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Integral geometry from Buffon to geometers of today

Rémi LANGEVIN

23

INTEGRAL GEOMETRY FROM BUFFON TO GEOMETERS OF TODAY

Rémi Langevin

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Cours Spécialisés
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
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