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ÉTALE COHOMOLOGY AND REDUCTION OF ABELIAN VARIETIES

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ABSTRACT. — In this paper we study the étale cohomology groups associated to abelian varieties. We obtain necessary and sufficient conditions for an abelian variety to have semistable reduction (or purely additive reduction which becomes semistable over a quadratic extension) in terms of the action of the absolute inertia group on the étale cohomology groups with finite coefficients.

RÉSUMÉ (*Cohomologie étale et réduction des variétés abéliennes*)

Nous étudions les groupes de cohomologies étales associés aux variétés abéliennes. Nous obtenons des conditions nécessaires et suffisantes pour qu'une variété abeliénne ait une réduction semistable (ou une réduction purement additive qui devient semistable sur une extension quadratique), en termes de l'action du groupe d'inertie absolu sur les groupes de cohomologies étales à coefficients finis.

1. Introduction

Suppose X is a smooth projective variety over a field F, v is a discrete valuation on F, and ℓ is a prime number not equal to the residue characteristic of v. Let F^s denote a separable closure of F, let $\bar{v}$ be an extension of v to F^s,

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let I denote the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/F)$, and let $\overline{X} = X \times_F F^s$. For every positive integer k, the group I acts naturally on the k-th ℓ-adic étale cohomology group $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$. Grothendieck proved the Monodromy Theorem (see the Appendix to [10], and 1.2 and 1.3 of [4]), which says that I acts on $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ via quasi-unipotent operators, *i.e.*, for every $\sigma \in I$ we have $(\sigma^m - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = 0$ for some positive integers m and r. It is known (see 3.7 of [4], and 3.5 and 3.6 of [5]) that if $k = 1$, then one may take $r = 2$. It easily follows (see Theorem 5.6 (i), (ii) below) that if X is an abelian variety, then one may take $r = k + 1$. It is shown in [6] (see 3.4 and 3.8 of [4], and p. vi of [3]) that one may take $r = k + 1$ whenever one knows the Purity Conjecture (3.1 of [4]) and resolution of singularities.

From now on, suppose X is a d-dimensional abelian variety. The Néron model $\mathcal{X}$ of X at v is a smooth separated model of X over the valuation ring R such that for every smooth scheme $\mathcal{Y}$ over R and morphism $\varphi : \mathcal{Y} \otimes_R F \to X$ over F there is a unique morphism $\mathcal{Y} \to \mathcal{X}$ over R which extends φ. The generic fiber of $\mathcal{X}$ can be canonically identified with X, and $\mathcal{X}$ is a commutative group scheme over R whose group structure extends that of X. Let X_v^0 denote the identity component of the special fiber of $\mathcal{X}$ at v. Over an algebraic closure of the residue field, there is an exact sequence of algebraic groups

$$0 \to U \times \mathcal{T} \longrightarrow X_v^0 \longrightarrow B \to 0,$$

where B is an abelian variety, $\mathcal{T}$ is the maximal algebraic torus in X_v^0, and U is a unipotent group. By definition, X is *semistable* at v if and only if $U = 0$. As I-modules, $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), \mathbb{Z}_\ell)$, where $T_\ell(X)$ is the ℓ-adic Tate module. Grothendieck's Galois Criterion for Semistability says that X is semistable at v if and only if every $\sigma \in I$ acts on $T_\ell(X)$ as a unipotent operator of echelon ≤ 2, *i.e.*, if and only if $(\sigma - 1)^2 H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = 0$ for every $\sigma \in I$.

Suppose n is a positive integer relatively prime to the residue characteristic of v. Raynaud's criterion says that if I acts trivially on the n-torsion X_n, and $n \geq 3$, then X is semistable at v. The authors (see [13] and [15]) proved that if $n \geq 5$, then X is semistable at v if and only if $(\sigma - 1)^2 X_n = 0$ for every $\sigma \in I$. In other words, necessary and sufficient conditions for semistability can be read off not only from the ℓ-adic representation, as shown by Grothendieck, but also from the mod n representation (for $n \geq 5$). The aim of this paper (see Theorem 5.10) is to generalize this result to the case of the higher étale cohomology groups H^k. Note that $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ and X_n are dual. Assume that $0 < k < 2d$, that $k < r \in \mathbb{Z}$, and that n does not belong to a certain finite set $N(r)$ of prime powers, defined explicitly in terms of r in § 2. (For example, $N(2) = \{1, 2, 3, 4\}$.) We show that if k is odd, then X is semistable at v if and only if $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$ for every $\sigma \in I$. If k is even, we show (under an additional assumption; see § 7) that $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$ for every $\sigma \in I$ if and only if either X is semistable at v or X has purely additive

reduction at v but is semistable over a (ramified) quadratic extension of F. We call the latter abelian varieties briefly unstable (see Definitions 5.2 and 5.3).

