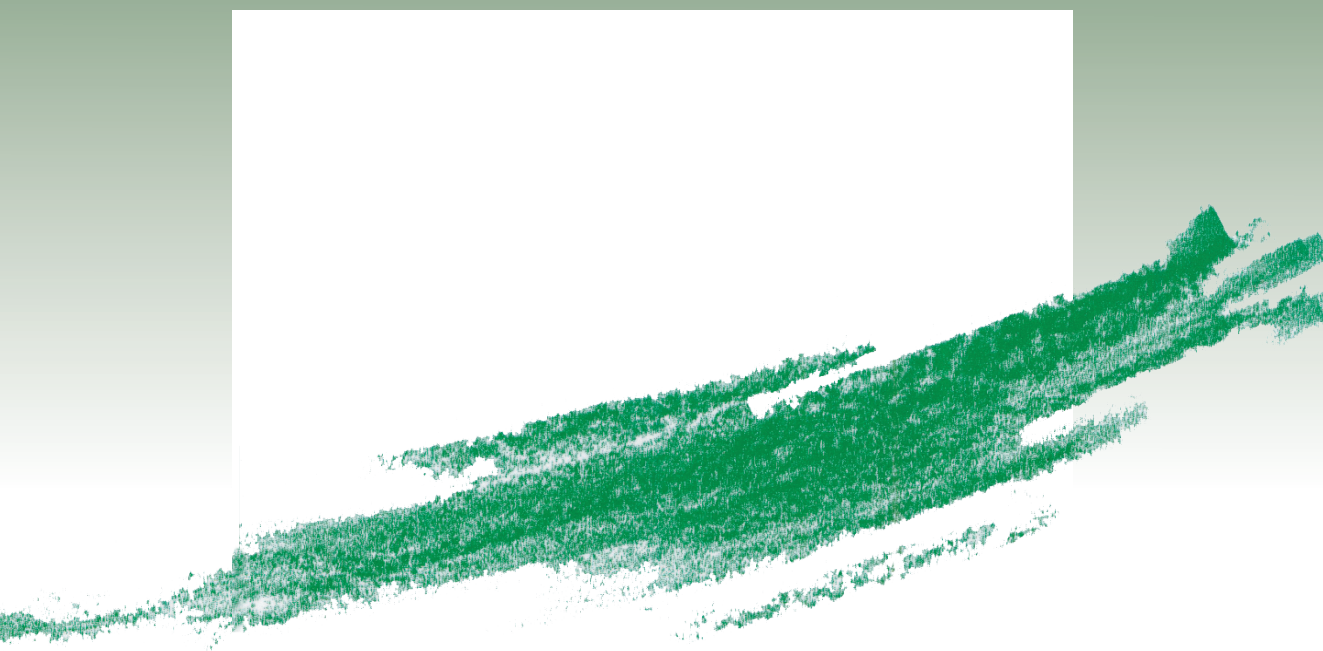


COURS SPÉCIALISÉS
COLLECTION SMF

Microlocal Analysis in Hyperbolic Dynamics and Geometry

Thibault LEFEUVRE

with a contributed chapter by Yann Chaubet



**MICROLOCAL ANALYSIS
IN
HYPERBOLIC DYNAMICS
AND
GEOMETRY**

Thibault Lefeuvre
with a contributed chapter
by
Yann Chaubet

Comité de rédaction

Oriane BLONDEL
Romain DUJARDIN
Céline ESSER
Cyril HOUDAYER
Cécilia LANCIEN

Thierry LÉVY
Simon RICHE
Gabriel RIVIÈRE
Barbara SCHAPIRA

Sophie GRIVAUX (Rédactrice en chef)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 93 € (\$ 140)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Cours Spécialisés
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
cours_specialises@smf.emath.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2025

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 1284-6090

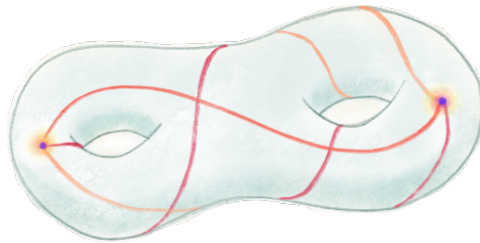
ISBN 978-2-37905-221-7

Directrice de la publication : Isabelle GALLAGHER

COURS SPÉCIALISÉS 32

**MICROLOCAL ANALYSIS
IN
HYPERBOLIC DYNAMICS
AND
GEOMETRY**

Thibault Lefeuvre
with a contributed chapter
by
Yann Chaubet



Société Mathématique de France 2025

Thibault Lefeuvre
Université Paris-Saclay
CNRS, Laboratoire de mathématique d'Orsay
91405, Orsay, France

Mathematical Subject Classification (2010). — 35S05, 35P05, 35P99, 35R30, 37-02, 37C30, 37D20, 37D30, 53C24, 53D25, 53C29, 58J40.

