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FLUX-LIMITED SOLUTIONS FOR QUASI-CONVEX HAMILTON-JACOBI EQUATIONS ON NETWORKS

BY CYRIL IMBERT AND RÉGIS MONNEAU

ABSTRACT. – We study Hamilton-Jacobi equations on networks in the case where Hamiltonians are quasi-convex with respect to the gradient variable and can be discontinuous with respect to the space variable at vertices. First, we prove that imposing a general *vertex condition* is equivalent to imposing a specific one which only depends on Hamiltonians and an additional free parameter, the *flux limiter*. Second, a general method for proving comparison principles is introduced. This method consists in constructing a *vertex test function* to be used in the doubling variable approach. With such a theory and such a method in hand, we present various applications, among which a very general existence and uniqueness result for quasi-convex Hamilton-Jacobi equations on networks.

RÉSUMÉ. – Nous étudions des équations de Hamilton-Jacobi posées sur des réseaux dans le cas d'Hamiltoniens quasi-convexes en la variable gradient et qui peuvent être discontinus en la variable d'espace au niveau des sommets. Nous prouvons d'une part qu'imposer une *condition de jonction* générale est équivalent à en imposer une de type contrôle optimal, qui ne dépend que des Hamiltoniens et d'un paramètre libre additionnel, le *limiteur de flux*. Nous introduisons d'autre part une méthode générale pour montrer des principes de comparaison. Cette méthode repose sur la construction d'une *fonction sommet* destinée à remplacer dans la méthode de dédoublement des variables la fonction quadratique habituelle. Nous présentons ensuite un large éventail d'applications, et notamment un résultat d'existence et d'unicité très général pour les équations de Hamilton-Jacobi quasi-convexes posées sur les réseaux.

1. Introduction

This paper is concerned with Hamilton-Jacobi (HJ) equations on networks associated with Hamiltonians that are quasi-convex and coercive in the gradient variable and possibly discontinuous at the vertices of the network in the space variable.

Space discontinuous Hamiltonians have been identified as both important/relevant and difficult to handle; in particular, a few theories/approaches (see below) were developed to study the associated HJ equations. In this paper, we show that if they are assumed to be quasi-convex and coercive in the gradient variable, then not only uniqueness can be proved for very

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general conditions at discontinuities (referred to as *junction conditions*), but such conditions can even be classified: imposing a general junction condition reduces to imposing a junction condition of optimal control type, referred to as *a flux-limited junction condition.* As far as uniqueness is concerned, a comparison principle is proved. We show that the doubling variable approach can be adapted to the discontinuous setting if we go beyond the classical test function $|x-y|^2/2$ by using a *vertex test function* instead. This vertex test function can be used to do much more, like dealing with second order terms [31] or getting error estimates for monotone schemes [33].

We point out that the present article is written in the one-dimensional setting for pedagogical reasons but our theory extends readily to higher dimensions [29].

1.1. The junction framework

We focus in this introduction and in most of the article on the simplest network, referred to as a *junction*, and on Hamiltonians which are constant with respect to the space variable on each edge. Indeed, this simple framework leads us to the main difficulties to be overcome and allows us to present the main contributions. We will see in Section 5 that the case of a general network with (t,x)-dependent Hamiltonians is only an extension of this special case.

A *junction* is a network made of one vertex and a finite number of infinite edges. It is endowed with a flat metric on each edge. It can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $i = 1,\dots,N$, each branch J_i is assumed to be isometric to $[0,+\infty)$ and

$$J = \bigcup_{i=1,\dots,N} J_i \qquad \text{with} \quad J_i \cap J_j = \{0\} \quad \text{for} \quad i \neq j \tag{1.1}$$

where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x,y)$ denotes the geodesic distance on J defined as

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same branch,} \\ |x|+|y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a smooth real-valued function u defined on J, $\partial_i u(x)$ denotes the (spatial) derivative of u at $x \in J_i$ and the "gradient" of u is defined as follows,

$$u_x(x) := \begin{cases} \partial_i u(x) & \text{if} \quad x \in J_i^* := J_i \setminus \{0\}, \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{if} \quad x = 0. \end{cases} \tag{1.2}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation on the junction J

$$\begin{cases} u_t + H_i(u_x) = 0 \text{ for} \quad t \in (0,+\infty) \quad \text{and} \quad x \in J_i^*, \\ u_t + F(u_x) = 0 \quad \text{for} \quad t \in (0,+\infty) \quad \text{and} \quad x = 0 \end{cases} \tag{1.3}$$

subject to the initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in J. \tag{1.4}$$

The second equation in (1.3) is referred to as *the junction condition*. In general, minimal assumptions are required in order to get a good notion of weak (i.e., viscosity) solutions.

We shed some light on the fact that Equation (1.3) can be thought as a *system* of Hamilton-Jacobi equations associated with H_i coupled through a "dynamical" boundary condition involving F. This point of view can be useful, see Subsection 1.5. As far as junction functions are concerned, we will construct below some special ones (denoted by F_A) from the Hamiltonians H_i ($i = 1, \dots, N$) and a real parameter A.

We consider the important case of Hamiltonians H_i satisfying the following structure condition:

(1.5) For $i = 1, \dots, N$, H_i continuous, quasi-convex and coercive.

We recall that H_i is quasi-convex if its sub-level sets $\{p : H_i(p) \leq \lambda\}$ are convex. In particular, since H_i is also assumed to be coercive, there exist numbers $p_i^0 \in \mathbb{R}$ such that

$$\begin{cases} H_i \text{ nonincreasing in } (-\infty, p_i^0] \\ H_i \text{ nondecreasing in } [p_i^0, +\infty). \end{cases}$$

1.2. First main new idea: classification of junction conditions

In the present paper, two notions of viscosity solutions are introduced: *relaxed (viscosity) solutions* (see Definition 2.1), which can be used to deal with all junction conditions, and *flux-limited (viscosity) solutions* (see Definition 2.2) which are associated with flux-limited junction conditions. Relaxed solutions are used to prove existence and ensure stability. Flux-limited solutions satisfy the junction condition in a stronger sense and are used in order to prove uniqueness. Our first main result states that relaxed solutions for general junction conditions are in fact flux-limited solutions for some junction conditions of optimal-control type.

We now introduce the notion of flux-limited junction condition. Given a *flux limiter* $A \in \mathbb{R} \cup \{-\infty\}$, the A-limited flux through the junction point is defined for $p = (p_1, \dots, p_N)$ as

$$F_A(p) = \max\left(A, \max_{i=1,\dots,N} H_i^-(p_i)\right) \tag{1.6}$$

where H_i^- is the nonincreasing part of H_i defined by

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if } q \leq p_i^0, \\ H_i(p_i^0) & \text{if } q > p_i^0. \end{cases}$$

We now consider the following important special case of (1.3),

$$\begin{cases} u_t + H_i(u_x) = 0 \text{ for } t \in (0, +\infty) \text{ and } x \in J_i^*, \\ u_t + F_A(u_x) = 0 \text{ for } t \in (0, +\infty) \text{ and } x = 0. \end{cases} \tag{1.7}$$

We point out that the flux functions F_A associated with $A \in [-\infty, A_0]$ coincide if one chooses

$$A_0 = \max_{i=1,\dots,N} \min_{\mathbb{R}} H_i. \tag{1.8}$$