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(974) *Algebraization of codimension one Webs*

Jorge Vitório PEREIRA

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
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ALGEBRAIZATION OF CODIMENSION ONE WEBS [after Trépreau, Hénaut, Pirio, Robert, ...]

by **Jorge Vitório PEREIRA**

Jean-Marie Trépreau, extending previous results by Bol and Chern-Griffiths, proved recently that codimension one webs with sufficiently many abelian relations are after a change of coordinates projectively dual to algebraic curves when the ambient dimension is at least three.

In sharp contrast, Luc Pirio and Gilles Robert, confirming a guess of Alain Hénaut, independently established that a certain planar 9-web is exceptional in the sense that it admits the maximal number of abelian relations and is non-algebraizable. After that a number of exceptional planar k-webs, for every $k \geq 5$, have been found by Pirio and others.

I will briefly review the subject history, sketch Trépreau's proof, describe some of the "new" exceptional webs and discuss related recent works.

Disclaimer. — This text does not pretend to survey all the literature on web geometry but to provide a bird's-eye view over the results related to codimension one webs and their abelian relations. For instance I do not touch the interface between web geometry and loops, quasi-groups, Poisson structures, singular holomorphic foliations, complex dynamics, singularity theory, ... For more information on these subjects the reader should consult **[6, 2, 27]** and references there within.

Acknowledgements. — There are a number of works containing introductions to web geometry that I have freely used while writing this text. Here I recognize the influence of **[4, 15, 30]** and specially **[40]**, which was my main source of historical references. I have also profited from discussions with C. Favre, H. Movasati, L. Pirio, F. Russo and P. Sad.

1. INTRODUCTION

A *germ of regular codimension one k-web* $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^n, 0)$ is a collection of k germs of smooth codimension one holomorphic foliations subjected to the condition that for any number m of these foliations, $m \leq n$, the corresponding tangent spaces at the origin have intersection of codimension m. Two webs $\mathcal{W}$ and $\mathcal{W}'$ are *equivalent* if there exists a germ of bihomolorphic map sending the foliations defining $\mathcal{W}$ to the ones defining $\mathcal{W}'$. Similar definitions can be made for webs of arbitrary (and even mixed) codimensions. Although most of the magic can be (and has already been) spelled in the $\mathcal{C}^\infty_{\mathbb{R}}$-category throughout I will restrict myself to the holomorphic category.

1.1. The origins

According to the first lines of **[6]** web geometry had its birth at the beaches of Italy in the years of 1926-27 when Blaschke and Thomsen realized that the configuration of three foliations of the plane has local invariants, see Figure 1.

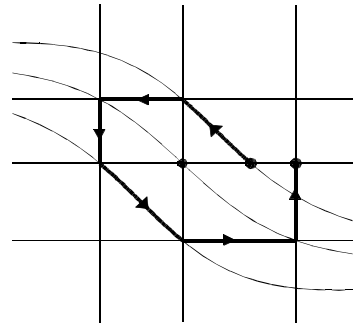

FIGURE 1. Following the leaves of foliations one obtains germs of diffeomorphisms in one variable whose equivalence class is a local invariant of the web. The web is called *hexagonal* if all the possible germs are the identity.

A more easily computable invariant was later introduced by Blaschke and Dubourdieu. If $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ is a planar web and the foliations $\mathcal{F}_i$ are defined by 1-forms ω_i satisfying $\omega_1 + \omega_2 + \omega_3 = 0$ then a simple computation shows that there exists an unique 1-form γ such that $d\omega_i = \gamma \wedge \omega_i$ for $i = 1, 2, 3$. Albeit the 1-form γ does depend on the choice of the ω_i its differential $d\gamma$ is intrinsically attached to $\mathcal{W}$, and is the so called *curvature* $\kappa(\mathcal{W})$ of $\mathcal{W}$.

Some early emblematic results of the theory developed by Blaschke and his collaborators are collected in the theorem below.

THEOREM 1.1. — *If $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ is a 3-web on $(\mathbb{C}^2, 0)$ then are equivalent:*

1. *$\mathcal{W}$ is hexagonal;*
2. *the 2-form $\kappa(\mathcal{W})$ vanishes identically;*
3. *there exists closed 1-forms η_i defining $\mathcal{F}_i$, $i = 1, 2, 3$, such that $\eta_1 + \eta_2 + \eta_3 = 0$;*
4. *$\mathcal{W}$ is equivalent to the web defined by the level sets of the functions x, y and $x - y$.*

Most of the results discussed in this text can be naïvely understood as attempts to generalize Theorem 1.1 to the broader context of arbitrary codimension one k-webs.

1.2. Abelian relations

The condition (3) in Theorem 1.1 suggests the definition of the *space of abelian relations* $\mathcal{A}(\mathcal{W})$ for an arbitrary k-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. If the foliations $\mathcal{F}_i$ are induced by integrable 1-forms ω_i then

$$\boxed{\mathcal{A}(\mathcal{W}) = \left\{ (\eta_i)_{i=1}^k \in (\Omega^1(\mathbb{C}^n, 0))^k \;\middle|\; \forall i \; d\eta_i = 0\,,\ \eta_i \wedge \omega_i = 0 \ \text{ and } \sum_{i=1}^k \eta_i = 0 \right\}.}$$

If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$ then, after integration, the abelian relations can be read as functional equations of the form $\sum_{i=1}^k g_i(u_i) = 0$ for some germs of holomorphic functions $g_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

Clearly $\mathcal{A}(\mathcal{W})$ is a vector space and its dimension is commonly called the *rank* of $\mathcal{W}$, denoted by $\mathrm{rk}(\mathcal{W})$. It is a theorem of Bol that the rank of a planar k-web is bounded from above by $\frac{1}{2}(k-1)(k-2)$. This bound was later generalized by Chern in his thesis (under the direction of Blaschke) for codimension one k-webs on $\mathbb{C}^n$ and reads as

$$\boxed{\mathrm{rk}(\mathcal{W}) \leq \pi(n, k) = \sum_{j=1}^{\infty} \max(0, k - j(n-1) - 1)\,.} \tag{1}$$

A k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$ is of *maximal rank* if $\mathrm{rk}(\mathcal{W}) = \pi(n, k)$. The integer $\pi(n, k)$ is the well-known Castelnuovo's bound for the arithmetic genus of irreducible and non-degenerated degree k curves in $\mathbb{P}^n$.

To establish these bounds first notice that $\mathcal{A}(\mathcal{W})$ admits a natural filtration

$$\mathcal{A}(\mathcal{W}) = \mathcal{A}^0(\mathcal{W}) \supseteq \mathcal{A}^1(\mathcal{W}) \supseteq \cdots \supseteq \mathcal{A}^j(\mathcal{W}) \supseteq \cdots,$$

where

$$\boxed{\mathcal{A}^j(\mathcal{W}) = \ker \left\{ \mathcal{A}(\mathcal{W}) \longrightarrow \left(\frac{\Omega^1(\mathbb{C}^n, 0)}{\mathfrak{m}^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\},}$$

with $\mathfrak{m}$ being the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_n\}$.

If the submersions u_i defining $\mathcal{F}_i$ have linear term ℓ_i, then

$$\dim \frac{\mathcal{C}^j(\mathcal{W})}{\mathcal{C}^{j+1}(\mathcal{W})} \leq k - \dim\left(\mathbb{C}\cdot\ell_1^{j+1} + \cdots + \mathbb{C}\cdot\ell_k^{j+1}\right). \tag{2}$$

Since the right-hand side is controlled by the inequality, cf. [**49**, Lemme 2.1],

$$k - \dim\left(\mathbb{C}\cdot\ell_1^{j+1} + \cdots + \mathbb{C}\cdot\ell_k^{j+1}\right) \leq \max(0, k-(j+1)(n-1)-1)$$

the bound (1) follows at once. Note that this bound is attained if, and only if, the partial bounds (2) are also attained. In particular,

$$\dim \mathcal{C}(\mathcal{W}) = \pi(n,k) \implies \dim \frac{\mathcal{C}^0(\mathcal{W})}{\mathcal{C}^2(\mathcal{W})} = 2k-3n+1. \tag{3}$$

It will be clear at the end of the next section that the appearance of Castelnuovo's bounds in web geometry is far from being a coincidence.