We gave such necessary and sufficient conditions for semistability in the case $k = 1$ and $r = 2$ in [13] and [15]. Poincaré duality then gives such criteria in the case $k = 2d - 1$. One can check that in the cases $k = 1$ or $2d - 1$, one cannot replace $N(r)$ by a smaller set. However, when $2 \leq k \leq 2d - 2$ one may replace $N(r)$ by an explicitly defined subset $N'(r)$ for which the result is sharp (see § 6).

In § 2 we introduce basic definitions and notation. Section 3 deals with multilinear algebra in characteristics 0 and ℓ and over $\mathbb{Z}_\ell$. We use the Jordan decompositions of exterior powers of linear operators to obtain a Minkowski-Serre type result. Section 4 contains some abelian variety results that will be used later. In §§ 5–6 we state and prove our main results. We give necessary and sufficient conditions for semistability, and also necessary and sufficient conditions for an abelian variety to either be semistable or have purely additive reduction which becomes semistable over a quadratic extension. In § 6 we shrink the exceptional set in the criteria when $2 \leq k \leq 2d - 2$. We prove that this exceptional set is minimal.

We hope that our results and/or methods will be useful in the study of semistability for the more general class of motives [9].

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2. Notation and definitions

If F is a field, let F^s denote a separable closure. Throughout this paper, X is a d-dimensional abelian variety defined over F, and v is a discrete valuation on F of residue characteristic $p \geq 0$. Let $\bar{v}$ denote an extension of v to F^s, and let I denote the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/F)$. If ℓ is a prime not equal to the characteristic of F, let

$$\rho_{\ell,X} : \mathrm{Gal}(F^s/F) \longrightarrow \mathrm{Aut}\big(T_\ell(X)\big) \cong \mathrm{GL}_{2d}(\mathbb{Z}_\ell)$$

denote the ℓ-adic representation on the Tate module $T_\ell(X) = \varprojlim X_{\ell^n}$. Let $V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, let $\overline{X} = X \times_F F^s$, and let

$$H^k_\ell = H^k_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = \mathrm{Hom}_{\mathbb{Z}_\ell}\big(\wedge^k(T_\ell(X)), \mathbb{Z}_\ell\big).$$

Then H_ℓ^k is a free $\mathbb{Z}_\ell$-module of rank $\binom{2d}{k} =: b$, and $H_\ell^k \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \wedge^k_{\mathbb{Q}_\ell}(V^*)$, where $V^* = \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ is the dual vector space. Let

$$\rho_{\ell,k} : I \longrightarrow \mathrm{Aut}(H_\ell^k) \cong \mathrm{GL}_b(\mathbb{Z}_\ell)$$

be the representation giving the action of I on H_ℓ^k.

DEFINITION 2.1. — If r is a positive integer, define a finite set of prime powers $N(r)$ by

$$N(r) = \{\text{prime powers } \ell^m \mid 0 \le m(\ell-1) \le r\}.$$

For example, $N(1) = \{1, 2\}$, $N(2) = \{1, 2, 3, 4\}$, $N(3) = \{1, 2, 3, 4, 8\}$, $N(4) = \{1, 2, 3, 4, 5, 8, 9, 16\}$.

3. Some linear algebra

LEMMA 3.1. — *Suppose g is a linear operator on a finite-dimensional vector space over a field of characteristic zero. Suppose $(g^m - 1)^2 = 0$ for some positive integer m. If g is unipotent, then $(g-1)^2 = 0$. If $-g$ is unipotent, then $(g+1)^2 = 0$.*

Proof. — If g is unipotent then all the eigenvalues of $g^{m-1} + \cdots + g + 1$ are $m \neq 0$. Therefore $g^{m-1} + \cdots + g + 1$ is an invertible operator. Since

$$0 = (g^m - 1)^2 = (g-1)^2(g^{m-1} + \cdots + g + 1)^2$$

we have $(g-1)^2 = 0$.

