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THE JACARANDA TREE IS STRONGLY APERIODIC AND HAS ZERO ENTROPY

by Alexandre Baraviera & Renaud Leplaideur

Abstract. — We prove that the Jacaranda tree obtained as a fixed point for a substreetution in previous work by the authors is strongly aperiodic and that the number of patches increases "slower than" $n^{\frac{\ln 3}{\ln 2}}$ with respect to the size n of the patch. As a consequence, the tree has zero entropy. This is also generalized to other Jacaranda-like trees.

Résumé (*Le Jacaranda est fortement apériodique et d'entropie nulle*). — Nous prouvons que l'arbre binaire colorié « Jacaranda », obtenu comme point fixe d'une substreetution dans un travail précédent, est fortement apériodique et que le nombre de motifs de profondeur n augmente moins vite que $n^{\frac{\ln 3}{\ln 2}}$. Une des conséquences est que l'entropie est nulle. Le résultat s'applique également à tous les cousins du Jacaranda, qui sont obtenus par une substreetution similaire.

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1. Introduction

1.1. Motivation for studying substreetutions. — In the present paper, we make a step forward in the ergodic study of the Jacaranda tree, which is obtained from a special example of substreetution as defined by authors in [2].

Substreetutions are substitutions acting on the set of colored binary trees $\{0,1\}^{\mathbb{F}_2^+}$, where $\mathbb{F}_2^+$ is the free semi-group with two generators a and b and an identity element e. They extend to $\{0,1\}^{\mathbb{F}_2^+}$ classical objects such as the Thue–Morse and the Fibonacci substitutions on $\{0,1\}^{\mathbb{N}}$.

Our aim is in line with a long work in progress to better understand similarities and differences between the ergodic and statistical mechanics viewpoints for thermodynamic formalism. The statistical mechanics viewpoint usually deals with $\mathbb{Z}^d$-action or even $\mathbb{F}_2^+$-actions (see [8, 12]). The principal motivation for studying substreetutions is thus the need to find dynamical systems driven by a (semi-)group action for which one can naturally define a *transfer operator*. We recall that the transfer operator is the key tool for most of the results in thermodynamical formalism for 1-dimensional actions. In particular, in the relatively new topic of *phase transition*[1], we find that in several examples (for 1-dimensional actions, see [3, 4, 5]), the Zeta function and its critical exponent 1 play an important role. We aim to investigate how the value "1" relates to the dimension (also $= 1$) of the action.

The lack of natural direction in $\mathbb{Z}^d$, $d \geq 2$, seems to be an irremediable obstacle to defining the transfer operator for these actions. Hence, $\mathbb{F}_2^+$ appears as the first and easiest example to be studied.

Along the way, several other motivations arose, as by-products of the initial goal. For instance, a notion of Sturmian trees has been introduced and studied (see, e.g., [9]). We recall that for the Fibonacci substitution the attractor is a Sturmian shift. It was thus quite natural to seek examples of such trees generated by substreetutions.

Substitutions are important objects, both from dynamical systems and combinatorics points of view. They are an important source of examples of quasi-periodic but chaotic systems. It is thus a natural question to inquire how they can be extended/generalized. Our work is also in that line. We point out the importance of the choice of the *grammar* for the substreetution (see below), and we show that different choices yield very different behaviors.

1.2. Presentation of the results. — In the present paper we continue the work in progress, advancing in the study of topological properties of the free semi-group action on the Jacaranda tree. Furthermore, we prove that the results hold for some *Jacaranda-like* substreetutions, and show that they do not hold for others (see Remarks A.2 and A.3, page 687).

1. for the ergodic viewpoint and 1-dimensional actions.

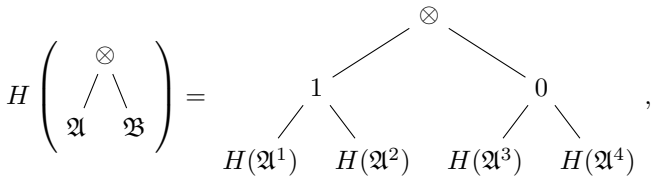
Any element in $\mathbb{F}_2^+$ can be seen either as the empty word e (with length 0) or as a finite word with letters a or b (with length the number of letters). A colored binary (rooted) tree $\mathfrak{A} \in \{0,1\}^{\mathbb{F}_2^+}$ will be seen as the collection of sites whose addresses are the ω's in $\mathbb{F}_2^+$, say $\mathfrak{A}_\omega$, and where $\mathfrak{A}_\omega$ has a color in $\{0,1\}$.

There is a natural action of $\mathbb{F}_2^+$ on $\{0,1\}^{\mathbb{F}_2^+}$ (see below). In [2] it was proved that the closure of $\mathbb{F}_2^+$-orbit of the fixed point for some special substreetution H, $X := \overline{\{T_\omega(\mathfrak{J}),\ \omega \in \mathbb{F}_2^+\}}$, is minimal and non-periodic, in the sense that it is not reduced to a periodic orbit. In other words, and following the terminology, it was proved that X is *weakly aperiodic*. Here we prove that X is actually *strongly aperiodic*.

The question of entropy is also of prime importance. Entropy measures the complexity of the system. For trees, the first natural question is to define the right normalization, since the increase is expected to be super exponential. Several notions of entropies with different normalizations have been defined in the literature (see [6, 11, 1]). We also point out that for usual $\mathbb{N}$-actions, entropy is also related to the number of preimages, and this turns out to be crucial for studying the transfer operator.

We prove here that the number of patches increases slower than $n^{\frac{\ln 3}{\ln 2}}$, where n is the number of lines. This upper bound is not sharp, and this remains an open question to know whether the increase is actually linear, and whether one can obtain a Sturmian tree with a substreetution. As a by-product, we find that X has zero entropy, whatever definition we use.

1.3. The Jacaranda-like trees. — More details on substreetutions are provided in Section 2. In the following, we study *Jacaranda-like* trees. They are the fixed points of substreetutions defined by



with $\otimes = 0, 1$ and 3 of the $\mathfrak{A}^i$'s are equal to $\mathfrak{B}$ and one is equal to $\mathfrak{A}$.

Exchanging the roles of 1 and 0, and using a symmetry (left↔right), the result will also hold for substreetutions where three of the $\mathfrak{A}^i$'s are equal to $\mathfrak{A}$ and one is equal to $\mathfrak{B}$.

We recall that the choice of which $\mathfrak{A}^i$ equals $\mathfrak{A}$ or $\mathfrak{B}$ is called the *grammar* in [2]. The Jacaranda tree is the one obtained with $\mathfrak{A}^1 = \mathfrak{A}^2 = \mathfrak{A}^4 = \mathfrak{B}$ and $\mathfrak{A}^3 = \mathfrak{A}$, i.e. , for the grammar $BBAB$.

It is clear that there is a one-to-one correspondence between the Jacaranda-like trees and the associated substreetution, since the fixed points (with root 0)