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DEFORMATIONS OF PATH ALGEBRAS OF QUIVERS WITH RELATIONS

Severin BARMEIER & Zhengfang WANG

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DEFORMATIONS OF PATH ALGEBRAS OF QUIVERS WITH RELATIONS

by Severin BARMEIER and Zhengfang WANG

Abstract. — Let $A = \Bbbk Q/I$ be the path algebra of a finite quiver Q modulo a two-sided ideal I of relations, and let R be any reduction system satisfying the diamond condition for I. We give a concrete description of the deformation theory of A by proving that it is equivalent to the deformation theory of R, the latter being controlled by a natural L_∞ algebra constructed explicitly from R. As a consequence, every formal deformation of the associative multiplication on A is, up to gauge equivalence, given by a combinatorial star product associated with a deformation of R.

For polynomial algebras, this combinatorial star product yields quantizations of algebraic Poisson structures on affine space. We give an explicit formula via bidifferential operators associated to graphs, closely paralleling Kontsevich's universal quantization formula, but without transcendental coefficients.

We also give criteria for the existence of algebraizations of formal deformations and interpret them geometrically in terms of algebraic varieties of reduction systems endowed with a natural groupoid action. In this context, the Maurer–Cartan equation of the L_∞ algebra appears as a natural generalization of the Braverman–Gaitsgory criterion for Poincaré–Birkhoff–Witt deformations of Koszul algebras.

Résumé. (**Déformations des algèbres de chemins de carquois et relations**) — Soit $A = \Bbbk Q/I$, où Q est un carquois fini et I un idéal bilatère de relations de l'algèbre de chemins $\Bbbk Q$, et soit R un système de réduction pour I satisfaisant la condition du diamant. Nous donnons une description concrète de la théorie des déformations de A en montrant qu'elle est équivalente à celle du système de réduction R. La théorie des déformations de R est gouvernée par une algèbre L_∞ naturelle, construite à partir de R. Il s'ensuit que toute déformation formelle de la multiplication associative de A est, à équivalence de jauge près, décrite par un produit étoile combinatoire associé à une déformation de R. Dans le cas des algèbres polynomiales, ce produit étoile fournit des quantifications de structures de Poisson algébriques sur l'espace affine.

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Nous en donnons une formule en termes d'opérateurs bidifférentiels indexés par des graphes, très proche de la formule universelle de quantification de Kontsevich, mais sans faire intervenir de coefficients transcendants.

Nous établissons également des critères d'existence d'algébrisations de déformations formelles et en donnons une interprétation géométrique au moyen de variétés algébriques de systèmes de réduction, munies d'une action naturelle d'un groupoïde. Dans ce cadre, l'équation de Maurer–Cartan de l'algèbre L_∞ apparaît comme une généralisation naturelle du critère de Braverman–Gaitsgory pour les déformations de type Poincaré–Birkhoff–Witt des algèbres de Koszul.

CONTENTS

1 Introduction 1
1.1 Background and motivation 1
1.2 Main results 3
1.3 Further applications 10
1.4 Outline of the monograph 11

2 Quivers, path algebras and relations 13
2.1 Quivers 13
2.2 Path algebra 13
2.3 Relations 14

3 Reduction systems and the Diamond Lemma 17
3.1 Reduction systems 17
3.2 The Diamond Lemma 21
3.3 Equivalence of reduction systems 25
3.4 Higher ambiguities 27

4 Projective resolutions 31
4.1 Bar resolutions 31
4.2 Projective resolutions from reduction systems 32

5 Homotopy deformation retract 37
5.1 Comparison morphisms and homotopy 37
5.2 Maps in low degrees 39

6 Deformation theory 43
6.1 Deformations of associative algebras 43
6.2 Deformation theory via DG Lie and L_∞ algebras 47
6.3 Algebraization of formal deformations 54

7 Deformations of path algebras of quivers with relations 57
7.1 Summary 57
7.2 An $L_\infty[1]$ algebra structure 61
7.3 Combinatorial star product 64
7.4 Combinatorial criterion for the Maurer–Cartan equation 71
7.5 Equivalences of deformation problems 74
7.A Computing Hochschild cohomology in degree 2 76

8 Deformations in the non-formal setting 79
8.1 Finiteness conditions 79
8.2 The Diamond Lemma from a deformation-theoretic viewpoint 83

9 Varieties of reduction systems and PBW deformations 85
9.1 Algebraic varieties of reduction systems 85
9.2 Poincaré–Birkhoff–Witt deformations 89
9.3 Algebraizations 95
9.4 Finite-dimensional algebras 96
9.5 Deformations of a local 4-dimensional algebra 97
9.6 Deformations of Brauer tree algebras 100

10 A graphical calculus 105
10.1 Deformation quantization and Kontsevich's Formality Theorem 109
10.2 Combinatorial deformation quantization 113

Acknowledgements 119

Bibliography 121

Index 131

CHAPTER 1

INTRODUCTION

1.1. Background and motivation

The deformation theory of associative algebras studies variations of the associative multiplication on an algebra A with respect to one or several parameters, first studied in a series of articles by Gerstenhaber [Ger64, Ger66, Ger68, Ger74].

