Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

HODGE THEORY FOR TROPICAL FANS

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Tome 154
Fascicule 3

2026

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 713-809

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 154, septembre 2026

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Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 175 € ($ 262),
avec supplément papier : Europe 266 €, hors Europe 307 € ($ 460)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
154 (3), 2026, p. 713-809

HODGE THEORY FOR TROPICAL FANS

BY OMID AMINI & MATTHIEU PIQUEREZ

ABSTRACT. — This work is devoted to the development of a Hodge theory for tropical fans. We introduce a notion of stability for tropical fans and prove that various geometric properties of tropical fans are stable. As a consequence, we establish Kähler properties for the Chow ring in a large class of tropical fans. We give, in particular, a new proof of the Kähler package for combinatorial geometries. The approach makes it possible to deal with simplicial tropical fans with general weights, going far beyond the case of matroids and their Bergman fans.

RÉSUMÉ (*Théorie de Hodge des éventails tropicaux*). — Ce travail est dédié au développement d'une théorie de Hodge pour les éventails tropicaux. Nous introduisons une notion de stabilité pour les éventails tropicaux et montrons que diverses propriétés géométriques de ces éventails sont stables. Comme conséquence, nous établissons des propriétés kählériennes pour une large classe d'éventails tropicaux. En particulier, nous en déduisons une nouvelle preuve des théorèmes fondamentaux en théorie de Hodge des matroïdes. Notre approche permet de travailler avec des éventails simpliciaux muni d'un poids tropical prenant des valeurs réelles positives arbitraires, dépassant largement le cas des matroïdes et leurs éventails de Bergman.

Texte reçu le 18 février 2025, accepté le 23 avril 2025.

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Mathematical subject classification (2010). — 05B35, 05E14, 05E40, 14C15, 14T20.

Key words and phrases. — Tropical fan, Hodge theory, Poincaré duality, Lefschetz theorem, Hodge–Riemann relations, matroid, Chow ring.

During the realization of this work, O.A. was part of the ANR project ANR-18-CE40-0009, and thanks Math+, the Berlin Mathematics Research Center, for support. M.P. has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 101001995).

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2026/713/$ 5.00
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doi:10.24033/bsmf.2931

1. Introduction

The work presented here is motivated by the recent development in combinatorial Hodge theory, which has undergone an expansion following the work of Adiprasito–Huh–Katz [4] and further follow-ups, leading to the resolution of several open problems in the theory of matroids and their applications. We refer to the expository papers by Huh [44, 45], Okounkov [62], Ardila [15, 16], and Baker [22] for an overview of these exciting developments in combinatorics.

Our starting point was the following question.

Question 1.1. — How are Chow rings of matroids and their Hodge theory linked to homological properties of tropical varieties? To what extent can the setting of combinatorial Hodge theory be extended beyond the setting of matroids and their Bergman fans? Is there a Kähler geometry in the tropical setting?

Our aim in this work and its companions is to propose answers to these questions by developing Hodge theoretic aspects of tropical geometry.

The specific work presented here is of local nature and concerns geometric properties of tropical fans captured in their Chow rings. Tropical fans and their support named tropical fanfolds are building blocks for the construction of more general tropical varieties. We establish the Kähler package for the Chow ring of a large class of tropical fans, which can be recursively constructed by using three basic operations that preserve the balancing condition (orientability in tropical geometry). As a by-product, we obtain a new proof of the Kähler package for combinatorial geometries that circumvents some of the main difficulties encountered in [4]. At the same time, our approach makes it possible to deal with tropical fans with more general weights.

The manuscript is written to be accessible with no previous knowledge of algebraic and tropical geometry. In particular, for algebraic geometric statements that play a crucial role in combinatorial and tropical Hodge theory, we provide new combinatorial proofs.

In the rest of this Introduction, we provide an overview of our results.

