Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

LOCAL MOTIVIC INVARIANTS OF RATIONAL FUNCTIONS IN TWO VARIABLES

Pierrette Cassou-Noguès & Michel Raibaut

Tome 154
Fascicule 2

2026

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 375-420

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 2, tome 154, juin 2026

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 175 € ($ 262),
avec supplément papier : Europe 266 €, hors Europe 307 € ($ 460)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
154 (2), 2026, p. 375-420

LOCAL MOTIVIC INVARIANTS OF RATIONAL FUNCTIONS IN TWO VARIABLES

by Pierrette Cassou-Noguès & Michel Raibaut

Abstract. — Let P and Q be two polynomials in two variables with coefficients in an algebraic closed field of characteristic zero. We consider the rational function $f = P/Q$. For an indeterminacy point x of f and a value c, we compute the motivic Milnor fiber $S_{f,\mathrm{x},c}$ in terms of some motives associated to the faces of the Newton polygons appearing in the Newton algorithms of $P - cQ$ and Q at x, without any condition of nondegeneracy or convenience. In the complex setting, assuming for any $(a, b) \in \mathbb{C}^2$ that x is a smooth or an isolated critical point of $aP + bQ$, and the curves $P = 0$ and $Q = 0$ do not have common irreducible component, we prove that the topological bifurcation set $\mathscr{B}^{\mathrm{top}}_{f,\mathrm{x}}$ is equal to the motivic bifurcation set $\mathscr{B}^{\mathrm{mot}}_{f,\mathrm{x}}$ and they are computed from the Newton algorithm.

Texte reçu le 28 août 2024, modifié le 7 mai 2025, accepté le 9 juin 2025.

Pierrette Cassou-Noguès, Institut de Mathématiques de Bordeaux, UMR 5251, Université de Bordeaux, 350, Cours de la Libération, 33405, Talence Cedex, France • *E-mail :* `Pierrette.Cassou-Nogues@math.u-bordeaux.fr`
Michel Raibaut, Laboratoire de Mathématiques, UMR 5127, Université Savoie Mont-Blanc, Bâtiment Chablais, Campus Scientifique, Le Bourget du Lac, 73376 Cedex, France • *E-mail :* `Michel.Raibaut@univ-smb.fr` • *Url :* `https://raibautm.perso.math.cnrs.fr/`

Mathematical subject classification (2020). — 14B05, 14B07, 14E18, 32S15, 32S30; 32S50, 32S55.

Key words and phrases. — Algebraic geometry, singularity theory, motivic integration, motivic Milnor fibers.

The first author is partially supported by the Spanish grants: PID2020-114750GB-C31 and PID2020-114750GB-C32. The second author was partially supported by the ANR grant ANR-15-CE40-0008 (Défigéo).

doi:10.24033/bsmf.2924

Résumé (*Invariants motiviques locaux des fonctions rationnelles en deux variables*). — Soient P et Q deux polynômes en deux variables à coefficients dans un corps algébriquement clos de caractéristique zéro. On considère la fraction rationnelle $f = P/Q$. Pour un point d'indétermination x de f et une valeur c, nous calculons la fibre de Milnor motivique $S_{f,\mathrm{x},c}$ en termes de motifs associés aux faces des polygones de Newton apparaissant dans les algorithmes de Newton de $P - cQ$ et Q en x, sans aucune condition de non-dégénérescence ou de commodité. Dans le cadre complexe, en supposant que pour tout $(a, b) \in \mathbb{C}^2$, le point x est un point lisse ou un point critique isolé de $aP + bQ$, et que les courbes $P = 0$ et $Q = 0$ n'ont pas de composantes irréductibles en commun, nous prouvons que l'ensemble de bifurcation topologique $\mathscr{B}^{\mathrm{top}}_{f,\mathrm{x}}$ est égal à l'ensemble de bifurcation motivique $\mathscr{B}^{\mathrm{mot}}_{f,\mathrm{x}}$ et qu'ils sont calculés par l'algorithme de Newton.

Introduction

Let P and Q be two polynomials in $\mathbb{C}[x_1, \ldots, x_d]$ with $d \geq 1$. We denote by I the common zero set of P and Q and by f the rational function P/Q well defined on $\mathbb{A}^d_{\mathbb{C}} \setminus I$ to $\mathbb{P}^1_{\mathbb{C}}$. For an indeterminacy point $\mathrm{x} \in I$ and a value c in $\mathbb{P}^1_{\mathbb{C}}$, Gusein-Zade, Luengo and Melle-Hernández in [22, 23], proved that f is a Milnor C^∞ locally trivial fibration on some $B(\mathrm{x}, \varepsilon) \setminus I$ for ε small enough, over a punctured neighborhood of c. They defined a *Milnor fiber of* f *at* x *for the value* c, denoted by $F_{\mathrm{x},c}$, endowed with a monodromy action $T_{\mathrm{x},c}$ induced by the fibration. If the fibration can be extended as a trivial fibration over a neighborhood of c, then c is called *typical value*, and *atypical value* otherwise. They proved that the set of atypical values is finite and called it, the *topological bifurcation set* $\mathscr{B}^{\mathrm{top}}_{f,\mathrm{x}}$ of the germ of f at x. Later, several authors (for instance [36, 37], [2, 3], [16], [40], [29], [30]) studied singularities of meromorphic functions, locally or at infinity. This is related to the study of pencils $aP + bQ = 0$ (for instance [27], [32], [36], [10]).