1.3. Algebraizable webs and Abel's Theorem

If C is a non-degenerated[(1)] reduced degree k algebraic curve on $\mathbb{P}^n$ then for every generic hyperplane H_0 a germ of codimension one k-web $\mathcal{W}_C$ is canonically defined on $(\check{\mathbb{P}}^n, H_0)$ by projective duality. This is the web induced by the levels of the holomorphic maps $p_i : (\check{\mathbb{P}}^n, H_0) \to C$ characterized by

$$H\cdot C = p_1(H) + p_2(H) + \cdots + p_k(H)$$

for every H sufficiently close to H_0.

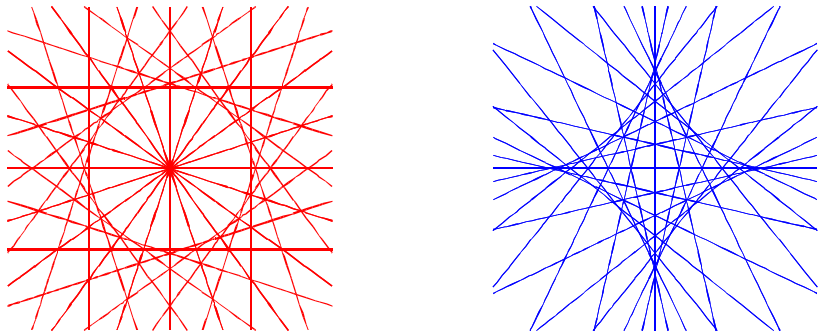

FIGURE 2. On the left $\mathcal{W}_C$ is pictured for a reduced cubic curve C formed by a line and a conic. On the right $\mathcal{W}_C$ is drawn for a rational quartic C.

(1) Throughout the term non-degenerated will be used in a stronger sense than usual in order to ensure that the dual web is smooth. It means that any collection of points in the intersection of C with a generic hyperplane, but not spanning the hyperplane, is formed by linearly independent points.

Abel's addition Theorem says that for every $p_0 \in C$ and every holomorphic[(2)] 1-form $\omega \in \mathrm{H}^0(C, \omega_C)$ the sum

$$\int_{p_0}^{p_1(H)} \omega + \int_{p_0}^{p_2(H)} \omega + \cdots + \int_{p_0}^{p_k(H)} \omega$$

does not depend of H. One can reformulate this statement as

$$\sum_{i=1}^{k} p_i^* \omega = 0\,.$$

It follows that the 1-forms on C can be interpreted as abelian relations of $\mathcal{W}_C$. In particular $\dim \mathcal{A}(\mathcal{W}_C) \geq h^0(C, \omega_C)$ and if C is an *extremal curve* — a non-degenerated reduced curve attaining Castelnuovo's bound — then $\mathcal{W}_C$ has maximal rank.

The key question dealt with in the works reviewed here is the characterization of the *algebraizable* codimension one webs. These are the webs equivalent to $\mathcal{W}_C$ for a suitable projective curve C.

1.4. A converse to Abel's Theorem

The ubiquitous tool for the algebraization of k-webs is the following theorem.

THEOREM 1.2. — *Let $C_1, \ldots, C_k$ be germs of curves on $\mathbb{P}^n$ all of them intersecting transversely a given hyperplane H_0 and write $p_i(H) = H \cap C_i$ for a hyperplane H sufficiently close to H_0. Let also ω_i be germs of non-identically zero 1-forms on the curves C_i and assume that the trace*

$$\sum_{i=1}^{k} p_i^* \omega_i$$

vanishes identically. Then there exist a degree k reduced curve $C \subset \mathbb{P}^n$ and a holomorphic 1-form ω on C such that $C_i \subset C$ and $\omega_{|C_i} = \omega_i$ for all i ranging from 1 to k.

Theorem 1.2 in the case of plane quartics was obtained by Lie in his investigations concerning double translation surfaces, cf. Figure 3. The general case follows from Darboux proof (following ideas of Poincaré) of Lie's Theorem. The result has been generalized to germs of arbitrary varieties carrying holomorphic forms of the maximum degree by Griffiths, cf. [**24**]. More recently Henkin and Henkin-Passare generalized the result even further by showing, in particular, that the rationality of the trace is sufficient to ensure the algebraicity of the data, see [**34**] and references therein.

(2) If C is singular then the holomorphicity of ω means that it is an 1-form of first kind with respect to system of hyperplanes, i.e., the expression $\left(\int_{p_0}^{p_1(H)} \omega + \int_{p_0}^{p_2(H)} \omega + \cdots + \int_{p_0}^{p_k(H)} \omega\right)$, seen as a holomorphic function of $H \in \check{\mathbb{P}}^n$, has no singularities. It turns out that the holomorphic 1-forms on C are precisely the sections of the dualizing sheaf ω_C.

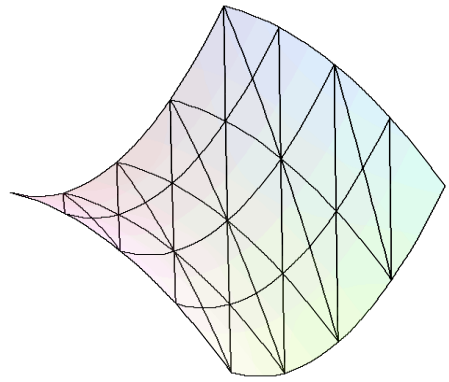

FIGURE 3. A double translation surface is a surface $S \subset \mathbb{R}^3$ that admits two independent parameterizations of the form $(x, y) \mapsto f(x) + g(y)$. S carries a natural 4-web $\mathcal{W}$. The lines tangent to leaves of $\mathcal{W}$ cut the hyperplane at infinity along 4 germs of curves. Lie's Theorem says that these 4 curves are contained in a degree 4 algebraic curve. This result was later generalized **[50]** to arbitrary double translation hypersurfaces.

The relevance of Theorem 1.2 to our subject is evident once one translates it — as Blaschke-Howe ($n = 2$) and Bol ($n \geq 3$) did — to the dual projective space. We recall that a *linear web* is a web whose leaves are pieces of hyperplanes.

THEOREM 1.3. — *A linear k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$ carrying an abelian relation that is not an abelian relation of any subweb extends to a global (but singular) web $\mathcal{W}_C$ on $\mathbb{P}^n$.*

With Theorem 1.3 at hand the algebraization of $2n$-webs on $(\mathbb{C}^n, 0)$ with $n + 1$ abelian relations follows from a beautiful argument of Blaschke — inspired in Poincaré's works on double translation surfaces — that goes as follows.

1.5. A first algebraization result

If $\mathcal{W}$ is a k-web on $(\mathbb{C}^n, 0)$ of maximal rank r then — mimicking the construction of the canonical map for algebraic curves — one defines, for $i = 1, \ldots, k$, the maps

$$\begin{array}{rcl} Z_i : (\mathbb{C}^n, 0) & \longrightarrow & \mathbb{P}^{r-1} \\ x & \mapsto & [\eta_i^1(x) : \ldots : \eta_i^r(x)] \end{array}$$

with $\left\{(\eta_1^\lambda, \ldots, \eta_k^\lambda)\right\}_{\lambda=1,\ldots,r}$ being a basis for $\mathcal{A}(\mathcal{W})$. Although the η_i^λ's are 1-forms the maps Z_i's are well-defined since, for a fixed i, any two of these forms differ by the multiplication of a meromorphic function constant along the leaves of $\mathcal{F}_i$. It is an immediate consequence that the image of each Z_i is a germ of curve. Note that the equivalence class under $\mathrm{Aut}(\mathbb{P}^r)$ of these germs is an analytic invariant of $\mathcal{W}$.

Since $\mathcal{W}$ has maximal rank then $\dim A^0(\mathcal{W})/\mathcal{C}^1(\mathcal{W}) = k-n$. Therefore the points $Z_1(x),\dots,Z_k(x)$ span a projective space $\mathbb{P}^{k-n-1} \subset \mathbb{P}^{r-1}$.

