Astérisque

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Astérisque, tome 191 (1990), p. 109-209

<http://www.numdam.org/item?id=AST_1990__191__109_0>

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André-Quillen Cohomology and the Bousfield-Kan Spectral Sequence

by Paul G. Goerss*

In this paper we investigate the Bousfield-Kan spectral sequence [6], [8]

$$Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) \Rightarrow \pi_{t-s}X_p \tag{1}$$

where $H^*X = H^*(X, \mathbf{F}_p)$ is cohomology with coefficients in the prime field $\mathbf{F}_p$, $\mathcal{UA}$ is the category of augmented unstable algebras over the Steenrod algebra, S^t is the t-sphere and, X_p is the p-completion of the pointed space X. This spectral sequence, an unstable Adams spectral sequence, is a major tool in attempts to compute or understand the homotopy groups of spaces.

Because $\mathcal{UA}$ is not an abelian category, $Ext^*_{\mathcal{UA}}(H^*X, H^*S^t)$ must be defined using a cotriple and, hence, a simplicial resolution of the algebra H^*X. Our first point is to notice that if $s\mathcal{UA}$ is the category of simplicial objects in $\mathcal{UA}$, then there is a contravariant functor

$$H^*_{\mathcal{QA}} : s\mathcal{UA} \to nn\mathbf{F}_p \tag{2}$$

to the category of bigraded $\mathbf{F}_p$ vector spaces that generalizes $Ext_{\mathcal{UA}}$ in the following sense: if Λ is an object in $\mathcal{UA}$, then we may regard Λ as the constant simplicial object that is Λ in every simplicial degree and every face and degeneracy operator is the identity. Then we will have the equation

$$[H^s_{\mathcal{QA}}\Lambda]_t \cong Ext^s_{\mathcal{UA}}(\Lambda, H^*S^t) \tag{3}$$

where $[H^s_{\mathcal{QA}}\Lambda]_t$ denotes the elements of degree t in the graded vector space $H^s_{\mathcal{QA}}\Lambda$.

* The author was partially supported by the National Science Foundation

For $A \in s\mathcal{UA}$, $H^*_{\mathcal{QA}}A$ is a cohomology of A in the sense of Quillen [17]; in fact, this is the sort of cohomology of algebras that has been studied extensively by André and Quillen [1], [18]. Hence the title of the paper.

We will take the observation of the existence of $H^*_{\mathcal{QA}}$ in two directions. The first is this: the category of simplicial objects $s\mathcal{UA}$ has a structure of a closed model category and, as such, we can do the usual homotopy theoretic constructions. In particular, if $f : A \to B$ is a morphism in $s\mathcal{UA}$, then f has a homotopy cofiber $M(f)$ and there is a long exact sequence in cohomology similar to the one that Quillen called a transitivity sequence [18]:

$$\cdots \to H^{n-1}_{\mathcal{QA}}A \to H^n_{\mathcal{QA}}M(f) \to H^n_{\mathcal{QA}}B \xrightarrow{H^*_{\mathcal{QA}}f} H^n_{\mathcal{QA}}A \to \cdots. \tag{4}$$

Here we need the full generality of $s\mathcal{UA}$. For, even if $f : \Lambda \to \Gamma$ is a morphism of constant simplicial algebras of the type considered in equation (3), $M(f)$ is not necessarily such an object. In fact, if we define the homotopy of an object $A \in s\mathcal{UA}$ by

$$\pi_* A = H_*(A, \partial)$$

where ∂ is the alternating sum of the face operators in A, then for a morphism $f : \Lambda \to \Gamma$ of constant simplicial algebras

$$\pi_* M(f) \cong Tor^{\Lambda}_*(\mathbf{F}_p, \Gamma).$$

We extend the long exact sequence in cohomology to the homotopy spectral sequence. The Bousfield-Kan spectral sequence is an example of the homotopy spectral sequence of a cosimplicial space. Given a fibrant cosimplicial space Z, there is a spectral sequence [8]

$$\pi^s \pi_t Z \Rightarrow \pi_{t-s} Tot(Z)$$

where $Tot(Z)$ is a kind of "codiagonal" of Z given by the mapping space of cosimplicial spaces

$$Tot(Z) = map(\Delta, Z)$$

where Δ is the cosimplicial space that in cosimplicial degree s is the standard s-simplex $\Delta[s]$. Now if Z is a cosimplicial space, then H^*Z is a simplicial

object in $\mathcal{UA}$ and we will notice that we can extend the Bousfield-Kan spectral sequence to a spectral sequence

$$[H^s_{\mathcal{QA}} H^* Z]_t \Rightarrow \pi_{t-s} Tot(Z)_p. \tag{5}$$

If X is a space and $Z = X$ is the constant simplicial space, then $H^*Z = H^*X$ is a constant simplicial algebra we can combine equation (3) with equation (5) to obtain the spectral sequence (1).

