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ULTRAMETRIC ANALOGUES OF ULAM STABILITY OF GROUPS

BY FRANCESCO FOURNIER-FACIO

ABSTRACT. — We study stability of metric approximations of countable groups with respect to groups endowed with ultrametrics, the main case study being a p-adic analogue of Ulam stability, where we take $\mathrm{GL}_n(\mathbb{Z}_p)$ as approximating groups instead of $\mathrm{U}(n)$. For finitely presented groups, the ultrametric nature implies equivalence of the pointwise and uniform stability problems, and the profinite one implies that the corresponding approximation property is equivalent to residual finiteness. Moreover, a group is uniformly stable if and only if its largest residually finite quotient is. We provide several examples of uniformly stable groups, including finite groups, virtually free groups, some groups acting on rooted trees, and certain lamplighter and (Generalised) Baumslag–Solitar groups. We construct a finitely generated group that is not uniformly stable. Finally, we prove and apply a (bounded) cohomological criterion for stability of a finitely presented group.

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Résumé (*Analogues ultramétriques de la stabilité des groupes au sense d'Ulam*). — Nous étudions la stabilité des approximations métriques de groupes dénombrables relativement à des groupes munis d'ultramétriques, le cas d'étude principal étant un analogue p-adique de la stabilité d'Ulam, où nous prenons $\mathrm{GL}_n(\mathbb{Z}_p)$ comme groupes approximants au lieu de $\mathrm{U}(n)$. Pour les groupes de présentation finie, la nature ultramétrique implique l'équivalence des problèmes de stabilité ponctuelle et uniforme, et le caractère profini implique que la propriété d'approximation correspondante est équivalente à la finitude résiduelle. De plus, un groupe est uniformément stable si et seulement si son plus grand quotient résiduellement fini l'est. Nous fournissons plusieurs exemples de groupes uniformément stables, y inclus les groupes finis, les groupes virtuellement libres, certains groupes agissant sur des arbres enracinés, ainsi que certains groupes lamplighter et groupes de Baumslag–Solitar (généralisés). Nous construisons un groupe de type fini qui n'est pas uniformément stable. Enfin, nous démontrons et appliquons un critère cohomologique (borné) pour la stabilité d'un groupe de présentation finie.

1. Introduction

Let Γ be a countable discrete group, and let $\mathcal{G}$ be a family of groups G equipped with bi-invariant metrics d_G. The question of *stability of* Γ *with respect to* $\mathcal{G}$ asks whether a map $\varphi\colon \Gamma \to G \in \mathcal{G}$ that is a homomorphism up to a small error is close to an actual homomorphism $\psi\colon \Gamma \to G$. This can be made rigorous in (at least) two different ways, which depend on whether one wants the errors and closeness to be pointwise or uniform. The notion of "error" is defined as follows:

Definition 1.1. — Let $\varphi\colon \Gamma \to G \in \mathcal{G}$ be a map. We define the *defect of* φ *at* $(g,h) \in \Gamma^2$ to be

$$\mathrm{def}_{g,h}(\varphi) \coloneqq d_G(\varphi(gh), \varphi(g)\varphi(h)).$$

The *defect of* φ is

$$\mathrm{def}(\varphi) \coloneqq \sup_{g,h\in\Gamma} \mathrm{def}_{g,h}(\varphi).$$

A sequence of maps $(\varphi_n\colon \Gamma \to G_n \in \mathcal{G})_{n\geq 1}$ is called a *pointwise asymptotic homomorphism* if $\mathrm{def}_{g,h}(\varphi_n) \xrightarrow{n\to\infty} 0$ for all $(g,h) \in \Gamma^2$; and a *uniform asymptotic homomorphism* if $\mathrm{def}(\varphi_n) \xrightarrow{n\to\infty} 0$.

Other commonly used terms are *almost-representation* [86] for the pointwise notion and *quasi-representation* [112] or *ε-representation* [75] for the uniform notion. The notion of "close" is defined as follows:

Definition 1.2. — Given two maps $\varphi, \psi\colon \Gamma \to G \in \mathcal{G}$, we define their *distance at* $g \in \Gamma$ to be

$$\mathrm{dist}_g(\varphi,\psi) \coloneqq d_G(\varphi(g), \psi(g));$$

and their *distance* to be

$$\mathrm{dist}(\varphi,\psi) \coloneqq \sup_{g\in\Gamma} \mathrm{dist}_g(\varphi,\psi).$$

Two sequences of maps $(\varphi_n,\psi_n\colon \Gamma \to G_n \in \mathcal{G})_{n\geq 1}$ are said to be *pointwise asymptotically close* if $\mathrm{dist}_g(\varphi_n,\psi_n) \xrightarrow{n\to\infty} 0$ for all $g\in\Gamma$; and *uniformly asymptotically close* if $\mathrm{dist}(\varphi_n,\psi_n) \xrightarrow{n\to\infty} 0$.

This leads to two notions of stability, which we attribute to Ulam, after [118]:

Definition 1.3 (Ulam). — The group Γ is *pointwise $\mathcal{G}$-stable* if every pointwise asymptotic homomorphism is pointwise asymptotically close to a sequence of homomorphisms. It is *uniformly $\mathcal{G}$-stable* if every uniform asymptotic homomorphism is uniformly asymptotically close to a sequence of homomorphisms.

Early mentions of these problems can be found in works of von Neumann [96] and Turing [117]. The problem of pointwise stability of $\mathbb{Z}^2$ with respect to certain families of matrices, for instance, self-adjoint and with the operator norm, received considerable attention during the second half of the twentieth century (see [79] and the references therein). In [118, Chapter 6], Ulam discusses more generally the question of stability of certain functional equations: because of this, the term *Ulam stability* was introduced in [31] to refer to uniform stability with respect to unitary groups equipped with the distance induced by the operator norm (see below). Some of the most common families $\mathcal{G}$ of approximating groups are the unitary groups $\mathrm{U}(n)$ or the symmetric groups S_n.

$\mathrm{U}(n)$ is typically considered with a metric induced by a norm defined on $\mathrm{M}_n(\mathbb{C})$. The first example is that of the *operator norm*, where pointwise stability has striking topological and K-theoretic interpretations [36, 39], all amenable groups are known to be uniformly stable [75], and groups with nontrivial quasimorphisms are known not to be uniformly stable [105, 31]. Recently, the list of so-called Ulam stable groups has been extended to include most high-rank lattices [56] and Thompson groups [50]. Another example is that of the *Frobenius norm* $\|A\|_{Frob} \coloneqq \sqrt{|A_{ij}|^2}$, that is, the norm induced by the embedding of $\mathrm{U}(n)$ into $\mathbb{C}^{n\times n}$: this has the advantage of allowing a cohomological criterion for pointwise stability [40]. The third main example is given by the *Hilbert–Schmidt norm* $\|A\|_{HS} \coloneqq \frac{1}{\sqrt{n}}\|A\|_{Frob}$, which is the normalisation of the Frobenius norm: this has the advantage of allowing a C^*-algebraic approach, which yields a characterisation of pointwise stability of amenable groups in terms of characters [67, 68], as well as a simple algebraic characterisation of uniform stability of finitely generated residually finite groups [2].

On the other hand the groups S_n are studied with the normalised Hamming distance $d_H(\sigma,\tau) \coloneqq 1 - \frac{1}{n}|Fix(\sigma^{-1}\tau)|$. Pointwise stability of equations in permutation was initially considered by Glebsky and Rivera [58], then by