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# THE CAUCHY PROBLEM FOR THE PERIODIC KADOMTSEV-PETVIASHVILI-II EQUATION BELOW $L^2$

BY Sebastian HERR, Robert SCHIPPA
AND Nikolay TZVETKOV

Abstract. – We extend Bourgain's $L^2$-well-posedness result for the KP-II equation on $\mathbb{T}^2$ to initial data with negative Sobolev regularity. The key ingredient is a new linear $L^4$-Strichartz estimate which is effective on frequency-dependent time scales. The $L^4$-Strichartz estimates follow from combining an $\ell^2$-decoupling inequality recently proved by Guth-Maldague-Oh with semiclassical Strichartz estimates. Moreover, we rely on a variant of Bourgain's bilinear Strichartz estimate on frequency-dependent times, which is proved via a biorthogonality argument akin to the proof of the Córdoba-Fefferman square function estimate.

Résumé. – On généralise le résultat de Bourgain démontrant le caractère bien posé dans $L^2$ de l'équation KP-II sur le tore $\mathbb{T}^2$ à données initiales appartenant à des espaces de Sobolev d'indices négatifs. Un élément clé de la preuve est une nouvelle estimation de Strichartz sur des intervalles de temps dépendant de la localisation en fréquence de la condition initiale. Cette nouvelle estimation de Strichartz est le résultat de la combinaison d'une inégalité récente de presque orthogonalité obtenue par Guth-Maldague-Oh et des estimées de Strichartz semi-classiques dans l'esprit des travaux de Staffilani-Tatatru et Burq-Gérard-Tzvetkov. Nous utilisons également une variante d'une estimation de Strichartz bilinéaire due à Bourgain, sur des intervalles de temps dépendant de la fréquence spatiale. Cette estimation utilise des arguments de bi-orthogonalité dans l'esprit de la preuve de l'estimée de la fonction carré de Córdoba-Fefferman.

## 1. Introduction

In this article we are concerned with the local well-posedness of the KP-II equation on the torus $\mathbb{T}^2$, where $\mathbb{T}$ is the circle $\mathbb{R}/(2\pi\mathbb{Z})$:

$$(1)\qquad \begin{cases} \partial_t u + \partial_x^3 u + \partial_y^2 \partial_x^{-1} u = u \partial_x u, & (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) = u_0 \in H^{s,0}(\mathbb{T}^2). \end{cases}$$


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Above $u_0$ denotes a real-valued distribution on $\mathbb{T}^2$ with $\hat{u}_0(0,\eta) = 0$ in the anisotropic Sobolev space $H^{s,0}(\mathbb{T}^2)$ with norm

$$\|f\|^2_{H^{s,0}(\mathbb{T}^2)} = \sum_{(\xi,\eta)\in\mathbb{Z}^2} \langle \xi \rangle^{2s} |\hat{f}(\xi,\eta)|^2. \tag{2}$$

With the vanishing condition on the Fourier coefficients, we can explain $\partial_x^{-1}$ as a Fourier multiplier

$$\widehat{\partial_x^{-1} f}(\xi,\eta) = (i\xi)^{-1} \hat{f}(\xi,\eta).$$

The vanishing mean condition is propagated by the nonlinear evolution.

Bourgain proved analytic local well-posedness of (1) for $u_0 \in L^2(\mathbb{T}^2)$ in [4]. Since the $L^2$-norm is conserved, global well-posedness in $L^2$ follows for real-valued initial data. Here we seek to improve the result on $\mathbb{T}^2$ in [4] by showing local well-posedness with regularity *below* $L^2$, i.e., $s < 0$ in (1).

By local well-posedness we refer to existence, uniqueness, and continuity of the data-to-solution mapping

$$S_T : H^{s,0}(\mathbb{T}^2) \to C([0,T], H^{s,0}(\mathbb{T}^2)),$$
$$S_T(u_0) = u$$

with $T = T(s, \|u_0\|_{H^{s,0}(\mathbb{T}^2))})$. We obtain the data-to-solution mapping as unique continuous extension of the data-to-solution mapping on $L^2(\mathbb{T}^2)$, which allows us to work with smooth solutions. The analytic local well-posedness in [4] is a consequence of the contraction mapping principle. Presently, we combine the Fourier restriction analysis [4] with the energy method on frequency-dependent time scales [23, 18]. We implement this argument via short-time Fourier restriction norms as introduced by Ionescu-Kenig-Tataru. Notably, Koch-Tzvetkov [23] previously combined energy arguments with Strichartz estimates on frequency-dependent time scales to show low regularity well-posedness of the Benjamin-Ono equation.

