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AUTOMORPHIC DENSITY ESTIMATES AND OPTIMAL APPROXIMATION EXPONENTS

BY Mikołaj FRĄCZYK, Alexander GORODNIK and Amos NEVO

Abstract. – The present paper is devoted to establishing an optimal approximation exponent for the action of an irreducible uniform lattice subgroup of a product group on its proper factors. Previously optimal approximation exponents for lattice actions on homogeneous spaces were established under the assumption that the restriction of the automorphic representation to the stability group is suitably tempered. However, for irreducible lattices in semisimple algebraic groups, either this property does not hold or it amounts to an instance of the Ramanujan-Petersson-Selberg conjecture. Sarnak's Density Hypothesis and its variants bounding the multiplicities of irreducible representations occurring in the decomposition of the automorphic representation can be viewed as a weakening of the temperedness property. A refined form of this hypothesis has recently been established for uniform irreducible arithmetic congruence lattices arising from quaternion algebras. We employ this result in order to establish, unconditionally, an optimal approximation exponent for the actions of these lattices on the associated symmetric spaces. We also give a general spectral criterion for the optimality of the approximation exponent for irreducible uniform lattices in a product of arbitrary Gelfand pairs. Our methods involve utilizing the multiplicity bounds in the pre-trace formula, establishing refined estimates of the spherical transforms, and carrying out an elaborate spectral analysis that bounds the Hilbert-Schmidt norms of carefully balanced geometric convolution operators.

Résumé. – Le présent article est consacré à l'établissement d'un exposant d'approximation optimal pour l'action d'un réseau irréductible cocompact d'un groupe produit sur ses facteurs propres. Auparavant, des exposants d'approximation optimaux pour les actions de réseaux sur des espaces homogènes avaient été établis sous l'hypothèse que la restriction de la représentation automorphe au groupe stabilisateur est convenablement tempérée. Toutefois, pour les réseaux irréductibles dans les groupes algébriques semi-simples, soit cette propriété n'est pas vérifiée, soit elle équivaut à un cas particulier de la conjecture de Ramanujan-Petersson-Selberg. L'hypothèse de densité de Sarnak et ses variantes, bornant les multiplicités des représentations irréductibles intervenant dans la décomposition de la représentation automorphe, peuvent être considérées comme un affaiblissement de la propriété de tempérance. Une forme raffinée de cette hypothèse a récemment été établie pour des réseaux de congruence arithmétiques, cocompacts et irréductibles provenant d'algèbres de quaternions. Nous utilisons ce résultat afin d'établir – inconditionnellement – un exposant d'approximation optimal pour les actions de ces réseaux sur les espaces symétriques associés. Nous donnons également un critère spectral général pour l'optimalité de l'exposant d'approximation pour des réseaux cocompacts irréductibles dans un

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produit de paires de Gelfand arbitraires. Nos méthodes reposent sur l'utilisation des bornes de multiplicité dans la formule de pré-trace, l'établissement d'estimations raffinées des transformées sphériques, et une analyse spectrale élaborée bornant les normes de Hilbert-Schmidt d'opérateurs de convolution géométrique soigneusement équilibrés.

1. Introduction

1.1. The approximation exponent problem for homogeneous spaces

This paper is devoted to some important instances of the following fundamental problem regarding quantitative density of orbits on homogeneous spaces. Let G be a locally compact second countable non-compact group, X a homogeneous space of G, and Γ a discrete subgroup of G. We consider the action of Γ on X, which we assume to have at least one dense orbit, and aim to establish quantitative gauges measuring the density of the corresponding Γ-orbits. For this purpose, we fix a metric D on X and a proper function $|\cdot| : G \to \mathbb{R}^+$. Given a dense orbit $\Gamma x \subset X$ and a point $y \in X$, we are interested in existence of solutions $\gamma \in \Gamma$ for the inequalities

$$D(\gamma\, x, y) \le \epsilon \qquad \text{and} \qquad |\gamma| \le R,$$

where $\epsilon \to 0$ and $R \to \infty$ with prescribed rates. Let us suppose that there exist $\zeta > 0$ and $\epsilon_0 = \epsilon_0(x, y, \zeta) > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, the system of inequalities

$$D(\gamma\, x, y) \le \epsilon \qquad \text{and} \qquad |\gamma| \le \zeta \, \log(1/\epsilon) \tag{1.1}$$

has solutions $\gamma \in \Gamma$. Following [13, Def. 2.1], we define the *approximation exponent* $\kappa_\Gamma(x, y)$ as the infimum of ζ such that the foregoing property holds. The exponent $\kappa_\Gamma(x, y)$ constitutes a natural gauge of the efficiency of approximating the point y by elements in the orbit Γx.

