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STABILIZATION OF NAVIER-STOKES EQUATION

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STABILIZATION OF NAVIER-STOKES EQUATION

by

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Abstract. — We survey here a few recent results and methods to stabilization of equilibrium solutions to Navier-Stokes in 2-D and 3-D.

1. The stabilization problem

Consider the Navier-Stokes equation in a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$,

$$(1.1) \quad \begin{aligned} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla)y &= f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \nabla \cdot y &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ y &= 0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ y(0) &= y_0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \end{aligned}$$

where $f_e \in (L^2(\mathcal{O}))^d$, $\nabla \cdot f_e = 0$, $f_e \cdot n = 0$.

Here n is the unit normal and is directed toward the exterior of $\partial\mathcal{O}$.

Let $y_e \in (H^2(\mathcal{O}))^d$ be an equilibrium solution to (1.1), that is,

$$(1.2) \quad \begin{aligned} -\nu \Delta y_e + (y_e \cdot \nabla)y_e &= f_e + \nabla p_e && \text{in } \mathcal{O}, \\ \nabla \cdot y_e &= 0 && \text{in } \mathcal{O}, \quad y_e = 0 \quad \text{on } \partial\mathcal{O}. \end{aligned}$$

1.1. Internal stabilization. — Let $\mathcal{O}_0 \subset \mathcal{O}$ be an open subdomain of $\mathcal{O}$ and consider the controlled system associated with (1.1)

$$(1.3) \quad \begin{aligned} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla)y &= f_e + \nabla p + \mathbb{1}_{\mathcal{O}_0} u \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \nabla \cdot y &\quad \text{in } \mathbb{R}^+ \times \mathcal{O}; \quad y = 0 \quad \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ y(0) &= y_0 \quad \text{in } \mathcal{O}, \end{aligned}$$

where the controller u is in $L^2(0, \infty; (L^2(\mathcal{O}))^d)$.

Problem 1.1. — Find the controller u in feedback form, that is $u(t) = \phi(y(t) - y_e)$ such that the solution to the corresponding solution y to the closed loop system (1.3) satisfies for all y_0 in a neighborhood of y_e

$$(1.4) \quad \|y(t) - y_e\|_{(L^2(\mathcal{O}))^d} \leq C e^{-\gamma t} \|y_0 - y_e\|_{(L^2(\mathcal{O}))^d}, \quad \forall t \geq 0,$$

where $\gamma > 0$.

If we set $y - y_e \rightarrow y$, Problem 1.1 reduces to find $u = \phi(y)$ such that the solution y to the equation

$$(1.5) \quad \begin{aligned} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla)y + (y_e \cdot \nabla)y + (y \cdot \nabla)y_e &= \nabla p + \mathbb{1}_{\mathcal{O}_0} u, \quad t \geq 0, \\ \nabla \cdot y &= 0 \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ y &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ y(0, x) &= y_0(x) - y_e(x) = y^0(x), \quad x \in \mathcal{O}. \end{aligned}$$

satisfies

$$(1.6) \quad \|y(t)\|_{(L^2(\mathcal{O}))^d} \leq C e^{-\gamma t} \|y^0\|_{(L^2(\mathcal{O}))^d}, \quad \forall t \geq 0.$$

We use the standard formalism to represent the Navier-Stokes equations as infinite-dimensional differential equations (see, e.g., [9], [20], [21]). That is we set

$$\begin{aligned} H &= \{y \in (L^2(\mathcal{O}))^d; \nabla \cdot y = 0 \quad \text{in } \mathcal{O}, \quad y \cdot n = 0 \quad \text{on } \partial\mathcal{O}\}, \\ Ay &= -P(\Delta y), \quad \forall y \in D(A) = y \in H \cap (H_0^1(\mathcal{O}))^d \cap (H^2(\mathcal{O}))^d, \\ A_0 y &= P((y_e \cdot \nabla)y + (y \cdot \nabla)y_e), \quad D(A_0) = H \cap (H_0^1(\mathcal{O}))^d, \\ By &= P((y \cdot \nabla)y), \end{aligned}$$

where $P : (L^2(\mathcal{O}))^d \rightarrow H$ is the Leray projector.

We may rewrite (1.5) as

$$(1.7) \quad \frac{dy}{dt} + \nu Ay + A_0 y + By = P(\mathbb{1}_{\mathcal{O}_0} u), \quad t \geq 0, \quad y(0) = y^0,$$

or, in a more compact form,

$$(1.8) \quad \frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0} u), \quad t \geq 0, \quad y(0) = y^0,$$

where $\mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H$ is the so called Oseen-Stokes operator

$$(1.9) \quad \mathcal{A} = \nu A + A_0, \quad D(\mathcal{A}) = D(A).$$

Then the *internal stabilization problem* reduces to find a feedback controller $u = \phi(y)$ such that the corresponding solution y to (1.8), that is,

$$(1.10) \quad \frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0} \phi(y)), \quad \forall t \geq 0, \quad y(0) = y_0,$$

satisfies

$$(1.11) \quad |y\tau|_H \leq Ce^{-\gamma t} |y_0|_H, \quad \forall t \geq 0,$$

for $\gamma > 0$ and all y_0 in a neighborhood of the origin. Here and everywhere in the following, $|\cdot|_H$ is the norm of the space H and $(\cdot, \cdot)_H$ is the corresponding scalar product.

