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INDUCTIVE SYSTEMS OF THE SYMMETRIC GROUP, POLYNOMIAL FUNCTORS AND TENSOR CATEGORIES

BY KEVIN COULEMBIER

ABSTRACT. – We initiate the systematic study of modular representations of symmetric groups that arise via the braiding in (symmetric) tensor categories over fields of positive characteristic. We determine what representations appear for certain examples of tensor categories, develop general principles and demonstrate how this question connects with the ongoing study of the structure theory of tensor categories. We also formalize a theory of polynomial functors as functors that act coherently on all tensor categories. We conclude that the classification of such functors is a different way of posing the above question of which representations of symmetric groups appear. Finally, we extend the classical notion of strict polynomial functors from the category of (super) vector spaces to arbitrary tensor categories, and show that this idea is also a different packaging of the same information.

RÉSUMÉ. – Nous entreprenons l'étude systématique des représentations modulaires des groupes symétriques qui apparaissent par le tressage des catégories tensorielles (symétriques) sur des corps de caractéristique positive. Nous déterminons quelles représentations interviennent pour certains exemples de catégories tensorielles, développons des principes généraux et montrons comment cette question s'articule avec l'étude en cours de la théorie structurelle des catégories tensorielles. Nous formalisons également une théorie des foncteurs polynomiaux en tant que foncteurs agissant de manière cohérente sur toutes les catégories tensorielles. Nous concluons que la classification de tels foncteurs constitue une autre manière de poser la question précédente de quelles représentations des groupes symétriques apparaissent. Enfin, nous généralisons la notion classique de foncteurs polynomiaux stricts, de la catégorie des (super)espaces vectoriels aux catégories tensorielles arbitraires, et montrons que cette idée constitue également une reformulation différente de la même information.

Introduction

The structure theory of tensor categories has been an active topic in the last decade. We use the term tensor category over a field $\mathbf{k}$ in the sense of [17, 24], for a *symmetric* rigid monoidal $\mathbf{k}$-linear abelian category with some finiteness assumptions.

Over fields of characteristic zero, classical results of Deligne [17, 18] show that tensor categories 'of moderate growth' must be representation categories of groups or supergroups.

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Without the moderate growth assumption, see for instance [30, 31], or over fields of positive characteristic, see for instance [4, 5, 8, 9, 13, 14, 15, 25, 37], exploring the structure theory remains ongoing.

To motivate the work in the current paper we give some background on the state of the art regarding tensor categories of moderate growth over fields of positive characteristic. The principle of tannakian reconstruction of [17] reduces the structure theory problem to the classification of 'incompressible' tensor categories, see [14, Theorem 5.2.1]. By [17, 18] the only incompressible categories of moderate growth over a field $\mathbf{k}$ of characteristic zero are thus the categories of vector spaces $\mathsf{Vec} = \mathsf{Vec}_{\mathbf{k}}$ and supervector spaces $\mathsf{sVec} = \mathsf{sVec}_{\mathbf{k}}$ over $\mathbf{k}$. In contrast, for characteristic $p > 0$, in [4, 5, 9], a chain of incompressible categories of moderate growth

$$\mathsf{Vec} \subset \mathsf{sVec} \subset \mathsf{Ver}_p \subset \mathsf{Ver}_{p^2} \subset \mathsf{Ver}_{p^3} \subset \cdots$$

was constructed. In [5] it was conjectured that every tensor category of moderate growth admits a tensor functor to $\mathsf{Ver}_{p^\infty} = \cup_n \mathsf{Ver}_{p^n}$; or equivalently that all incompressible categories of moderate growth are subcategories of Ver_{p^∞}. Working towards this conjecture, in [13] it was proved that a tensor category of moderate growth admits a tensor functor to Ver_p if and only if it is 'Frobenius-exact', meaning the Frobenius functor

$$\mathrm{Fr} : \mathcal{C} \to \mathcal{C} \boxtimes \mathsf{Ver}_p \tag{0.1}$$

is exact. The functor Fr is defined in [37, 25, 8] as follows. One sends $X \in \mathcal{C}$ to $X^{\otimes p}$, which can be interpreted as an S_p-representation internal to $\mathcal{C}$ via the braiding, and subsequently one manipulates $X^{\otimes p}$ via a non-exact symmetric monoidal functor $\mathsf{Rep}S_p \to \mathsf{Ver}_p$. These results lead to three (interrelated) questions:

(Q1) By Takeuchi's theorem, for a tensor category $\mathcal{C}$ there exists a faithful exact (non-monoidal) functor $\omega : \mathcal{C} \to \mathsf{Vec}$, so that $\omega(X^{\otimes d})$ is an ordinary S_d-representation over $\mathbf{k}$. Attempts at constructing 'higher' versions of Fr in [15], relating to Ver_{p^n} rather than Ver_p, lead to the question of which S_d-representations can occur in this way for $d = p^n$. A concrete example of such questions for $d = p$ already appeared in [13, Question 7.3].

(Q2) The functor Fr in (0.1) 'commutes with tensor functors'. By taking direct summands of Fr, one obtains functors $\mathcal{C} \to \mathcal{C}$ that commute with tensor functors. More familiar examples are the (skew) symmetric powers Sym^d and $\bigwedge^d$, and all of the above functors are subquotients of the functor $X \mapsto X^{\otimes d}$. In characteristic zero, such functors simply produce (direct sums of) Schur functors $\mathbf{S}_\lambda$, see [18]. In positive characteristic, many interesting questions, such as their 'classification', are open. We will start by developing a rigorous theory and definition of such functors, and prove that the classification question is just a reformulation of (Q1).