Keywords. — Microlocal analysis, hyperbolic flows, spectral theory, inverse problems, geodesic flows, rigidity.

Mots-clefs. — Analyse microlocale, flots hyperboliques, théorie spectrale, problèmes inverses, flot géodésique, rigidité.

MICROLOCAL ANALYSIS
IN
HYPERBOLIC DYNAMICS
AND
GEOMETRY

Thibault Lefeuvre
with a contributed chapter
by
Yann Chaubet

Abstract. — The statistical properties of hyperbolic dynamical systems—such as ergodicity and mixing—can be studied through spectral theory, in particular via anisotropic Sobolev spaces of distributions. In settings where the geodesic flow exhibits hyperbolic features, rigidity phenomena in Riemannian geometry—showing that certain spectral or geometric invariants determine the underlying geometry—can likewise be addressed using microlocal analysis. This book offers a comprehensive introduction to microlocal analysis and its applications to hyperbolic dynamics and Riemannian rigidity. It is intended for graduate students and researchers seeking to familiarize themselves with these powerful techniques.

Résumé (Analyse micro-locale en dynamique hyperbolique et en géométrie). — Les propriétés statistiques des systèmes dynamiques hyperboliques, telles que l'ergodicité et le mélange, peuvent être étudiées à l'aide de la théorie spectrale, en particulier via les espaces de Sobolev anisotropes de distributions. Dans des contextes où le flot géodésique présente des caractéristiques hyperboliques, les phénomènes de rigidité en géométrie riemannienne, qui montrent que certains invariants spectraux ou géométriques déterminent la géométrie sous-jacente, peuvent également être abordés à l'aide de l'analyse microlocale. Cet ouvrage offre une introduction complète à l'analyse microlocale et à ses applications à la dynamique hyperbolique et à la rigidité riemannienne. Il s'adresse aux étudiants de troisième cycle et aux chercheurs qui souhaitent se familiariser avec ces techniques puissantes.

À la mémoire de mon père

“In the East they swore that by three sides
was the decent way across a square.”

T. E. LAWRENCE, *Seven Pillars of Wisdom*

CONTENTS

Foreword	xxi
1. Introduction	1
1.1. Why this book?	1
1.2. Content of the book	2
1.2.1. Spectral theory of Anosov flows	2
1.2.2. Geometric inverse problems, rigidity of Riemannian manifolds	5
1.2.3. Background material	8
1.2.4. Other topics covered	8
1.2.5. Topics not covered in this monograph	9
1.3. Book organization	9
1.3.1. Lecturing with this textbook	9
1.3.2. Chapters dependency graph	11
1.4. Notations and conventions	12
1.4.1. Analysis	12
1.4.2. Dynamics	13
1.4.3. Geometry	13
Part I. Microlocal analysis	15
2. What is microlocal analysis?	17
2.1. Partial differential equations	17
2.2. Quantization	19
3. Oscillatory integrals	21
3.1. Definition and first properties	21
3.1.1. Symbols	22
3.1.2. Phase functions	27
3.1.3. Definition and regularization of an oscillatory integral	27
3.1.4. Exercises	32
3.2. The Schwartz kernel theorem	33
3.2.1. Main theorem	34
3.2.2. Operators with oscillatory kernels	39
3.3. Stationary phase lemma	41

