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TIME QUASI-PERIODIC VORTEX PATCHES FOR QUASI-GEOSTROPHIC SHALLOW-WATER EQUATIONS

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Société Mathématique de France 2026

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TIME QUASI-PERIODIC VORTEX PATCHES FOR QUASI-GEOSTROPHIC SHALLOW-WATER EQUATIONS

Taoufik Hmidi, Émeric Roulley

Abstract. – In this paper, we are interested in the construction of time quasi-periodic solutions for the shallow-water quasi-geostrophic model. We look for the solutions in the vortex patch form near Rankine vortices, and whose dynamics can be recast in terms of a Hamiltonian system on the patch radius. We show the existence of quasi-periodic solutions when the Rossby deformation length belongs to a massive Cantor set. As a consequence, the patch boundary is localized within a narrow annulus for any time. The proof rests upon the application of KAM theory and Nash-Moser scheme which aligns with the approach developed in the papers [**5, 12**]. One of the main difficulties pertains to establishing an approximate right inverse with tame estimates for the linearized operator close to the equilibrium state. This operator takes the form of a quasilinear transport operator with a perturbation of order zero. To ensure its invertibility, we perform several reducibility steps in order to conjugate it to a Fourier multiplier. Throughout this long process, we must carefully curtail the range of the external parameter to circumvent resonances and grapple with a small divisors problem.

Résumé **(Poches de tourbillon quasi-périodiques en temps pour les équations de Saint-Venant quasi-géostrophiques)**

Dans cet article, nous nous intéressons à la construction de solutions quasi-périodiques en temps pour le modèle de Saint-Venant quasi-géostrophique. Nous recherchons les solutions sous forme d'une poche de tourbillon à proximité des tourbillons de Rankine, et dont la dynamique peut être retranscrite en un système hamiltonien portant sur le rayon de la poche. Nous montrons l'existence de solutions quasi-périodiques lorsque la longueur de déformation de Rossby appartient à un ensemble de Cantor massif. En conséquence, le bord de la poche est localisé à tout moment dans un anneau étroit. La preuve repose sur l'application de la théorie KAM et du schéma de Nash-Moser qui s'aligne sur l'approche développée dans les articles [**5, 12**]. L'une des principales difficultés réside dans l'établissement d'un inverse approché à droite du linéarisé proche de l'état d'équilibre avec des estimations

douces. Cet opérateur prend la forme d'un opérateur de transport quasi-linéaire avec une perturbation d'ordre zéro. Pour assurer son inversibilité, nous effectuons plusieurs étapes de réductibilité afin de le conjuguer à un multiplicateur de Fourier. Tout au long de ce long processus, nous devons soigneusement réduire la plage du paramètre externe pour contourner les résonances et lutter contre le problème des petits diviseurs.

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CHAPTER 1

INTRODUCTION

We shall discuss in this introduction the quasi-geostrophic shallow-water equation which is a nonlinear and nonlocal transport equation generalizing 2D Euler equations and used to describe large scale motion for the atmosphere and the ocean circulation. A particular concern will be addressed to the long time dynamics behavior and especially to the emergence of long-lived structures in the vortex patch setting. The main novelty in this work is to explore quasi-periodic solutions around the Rankine vortices using KAM theory and Nash-Moser scheme in the spirit of the recent work of Baldi, Berti, Haus and Montalto [**5**].

1.1. Model, relative equilibria from periodic to quasi-periodic solutions

The current paper deals with the quasi-geostrophic shallow-water equations $(\mathrm{QGSW})_\lambda$, which is considered as one of the most common asymptotic models used to describe the large scale motion of the atmospheric and oceanic circulation and can be derived asymptotically from the rotating shallow-water equations when Rossby and Froude numbers are small enough, for more details we refer to [**28, 55**] and the references therein. This model is planar and the evolution of the potential vorticity $\boldsymbol{q}$ takes the form of a nonlinear and nonlocal transport equation,

$$(\mathrm{QGSW})_\lambda \begin{cases} \partial_t \boldsymbol{q} + \mathbf{v}\cdot\nabla\boldsymbol{q} = 0, & \text{in } \mathbb{R}_+\times\mathbb{R}^2 \\ \mathbf{v} = \nabla^\perp(\Delta-\lambda^2)^{-1}\boldsymbol{q}, & \\ \boldsymbol{q}(0,\cdot) = \boldsymbol{q}_0, & \end{cases} \qquad \text{where} \quad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}.$$

Here $\mathbf{v}$ denotes the velocity field which is solenoidal and $\boldsymbol{q}$ is a scalar function. Physically, the parameter λ is defined by

$$\lambda = \frac{\omega_c}{\sqrt{gH}},$$

where g is the gravity constant, H is the mean active layer depth and ω_c is the Coriolis frequency, assumed to be constant. In the literature, the number $\frac{1}{\lambda}$ is called the *Rossby deformation length* or *Rossby radius* and measures the length scale at which the rotation effects are balanced by the stratification. Notice that small values

of λ corresponds to a free surface which is nearly rigid and when $\lambda = 0$ we get Euler equations written in the formulation velocity-vorticity.

The main purpose of this paper is to explore the emergence of time quasi-periodic solutions for $(\mathrm{QGSW})_\lambda$ when λ belongs to a massive Cantor like set. This goal will be accomplished in the special class of vortex patches. Before coming to the details we shall first discuss the existence of relative equilibria which are solutions that do not change their shape during the motion. This subject is well-explored for Euler equations and new results have been established recently for $(\mathrm{QGSW})_\lambda$ in [**26**]. Next we shall discuss the vortex patch problem and explore some of its specific feature.

Vortex patches/relative equilibria. – We consider an initial condition taking the form of a vortex patch, that is, $\boldsymbol{q}_0 = \mathbf{1}_{D_0}$ where D_0 is a bounded domain in $\mathbb{R}^2$. Since $\mathbf{1}_{D_0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then in a similar way to Euler equations Yudovich's theory applies and implies that this system admits a unique global in time weak solution. Then from the transport equation governing the potential vorticity we find that the patch structure is preserved during the motion and one gets

$$\boldsymbol{q}(t,\cdot) = \mathbf{1}_{D_t}, \quad \text{where} \quad D_t := \boldsymbol{\Phi}_t(D_0),$$

where $\boldsymbol{\Phi}_t : \mathbb{C} \to \mathbb{C}$ stands for the flow map associated to the velocity field $\mathbf{v}$ and defined by

$$\partial_t \boldsymbol{\Phi}_t(z) = \mathbf{v}\big(t, \boldsymbol{\Phi}_t(z)\big) \quad \text{and} \quad \boldsymbol{\Phi}_0 = \mathrm{Id}_{\mathbb{C}}. \tag{1.1}$$