A cornerstone of the theory is the Gerstenhaber bracket, which endows the (shifted) Hochschild cochain complex $\mathrm{Hom}_{A^e}(\mathrm{Bar}_{\bullet+1}, A)$ with the structure of a DG Lie algebra. This is precisely the algebraic structure used in the Maurer–Cartan formalism of deformation theory via DG Lie and L_∞ algebras. A fundamental observation is that first-order deformations of the associative multiplication of A correspond to Hochschild 2-cocycles and two first-order deformations are equivalent precisely when the corresponding 2-cocycles differ by a 2-coboundary. As a consequence, the second Hochschild cohomology $\mathrm{HH}^2(A, A)$ can be identified with the set of first-order deformations up to equivalence, and obstructions to extending such deformations to higher orders are seen to lie in $\mathrm{HH}^3(A, A)$.

This classical description is both powerful and elegant. However, the classical theory works with multilinear maps and with the higher structure on the complex $\mathrm{Hom}_{A^e}(\mathrm{Bar}_{\bullet+1}, A)$, where $\mathrm{Bar}_\bullet$ is the bar resolution of A

$$\cdots \longrightarrow A \otimes_{\Bbbk} A^{\otimes_{\Bbbk} n} \otimes_{\Bbbk} A \longrightarrow \cdots \longrightarrow A \otimes_{\Bbbk} A \otimes_{\Bbbk} A \longrightarrow A \otimes_{\Bbbk} A \longrightarrow 0.$$

The enormous size of the bar resolution is often the main obstacle to obtaining a concrete description of the deformations of A. Indeed, such a concrete description is difficult to come by, even in deceptively simple cases such as finite-dimensional algebras [Gab74] or commutative polynomial algebras [Kon03].

Associative deformations can also be related to deformations in a categorical setting. For instance, from a representation-theoretic point of view, deformations of A can be identified with deformations of the module category $\mathrm{Mod}(A)$ as an Abelian

category [LVdB05, LVdB06]. The effect of deformations of A on $\mathrm{Mod}(A)$ is of interest both in representation theory and (noncommutative) algebraic geometry, where $\mathrm{Mod}(A)$ can be viewed as the category of quasi-coherent sheaves on a (noncommutative) affine scheme [Gab62, Kra01]. In the context of (DG enhanced) triangulated categories, deformations of A can be viewed as deformations of the derived category $\mathrm{D}(A)$ [Kel03]. Via derived equivalence, these deformations can be related to many different algebraic or geometric objects, for example to algebraic varieties via the derived category of quasi-coherent sheaves, or to symplectic manifolds via the Fukaya category [Kon94a]. It is a compelling and difficult problem to understand concretely how algebraic deformations of A change the representation-theoretic or geometric properties of these associated objects. For example, deformations of (Abelian or derived) categories of quasi-coherent sheaves are at the heart of the study of noncommutative algebraic geometry [LVdB06, Kal08, Sch16b, BW24]. In symplectic geometry, deformations of (wrapped or partially wrapped) Fukaya categories are closely related to partial compactifications of the underlying symplectic manifold as advocated in Seidel's ICM 2002 address [Sei02].

In this monograph we develop a general combinatorial method to study the deformation theory of path algebras of quivers with relations. More precisely, when A can be written as the path algebra $\Bbbk Q/I$ of a finite quiver Q modulo a two-sided ideal I of relations, we study the deformations of A via the combinatorics of a *reduction system* used by Bergman [Ber78] to state and prove his Diamond Lemma (see §3.1). Our method can be applied using any reduction system satisfying the diamond condition. Such a reduction system always exists (Proposition 3.14) and can be computed algorithmically — often by hand — or obtained from a noncommutative Gröbner basis, if available. This combinatorial approach to deformations of associative algebras turns out to be of considerable practical value, allowing one to easily compute particular classes of examples, as well as give criteria for the existence of algebraizations of formal deformations.

As every finitely generated associative algebra can be written as $\Bbbk Q/I$ where Q has a single vertex with a finite number of loops at this vertex, this method can, for instance, be used to study deformations of

(*i*) commutative algebras such as the polynomial algebra $\Bbbk[x_1, \ldots, x_d]$ whose deformation theory is related to quantizations of Poisson structures

(*ii*) graded associative algebras such as N-Koszul algebras.

