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ON CUBIC BIRATIONAL MAPS OF $\mathbb{P}_{\mathbb{C}}^3$

BY JULIE DÉSERTI & FRÉDÉRIC HAN

ABSTRACT. — We study the birational maps of $\mathbb{P}_{\mathbb{C}}^3$. More precisely we describe the irreducible components of the set of birational maps of bidegree $(3, 3)$ (resp. $(3, 4)$, resp. $(3, 5)$).

RÉSUMÉ (*Sur les transformations birationnelles cubiques de $\mathbb{P}_{\mathbb{C}}^3$*)

Nous étudions les transformations birationnelles de $\mathbb{P}_{\mathbb{C}}^3$. Plus précisément nous décrivons les composantes irréductibles de l'ensemble des transformations birationnelles de $\mathbb{P}_{\mathbb{C}}^3$ de bidegré $(3, 3)$ (resp. $(3, 4)$, resp. $(3, 5)$).

1. Introduction

The Cremona group, denoted $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$, is the group of birational maps of $\mathbb{P}_{\mathbb{C}}^n$ into itself. If $n = 2$ a lot of properties have been established (see [4, 9] for example). As far as we know the situation is much more different for $n \geq 3$ (see [14, 5] for example). If ψ is an element of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ then $\deg \psi = \deg \psi^{-1}$. It is not the case in higher dimensions; if ψ belongs to $\text{Bir}(\mathbb{P}_{\mathbb{C}}^3)$ we only have the inequality $\deg \psi^{-1} \leq (\deg \psi)^2$ so one introduces the bidegree of ψ as the pair $(\deg \psi, \deg \psi^{-1})$. For $n = 2$, $\mathfrak{Bir}_d(\mathbb{P}_{\mathbb{C}}^2)$ is the set of birational maps of the complex projective plane of degree d ; for $n \geq 3$ denote by $\text{Bir}_{d,d'}(\mathbb{P}_{\mathbb{C}}^n)$

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the set of elements of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ of bidegree (d, d') , and by $\mathfrak{Bir}_d(\mathbb{P}_{\mathbb{C}}^n)$ the union $\bigcup_{d'} \text{Bir}_{d,d'}(\mathbb{P}_{\mathbb{C}}^n)$. The set $\mathfrak{Bir}_d(\mathbb{P}_{\mathbb{C}}^n)$ inherits a structure of algebraic variety as a locally closed subspace a projective space ([3, Lemma 2.4, Proposition 2.15]), and we will always consider it with the Zariski topology ([8, 17]).

The varieties $\mathfrak{Bir}_2(\mathbb{P}_{\mathbb{C}}^2)$ and $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^2)$ are described in [6]: $\mathfrak{Bir}_2(\mathbb{P}_{\mathbb{C}}^2)$ is smooth, and irreducible in the space of quadratic rational maps of the complex projective plane whereas $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^2)$ is irreducible, and rationally connected. Besides, $\mathfrak{Bir}_d(\mathbb{P}_{\mathbb{C}}^2)$ is not irreducible as soon as $d > 3$ (see [2]). In [7] Cremona studies three types of generic elements of $\mathfrak{Bir}_2(\mathbb{P}_{\mathbb{C}}^3)$. Then there were some articles on the subject, and finally a precise description of $\mathfrak{Bir}_2(\mathbb{P}_{\mathbb{C}}^3)$; the left-right conjugacy is the following one

$$\text{PGL}(4; \mathbb{C}) \times \text{Bir}(\mathbb{P}_{\mathbb{C}}^3) \times \text{PGL}(4; \mathbb{C}) \rightarrow \text{Bir}(\mathbb{P}_{\mathbb{C}}^3), \quad (A, \psi, B) \mapsto A\psi B^{-1}.$$

Pan, Ronga and Vust give quadratic birational maps of $\mathbb{P}_{\mathbb{C}}^3$ up to left-right conjugacy, and show that there are only finitely many biclasses ([15, Theorems 3.1.1, 3.2.1, 3.2.2, 3.3.1]). In particular they show that $\mathfrak{Bir}_2(\mathbb{P}_{\mathbb{C}}^3)$ has three irreducible components of dimension 26, 28, 29; the component of dimension 26 (resp. 28, resp. 29) corresponds to birational maps of bidegree $(2, 4)$ (resp. $(2, 3)$, resp. $(2, 2)$). We will see that the situation is slightly different for $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^3)$; in particular we cannot expect such an explicit list of biclasses because there are infinitely many of biclasses (already the dimension of the family $\mathcal{E}_2$ of the classic cubo-cubic example is 39 that is strictly larger than $\dim(\text{PGL}(4; \mathbb{C}) \times \text{PGL}(4; \mathbb{C})) = 30$). That's why the approach is different.

We do not have such a precise description of $\mathfrak{Bir}_d(\mathbb{P}_{\mathbb{C}}^3)$ for $d \geq 4$. Nevertheless we can find a very fine and classical contribution for $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^3)$ due to Hudson ([11]); in the appendix we reproduce Table VI of [11]. Hudson introduces there some invariants to establish her classification. But it gives rise to many cases, and we also find examples where invariants take values that do not appear in her table. We do not know references explaining how her families fall into irreducible components of $\text{Bir}_{3,d}(\mathbb{P}_{\mathbb{C}}^3)$ so we focus on this natural question.