The boundary motion in the smooth case reduces to tackle the evolution of a curve in the complex plane surrounding a constant area domain and subject to the deformation induced by its own effect. Local/global in time persistence of the boundary regularity is a relevant subject in fluid dynamics and has attracted a lot of attention during the past decades, not only for Euler equations but also for similar active scalar equations such as generalized surface quasi-geostrophic equations, the aggregation equation. The literature is huge and we shall restrict the discussion to some suitable contributions that fit with our main task. Let us now briefly see how to write down the contour dynamics equations, more details can be found in [**44, 45**]. Given a smooth parametrization $z(t,\cdot) : \mathbb{T} \to \partial D_t$ of the boundary of the patch, then as particles located at the boundary move with the boundary then we get the evolution equation

$$\Big[\partial_t z(t,\theta) - \mathbf{v}\big(t, z(t,\theta)\big)\Big] \cdot \mathbf{n}\big(t, z(t,\theta)\big) = 0, \tag{1.2}$$

where $\mathbf{n}(t,\cdot)$ is the outward normal vector to the boundary ∂D_t of D_t. This equation reflects the fact that the particle velocity and the boundary velocity admit the same normal components. As we shall see later in Section 2.1, one may deduce from the preceding equation that the boundary equation will evolve through a nonlinear integro-differential equation. Looking for particular solutions where the domain moves without any shape deformation is a traditional subject in fluid dynamics and important developments have been performed for long time ago. In the literature, these structures appear under different names: relative equilibria, V-states, long-lived

structures, vortex crystals, etc. A particular case is given by rotating patches which are solutions rotating uniformly with constant angular velocity about their center of mass that can be fixed at the origin, namely,

$$\boldsymbol{q}(t,\cdot) = \mathbf{1}_{D_t} \quad \text{with} \quad D_t = e^{\mathrm{i}t\Omega} D_0.$$

These solutions are periodic in time with period $\frac{2\pi}{\Omega}$ or equivalently with frequency Ω. For Euler equations, we know two explicit examples of rotating patches. The first one is the so-called Rankine vortices given by the disks which are stationary solutions not only for Euler but also for $(\mathrm{QGSW})_\lambda$. The second example is called Kirchhoff ellipses which rotate with the angular velocity $\Omega = \frac{ab}{(a+b)^2}$ where a and b are the semi-axes of the ellipse, we refer to [**49**] and [**19**, p. 304] for more explanations. Numerical experiments achieved by Deem and Zabusky in [**25**] show the existence of rotating solutions with $\mathbf{m}$-fold symmetry for $\mathbf{m} = 3, 4, 5$. An analytical proof based on the bifurcation theory and complex analysis tools was devised by Burbea in [**21**] who proved the existence of $\mathbf{m}$-fold (for any $\mathbf{m} \in \mathbb{N}^*$) symmetric V-states bifurcating from Rankine vortices with angular velocity $\Omega_{\mathbf{m}} := \frac{\mathbf{m}-1}{2\mathbf{m}}$. More investigations on the V-states in different settings like the doubly-connected patches, vortex pairs, boundary effects or for different models has been implemented during the past decade by several authors, for more details we refer to [**23, 22, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43**]. Concerning $(\mathrm{QGSW})_\lambda$ there are a few results dealing with relative equilibria. Interesting numerical simulations showing the complexity and the richness of the bifurcation diagram with respect to the parameter λ was studied in [**27, 26**]. In [**26**], using bifurcation tools the authors proved analogous results to those of Burbea. They show in particular the existence of branches of $\mathbf{m}$-fold symmetric V-states ($\mathbf{m} \geqslant 2$) bifurcating from Rankine vortex $\mathbf{1}_{\mathbb{D}}$ with the angular velocity

$$\Omega_{\mathbf{m}}(\lambda) = I_1(\lambda)K_1(\lambda) - I_{\mathbf{m}}(\lambda)K_{\mathbf{m}}(\lambda),$$

where I_m and K_m are the modified Bessel functions of first and second kind. For more details about these functions, we refer to the appendix. Notice that in the same paper the authors explored the two-fold branch when λ is small and proved first that it is located close to the ellipse branch of Euler equations and second it is not connected and from numerical simulations they put in evidence the fragmentation of this branch in multiple connected pieces. The second bifurcation from this branch was also analyzed leading to similar results as for Euler equations.

It is worthy to point out that the model under consideration is typically a reversible Hamiltonian equation with one degree of freedom given by the external parameter λ. Therefore, it is a natural issue to explore whether or not quasi-periodic solutions constructed from the linearized operator can survive at the nonlinear level when λ is selected in a suitable Cantor set. Studying the persistence of invariant tori is a relevant subject from KAM theory where a lot of important developments has been done in different directions. In the next paragraph we shall review and recall some basic notions and results in this subject.

Quasi-periodic solutions. – A function $f : \mathbb{R} \to \mathbb{R}$ is called quasi-periodic if there exists $F : \mathbb{T}^d \to \mathbb{R}$ such that

$$\forall\, t \in \mathbb{R}, \quad f(t) = F(\omega t)$$

for some frequency vector $\omega \in \mathbb{R}^d$ ($d \in \mathbb{N}^*$) which is non-resonant, that is

$$\forall\, l \in \mathbb{Z}^d \setminus \{0\}, \quad \omega \cdot l \neq 0. \tag{1.3}$$

Here and in the sequel, $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ denotes the flat torus of dimension d. In the case $d = 1$, we recover from this definition the periodic functions with frequency $\omega \in \mathbb{R}^*$.

The study of quasi-periodic solutions to Hamiltonian systems goes back to the pioneering works of Kolmogorov [**50**], Arnold [**3**] and Moser [**52**] where they proved, in finite dimensions and under suitable non degeneracy and smoothness conditions, the persistence of invariant tori for small perturbation of integrable Hamiltonian systems taking the form

$$H(I, \vartheta) = h(I) + \varepsilon P(I, \vartheta).$$

For the integrable case $\varepsilon = 0$, the whole phase space is foliated by Lagrangian invariant tori (isotropic tori with maximal dimension) on which the flow is linear. The quasi-periodic tori associated to the incommensurable vector frequency $\omega = \nabla h(I)$ is filled up by the orbits contained therein. Roughly speaking, KAM theorem asserts that, under analytic perturbation and a non-degeneracy condition, most of strongly non-resonant (the frequency ω is Diophantine) Lagrangian invariant tori will survive. Notice that a huge literature on the improvement of KAM theory and its extension has been generated in the past few decades. For a general survey we refer the reader to the monographs [**4, 14, 20, 24, 48**].

Later on, KAM theory was explored and developed in partial differential equations by several authors leading to important contributions and opening new perspectives. The complexity of the problem depends on the space dimension and on the structure of the equations. For example in the semi-linear case the nonlinearity can be seen as a bounded perturbation of the linear problem and this simplifies a lot the problem of finding a right inverse of the linearized operator around a state close to the equilibrium. However in the quasi-linear case where the nonlinearity is unbounded and has the same order as the linear part the situation turns to be much more tricky. This is the case for instance in the water-wave equations where several results has been obtained in the past few years on the periodic or quasi-periodic, standing or traveling settings, see [**2, 5, 15, 16, 18, 47, 53**].