(Q3) A logical term for the functors in (Q2) would be 'polynomial' functors. Classical strict polynomial functors were introduced in [28], as a tool for proving finite generation of the cohomology ring of finite group schemes. In the description of [36], the category of strict polynomial functors of degree d is the category of $\mathbf{k}$-linear functors

$$\mathsf{Fun}_{\mathbf{k}}(\Gamma^d\mathsf{Vec}, \mathsf{Vec}) \simeq \mathsf{Rep}^d\, \mathrm{GL}_V, \tag{0.2}$$

where the equivalence is [28, Theorem 3.2] for a vector space V of dimension at least d and $\mathsf{Rep}^d\, \mathrm{GL}_V$ is the category of polynomial representations of degree d. Here $\Gamma^d\mathsf{Vec}$ is the category with the same class of objects as Vec, but with morphism spaces

$$\Gamma^d\mathsf{Vec}(U,V) := \mathrm{Hom}_{\mathbf{k}S_d}(U^{\otimes d}, V^{\otimes d}),$$

so that linear functors out of $\Gamma^d\mathsf{Vec}$ indeed correspond to 'polynomial of degree d' functors out of Vec. In [1], a 'super' version of polynomial functors was introduced, and this was used in [21] to prove cohomological finite-generation for finite supergroup schemes. One can follow this template for Vec and sVec and define a category of strict polynomial functors based on every (incompressible) tensor category, and search for equivalences as in (0.2). In the long term, one would hope to use this towards proving [27, Conjecture 2.18] regarding cohomological finite-generation for finite tensor categories. In the current paper we only show that this study is also equivalent to (Q1) and (Q2).

Prelude: Schur-Weyl duality

The double centraliser property between the symmetric group S_d and the general linear group GL_V of a complex vector space V, say with $\dim_{\mathbb{C}} V \geq d$, acting on $V^{\otimes d}$, leads to an equivalence of $\mathbb{C}$-linear categories

$$\mathsf{Rep}_{\mathbb{C}} S_d \xrightarrow{\sim} \mathsf{Rep}^d_{\mathbb{C}}\,\mathrm{GL}_V, \qquad M \mapsto V^{\otimes d} \otimes_{\mathbb{C}S_d} M. \tag{0.3}$$

Since $\mathsf{Rep}_{\mathbb{C}} S_d$, as a $\mathbb{C}$-linear category, is a finite direct sum of $\mathsf{Vec}_{\mathbb{C}}$, it is equivalent to the category of $\mathbb{C}$-linear functors from $\mathsf{Rep}_{\mathbb{C}} S_d$ to $\mathsf{Vec}_{\mathbb{C}}$. The resulting equivalence

$$\mathsf{Fun}_{\mathbb{C}}(\mathsf{Rep}_{\mathbb{C}} S_d, \mathsf{Vec}_{\mathbb{C}}) \xrightarrow{\sim} \mathsf{Rep}^d_{\mathbb{C}}\,\mathrm{GL}_V, \qquad F \mapsto F(V^{\otimes d}) = \bigoplus_{\lambda\vdash d} F(S^\lambda)\otimes \mathbf{S}_\lambda(V) \tag{0.4}$$

is less common, but perhaps more natural as we explain below. Firstly, note that S^λ denotes the (simple) Specht module of S_d, so that, as a bimodule,

$$V^{\otimes d} \simeq \bigoplus_{\lambda\vdash d} S^\lambda \boxtimes \mathbf{S}_\lambda(V).$$

While equivalence (0.3) does not extend to positive characteristic, its incarnation (0.4) can be extended very neatly. Denote by $\mathbf{Young}^d \subset \mathsf{Rep} S_d$ the full subcategory of direct sums of Young modules, see [23]. Then over any field $\mathbf{k}$ we do get an equivalence

$$\mathsf{Fun}_{\mathbf{k}}(\mathbf{Young}^d, \mathsf{Vec}) \xrightarrow{\sim} \mathsf{Rep}^d\,\mathrm{GL}_V, \quad F \mapsto F(V^{\otimes d}). \tag{0.5}$$

We can observe that $\mathbf{k}$-linear functors out of $\mathbf{Young}^d$ are just modules over the Schur algebra $S(n,d)$, for $n \geq d$, which yields the more standard interpretations of (0.5).

Since the functor $\Gamma^d\mathsf{Vec} \to \mathbf{Young}^d$, $U \mapsto U^{\otimes d}$ corresponds to the inclusion of $\Gamma^d\mathsf{Vec}$ into its idempotent completion, the left-hand sides of (0.2) and (0.5) can be canonically identified. Moreover, while we used Schur functors to make (0.4) precise, the assignment $V \mapsto F(V^{\otimes d})$ for some functor F from $\mathbf{Young}^d$ to Vec, as appears in (0.5), paves the way for more general definitions of polynomial functors in the sense of (Q2). Finally, $\mathbf{Young}^d$ is clearly the category of S_d-representations that arise via the braid action from the tensor category Vec as in (Q1). It thus follows that (0.5) gives an ansatz for connecting (Q1), (Q2) and (Q3).

To explain our results, we henceforth fix an algebraically closed field $\mathbf{k}$ of prime characteristic p and only consider tensor categories $\mathcal{C}$ over $\mathbf{k}$.