One can thus define the *Poincaré map* $\mathcal{P} : (\mathbb{C}^n,0) \longrightarrow \mathbb{G}_{k-n-1}(\mathbb{P}^{r-1})$ by setting $\mathcal{P}(x) = \mathrm{Span}(Z_1(x),\dots,Z_k(x))$. It is a simple matter to prove that $\mathcal{P}$ is an immersion.

If $k = 2n$ then the Poincaré map takes values in $\mathbb{G}_{n-1}(\mathbb{P}^n) = \check{\mathbb{P}}^n$. The image of the leaf through x of the foliation $\mathcal{F}_i$ lies on the hyperplane of $\check{\mathbb{P}}^n$ determined by $Z_i(x)$. Thus $\mathcal{P}_*\mathcal{W}$ is a linear web and its algebraicity follows from Theorem 1.3. □

1.6. Bol's Algebraization Theorem and further developments

Most of the material so far exposed can be found in [**6**]. This outstanding volume summarizes most of the works of the Blaschke School written during the period 1927-1938. One of its deepest result is Bol's *Hauptsatz für Flächengewebe* (main theorem for webs by surfaces) presented in §32–35 and originally published in [**7**]. It says that *for $k \neq 5$, every codimension one k-web on $(\mathbb{C}^3,0)$ of maximal rank is algebraizable.*

For $k \leq 4$ the result is an easy exercise and the case $k = 6$ has just been treated in §1.5. Every 5-web on $(\mathbb{C}^3,0)$ of the form $\mathcal{W}(x,y,z,x+y+z,f(x)+g(y)+h(z))$[(3)] has maximal rank but for almost every choice of the functions f,g,h it is not algebraizable, see for instance [**4, 49**].

In the remaining cases, $k \geq 2n+1$, Bol's proof explores an analogy between the equations satisfied by the defining 1-forms of maximal rank webs and geodesics on semi-riemannian manifolds. Latter in [**15**] Chern and Griffiths attempted to generalize Bol's result to arbitrary dimensions. Their strategy consisted in defining a path geometry in which the leaves of the web turn out to be totally geodesic hypersurfaces. The linearization follows from the flatness of such path geometry. Unfortunately there was a gap in the proof, cf. [**17**], that forced the authors to include an ad-hoc hypothesis on the web to ensure its algebraization.

2. ALGEBRAIZATION OF CODIMENSION ONE WEBS ON $(\mathbb{C}^n,0)$, $n \geq 3$

The purpose of this section is to sketch Trépreau's Algebraization Theorem stated below. An immediate corollary is the algebraization of maximal rank k-webs on dimension at least three for $k \geq 2n$. One has just to combine Trépreau's result with the displayed equation (3). In particular the ad-hoc hypothesis in Chern-Griffiths Theorem is not necessary.

(3) $\mathcal{W}(u_1,\dots,u_k)$ is the k-web induced by the levels of the functions $u_1,\dots,u_k$.

THEOREM 2.1 ([**49**]). — *Let $n \geq 3$ and $k \geq 2n$ or $k \leq n+1$. If $\mathcal{W}$ is a k-web on $(\mathbb{C}^n, 0)$ satisfying*

$$\dim \frac{\mathcal{C}^0(\mathcal{W})}{\mathcal{C}^2(\mathcal{W})} = 2k - 3n + 1$$

then $\mathcal{W}$ is algebraizable.

Like Bol's Theorem the result is true for $k \leq n+1$ and false for $n+1 < k < 2n$ thanks to fairly elementary reasons.

Trépreau pointed out [**49**] that the general strategy has a high order of contact with Bol's proof and that [**6**, §35.3] suggests that the result should hold true for webs by surfaces on $(\mathbb{C}^3, 0)$.

It has also to be remarked that Theorem 2.1 does not completely characterize the algebraizable webs on $(\mathbb{C}^n, 0), n \geq 3$. In contrast with the planar case — where all the algebraic webs have maximal rank — the algebraic webs on higher dimensions satisfying the hypothesis of Theorem 2.1 are dual to rather special curves. One distinguished feature of these curves is that they are contained in surfaces of minimal degree. For instance, in the simplest case where the curve is an union of lines through a certain point $x \in \mathbb{P}^n$ the dual web satisfies the hypothesis if, and only if, the corresponding points in $\mathbb{P}(\mathrm{T}_x\mathbb{P}^n)$ lie on a rational normal curve of degree $n-1$.

Since Trépreau's argument is fairly detailed and self-contained I will avoid the technicalities to focus on the general lines of the proof.

2.1. A field of rational normal curves on $\mathbb{P}\mathrm{T}^*(\mathbb{C}^n, 0)$

When $k = 2n$ the hypothesis of Theorem 2.1 implies that $\mathcal{W}$ has maximal rank. The argument presented in §1.5 suffices to prove the theorem in this particular case. Until the end of the proof it will be assumed that $k \geq 2n+1$.

LEMMA 2.2. — *If $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$ is a k-web on $(\mathbb{C}^n, 0)$, $n \geq 2$ and $k \geq 2n+1$ such that $\dim \mathcal{C}^0(\mathcal{W})/\mathcal{C}^2(\mathcal{W}) = 2k - 3n + 1$ then there exists a basis $\omega_0, \ldots, \omega_{n-1}$ of the $\mathcal{O}$-module $\Omega^1_{(\mathbb{C}^n,0)}$ such that the defining submersions $u_1, \ldots, u_k$ of $\mathcal{W}$ satisfy $du_\alpha = k_\alpha \sum_{\mu=0}^{n-1} (\theta_\alpha)^\mu \omega_\mu$ for suitable functions $k_\alpha, \theta_\alpha : (\mathbb{C}^n, 0) \to \mathbb{C}$.*

Geometrically speaking the lemma says that for every $x \in (\mathbb{C}^n, 0)$ the points on $\mathbb{P}T^*_x(\mathbb{C}^n, 0)$ determined by $T_x\mathcal{F}_1, \ldots, T_x\mathcal{F}_k$ lie on a degree $(n-1)$ rational normal curve $C(x)$ parameterized as $[s:t] \mapsto \left[\sum_{i=0}^{n-1} s^{n-i} t^i \omega_i\right]$. The basis $\omega_0, \ldots, \omega_{n-1}$ as in the statement of Lemma 2.2 is called an *adapted basis* for $\mathcal{W}$.

The details are in [**49**, Lemme 3.1] or [**15**, p. 61-62]. Here I will just remark that once one realizes that

$$\dim \frac{\mathcal{C}^0(\mathcal{W})}{\mathcal{C}^2(\mathcal{W})} = 2k - 3n + 1 \implies \dim \frac{\mathcal{C}^1(\mathcal{W})}{\mathcal{C}^2(\mathcal{W})} = k - 2n + 1$$

and that the latter equality implies that the space of quadrics on $\mathbb{P}\mathrm{T}^*_x(\mathbb{C}^n,0)$ containing $T_x\mathcal{F}_1,\ldots,T_x\mathcal{F}_k$ has codimension $(2n-1)$ then the proof of the lemma follows immediately from the Lemma of Castelnuovo: *If $k\geq 2(n-1)+3$ points in general position on $\mathbb{P}^{n-1}$ impose just $2(n-1)+1$ conditions on the space of quadrics then these points belong to a rational normal curve of degree $(n-1)$.*

2.2. Rational normal curves on $\mathbb{P}^{2k-3n}$

Let $\mathcal{W}$ be a k-web on $(\mathbb{C}^n,0)$ satisfying the hypothesis of Lemma 2.2. Fix local submersions $u_1,\ldots,u_k:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ defining $\mathcal{W}$ and $(\eta_1^\lambda,\ldots,\eta_k^\lambda)$, $\lambda=1,\ldots,2k-3n+1$, elements in $\mathcal{A}(\mathcal{W})$ with classes generating $\mathcal{A}^0(\mathcal{W})/\mathcal{A}^2(\mathcal{W})$.