1.1. Fans and Their Chow rings. — Consider a lattice of finite rank $N \simeq \mathbb{Z}^n$ and let $N_{\mathbb{R}}$ be the vector space generated by N. A *rational fan* in $N_{\mathbb{R}}$ is a non-empty collection Σ of strongly convex rational polyhedral cones that verifies the following two properties:

(1) If σ is a cone in Σ, then any face τ of σ belongs to Σ.
(2) For a pair of cones σ and η in Σ, their intersection $\sigma \cap \eta$ is a common face of both σ and η.

We denote the support of Σ by $\left|\Sigma\right|$ and call it a *fanfold*. For each integer k, Σ_k denotes the set of k-dimensional cones in Σ.

Let Σ be a rational fan that we assume to be simplicial, meaning that each cone in Σ is generated by as many rays as its dimension. The *Chow ring* of Σ denoted by $A^\bullet(\Sigma)$ is defined by generators and relations. Consider the polynomial ring $\mathbb{Z}[\mathrm{x}_\zeta]_{\zeta\in\Sigma_1}$ with indeterminate variables x_ζ associated to rays ζ in Σ_1. Then, $A^\bullet(\Sigma)$ is the quotient ring

$$A^\bullet(\Sigma) \coloneqq \mathbb{Z}[\mathrm{x}_\zeta]_{\zeta\in\Sigma_1}\big/\big(I+J\big),$$

where

- I is the ideal generated by the products $\mathrm{x}_{\rho_1}\cdots\mathrm{x}_{\rho_k}$, for $k\in\mathbb{N}$, such that $\rho_1,\dots,\rho_k$ do not form a cone in Σ, and
- J is the ideal generated by the elements of the form

$$\sum_{\zeta\in\Sigma_1} m(\mathfrak{e}_\zeta)\mathrm{x}_\zeta, \qquad m\in M\coloneqq N^\star,$$

  with $\mathfrak{e}_\zeta$ the primitive vector of the ray ζ.

The ideal $I+J$ is homogeneous, and the Chow ring inherits a graded ring structure. Moreover, for degree larger than the dimension of Σ, the corresponding graded piece vanishes. Denoting by d the dimension of Σ, we can thus write

$$A^\bullet(\Sigma) = \bigoplus_{k=0}^{d} A^k(\Sigma),$$

with the k-th degree piece $A^k(\Sigma)$, $0\leqslant k\leqslant d$, generated as $\mathbb{Z}$-module by degree k monomials. When the fan is unimodular, the Chow ring $A^\bullet(\Sigma)$ coincides with the Chow ring of the toric variety $\mathbb{P}_\Sigma$ associated to Σ, see [29] and [32, 25, 39].

Similarly as above, we define the Chow ring with rational and real coefficients that we denote by $A^\bullet(\Sigma,\mathbb{Q})$ and $A^\bullet(\Sigma,\mathbb{R})$, respectively.

Chow rings of fans have a rich combinatorics. A discussion of these properties is provided in Section 3.

We fix some terminology before proceeding. Each cone σ in a rational fan Σ defines a sublattice of N denoted by N_σ that has rank equal to the dimension of σ. The sublattice N_σ is given by the integral points of the vector space generated by σ. We denote by $|\sigma|$ the dimension of a face σ. A cone σ is called *simplicial* if it is generated by $|\sigma|$ rays, equivalently, if σ is of the form $\sum_i \mathbb{R}_{\geqslant 0}\mathfrak{e}_i$ for a family of independent vectors $\mathfrak{e}_1,\dots,\mathfrak{e}_{|\sigma|}$ in $N_\mathbb{R}$. A rational cone σ is called *unimodular* if it is generated by $|\sigma|$ vectors that form a basis of N_σ. A fan Σ is called simplicial, resp. unimodular, if all its cones are simplicial, resp. unimodular. A *facet* is a maximal cone. A fan is *pure dimensional* if all its facets have the same dimension. We denote the dimension of Σ by d.

In this work, the star fan Σ^σ of Σ at a cone $\sigma\in\Sigma$ refers to the fan in $N_\mathbb{R}/N_{\sigma,\mathbb{R}}$ induced by the cones η in Σ, which contain σ as a face.