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RENORMALIZATION, FREEZING PHASE TRANSITIONS AND FIBONACCI QUASICRYSTALS

BY HENK BRUIN AND RENAUD LEPLAIDEUR

ABSTRACT. – We examine the renormalization operator determined by the Fibonacci substitution within the full shift on two symbols $\Sigma := \{0,1\}^{\mathbb{N}}$. We exhibit a fixed point and determine its stable leaf (under iteration of the operator acting on potentials $V : \Sigma \to \mathbb{R}$), which is completely determined by the germ near the attractor of the substitution. Then we study the thermodynamic formalism for potentials in this stable leaf, and prove they have a freezing phase transition at finite temperature, with ground state supported on the attracting quasi-crystal associated to the Fibonacci substitution.

RÉSUMÉ. – Nous étudions les relations entre renormalisation, substitutions et transitions de phase : nous montrons que la substitution de Fibonacci dans le shift plein à deux symboles $\Sigma := \{0,1\}^{\mathbb{N}}$ génère un opérateur de renormalisation sur les potentiels $V : \Sigma \to \mathbb{R}$. Nous montrons que cet opérateur possède un point fixe, uniquement déterminé par son germe proche de l'attracteur associé à la substitution de Fibonacci. Nous déterminons aussi la feuille stable de ce point fixe. Dans un second temps, nous montrons que tous les potentiels dans cette feuille stable présentent une transition de phase congelante. En particulier, cela donne un nouvel exemple d'obtention d'un état fondamental porté par un quasi-cristal avant le zéro absolu.

1. Introduction

1.1. Background

The present paper studies phase transitions from an ergodic theory and dynamical systems point of view. It investigates renormalization, substitutions and phase transition initiated in [2] and continued in [5].

Phase transitions are an important topic in statistical mechanics and also in probability theory (see e.g., [11, 12, 23, 27]). The viewpoint presented here is different for several reasons. One of them is that, here, the geometry of the lattice is not relevant[(1)], whereas in statistical mechanics, the geometry of the lattice is the most important part.

[(1)] and we only consider a one-dimensional lattice.

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During the 1970's, motivated by problems in statistical mechanics, Bowen, Ruelle and Sinai (see [3, 24, 26]) introduced thermodynamic formalism into ergodic theory. Given a dynamical system (X,T) and a potential $\varphi : X \to \mathbb{R}$, the pressure function is given by

$$\mathscr{P}(\beta) := \sup \left\{ h_\mu(T) + \beta \int \varphi \, d\mu \right\},$$

where the supremum is taken over the invariant probability measures μ, $h_\mu(T)$ is the Kolmogorov entropy and β is a real parameter. Any measure realizing the supremum is then called *an equilibrium state* for $\beta\varphi$.

For a uniformly hyperbolic dynamical system (X,T) and a Hölder continuous potential φ, the pressure function $\beta \mapsto \mathscr{P}(\beta)$ is analytic (see e.g., [3, 24, 15]) and there is a unique equilibrium state $\mu_{\beta\varphi}$ (for every β). This equilibrium also satisfies a *Gibbs property*; in the dynamical systems language this condition expresses how the measure of n-cylinders scales: There is $K > 0$ such that

$$K^{-1} \le \frac{\mu_{\beta\varphi}(Z_n)}{\exp(\beta \sum_{i=0}^{n-1} \varphi \circ T^i(x) - n\mathscr{P}(\beta))} \le K \tag{1}$$

for all $n \ge 1$, all n-cylinders Z_n and $x \in Z_n$.

Since the late 1970s, people in dynamical systems focused on extending the notions and results of thermodynamics to non-uniformly hyperbolic dynamical systems. This started with the work of Hofbauer [13, 22] proving non-analyticity of pressure for a non-Hölder potential φ on the shift-space $(\{0,1\}^{\mathbb{N}}, \sigma)$. This example is closely related to the Manneville-Pomeau map, and an associated renormalization procedure, presented in [2], was the starting point of the project this paper is part of. Ledrappier [17] showed that any finite number of equilibrium can co-exist in similar examples (cf. also [4, 9] for the co-existence of multiple equilibirum states in other settings). Weakening the Gibbs property may be necessary as well. For instance, Yuri [29], in the setting of maps with neutral fixed points, used a version of *weak Gibbs* in which the K in (1) is replaced by K_n with $\lim_n \frac{1}{n} \log K_n = 0$. See [6, Section 3.1] for similar results to smooth interval maps with critical points.

