Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

NONLOCAL QUASILINEAR SINGULAR SPDES

Ismael Bailleul & Hugo Eulry

Tome 154
Fascicule 3

2026

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 689-711

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 154, septembre 2026

Comité de rédaction

Boris ADAMCZEWSKI
Valeria BANICA
Julie DÉSERTI
Gabriel DOSPINESCU
Dorothee FREY
Youness LAMZOURI
Wendy LOWEN
Ludovic RIFFORD
Erwan ROUSSEAU
Béatrice de TILIÈRE

François DAHMANI (Dir.)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 175 € ($ 262),
avec supplément papier : Europe 266 €, hors Europe 307 € ($ 460)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

© *Société Mathématique de France 2026*

Tous droits réservés (article L 122–4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335–2 et suivants du CPI.

ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
154 (3), 2026, p. 689-711

NONLOCAL QUASILINEAR SINGULAR SPDES

BY ISMAEL BAILLEUL & HUGO EULRY

ABSTRACT. — In this short note we study a counterpart to the quasilinear generalized parabolic Anderson model (gPAM) on the 2-dimensional torus, where the coefficients are nonlocal functionals of the solution. Under a positivity assumption on the diffusion coefficient, we develop a local-in-time solution theory within the framework of paracontrolled calculus.

RÉSUMÉ (*Équations quasi-linéaires singulières non locales*). — Nous étudions dans cette note une équation parabolique de type Anderson quasi-linéaire dont les coefficients sont des fonctionnelles non locales de la solution, en dimension 2. Nous démontrons que l'équation est localement bien posée en temps sous une hypothèse de positivité sur le coefficient de diffusion.

1. Introduction

In this work, we are interested in developing a solution theory for some quasilinear singular stochastic partial differential equations

$$\partial_t u - A\big(f(u)\big)\Delta u = B\big(g(u)\big)\,\xi \tag{1.1}$$

Texte reçu le 31 mai 2024, modifié le 10 septembre 2025, accepté le 5 janvier 2026.

ISMAEL BAILLEUL, Univ Brest, CNRS, LMBA – UMR 6205, F-29238 Brest, France •
E-mail : ismael.bailleul@univ-brest.fr
HUGO EULRY, ENS Lyon, CNRS, UMPA – UMR 5669, F-69364 Lyon, France •
E-mail : hugo.eulry@ens-lyon.fr

Mathematical subject classification (2010). — 60H17.

Key words and phrases. — Singular stochastic PDEs, paracontrolled calculus, nonlocal operators.

© Société Mathématique de France
0037-9484/2026/689/$ 5.00
doi:10.24033/bsmf.2930

whose coefficients $A\big(f(u)\big), B\big(g(u)\big)$ are *nonlocal functionals* of some functions of the solution u. We do so here in the mildly singular setting of the 2-dimensional torus $\mathbb{T}^2$ for a spatial white noise ξ. We need some notations to set the stage for our main result, Theorem 1.1, below.

Denote by (e_1, e_2) the canonical basis of $\mathbb{Z}^2 \subset \mathbb{R}^2$, and for $i \in \{1,2\}$ and a function $\sigma\colon \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ set

$$(D_i\sigma)(x,k) := \sigma(x, k+e_i) - \sigma(x,k),$$

and $\mathbf{D} := (D_1, D_2)$. Write $\mathcal{F}$ for the Fourier transform on $\mathbb{T}^2$ and denote by $\widehat{\sigma}(n,k) = (\mathcal{F}\sigma)(n,k)$ the Fourier transform of the function $\sigma(\cdot,k)$ for each $k \in \mathbb{Z}^2$. Write $\langle k\rangle$ for $1+|k|$. Given $s \in \mathbb{R}$ and $\alpha > 0$, we define the class Σ^s_α of functions $\sigma\colon \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ such that

$$\big\|\partial_x^i \mathbf{D}^j \sigma(\cdot,k)\big\|_{\mathcal{C}^\alpha} \lesssim \langle k\rangle^{s-|j|} \tag{1.2}$$

for all multi-indices i, j in $\mathbb{N}^2$, and for which there exists a constant $0 < \mu < 1$ such that

$$(\mathcal{F}\sigma)(n,k) = 0 \quad \text{whenever} \quad |n| > \mu\langle k\rangle. \tag{1.3}$$

For such a symbol, we can easily bound $\widehat{\sigma}$ uniformly in (n,k) by

$$\big|\widehat{\sigma}(n,k)\big| \lesssim \langle n\rangle^{-4}\, \langle k\rangle^s\, \mathbf{1}_{|n|\leq \mu\langle k\rangle}. \tag{1.4}$$

The convolution operator with a function $\mathcal{F}^{-1}(\sigma)$ on $\mathbb{T}^2$ whose Fourier transform $\sigma(k)$ satisfies (1.2) provides an elementary example of a function $\sigma \in \Sigma^0_\alpha$ – here it is independent of x and σ is supported on the diagonal $\{n = k\}$. Similarly, ad hoc conditions on a kernel K ensure that the integral operator $v \mapsto \int K(\cdot,y)v(y)dy$ is associated with a symbol $\sigma \in \Sigma^s_\alpha$ for some s. A negative s means an operator that regularizes. For a and b in Σ^s_α, we denote by A and B the pseudodifferential operators with symbols a and b, respectively.