If $-g$ is unipotent then all the eigenvalues of g^m are $(-1)^m$. Since g^m is unipotent, $m = 2r$ is even. We have

$$0 = (g^m - 1)^2 = (g^{2r} - 1)^2 = (g+1)^2(g-1)^2(g^{2(r-1)} + \cdots + g^2 + 1)^2.$$

All the eigenvalues of $g-1$ are -2 and all the eigenvalues of $g^{2(r-1)} + \cdots + g^2 + 1$ are $r \neq 0$. Therefore $g-1$ and $g^{2(r-1)} + \cdots + g^2 + 1$ are invertible, and $(g+1)^2 = 0$. □

THEOREM 3.2. — *Suppose m, r and g are positive integers, ℓ is a prime number, and $A \in \mathrm{M}_g(\mathbb{Z}_\ell)$ satisfies $(A-1)^r \in \ell^m \mathrm{M}_g(\mathbb{Z}_\ell)$. Suppose λ is an eigenvalue of A which is a root of unity. If $r = m(\ell-1)$ then $\lambda^\ell = 1$. If $r < m(\ell-1)$ then $\lambda = 1$.*

Proof. — This is a special case of Theorem 6.7 of [12]. □

THEOREM 3.3. — *Suppose $\bar{\mathbb{Z}}$ is the ring of algebraic integers in $\bar{\mathbb{Q}}$, n and r are positive integers, λ is a root of unity in $\bar{\mathbb{Z}}$, and $(\lambda - 1)^r \in n\bar{\mathbb{Z}}$. Then $\lambda = 1$ if $n \notin N(r)$.*

Proof. — This is a special case of Corollary 3.3 of [12]. □

LEMMA 3.4 (see Lemma 4.3 of [13]). — *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, and ℓ is a prime number. Suppose r, s, and m are positive integers such that $r \geq m\ell^{s-1}(\ell-1)$. Suppose $\alpha \in \mathcal{O}$ and $\alpha^{\ell^s} = 1$. Then $(\alpha-1)^r \in \ell^m\mathbb{Z}[\alpha]$.*

Suppose V is a finite-dimensional vector space over $\mathbb{Q}_\ell$. Suppose k is an integer and $0 < k < \dim(V)$. Let

$$f_k : \mathrm{GL}(V) \longrightarrow \mathrm{GL}\big(\wedge^k(V)\big)$$

denote the natural representation defined by $f_k(g) = \wedge^k(g)$. For $g \in \mathrm{GL}(V)$, let $g = s_g u_g = u_g s_g$ be the Jordan decomposition, where u_g, $s_g \in \mathrm{GL}(V)$, u_g is unipotent, and s_g is semisimple. Then $f_k(g) = f_k(s_g)f_k(u_g) = f_k(u_g)f_k(s_g)$, $f_k(u_g)$ is unipotent, and $f_k(s_g)$ is semisimple.

REMARK 3.5. — In the notation of § 2, with "$*$" denoting the dual map, for every $\sigma \in I$ we have

$$\rho_{\ell,k}(\sigma) = \big(f_k(\rho_{\ell,X}(\sigma^{-1}))\big)^*.$$

REMARK 3.6. — The kernel of f_k is $\{\gamma \in \mathbb{Z}_\ell \mid \gamma^k = 1\}$. In particular, if $(k, \ell-1) = 1$ then f_k is injective.

LEMMA 3.7. — *The element $\wedge^k(g)$ is unipotent if and only if there exists a k-th root of unity $\gamma \in \mathbb{Z}_\ell$ such that γg is unipotent.*

Proof. — If $\wedge^k(g)$ is unipotent, then $\wedge^k(s_g) = 1$. By Remark 3.6, s_g is a k-th root of unity in $\mathbb{Z}_\ell$. Let $\gamma = s_g^{-1}$. □

Now assume that V is even-dimensional, choose a non-degenerate alternating bilinear form on V, and let $\mathrm{Sp}(V) \subset \mathrm{GL}(V)$ be the corresponding symplectic group. Write

$$\rho_k : \mathrm{Sp}(V) \longrightarrow \mathrm{GL}(\wedge^k(V))$$

for the restriction of f_k to $\mathrm{Sp}(V)$. It is well-known that $u_g, s_g \in \mathrm{Sp}(V)$ for every $g \in \mathrm{Sp}(V)$.

REMARK 3.8. — Remark 3.6 easily implies that the kernel of ρ_k is $\{1\}$ if k is odd and is $\{1, -1\}$ if k is even.

LEMMA 3.9. — *Suppose that $g \in \mathrm{Sp}(V)$.*

(i) *If $(g-1)^2 = 0$, then $(\rho_k(g)-1)^{k+1} = 0$.*
(ii) *If k is even and $(g+1)^2 = 0$, then $(\rho_k(g)-1)^{k+1} = 0$.*
(iii) *If γg is unipotent for some $\gamma \in \mathbb{Q}_\ell$, then $\gamma \in \{\pm 1\}$.*
(iv) *Suppose $\wedge^k(g)$ is unipotent. If k is odd then g is unipotent. If k is even then either g or $-g$ is unipotent.*