Associative algebras of the form $\Bbbk Q/I$, where the quiver Q has multiple vertices, also arise naturally in representation theory, commutative and noncommutative algebraic geometry, and symplectic geometry. Examples include

(*iii*) finite-dimensional associative algebras

(*iv*) endomorphism algebras of tilting bundles on algebraic varieties

(*v*) diagram algebras associated to a diagram (presheaf) of algebras

(*vi*) noncommutative resolutions of singularities

(*vii*) derived endomorphism algebras of (formal) generators of Fukaya categories.

We study examples of type (*i*)–(*iii*) in Chapters 9 and 10 of this monograph. Types (*iv*)–(*vi*), which describe the deformation theory of the Abelian category of (quasi)coherent sheaves, are studied in [BW24]. In case (*vii*), the derived endomorphism algebras of formal generators are naturally graded, and generalizations of the present method to A_∞ deformations of graded algebras are studied in [BW26, BSW^b].

A very short summary of the contents and main applications of this work can also be found in the Oberwolfach Report [Bar24].

1.2. Main results

Throughout $\Bbbk$ denotes a field of characteristic 0.

1.2.1. Equivalences of deformation problems. — Our central theoretical result is the following.

***Theorem 1.1* (*Theorem 7.1*).** — *Let Q be a finite quiver, let $I \subset \Bbbk Q$ be a two-sided ideal of relations and let $A = \Bbbk Q/I$ denote the quotient algebra. Let R be any reduction system satisfying the diamond condition ($\diamond$) for I.*

There is an equivalence of formal deformation problems between

(*i*) *deformations of the associative algebra A*

(*ii*) *deformations of the reduction system R*

(*iii*) *deformations of the relations I.*

The equivalences in Theorem 1.1 are given by isomorphisms of the corresponding deformation functors. To formulate the second deformation problem, we introduce a new notion of equivalence intrinsic to the theory of reduction systems (see Definitions 3.21 and 7.4).

From the equivalence of (*i*) and (*ii*) in Theorem 1.1 we obtain a bijection between

$$\begin{array}{c}\text{formal deformations of } A\\ \text{up to equivalence}\end{array} \quad \longleftrightarrow \quad \begin{array}{c}\text{formal deformations of } R\\ \text{up to equivalence.}\end{array} \tag{1.2}$$

The right-hand side is often easier to determine in practice and we give many applications and examples in this monograph. Further applications are developed in other works [BS23, BW24, Qiu24, BW26, BSW^a, BSW^b] (see §1.3 for a brief outline).

The equivalence between (i) and (ii) in Theorem 1.1 follows from an $\mathrm{L}_\infty[1]$ quasi-isomorphism between the (shifted) classical DG Lie algebra

$$\mathbf{h}(A) = (\mathrm{Hom}_{A^e}(\mathrm{Bar}_{\bullet+2}, A), d, \langle -,- \rangle_{\mathrm{G}}) \tag{1.3}$$

formed by the (doubly shifted) Hochschild complex of A equipped with the (shifted) Gerstenhaber bracket, and a new $\mathrm{L}_\infty[1]$ algebra

$$\mathbf{p}(Q,R) = (\mathrm{Hom}_{A^e}(P_{\bullet+2}, A), \partial, \langle -,- \rangle, \langle -,-,- \rangle, \dots) \tag{1.4}$$

that we construct explicitly (see §1.2.2 for a summary).

***Remark 1.5* (*Shifts and signs*). —** Here, and indeed throughout most of the monograph, we work with the notions of $\mathrm{L}_\infty[1]$ algebras and DG Lie[1] algebras. We have been careful in working with a consistent choice of sign conventions. The notions of $\mathrm{L}_\infty[1]$ and DG Lie[1] algebras — which are simply shifted versions of the more classical notions of L_∞ algebras and DG Lie algebras (see Remark 6.20) — allow us to avoid otherwise rather complicated signs that would appear in the various formulas involving higher brackets. Except when taking a close look at signs, the reader may safely ignore the shift [1].

1.2.2. Homotopy theory of reduction systems. — Reduction systems go back to the theory of abstract rewriting systems which are used in a variety of disciplines, including computer science, logic and linguistics (see e.g. [BN98]). In algebra, they were popularized by Bergman's 1978 paper titled "The Diamond Lemma in ring theory" [Ber78]. The Diamond Lemma relates a quotient algebra $A = \Bbbk Q/I$ to the word combinatorics of a reduction system R satisfying the diamond condition $(\diamond)$ for I. A key notion is that of an *overlap ambiguity* (see Definition 3.9) for R. Iterated overlaps, called *higher ambiguities*, can be used to produce a projective A-bimodule resolution $P_\bullet$ of A. This resolution was constructed by Chouhy–Solotar [CS15], generalizing Bardzell's resolution for monomial algebras [Bar96, Bar97].