If $z^\lambda(x)=(z_1^\lambda(x),\ldots,z_k^\lambda(x))$ are vector functions for which

$$\eta^\lambda(x)=z^\lambda(x)\cdot(du_1(x),\ldots,du_k(x))^T$$

then the maps $Z_i:(\mathbb{C}^n,0)\to\mathbb{P}^{2k-3n}$ — natural variant of the maps under the same label defined in §1.5 — can be explicitly written as the projectivization of the maps

$$\begin{array}{rcll}\widetilde{Z}_i:(\mathbb{C}^n,0) & \longrightarrow & \mathbb{C}^{2k-3n+1} & (i=1,\ldots,k)\\ x & \mapsto & (z_i^1(x),z_i^2(x),\ldots,z_i^{2k-3n+1}(x))\,.\end{array}$$

For a fixed $x\in(\mathbb{C}^n,0)$, like in §1.5, the span of $Z_1(x),\ldots,Z_k(x)$ has dimension $k-n-1$. It will be denoted by $\mathbb{P}^{k-n-1}(x)$.

Using the notation of Lemma 2.2 one can introduce the map

$$\begin{array}{rcl}\widetilde{Z}_*:(\mathbb{C}^n,0)\times\mathbb{C} & \to & \mathbb{C}^{2k-3n+1}\\ (x,t) & \mapsto & \displaystyle\sum_{i=1}^{d}\left(\prod_{j\neq i}(t-\theta_j(x))\right)k_j(x)\widetilde{Z}_j(x)\end{array}$$

and its projectivization $Z_*:(\mathbb{C}^n,0)\times\mathbb{P}^1\to\mathbb{P}^{2k-3n}$. Expanding the entries of $Z_*(x,t)$ as polynomials on t one verifies that these have degree $(k-n-1)$. Thus the points $Z_1(x),\ldots,Z_k(x)$ lie on a unique degree $(k-n-1)$ rational normal curve $C(x)$ contained in $\mathbb{P}^{k-n-1}(x)$, see [**49**, Lemme 4.3].

It can also be shown that the *Poincaré map* $x\mapsto\mathbb{P}^{k-n-1}(x)$ is an immersion. Moreover, if x and x' are distinct points then $\mathbb{P}^{k-n-1}(x)$ and $\mathbb{P}^{k-n-1}(x')$ intersect along a projective space $\mathbb{P}^{n-2}(x,x')$. Since any number of distinct points on a degree $(n-1)$ rational normal curve contained in $\mathbb{P}^{n-1}$ are in general position it follows that the curves $C(x)$ and $C(x')$ intersect in at most $(n-1)$ points, cf. [**49**, Lemme 4.2].

2.3. The rational normal curves $C(x)$ define an algebraic surface $S \subset \mathbb{P}^{2k-3n}$

The main novelty of Trépreau's argument is his elementary proof that, when $n \geq 3$,

$$Z_* : (\mathbb{C}^n, 0) \times \mathbb{P}^1 \to \mathbb{P}^{2k-3n} \quad \text{has rank two for every } (x,t) \in (\mathbb{C}^n, 0) \times \mathbb{P}^1.$$

Besides ingenuity the key ingredient is [**49**, Lemme 3.2] stated below. It is deduced from a careful analysis of second order differential conditions imposed by the maximality of the dimension of $\mathscr{C}^0(\mathcal{W})/\mathscr{C}^2(\mathcal{W})$.

LEMMA 2.3. — *If we write a* 1*-form* α *as* $\alpha = \sum(\alpha)_\mu \omega_\mu$ *and use the same hypothesis and notations of Lemma 2.2 then for every* $\mu \in \{0, \ldots, n-2\}$ *there exist holomorphic functions* $m_{\mu 0}, \ldots, m_{\mu(n-1)}$ *satisfying* $\left(d(k_\alpha \theta_\alpha)\right)_\mu - (dk_\alpha)_{\mu+1} = k_\alpha \sum_{\lambda=0}^{n-1} m_{\mu\lambda}(\theta_\alpha)^\lambda$. *Moreover, if* $n \geq 3$ *then* $\theta_\alpha (d\theta_\alpha)_\mu - (d\theta_\alpha)_{\mu+1} = \sum_{\lambda=0}^{n} n_{\mu\lambda}(\theta_\alpha)^\lambda$ *for suitable functions* $n_{\mu 0}, \ldots, n_{\mu n}$.

Only in the proof of this lemma the hypothesis on the dimension of the ambient space is used. In particular the algebraization of maximal rank planar webs for which the conclusion of the lemma holds will also follow.

For every $x \in (\mathbb{C}^n, 0)$ the map $t \mapsto Z_*(x,t)$ is an isomorphism from $\mathbb{P}^1$ to $C(x)$. Combining this with the fact that Z_* has rank two everywhere it follows that the image of Z_* is a smooth analytic open surface $S_0 \subset \mathbb{P}^{2k-3n}$.

If x and x' are distinct points laying on the same leaf of $(n-1)$ foliations defining $\mathcal{W}$ then $C(x)$ and $C(x')$ will intersect in exactly $n-1$ points. This is sufficient to ensure that the curve $C(0)$ has self-intersection (in the surface S_0) equal to $n-1$.

To prove that S_0 is an open subset of an (eventually singular) algebraic surface $S \subset \mathbb{P}^{2k-3n}$ consider the subset $\mathfrak{X}$ of $\mathrm{Mor}_{k-n-1}(\mathbb{P}^1, \mathbb{P}^{2k-3n})$[(4)] consisting of morphisms ϕ with image contained in S_0 and $\phi(0:1) = x_0$. It follows that $\mathfrak{X}$ is algebraic — just expand $f_i(\phi(t:1))$ for every defining equations f_i of S_0 in a suitable neighborhood of x_0. To conclude one has just to notice that the Zariski closure of the natural projection to $\mathbb{P}^{2k-3n}$ — the evaluation morphism — sends $\mathfrak{X}$ to an algebraic surface S of $\mathbb{P}^{2k-3n}$ containing S_0.

2.4. The curves $C(x)$ belong to a linear system of projective dimension n

The proof presented in [**49**] is based on a classical Theorem of Enriques [**20**] concerning the linearity of families of divisors. For a modern proof and generalizations of Enriques Theorem, see [**18**, Theorem 5.10]. Here an alternative approach, following [**17**, p. 82], is presented.

(4) This is just the set of morphisms from $\mathbb{P}^1$ to $\mathbb{P}^n$ of degree $k-n-1$ which can be naturally identified with a Zariski open subset of $\mathbb{P}\left(\mathbb{C}_{k-n-1}[s,t]^{2k-3n+1}\right)$.

Since $S_0 \subset S$ is smooth we can replace S by one of its desingularizations in such a way that S_0 will still be an open subset. Moreover being the curves $C(x)$ pairwise homologous in S_0 the same will hold true for their strict transforms. Summarizing, for all that matters, we can assume that S is itself a smooth surface.

Because S is covered by rational curves of positive self-intersection it is a rational surface. Therefore $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ and homologous curves are linearly equivalent. Consequently if we set $C = C(0)$ then all the curves $C(x)$ belong to $\mathbb{P}\mathrm{H}^0(S, \mathcal{O}_S(C))$.

The exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to N_C \to 0$ immediately implies that

$$\mathrm{h}^0(S, \mathcal{O}_S(C)) = 1 + \mathrm{h}^0(C, N_C) = 1 + \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(C^2)) = n + 1.$$

Consequently $\dim \mathbb{P}\mathrm{H}^0(S, \mathcal{O}_S(C)) = n$.

2.5. The algebraization map

The map $x \mapsto C(x)$ takes values on the projective space $\mathbb{P}^n = \mathbb{P}\mathrm{H}^0(S, \mathcal{O}_S(C))$. It is a holomorphic map and the leaf through x of one of the defining foliations $\mathcal{F}_i$ is mapped to the hyperplane contained in $\mathbb{P}\mathrm{H}^0(S, \mathcal{O}_S(C))$ corresponding to the divisors through $Z_i(x) \in S$.

The common intersection of the hyperplanes corresponding to the leaves of $\mathcal{W}$ through 0 reduces to the point corresponding to C. Otherwise there would be an element in $\mathbb{P}\mathrm{H}^0(S, \mathcal{O}_S(C))$ intersecting C in at least n points contradicting $C^2 = n-1$. Therefore the map is an immersion and the image of $\mathcal{W}$ is a linear web. Trépreau's Algebraization Theorem follows from Theorem 1.3. □

3. EXCEPTIONAL PLANAR WEBS I: THE HISTORY

The webs of maximal rank that are not algebraizable are usually called *exceptional webs*. Trépreau's Theorem says that on dimension $n \geq 3$ there are no exceptional codimension one k-webs for $k \geq 2n$. The next three sections, including this one, discuss the planar case. On the first I will draw the general plot of the quest for exceptional webs on $(\mathbb{C}^2, 0)$ — as I have learned from [**40**, Chapitre 8] and references therein. The second will survey the methods to prove that a given web is exceptional while the third will be completely devoted to examples.