1.2. Boundary stabilization. — Consider the boundary control system associated with (1.1)

$$(1.12) \quad \begin{aligned} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y &= f_e + \nabla p \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \nabla \cdot y &= 0 \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \quad y = u \quad \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ y(0) &= y_0 \quad \text{in } \mathcal{O}. \end{aligned}$$

Problem 1.2. — Find a boundary controller u in the feedback form $u = \psi(y - y_e)$ such that the corresponding solution y to (1.5) satisfies (1.4) for all y_0 in a neighborhood of y_e .

Equivalently, the solution y to

$$(1.13) \quad \begin{aligned} \frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + (y \cdot \nabla) y_e + (y_e \cdot \nabla) y &= \nabla p \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ y &= u \quad \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \quad y(0) = y^0 - y_e, \\ \nabla \cdot y &= 0 \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \end{aligned}$$

where $u = \psi(y)$, satisfies (1.4).

If u is tangential, that is $u \cdot n = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$, then the stabilization is said to be *tangential* while, if $u \cdot \tau = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$ (where τ is the tangent vector to $\partial\mathcal{O}$), the stabilization is called *normal*.

Denote by $D : (L^2(\partial\mathcal{O}))^d \rightarrow H$ the Dirichlet map defined by

$$(1.14) \quad -\nu \Delta(Du) + (y_e \cdot \nabla) Du + (Du \cdot \nabla) y_e + k Du = \nabla p \quad \text{in } \mathcal{O}, \quad Du = u \quad \text{on } \partial\mathcal{O},$$

where $k > 0$ is sufficiently large but fixed.

It turns out that D is well defined on the space of all $u \in (L^2(\partial\mathcal{O}))^d$ such that $u \cdot n = 0$ on $\partial\mathcal{O}$ and that D is continuous from $(H^s(\partial\mathcal{O}))^d \rightarrow (H^{s+\frac{1}{2}}(\mathcal{O}))^d \cap H$ if $s \geq \frac{1}{2}$. (See Theorem A.2.1 in [1].) Then (1.13) reduces to

$$(1.15) \quad \frac{dy}{dt} + \mathcal{A}(y - Du) + By = kDu, \quad t \geq 0, \quad y(0) = y^0.$$

If we denote by $\widetilde{\mathcal{A}}$ the extension, by transposition, $\widetilde{\mathcal{A}} : H \rightarrow (D(\mathcal{A}^*))'$ with respect to H as pivot space of the original operator $\mathcal{A}$, that is, $(\widetilde{\mathcal{A}}y, z) = (y, \mathcal{A}^*z)$, for all $z \in D(\mathcal{A})$, we can write (1.15) as

$$(1.16) \quad \frac{dy}{dt} + \widetilde{\mathcal{A}}y + By = kDu + \widetilde{\mathcal{A}}Du, \quad t \geq 0, \quad y(0) = y^0,$$

and so, the tangential stabilization problem reduces to find a feedback controller $u = \psi(y)$ such that the solution y to (1.16) satisfies (1.4) for all y in a neighborhood of the origin.

It is obvious that the solution y to the Cauchy problem is taken here in a mild sense

$$(1.17) \quad y\tau = e^{-\mathcal{A}\tau}y^0 - \int_0^\tau e^{-\widetilde{\mathcal{A}}(\tau-s)}(By(s) + kDu + \widetilde{\mathcal{A}}Du(s))ds, \quad t \geq 0.$$

Of course, if $\frac{d}{dt} Du \in L^2_{\text{loc}}(0, \infty; H)$, we may rewrite (1.17) as

$$(1.18) \quad \begin{aligned} y\tau &= Du\tau + e^{-\mathcal{A}\tau}(y^0 - Du(0)) \\ &\quad - \int_0^\tau e^{-\mathcal{A}(\tau-s)} \left(By(s) + kDu(s) - \frac{d}{ds} Du(s) \right) ds, \quad \forall t \geq 0. \end{aligned}$$

The functional representation of system (1.13) with normal boundary controller is a more delicate problem.

1.3. Main results

Theorem 1.3 (Barbu & Triggiani 2004). — *There is a feedback controller*

$$(1.19) \quad u = \sum_{i=1}^M (R(y - y_e), \psi_i)_{(L^2(\mathcal{O}_0))^d} \psi_i, \quad R \in (L^2(\mathcal{O})),$$

which stabilizes exponentially y_e for

$$\|y_0 - y_e\|_W \leq \rho, \quad W = (H^{\frac{1}{2}}(\mathcal{O}))^d.$$

Here M^* is dependent of the multiplicity of eigenvalues λ_j of the Oseen-Stokes operator $\text{Re } \lambda_j \leq 0$, $j = 1, \dots, N$. The functions ψ_j are linear combinations of eigenfunctions φ_j^* .