

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

A HENSTOCK-KURZWEIL TYPE INTEGRAL ON ONE-DIMENSIONAL INTEGRAL CURRENTS

Antoine Julia

Tome 148
Fascicule 2

2020

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 283-319

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 2, tome 148, juin 2020

Comité de rédaction

Christine BACHOC	Laurent MANIVEL
Yann BUGEAUD	Julien MARCHÉ
Jean-François DAT	Kieran O'GRADY
Clothilde FERMANIAN	Emmanuel RUSS
Pascal HUBERT	Christophe SABOT

Marc HERZLICH (Dir.)

Diffusion

Maison de la SMF	AMS
Case 916 - Luminy	P.O. Box 6248
13288 Marseille Cedex 9	Providence RI 02940
France	USA
commandes@smf.emath.fr	www.ams.org

Tarifs

Vente au numéro : 43 € (\$ 64)

Abonnement électronique : 135 € (\$ 202),

avec supplément papier : Europe 179 €, hors Europe 197 € (\$ 296)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie

75231 Paris Cedex 05, France

Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96

bulletin@smf.emath.fr • smf.emath.fr

© Société Mathématique de France 2020

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

A HENSTOCK-KURZWEIL TYPE INTEGRAL ON ONE-DIMENSIONAL INTEGRAL CURRENTS

BY ANTOINE JULIA

ABSTRACT. — We define a non-absolutely convergent integration method on integral currents of dimension 1 in Euclidean space. This integral is closely related to the Henstock-Kurzweil and Pfeffer integrals. Using it, we prove a generalized fundamental theorem of calculus on these currents. A detailed presentation of Henstock-Kurzweil integration is given in order to make the paper accessible to non-specialists.

RÉSUMÉ (*Une intégrale à la Henstock-Kurzweil sur les courants entiers de dimension 1*). — On définit une intégrale non absolument convergente sur les courants entiers euclidiens de dimension 1. Cette intégrale est inspirée des intégrales de Henstock-Kurzweil et de Pfeffer. Dans ce contexte, on démontre un théorème fondamental généralisé sur ces courants. On donne aussi une présentation détaillée de l'intégrale de Henstock et Kurzweil, pour les non-spécialistes.

1. Introduction

The goal of this paper is to present an integration method for functions defined on the support of an integral current of dimension 1 in Euclidean space. This method is inspired from the Henstock-Kurzweil (HK) and Pfeffer integrals

Texte reçu le 17 mai 2019, modifié le 10 octobre 2019, accepté le 29 octobre 2019.

ANTOINE JULIA, Dipartimento di Matematica, via Trieste 63, 35121 Padova, Italy •
E-mail : ajulia@math.unipd.it

Mathematical subject classification (2010). — 28A25, 26B20, 49Q15.

Key words and phrases. — Integration, Henstock-Kurzweil, Integral currents.

The author is supported by the University of Padova STARS Project “Sub-Riemannian Geometry and Geometric Measure Theory Issues: Old and New” (SUGGESTION), and by GNAMPA of INdAM (Italy) through the project “Rectifiability in Carnot Groups”.

[23, 20, 32], and, like them, tailored to the study of the fundamental theorem of calculus. The HK integral is a variant of the Riemann integral; it is more general than the Lebesgue integral, in the sense that all Lebesgue integrable functions are HK integrable, but *non-absolutely convergent*; there exist functions that are HK integrable, while their absolute value is not; in the same way that the series $\sum_k (-1)^k k^{-1}$ converges, while $\sum_k k^{-1}$ does not. A classical example of such functions is the derivative of $x \mapsto x^2 \sin(x^{-2})$ for $x \in (0, 1]$.

An integral that solves this problem was defined in two different ways by A. Denjoy and O. Perron in the early twentieth century, see [36]. R. Henstock and J. Kurzweil independently found a simpler, equivalent definition of this integral; we will use the HK formalism, but refer the reader to [18] for a comparison of these three approaches. For functions defined on a bounded interval $[a, b]$, the fundamental theorem of calculus of the integral of HK is the following:

THEOREM A ([33, Theorem 6.1.2]). — *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function that is differentiable everywhere, then its derivative f' is HK integrable on $[a, b]$ and there holds:*

$$(1) \quad (HK) \int_a^b f' = f(b) - f(a).$$

Note that several other integration methods have been defined for which Theorem A holds, see, in particular, [2, 3, 7] and a “minimal” theory in [6]. The (Denjoy-Perron-)Henstock-Kurzweil integral is now well understood and its integrable functions and their primitives have been completely characterized [8]. It is also interesting to note that a small variation in the definition of the HK integral yields the McShane integral [28], which is equivalent to the Lebesgue integral. Lastly, although the present paper focuses on scalar valued functions, we mention that HK-like integration of Banach space valued functions has raised considerable interest [15, 37, 5], in particular with an application to the Cauchy problem in [10].