We give a recursive formula for the differential of $P_\bullet$ (Proposition 4.3) and prove the following theorem based on an explicit homotopy deformation retract.

***Theorem 1.6* (*Theorems 5.11 and 7.10*). —** *Let Q be a finite quiver, let $I \subset \Bbbk Q$ be a two-sided ideal of relations and let $A = \Bbbk Q/I$ denote the quotient algebra. Let R be any reduction system satisfying the diamond condition $(\diamond)$ for I.*

(i) *There is a special homotopy deformation retract*

$$(P_\bullet, \partial_\bullet) \underset{G_\bullet}{\overset{F_\bullet}{\rightleftarrows}} (\mathrm{Bar}_\bullet, d_\bullet) \circlearrowleft h_\bullet$$

between the resolution $P_\bullet$ associated to the reduction system R and the bar resolution $\mathrm{Bar}_\bullet$ of A.

(*ii*) *The (doubly shifted) cochain complex* $\mathrm{Hom}_{A^e}(P_{\bullet+2}, A)$ *associated to* $P_\bullet$ *naturally carries the structure of an* $L_\infty[1]$ *algebra*

$$\mathbf{p}(Q,R) = (\mathrm{Hom}_{A^e}(P_{\bullet+2}, A), \partial, \langle -,- \rangle, \langle -,-,- \rangle, \dots)$$

obtained by homotopy transfer from the DG Lie$[1]$ *algebra structure on the Hochschild complex* $\mathbf{h}(A)$ *given in* (1.3).

1.2.3. **Deformations of reduction systems.** — The equivalence between deformations of $A = \Bbbk Q/I$ and deformations of R stated in Theorem 1.1 follows from the following result.

***Theorem 1.7* (*Theorem 7.45*).** — *Let Q be a finite quiver, let $I \subset \Bbbk Q$ be a two-sided ideal of relations and let $A = \Bbbk Q/I$ be the quotient algebra. Let R be any reduction system satisfying the diamond condition $(\diamond)$ for I.*

(*i*) *The $L_\infty[1]$ algebra $\mathbf{p}(Q,R)$ naturally controls the deformation theory of R.*

(*ii*) *There exists an $L_\infty[1]$ quasi-isomorphism*

$$\mathbf{p}(Q,R) \longrightarrow \mathbf{h}(A).$$

Therefore, $\mathbf{p}(Q,R)$ also controls the deformation theory of A.

1.2.4. **Combinatorial star products.** — Theorem 1.1 gives an abstract equivalence between (formal) deformation problems associated to an associative algebra $A = \Bbbk Q/I$, and to a reduction system R for I. We show how to use this equivalence to give concrete formulas for deforming the associative multiplication on A. A central role is played by the *combinatorial star product* $\star$ which is given by performing rightmost reductions with respect to the deformation of R.

More precisely, a formal deformation of R is given by an element $\widetilde{\varphi} \in P^2 \mathbin{\widehat{\otimes}} \mathfrak{m}$ whose associated formal reduction system $R_{\varphi+\widetilde{\varphi}}$ satisfies a formal analogue of the diamond condition (see §7.1.1.2). Here $(B, \mathfrak{m})$ is a complete local Noetherian $\Bbbk$-algebra, for example $B = \Bbbk[\![t]\!]$ with $\mathfrak{m} = (t)$. The combinatorial star product associated to $\widetilde{\varphi}$ is a combinatorially defined operation $\star\colon A \otimes A \longrightarrow A \mathbin{\widehat{\otimes}} B$ (see Definition 7.21) given by performing rightmost reductions with respect to the deformed reduction system $R_{\varphi+\widetilde{\varphi}}$.

We show that $\star$ coincides with the bilinear operation obtained from homotopy transfer along the $L_\infty[1]$ quasi-isomorphism of Theorem 1.7 (see Theorem 7.24). This has the following consequences.

***Theorem 1.8* (*Theorem 7.40 and Corollary 7.34*).** — *Let $A = \Bbbk Q/I$ and let R be a reduction system satisfying the diamond condition $(\diamond)$ for I. Let $\star$ be the combinatorial star product associated to an element $\widetilde{\varphi} \in P^2 \mathbin{\widehat{\otimes}} \mathfrak{m}$.*

(*i*) *$\widetilde{\varphi}$ is a Maurer–Cartan element of $\mathbf{p}(Q,R)\mathbin{\widehat{\otimes}}\mathfrak{m}$ if and only if $\star$ is associative on overlap ambiguities. In this case, $\star$ defines an associative multiplication on $A\mathbin{\widehat{\otimes}}B$.*

(*ii*) *Any deformation of A over a complete local Noetherian $\Bbbk$-algebra $(B,\mathfrak{m})$ is gauge equivalent to $(A\mathbin{\widehat{\otimes}}B,\star)$ for some Maurer–Cartan element $\widetilde{\varphi}$ of $\mathbf{p}(Q,R)\mathbin{\widehat{\otimes}}\mathfrak{m}$.*

Thus $\star$ gives both a combinatorial criterion for the Maurer–Cartan equation of $\mathbf{p}(Q,R)\mathbin{\widehat{\otimes}}\mathfrak{m}$, and an explicit formula for the corresponding formal deformation of the associative multiplication on A.