3.3.1. Non-stationary phase lemma	41
3.3.2. Stationary phase lemma	41
3.4. Bibliographical notes	47
4. Wavefront set calculus	49
4.1. The wavefront set	49
4.1.1. Definition. First properties	49
4.1.2. Wavefront set of an oscillatory integral	57
4.1.3. Exercises	59
4.2. Extension of linear operators to distributions	60
4.2.1. Bilinear integration of distributions	60
4.2.2. Extension theorem	63
4.3. Applications	67
4.3.1. Elementary operations on distributions	67
4.3.2. Wavefront set on manifolds	72
4.3.3. Flat trace	73
4.3.4. Exercises	75
4.4. Bibliographical notes	77
5. Pseudodifferential operators	79
5.1. Definition and first properties	79
5.1.1. Definition	79
5.1.2. Rough properties	80
5.1.3. Properly supported Ψ DOs	82
5.1.4. Bijection between symbols and quantization	84
5.2. Pseudodifferential calculus	89
5.2.1. Basic operations	89
5.2.2. Principal symbol	93
5.2.3. Pseudodifferential operators on closed manifolds	95
5.2.4. Exercises	100
5.3. Ellipticity and parametrices	102
5.3.1. Elliptic operators	102
5.3.2. Wavefront set revisited	106
5.4. Sobolev continuity	108
5.4.1. Boundedness on $L^2(M)$	109
5.4.2. Boundedness on Sobolev spaces	113
5.4.3. Exercises	118
5.5. Bibliographical notes	119
6. Advanced properties of pseudodifferential operators	121
6.1. Elliptic operators: Fredholmness and elliptic estimates	121
6.1.1. Index of an elliptic operator	121
6.1.2. Inverse and sharp parametrix	122
6.1.3. Elliptic regularity	124

6.1.4. Gårding's inequalities	125
6.2. Propagation of singularities	127
6.2.1. Propagation of singularities for flows	127
6.2.2. Egorov's theorem	129
6.2.3. Real principal type operators	134
6.2.4. Exercises	135
6.3. Spectral theory	136
6.3.1. Spectral description	136
6.3.2. Mollifiers	139
6.3.3. Exercises	139
6.4. Anisotropic pseudodifferential operators	141
6.4.1. Definitions. First properties	141
6.4.2. Ellipticity. Existence of a parametrix	143
6.4.3. Anisotropic Sobolev spaces	144
6.5. Bibliographical notes	144
Part II. Spectral theory of Anosov flows	147
7. What is chaos?	149
7.1. Chaos in dynamical systems	149
7.1.1. Taming chaos	149
7.1.2. Anosov flows	150
7.1.3. Sinai billiards and negatively curved Riemannian manifolds	151
7.2. Spectral description of Anosov flows	153
7.2.1. Transfer operator	153
7.2.2. Spectrum of volume-preserving Anosov flows	155
8. Anosov flows	157
8.1. Elementary properties of Anosov flows	157
8.1.1. Definition	157
8.1.2. Alekseev's Cone Field Criterion	159
8.1.3. Stable and unstable manifolds	161
8.1.4. Local product structure	163
8.2. First examples	163
8.2.1. Geodesic flow on hyperbolic surfaces	163
8.2.2. Time reparametrization of Anosov flows	165
8.2.3. Suspensions of Anosov diffeomorphisms	166
8.2.4. Exercises	168
8.3. Orbit structure	169
8.3.1. Topological structure	169
8.3.2. Structural stability	171
8.3.3. Shadowing lemmas	176
8.3.4. Homoclinic orbits	177

8.4. Ergodic theory	177
8.4.1. Recurrence	178
8.4.2. Ergodicity	180
8.4.3. Mixing	183
8.5. Bibliographical notes	184
9. Spectral theory of Anosov flows	185
9.1. Pollicott-Ruelle resonances	185
9.1.1. Spectral theory of the flow generator	185
9.1.2. Escape function and anisotropic spaces	192
9.1.3. Meromorphic extension of the resolvent	197
9.1.4. Exercises	202
9.2. Statistics of volume-preserving Anosov flows	205
9.2.1. Description of the spectral measure	205
9.2.2. Ergodicity and mixing	210
9.2.3. The Anosov alternative	212
9.2.4. Spectral measure at 0	213
9.2.5. Exercises	215
9.3. Bibliographical notes	217
10. Dynamical zeta functions (by Y. Chaubet)	219
10.1. Introduction	219
10.2. Currents	220
10.2.1. Definition and first properties	220
10.2.2. Operations on currents	224
10.2.3. Periodic orbits	230
10.3. Ruelle zeta function	232
10.3.1. Resonances of the Lie derivative	233
10.3.2. Wavefront set of the resolvent	235
10.3.3. The super flat trace and the Ruelle zeta function	244
10.4. Ruelle zeta function at zero for surfaces	248
10.4.1. Topology of surfaces	249
10.4.2. Resonant forms for the resonance zero	254
10.5. Counting closed geodesics on hyperbolic surfaces	260
10.5.1. The first resonance on forms	260
10.5.2. Proof of the prime geodesic theorem	261
10.6. Bibliographical notes	263
11. Livšic's theory	265
11.1. The Abelian Livšic theory	265
11.1.1. The exact Abelian Livšic Theorem	266
11.1.2. The approximate Abelian Livšic Theorem	268
11.1.3. Conjugacy of flows	274
11.1.4. Exercises	274