Next we shall give some insights on the general scheme performed to construct quasi-periodic solutions in the quasi-linear setting that was developed in particular by Berti and Bolle in [**13**]. This approach is robust and flexible and will be adapted to our framework with the suitable modifications. The first step is to write in a standard way the equations using the action-angle variables for the tangential part. When we linearize the nonlinear functional around a state near the equilibrium we end up with an operator with variable coefficients that we should invert approximately up to

small errors provided the external parameters belong to a suitable Cantor set defined through various Diophantine conditions. To do that we first look for an approximate inverse using an isotropic torus built around the initial one. It has the advantage to transform the linearized operator via symplectic change of coordinates into a triangular system up to errors that vanish when tested against an invariant torus. Notice that the outcome is that the Hamiltonian has a good normal form structure such that we can almost decouple the dynamics in the phase space in tangential and normal modes. On the tangential part the system can be solved in a triangular way provided we can invert the linearized operator on the normal part up to a small coupling error term. This is more or less a finite dimensional KAM theory appearing here. Then, the analysis reduces to invert the linearized operator on the normal part which is a small perturbation of a diagonal infinite dimensional matrix. This is done by conjugating the linearized operator to a diagonal one with constant coefficients. This step is long and technical and most of the non-resonance conditions in the Cantor set arise during this process. This allows the construction of an approximate inverse for the linearized operator with adequate tame estimates required along Nash-Moser scheme. We point out that this approach has been successfully implemented to generate quasi-periodic solutions to several quasi-linear and fully nonlinear autonomous PDE's, see for instance [**5, 6, 7, 18**]. Recent progress towards applying KAM theory for the vortex dynamics has been performed for gSQG equation [**37**] and Euler equation [**17**]. Finally, we point out that the use of suitable isotropic tori is a commodity but it is not essential to get the triangular structure up to small errors. This point will be discussed in Section 5, see also [**37**].

1.2. Main result and sketch of the proof

The contour dynamics equation stated in (1.2) can be written in a more tractable way using polar coordinates. This is meaningful at least for short time when the initial patch is close to the equilibrium state given by the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ where $\mathbb{D}$ is the unit disk of $\mathbb{R}^2$. Thus the boundary ∂D_t will be parametrized as follows

$$z(t,\theta) := R(t,\theta)e^{i\theta} \quad \text{with} \quad R(t,\theta) := \big(1+2r(t,\theta)\big)^{\frac{1}{2}}. \tag{1.4}$$

We shall prove in Section 2.1 that the function r satisfies a nonlinear and non-local transport equation taking the form

$$\partial_t r + F_\lambda[r] = 0, \tag{1.5}$$

with

$$F_\lambda[r](t,\theta) := \int_{\mathbb{T}} K_0\big(\lambda A_r(t,\theta,\eta)\big)\partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big)d\eta$$

and

$$A_r(t,\theta,\eta) := \big|R(t,\theta)e^{i\theta} - R(t,\eta)e^{i\eta}\big|.$$

The function K_0 is a Bessel function of imaginary argument and it is defined in the appendix. Next, we take a parameter $\Omega \neq 0$ and look for the solutions in the form

$$r(t,\theta) = \tilde{r}(t,\theta+\Omega t), \tag{1.6}$$

then the Equation (1.5) is equivalent to (to alleviate the notation $\tilde{r}$ will be denoted by r)

$$\partial_t r + \Omega\partial_\theta r + F_\lambda[r] = 0. \tag{1.7}$$

We point out that the introduction of the parameter Ω seems at this level artificial but it will be used later to fix the degeneracy of the first eigenvalue associated with the linearized operator at the equilibrium state. As we shall see in Proposition 2.1, the Equation (1.7) has a Hamiltonian structure

$$\partial_t r = \partial_\theta \nabla H(r), \tag{1.8}$$

where the Hamiltonian H is related to the kinetic energy and the angular momentum which are prime integrals of the system. In the quasi-periodic setting, we should find a frequency vector $\omega \in \mathbb{R}^d$, such that the Equation (1.7) admits solutions in the form $r(t,\theta) = \widehat{r}(\omega t,\theta)$ with $\widehat{r}$ being a smooth $(2\pi)^{d+1}-$periodic function. Then $\widehat{r}$ satisfies (to alleviate the notation we keep the notation r for $\widehat{r}$)

$$\omega\cdot\partial_\varphi r + \Omega\partial_\theta r + F_\lambda[r] = 0. \tag{1.9}$$

To explore quasi-periodic solutions for the nonlinear model, a good hint (albeit not necessary) is to check their existence at the linear level, which we plan to briefly discuss in the next few lines. According to Lemma 3.1 the linearized operator to (1.9) around a given small state r is given by the linear Hamiltonian equation,

$$\mathcal{L}_r\rho = 0 \quad \text{with} \quad \mathcal{L}_r := \partial_t + \partial_\theta\left[V_r - \mathbf{L}_r\right], \tag{1.10}$$

where V_r is a scalar function defined by

$$V_r(\lambda,t,\theta) := \Omega + \frac{1}{R(t,\theta)}\int_{\mathbb{T}} K_0\big(\lambda A_r(t,\theta,\eta)\big)\partial_\eta\big(R(t,\eta)\sin(\eta-\theta)\big)d\eta \tag{1.11}$$

and $\mathbf{L}_r$ is a non-local operator in the form

$$\mathbf{L}_r(\rho)(\lambda,t,\theta) := \int_{\mathbb{T}} K_0\big(\lambda A_r(t,\theta,\eta)\big)\rho(t,\eta)d\eta. \tag{1.12}$$

At the equilibrium state $r\equiv 0$, we find that the linearized operator is a Fourier multiplier, see Lemma 3.2,

$$\mathcal{L}_0\rho = \partial_t\rho + V_0(\lambda)\partial_\theta\rho - \partial_\theta\mathcal{K}_\lambda * \rho, \tag{1.13}$$

where $*$ denotes the convolution product in the variable θ and

$$V_0(\lambda) := \Omega + I_1(\lambda)K_1(\lambda) \quad \text{and} \quad \mathcal{K}_\lambda(\theta) := K_0\big(2\lambda\left|\sin\left(\frac{\theta}{2}\right)\right|\big).$$