In a general L_∞ or $\mathrm{L}_\infty[1]$ algebra, the notion of equivalence between Maurer–Cartan elements is that of homotopy equivalence, which can be rather cumbersome in practice. Theorems 1.7 and 1.8 imply in particular that we may instead work with the intrinsic and practicable notion of equivalence between reduction systems, also for understanding gauge equivalences between formal deformations of associative algebras.

1.2.5. Hochschild cohomology. — Recall from §1.1 that $\mathrm{HH}^2(A,A)$ parametrizes the (first-order) deformations of A. When we restrict to deformations over the dual numbers $B=\Bbbk[\varepsilon]/(\varepsilon^2)$, the combinatorial star product provides a simple combinatorial method for computing $\mathrm{HH}^2(A,A)$.

***Proposition 1.9* (*Proposition 7.47*).** — *Let $A=\Bbbk Q/I$ and let R be any reduction system satisfying the diamond condition $(\diamond)$ for I. Then $\mathrm{HH}^2(A,A)$ is isomorphic to the space of first-order deformations of R modulo equivalence.*

This perspective often proves very useful in practice. In [BW26] we use this approach to show that $\mathrm{HH}^2(A,A)\neq 0$ for algebras A given by the Koszul duals of extended Khovanov arc algebras, settling Stroppel's conjecture from her ICM 2010 address [Str10].

1.2.6. Generalization of the Diamond Lemma. — Theorem 1.1 concerns formal deformations. In Chapter 8 we consider certain finiteness conditions under which deformations of reduction systems can be understood in the non-formal setting, i.e. when considering *strict* deformations.

We introduce two finiteness conditions (FC0) and (FC1) which are weaker than reduction-uniqueness (FC2) used in the classical Diamond Lemma (see §8.1). Under the assumptions (FC0) or (FC1) we obtain the following generalization of the Diamond Lemma (see Theorem 8.3 and Remark 8.4).

***Theorem 1.10* (*Theorem 8.3*).** — *Let $A=\Bbbk Q/I$ and let $R=\{(s,\varphi_s)\}_{s\in S}$ be a reduction system satisfying the diamond condition $(\diamond)$ for I. Let $\widetilde{\varphi}\in P^2\simeq\mathrm{Hom}(\Bbbk S,A)$ and consider the following four statements*

(1) *$\widetilde{\varphi}$ satisfies the Maurer–Cartan equation*

(2) *all overlaps of $R_{\varphi+\widetilde{\varphi}} = \{(s, \varphi_s + \widetilde{\varphi}_s)\}_{s\in S}$ resolve via rightmost reductions*

(3) *the combinatorial star product $\star$ associated to $\widetilde{\varphi}$ is associative*

(4) *the image of the set Irr_S under the projection*

$$\Bbbk Q \longrightarrow \Bbbk Q / \langle s - \varphi_s - \widetilde{\varphi}_s \rangle_{s\in S} =: A_{\varphi+\widetilde{\varphi}}$$

forms a $\Bbbk$-basis of $A_{\varphi+\widetilde{\varphi}}$.

We then have the following:

(*i*) *If $\widetilde{\varphi}$ satisfies* (FC0)*, then* (1) *and* (2) *are equivalent.*

(*ii*) *If $\widetilde{\varphi}$ satisfies* (FC1)*, then* (1)–(3) *are equivalent.*

(*iii*) *If $\widetilde{\varphi}$ satisfies* (FC2)*, then* (1)–(4) *are equivalent.*

Here and throughout, $S \subset Q_{\geqslant 2}$ is a set of paths (which can be viewed as a set of "leading terms" of the ideal I) which together with the map $\varphi \in \mathrm{Hom}(\Bbbk S, A)$ completely determines the reduction system R (see Chapter 3).

Under the assumption of reduction-uniqueness, i.e. under the finiteness condition (FC2), the classical Diamond Lemma can also be understood in deformation-theoretic terms (see §8.2) as follows.