3.1. Blaschke's approach to the algebraization of planar 5-webs

In the five pages paper [**5**] the *proof* that all 5-webs on $(\mathbb{C}^2, 0)$ of maximal rank are algebraizable is sketched. Although wrong Blaschke's paper turned out to be a rather influential piece of mathematics. For instance, the starting point of Bol's proof of the *Hauptsatz für Flächengewebe* can be found there.

For a 5-web of maximal rank Blaschke defines a variation of the Poincaré map — the *Poincaré-Blaschke map* — as follows

$$\begin{aligned} \mathscr{PB} : (\mathbb{C}^2, 0) &\longrightarrow \mathbb{G}_4(\mathbb{P}^5) = \check{\mathbb{P}}^5 \\ x &\mapsto \mathrm{Span}\big(Z_1(x), \ldots, Z_5(x), {Z_1}'(x), \ldots, {Z_5}'(x)\big)\,, \end{aligned}$$

where Z_i is defined as in §1.5 and ${Z_i}'$ makes sense since the image of the map Z_i has dimension one. The fact that the spanned projective subspace has dimension 4 follows from a reasoning similar to the one presented in §1.5.

The main mistake in loc. cit. is Satz 2 that, combined with a result of Darboux, implies that the image of $\mathscr{PB}$ is contained in a Veronese surface. If this is the case then it is indeed true that the 5-web is algebraizable. For a detailed proof of the latter statement, see [**40**, Proposition 8.4.6].

3.2. Bol's counter-example and Blaschke-Segre surfaces on $\mathbb{P}^5$

Blaschke's mistake was pointed out by Bol in [**8**]. There he provided a counterexample by proving that the 5-web $\mathscr{B}_5$ had rank 6, see Figure 4. Besides 5 linearly independent *obvious* abelian relations coming from the hexagonal 3-subwebs he found another one of the form $\sum_{i=1}^{5} \left(\frac{\log 1-t_i}{t_i} + \frac{\log t_i}{1-t_i} \right) dt_i = 0$, where $t_1 = \frac{y}{x}$, $t_2 = \frac{x+y-1}{y}$, $t_3 = \frac{x-y}{1-y}$, $t_4 = \frac{1-y}{x}$ and $t_5 = \frac{x(1-x)}{y(1-y)}$ are rational functions defining $\mathscr{B}_5$. The integration of this expression leads to Abel's functional equation

$$\boxed{\sum_{i=1}^{5} Li_2(t_i) + Li_2(1-t_i) = 0\,,}$$

for Euler's dilogarithm $Li_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$.

Bol studies the image of the Poincaré-Blaschke map for $\mathscr{B}_5$ and shows that it is a germ of (transcendental) surface with the remarkable property: *it is non-degenerated and has five families of curves such that the tangent spaces of S along each of these curves lie on a hyperplane of $\mathbb{P}^5$ that depends just on the curve.* To make the further reference easier let us adopt the (non-standard) terminology *Blaschke-Segre surfaces* to describe the surfaces with this property. The choice of terminology follows from the fact that the tangents of the curves in the five families must coincide with Segre's principal directions of S. We recall that at a point $p \in S$ (S non-degenerated and not contained in a Veronese surface) these are the five directions (multiplicities taken into account) determined by the tangent cones of the intersection of S with one of the five hyperplanes that intersects S in a tacnode (or worst singularity) at p.

The relation between exceptional 5-webs and Blaschke-Segre surfaces was noticed by Bol. In his own words: "*Im übrigen sieht man, daß die Bestimmung von allen*

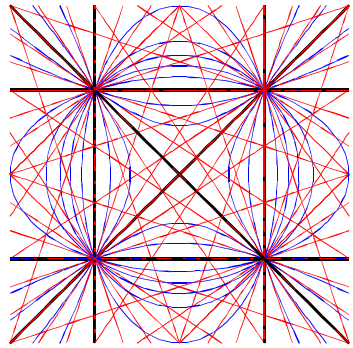

FIGURE 4. *Bol's Exceptional* 5*-web* $\mathscr{B}_5$ is the web induced by four pencils of lines with base points in general position and a pencil of conics through these four base points. It is the unique non-linearizable web for which all its 3-subwebs are hexagonal. For almost 70 years it remained the only known example of non-algebraizable 5-web of maximal rank.

Fünfgeweben höchsten Ranges hinausläuft auf die Angabe aller Flächen mit Segreschen Kurvenscharen, und umgekehrt; (. . .)", in [**8**, pp. 392–393]

The beautiful underlying geometry of the Blaschke-Segre surfaces caught the eyes of some Italian geometers including Bompiani, Buzano and Terracini. In the first lines of [**9**] it is remarked that the exceptional 5-webs give rise to Blaschke-Segre surfaces echoing the above quote by Bol. Buzano and Terracini pursued the task of determining/classifying other germs of Blaschke-Segre surfaces in [**48, 10**]. Their approach was mainly analytic and quickly led to the study of certain non-linear system of PDEs. They were able to classify, under rather strong geometric assumptions on the families of curves, some classes of Blaschke-Segre surfaces. At the end they obtained a small number of previously unknown examples. Apparently, the determination of the rank of the naturally associated 5-webs was not pursued at that time, cf. [**6**, page 261].

Buzano pointed out that two of his Blaschke-Segre surfaces induced quite remarkable 5-webs: both are of the form $\mathscr{W}(x, y, x + y, x - y, f(x, y))$ and, moreover, the 3-subwebs $\mathscr{W}(x, y, f(x, y))$ and $\mathscr{W}(x + y, x - y, f(x, y))$ are hexagonal. The complete classification of 5-webs with these properties is carried out in [**11**]. Nevertheless he did not wonder whether the obtained 5-webs come from Blaschke-Segre surfaces or if they are of maximal rank.

After the 1940's the study of webs of maximal rank seemed to be forgotten until the late seventies when Chern and Griffiths — apparently motivated by Griffiths' project aiming at the understanding of rational equivalence of cycles in algebraic varieties — pursued the task of extending Bol's Theorem for dimensions greater than three, cf. §1.6.

In a number of different opportunities Chern emphasized that a better understanding of the exceptional planar 5-webs, or more generally of the exceptional webs, should be pursued. For instance, after a quick browsing of the papers by Chern on web geometry and Blaschke's work one collects the following quotes (see also [**12**, Unsolved Problems], [**14**, Problem 6]):

- *"At this low-dimensional level an important unsolved problem is whether there are other 5-webs of rank 6, besides algebraic ones and Bol's example."*, [**13**].
- *"In general, the determination of all webs of maximum rank will remain a fundamental problem in web geometry and the non-algebraic ones, if there are any, will be most interesting."*, [**14**].
- *"(...) we cannot refrain from mentioning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many. It may not be unreasonable to compare the situation with the exceptional simple Lie groups."*, [**17**].

Chern's insistence can be easily justified. The exceptional planar webs are, in a certain sense, generalizations of algebraic plane curves and a better understanding of these objects is highly desirable.

The questions of Chern had to wait around 20 years to receive a first answer. In [**30**], Hénaut recognizes that 9-web induced by the rational functions figuring in Spence-Kummer 9-terms functional equation for the trilogarithm as a good candidate for exceptionality. In 2002, Pirio and Robert independently settled that this 9-web is indeed exceptional.

In [**25**] Griffiths suggests that exceptionality is in strict relation with the polylogarithms. In particular he asks if all the exceptional webs are somehow related to functional equations for polylogarithms.