DEFINITION. — An element ψ of $\text{Bir}_{3,d}(\mathbb{P}_{\mathbb{C}}^3)$ is *ruled* if the strict transform of a generic plane under ψ^{-1} is a ruled cubic surface.

Denote by $\text{ruled}_{3,d}$ the set of $(3, d)$ ruled maps; we detail it in Lemma 2.3. Let us remark that there are no ruled birational maps of bidegree $(3, d)$ with $d \geq 6$.

We describe the irreducible components of $\text{Bir}_{3,d}(\mathbb{P}_{\mathbb{C}}^3)$ for $3 \leq d \leq 5$. Let us recall that the inverse of an element of $\text{Bir}_{3,2}(\mathbb{P}_{\mathbb{C}}^3)$ is quadratic and so treated in [15].

THEOREM A. — Assume that $2 \leq d \leq 5$. The set $\text{ruled}_{3,d}$ is an irreducible component of $\text{Bir}_{3,d}(\mathbb{P}_{\mathbb{C}}^3)$.

In bidegree $(3, 3)$ (resp. $(3, 4)$) there is only an other irreducible component; in bidegree $(3, 5)$ there are three others.

The set $\overline{\text{ruled}_{3,3}}$ intersects the closure of any irreducible component of $\text{Bir}_{3,4}(\mathbb{P}_{\mathbb{C}}^3)$ (the closures being taken in $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^3)$).

NOTATIONS 1.1. — Consider a dominant rational map ψ from $\mathbb{P}_{\mathbb{C}}^3$ into itself. For a generic line ℓ , the preimage of ℓ by ψ is a complete intersection Γ_{ℓ} ; define the scheme $\mathcal{C}_2$ to be the union of the irreducible components of Γ_{ℓ} supported in the base locus of ψ . Define $\mathcal{C}_1$ by liaison from $\mathcal{C}_2$ in Γ_{ℓ} . Remark that if ψ is birational, then $\mathcal{C}_1 = \psi_*^{-1}(\ell)$. Let us denote by $\mathbf{p}_a(\mathcal{C}_i)$ the arithmetic genus of $\mathcal{C}_i$.

It is difficult to find a uniform approach to classify elements of $\mathfrak{Bir}_3(\mathbb{P}_{\mathbb{C}}^3)$. Nevertheless in small genus we succeed to obtain some common detailed results; before stating them, let us introduce some notations.

Let us remark that the inequality $\deg \psi^{-1} \leq (\deg \psi)^2$ mentioned previously directly follows from

$$(\deg \psi)^2 = \deg \psi^{-1} + \deg \mathcal{C}_2.$$

PROPOSITION B. — Let ψ be a $(3, d)$ birational map of $\mathbb{P}_{\mathbb{C}}^3$.

Assume that ψ is not ruled, and $\mathbf{p}_a(\mathcal{C}_1) = 0$, i.e., $\mathcal{C}_1$ is smooth. Then

- $d \leq 6$;
- and $\mathcal{C}_2$ is a curve of degree $9 - d$, and arithmetic genus $9 - 2d$.

Suppose $\mathbf{p}_a(\mathcal{C}_1) = 1$, and $2 \leq d \leq 6$. Then

- there exists a singular point p of $\mathcal{C}_1$ independent of the choice of $\mathcal{C}_1$;
- if $d \leq 4$, all the cubic surfaces of the linear system Λ_{ψ} are singular at p ;
- the curve $\mathcal{C}_2$ is of degree $9 - d$, of arithmetic genus $10 - 2d$, and lies on a unique quadric Q ; more precisely $\mathcal{I}_{\mathcal{C}_2} = (Q, S_1, \dots, S_{d-2})$ where the S_i 's are independent cubics modulo Q .

We denote by $\text{Bir}_{3,d,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ the subset of non-ruled $(3, d)$ birational maps such that $\mathcal{C}_2$ is of degree $9 - d$, and arithmetic genus $\mathbf{p}_2$. One has the following statement:

THEOREM C. — If $\mathbf{p}_2 \in \{3, 4\}$, then $\text{Bir}_{3,3,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is non-empty, and irreducible; $\text{Bir}_{3,3,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is empty as soon as $\mathbf{p}_2 \notin \{3, 4\}$.

If $\mathbf{p}_2 \in \{1, 2\}$, then $\text{Bir}_{3,4,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is non-empty, and irreducible; $\text{Bir}_{3,4,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is empty as soon as $\mathbf{p}_2 \notin \{1, 2\}$.

The set $\text{Bir}_{3,5,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is empty as soon as $\mathbf{p}_2 \notin \{-1, 0, 1\}$ and

- if $\mathbf{p}_2 = -1$, then $\text{Bir}_{3,5,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is non-empty, and irreducible;
- if $\mathbf{p}_2 = 0$, then $\text{Bir}_{3,5,\mathbf{p}_2}(\mathbb{P}_{\mathbb{C}}^3)$ is non-empty, and has two irreducible components;