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NORMED MOTIVIC SPECTRA AND POWER OPERATIONS

BY Tom BACHMANN, Elden ELMANTO AND Jeremiah HELLER

ABSTRACT. – We use motivic colimits to construct power operations on the homotopy groups of normed motivic spectra admitting a (normed) map from $H\mathbb{F}_2$. We establish enough of their standard properties to prove that the motivic dual Steenrod algebra is generated by one element under ring and power operations, establishing a motivic analog of Steinberger's theorem.

RÉSUMÉ. – Nous utilisons les colimites motiviques pour construire des opérations de puissance sur les groupes d'homotopie des spectres motiviques normés admettant une application (normée) depuis $H\mathbb{F}_2$. Nous établissons suffisamment de leurs propriétés standard pour prouver que l'algèbre de Steenrod duale motivique est générée par un seul élément sous les opérations d'anneau et de puissance, établissant ainsi un analogue motivique du théorème de Steinberger.

1. Introduction

Power operations have a long and well established history within classical algebraic topology. They are an essential computational tool for making effective use of generalized homology theories. Essential to the definition of power operations is some amount of extra structure. In order to construct power operations on a spectrum X, one requires at minimum a "coherently commutative binary multiplication," also known as an $\mathcal{H}_2$ structure. Such a structure is encoded as a map $(X^{\otimes 2})_{h\Sigma_2} \to X$ which satisfies some basic properties. Given such an X, there are operations on its homology which take the form

$$Q^i : H_*(X) \to H_{*+i}(X),$$

which are sometimes re-indexed as Q_s defined by $Q_s(x) := Q^{s+|x|}(x)$.

Consider the following important classical theorem of Steinberger.

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THEOREM 1.1 (Steinberger, [8, III.2.2-III.2.4]). – *Let* $\mathcal{A}_* = \mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$ *be the dual Steenrod algebra which is a free polynomial algebra* [21] *of the form*

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots, \xi_j, \dots],$$

where the degrees of the generators are $|\xi_j| = 2^j - 1$. *Applying the power operation* Q_1 *repeatedly to the generator* ξ_1 *yields elements in the dual Steenrod algebra which also generate it freely:*

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, Q_1(\xi_1), (Q_1)^2(\xi_1), \dots]. \tag{1.2}$$

Theorem 1.1 is an "economical" presentation of the dual Steenrod algebra — it expresses other elements in this very large ring, corresponding to the duals of Steenrod's operations, in terms of ξ_1 (which corresponds to the Bockstein operation) and the Q_1-operation. This generation by ξ_1 under the Q_1-operation underlies and/or foreshadows many important results in homotopy theory. Among them is the celebrated Hopkins-Mahowald theorem [20], expressing $H\mathbb{F}_2$ as a Thom spectrum, as well as Bökstedt's famous periodicity theorem computing the topological Hochschild homology (THH) of $\mathbb{F}_p$ (see [18, Theorem 1.1, Appendix A] for a proof and an extensive discussion on how these results are intimately related).

In the same vein, the Steenrod algebra had previously entered algebraic geometry in another crucial way via the Rost-Voevodsky proofs of Milnor and Bloch-Kato conjectures [30, 29, 32]. One of the crucial ingredients of the proof is the construction of the motivic analog of Margolis homology, which is a certain chain complex whose differentials are the (motivic analogs of the) Milnor operations. We refer the interested reader to [14, Section 1.6] for a brief explanation on the role of power operations for motivic cohomology in the proof of the Milnor of Bloch-Kato conjectures, and [14, Chapter 13] for a more extensive exposition.

Now we turn to our results. Let S be a base scheme where 2 is invertible and $\mathrm{H}\mathbb{Z}/2 \in \mathcal{SH}(S)$ is the motivic spectrum representing mod-2 motivic cohomology as constructed by Spitzweck. Write $\mathcal{A}_{**} = \pi_{**}(\mathrm{H}\mathbb{Z}/2 \otimes \mathrm{H}\mathbb{Z}/2)$ for the dual motivic Steenrod algebra. By work of Voevodsky [30], Hoyois-Kelly-Østvær [16], and Spitzweck [27] this is a Hopf algebroid with underlying algebra

$$\mathcal{A}_{**} \cong \mathrm{H}\mathbb{Z}/2_{**}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1}),$$

where $|\tau_i| = (2^{i+1} - 1, 2^i - 1)$ and $|\xi_i| = (2(2^i - 1), 2^i - 1)$; see Appendix D.3 for details and the full Hopf algebroid structure.

The bulk of this paper concerns the construction and analysis of motivic power operations on the homotopy of "quadratically normed" $\mathrm{H}\mathbb{Z}/2$-modules. We discuss what this means shortly, for the moment we simply note that our constructions apply to $\mathrm{H}\mathbb{Z}/2 \otimes \mathrm{H}\mathbb{Z}/2$. In particular, we obtain power operations acting on $\mathcal{A}_{**}$. Our main result is the following motivic version of Steinberger's theorem.

