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STRONG DISORDER AND VERY STRONG DISORDER ARE EQUIVALENT FOR DIRECTED POLYMERS

BY STEFAN JUNK AND HUBERT LACOIN

ABSTRACT. – We show that if the normalized partition function W_n^β of the directed polymer model on $\mathbb{Z}^d$ converges to zero, then it does so exponentially fast. This implies that there exists a critical value β_c for the inverse temperature such that the normalized partition function has a non-degenerate limit for all $\beta \in [0, \beta_c]$—weak disorder holds—while for $\beta \in (\beta_c, \infty)$ it converges exponentially fast to zero—very strong disorder holds. This solves a twenty-years-old conjecture formulated by Comets, Yoshida, Carmona and Hu. Our proof requires a technical assumption on the environment, namely, that it is bounded from above.

RÉSUMÉ. – Nous prouvons que si la fonction de partition normalisée W_n^β du modèle de polymère dirigé sur $\mathbb{Z}^d$ tend vers zéro, alors la convergence est exponentiellement rapide. Cela implique qu'il existe une valeur β_c pour la température inverse telle que la fonction de partition normalisée possède une limite non dégénérée pour tout $\beta \in [0, \beta_c]$ – il y a désordre faible – alors que pour $\beta \in (\beta_c, \infty)$ elle converge à rythme exponentiel vers zéro – il y a désordre très fort. Ce résultat résout une conjecture formulée par Comets, Yoshida, Carmona and Hu il y a vingt ans. Notre démonstration est réalisée sous l'hypothèse que l'environnement est borné supérieurement.

1. Introduction

The *directed polymer in a random environment* on $\mathbb{Z}^d$ is a statistical mechanics model, which has been introduced, for $d = 1$, in [26] as a toy model to study the interfaces of the planar Ising model with random coupling constants. The model was shortly afterwards generalized to higher dimension (see for instance [11, 27]). When $d \geq 2$, rather than an effective interface model, the directed polymer in a random environment can be interpreted as modeling the behavior of a stretched polymer in a solution with impurities. The interest in the model, triggered by its rich phenomenology (including connections with the Stochastic heat equation and the Kardar-Parisi-Zhang (KPZ) equation, see [15, Section 8]) has since then generated a plentiful literature in theoretical physics and mathematics. We refer to [15, 45] for recent reviews on the subject.

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An important topic for the directed polymer is the so-called localization transition, which can be described as follows. At low temperature, the polymer trajectories are localized in a narrow corridor where the environment is most favorable [3, 4, 5, 13, 14, 16]. At high temperature, the effect of the disorder disappears on large scale, and the rescaled polymer trajectories have the same scaling limit as the simple random walk, that is to say, standard Brownian motion [11, 18, 27, 42].

This transition can be defined in terms of the asymptotic behavior of the renormalized partition function of the model W_n^β (see (4) in the next section). If W_n^β converges to an almost surely positive limit W_∞^β we say that *weak disorder* holds. On the other hand if W_n^β converges to zero almost surely, we say that *strong disorder* holds. It has been proved in [18] that weak disorder implies that the distribution of the polymer converges after diffusive rescaling to that of standard Brownian motion (see also [35] for an alternative short and self-contained proof). While some localization results have been proved under the strong disorder assumptions, for instance in [14], much stronger characterizations of disorder-induced localization have been obtained under the stronger assumption that W_n^β converges to zero exponentially fast ($\lim_{n\to\infty} n^{-1} \log W_n^\beta < 0$), see for instance [3, 5]. This latter regime is known as the *very strong disorder* regime.

When $d = 1$ or 2 it has been proved that very strong disorder holds for every $\beta > 0$ [17, 34] (it was proved earlier in [13, 16] that strong disorder holds for every $\beta > 0$). On the other hand, when $d \geq 3$, it has been shown [18] that there exist $\beta_c, \bar{\beta}_c \in (0, \infty]$ such that:

— weak disorder holds for $\beta \in [0, \beta_c)$;

— strong disorder holds for $\beta > \beta_c$;

— very strong disorder holds if and only if $\beta > \bar{\beta}_c$.

It has been conjectured [5, 14, 15, 18, 34, 43, 45] that $\beta_c = \bar{\beta}_c$. This conjecture implies that, at least in a mild sense, the two notions of strong disorder are equivalent. The present paper proves this conjecture and also establishes that weak disorder holds at β_c.

