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ON THE DENSITY OF STRONGLY MINIMAL ALGEBRAIC VECTOR FIELDS

BY Rémi JAOUI

Abstract. – Two theorems witnessing the abundance of geometrically trivial strongly minimal autonomous differential equations of arbitrary order are shown. The first one states that a generic algebraic vector field of degree $d \geq 2$ on the affine space of dimension at least two is strongly minimal and geometrically trivial. The second one states that if X_0 is the complement of a smooth hyperplane section H in a smooth projective variety, then for d large enough, the system of differential equations associated with a generic vector field on X_0 with a pole of order at most d along H is strongly minimal and geometrically trivial.

This produces the first examples of meromorphic functions that are new in the sense of Painlevé and satisfy autonomous differential equations of order at least four.

Résumé. – Deux théorèmes témoignant de l'abondance des équations différentielles autonomes fortement minimales et géométriquement triviales sont démontrés dans cet article. Le premier affirme qu'un champ de vecteurs algébrique générique de degré $d \geq 2$ sur l'espace affine de dimension au moins deux est fortement minimal et géométriquement trivial. Le second affirme que si X_0 est le complémentaire d'une hypersurface lisse H dans une variété projective lisse, alors pour tout d suffisamment grand, le système d'équations différentielles associé à un champ de vecteurs générique sur X_0 avec un pôle d'ordre au plus d le long de H est fortement minimal et géométriquement trivial.

Cela produit les premiers exemples de fonctions méromorphes qui sont nouvelles au sens de Painlevé et vérifient des équations différentielles autonomes d'ordre au moins égal à quatre.

1. Introduction

Following the pioneering work of Hrushovski [21] on Manin kernels, a large body of results has emerged in the model-theoretic literature concerning the study of the algebraic relations between solutions of algebraic *nonlinear* differential equations for which the classical Picard-Vessiot Galois theory is not applicable.

The effectiveness of these model-theoretic techniques has been demonstrated both from a practical and from a theoretical perspective. On the practical side, they led to a full understanding of the algebraic relations between solutions of various classical algebraic differential

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equations. This includes the families of *Painlevé equations* [33, 34, 16], families of *Schwarzian equations* [17, 8, 2] and certain families of *Liénard equations* [14]. From a theoretical perspective, they provided new insights on the structure of algebraic relations between solutions of arbitrary algebraic differential equations [15].

1.1. The model-theoretic strategy

Given an algebraic family of differential equations depending on some parameters α in $\mathbb{C}^n$ and written as

$$E(\alpha) : F_\alpha(y, y', y'', \dots, y^{(n)}) = 0,$$

the core of the model-theoretic strategy is to first establish a "*coarse classification*" of the algebraic relations among solutions of the equations $E(\alpha)$. This coarse classification amounts to describing the structures of the solution sets of the equations $E(\alpha)$ for fixed values of α. It is phrased in the language of geometric stability theory [39] and called the *semi-minimal analysis* of the family.

Among all the possible semi-minimal analysis, a key feature of nonlinear differential algebra is that, after discarding a "small" subset $\mathcal{E}$ of parameters, one of them appears to have an *ubiquitous character* shared, for instance, by all the classical families of equations above. This ubiquitous semi-minimal analysis can be summarized as follows:

1. *strong minimality*: outside of the exceptional set $\mathcal{E}$, the equations $E(\alpha)$ are strongly minimal: for every solution y of $E(\alpha)$ and every differential field L containing the parameters of $E(\alpha)$, we have:

   $$\operatorname{trdeg}(L\langle y\rangle/L) = 0 \text{ or } n,$$

   where n denotes the order of $E(\alpha)$ and $L\langle y\rangle$ denotes the differential field generated by L and y;

2. *geometric triviality:* for every $r \geq 2$, every $\alpha_1, \dots, \alpha_r$ outside of the exceptional set $\mathcal{E}$ and every solutions $y_1, \dots, y_r$ of $E(\alpha_1), \dots, E(\alpha_r)$ respectively, if $\operatorname{trdeg}(L\langle y_1, \dots, y_r\rangle/L) < r \cdot n$ then

   $$\text{for some } i \neq j, \operatorname{trdeg}(L\langle y_i, y_j\rangle/L) = 0 \text{ or } n.$$