The Riemann-like formulation of the HK integral makes it straightforward to allow for singularities in the above theorem; if f is only differentiable at all but countably many points of $[a, b]$, the result still holds. This statement is, in some sense, optimal. Indeed, as shown by Z. Zahorsky in [41], the set of non-differentiability points of a continuous function is a countable union of G_δ sets. In particular, if it is uncountable, it must contain a Cantor subset by [30, Lemma 5.1], and to any Cantor subset of an interval having zero Lebesgue measure, one can associate a “Devil’s Staircase,” which has derivative equal to 0 outside of the set and is non-constant.

However, the differentiability condition can be relaxed and replaced by a pointwise Lipschitz condition. Thus, a more general statement is

THEOREM B ([33, Theorem 6.6.9]). — *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function that is pointwise Lipschitz at all but countably many points, then, it is differentiable almost everywhere in $[a, b]$, its derivative f' is HK integrable on $[a, b]$, and identity (1) holds.*

Natural extensions of the fundamental theorem of calculus include the Gauss-Green (or divergence) theorem and Stokes' theorem. For the former in bounded sets of finite perimeter an integral was developed by W.F. Pfeffer in [32], following, in particular, J. Mawhin [26] and J. Mařík [25]. The results naturally extend to Stokes' theorem on smooth oriented manifolds. For singular varieties, an integral adapted to Stokes' theorem was defined by the author on certain types of integral currents in Euclidean spaces [21, 22]. Let us also mention works on integration on more fractal objects with different methods [40, 19, 42].

The present paper corresponds to the second chapter of the author's thesis [21], which focuses on one-dimensional integral currents. These are treated separately from the higher-dimensional ones, as they can be decomposed into a countable family of curves. We, thus, define an integral closer to the Henstock-Kurzweil one, which we call the $\mathcal{R}_1$ integral.

Given an integral current T of dimension 1 in $\mathbb{R}^n$, define $\text{Indec}(T)$ to be the subset of $\text{spt } T$ containing the points in the support of an indecomposable piece of T (see Section 3 for the notations on currents). Denote by $\|T\|$ the carrying measure of T , by $\vec{T}$ its tangent vector field and by $\text{spt } T$ its support. The main result of this paper is the following:

THEOREM 1.1 (Fundamental theorem of calculus for 1-currents). — *Let T be a fixed integral current of dimension 1 in $\mathbb{R}^n$ and u be a continuous function on $\text{spt } T$. Suppose that u is pointwise Lipschitz at all but countably many points in $\text{Indec}(T)$ and that u is differentiable $\|T\|$ almost everywhere, then $x \mapsto \langle Du(x), \vec{T}(x) \rangle$ is $\mathcal{R}_1$ integrable on T and*

$$(\partial T)(u) = (\mathcal{R}_1) \int_T \langle Du, \vec{T} \rangle.$$

This theorem is equivalent to Theorem B when T represents a bounded interval.

Summary of the paper. — In Section 2, we define the integral of Henstock and Kurzweil and its main properties along with schemes of proofs of the main theorems. We also give an equivalent definition of integrability — inspired from the Pfeffer integral — which will be useful in the sequel. It is important to note that the Pfeffer integral is not equivalent to the HK integral.

In Section 3, we recall the definition of integral currents of dimension 1 in Euclidean spaces and define the main ingredients of $\mathcal{R}_1$ integration: pieces of a current and functions on the space of pieces of a current; we also study the derivation of these functions, following H. Federer [16, Section 2.9] and W.F. Pfeffer [33, Section 9.3] and [34]. Section 4 contains the definition of $\mathcal{R}_1$ integration and the proof of its main properties, as well as the proof of Theorem 1.1.

Possible generalizations. — First, one can ask if u could be allowed to be discontinuous outside of the set of positive lower-density points of $\|T\|$, yet remain bounded. Proposition 3.5 and Example 2.10 show that this is not straightforward.