THEOREM 1.3 (see Theorem 5.33). – *Let* S *be a scheme where* $\frac{1}{2} \in \mathcal{O}_S$. *Then:*

1. *there are* motivic power operations*:*

$$Q^i : \mathcal{A}_{**} \to \mathcal{A}_{*+i,*+\lceil i/2 \rceil} \qquad i \in \mathbb{Z};$$

2. *the smallest* $\mathrm{H}\mathbb{Z}/2_{**}$*-subalgebra of* $\mathcal{A}_{**}$ *containing* τ_0 *and stable under the operations* Q^i*, is* $\mathcal{A}_{**}$ *itself.*

We note that one of the themes of this project is the indispensible nature of *genuine* structures in motivic homotopy theory. Indeed, one can easily construct *naive* or *categorical* power operations $Q^i_{\mathrm{naive}} : \mathrm{H}\mathbb{Z}/2_{**}\mathrm{H}\mathbb{Z}/2 \to \mathrm{H}\mathbb{Z}/2_{*+i,2*}\mathrm{H}\mathbb{Z}/2$ which will not interact correctly with the Tate twists, which are so central to the deeper properties of motivic cohomology. In particular, the element τ and applications of Q^i_{naive} on it will not be able to generate the dual Steenrod algebra.

1.1. Motivic power operations

The first author together with Marc Hoyois, introduced the theory of normed motivic spectra in [6]. Given $E \in \mathcal{SH}(S)$ and a finite étale morphism $p : T \to S$, a parametrized smash power $E^{\otimes T}$ is constructed in loc. cit. (where it is denoted $p_{\otimes} p^* E$). Now, informally speaking a normed motivic spectrum over S is a motivic spectrum $E \in \mathcal{SH}(S)$ equipped with multiplications $(E_{|_X})^{\otimes Y} \to E_{|_X}$ parametrized by finite étale morphisms $Y \to X$ over S, which are required to satisfy coherency conditions. Examples of normed motivic spectrum include familiar motivic spectra such as the sphere spectrum, MGL, KGL, and importantly $\mathrm{H}\mathbb{Z}$ and $\mathrm{H}\mathbb{Z}/n$.

Even in the informal description just given, it is clear that the concept of a normed motivic spectrum encodes an incredible amount of structure. We prefer to rely on as little structure as possible for our construction of power operations. To this end we introduce the category of “quadratically normed” spectra. In brief a quadratically normed spectrum is one equipped with multiplications parametrized by finite étale morphisms of degree 2. This is formalized as a map $D_2^{\mathrm{mot}}(X) \to X$, where $D_2^{\mathrm{mot}}(-)$ is the motivic extended powers functor [5].

Our construction relies on the interplay of two viewpoints: categorical and equivariant. On the one hand the motivic extended powers functor is a motivic colimit $\mathrm{D}_2^{\mathrm{mot}}(E) \simeq \mathrm{colim}_{p:Y \to X}(E_{|_X})^{\otimes Y}$. On the other hand we retain computational control via an equivariant model $\mathbf{D}_2(E) \simeq \mathbf{E}\Sigma_{2+} \otimes_{\Sigma_2} E^{\otimes \underline{2}}$, again see [5].

We construct a spectrum $\mathrm{Ops}_{\mathrm{H}\mathbb{Z}/2}$ which parametrizes power operations on quadratically normed $\mathrm{H}\mathbb{Z}/2$-modules; these form a category denoted by $\mathrm{HNAlg}_2(\mathcal{SH}(S)_{\mathrm{H}\mathbb{Z}/2/}$. In Theorem 4.3 we compute that if S is essentially smooth over a Dedekind domain with residue characteristics $\neq 2$, then

$$\pi_{**}\mathrm{Ops}_{\mathrm{H}\mathbb{Z}/2} \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{H}\mathbb{Z}/2_{**}\{e_{2n}\} \oplus \bigoplus_{n\in\mathbb{Z}} \mathrm{H}\mathbb{Z}/2_{**}\{e_{2n+1}\},$$

where e_i are classes with $|e_{2n}| = (2n, n)$ and $|e_{2n+1}| = (2n+1, n+1)$. These give rise to operations for a quadratically normed $\mathrm{H}\mathbb{Z}/2$-module E (over any base containing $1/2$),

$$Q^i : \pi_{**}E \to \pi_{*+i,*+\lceil i/2 \rceil}E.$$

The following proposition summarizes the results of Section 5.2, establishing the basic properties of these operations.

Proposition 1.4. – *Let* $E \in \mathrm{HNAlg}_2(\mathcal{SH}(S)_{\mathrm{H}\mathbb{Z}/2/}$ *and* $x, y \in \pi_{**}E$.

(1) Naturality: *If* $\alpha : E \to F$ *is a map of quadratically normed* $\mathrm{H}\mathbb{Z}/2$*-modules, then* $Q^i\alpha_* = \alpha_* Q^i$.