Expanding into Fourier series

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}}\rho_j(t)e^{\mathrm{i}j\theta},$$

yields to

$$\rho\in\ker(\mathcal{L}_0)\quad\Longleftrightarrow\quad\rho(t,\theta)=\sum_{j\in\mathbb{Z}}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(\lambda)t)},\tag{1.14}$$

where the eigenvalues Ω_j are defined by

$$\Omega_j(\lambda):=j\Big(\Omega+I_1(\lambda)K_1(\lambda)-I_j(\lambda)K_j(\lambda)\Big)\tag{1.15}$$

and the Bessel functions of imaginary argument I_n and K_n are given by (A.2). It is worthy to point out that the frequency associated to the mode $j=0$ is vanishing and therefore it creates trivial resonance. This can be fixed by imposing a zero space average which can be maintained at the nonlinear level by virtue of the structure of (1.8). Hence we shall work with the phase space of real functions enjoying this property, namely,

$$H^s_0:=H^s_0(\mathbb{T},\mathbb{R})=\Big\{r(\theta)=\sum_{j\in\mathbb{Z}^*}r_je^{\mathrm{i}j\theta}\text{ s.t. }r_{-j}=\overline{r_j}\text{ and }\|r\|_s^2=\sum_{j\in\mathbb{Z}^*}|r_j|^2|j|^{2s}<\infty\Big\}.$$

Another similar comment concerns the trivial resonances corresponding to the modes $j=\pm1$ where $\Omega_{\pm1}$ vanish for any λ when $\Omega=0$. To circumvent this issue it is enough to work in (1.15) with any strictly positive value of Ω. Another important feature of the Equation (1.8) is its reversibility, namely its invariance with respect to the involution $(\mathscr{S}r)(\theta)=r(-\theta)$, which implies in particular that if $(t,\theta)\mapsto r(t,\theta)$ is a solution then $(t,\theta)\mapsto r(-t,-\theta)$ is a solution too. Thus, we shall look for reversible solutions, that is, $r(t,\theta)=r(-t,-\theta)$. Notice that this latter property is crucial to reduce the size of the eigenspaces of the linearized operator. Therefore, the solutions to the linear problem with this symmetry are in the form

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j\cos\big(j\theta-\Omega_j(\lambda)t\big).\tag{1.16}$$

Now, in order to generate quasi-periodic solutions to the linear problem it suffices to excite a finite number of frequencies from the linear spectrum. We shall then consider the following frequency vector

$$\omega_{\mathrm{Eq}}(\lambda):=\big(\Omega_j(\lambda)\big)_{j\in\mathbb{S}}\quad\text{with}\quad\mathbb{S}:=\{j_1,\dots,j_d\}\subset\mathbb{N}^*.$$

Note that throughout the paper, we use the notation

$$\mathbb{N}:=\{0,1,2,\dots\}\quad\text{and}\quad\mathbb{N}^*:=\{1,2,\dots\}.$$

Notice that the vector $\omega_{\mathrm{Eq}}(\lambda)$ gives quasi-periodic solutions provided that it satisfies the non-resonant condition (1.3). This property holds true for almost all the values of λ as it is proved in Proposition 3.1. Our main result concerns the persistence of quasi-periodic solutions for the nonlinear model (1.8) when the perturbation is small enough and the parameter λ is subject to be in a massive Cantor set.

THEOREM 1.1. – *Let $\lambda_1 > \lambda_0 > 0$, $\Omega > 0$, $d \in \mathbb{N}^*$ and $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. There exists $\varepsilon_0 \in (0,1)$ small enough with the following properties : For every amplitudes $\mathtt{a} = (\mathtt{a}_j)_{j\in\mathbb{S}} \in (\mathbb{R}_+^*)^d$ satisfying*

$$|\mathtt{a}| \leqslant \varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty \subset (\lambda_0, \lambda_1)$ with asymptotically full Lebesgue measure as $\mathtt{a} \to 0$, i.e.,

$$\lim_{\mathtt{a}\to 0} |\mathcal{C}_\infty| = \lambda_1 - \lambda_0,$$

such that for any $\lambda \in \mathcal{C}_\infty$, the Equation (1.8) admits a time quasi-periodic solution with Diophantine frequency vector $\omega_{pe}(\lambda, \mathtt{a}) := \big(\omega_j(\lambda, \mathtt{a})\big)_{j\in\mathbb{S}} \in \mathbb{R}^d$ and taking the form

$$r(t,\theta) = \sum_{j\in\mathbb{S}} \mathtt{a}_j \cos\big(j\theta + \omega_j(\lambda,\mathtt{a})t\big) + \mathtt{p}\big(\omega_{pe}(\lambda,\mathtt{a})t, \theta\big),$$

with

$$\omega_{pe}(\lambda,\mathtt{a}) \underset{\mathtt{a}\to 0}{\longrightarrow} \big(-\Omega_j(\lambda)\big)_{j\in\mathbb{S}},$$

where $\Omega_j(\lambda)$ are the equilibrium frequencies defined in (1.15) and the perturbation $\mathtt{p} : \mathbb{T}^{d+1} \to \mathbb{R}$ is an even function satisfying for some large index of regularity $s > \frac{d+1}{2}$

$$\|\mathtt{p}\|_{H^s(\mathbb{T}^{d+1},\mathbb{R})} \underset{\mathtt{a}\to 0}{=} o(|\mathtt{a}|).$$

Before discussing the main steps of the proof, some remarks are in order.

REMARK 1.1. – — Combining this theorem with (1.6) and (1.4) we find that the boundary shape of the patch can be parametrized in polar coordinates as follows

$$z(t,\theta) = R(t,\theta+\Omega t)e^{\mathrm{i}\theta} \quad \text{with} \quad R(t,\theta) = \big(1 + 2r(t,\theta)\big)^{\frac{1}{2}}$$

and r is described as in the theorem. The time evolution of the shape is given by small pulsation around the unit disk and the boundary is localized in an annulus around the unit circle.

— It is not at clear what could happen when λ is in the complement of the Cantor set. We expect the linear invariant tori to be destroyed by the nonlinearity and filamentation may take place generating fast increase of the curvature and the boundary length.

— The linear stability of the solutions constructed in the preceding theorem is a consequence of the proof. Indeed, the spectrum of the linearized operator at these solutions are purely imaginary.

— The dependence of the Cantor set with respect to Ω is not trivial. This dependence is linear for the linearized operator at the equilibrium state, however it becomes nonlinear along Nash-Moser scheme due to the use of an approximate inverse which depends nonlinearly on Ω. As a consequence, the eigenvalues describing the final Cantor sets might depend in a complicated way with respect to Ω.

— The Sobolev index of regularity $s > \frac{d+1}{2}$ describing the regularity of quasi-periodic solutions depends on the transversality index q of the linear frequencies, see Lemma 3.5, and the geometry of the Cantor sets. This regularity will be fixed during Nash-Moser scheme, see Proposition 7.1. Actually, we expect the solutions to be C^∞ or analytic but it is not clear for us whether we can derive it from our construction. The reason is that all the constants in Nash-Moser scheme may depend on the regularity and this may affect the size of ε_0 which might shrink to zero as s grows to infinity. We point out that this problem has been analyzed in the periodic setting for Burbea curves constructed through the bifurcation tools, see **[23, 38, 44]**. One approach that could be implemented here is to change the function spaces in order to include analytic functions and implement Nash-Moser scheme in this setting.

We shall now outline the main steps of the proof which will be developed following standard scheme as in the preceding works **[5, 7, 13, 18]** with different variations connected to the models' structure. We mainly use techniques from KAM theory combined with Nash Moser scheme. This will be implemented along several steps which are detailed below.