In view of all these questions, it was a surprise when Pirio showed that $\mathcal{W}(x, y, x+y, x-y, x^2+y^2)$ is an exceptional 5-web and its space of abelian relations is generated by the elementary polynomial identities (cf. [**41**] and also [**40**])

$$\begin{array}{lcllcl} (x^2+y^2) & = & x^2+y^2 & 0 & = & x-y-(x-y) \\ 6(x^2+y^2)^2 & = & 4x^4+4y^4+(x+y)^4+(x-y)^4 & 0 & = & (x-y)^2+(x+y)^2-2x^2-2y^2 \\ 10(x^2+y^2)^3 & = & 8x^6+8y^6+(x+y)^6-(x-y)^6 & 0 & = & x+y-(x+y). \end{array}$$

In loc. cit. other exceptional webs are determined, e.g. $\mathcal{W}(x, y, x+y, x-y, xy)$ and $\mathcal{W}(x, y, x+y, x-y, x^2+y^2, xy)$. In section §5 most of the exceptional webs found by Pirio, Robert and others are described.

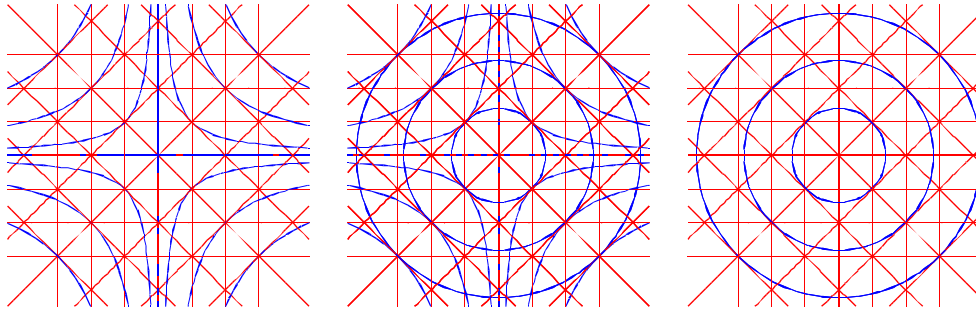

FIGURE 5. Three of the *new* examples of exceptional webs found by Pirio. The 6-web in the middle is the superposition of the other two 5-webs.

4. EXCEPTIONAL PLANAR WEBS II: THE METHODS

To put in evidence the exceptionality of a k-web on $(\mathbb{C}^2,0)$ one has to check that the web is non-linearizable and that it has maximal rank. Here I will briefly survey some of the methods to deal with both problems.

4.1. Linearization conditions for planar webs

If $\mathcal{W}=\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_k$ is a k-web on $(\mathbb{C}^2,0)$ and the foliations $\mathcal{F}_i$ are induced by vector fields $X_i=\frac{\partial}{\partial x}+p_i(x,y)\frac{\partial}{\partial y}$ then there exists an unique polynomial

$$P_{\mathcal{W}}(x,y,p)=l_1(x,y)p^{k-1}+l_2(x,y)p^{k-2}+\cdots+l_k(x,y)$$

in $\mathbb{C}\{x,y\}[p]$ of degree at most $(k-1)$ such that $X_i(p_i)=\frac{\partial p_i}{\partial x}+p_i\frac{\partial p_i}{\partial y}=P_{\mathcal{W}}(x,y,p_i(x,y))$ for every $i\in\{1,\ldots,k\}$.

One can verify that the leaves of the web $\mathcal{W}$ can be presented as the graphs of the solutions of $y''=P_{\mathcal{W}}(x,y,y')$. In **[28]** (see also **[6]**, §29]) it is proven that a k-web $\mathcal{W}$ is linearizable if, and only if, there exists a local change of the coordinates (x,y) that simultaneously linearizes all the solutions of the second order differential equation above. A classical result of Liouville says that this is the case if, and only if, (a) $\deg_p P_{\mathcal{W}}\leq 3$; and (b) the coefficients $(l_k,l_{k-1},l_{k-2},l_{k-3})$ satisfy a certain (explicit) system of differential equations, cf. **[28]** for details.

Notice that all the computations involved can be explicitly carried out. Moreover, if the web is given in implicit form $F(x,y,y')$ then the polynomial $\mathcal{P}_{\mathcal{W}}$ can also be explicitly computed from the coefficients of F, see **[45]**, Chapitre 2].

For our purposes, a particularly useful consequence of this criterium is the following corollary **[28]**, **[6]**, p. 247]: *If $\mathcal{W}$ is a k-web on $(\mathbb{C}^2,0)$ with $k\geq 4$ then, modulo projective transformations, $\mathcal{W}$ admits at most one linearization.*

As a side remark we mention a related result due to Nakai [**38**, Theorem 2.1.3]: if $\mathcal{W}_C$ and $\mathcal{W}_{C'}$ are two algebraic webs associated to irreducible curves on $\mathbb{P}^n$ of degree at least $n+2$ then every orientation preserving homeomorphisms of $\check{\mathbb{P}}^n$ conjugating $\mathcal{W}_C$ and $\mathcal{W}_{C'}$ is an automorphism of $\mathbb{P}^n$. An amusing corollary is in Nakai's own words: *"the complex structure of a line bundle $L \to C$ on a Riemann surface is determined by the topological structure of a net of effective divisors determining L."*

There are other criteria for linearizability of d-webs, $d \geq 4$, cf. [**3**]. Concerning the linearization of planar 3-webs there is Gronwall's conjecture: a non-algebraizable 3-web on $(\mathbb{C}^2, 0)$ admits at most one linearization. Bol proved that the number of linearizations is at most 16. In [**26**] an approach suggested by Akivis to obtain relative differential invariants characterizing the linearization of 3-webs is followed. The authors succeeded in reducing Bol's bound to 15. Similar results have been recently reobtained in [**23**].

4.2. Detecting the maximality of the rank

The methods to check the maximality of the rank can be naturally divided into two types. The ones of the first type — Methods 1, 2 and 3 below — aim at the determination of the space of abelian relations. Methods 4, 5 and 6 do not determine the abelian relations explicitly but in turn characterize the webs of maximal rank by the vanishing of certain algebraic functions on the data (and their derivatives) defining it. These characterizations can be interpreted as generalizations of the equivalence (2) $\iff$ (3) in Theorem 1.1.

4.2.1. *Method 1: differential elimination (Abel's method).* — If a k-web $\mathcal{W} = \mathcal{W}(u_1, \ldots, u_k)$ is defined by germs of submersions $u_i : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ then the determination of $\mathcal{A}(\mathcal{W})$ is equivalent to finding the germs of functions $f_1, \ldots, f_k : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfying

$$f_1(u_1) + f_2(u_2) + \cdots + f_k(u_k) = 0.$$

Abel, in his first published paper — *Méthode générale pour trouver des fonctions d'une seule quantité variable lorsqu'une propriété de ces fonctions est exprimée par une équation entre deux variables* [**1**] — furnished an algorithmic solution to it. The key idea consists in eliminating the dependence in the functions $u_2, \ldots, u_k$ by means of successive differentiations in order to obtain a linear differential equation of the form

$$\boxed{f_1^{(l)}(u_1) + c_{l-1}(u_1) f_1^{(l-1)}(u_1) + \cdots + c_0(u_1) f_1(u_1) = 0}$$

satisfied by the f_1. The coefficients c_i are expressed as rational functions of $u_1, u_2, \ldots, u_k$ and their derivatives. After solving this linear differential equation

and the similar ones for $f_2, \dots, f_k$ the determination of the abelian relations reduces to plain linear algebra.

Abel's method has been revisited by Pirio — cf. [**40**, Chapitre 2], [**42**] — and after implementing it he was able to determine the rank of a number of planar webs including the ones induced by the Blaschke-Segre surfaces found by Buzano and Terracini. They all turned out to be exceptional.

Notice that the computations involved tend to be rather lengthy and this, perhaps, explains why the use of such method to determine *new* exceptional webs had to wait until 2002.

4.2.2. *Method 2: polylogarithmic functional relations.* — Another approach to determine some of the abelian relations of a given particular web was proposed by Robert in [**47**]. Instead of looking for all possible abelian relations he aims at the ones involving polylogarithms. He uses a variant of a criterium due to Zagier [**51**] that reduces the problem to linear algebra. In contrast with Abel's method this one has a narrower scope but tends to be more efficient since it bypasses the solution of differential equations.