Diamond Lemma	*Deformation-theoretic viewpoint*
(*i*) all overlap ambiguities of R are resolvable	(*i*) φ satisfies the Maurer–Cartan equation of $\mathbf{p}(Q, R_{\mathrm{mon}})$
(*ii*) R satisfies $(\diamond)$ for I	(*ii*) R is a strict deformation of R_{mon}
(*iii*) the image of the set Irr_S of irreducible paths under the projection $\Bbbk Q \xrightarrow{\pi} A$ is a $\Bbbk$-basis of A.	(*iii*) A is a strict deformation of A_{mon}.

It gives a precise sense in which an algebra A can be viewed as a strict deformation of a monomial algebra.

1.2.7. Algebraic varieties of reduction systems. — It is often desirable to obtain "strict" deformations of an algebra by finding an algebraization of a formal deformation and evaluating the deformation parameters to constants — if possible. In this case, the original algebra and its strict deformation have the *same* basis, and may thus be compared on equal footing. In Chapter 9 we study this problem of algebraization from a geometric point of view: by considering certain subspaces of P^2 defined by the degree conditions $(<)$ and $(\prec)$ (see Definitions 9.1 and 9.9).

In particular, the degree conditions $(<)$ and $(\prec)$ allow us to define affine algebraic varieties (or, more naturally, affine *schemes*) $V_<$ and $V_\prec$ of reduction systems, whose equations are given by the Maurer–Cartan equation of $\mathbf{p}(Q,R)$. These varieties were studied by Green–Hille–Schroll [GHS21] from the viewpoint of noncommutative Gröbner bases.

We show that our notion of equivalence of reduction systems introduced in Definition 3.21 gives rise to a natural new groupoid action on these varieties, where two reduction systems lie in the same orbit if and only if their associated algebras are isomorphic. These varieties can provide algebraizations of formal deformations, even for possibly infinite-dimensional algebras (see also §1.2.9).

***Theorem 1.11* (*Theorem 9.4*). —** *Let $A = \Bbbk Q/I$ and let $R = \{(s,\varphi_s)\}_{s\in S}$ be a reduction system satisfying the diamond condition $(\diamond)$ for I. Given any admissible order $\prec$ on Q, let $\dim \mathrm{Hom}(\Bbbk S, A)_\prec = N < \infty$ and let $V_\prec$ be the affine scheme of algebras given by reduction-unique reduction systems satisfying the degree condition $(\prec)$.*

- *(i) The Maurer–Cartan equation of $\mathbf{p}(Q,R)$ gives the explicit algebraic equations for the scheme $V_\prec \subset \mathbb{A}^N$.*
- *(ii) The groupoid $G \rightrightarrows \mathrm{MC}$ of reduction-unique reduction systems restricts to a groupoid $G_\prec \rightrightarrows V_\prec$ and two reduction systems in $V_\prec$ are equivalent if and only if they lie in the same orbit of $G_\prec \rightrightarrows V_\prec$.*
- *(iii) The Zariski tangent space of $V_\prec$ at the point $\widetilde{\varphi} \in V_\prec$ is isomorphic to the space of 2-cocycles in $\mathrm{Hom}(\Bbbk S, A_{\varphi+\widetilde{\varphi}})_\prec$ where $A_{\varphi+\widetilde{\varphi}} = \Bbbk Q/\langle s - \varphi_s - \widetilde{\varphi}_s\rangle_{s\in S}$. In particular, we have a* Kodaira–Spencer map $\mathrm{KS}\colon \mathrm{T}_{\widetilde{\varphi}}V_\prec \longrightarrow \mathrm{HH}^2(A_{\varphi+\widetilde{\varphi}}, A_{\varphi+\widetilde{\varphi}})$.
- *(iv) The Zariski tangent space to the orbit of the groupoid $G_\prec \rightrightarrows V_\prec$ at $\widetilde{\varphi}$ is contained in the space of 2-coboundaries in $\mathrm{Hom}(\Bbbk S, A_{\varphi+\widetilde{\varphi}})_\prec$.*

1.2.8. PBW deformations. — When considering the degree condition $(<)$ (see Definition 9.9) given by compatibility with the path length, the Maurer–Cartan equation of $\mathbf{p}(Q,R)$ can be viewed as a natural analogue of the Braverman–Gaitsgory criterion for PBW deformations of Koszul algebras, thereby extending this criterion to non-Koszul algebras and non-PBW deformations.

***Theorem 1.12* (*Theorem 9.12*). —** *Let $A = \Bbbk Q/I = \mathrm{T}_{\Bbbk Q_0}(\Bbbk Q_1)/I$ be a finitely generated algebra which is graded by path length. Let $R = \{(s,\varphi_s)\}_{s\in S}$ be a reduction system satisfying $(\diamond)$ for I such that $S \subset Q_N$ and $\varphi_s \in \Bbbk Q_N$ for all $s \in S$ and assume that $S_3 \subset Q_{N+1}$.*

Then A is an N-Koszul algebra and the degree components of the Maurer–Cartan equation of $\mathbf{p}(Q,R)$ can naturally be identified with the Braverman–Gaitsgory criterion.

