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# EXTERIOR STABILITY OF MINKOWSKI SPACETIME WITH BORDERLINE DECAY

BY Dawei SHEN

ABSTRACT. – In 1993, the global stability of Minkowski spacetime was proved in the celebrated work of Christodoulou and Klainerman. In 2003, Klainerman and Nicolò revisited Minkowski stability in the exterior of an outgoing null cone. In 2023, the author extended the results of Christodoulou-Klainerman to minimal decay assumptions. In this paper, we prove that the exterior stability of Minkowski holds with decay that is borderline compared to the minimal decay considered in 2023.

RÉSUMÉ. – En 1993, la stabilité globale de l'espace-temps de Minkowski a été prouvée dans les célèbres travaux de Christodoulou et Klainerman. En 2003, Klainerman et Nicolò ont revisité la stabilité de Minkowski à l'extérieur d'un cône de lumière sortant. En 2023, l'auteur a étendu les résultats de Christodoulou et Klainerman aux hypothèses de décroissance minimale. Dans cet article, nous prouvons que la stabilité extérieure de Minkowski tient avec une décroissance qui est limite par rapport à la décroissance minimale considérée par l'auteur en 2023.

## 1. Introduction

### 1.1. Einstein vacuum equations and the Cauchy problem

A Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$ is called a vacuum spacetime if it solves the Einstein vacuum equations:

$$\mathbf{Ric}(\mathbf{g}) = 0 \qquad \text{in } \mathcal{M}, \tag{1}$$

where **Ric** denotes the Ricci tensor of the Lorentzian metric **g**. The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1) is a non-linear geometric coupled system of partial differential equations of order 2 for **g**. In suitable coordinates, for example so-called wave coordinates, it can be shown that (1) is hyperbolic and hence admits an initial value formulation.


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The corresponding initial data for the Einstein vacuum equations is given by specifying a triplet $(\Sigma, g, k)$ where $(\Sigma, g)$ is a Riemannian 3-manifold and $k$ is the traceless symmetric 2-tensor on $\Sigma$ satisfying the constraint equations:

$$R = |k|^2 - (\operatorname{tr} k)^2, \qquad D^j k_{ij} = D_i(\operatorname{tr} k), \tag{2}$$

where $R$ denotes the scalar curvature of $g$, $D$ denotes the Levi-Civita connection of $g$ and

$$|k|^2 := g^{ad} g^{bc} k_{ab} k_{cd}, \qquad \operatorname{tr} k := g^{ij} k_{ij}.$$

In the future development $(\mathcal{M}, \mathbf{g})$ of such initial data $(\Sigma, g, k)$, $\Sigma \subset \mathcal{M}$ is a spacelike hypersurface with an induced metric $g$ and a second fundamental form $k$.

The seminal well-posedness results for the Cauchy problem obtained in [7, 4] ensure that for any smooth Cauchy data, there exists a unique smooth maximal globally hyperbolic development $(\mathcal{M}, \mathbf{g})$ solution of Einstein Equations (1) such that $\Sigma \subset \mathcal{M}$ and $g$, $k$ are respectively the first and second fundamental forms of $\Sigma$ in $\mathcal{M}$.

The prime example of a vacuum spacetime is Minkowski spacetime:

$$\mathcal{M} = \mathbb{R}^4, \qquad \mathbf{g} = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \qquad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \qquad k = 0.$$

In the present work, we consider the problem of the stability of Minkowski spacetime and start with reviewing the literature on this problem.

### 1.2. Previous works of the stability of Minkowski spacetime

In 1993, Christodoulou and Klainerman [5] proved the global stability of Minkowski for the Einstein-vacuum equations, a milestone in the domain of mathematical general relativity. In 2003, Klainerman and Nicolò [14] proved the Minkowski stability in the exterior of an outgoing cone using the double null foliation. Moreover, Klainerman and Nicolò [15] showed that under stronger asymptotic decay and regularity properties than those used in [5, 14], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties. Given that the goal of this paper is to extend the result of [14], we will state the results of [5, 14] in Section 1.3.