   Here, L denotes any differential field containing the parameters of the $E(\alpha_i)$. This condition is equivalent to geometric triviality of each individual fiber outside of the exceptional set $\mathcal{E}$;

3. *multidimensionality:* for almost every pair (α, β) of parameters, the equations $E(\alpha)$ and $E(\beta)$ are *orthogonal*:

   $$\operatorname{Corr}(E(\alpha), E(\beta)) = \emptyset,$$

   where $\operatorname{Corr}(E(\alpha), E(\beta))$ denotes the set of generically finite-to-finite correspondences between the solution sets of $E(\alpha)$ and $E(\beta)$.

The problem addressed in this article is to confirm the ubiquitous character of this semi-minimal analysis by showing that the families of algebraic differential equations satisfying (1) and (2) are indeed abundant in the universe of *autonomous* algebraic differential equations. The conjectural property (3) as well as other related problems are discussed in Subsection 1.5.

1.2. State of the art

Before Nagloo and Pillay clarified in [33] the relationship between (1) and classical ideas of Painlevé from [35, Vingt-et-unième leçon, pp.487] on “highly irreducible” differential equations, the scope of this model-theoretic strategy was essentially limited to order one equations. Nevertheless, it was already clear since the beginning of the eighties that the properties (1)-(3) should hold somewhat generically for algebraic differential equations of higher order as well. For instance, after showing that the Poizat equation $y'' \cdot y = y'$ and $y' \neq 0$ satisfies (1), Poizat conjectures in [43] that *for every $n \geq 2$, a “general” order n algebraic (nonlinear) differential equation should be strongly minimal.*

The first result of this kind was established in [28] and only came after substantial progress was made on the study of the classical examples mentioned above. It states that the family $\Xi(2, d)$ of planar systems of the form

$$(S) : \begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} \quad \text{where } f, g \in \mathbb{C}[X, Y]_{\leq d},$$

satisfies (1) and (2) whenever $d \geq 3$. Allowing the coefficients of the equations to be differentially transcendental instead, Devilbiss and Freitag [11] then established a similar result valid in arbitrary dimension. Their result states that for every $n \geq 1$ and every $d \geq 6$, the “highly non-autonomous” equations

$$F(y, y', \ldots, y^{(n)}) = 0,$$

where $F \in \mathcal{M}(U)[Y_0, \ldots, Y_n]$ is a (multivariate) polynomial of total degree d with *differentially independent and differentially transcendental* coefficients are strongly minimal. Together with the results of this article, they provide a complete answer to Poizat’s conjecture.

1.3. Statement of the main results

The first result concerns systems of polynomial differential equations. In contrast with the previous results, it already applies to quadratic systems of polynomial differential equations.

THEOREM A. – *Let $n, d \in \mathbb{N}$. Consider the family $\Xi(n, d)$ of systems of differential equations of the form*

$$(S) : \begin{cases} y_1' &= f_1(y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(y_1, \ldots, y_n) \end{cases}$$

parametrized by n-tuples of complex polynomials $f_1, \ldots, f_n$ of degree bounded by d. If $n, d \geq 2$ then the family $\Xi(n, d)$ satisfies (1) and (2):

(1) *a* generic *system (S) from $\Xi(n, d)$ is strongly minimal,*

(2) *if $y(1), \ldots, y(r)$ are r (n-tuples of) solutions of* generic *systems $(S_1), \ldots, (S_r)$ from $\Xi(n, d)$ and L is a differential field extension of $\mathbb{C}$ then*

$$\operatorname{trdeg}(L(y(1), \ldots, y(r))/L) = r \cdot n$$

unless for some $i \neq j$, $\operatorname{trdeg}(L(y(i), y(j))/L) = 0$ *or n.*