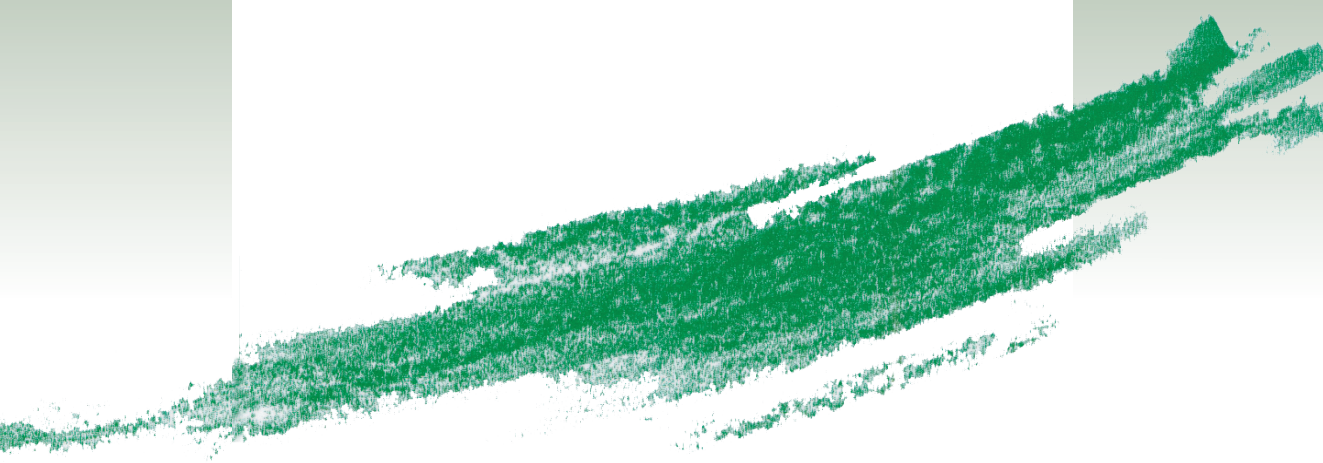
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COLLECTION SMF

# Some aspects of topological dynamics of Polish groups

## With an introduction to descriptive set theory

Julien MELLERAY



34

# SOME ASPECTS OF TOPOLOGICAL DYNAMICS OF POLISH GROUPS

## WITH AN INTRODUCTION TO DESCRIPTIVE SET THEORY

**Julien Melleray**

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# SOME ASPECTS OF TOPOLOGICAL DYNAMICS OF POLISH GROUPS

## WITH AN INTRODUCTION TO DESCRIPTIVE SET THEORY

**Julien Melleray**

**Société Mathématique de France 2026**

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# SOME ASPECTS OF TOPOLOGICAL DYNAMICS OF POLISH GROUPS

## With an introduction to descriptive set theory

**Julien Melleray**

***Abstract.*** — In the first part of the text we focus on continuous actions of *Polish groups* (separable, completely metrizable topological groups) on compact Hausdorff spaces. We begin by presenting some properties of this class of groups as well as some basic tools used in that area, notably based on the Baire category theorem. We then present some important examples linked to model theory. After that we move on to structural properties in topological dynamics, leading to the notion of *universal minimal flow* before presenting a fundational theorem due to Kechris, Pestov and Todorčevic which establishes a link between topological dynamics and combinatorics (Ramsey theory in particular). We conclude this part by a discussion of the metrizability of universal minimal flows of Polish groups.

A few results presented in the first part require some knowledge of descriptive set theory. The second part of the text gives the necessary background and presents some important results in that area. This discussion culminates in a proof, due to B. Miller, of the $G_0$-dichotomy theorem of Kechris, Solecki and Todorčevic.

***Résumé*** **(*Quelques aspects de la dynamique topologique des groupes polonais*).** — Dans la première partie de ce texte, nous nous intéressons aux actions continues de *groupes polonais* (groupes topologiques séparables et complètement métrisables) sur des espaces compacts. Nous commençons par dégager quelques propriétés de cette classe de groupes ainsi que quelques outils fondamentaux pour les manipuler, notamment basés sur des méthodes de Baire. Ensuite nous présentons des exemples importants liés à la théorie des modèles. Puis nous passons à des propriétés structurelles en dynamique topologique, dégageant la notion de *flot minimal universel*, avant d'établir un théorème fondamental de Kechris, Pestov et Todorčevic qui crée un lien entre combinatoire (théorie de Ramsey en particulier) et dynamique topologique.