More recently, the original motivation came back into focus, and the question of phase transitions is now a very active theme in ergodic theory. Nevertheless, due to equivalences or interdependences in the "classical" settings between unique existence of Gibbs measures, unique existence of equilibrium states and regularity of the pressure function, and also due to historical or inspiration models (e.g., the Ising model in probability or Erhenfest vs. Gibbs classification in statistical mechanics), the notion of phase transition may vary in the literature. In this paper, we adopt a largely accepted definition now in dynamical systems: a *phase transition* is characterized by a lack of analyticity of the pressure function.

Although analyticity is usually considered as a very rigid property and thus quite rare, it turns out that proving non-analyticity for the pressure function is not so easy. Currently, this has become an important challenge in smooth ergodic theory to produce and study phase transitions, see e.g., [19, 8, 5] and also [14, Sec. 6] for the possible shapes of the pressure function. We also refer to [25] for results on the regularity of the pressure in the non-compact setting and [29, 21] for uniqueness of the equilibrium state, again in the non-compact case.

To observe phase transitions, one has to weaken hyperbolicity of the system or of regularity of the potential; it is the latter one that we continue to investigate here. Our dynamical

system is the full shift, which is uniformly hyperbolic. The first main question we want to investigate is thus which potentials φ will produce phase transitions. More precisely, we are looking for a machinery to produce potentials with phase transitions.

The main purpose of [2] was to investigate possible relation between *renormalization* and phase transitions. In the shift space $(\{0,1\}^{\mathbb{N}}, \sigma)$, a renormalization is a function H for which there is an integer $k \geq 2$ such that

$$\sigma^k \circ H = H \circ \sigma. \tag{2}$$

The link with potentials was made in [2] by introducing a renormalization operator $\mathscr{R}$ acting on potentials and related to a solution H for (2). It is easy to check that constant length k substitutions are solutions to (2). In [5], we studied the Thue-Morse substitution, which has constant length 2. Here we investigate the Fibonacci substitution, which is not of constant length. Several reasons led us to study the Fibonacci case:

Together with the Thue-Morse substitution, the Fibonacci substitution is the most "famous" substitution and it has been well-studied. In particular, the dynamical properties of their respective attracting sets are well-known and this will be used extensively in this paper. As a result, we were able to describe the relevant fixed point of renormalization exactly. Information of the left and right-special words in these attractors is a key ingredient to prove existence of a phase transition; it is a crucial issue in the relations between substitutions and phase transitions.

The type of phase transition we establish is a *freezing phase transition*. This means that beyond the phase transition (i.e., for large β), the pressure function is affine and equal to its asymptote, and the equilibrium state (i.e., ground state) is the unique shift-invariant measure supported on an aperiodic subshift space, sometimes called *quasi-crystal*. One open question in statistical mechanics (see [10]) is whether freezing phase transitions can happen and whether *quasi-crystal ground state* can be reached at *positive temperature*. An affirmative answer was given for the Thue-Morse quasi-crystal in [5]; here we show that this also holds for the Fibonacci quasi-crystal.

We think that Fibonacci shift opens the door to study more general cases. One natural question is whether any quasi-crystal can be reached as a ground state at positive temperature. In this context we emphasize that the Fibonacci substitution space is also Sturmian shift, that is, it encodes the irrational rotation (with angle the golden mean $\gamma := \frac{1+\sqrt{5}}{2}$). We expect that the machinery developed here can be extended to the Sturmian shift associated to general irrational rotation numbers (although those with bounded entries in the continued fraction expansion will be the easiest), possibly to rotations on higher-dimensional tori, and also to more general substitutions.

1.2. Results

Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the full shift space; points in Σ are sequences $x := (x_n)_{n\geq 0}$ or equivalently infinite *words* $x_0x_1\ldots$ Throughout, let $\overline{x}_j = 1 - x_j$ denote the opposite symbol. The dynamics is the left-shift

$$\sigma : x = x_0x_1x_2\ldots \mapsto x_1x_2\ldots$$