Step 1. Action-angle reformulation. – We first notice that the Equation (1.8) can be seen as a perturbation of the integrable system given by the linear dynamics at the equilibrium state. Indeed, by combining (1.13), (1.15) and (1.8) we may write

$$\partial_t r = \partial_\theta \mathrm{L}(\lambda)(r) + X_P(r),$$

where $\mathrm{L}(\lambda)$ and the perturbed Hamiltonian vector field X_P are defined by

$$\mathrm{L}(\lambda)(r) := -\big(\Omega + (I_1 K_1)(\lambda)\big)r + \mathcal{K}_\lambda * r \quad \text{and} \quad X_P(r) := I_1(\lambda)K_1(\lambda)\partial_\theta r - \partial_\theta \mathcal{K}_\lambda * r - F_\lambda[r].$$

Since we are looking for small solutions then we find it convenient to rescale the solution $r \rightsquigarrow \varepsilon r$ with ε a small positive number and consequently the new unknown still denoted by r satisfies

$$\partial_t r = \partial_\theta \mathrm{L}(\lambda)(r) + \varepsilon X_{P_\varepsilon}(r),$$

where X_{P_ε} is the Hamiltonian vector field defined by $X_{P_\varepsilon}(r) := \varepsilon^{-2} X_P(\varepsilon r)$. Then finding quasi-periodic solutions with frequency $\omega \in \mathbb{R}^d$ amounts to solving the equation

$$\omega \cdot \partial_\varphi r = \partial_\theta \mathrm{L}(\lambda)(r) + \varepsilon X_{P_\varepsilon}(r).$$

Here we still use the same notation r for the new profile which depends in the variables $(\varphi, \theta) \in \mathbb{T}^{d+1}$. The next step consists in splitting the phase space H_0^s into an orthogonal sum of tangential and normal subspaces as follows

$$H_0^s = H_{\overline{\mathbb{S}}} \overset{\perp}{\oplus} H_\perp^s,$$

where $H_{\overline{\mathbb{S}}}$ is the finite dimensional subspace of real functions generated by $\{e^{\mathrm{i}j\theta}, j \in \overline{\mathbb{S}}\}$ with $\overline{\mathbb{S}} := \mathbb{S} \cup (-\mathbb{S})$. For more details on this description we refer to Section 5.1. To track the dynamics it seems to be more suitable to use the action-angle variables

(I,ϑ) seen as symplectic polar variables for the Fourier coefficients of the tangential part in $H_{\overline{\mathbb{S}}}$. This leads to reformulate the problem in terms of the embedded torus,

$$\begin{aligned} i\colon \mathbb{T}^d &\to \mathbb{T}^d \times \mathbb{R}^d \times H^s_\perp \\ \varphi &\mapsto \big(\vartheta(\varphi), I(\varphi), z(\varphi)\big), \end{aligned}$$

with

$$r(\varphi,\theta) = \underbrace{v\big(\vartheta(\varphi), I(\varphi)\big)(\theta)}_{\in H_{\overline{\mathbb{S}}}} + \underbrace{z(\varphi,\theta)}_{\in H^s_\perp} := A\big(i(\varphi)\big)(\theta)$$

and

$$v(\vartheta, I) := \sum_{j\in\overline{\mathbb{S}}} \sqrt{\mathtt{a}_j^2 + |j|I_j}\, e^{\mathrm{i}\vartheta_j} e_j, \quad e_j(\theta) := e^{\mathrm{i}j\theta}.$$

Notice that the action and angle variables should satisfy the symmetry properties

$$\forall j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{R}.$$

Therefore we reduce the problem in the new variables to construct invariant tori with non-resonant frequency vector ω to the system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_\varepsilon}\big(i(\varphi)\big), \tag{1.17}$$

where X_{H_ε} is the Hamiltonian vector field associated to the Hamiltonian H_ε given by

$$H_\varepsilon = -\omega_{\mathrm{Eq}}(\lambda)\cdot I + \frac{1}{2}\langle \mathrm{L}(\lambda) z, z\rangle_{L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon,$$

with $\mathcal{P}_\varepsilon$ defined by $\mathcal{P}_\varepsilon := P_\varepsilon \circ A$. A useful trick used by Berti and Bolle in **[13]** consists to solve first the relaxed problem

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H^\alpha_\varepsilon}\big(i(\varphi)\big),$$

where the vector field $X_{H^\alpha_\varepsilon}$ is associated to the modified Hamiltonian H^α_ε given by

$$H^\alpha_\varepsilon := \alpha \cdot I + \frac{1}{2}\langle \mathrm{L}(\lambda) z, z\rangle_{L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon.$$

The advantage of this procedure is to get one degree of freedom with the vector $\alpha \in \mathbb{R}^d$ that will be used to ensure some compatibility assumptions during the construction of an approximate inverse of the linearized operator. At the end of Nash-Moser scheme we shall adjust implicitly the frequency ω so that α coincides with the equilibrium frequency $-\omega_{\mathrm{Eq}}(\lambda)$, which enables to finally get solutions to the original Hamiltonian equation. The relaxed problem can be written in the following form

$$\mathcal{F}(i, \alpha, \lambda, \omega, \varepsilon) = 0,$$

with

$$\begin{aligned}\mathcal{F}(i,\alpha,\lambda,\omega,\varepsilon) &:= \omega\cdot\partial_\varphi i(\varphi) - X_{H_\varepsilon^\alpha}\big(i(\varphi)\big)\\ &= \begin{pmatrix} \omega\cdot\partial_\varphi\vartheta(\varphi) - \alpha - \varepsilon\partial_I\mathcal{P}_\varepsilon\big(i(\varphi)\big)\\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\vartheta\mathcal{P}_\varepsilon\big(i(\varphi)\big)\\ \omega\cdot\partial_\varphi z(\varphi) - \partial_\theta\Big(\mathrm{L}(\lambda)z(\varphi) + \varepsilon\nabla_z\mathcal{P}_\varepsilon\big(i(\varphi)\big)\Big)\end{pmatrix}.\end{aligned}\tag{1.18}$$

We point out that the linear torus corresponding to the linear solution

$$r(\varphi,\theta) = \sum_{j\in\bar{\mathbb{S}}} \mathtt{a}_j\, e^{\mathrm{i}\varphi_j} e^{\mathrm{i}j\theta}$$

is given in the new coordinates system by $i_{\mathrm{flat}}(\varphi) = (\varphi, 0, 0)$ and it is obvious that

$$\mathcal{F}\big(i_{\mathrm{flat}}, -\omega_{\mathrm{Eq}}(\lambda), \lambda, -\omega_{\mathrm{Eq}}(\lambda), 0\big) = 0.$$

We emphasize that at this stage the classical implicit function theorem does not work because the linearized operator at the equilibrium state is not invertible due to resonances. One can avoid resonances by restricting the parameter λ to a suitable Cantor set according to some Diophantine conditions on the linear frequency $\omega_{\mathrm{Eq}}(\lambda)$ allowing in particular to control the small divisors problem. By this way we get an inverse at the equilibrium state but with algebraic loss of regularity. Unfortunately, this is not enough to apply Nash-Moser scheme which requires to construct a right inverse with tame estimates in a small neighborhood of the equilibrium and this is the challenging deal in this topic. Indeed, the linearized operator is no longer with constant coefficients as for the integrable case and its main part is not a Fourier multiplier. At this level we are dealing with a quasilinear problem where the perturbation is unbounded.