We remark that on the Euclidean space the KP-II equation has the scaling symmetry

$$u \to u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y).$$

On $\mathbb{R}^2$ this distinguishes the scaling-critical Sobolev space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$. Hadac-Herr-Koch [14] showed global well-posedness and scattering for initial data in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ under suitable smallness assumption.

A few more remarks on the model are in order: The KP-II equation was proposed by Kadomtsev and Petviashvili [19] as model for two-dimensional water waves (see also Ablowitz-Segur [1]). Solutions to (1) which do not depend on the $y$-variable solve the KdV equation:

$$\partial_t u + \partial_{xxx} u = u \partial_x u.$$

Understanding the stability of solitons to the KdV equation under transverse perturbations was a central question upon the proposal of KP-equations.

Like the KdV equation, the KP-equations are known to be completely integrable. A Lax pair formulation was obtained by Dryuma [9] and as a consequence an infinite number of conserved quantities. Two conserved quantities for (1) read

$$M = \frac{1}{2}\int_{\mathbb{T}^2} u^2 dxdy,$$
$$E = \frac{1}{2}\int_{\mathbb{T}^2} \big((\partial_x u)^2 - \frac{1}{3}u^3 - (\partial_x^{-1}\partial_y u)^2\big) dxdy.$$

Due to a lack of coercivity, which is evident for $E$, the mass $M$ seems to be the only conservation law, which can be used to show global well-posedness. Recently, complete integrability techniques have been successfully employed to establish low regularity well-posedness in Sobolev spaces for various equations: Firstly, a priori estimates up to the scaling critical regularity [22, 21] were proved and later even global well-posedness [20, 15] via the method of commuting flows introduced by Killip-Vişan. Whether the arguments extend to two-dimensional models remains open. Our approach does not rely on the complete integrability; the symmetries it depends on are the real-valuedness of solutions and a conservative derivative nonlinearity.

In [4] Fourier restriction spaces were used (previously introduced in [2, 3]) to show analytic well-posedness in $L^2$. We use short-time Fourier restriction spaces introduced in [18]. A second key ingredient is an $L^4_{t,x,y}$-Strichartz estimate with sharp (up to endpoints) derivative loss. This is based on the very recent work by Guth-Maldague-Oh [13] on $\ell^2$-Fourier decoupling. It is combined with the dispersive inequality on a semiclassical time scale. This novel approach to low-regularity well-posedness of quasilinear dispersive equations does apply to other models as well, see e.g., the recent paper of the second author [27].

In the non-periodic setting the Strichartz estimate

$$\|e^{-t(\partial_x^3+\partial_y^2\partial_x^{-1})}u_0\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)},$$

which was obtained by Saut in [26], plays a fundamental role [14]. The example in Subsection 3.4 shows that in the periodic setting the estimate

$$\|e^{-t(\partial_x^3+\partial_y^2\partial_x^{-1})}u_0\|_{L^4([0,1]\times\mathbb{T}^2)} \lesssim \|u_0\|_{H^{s,0}(\mathbb{T}^2)}$$

fails for $s < 1/8$, while in Theorem 3.6 we prove that this estimate holds true for $s > 1/8$ if the transversal frequencies are not too large. In the following we survey the key ingredients of our analysis. Denote the dispersion relation by

$$\omega(\xi,\eta) = \xi^3 - \frac{\eta^2}{\xi}.$$

A key quantity is the resonance function:

$$\begin{aligned}\Omega(\xi_1,\eta_1,\xi_2,\eta_2) &= \omega(\xi_1+\xi_2,\eta_1+\eta_2) - \omega(\xi_1,\eta_1) - \omega(\xi_2,\eta_2)\\ &= 3\xi_1\xi_2(\xi_1+\xi_2) + \frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}.\end{aligned}$$

Since we impose a vanishing-mean condition on the initial data, which is preserved under the nonlinear evolution, the frequencies satisfy $\xi_i \neq 0$, $\xi_1+\xi_2 \neq 0$.

We let $\Omega_{\mathrm{KdV}}(\xi_1,\xi_2) = 3\xi_1\xi_2(\xi_1+\xi_2)$. Suppose that $|\xi_i| \sim N_i$ and $|\xi_1+\xi_2| \sim N_3$ for dyadic numbers $N_i \in 2^{\mathbb{N}_0}$. We note that we always have

$$|\Omega| \gtrsim N_{\max}^2 N_{\min}, \qquad N_{\max} = \max_{i=1,2,3} N_i, \qquad N_{\min} = \min_{i=1,2,3} N_i, \tag{3}$$