Our result is an example of a *sharp phase transition*, since W_n^β either converges to some positive (random) limit—if $\beta \in [0, \beta_c]$—or decays exponentially fast—if $\beta \in (\beta_c, \infty)$. This excludes the existence of an intermediate phase. This behavior is similar to that of the long-range connection probability for Bernoulli percolation [1, 40] and to that of spin-spin correlation for the Ising model [2]. We refer to [23] for a more recent shorter proof and to [22] and to [21] for recently obtained sharpness results concerning Potts model and Random interlacements, respectively.

Combined with the recent observations made on the tail distribution of W_∞^β in the weak disorder phase [29, 30, 28, 31], our result provides a new perspective on the phase transition from weak to strong disorder for directed polymers when $d \geq 3$. The latter is, for the time being, largely unchartered territory. Indeed, our result identifies exactly the tail distribution of the limiting partition function at criticality $W_\infty^{\beta_c}$ (see Corollary 2.3 and [31, Theorem 2.1]). This information might allow an analysis of the behavior of the directed polymer at β_c by identifying features that are specific to the model at criticality. Furthermore, it should help in controlling the rate of decay of W_n^β for β just slightly above β_c, hence establishing regularity estimates for the free energy at the vicinity of the critical point similar to those known to hold in dimensions $d = 1$ and $d = 2$ (see [6, 41] for the sharpest established results).

Our proof relies on two main technical innovations. The first is to reduce the problem to a question about end-point localization. More precisely, we reduce the proof of the equivalence of strong and very strong disorder to the following claim: when W_n^β reaches its supremum, then, most likely, the endpoint polymer measure is localized (see Theorem 4.4). This implication is obtained by chaining a variety of arguments. Some of these (e.g., fractional moments, spine constructions) have already been used in the study of directed polymer and disordered system, while others are novel. The reasoning is explained in Sections 3 to 6. The second innovation is the proof of this localization statement, based on discrete-time stochastic calculus. It is detailed in Sections 7–8.

2. Model and results

2.1. Weak, strong and very strong disorder for directed polymer

We let $X = (X_k)_{k\geq 0}$ be the nearest neighbor simple random walk on $\mathbb{Z}^d$ starting from the origin, and P the associated probability distribution. It is the Markov chain satisfying $P(X_0 = 0) = 1$ and

$$P(X_{n+1} = y \mid X_n = y) = \frac{1}{2d}\mathbb{1}_{\{|y-x|=1\}} =: D(x, y), \tag{1}$$

where $|\cdot|$ denotes the ℓ_1 distance on $\mathbb{Z}^d$. Given a collection $\omega = (\omega_{k,x})_{k\geq 1, x\in\mathbb{Z}^d}$ of real-valued weights (the environment), $\beta \geq 0$ (the inverse temperature) and $n \geq 1$ (the polymer length), we define the polymer measure $P_{\omega,n}^\beta$ as a modification of the distribution P which favors trajectories that visit sites where ω is large. More precisely, to each path $\pi\colon \mathbb{N} \to \mathbb{Z}^d$ we associate an energy $H_n(\omega, \pi) := \sum_{i=1}^n \omega_{i,\pi(i)}$, and we set

$$P_{\omega,n}^\beta(\mathrm{d}X) := \frac{1}{Z_{\omega,n}^\beta} e^{\beta H_n(\omega,X)} P(\mathrm{d}X) \quad \text{where} \quad Z_{\omega,n}^\beta := E\left[e^{\beta H_n(\omega,X)}\right]. \tag{2}$$

The quantity $Z_{\omega,n}^\beta$ is referred to as the partition function of the model. In what follows, we assume that the environment ω is given by a fixed realization of an i.i.d. random field on $\mathbb{Z}^d$ and we let $\mathbb{P}$ denote the associated probability distribution. We assume that ω has finite exponential moments of all orders, that is

$$\forall \beta \in \mathbb{R},\ \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,0}}] < \infty. \tag{3}$$

The reader can readily check that in this setup we have $\mathbb{E}[Z_{\omega,n}^\beta] = e^{n\lambda(\beta)}$. We thus define the normalized partition function W_n^β (the dependence in ω is dropped from the notation for convenience) by setting

$$W_n^\beta = \frac{Z_{\omega,n}^\beta}{\mathbb{E}[Z_{\omega,n}^\beta]} = E\left[e^{\beta H_n(\omega,X) - n\lambda(\beta)}\right]. \tag{4}$$

The normalized partition function W_n^β encodes essential information on the typical behavior of X under $P_{\omega,n}^\omega$ and has thus been a central object of attention in polymer studies. A crucial observation made in [11] is that W_n^β is a non-negative martingale with respect to the natural filtration $(\mathcal{F}_n)$ associated with ω ($\mathcal{F}_n := \sigma((\omega_{k,x})_{k\leq n, x\in\mathbb{Z}^d})$) and hence