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# ON A CONJECTURE OF KAZHDAN AND POLISHCHUK

BY OLIVIER DEBARRE

ABSTRACT. — We discuss a conjecture made by David Kazhdan and Alexander Polishchuk at the 2022 ICM about a variety naturally attached to any stable vector bundle of rank 2 and degree $2g-1$ on a smooth projective complex curve of genus $g$.

RÉSUMÉ (*Sur une conjecture de Kazhdan et Polishchuk*). — Nous discutons une conjecture énoncée par David Kazhdan et Alexander Polishchuk au ICM 2022 portant sur une variété naturellement attachée à tout fibré vectoriel stable de rang 2 et degré $2g-1$ sur une courbe projective lisse complexe de genre $g$.

## 1. The conjecture

The aim of this note is to discuss the following conjecture made by David Kazhdan and Alexander Polishchuk. It appeared in the survey paper [5] as Conjecture 3.6. Everything is over the field of complex numbers.

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CONJECTURE 1.1 (Kazhdan–Polishchuk). — *Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $E$ be a stable vector bundle on $C$ of rank 2 and degree $2g-1$. Let*

$$\mathscr{F}_E \coloneqq \{([L], s) \mid [L] \in \mathrm{Pic}^0(C),\ s \in \mathbf{P}(H^0(C, E \otimes L))\}. \tag{1}$$

*Then,*

(a) *$\mathscr{F}_E$ is irreducible,*
(b) *$\dim(\mathscr{F}_E) = g$,*
(c) *$\mathscr{F}_E$ has rational singularities.*

When $E$ is a *general* stable vector bundle on $C$ of rank 2 and degree $2g-1$, the scheme $\mathscr{F}_E$ is smooth irreducible of dimension $g$ (Proposition 2.2). The aim of this note is to prove items (a) and (b) above for all stable $E$ and all $g \geq 2$.

THEOREM 1.2. — *Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $E$ be a stable vector bundle on $C$ of rank 2 and degree $2g-1$. Then $\mathscr{F}_E$ is an irreducible, normal, local complete intersection of dimension $g$.*

Conjecture 1.1 has a natural extension to stable vector bundles $E$ of rank $n$ and degree $n(g-1)+1$. When $E$ is general on a given curve $C$ of genus $g \geq 2$, the scheme $\mathscr{F}_E$ is still smooth irreducible of dimension $g$ (Proposition 2.2). However, in Section 4.3, we construct stable vector bundles $E$ for which the scheme $\mathscr{F}_E$ is reducible, in any rank at least $g+1$ and, when $C$ is hyperelliptic, in any rank at least 3.

After a first version of this note was written, I sent it to Alexander Braverman, who directed me to the eprint [11], where some progress had been made on Conjecture 1.1 (called Conjecture B in that paper). In particular, the conjecture is proved there for any curve of genus 2 ([11, Theorem 4.20]) and any nonhyperelliptic curve of genus 3 ([11, Theorem 4.34]). This first version of the present note also contained a proof of the conjecture for $g = 2$ but, since it was essentially the same as in [11, Theorem 4.20], I decided in this second version to include instead a slightly different proof that makes use of Theorem 1.2.

To prove Theorem 1.2, we establish two conjectures from [11]: their Conjecture C(i) is our Lemma 3.2 (proved by Brivio and Verra) and their Conjecture C(ii) is our Lemma 3.4. Conjecture C(ii) implies, by [11, Lemma 4.9], the normality of $\mathscr{F}_E$.

## 2. Properties of the scheme $\mathscr{F}_E$ and the conjecture for general $E$

In this section, $C$ is a smooth projective curve of genus $g \geq 2$ and $E$ is a vector bundle on $C$ of rank $n \geq 1$ and degree $d \coloneqq n(g-1)+1$. The set $\mathscr{F}_E$ was defined in (1).

**2.1. General properties of the scheme $\mathscr{F}_E$.** — For all $[L] \in \mathrm{Pic}^0(C)$, one has $h^0(C, E \otimes L) \geq \chi(C, E \otimes L) = \chi(C, E) = 1$, hence the first projection

$$\mathsf{pr}_1 \colon \mathscr{F}_E \longrightarrow \mathrm{Pic}^0(C)$$

is dominant. Its fibers are projective spaces, hence $\mathscr{F}_E$ is connected.

Using a standard argument, we define a scheme structure on $\mathscr{F}_E$ and show that it has dimension at least $g$ at every point.

PROPOSITION 2.1. — *Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $E$ be a vector bundle on $C$ of rank $n$ and degree $n(g-1)+1$. The scheme $\mathscr{F}_E$ has dimension at least $g$ everywhere and is a local complete intersection at every point where its dimension is exactly $g$.*

*Proof.* — Let $D$ be an effective divisor on $C$ of very large degree $e$, so that $H^1(C, E \otimes L(D)) = 0$ for all $[L] \in \mathrm{Pic}^0(C)$. When $[L]$ varies in $\mathrm{Pic}^0(C)$, the vector spaces $H^0(C, E \otimes L(D))$ are then the fibers of a vector bundle $\mathscr{F}$ on $\mathrm{Pic}^0(C)$ of rank $\chi(C, E(D)) = 1 + ne$, whereas the vector spaces $H^0(C, E \otimes L(D)|_D)$ are the fibers of a vector bundle $\mathscr{G}$ on $\mathrm{Pic}^0(C)$ of rank $m \coloneqq ne$. The kernel at a point $[L]$ of the restriction morphism

$$\alpha \colon \mathscr{F} \longrightarrow \mathscr{G} \tag{2}$$

is $H^0(C, E \otimes L)$. The vector bundles $\mathscr{F}$ and $\mathscr{G}$ have respective ranks $m+1$ and $m$.

Set

$$\mathscr{X} \coloneqq \{(u, [x]) \in \mathrm{Hom}(\mathbf{C}^{m+1}, \mathbf{C}^m) \times \mathbf{P}(\mathbf{C}^{m+1}) \mid u(x) = 0\}.$$

Looking at the projection $\mathscr{X} \to \mathbf{P}(\mathbf{C}^{m+1})$, we see that $\mathscr{X}$ is smooth of codimension $m$ in $\mathrm{Hom}(\mathbf{C}^{m+1}, \mathbf{C}^m) \times \mathbf{P}(\mathbf{C}^{m+1})$.

The variety $\mathrm{Pic}^0(C)$ is covered by open subsets $U$ over which the vector bundles $\mathscr{F}$ and $\mathscr{G}$ are trivial. Choices of trivializations define a morphism $U \to \mathrm{Hom}(\mathbf{C}^{m+1}, \mathbf{C}^m)$ such that

$$\mathsf{pr}_1^{-1}(U) = U \times_{\mathrm{Hom}(\mathbf{C}^{m+1}, \mathbf{C}^m)} \mathscr{X}.$$

This defines a scheme structure on $\mathscr{F}_E$ (independent of the choices made) that makes it into a subscheme of $\mathbf{P}(\mathscr{F})$.

Since $\mathscr{F}_E$ is not empty, it follows that it has codimension at most $m$ in $\mathbf{P}(\mathscr{F})$ everywhere and that it is a local complete intersection at every point where its codimension is exactly $m$. □

**2.2. The conjecture for general vector bundles.** — Stable vector bundles $E$ considered here have a fine moduli space $U_C(n, d)$ (since $n$ and $d$ are relatively prime, stability and semistability are equivalent). This is a smooth projective irreducible variety of dimension $n^2(g-1)+1$. When one fixes the