Step 2. Approximate inverse. – Let $\alpha_0 \in \mathbb{R}^d$ (actually α_0 is a function of the parameters ω and λ) and consider an embedded torus $i_0 = (\vartheta_0, I_0, z_0)$ near the flat one with the reversible structure,

$$\vartheta_0(-\varphi) = -\vartheta_0(\varphi), \quad I_0(-\varphi) = I_0(\varphi) \quad \text{and} \quad z_0(-\varphi,-\theta) = z_0(\varphi,\theta).$$

To deal with the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$, which exhibits complicated structure, and see whether we can construct an approximate inverse we should fix two important issues. One is related to the coupling structure in the new coordinates system and the second is that the linearized operator is with variable coefficients. For this aim we shall follow the approach conceived by Berti and Bolle in [**13**] with making suitable modifications. This approach consists in linearizing around an isotropic torus close enough to the original one and then use a symplectic change of coordinates leading to a triangular system up to small errors, essentially of "type Z" or highly decaying in frequency, that can be incorporated in Nash-Moser scheme. Therefore to invert this triangular system it suffices to get an approximate right inverse for the linearized operator in the normal direction, denoted in what follows by $\widehat{\mathcal{L}}_\omega$. We notice that in Section 5.3, and similarly to [**37**], we can bypass the use of isotropic torus by

a slight modification of Berti-Bolle approach. Actually, according to Proposition 5.1, we can conjugate the linearized operator with the transformation described by (5.59) computed at the torus i_0 and get a triangular system with small errors mainly of "type Z". The computations are performed in a straightforward way using in a crucial way the Hamiltonian structure of the original system. The main advantage that simplifies some arguments is to require the invertibility for the linearized operator only at the torus itself and not necessarily at a closer isotropic one. By this way, we can avoid the accumulation of different errors induced by the isotropic torus that one encounters for example in the estimates of the approximate inverse or in the multiple Cantor sets generated along the different reduction steps where the coefficients should be computed at the isotropic torus. The final outcome of this first step is to reduce the invertibility to finding an approximate inverse of $\widehat{\mathcal{L}}_\omega$ which takes, according to Proposition 6.1, the form

$$\widehat{\mathcal{L}}_\omega = \Pi^{\perp}_{\mathbb{S}_0}\left(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\right)\Pi^{\perp}_{\mathbb{S}_0} \quad \text{with} \quad \mathcal{L}_{\varepsilon r} = \omega\cdot\partial_\varphi + \partial_\theta\left[V_{\varepsilon r} - \mathbf{L}_{\varepsilon r}\right],$$

where $\varepsilon\partial_\theta\mathcal{R}$ is a perturbation of finite rank, the function $V_{\varepsilon r}$ and the nonlocal operator $\mathbf{L}_{\varepsilon r}$ are defined in (1.11) and (1.12), respectively. At the equilibrium state (corresponding to $\varepsilon = 0$) $\widehat{\mathcal{L}}_\omega$ is diagonal and we shall see that the set of parameters (λ,ω) leading to the existence of a right inverse is almost full. Now remark that even for ε small, the perturbation affects the main part of the operator in a similar way to water waves **[5, 18]** or generalized SQG equation **[37]** and then we should construct the suitable change of coordinates in order to reduce the positive part of the operator to a diagonal operator. Later we should implement KAM scheme to diagonalize the zero-order part. This will be done in three steps.

Step 2a. Reduction of the transport part. – This procedure will be discussed in Proposition 6.2 and Proposition 6.3. We basically use KAM techniques as in **[15, 29]** in order to conjugate the operator $\mathcal{L}_{\varepsilon r}$, through a suitable quasi-periodic symplectic change of coordinates $\mathscr{B}$, to a transport operator with constant coefficients. Indeed, we may construct an invertible transformation

$$\mathscr{B}\rho(\varphi,\theta) := \big(1+\partial_\theta\beta(\varphi,\theta)\big)\rho\big(\varphi,\theta+\beta(\varphi,\theta)\big)$$

and a constant $c_{i_0}(\lambda,\omega)$ such that for any given number $n\in\mathbb{N}$, if the parameter (λ,ω) belongs to the truncated set defined through the first order Melnikov condition

$$\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0) := \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \left\{(\lambda,\omega)\in\mathcal{O} \quad \text{s.t.} \quad \left|\omega\cdot l + jc_{i_0}(\lambda,\omega)\right| > \frac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\},$$

then we have

$$\mathfrak{L}_{\varepsilon r} := \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}(\lambda,\omega)\partial_\theta - \partial_\theta\mathcal{K}_\lambda * + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathtt{E}^0_n, \tag{1.19}$$

with $N_n := N_0^{(\frac{3}{2})^n}, N_0 \geqslant 2$, $\upsilon\in(0,1)$, $\mathcal{O} := (\lambda_0,\lambda_1)\times\mathscr{U}$, $0<\lambda_0<\lambda_1$ and $\mathscr{U} := B(0,R_0)$ being an open ball of $\mathbb{R}^d$ containing the curve of the linear vector

frequency $\lambda \in (\lambda_0, \lambda_1) \mapsto -\omega_{\mathrm{Eq}}(\lambda)$. The operator $\mathfrak{R}_{\varepsilon r}$ is a self-adjoint Toeplitz integral operator satisfying the estimates

$$\forall s \in [s_0, S], \quad \max_{k\in\{0,1,2\}} \|\partial_\theta^k \mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\right),$$

where the off-diagonal norm $\|\cdot\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}$ is defined in (4.10) and the loss of regularity σ is connected to τ_1 and d but it is independent of the index regularity s. Concerning the operator $\mathtt{E}^0_n$, we can show that it is a small fast decaying remainder with the following estimate for low regularity

$$\|\mathtt{E}^0_n \rho\|^{\gamma,\mathcal{O}}_{q,s_0} \lesssim \varepsilon\, N_0^{\mu_2} N_n^{-\mu_2} \|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}, \tag{1.20}$$

where the weighted norms $\|\cdot\|^{\gamma,\mathcal{O}}_{q,s_0}$ are defined in (4.5). For the number μ_2, it is connected to the regularity of the torus i_0 and can be taken large enough allowing to identify the contributions of $\mathtt{E}^0_n$ as small errors in the construction of the approximate inverse. The next step will be discussed in Proposition 6.4 where we explore the effect of the transport reduction on the original operator $\widehat{\mathcal{L}}_\omega$ which is localized to the normal direction. We prove that with the localized transformation $\mathscr{B}_\perp$ defined by