A natural question would be whether Theorem 1.1 could be generalized to normal currents in Euclidean spaces. Indeed, by a Theorem of S.K. Smirnov [38], normal currents of dimension 1 also admit a decomposition into Lipschitz curves. More precisely, given a current T of dimension 1, with finite mass and finite boundary mass in $\mathbb{R}^n$, there exists a finite measure μ on the space of finite length Lipschitz curves in $\mathbb{R}^n$ such that

$$(2) \quad T = \int \llbracket \gamma \rrbracket d\mu(\gamma),$$

where $\llbracket \gamma \rrbracket$ is the integral current of dimension 1 associated to the Lipschitz curve γ with multiplicity 1 and orientation given by the parameterization. However, there is no a-priori constraint on the measure μ ; it can be somewhat diffuse, as the carrying measure of a normal current can be absolutely continuous with respect to the Lebesgue measure. It is, therefore, impossible to work with countable sums of pieces, and one would probably need another notion of piece of a normal current to define suitable Riemann sums. Recall that Fubini-type arguments do not work well with non-absolutely convergent integrals, as shown in [33, Section 11.1]. Note also that the space of curves, on which we would have to integrate is far from Euclidean.

Another natural idea would be to consider integral currents of dimension 1 in Banach spaces or complete metric spaces, following [1] or [13].

Acknowledgements. — I wish to thank my PhD advisor Thierry De Pauw and my academic older brother Laurent Moonens, for their help during this thesis work, as well as Marianna Csörnyei for a helpful conversation on the Besicovitch covering theorem. I also thank the anonymous referee for her or his corrections and suggestions. I wrote my PhD thesis at the Institut de Mathématiques de Jussieu, Université Paris Diderot USPC.

Finally, I wish to mention that the notions of pieces and subcurrents studied in this paper and in [21, 22] are very close to the subcurrents defined by E. Paolini and E. Stepanov in [31] for normal currents in metric spaces.

2. The integral of Kurzweil and Henstock

Here, we give a short presentation of the Henstock-Kurzweil integral. The proofs of these results can be found in any treaty on the subject, such as [27, 33, 18, 17]. Let us also mention the very detailed recent book [29] — in French, and Appendix H to [11].

2.1. Definition and classical properties. — A non-negative function defined on a set $E \subseteq \mathbb{R}$ is called a **gauge** if its zero set is countable. In the classical definition of the Henstock-Kurzweil integral, gauges are always positive, but for our purposes it makes sense to allow the gauge to take on the value zero in a countable set. A **tagged family** in an interval $[a, b]$ is a finite collection of pairs $([a_j, b_j], x_j)_{j=1,2,\dots,p}$, where one has $a \leq a_1 < b_1 \leq a_2 < \dots \leq a_p < b_p \leq b$ and for all j , $x_j \in [a_{j-1}, a_j]$. The **body** of a family $\mathcal{P}$ is the union denoted by $[\mathcal{P}]$ of all the intervals in $\mathcal{P}$. A **tagged partition** in $[a, b]$ is a tagged family whose body is $[a, b]$. If δ is a gauge on $[a, b]$, we say that a tagged family (or a tagged partition) is δ -fine, when for all j , $b_j - a_j < \delta(x_j)$. In particular, it holds that $\delta(x_j) > 0$, for all j .

DEFINITION 2.1. — A function f defined on a compact interval $[a, b]$ is **Henstock-Kurzweil integrable on** $[a, b]$ if there exists a real number α , such that for all $\epsilon > 0$, there exists a positive gauge δ on $[a, b]$, such that for each δ -fine tagged partition $\mathcal{P} = \{([a_{j-1}, a_j], x_j)\}_{j=1,\dots,p}$, it holds that:

$$\left| \sum_{j=1}^p f(x_j)(a_j - a_{j-1}) - \alpha \right| < \epsilon.$$

In the following, we will write $\sigma(f, \mathcal{P})$ for the sum on the left-hand side, whenever $\mathcal{P}$ is a tagged family. If α exists as above, we denote it by $(HK) \int_a^b f$. This definition is well posed as a consequence of the following key result.

LEMMA 2.2 (Cousin's lemma [18, Lemma 9.2]). — *If I is a closed bounded interval, and δ is a positive gauge on I , then a δ -fine tagged partition of I exists.*

To characterize integrability, the following proposition is useful:

PROPOSITION 2.3 (Cauchy criterion for integrability [18, Theorem 9.7]). — *A function f is HK integrable on the interval $[a, b]$, if and only if for each $\epsilon > 0$ there exists a positive gauge δ on $[a, b]$, such that whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are δ -fine tagged partitions of $[a, b]$, it holds that*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \epsilon.$$

Let us also list some fundamental properties of HK integrable functions.