We say that A *preserves positivity* if $A(v) > 0$ whenever v is bounded below by a positive constant. Positivity-preserving pseudodifferential operators with a symbol in Σ^0_α for some $\alpha > 0$ are given by some kernels $\mu(x,\cdot)$ that are finite non-negative measures for all $x \in \mathbb{T}^2$ such that testing $A(v)$ against a smooth test function ℓ on $\mathbb{T}^2$ one has

$$A(v)(\ell) = \iint v(y)\mu(x,dy)\ell(x)dx. \tag{1.5}$$

This is the case of the integral operator with kernel K when K is bounded below by a positive constant. In the renormalized Equation (1.6) below, we regularize the spatial white noise ξ into a smooth function

$$\xi^\varepsilon := \mathcal{F}^{-1}\big(\chi(\varepsilon|\cdot|)\,\mathcal{F}(\xi)(\cdot)\big)$$

using a 'smooth Fourier cut-off' $\chi\colon [0,\infty) \to [0,1]$ of class C^∞, equal to 1 on $[0,1]$ and 0 on $[2,\infty)$.

THEOREM 1.1. — *Pick two regularity exponents $2/3 < \beta < \alpha < 1$ and $s \leq 0$. Take two symbols $a, b \in \Sigma^s_\alpha$ and assume that the operator A preserves positivity. Take further $f, g \in C^3_b(\mathbb{R}, \mathbb{R})$ and assume $\inf_{\mathbb{R}} f > 0$.*

(a) *There exist two deterministic diverging functions $c_a, c_b \colon (0,1] \to C^\infty(\mathbb{T}^2)$ and a positive random time T such that the solution u^ε to the equation*

$$\partial_t u^\varepsilon - A\big(f(u^\varepsilon)\big)\Delta u^\varepsilon = B\big(g(u^\varepsilon)\big)\,\xi^\varepsilon + c_a(\varepsilon)\left(\frac{B\big(g(u^\varepsilon)\big)}{A\big(f(u^\varepsilon)\big)}\right)^2 f'(u^\varepsilon) - c_b(\varepsilon)\frac{B\big(g(u^\varepsilon)\big)}{A\big(f(u^\varepsilon)\big)}\,g'(u^\varepsilon) \tag{1.6}$$

with initial condition $u_0 \in \mathcal{C}^\alpha(\mathbb{T}^2)$ converges in probability in the parabolic α-Hölder space on $[0,T] \times \mathbb{T}^2$, as $\varepsilon \in (0,1]$ goes to 0.

(b) *The singular PDE*

$$\partial_t u - A\big(f(u)\big)\Delta u = B\big(g(u)\big)\,\xi$$

has a well-defined formulation in a space of paracontrolled functions, where it has a unique solution (u, u'). The limit of the u^ε is given by u.

This statement gives back the result first proved in [14, 8, 5] by Otto & Weber, Furlan & Gubinelli and Bailleul, Debussche & Hofmanová when A and B are the identity operators. The approach of Otto & Weber was later fully developed in [15, 11, 12] to provide a variant of regularity structures well-adapted to a certain class of quasilinear singular stochastic PDEs. It would be interesting to see whether their approach can be tweaked to handle nonlocal equations of the type considered here. The approach of [5] was developed in a paracontrolled setting in Bailleul & Mouzard's work [7] and fully developed in a regularity structure setting in Bailleul, Hoshino & Kusuoka's work [6].

In the proof of Theorem 1.1, we obtain point (a) as a corollary of point (b). The proof of point (b) involves two different tasks:

- Build an analytical setting wherein to formulate the equation as a well-defined fixed-point problem.
- Construct a random variable that plays the role of an ill-defined polynomial functional of the noise. It is required as an independent ingredient in the analytical setting.

We formulate the equation as a fixed point in a space of paracontrolled functions in Section 2 . We prove therein that it is well-posed locally in time. The construction of the polynomial functional of the noise is done in Section 3. A quick reminder on paraproducts as well as a collection of useful estimates in provided in Appendix A.

NOTATION. — Throughout, we denote by $\mathcal{C}^\gamma(\mathbb{T}^2)$ the Besov space $B^\gamma_{\infty,\infty}(\mathbb{T}^2)$, for any $\gamma \in \mathbb{R}$. We will, however, keep the notation $C^\gamma(\mathbb{R})$ for functions