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LIOUVILLE POLARIZATIONS AND THE RIGIDITY OF THEIR LAGRANGIAN SKELETA IN DIMENSION 4

BY Emmanuel OPSHTEIN AND Felix SCHLENK

ABSTRACT. – In 2000, Biran introduced polarizations of closed symplectic manifolds and showed that their Lagrangian skeleta exhibit remarkable rigidity properties. He found, in particular, that their complements contain only small balls. In this paper, we introduce so-called Liouville polarizations of certain open 4-dimensional symplectic manifolds. This leads to several symplectic embedding results, that in turn lead to new Lagrangian non-removable intersections and a novel phenomenon of Legendrian barriers.

We show, for instance, that given any connected symplectic 4-manifold (M,ω) and a 4-ball of smaller volume, there exists an explicit finite union of Lagrangian disks in the 4-ball such that their complement symplectically embeds into (M,ω), extending a result by Sackel-Song-Varolgunes-Zhu and Brendel. Other applications are new Lagrangian intersection results and relative versions of the Arnold chord conjecture.

RÉSUMÉ. – En 2000, Biran a introduit les polarisations de variétés symplectiques compactes et a montré que leurs squelettes lagrangiens présentent des propriétés remarquables de rigidité. Il a notamment constaté que leurs complémentaires ne contiennent que des petites boules. Dans cet article, nous introduisons des polarisations dites de Liouville de certaines variétés symplectiques non compactes de dimension 4. Cela conduit à plusieurs résultats de plongements symplectiques, qui à leur tour conduisent à de nouvelles intersections lagrangiennes persistantes et un nouveau phénomène de barrières legendriennes.

Nous montrons par exemple qu'étant donné une variété symplectique connexe (M,ω) de dimension 4 et une boule de dimension 4 de plus petit volume, il existe une union finie explicite de disques lagrangiens dans la 4-boule dont le complémentaire se plonge symplectiquement dans (M,ω), ce qui étend un résultat de Sackel-Song-Varolgunes-Zhu et Brendel. D'autres applications sont de nouveaux résultats d'intersections lagrangiennes et des versions relatives de la conjecture des cordes d'Arnold.

1. Introduction and main results

A polarization of a *closed* symplectic manifold (M,ω) is a decomposition of the symplectic class $[\omega]$ along positive real multiples of the Poincaré-dual classes of codimension 2 symplectic submanifolds. Closed symplectic manifolds always admit polarizations,

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and these have proven useful to study the flexible side of symplectic embedding problems [6, 25]. In this work, we introduce polarizations of *open* symplectic manifolds and show that this concept offers new opportunities, at least in dimension 4. Referring to §2.3 for the definition, we focus in this introduction on the applications.

1.1. Symplectic embeddings

As usual, $D(a)$, $B^4(a)$, $C^4(a)$, and $Z^4(a)$ denote the open round disk of area a, the open 4-ball of capacity $\pi r^2 = a$, the open "cube" $D(a) \times D(a)$, and the open cylinder $D(a) \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$, all endowed with the standard symplectic form $\omega_{\rm st} = \sum_{j=1}^2 dx_j \wedge dy_j$ on $\mathbb{R}^4$ with primitive $\alpha_{\rm st} = \frac{1}{2} \sum_{j=1}^2 x_j dy_j - y_j dx_j$.

A cornerstone of modern symplectic geometry is Gromov's non-squeezing theorem from [16], stating in dimension four that $B^4(a)$ symplectically embeds into $Z^4(1)$ only for $a \leq 1$. In [30], Sackel, Song, Varolgunes, and Zhu investigated a refinement of this result, asking how large a set (in the sense of the Hausdorff dimension, for instance) one has to remove from $B^4(a)$ such that the complement symplectically embeds into $Z^4(1)$. They first proved that it does not suffice to remove a set of lower Minkowski dimension < 2, and then showed optimality of this result by proving that for $a \leq 2$ it suffices to remove a Lagrangian coordinate disk from $B^4(a)$. Later, Brendel in the appendix of [30] showed that for $a < 3$ it suffices to remove three Lagrangian pinwheels (certain simple Lagrangian CW-complexes) and a symplectic torus. In their Question 3, these authors then asked:

QUESTION 1.1. – *What is the largest a such that the complement of a 2-dimensional subset of $B^4(a)$ symplectically embeds into $Z^4(1)$?*

We denote by $\Gamma_{\frac{1}{k}}$ the union of the k half-lines emanating from the origin in $\mathbb{C}$ and cutting the plane into isometric sectors, as partly shown in (i), (ii), and (iii) of Figure 1.1 on page 912 for $k = 2, 3, 4$, and write

$$\Delta_k := \Gamma_{\frac{1}{k}} \times \Gamma_{\frac{1}{k}} \subset \mathbb{C}^2.$$

Then Δ_k is the orbit of the Lagrangian quadrant $\mathbb{R}^2_{\geq 0} \subset \mathbb{C}^2$ under the subgroup G of U(2) generated by $(e^{2\pi i/k}, 1)$ and $(1, e^{2\pi i/k})$. When k is even, then Δ_k is also the G-orbit of the standard Lagrangian plane $\mathbb{R}^2 \subset \mathbb{C}^2$.