An extensive class of Diophantine approximation problems, and in particular the approximation exponent, were originally raised by Lang [24], in the setting of actions of arithmetic groups on homogeneous varieties of linear algebraic groups. These problems include analogs of the classical Dirichlet's, Khinchine's and Schmidt's theorems in Diophantine approximation, extended to the context of lattice orbits on homogeneous spaces. A systematic approach to investigate these problem has been developed in recent years, and for a detailed discussion we refer to [11, 12, 13, 15]. In the present paper, however, our focus is on establishing optimal results for the approximation exponent.

Let us assume further that G is a unimodular group and $X = G/H$ for a closed unimodular subgroup H. We denote the invariant measures on the groups G and H by m_G and m_H, respectively, and the G-invariant measure on the space X by m_X. There is a natural upper bound governing how many approximations as in (1.1) can be constructed for a generic point y. Namely, the number of solutions of (1.1) can be expected to be bounded by the number of points of the orbit Γx which are contained in compact neighborhoods of y in X. Let us assume, for simplicity, that the limit

$$a(H) := \lim_{R\to\infty} \frac{1}{R} \log m_H\big(\{h \in H : |h| \le R\}\big)$$

exists and is positive. Under certain mild regularity conditions on the gauge function $|\cdot|$ and the metric D, one can show that the number of points γx lying in a compact subset $\Omega \subset X$ with $\gamma \in \Gamma$ satisfying $|\gamma| \le R$ is at most $O_{\Omega,x,\delta}\big(e^{(a(H)+\delta)R}\big)$, with $\delta > 0$ arbitrarily small. This upper bound on the number of points available for the approximation can be expected to constrain the quality of approximations by elements of $\gamma \in \Gamma$ satisfying $|\gamma| \le R$. Developing this idea, it was shown in [13, Th. 3.1] that under the assumption of ergodicity for the action of Γ on X, for every $x \in X$ with a dense Γ-orbit and for almost every $y \in X$, the following pigeonhole lower bound holds:

$$\kappa_\Gamma(x,y) \ge \frac{\dim(X)}{a(H)}. \tag{1.2}$$

Therefore, the challenge is to establish a comparable upper bound for $\kappa_\Gamma(x,y)$.

When Γ is a lattice subgroup in G, an approach to establishing upper bounds for the approximation exponents $\kappa_\Gamma(x,y)$ was developed in [13] using spectral estimates in the automorphic representation $\lambda_{\Gamma\backslash G}$ of G on $L^2(\Gamma\backslash G)$. The key to the method is to establish a quantitative mean ergodic theorem for the action of the subgroup H on the space $\Gamma\backslash G$ equipped with the invariant probability measure $m_{\Gamma\backslash G}$ induced by the measure m_G. This provides an effective speed of distribution for H-orbits in $\Gamma\backslash G$, and allows the use of dynamical considerations in the form of a "shrinking target argument". These considerations are combined with a quantitative duality argument, which translates an equidistribution rate for H-orbits on $\Gamma\backslash G$ to a quantitative density estimate for the Γ-orbits in $X = G/H$.

To formulate the relevant spectral estimate, define

$$H_R := \{h \in H \,:\, |h| \le R\}$$

and assume that $m_H(H_R) \ge 1$ for all $R \ge R_1$. Introduce the associated averaging operators

$$\lambda_{\Gamma\backslash G}(\beta_R) : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)$$

defined by

$$\lambda_{\Gamma\backslash G}(\beta_R)\phi(z) := \frac{1}{m_H(H_R)}\int_{H_R} \phi(zh)\, dm_H(h) \quad \text{for } \phi \in L^2(\Gamma \setminus G) \text{ and } z \in \Gamma\backslash G.$$

The *quantitative mean ergodic theorem* for the operators $\lambda_{\Gamma\backslash G}(\beta_R)$ states that there exists $\theta > 0$ such that for all $\delta > 0$ and all $R \ge R_1$

$$\left\|\lambda_{\Gamma\backslash G}(\beta_R)\phi - \int_{\Gamma\backslash G} \phi\, dm_{\Gamma\backslash G}\right\|_{L^2(\Gamma\backslash G)} \le C_\delta m_H(H_R)^{-\theta+\delta}\, \|\phi\|_{L^2(\Gamma\backslash G)}, \quad \phi \in L^2(\Gamma\backslash G), \tag{1.3}$$

Let us denote by $\theta = \theta(G,H,\Gamma) > 0$ the maximal θ for which property (1.3) holds. It was shown in [13, Th. 3.3] that then for almost every $x \in X$ and every $y \in X$,

$$\kappa_\Gamma(x,y) \le \frac{1}{2\theta(G,H,\Gamma)} \cdot \frac{\dim(X)}{a(H)}. \tag{1.4}$$

In particular, when $\theta(G,H,\Gamma) = 1/2$, the lower bound (1.2) coincides with the upper bound (1.4), establishing the optimal approximation exponents for almost every pair $(x,y) \in X^2$.

We note that the estimate (1.3) with $\theta = 1/2$ holds when H is a simple non-compact Lie group, and the representation of H on the space $L^2_0(\Gamma\backslash G)$, the space of functions with zero integral, is weakly contained in the regular representation of H. In this case the required