Working over a characteristic zero field $\mathbf{k}$, with P and Q polynomials with coefficients in $\mathbf{k}$, using the motivic integration theory, introduced by Kontsevich in [24], and more precisely constructions of Denef–Loeser in [11, 12, 14] and Guibert–Loeser–Merle in [19], the second author in [34] and Nguyen–Takeuchi in [31] (see also [17] and [40]) defined a *motivic Milnor fiber* of f at x and a value c, denoted by $S_{f,\mathrm{x},c}$ (Section 3.3). It is an element of $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{G}_m}$, a modified Grothendieck ring of varieties over $\mathbf{k}$ endowed with an action of the multiplicative group $\mathbb{G}_m$ of $\mathbf{k}$. When $\mathbf{k}$ is the field of complex numbers, it follows from Denef–Loeser results that the motive $S_{f,\mathrm{x},c}$ is a "motivic" incarnation of the topological Milnor fiber $F_{\mathrm{x},c}$ endowed with its monodromy action $T_{\mathrm{x},c}$. For instance, the motive $S_{f,\mathrm{x},c}$ realizes on the Euler characteristic of $F_{\mathrm{x},c}$ and the classes in appropriate Grothendieck rings of the associated mixed Hodge modules and limit mixed Hodge structures (Theorem 3.8). In [34], the second author proved that the set of values c such that $S_{f,\mathrm{x},c} \neq 0$ is a finite set, denoted by $\mathscr{B}_{f,\mathrm{x}}^{\mathrm{mot}}$ and called *motivic bifurcation set* (Definition 4.8).

In this article, similarly to [4, 5], assuming $\mathbf{k}$ algebraically closed, we investigate the case $d = 2$ in full generality (namely without any assumptions of convenience or nondegeneracy w.r.t any Newton polygon) using ideas of Guibert in [18], Guibert–Loeser–Merle in [21, 20], and the works of the first author and Veys in the case of an ideal of $\mathbf{k}[[x, y]]$ in [6, 7].

More precisely, in Theorem 3.9, we give an inductive expression of the motive $S_{f,\mathrm{x},c}$, in terms of some motives associated to faces of the Newton polygons appearing in the Newton algorithm of $P - cQ$ and Q recalled in Section 1. Furthermore, in Theorem 4.7 and Theorem 4.10, in the smooth case or isolated singularity case, assuming that the curves $P = 0$ and $Q = 0$ do not have a common irreducible component, we have the equality $\mathscr{B}_{f,\mathrm{x}}^{\mathrm{top}} = \mathscr{B}_{f,\mathrm{x}}^{\mathrm{Newton}} = \mathscr{B}_{f,\mathrm{x}}^{\mathrm{mot}}$, where $\mathscr{B}_{f,\mathrm{x}}^{\mathrm{Newton}}$ is the Newton bifurcation set of f, which is defined in an algorithmic way (Definition 4.4) using the Newton algorithm of $P - \mathbf{c}Q$ and Q with $\mathbf{c}$ an indeterminate. Up to factorization of polynomials, this gives, in particular, an algorithm to compute the usual topological bifurcation set $\mathscr{B}_{f,\mathrm{x}}^{\mathrm{top}}$ without assumptions of convenience or nondegeneracy toward any Newton polygon.

1. Newton algorithm

Let $\mathbb{N}$ be the set of non-negative integers. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, with $\mathbb{G}_m$ its multiplicative group. Recall first standard notation used throughout this article.

Definition 1.1. — For any $E \subset \mathbb{N}^2$, we denote by $\Delta(E)$ the smallest convex set containing

$$E + (\mathbb{R}_{\geq 0})^2 = \left\{ \mathrm{v} + \mathrm{w} \,\middle|\, \mathrm{v} \in E, \mathrm{w} \in (\mathbb{R}_{\geq 0})^2 \right\}.$$

A subset $\Delta \subset (\mathbb{R}_{\geq 0})^2$ is called *Newton diagram* if $\Delta = \Delta(E)$ for some subset $E \subset \mathbb{N}^2$. The smallest set E_0 of $\mathbb{N}^2$ such that $\Delta = \Delta(E_0)$ is called *set of vertices* of Δ; it is a finite set.

Let $\Delta \subset (\mathbb{R}_{\geq 0})^2$ be a Newton diagram and $E_0 = \{\mathrm{v}_0, \ldots, \mathrm{v}_d\}$ be its set of vertices with for any $i \in [\![0, d]\!] := \{0, \ldots, d\}$, $\mathrm{v}_i = (a_i, b_i) \in \mathbb{N}^2$. By convexity, we order E_0 such that for any $i \in [\![1, d]\!]$, we have $a_{i-1} < a_i$ and $b_{i-1} > b_i$ and we write $v_i > v_{i-1}$. For such i, we denote by S_i the segment $[\mathrm{v}_{i-1}, \mathrm{v}_i]$ and by l_{S_i} the line supporting S_i. We define the *Newton polygon* of Δ as

$$\mathcal{N}(\Delta) = \{S_i\}_{i\in[\![1,d]\!]} \cup \{\mathrm{v}_i\}_{i\in[\![0,d]\!]},$$