We show that the varieties of Theorem 1.11 give rise to a larger variety of filtered deformations which generally strictly contains the variety of PBW deformations.

***Theorem 1.13* (*Theorem 9.16*). —** *Let $A = \Bbbk Q/\langle S\rangle$. Let T be the set of admissible orders compatible with the path length.*

(i) $\bigcup_{\prec\in T} V_\prec$ *is an algebraic variety.*

(ii) $\bigcap_{\prec\in T} V_\prec = V_<$ *is the variety of PBW deformations of A.*

Note that the groupoid action of Theorem 1.11 restricts to the subvariety of PBW deformations, so the notion of equivalence of reduction systems gives a natural way of studying the isomorphism classes of PBW deformations. Note that A need not be finite-dimensional. In fact, the varieties of reduction systems can also detect strict deformations of formally rigid algebras such as the Weyl algebra (see Example 9.7).

1.2.9. **Algebraization of formal deformations. —** From a more algebraic perspective, the varieties of Theorems 1.11 and 1.13 of strict deformations can be viewed as providing sufficient criteria for certain formal deformations to admit an algebraization.

***Proposition 1.14* (*Proposition 9.20*). —** *Let $A = \Bbbk Q/I$ and let $R = \{(s,\varphi_s)\}_{s\in S}$ be a reduction system satisfying the diamond condition for I.*

Let $(B,\mathfrak{m})$ be a complete local Noetherian $\Bbbk$-algebra which is the adic completion of a finitely generated commutative algebra $(B_0,\mathfrak{m}_0)$. Let $\widetilde{\varphi}$ be an element of $\mathrm{Hom}(\Bbbk S, A)_\prec \otimes \mathfrak{m}_0$ *or* $\mathrm{Hom}(\Bbbk S, A)_< \otimes \mathfrak{m}_0$ *and let $\star$ be the corresponding combinatorial star product.*

(i) For any $a,b \in A$ the product $a\star b$ is a finite sum. In particular, $\star$ is a well-defined B_0-bilinear operation on $A\otimes B_0$.

(ii) If $\widetilde{\varphi}$ satisfies the Maurer–Cartan equation of $\mathbf{p}(Q,R)\mathbin{\widehat{\otimes}}\mathfrak{m}$, then the formal deformation $(A\mathbin{\widehat{\otimes}}B,\star)$ admits $(A\otimes B_0,\star)$ as an algebraization.

1.2.10. **A graphical description of the combinatorial star product. —** In Chapter 10 we show that for the polynomial algebra $A = \Bbbk[x_1,\dots,x_d]$ the combinatorial star product $\star$ admits the following graphical description.

***Theorem 1.15* (*Theorem 10.3 and Proposition 10.17*). —** *Let $A = \Bbbk[x_1,\dots,x_d]$ be the polynomial algebra and consider the reduction system $R = \{(x_jx_i, x_ix_j)\}_{1\leqslant i<j\leqslant d}$ which satisfies the diamond condition for the ideal $I = \langle x_jx_i - x_ix_j\rangle_{1\leqslant i<j\leqslant d}$ of commutativity relations in the free algebra in d variables viewed as the path algebra of a quiver with one vertex and d loops.*

Given any degree 0 element $\widetilde{\varphi}\in\mathbf{p}(Q,R)\mathbin{\widehat{\otimes}}\mathfrak{m}$, the associated combinatorial star product $\star$ can be given as

$$f\star g = \sum_{k\geqslant 0}\sum_{\Pi\in\mathfrak{G}^{\mathrm{C}}_{k,2}} \mathrm{C}_\Pi(f,g) \qquad f,g\in A \tag{1.16}$$

where $\mathfrak{G}^{\mathrm{C}}_{k,2}$ is a set of graphs which is in bijection with the set of Kontsevich graphs without cycles equipped with an ordering of the incoming edges at each vertex, and each C_Π is a bidifferential operator associated to a graph Π.

1.2.11. Deformation quantization. — Deformations of the polynomial algebra $A = \Bbbk[x_1, \dots, x_d]$ as in §1.2.10 are of central importance in deformation quantization. Working with deformations over $\Bbbk[\![\hbar]\!]$ with maximal ideal $\mathfrak{m} = (\hbar)$, we have the following result.

