dans sa direction normale, et se maintient sur un intervalle de temps arbitrairement long dans l'échelle de temps naturelle. Afin de prouver ce résultat, nous obtenons une nouvelle formule explicite pour la partie singulière du noyau de Biot-Savart dans une reformulation bidimensionnelle du problème. Cela nous permet d'obtenir une décomposition appropriée du champ de vitesse permettant d'étendre les méthodes récentes utilisées pour décrire la dynamique des anneaux de tourbillons ou des tourbillons ponctuels pour l'équation des lacs.

1. Introduction

The purpose of this paper is to study the time evolution for 3D inviscid flows for which the vorticity is initially concentrated around helical curves.

We consider the Euler equations governing the dynamics of a three-dimensional inviscid, incompressible fluid in a domain Ω:

$$\text{(E)} \qquad \begin{cases} \partial_t U + (U\cdot\nabla)U = -\nabla P & \text{in } \Omega\times\mathbb{R}_+^*, \\ \operatorname{div}(U) = 0 & \text{in } \Omega\times\mathbb{R}_+, \\ U\cdot n = 0 & \text{on } \partial\Omega\times\mathbb{R}_+, \end{cases}$$

where n is the outward normal vector, and $U : \Omega\times\mathbb{R}_+^* \to \mathbb{R}^3$ denotes the velocity of the fluid and P the pressure. We shall focus on particular flows, called vortex filaments, for which the vorticity $\operatorname{curl}(U)$ is sharply concentrated in a thin tube around a curve in $\mathbb{R}^3$. Understanding the stability (namely, whether the concentration of the filament around a curve persists in time) and the dynamics of vortex filaments in three-dimensional flows are a longstanding issue in mathematical physics. Da Rios formally derived in [13] that, to leading order, the asymptotic motion law for one single vortex filament in a tube of size ε around a curve parametrized by $\chi(\cdot,t)$, with arc-length parameter σ, is governed by the binormal curvature flow:

$$\text{(BF)} \qquad \partial_t\chi = |\ln\varepsilon|(\partial_\sigma\chi\times\partial_{\sigma\sigma}\chi),$$

where the curvature is embedded in $\partial_{\sigma\sigma}\chi$. Note that (BF) exhibits some trivial solutions: the stationary vortex line, the uniformly translating circle (known as "vortex ring") and the translating-rotating helix (referred to as "helical filaments" in the remainder of the paper). We refer, e.g., to [27] and to references therein for a general introduction on the subject. The "vortex filament conjecture" is the conjecture that vorticity initially concentrated around a curve remains close to a curve evolving according to (BF) to leading order for at least a certain interval of time. While it is completely settled in the 2D case (where vortex filaments reduce to point vortices), see [28], it is open in general. Jerrard and Seis [25] provided a rigorous derivation of (BF) *assuming* the vorticity remains concentrated around the curve.

Without assuming *a priori* concentration, further results have been obtained under supplementary symmetry assumptions. For axisymmetric flows without swirl, Buttà, Cavallaro and Marchioro [4] recently rigorously justified the dynamics of several vortex rings of different radii. They also established a "semi-strong" localization result: the filaments remains for all time sharply localized in the radial direction (namely with respect to the distance to the symmetry axis). Their approach inspired a recent work by Hientzsch, Lacave, and Miot [24] in the setting of point vortices for the lake equations, which is a 2D model for incompressible flows inheriting an anelastic constraint from the 3D case.

In the special case of vortex rings, Fraenkel [20] exhibited a family of solutions of (E) such that the corresponding vorticity concentrates for all time on a curve solution of (BF). In [14], Davila, Pino, Musso, and Wei constructed such a family of solutions that do not change form, concentrating to one or several polygonally distributed rotating-translating helical filaments, by means of elliptic singular perturbation techniques. In a series of papers [8, 9, 6, 11, 10, 7], several desingularization results were obtained for helical vortex filaments by means of various variational techniques, enabling us, in particular, to construct helical vortex patches. In particular, [8] obtains helical vortices with compact support, [11] obtains helical vortex patches, and [10] covers both cases and some other profiles by use of rearrangements techniques. We also mention the recent work by Guerra and Musso [22], which constructs a special family of solutions concentrating to a collapsing configuration of helical filaments. We emphasize the fact that in all the works mentioned above, the solutions always belong to a restricted class of functions. Indeed, the approach is to construct a specific class of solutions with a prescribed vorticity profile, except in [10] where a general initial vortex profile is chosen, but the solution for all time is then constructed by means of a rearrangement of this profile, restricting the actual choice of the initial data.

Our objective here is to establish the dynamics of helical filaments starting from generic initial data, with very few and natural assumptions relating only to the initial concentration, and with other techniques.

The helical symmetry is a physically relevant framework since this symmetry is obtained in many different contexts, in particular in the wake of rotors. This covers a wide range of situations from the study of wind or water turbine to vertical flight of helicopters, for instance. Moreover, in the wake of each wing of an airplane, vortices of the same sign are created on straight lines, but immediately start interacting with each other, inducing a rotation that creates