We now mention other proofs of Minkowski stability. In 2007, Bieri [2] gave a new proof of global stability of Minkowski requiring one less derivative and fewer vectorfields compared to [5]. Lindblad and Rodnianski [18, 19] gave a new proof of the stability of the Minkowski spacetime using *wave-coordinates* and showing that the Einstein equations verify the so called *weak null structure* in that gauge. Huneau [12] proved the nonlinear stability of Minkowski spacetime with a translation Killing field using generalized wave-coordinates. Using the framework of Melroses b-analysis, Hintz and Vasy [10] reproved the stability of Minkowski space. Graf [8] proved the global nonlinear stability of Minkowski space in the context of the spacelike-characteristic Cauchy problem for Einstein vacuum equations, which together with [14] allows reobtaining [5]. Under the framework of [14] and using $r^p$-weighted estimates of Dafermos and Rodnianski [6], the author [21] reproved the Minkowski stability in exterior regions. More recently, Hintz [9] reproved the Minkowski stability in exterior regions by using the framework of [10]. Using $r^p$-weighted estimates, the author [3] extends the results of [2] to minimal decay assumption.

There are also stability results concerning Einstein's equations coupled with non trivial matter fields, see for example the introduction of [21, 3].

### 1.3. Minkowski Stability in [5, 14]

We recall in this section the results in [5, 14]. First, we recall the definition of a *maximal hypersurface*, which plays an important role in the statements of the main theorems in [5, 14].

DEFINITION 1.1. – *An initial data* $(\Sigma, g, k)$ *is posed on a maximal hypersurface if it satisfies*

$$\operatorname{tr} k = 0. \tag{3}$$

*In this case, we say that* $(\Sigma, g, k)$ *is a maximal initial data set, and the constraint Equations* (2) *reduce to*

$$R = |k|^2, \qquad \operatorname{div} k = 0, \qquad \operatorname{tr} k = 0. \tag{4}$$

We introduce the notion of $(s, q)$-*asymptotically flat initial data.*

DEFINITION 1.2. – *Given* $s \geq 1$ *and* $q \geq 0$*, we say that a data set* $(\Sigma_0, g, k)$ *is* $(s, q)$*-asymptotically flat if there exists a coordinate system* $(x^1, x^2, x^3)$ *defined outside a sufficiently large compact set such that:*

— *In the case of* $s \geq 3$*,*

$$g_{ij} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_{q+1}(r^{-\frac{s-1}{2}}), \qquad k_{ij} = o_q(r^{-\frac{s+1}{2}}). \tag{5}$$

— *In the case of* $1 \leq s < 3$*,*

$$g_{ij} = \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \qquad k_{ij} = o_q(r^{-\frac{s+1}{2}}). \tag{6}$$

*Here, the notation* $f = o_l(r^{-m})$ *means that* $\partial^\alpha f = o(r^{-m-|\alpha|})$ *holds for all* $|\alpha| \leq l$.

We also introduce the following functional:

$$\begin{aligned} J_0(\Sigma_0, g, k) := \sup_{\Sigma_0} \left((d_0^2 + 1)^3 |\operatorname{Ric}|^2\right) + \int_{\Sigma_0} \sum_{l=0}^{3} (d_0^2 + 1)^{l+1} |D^l k|^2 \\ + \int_{\Sigma_0} \sum_{l=0}^{1} (d_0^2 + 1)^{l+3} |D^l B|^2, \end{aligned}$$

where $d_0$ is the geodesic distance from a fixed point $O \in \Sigma_0$, and $B_{ij} := (\operatorname{curl} \widehat{R})_{ij}$ is the *Bach tensor*, $\widehat{R}$ is the traceless part of Ric. Now, we can state the main theorems of [5] and [14].

THEOREM 1.3 (Global stability of Minkowski space [5]). – *There exists an* $\varepsilon_0 > 0$ *sufficiently small such that if* $J_0(\Sigma_0, g, k) \leq \varepsilon_0^2$*, then the initial data set* $(\Sigma_0, g, k)$*,* $(4, 3)$*-asymptotically flat (in the sense of Definition 1.2) and maximal, has a unique, globally hyperbolic, smooth, geodesically complete solution. This development is globally asymptotically flat, i.e., the Riemann curvature tensor tends to zero along any causal or spacelike geodesic. Moreover, there exists a global maximal time function* $t$ *and an optical function* $u$*, which satisfies* $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$*, defined everywhere in an external region.*