$$\mathscr{B}_\perp := \Pi^\perp_{\mathbb{S}_0} \mathscr{B} \Pi^\perp_{\mathbb{S}_0},$$

one obtains in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)$,

$$\mathscr{B}_\perp^{-1} \widehat{\mathcal{L}}_\omega \mathscr{B}_\perp = \omega\cdot\partial_\varphi \Pi^\perp_{\mathbb{S}_0} + \mathscr{D}_0 + \mathscr{R}_0 + \mathtt{E}^1_n, \tag{1.21}$$

where $\mathscr{D}_0$ is a diagonal operator whose spectrum $\{\mathrm{i}\mu^0_j, j \in \mathbb{S}^c_0\}$ satisfies

$$\mu^0_j(\lambda,\omega,i_0) = \Omega_j(\lambda) + j r^1(\lambda,\omega,i_0) \quad \text{ with } \quad \|r^1\|^{\gamma,\mathcal{O}}_q \lesssim \varepsilon$$

and $\mathscr{R}_0$ is a remainder term taking the form of an integral operator with Toeplitz and reversibility structures with the estimates

$$\forall s \in [s_0, S], \quad \max_{k\in\{0,1\}} \|\partial_\theta^k \mathscr{R}_0\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\right).$$

We remark that the operator $\mathtt{E}^1_n$ satisfies similar estimates as for $\mathtt{E}^0_n$ seen before in (1.20). Finally, we want to emphasize that the derivation of the asymptotic structure of the operator $\mathfrak{L}_{\varepsilon r}$ seen before in (1.19) requires some refined analysis. The delicate point concerns the expansion of the operator $\mathbf{L}_r$ defined in (1.12) and for this part we use the kernel structure detailed in (A.6)

$$K_0(z) = -\log\left(\frac{z}{2}\right) I_0(z) + \sum_{m=0}^{\infty} \frac{\psi(m+1)}{(m!)^2}\left(\frac{z}{2}\right)^{2m},$$

with I_0 being analytic. This is different from the cases discussed before as for the water waves in **[5, 18]** where the kernel is given by that of Euler equations (corresponding to $\lambda = 0$), that is, $K(z) = -\log\left(\frac{z}{2}\right)$. In this latter case the deformed kernel enjoys the structure

$$-\log\big(2A_r(t,\theta,\eta)\big) = -\log\left|\sin\left(\frac{\theta-\eta}{2}\right)\right| + \text{smooth nonhomogeneous kernel.}$$

This means that the associated operator is given by a diagonal operator of order -1 up to a smoothing non diagonal pseudo-differential operator in $OPS^{-\infty}$. In our context, this decomposition fails for $\lambda > 0$ and we get a similar one but with less smoothing operator. Actually we obtain from (5.40) the splitting

(1.22)

$$K_0\big(2\lambda A_r(t,\theta,\eta)\big) = K_0\left(2\lambda \sin\left(\frac{\theta-\eta}{2}\right)\right) + \mathscr{K}(\eta-\theta)\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta) + \mathscr{K}^2_{r,1}(\lambda,\varphi,\theta,\eta),$$

where the kernels $\mathscr{K}^1_{r,1}$ and $\mathscr{K}^2_{r,1}$ are smooth whereas $\mathscr{K}$ is slightly singular taking the form

$$\mathscr{K}(\theta) := \sin^2\left(\frac{\theta}{2}\right)\log\left(\left|\sin\left(\frac{\theta}{2}\right)\right|\right).$$

Step 2b. KAM reduction of the remainder term. – This is the main target of Section 6.3.2 and the result is stated in Proposition 6.5. The goal is to conjugate the remainder $\mathscr{R}_0$ of (1.21) and transform it into a diagonal operator. This will be developed in a standard way by constructing successive transformations through the KAM reduction allowing to replace at each step the old remainder by a new one which is much smaller provided that we make the suitable parameters extraction. This scheme can be achieved unless we solve the associated *homological equation*. To avoid resonances we should at each step make an extraction from the parameters set through the second order Melnikov conditions and the final outcome is as follows,

$$\mathscr{L}_\infty := \Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty = \omega\cdot\partial_\varphi\Pi^\perp_{\mathbb{S}_0} + \mathscr{D}_\infty,$$

where $\mathscr{D}_\infty := \big(\mathrm{i}\mu_j^\infty(\lambda,\omega,i_0)\big)_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}$ is a diagonal operator with pure imaginary spectrum and Φ_∞ is a reversible invertible operator. This reduction is possible when the parameters (λ,ω) belong to the following Cantor-like set,

$$\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0) := \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n}} \big\{(\lambda,\omega)\in\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0) \\ \text{s.t.} \quad \big|\omega\cdot l + \mu_j^\infty(\lambda,\omega,i_0) - \mu_{j_0}^\infty(\lambda,\omega,i_0)\big| > \frac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\big\}.$$

The eigenvalues admit the following asymptotic

$$\mu_j^\infty(\lambda,\omega,i_0) = \Omega_j(\lambda) + jr^1(\lambda,\omega,i_0) + r_j^\infty(\lambda,\omega,i_0),$$

where r^1 and r_j^∞ are real small coefficients with Lipschitz dependence with respect to the torus. Indeed, we have

$$\|r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}|j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1},$$

$$\|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}\|\Delta_{12}r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma},$$

for some index regularity $\overline{s}_h+\sigma$ and $\Delta_{12}r^1 = r^1(\lambda,\omega,i_1) - r^1(\lambda,\omega,i_2)$.

Step 2c. Construction of the approximate inverse. – The next step is to invert approximately the operator $\widehat{\mathcal{L}}_\omega$ detailed in Proposition 6.6. First we establish an approximate inverse for $\mathscr{L}_\infty$, on the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) := \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}} \left\{(\lambda,\omega)\in\mathcal{O} \quad \text{s.t.} \quad \left|\omega\cdot l+\mu_j^\infty(\lambda,\omega)\right| > \frac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\}.$$

Then, introducing the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0) := \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),$$

we are able to construct an approximate inverse of $\widehat{\mathcal{L}}_\omega$ in the following sense,

$$\widehat{\mathcal{L}}_\omega \mathtt{T}_{\omega,n} = \mathrm{Id} + \mathtt{E}_n \quad \text{in } \mathcal{G}_n,$$

where $\mathtt{E}_n$ is a fast frequency decaying operator as in (1.20) and $\mathtt{T}_{\omega,n}$ satisfies tame estimates uniformly in n. Therefore coming back to Section 5.3, more precisely to Theorem 5.1, this enables to construct an approximate right inverse T_0 for the full differential $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ enjoying suitable tame estimates.