More precisely, if $u_1, \dots, u_k \in \mathbb{C}(x,y)$ are rational functions on $\mathbb{C}^2$ and $U \subset \mathbb{C}^2$ is a suitably chosen open subset then the existence of abelian relations of the form

$$\boxed{\sum_{i=1}^{k} \lambda_i \mathrm{Li}_r(u_i) + \sum_{i=1}^{k} \sum_{l=1}^{r-1} P_{i,l}(\log u_i) \mathrm{Li}_{r-l}(u_i) = 0\,,}$$

with $P_{i,j} \in \mathbb{C}[x,y]$ and $\lambda_i \in \mathbb{C}$, is equivalent to the symmetry of the tensor

$$\boxed{\sum_{i=1}^{k} \lambda_i \left(\left(\frac{du_i}{u_i} \right)^{\otimes k-1} \otimes \frac{du_i}{1-u_i} \right).}$$

in $\bigotimes_{\mathbb{C}}^{r} \Omega^1(U)$, cf. [**47**, Théorème 1.3].

4.2.3. *Method 3: Abelian relations in the presence of automorphisms.* — Let $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ denote a k-web in $(\mathbb{C}^2, 0)$ which admits an infinitesimal automorphism X, regular and transverse to the foliations $\mathcal{F}_i$ in a neighborhood of the origin.

Clearly the Lie derivative of L_X acts on $\mathcal{A}(\mathcal{W})$ and an analysis of such action allows one to infer that the abelian relations of $\mathcal{W}$ can be written in the form, cf. [**36**, Proposition 3.1],

$$\boxed{P_1(u_1)\, e^{\lambda_i\, u_1}\, du_1 + \cdots + P_k(u_k)\, e^{\lambda_i\, u_k}\, du_k = 0}$$

where $P_1, \dots, P_k$ are polynomials of degree less than or equal to the size of the i-th Jordan block of $L_X : \mathcal{C}(\mathcal{W}) \circlearrowleft$, λ_i are the eigenvalues and $u_i = \int \frac{\omega_i}{\omega_i(X)}$.

The rank of the web $\mathcal{W} \boxtimes \mathcal{F}_X$ obtained from $\mathcal{W}$ by superposing the foliation induced by X is related to the rank of $\mathcal{W}$ [**36**, Theorem 1] by the formula

$$\boxed{\mathrm{rk}(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathrm{rk}(\mathcal{W}) + (k-1)\,.}$$

In particular, $\mathcal{W}$ is of maximal rank if, and only if, $\mathcal{W} \boxtimes \mathcal{F}_X$ is also of maximal rank. Once one realizes that the Lie derivative L_X induces linear operators on $\mathcal{C}(\mathcal{W})$ and $\mathcal{C}(\mathcal{W} \boxtimes \mathcal{F}_X)$ then the proof of this result boils down to linear algebra.

4.2.4. *Method 4: Pantazi's Method.* — In [**39**], Pantazi explains a method to determine the rank of a k-web defined by k holomorphic 1-forms $\omega_1, \dots, \omega_k$. He introduced $N = (k-1)(k-2)/2$ expressions — algebraically and explicitly constructed from the coefficients of the ω_i and their derivatives — which are identically zero if, and only if, the web is of maximal rank.

Building on Pantazi's method Mihăileanu obtains in [**37**] a necessary condition for the maximality of the rank: the sum of the curvatures of all 3-subwebs of $\mathcal{W}$ must vanish.

4.2.5. *Method 5: the implicit approach (Hénaut's Method* [**32**]*).* — If $\mathcal{W}$ is a regular k-web defined on $(\mathbb{C}^2, 0)$ by an implicit differential equation $f(x, y, y')$ of degree k on y' then the contact 1-form $dy - pdx$ on $(\mathbb{C}^2, 0) \times \mathbb{C}$ defines a foliation $\mathcal{F}_{\mathcal{W}}$ on the surface S cut out by $f(x, y, p)$ such that $\pi_* \mathcal{F} = \mathcal{W}$, with $\pi : S \to (\mathbb{C}^2, 0)$ being the natural projection.

On this implicit framework the abelian relations of $\mathcal{W}$ can be interpreted as 1-forms $\eta = \left(b_3 p^{d-3} + \cdots + b_d\right) \cdot \frac{dy - pdx}{\frac{\partial f}{\partial p}} \in \pi_* \Omega^1_S$ which are closed. It follows that there exists a linear system of differential equations $\mathcal{M}_{\mathcal{W}}$ with space of solutions isomorphic to $\mathcal{C}(\mathcal{W})$. The system $\mathcal{M}_{\mathcal{W}}$ is completely determined by f.

Using Cartan-Spencer theory, Hénaut builds a rank $N = (k-1)(k-2)/2$ vector bundle E contained in the jet bundle $J_{k-2}(\mathcal{O}^{k-2})$ and a holomorphic connection $\nabla : E \to E \otimes \Omega^1$ such that the local system of solutions of ∇ is naturally isomorphic to $\mathcal{M}_{\mathcal{W}}$. It follows that $\mathcal{W}$ has maximal rank if, and only if, the curvature form of ∇ is identically zero.

Although not explicit in principle, this construction has been untangled by Ripoll, who implemented in a symbolic computation system the curvature matrix determination for 3, 4 and 5-webs, cf. [**45**].

An interpretation for the induced connection $(\det E, \det \nabla)$ is provided by [**45**, Théorème 5.2] when $k \leq 6$ and in [**33**, p. 281],[**46**] for arbitrary k. After multiplying

f by a suitable unit there exists a connection isomorphism

$$(\det E, \det \nabla) \simeq \left(\bigotimes_{k=1}^{\binom{d}{3}} L_k, \bigotimes_{k=1}^{\binom{d}{3}} \nabla_k \right)$$

where (L_k, ∇_k) are (suitably chosen) connections of all 3-subwebs of $\mathcal{W}$. As a corollary they reobtain Mihăileanu necessary condition for the rank maximality.

An extensive study of the connection ∇ and its invariants remains to be done. For a number of interesting questions and perspectives, see [**33**]. Here I will just point out that due to their complementary nature it would be interesting to clarify the relation between Pantazi's and Hénaut's method.

4.2.6. *Method 6: Goldberg-Lychagin's Method.* — A variant of the previous two methods has been proposed in [**22**]. The equations imposing the maximality of the rank are expressed in terms of relative differential invariants of the web.

5. EXCEPTIONAL PLANAR WEBS III: THE EXAMPLES

In this section, I will briefly describe new exceptional webs that have come to light since 2002. The list below is not extensive. To the best of my knowledge all the other new examples available in the literature can be found in [**40**] and [**36**]. The non-linearizability of all the examples below can be inferred from the fact they are non-linear webs but contain a linear k-subweb with $k \geq 4$, see §4.1.

5.1. Polylogarithmics webs

If $Li_3(z) = \sum \frac{z^n}{n^3}$ is the trilogarithm then the Spence-Kummer functional equation for it is

$$\begin{aligned} & 2Li_3(x) + 2Li_3(y) - Li_3\left(\frac{x}{y}\right) + 2Li_3\left(\frac{1-x}{1-y}\right) + 2Li_3\left(\frac{x(1-y)}{y(1-x)}\right) \\ - \quad & Li_3\left(xy\right) + 2Li_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2Li_3\left(-\frac{1-y}{y(1-x)}\right) - Li_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) \\ = \quad & 2Li_3\left(1\right) - \log(y)^2 \log\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\log(y) + \frac{1}{3}\log(y)^3 . \end{aligned}$$

The naturally associated 9-web, after the change $(x,y)\mapsto\left(\frac{1+x}{x},\frac{1+y}{y}\right)$, is

$$\mathcal{W}_{\mathscr{JK}}=\mathcal{W}\left(\underbrace{\frac{x}{1+y},\overbrace{\frac{1+x}{y},\underbrace{x,y,\frac{x}{y},\frac{1+x}{1+y},\frac{y(1+x)}{x(1+y)}}_{\mathcal{B}_5},\frac{(1+x)(1+y)}{xy}}^{\mathcal{B}_6},\frac{x(1+x)}{y(1+y)}}_{\mathcal{B}_7}\right).$$

$\mathcal{W}_{\mathscr{JK}}$ was recognized as a good candidate for exceptionality in **[30]**. It was later shown to be exceptional by two different methods. Robert apparently developed method 2 for this purpose and Pirio used Abel's method. The subweb $\mathcal{B}_5$ is clearly an isomorphic copy of Bol's 5-web. The subwebs $\mathcal{B}_6$ and $\mathcal{B}_7$ (see displayed equation) are also exceptional. Notice that $\mathcal{B}_5\subset\mathcal{B}_6\subset\mathcal{B}_7\subset\mathcal{W}_{\mathscr{JK}}$.