11.2. The non-Abelian Livšic theory	275
11.2.1. Parry's free monoid. Transitivity group	275
11.2.2. The non-Abelian Livšic Theorem	277
11.2.3. Exercises	280
11.3. Bibliographical notes	282
12. Isometric extensions of Anosov flows	283
12.1. Geometry and dynamics of isometric extensions	283
12.1.1. Isometric extensions	283
12.1.2. Holonomies	286
12.1.3. Transitivity group	288
12.2. Ergodicity	289
12.2.1. Main result	289
12.2.2. Orbit space structure	289
12.2.3. Proof of ergodicity	294
12.3. Mixing	297
12.3.1. Main result	297
12.3.2. Proof of mixing	298
12.4. Principal bundles	300
12.4.1. General statement	300
12.4.2. Finite covers	302
12.5. Bibliographical notes	304
Part III. Riemannian geometry and geodesic flows	305
13. Geodesic flows	307
13.1. Geometry of the unit tangent bundle	307
13.1.1. Geometry of the unit tangent bundle	307
13.1.2. Differential operators	313
13.2. Conjugate points and related notions	317
13.2.1. Twist property of the vertical bundle	317
13.2.2. Conjugate points	318
13.3. Anosov geodesic flows	321
13.3.1. Negatively curved metrics	321
13.3.2. Anosov metrics	323
13.3.3. Exercises	333
13.4. Bibliographical notes	333
14. Fourier analysis on the unit sphere bundle	335
14.1. Symmetric tensors	335
14.1.1. Symmetric tensors in a Euclidean vector space	335
14.1.2. Symmetric tensors on a Riemannian manifold	343
14.1.3. Exercises	346
14.2. Fourier analysis in the fibers	347

14.2.1. Trivial line bundle	347
14.2.2. Twisted Fourier analysis	349
14.3. The Pestov identity	352
14.3.1. Non-twisted Pestov identity	352
14.3.2. Twisted Pestov identity	354
14.4. Bibliographical notes	355
15. Geometry of surfaces	357
15.1. Complex structure	357
15.1.1. Riemann surfaces	357
15.1.2. Isothermal coordinates	360
15.1.3. Teichmüller and moduli spaces	363
15.1.4. Exercises	365
15.2. Unit tangent bundle	365
15.2.1. Geometry of the unit tangent bundle	365
15.2.2. The transport algebra	372
15.2.3. Exercises	374
15.3. Bibliographical notes	375
Part IV. Applications	377
16. Geodesic X-ray transform	379
16.1. Geodesic X-ray transform	379
16.1.1. Definition. First properties	379
16.1.2. Cohomological equations. Tensor tomography	381
16.1.3. Twisted cohomological equations	384
16.2. The normal operator	385
16.2.1. Definition. First properties	385
16.2.2. Main properties of the normal operator	391
16.2.3. Stability estimates for the X-ray transform	395
16.3. Bibliographical notes	396
17. The marked length spectrum	397
17.1. The Burns-Katok conjecture	397
17.1.1. Marked length spectrum rigidity	397
17.1.2. Isospectral problem. Spectral and linear rigidity	399
17.1.3. Conformal case	401
17.2. Local rigidity in higher dimensions	402
17.2.1. Local geometry of the space of metrics	402
17.2.2. Geodesic stretch	405
17.2.3. Proof of local rigidity	407
17.3. Global rigidity on surfaces	408
17.3.1. Surjectivity of the transport algebra	409
17.3.2. Recovering the unmarked complex structure	410

17.3.3. Marked complex structure. End of the proof	416
17.4. Bibliographical notes	417
18. The Wilson loop operator	419
18.1. Geodesic Wilson loop operator	419
18.1.1. Definition. Main result	419
18.1.2. Application to isospectral connections	421
18.2. Polynomial maps between spheres	422
18.2.1. Definition	423
18.2.2. Hopf's construction	423
18.2.3. Wood's theorem	424
18.3. Injectivity of the Wilson loop operator	425
18.3.1. Injectivity on line bundles	425
18.3.2. Global injectivity under low-rank assumption	428
18.3.3. Counter-example on even-dimensional manifolds	429
18.4. Bibliographical notes	432
19. Frame flows in negative curvature	435
19.1. Frame flow ergodicity	435
19.1.1. Definition. First properties	435
19.1.2. Ergodicity	436
19.1.3. Exercises	439
19.2. Curvature tensors	439
19.2.1. Definition	439
19.2.2. Pinched curvature	440
19.3. Flow-invariant sections	442
19.3.1. Topological reductions	442
19.3.2. Invariant sections on the normal bundle	445
19.4. Absence of invariant geometric structures	447
19.4.1. Normal twisted conformal Killing tensors	448
19.4.2. The Pestov inequality	448
19.4.3. Non-existence of invariant structures	452
19.5. Bibliographical notes	454
Part V. Appendix	455
20. Elements of geometry	457
20.1. Connections on vector bundles	457
20.1.1. Definition. Elementary properties	457
20.1.2. Functoriality	462
20.1.3. The Ambrose-Singer formula	464
20.1.4. Exercises	466
20.2. Riemannian geometry	467
20.2.1. Basic notions	467