In what follows we want to make some comments. The first one concerns the Lipschitz dependence of the eigenvalues with respect to the torus. This is required in studying the stability of Cantor sets in Nash-Moser scheme and allows to construct a final massive Cantor set. As for the second one, it concerns the KAM reduction which allows to diagonalize the operator when the parameters belong to a Cantor set like, even though all the involved transformations and operators can be extended to the whole set of parameters using standard cut-off functions for the Fourier coefficients. This extension with adequate estimates will be needed later during the implementation of Nash-Moser scheme. This is not the only way to produce suitable extensions and one expects Whitney extension to be also well adapted as in **[5, 18]**. In our case we adopt the first procedure which can be easily set up and manipulated using classical functional tools. The last comment is related to a technical point in KAM reduction. In contrast with the preceding papers such as **[5, 18]**, we do not need to use pseudo-differential operators techniques in the description of the aforementioned asymptotic structures of $\mathfrak{L}_{\varepsilon r}$ and $\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp$. In fact, they can be avoided since all the involved operators can be described through their kernels and therefore instead of splitting the symbols we simply expand the kernels as in (1.22) which sounds to be more appropriate in our context.

Step 3. Nash-Moser scheme. – This is the main purpose of Section 7.1 where we construct zeros for the nonlinear function $\mathcal{F}$ defined in (1.18) for small ε following Nash-Moser scheme in the spirit of the papers **[5, 18]**. Let us quickly sketch this scheme. We build by induction a sequence of approximate solutions U_n

$$U_{n+1} := U_n + H_{n+1} \quad \text{with} \quad H_{n+1} := -\Pi_{N_n}\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n),$$

with T_n an approximate inverse of $d_{i,\alpha}\mathcal{F}(U_n)$ constructed in **Step 2**. Thus using Taylor Formula we may write

$$\begin{aligned}\mathcal{F}(U_{n+1}) = \Pi_{N_n}^{\perp}\mathcal{F}(U_n) - \Pi_{N_n}\big(L_n\mathrm{T}_n - \mathrm{Id}\big)\Pi_{N_n}\mathcal{F}(U_n)\\ + \big(L_n\Pi_n^{\perp} - \Pi_{N_n}^{\perp}L_n\big)\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n) + Q_n,\end{aligned}$$

where Q_n is a quadratic functional. Consider the Cantor set

$$\mathcal{A}_n^{\gamma} := \bigcap_{k=0}^{n-1}\mathcal{G}_k(\gamma_{k+1},\tau_1,\tau_2,i_k),$$

with $\gamma_n = \gamma(1+2^{-n})$, then we show by induction that

(1.23)
$$\|U_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}, \quad \|U_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_{n-1}^{\mu}, \quad \|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} \lesssim \varepsilon N_{n-1}^{-a_1},$$

for a suitable choice of the parameters $a_1, b_1, \overline{a}, \mu, \overline{\sigma} > 0$ and O_n^{γ}, is an open enlargement of $\mathcal{A}_n^{\gamma}$ needed to construct classical extensions to the whole set of parameters $\mathcal{O}$. Actually, we get a precise statement in Proposition 7.1 allowing to deduce that the sequence $(U_n)_n$ converges in a strong topology towards a sufficiently smooth profile $(\lambda,\omega)\in\mathcal{O}\mapsto U_{\infty}(\lambda,\omega) = \big(i_{\infty}(\lambda,\omega),\alpha_{\infty}(\lambda,\omega),(\lambda,\omega)\big)$ with

$$\forall(\lambda,\omega)\in\mathcal{G}_{\infty}^{\gamma}, \quad \mathcal{F}\big(U_{\infty}(\lambda,\omega)\big) = 0, \quad \mathcal{G}_{\infty}^{\gamma} := \bigcap_{n\in\mathbb{N}}\mathcal{A}_n^{\gamma}.$$

Moreover, we get in view of Corollary 7.1 a smooth function

$$\lambda\in(\lambda_0,\lambda_1)\mapsto\big(\lambda,\omega(\lambda,\varepsilon)\big)$$

with

$$\omega(\lambda,\varepsilon) = -\omega_{\mathrm{Eq}}(\lambda) + \bar{r}_{\varepsilon}(\lambda), \quad \|\bar{r}_{\varepsilon}\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \tag{1.24}$$

and

$$\forall\lambda\in\mathcal{C}_{\infty}^{\varepsilon}, \quad \mathcal{F}\Big(U_{\infty}\big(\lambda,\omega(\lambda,\varepsilon)\big)\Big) = 0 \quad \text{with} \quad \alpha_{\infty}\big(\lambda,\omega(\lambda,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(\lambda),$$

where the Cantor set $\mathcal{C}_{\infty}^{\varepsilon}$ is defined by

$$\mathcal{C}_{\infty}^{\varepsilon} := \Big\{\lambda\in(\lambda_0,\lambda_1) \quad \text{s.t.} \quad \big(\lambda,\omega(\lambda,\varepsilon)\big)\in\mathcal{G}_{\infty}^{\gamma}\Big\}. \tag{1.25}$$

This gives solutions to the original Equation (1.17) provided that λ belongs to the final Cantor set $\mathcal{C}_{\infty}^{\varepsilon}$ and the last point to deal with aims to measure this set.

Step 4. Measure estimates. – The measure of the final Cantor set $\mathcal{C}_{\infty}^{\varepsilon}$ will be explored in Section 7.2. We show in Proposition 7.2 that by fixing $\gamma = \varepsilon^a$ for some small a we get

$$|\mathcal{C}_{\infty}^{\varepsilon}| \geqslant (\lambda_1-\lambda_0) - C\varepsilon^{\delta},$$

with small δ connected to the geometry of the Cantor set and the non degeneracy of the equilibrium spectrum. There are two main ingredients to get this result. The first one is the stability of the intermediate Cantor sets $(\mathcal{A}_n^{\gamma})_n$ following from the fast convergence of Nash-Moser scheme. However the second one is the transversality

property stated in Lemma 7.3 used in the spirit of [**9**] and [**54**]. This property will first be established for the linear frequencies in Proposition 3.5, using the analyticity of the eigenvalues and their asymptotic behavior. Then the extension of the transversality assumption to the perturbed frequencies is done using perturbative arguments together with the asymptotic description of the approximate eigenvalues detailed in (7.68), (7.83) and (7.82).

We emphasize that the transversality is strongly related to the non-degeneracy of the eigenvalues in the sense of the Definition 3.1. For instance, we show that the curve $\lambda \in [\lambda_0, \lambda_1] \mapsto \big(\Omega_{j_1}(\lambda), \dots, \Omega_{j_d}(\lambda)\big)$ is not contained in any vectorial plane, that is, if there exists a constant vector $c = (c_1, .., c_d)$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad \sum_{j=1}^{d} c_k \Omega_{j_k}(\lambda) = 0,$$

then $c = 0$. This is proved in Lemma 3.4 and follows from the asymptotic of the eigenvalues for large values of λ according to the law (A.11) combined with the invertibility of Vandermonde matrices.