In particular, if $f : Z \to Y$ is a morphism of cosimplicial spaces, we will define a new cosimplicial space F so that $H^*F \cong M(f^*)$ — the homotopy cofiber of f^* in $s\mathcal{UA}$ — and so that there is a homotopy fibration sequence of spaces

$$Tot(F) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p.$$

Further, there will be a diagram of spectral sequences

$$\begin{array}{ccccccc}
\to & [H^s_{\mathcal{QA}} H^* Z]_t & \to & [H^s_{\mathcal{QA}} H^* Y]_t & \xrightarrow{\partial} & [H^{s+1}_{\mathcal{QA}} H^* F]_t & \to \\
 & \Downarrow & & \Downarrow & & \Downarrow & \\
\to & \pi_{t-s} Tot(Z)_p & \to & Tot_{t-s} Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1} Tot(F) & \to
\end{array}$$

where the top row comes from the long exact sequence (4) and the bottom row is the long exact sequence of the fibration sequence. The hard work here is to produce the commutative diagram of spectral sequences

$$\begin{array}{ccc}
[H^s_{\mathcal{QA}} H^* Y]_t & \xrightarrow{\partial} & [H^{s+1}_{\mathcal{QA}} H^* F]_t \\
\Downarrow & & \Downarrow \\
\pi_{t-s} Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1} Tot(F).
\end{array}$$

This is done in section 5.

There are other ramifications to the idea that $s\mathcal{UA}$ is a closed model category. Among them are the notions of universal infinite cycles and universal differentials for the spectral sequence (5). Although these were noted by Bousfield and Kan [6] and have been extended by the work of Bousfield [3], and are related to Barratt's desuspension spectral sequence, as rediscovered by Hopkins, they have not been systematically studied from our point of view. We undertake this study, beginning in section 3, but extending our computations into further sections.

The universal cycle, for example, is a cosimplicial space $\bar{\mathbf{F}}_p^{\cdot}S(s,t)$ whose cohomology is relevant to the Quillen cohomology of equation (2) in the following way. There is an object $K(s,t)_+ \in s\mathcal{UA}$ that corepresents cohomology in the usual way: there is a universal class $\iota \in [H^s_{Q\mathcal{A}}K(s,t)_+]_t$ and an isomorphism

$$[A, K(s,t)_+]_{s\mathcal{UA}} \xrightarrow{\cong} [H^s_{Q\mathcal{A}}A]_t$$

given by

$$f \longmapsto f^*(\iota).$$

$[\ ,\ ]_{s\mathcal{UA}}$ denotes the morphisms in the homotopy category associated to the closed model category on $s\mathcal{UA}$. Then, we have for the universal cycle $\bar{\mathbf{F}}_p^{\cdot}S(s,t)$, a weak equivalence in $s\mathcal{UA}$

$$H^*\bar{\mathbf{F}}_p^{\cdot}S(s,t) \xrightarrow{\simeq} K(s,t)_+$$

and, hence, an equation

$$H^*_{Q\mathcal{A}}H^*\bar{\mathbf{F}}_p^{\cdot}S(s,t) \cong H^*_{Q\mathcal{A}}K(s,t)_+.$$

Furthermore $Tot(\bar{\mathbf{F}}_p^{\cdot}S(s,t)) \simeq S_p^{t-s}$. Hence we get a spectral sequence

$$H^*_{Q\mathcal{A}}K(s,t)_+ \Rightarrow \pi_* S_p^{t-s}$$

that is universal in the following sense. Suppose that, for a fibrant cosimplicial space Z,

$$\alpha \in [H^s_{Q\mathcal{A}}H^*Z]_t$$

survives to E_∞ in the spectral sequence (5), and detects

$$x \in \pi_{t-s}Tot(Z)_p.$$

Then there is a morphism in $s\mathcal{UA}$

$$f : H^*Z \to K(p,q)_+$$

corepresenting α and the resulting map

$$H^*_{Q\mathcal{A}}f : H^*_{Q\mathcal{A}}K(s,t)_+ \to H^*_{Q\mathcal{A}}H^*Z$$