20.2.2. Special structures in negative curvature	472
20.2.3. Isometric actions	473
20.3. Symplectic geometry	476
20.3.1. Symplectic vector spaces	476
20.3.2. Symplectic manifolds. Definition and first properties	478
20.3.3. Hamiltonian vector field	479
20.3.4. Poisson bracket	480
21. Elements of analysis	483
21.1. Abstract functional analysis	483
21.1.1. Fréchet spaces	483
21.1.2. Main theorems	484
21.1.3. Compact operators	485
21.1.4. Fredholm operators	489
21.1.5. Exercises	494
21.2. Distribution theory	495
21.2.1. Distributions	495
21.2.2. Convergence of distributions	497
21.2.3. Convolution	499
21.2.4. Fourier transform	500
21.2.5. Integration theory	501
21.3. Spectral theory	502
Index	505
Bibliography	507

FOREWORD

I initiated this book project during the second Covid lockdown in France, in the fall of 2020, at a time when the world seemed to have come to a halt. Universities, libraries, and offices were closed, and academic life had largely shifted to virtual platforms. With only online seminars taking place and travel restrictions in effect, I found myself with an unexpected amount of uninterrupted time for reflection and writing. My initial goal was to produce a concise, 100-page survey on geometric inverse problems and microlocal analysis, intended as a reference for researchers and graduate students interested in these topics. However, I must confess that I largely failed in this task, as the so-called survey quickly grew out of control and exceeded 300 pages. I then decided to turn it into book which, if you are reading these lines, is now likely in your hands.

Beyond the unusual circumstances of its inception, this book also owes much to the series of lectures I delivered on microlocal analysis and hyperbolic dynamics at Sorbonne Université for master's students and at the Collège de France between 2021 and 2023. These lectures provided an opportunity to refine many of the ideas presented here and to shape their exposition in a way that would be accessible to a broad mathematical audience. Later, I had the privilege of further developing these topics in courses and seminars at Northwestern University, the University of Zurich, Fortaleza, and Montevideo. I am deeply grateful to the institutions and individuals who extended these invitations. Some of these lectures were recorded and are available online on my webpage. I will also maintain on the same page a list of errata.

I am grateful to all those who contributed, in one way or another to this book, whether in informal discussions or by providing me with proofs or exercises. I thank Louis Beaufort, Jan Bohr, Colin Guillarmou, Hedong Hou, Tristan Humbert, Maxime Ingremeau, Alessandro Morescalchi, Sebastián Muñoz-Thon, Gabriel Paternain, Amie Wilkinson and Yuhui Zhu for valuable feedback on this manuscript. This book project was also made possible through the support of the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement no. 101162990—ADG).

I am deeply indebted to my long-time collaborator, Mihajlo Cekić, for the fruitful projects we have worked on together, some of which are presented in the latter part of this book. He also graciously provided proofs for certain lemmas, for which I am sincerely grateful. I am thankful to Yann Chaubet for contributing a chapter on dynamical zeta functions and for his valuable feedback on earlier drafts. I would also like to

acknowledge the anonymous referees, whose meticulous review of the manuscript was nothing short of remarkable. Their extensive 48-page report (!), filled with detailed suggestions and constructive criticism, has undoubtedly helped improve the presentation and clarity of this book. On a more personal note, I owe a profound debt of gratitude to Marie, whose unwavering support and encouragement have been invaluable throughout this long journey—none of this would have been possible without you.

Finally, I dedicate this book to the memory of my father, who passed away during the final stages of its writing in October 2024. Though he will never hold it in his hands, I have no doubt he would have been proud. I hope this book stands as a small tribute to his memory.

Paris, October 2020 – June 2025.