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FROM HOMOTOPY LINKS TO STRATIFIED HOMOTOPY THEORIES

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Abstract. **–** In previous work, the first author defined homotopy theories for stratified spaces from a simplicial and a topological perspective. In both frameworks stratified weak-equivalences are detected by suitable generalizations of homotopy links. These two frameworks are connected through a stratified version of the classical adjunction between the realization and the functor of singular simplices. Using a modified version of this adjunction, the first author showed that over a fixed poset of strata the two homotopy theories were equivalent. Building on this result we now show that the unmodified adjunction induces an equivalence between the global homotopy theories of stratified spaces and of stratified simplicial sets. We do so through an in depth study of the homotopy links. As a consequence, we prove that the classical homotopy theory of conically stratified spaces embeds fully-faithfully into the homotopy theory of all stratified spaces.

Résumé **(Des liens homotopiques aux théories homotopiques stratifiées)**

Dans des travaux précédents, le premier auteur a défini des théories homotopiques pour les espaces stratifiés, d'un point de vue simplicial, et d'un point de vue topologique. Dans ces deux cas, les équivalences faibles stratifiés sont détectées par des généralisations appropriées des entrelacs homotopiques. Ces deux points de vue sont reliés par une version stratifiée de l'adjonction classique donnée par la réalisation et le foncteur des simplexes singuliers. En utilisant une version modifiée de cette adjonction, le premier auteur a montré que – au dessus d'un ensemble ordonné de strates fixé – les deux théories homotopiques sont équivalentes. Dans cet article, nous allons au-delà de ce résultat et montrons que l'adjonction originale induit une équivalence entre les théories homotopiques globales des espaces stratifiés et des ensembles simpliciaux stratifiés. Ce résultat découle d'une étude approfondie des entrelacs homotopiques. On en déduit que la théorie homotopique classique des espaces coniquement stratifiés se plonge, de façon pleinement fidèle, dans la théorie homotopique des espaces stratifiés.

CHAPTER 1

INTRODUCTION

Stratifications and stratified spaces were first introduced by Whitney [**38**], Thom [**35**] and Mather to describe manifolds with singularities. In this context, stratifications correspond to decompositions into strata that are themselves manifolds and satisfy certain compatibility conditions. Stratified spaces of this kind are usually called pseudomanifolds, since they provide an extension of the class of manifolds over which many invariants can be extended while retaining their key properties. Most notably, intersection cohomology, introduced by Goresky and MacPherson in [**19**] is an extension of singular cohomology to pseudomanifolds that still satisfies Poincaré Duality. Intersection cohomology is only an invariant up to stratum-preserving homotopy equivalence, not arbitrary homotopy equivalences, which motivates the developpment of a homotopy theory for stratified spaces. This question was already asked by Goresky and MacPherson (see for example [**3**, Problems 4 and 11]).

Motivated by the study of the homotopical properties of stratified objects, Quinn [**32**] introduced a more general notion of stratified spaces: Homotopically stratified sets. While for pseudomanifolds compatibility conditions between the strata are characterized by geometric links, homotopically stratified sets make use of homotopy links (or holinks), which are spaces of paths going from one stratum to another. As Miller later showed in [**27**, Theorem 6.3], stratified homotopy equivalences between homotopically stratified sets can be fully characterized as those maps inducing homotopy equivalences on all strata and homotopy link. This inspired the idea that homotopy links and strata should be the basic blocks for defining a stratified homotopy type.

Let us now recall a few classical results of homotopy theory: In [**31**] Quillen introduced the notion of a model category and immediately gave two seminal examples. The model category of topological spaces, Top, and the model category of simplicial sets, sSet. He also showed that the adjunction relating those two categories $|-|\colon \mathrm{sSet} \leftrightarrow \mathrm{Top}\colon \mathrm{Sing}$ was in fact a Quillen equivalence, meaning that the two homotopy theories are equivalent. This is a key result in homotopy theory since it allows one to prove results about the homotopy theory of spaces while working in a purely combinatorial setting.