***Theorem 1.17* (*Theorem 10.19 and Proposition 10.25*). —** (*i*) *Any algebraic Poisson structure on d-dimensional affine space $\mathbb{A}^d$ can be quantized by a combinatorial star product.*

(*ii*) *Given any Maurer–Cartan element $\widetilde{\varphi} = \widetilde{\varphi}_1\hbar + \widetilde{\varphi}_2\hbar^2 + \cdots$ of $\mathbf{p}(Q, R) \mathbin{\widehat{\otimes}} (\hbar)$, the combinatorial star product gives an explicit formula for the quantization of the Poisson structure corresponding to $\widetilde{\varphi}_1$. Moreover, any algebraic Poisson structure can be obtained in this way.*

Note that the star product in Theorem 1.17 admits an explicit graphical formula by Theorem 1.15. A notable feature of the purely combinatorial approach we take in this work is that it recovers the graphs without cycles used in Kontsevich's universal quantization formula (10.14), which can be motivated from Feynman diagrams [CF00]. Whereas Kontsevich's quantization formula uses certain weights related to multiple zeta values, no extra weights enter in the combinatorial formula (1.16), which may prove useful in obtaining a quantization formula defined over $\mathbb{Q}$ (for examples, see §10.2). The combinatorial star product has also proved useful in studying convergence and continuity properties of star products needed to produce strict quantizations [BS23], where the formal parameter $\hbar$ (standing in for Planck's constant) may be evaluated to a complex number.

1.3. Further applications

The methods developed in this monograph can also be applied to study deformations arising in algebraic geometry, singularity theory, representation theory and symplectic geometry. In [BW24] we show how deformations of path algebras of quivers with relations can be used to study deformations of the Abelian category of coherent sheaves on any separated Noetherian scheme of finite type over $\Bbbk$. By results of Lowen–Van den Bergh [LVdB05, LVdB06], deformations of $\mathrm{coh}(X)$ for some separated Noetherian scheme X are equivalent to deformations of the diagram algebra $\mathscr{O}_X|_{\mathfrak{U}}$! obtained from restricting the structure sheaf of X to an affine open cover $\mathfrak{U}$ closed under intersections. When X is of finite type over $\Bbbk$, this diagram algebra can naturally be written as the path algebra of a finite quiver with relations which admits a finite reduction system. In [BW24] we also study deformations of $\mathrm{coh}(X)$ for varieties with a tilting bundle, whose endomorphism algebra is again of the form $\Bbbk Q/I$.

This method also allows us to study the effect of associative, or more generally, A_∞ deformations of graded gentle algebras on their representation theory and on the symplectic geometry of the associated surface models [BSW^a, BSW^b]. In [BW26] we also apply our results in this monograph to settle Stroppel's Conjecture from her ICM 2010 address [Str10] on the Hochschild cohomology and A_∞ deformations of extended Khovanov arc algebras appearing in geometric representation theory, link homology and symplectic geometry [Str09], which also describe algebraic deformations of Fukaya–Seidel categories associated to certain Hilbert schemes of points on algebraic surfaces.

1.4. Outline of the monograph

Chapter 2 recalls some basic notions on quivers and their path algebras. Chapter 3 introduces reduction systems and Bergman's Diamond Lemma. We define an intrinsic notion of equivalence between reduction systems (Definition 3.21), recall the notion of a noncommutative Gröbner basis and explain how it naturally gives rise to a reduction system. We also give a detailed account of the combinatorics of reductions and of the higher ambiguities generalizing overlap ambiguities.

Chapter 4 recalls the projective resolution $P_\bullet$ associated to a reduction system and gives explicit recursive formulas for its differential and contracting homotopy (see Proposition 4.3). Chapter 5 constructs an explicit homotopy deformation retract between the projective resolution $P_\bullet$ and the bar resolution $\mathrm{Bar}_\bullet$. We describe the maps in low degrees explicitly, as they will later be used in our description of the deformation theory of A.

Chapter 6 contains a brief review of deformation theory via DG Lie and $L_\infty[1]$ algebras, including the classical theory of strict and formal deformations of associative algebras. In Chapter 7 we prove our main result on the equivalences between the formal deformations of reduction systems, ideals and path algebras of quivers with relations. In §7.A we use this equivalence to give a combinatorial description of $\mathrm{HH}^2(A, A)$ by restricting to deformations over $\Bbbk[\varepsilon]/(\varepsilon^2)$.

Chapter 8 studies deformations in the non-formal setting and gives a deformation-theoretic interpretation of the Diamond Lemma. Chapter 9 studies algebraic varieties of reduction systems whose points may be viewed as strict deformations of A, by introducing suitable degree conditions. We show that the Maurer–Cartan equation of $\mathbf{p}(Q, R)$ may be viewed as a generalization of the classical Braverman–Gaitsgory criterion for PBW deformations of N-Koszul algebras (see Theorem 9.12), and we give several examples.

Finally, Chapter 10 gives a graphical description of the combinatorial star product for the polynomial algebra. This star product can be used to obtain explicit formulas for deformation quantizations of algebraic Poisson structures on $\mathbb{A}^d$.