Nous concluons cette partie par une discussion de la métrisabilité des flots minimaux universels de groupes polonais.

Certains résultats de la première partie nécessitent des connaissances en théorie descriptive des ensembles; la deuxième partie du texte fournit tous les détails nécessaires et propose une introduction à quelques résultats importants de ce domaine. Elle se conclut par une démonstration, due à B. Miller, du théorème de $G_0$-dichotomie de Kechris, Solecki et Todorčevic.

# FOREWORD

These notes grew out of a semester course given at the university of Turin during the spring of 2024 (the content of that course corresponds roughly to Chapters 1–8). They are aimed at graduate (or advanced undergraduate) students and researchers who want an introduction to some topics around the theory of *Polish groups*, i.e., completely metrizable second-countable topological groups. This class of groups contains all second-countable locally compact Hausdorff topological groups; Polish groups are ubiquitous in functional analysis, ergodic theory, logic, etc. The primary focus of the first part of the book is an introduction to some aspects of topological dynamics of Polish group actions. We are concerned with continuous actions of Polish groups by homeomorphisms on Hausdorff compact spaces, and develop some of the general theory. One of our main objectives in that part of the book is to establish a theorem due to Kechris, Pestov and Todorčevic, which characterizes automorphism groups of countable structures which are *extremely amenable*, which means that every continuous action of such a group on a compact Hausdorff space has a fixed point. This theorem establishes a connection between topological dynamics and combinatorics and has proved to be extremely influential.

We do not take the shortest route to this result, instead taking time to discuss some of the background and present various methods, in particular based on Baire-category-theoretic notions, that are useful when working with Polish groups. In the first part of the book we sometimes need standard results from *descriptive set theory*, which we use as black boxes; in the second part all those results (and more) are proved.

Descriptive set theory is, in a nutshell, the study of the properties of "definable" subsets of Polish spaces; this classical theory goes back more than a century and many topics are not covered here. I chose which topics to present based on personal taste and ease of presentation, with an eye on applicability in various areas of mathematics. The culminating point of the second part of the book is a proof, due to B. Miller, of a fundamental dichotomy theorem of Kechris, Solecki and Todorčevic; this dichotomy theorem has striking structural consequences, some of which we briefly discuss. My objective in this part is twofold: developing some useful tools, and hopefully motivating some readers to dig deeper into descriptive set theory and its recent developments. Several classical, important and beautiful parts of descriptive set theory are not covered at all in these notes; I hope the reader will want to know more and use this book as a first step towards more thorough presentations of the topic such as [**34**].

This text uses some set-theoretic notations (notably, $\omega$ for the set of non negative integers); but there is little actual set-theoretic background, and I give a quick presentation of that background (as well as the axiomatic framework which we are using; notably, we freely use the axiom of choice) in an appendix. I tried to make this text accessible to readers without a background in logic, set theory or dynamics—some familiarity with point set topology (including the Baire category theorem) as well as group actions is however required.

The text is sprinkled with exercises of varying difficulty (some of them should be very easy, while others seem quite difficult to me); solutions to these exercises are given at the end of the text. Sometimes the reader is expected to fill in some of the details of the argument. It is my belief that at least attempting to solve many of these exercises while going through the text will help the reader get a good grasp on the material.

Each chapter ends with a few comments, mostly providing references for the material presented and proposing some further reading.

I would like to thank the students who attended my lectures in Turin and worked on many of the exercises; in particular Paolo Boldrini, Fabio Carrozzo and Laura Paglietta deserve a mention here. I am also grateful to Raphaël Carroy for organizing my visit, to Luca Motto Ros for graciously sharing his office with me, and to the whole logic group for their welcome. This visit was made possible by the Unito Visiting Professor Program 2023/2024.

Bruno Duchesne suggested turning my lecture notes into a publishable text; I am not sure I should thank him for all the additional work that this entailed, but will do so nevertheless. I am beholden to Sophie Grivaux for her encouragement and her very efficient handling of the manuscript.

I owe a debt of gratitude to Todor Tsankov, who explained to me much of what I know about topological dynamics of Polish groups. I am also indebted to an anonymous reviewer, who made many useful suggestions and caught an erroneous argument of mine; and to Andy Zucker for helping me understand why that argument was in fact incorrect. François Le Maître was of great help when I was working on the final version of this manuscript; for this and our many discussions over the years I thank him. Finally, I am much obliged to Johan Koschall and Flore Le Roux for their comments and feedback.

There is unfortunately little doubt that some errors are still lurking—finding and correcting them is left as an exercise for the reader.

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