A symplectic embedding $\varphi\colon (M, d\alpha) \to (M', d\alpha')$ between exact symplectic manifolds is (α, α')-*exact* if $\varphi^*\alpha' = \alpha + df$ for a smooth function f on M. Since $B^4(1) \subset C^4(1)$, our first symplectic embedding result gives a positive answer to Question 1.1 for every a.

THEOREM 1. – *There exists an* $(\alpha_{\rm st}, \alpha_{\rm st})$*-exact symplectic embedding*

$$C^4(1) \backslash \Delta_k \;\to\; Z^4\left(\tfrac{2}{k}\right).$$

REMARKS 1.2. – (i) There exists a symplectic embedding of $D(\frac{1}{k}) \times D(\frac{1}{k})$ into $C^4(1) \setminus \Delta_k$, so $C^4(1) \setminus \Delta_k$ does not symplectically embed into $Z^4(A)$ for $A < \frac{1}{k}$ by the nonsqueezing theorem. Thus Theorem 1 is sharp up to a possible factor of 2. We do not know the exact size of the thinnest cylinder into which $C^4(1) \backslash \Delta_k$ or $B^4(1) \backslash \Delta_k$ symplectically embed, except for the optimal embedding $B^4(1) \setminus \Delta_2 \to Z^4(\frac{1}{2})$ from [30]. For a different such embedding see Remark 4.9.

(ii) A different positive answer to Question 1.1 was independently given by Haim-Kislev, Hind, and Ostrover in [17], and it is interesting to compare the two constructions, see § 1.5.

Our construction also applies to unbounded domains. For instance, consider the union of Lagrangian planes $\Gamma \times \Gamma$ in $\mathbb{R}^4$, where $\Gamma \subset \mathbb{R}^2$ is the square grid

$$\Gamma = \bigcup_{(n,m)\in\mathbb{Z}^2} \{n\} \times \mathbb{R} \cup \mathbb{R} \times \{m\}.$$

THEOREM 2. – *There exists an* $(\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$*-exact symplectic embedding*

$$\mathbb{R}^4 \setminus (\Gamma \times \Gamma) \to Z^4(1).$$

This answers in particular the question asked by Claude Viterbo to Paul Biran in 1998, whether the Gromov width of $\mathbb{R}^4 \setminus (\Gamma \times \Gamma)$ is infinite or not, [33].

We can construct similar embeddings when the source or the target has a topology. If we take the 4-ball as domain, we can fill "anything" after removing sufficiently many Lagrangian planes:

THEOREM 3. – *Let* (M, ω) *be a connected symplectic 4-manifold of finite volume. Let* $B^4(a)$ *be the ball of the same volume. Then for every* $\varepsilon > 0$ *there exists an even number* $k \in \mathbb{N}$ *such that* $B^4(a-\varepsilon) \setminus \Delta_k$ *symplectically embeds into* (M, ω).

In fact a much more general statement holds. Any affine part of a closed symplectic 4-manifold (defined as the complement of a polarizing divisor, see Definition 2.2) verifies the same kind of flexibility: After removing a suitable Lagrangian CW-complex, it can be embedded into any other affine part of a closed symplectic 4-manifold of larger volume. We shall give the proof of this result in a separate paper, [28], because in addition to the embedding method developed here the proof relies on a relative version of Giroux's result in [15] and requires specific techniques. In contrast to the results of this paper, the removed CW-complex cannot be explicitly described, however.

1.2. Rigidity properties of Lagrangian skeleta

Following Biran [2], we notice that if the complement of a Lagrangian CW-complex Δ in a domain U symplectically embeds into a cylinder, then any domain $V \subset U$ that does not symplectically embed into that cylinder must intersect Δ. The results of the previous paragraph therefore have implications on non-removable intersections of Δ with various subsets, and even with Lagrangian submanifolds. We shall remove Lagrangian CW-complexes more general than the Δ_k, namely products of arbitrary connected graphs.

DEFINITION 1.3. – *A* grid Γ *in* $D(A)$ *is the part in* $D(A)$ *of a connected graph* $\Gamma \cup \partial D(A)$ *in* $\overline{D(A)}$ *that has no 1-valent vertex, contains the boundary of* $D(A)$*, and has smooth edges.*

Figure 1.1 shows examples of grids. Note that the complement of a grid consists of topological disks, and that for two grids Γ and Γ' the Lagrangian CW-complex $\Gamma \times \Gamma'$ is smooth if and only if both Γ and Γ' have no vertex, as in (i) and (iv).