Due to the rich automorphism group of $\mathcal{W}_{\mathscr{JK}}$ one can easily recognize other subwebs isomorphic to $\mathcal{B}_5,\mathcal{B}_6$ and $\mathcal{B}_7$ contained in $\mathcal{W}_{\mathscr{JK}}$. Besides these there are one exceptional 5-subweb ([**40**, Théorème 7.2.5]) and one exceptional 6-subweb ([**40**, Théorème 7.2.5], [**47**, §3.2]) of $\mathcal{W}_{\mathscr{JK}}$ that are non-isomorphic to $\mathcal{B}_5$ and $\mathcal{B}_6$ respectively.

Robert has also determined an exceptional 8-web $\mathcal{B}_8$ containing $\mathcal{B}_7$ but not isomorphic to any 8-subweb of $\mathcal{W}_{\mathscr{JK}}$. It is obtained from $\mathcal{B}_7$ by adding the pencil of lines $\frac{2x-1}{2y-1}$, [**47**, Théorème 3.1].

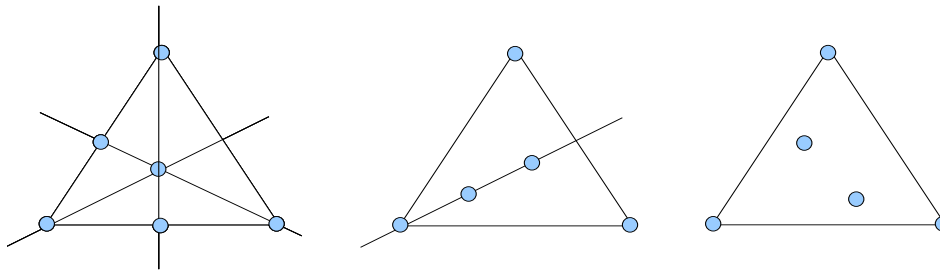

FIGURE 6. The configuration in the left naturally induces $\mathcal{W}_{\mathscr{JK}}$ while the one in the middle induces a 1-parameter family of exceptional webs.

$\mathcal{W}_{\mathscr{JK}}$ admits a description analogous to Bol's 5-web. If one considers the configuration of six points on $\mathbb{P}^2$ schematically represented in the left of Figure 6 then $\mathcal{W}_{\mathscr{JK}}$ is formed by the six pencils of lines through the points and three pencils of conics through any four of the six points that are in general position. If one considers exactly the same construction using the other two configurations of five points represented in Figure 6 then the configuration in the middle induces a 1-parameter family of 8-webs while the one in the right induces a 2-parameters family of 10-webs. The first turns

out to be a family of exceptional 8-webs, cf. [**40**, Théorème 7.3.1]. The second remains a good candidate for a family of exceptional 10-webs since all the members satisfy Mihăileanu necessary condition for the rank maximality.

All the other possible configurations of five points on $\mathbb{P}^2$ induce exceptional webs. On the other hand, see [**40**, p. 182], the web associated to a generic configuration of 6 points on $\mathbb{P}^2$ does not satisfy Mihăileanu condition and therefore is not exceptional.

Webs naturally associated to Kummer's equations for the tetralogarithm and the pentalogarithm have also been studied in [**40**, Chapitre 7]. They do not satisfy Mihăileanu condition and therefore are not exceptional. Nevertheless they do contain some previously unknown exceptional 5 and 6-subwebs.

5.2. Quasi-parallel webs

In [**41**] a number of 5-webs on $(\mathbb{C}^2,0)$ have been determined with the help of Abel's method. They are all of the form $\mathcal{W}(x,y,x+y,x-y,u(x,y))$ for some germ of holomorphic function $u(x,y)=v(x)+w(y)$.

Later in [**43**] the classification of the 5-webs of this particular form was pursued. At the end they obtained that all 5-webs on $(\mathbb{C}^2,0)$ of the form $\mathcal{W}[v(x)+w(y)] = \mathcal{W}(x,y,z+y,x-y,v(x)+w(y))$ are equivalent to one of the following

$$(a)\quad \mathcal{W}[\log(\sin(x)\sin(y))] \qquad (b)\quad \mathcal{W}[x^2-y^2] \qquad (c)\quad \mathcal{W}[x^2+y^2]$$
$$(d)\quad \mathcal{W}[\log(\tanh(x)\tanh(y))] \quad (e)\quad \mathcal{W}[\exp(x)+\exp(y)]$$
$$(f)_k\quad \mathcal{W}[\log(\mathrm{sn}_k(x)\mathrm{sn}_k(y))]$$

with sn_k being the Jacobi's elliptic functions of module $k\in\mathbb{C}\setminus\{-1,0,1\}$. The webs $(a),(b),(c),(d)$ and (e) can all be interpreted as *limits* of the webs $(f)_k$ under suitable *renormalizations*.

The abelian relations are either polynomial ones or follow from well-known identities involving theta functions and classical functions.

Notice that all 3-webs of the form $\mathcal{W}(x,y,v(x)+w(y))$ are hexagonal. In the course of the classification it is proved that the maximality of the rank of $\mathcal{W}[v(x)+w(y)]$ implies that the 3-subweb $\mathcal{W}(x+y,x-y,v(x)+w(y))$ is hexagonal. Coincidentally this reduces the problem to the one considered in [**11**].

5.3. Webs admitting infinitesimal automorphisms

Method 3 implies that every reduced degree k curve $C\subset\mathbb{P}^2$ left invariant by a $\mathbb{C}^*$-action induces, on the dual projective plane, an exceptional $(k+1)$-web formed by the superposition of $\mathcal{W}_C$ and the orbits of the dual $\mathbb{C}^*$-action [**36**].

If one considers the curves cut out by polynomials of the form

$$\prod_{i=1}^{\lfloor k/2 \rfloor} \left(xy - \lambda_i z^2 \right) , \quad \lambda_i \neq \lambda_j \in \mathbb{C}^*,$$

then it follows that for every $k \geq 5$ there exists a family of dimension at least $\lfloor k/2 \rfloor - 1$ of pairwise non-equivalent exceptional global k-webs on $\mathbb{P}^2$.

6. WEBS OF ARBITRARY CODIMENSION

There are a number of works dealing with webs of arbitrary codimension and their abelian relations. In the next few lines I will try to briefly review some of the most recent advances. Although, even more than in the previous paragraphs, I do not aim at completeness and, probably, a number of important omissions are made.

A k-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ of codimension r on $(\mathbb{C}^n, 0)$ is a collection of k foliations of codimension r such that the tangent spaces $T_0\mathcal{F}_1, \ldots, T_0\mathcal{F}_k$ are in general position, i.e., the intersection of any number m of these subspaces has the minimal possible dimension while the union has the maximal possible dimension.

For every non-negative integer $\ell \leq r$ one can define the space of degree ℓ abelian relations of $\mathcal{W}$ in terms of closed ℓ-forms vanishing along the leaves of the defining foliations.

If V is a reduced non-degenerated[(5)] subvariety of $\mathbb{P}^{n+r-1}$ of degree k and dimension r and Π is a generic $(n-1)$-plan then, analogously to the case of curves, V induces a k-web $\mathcal{W}_V$ on $(\mathbb{G}_{n-1}(\mathbb{P}^{n+r-1}), \Pi)$, where $\mathbb{G}_{n-1}(\mathbb{P}^{n+r-1})$ is the Grassmanian of $(n-1)$ planes on $\mathbb{P}^{n+r-1}$. Using a natural affine chart around Π one sees that $(\mathbb{G}_{n-1}(\mathbb{P}^{n+r-1}), \Pi) \cong (\mathbb{C}^{nr}, 0)$ and that $\mathcal{W}_V$ is equivalent to a k-web of codimension r on $(\mathbb{C}^{nr}, 0)$ with linear leaves. The k-webs of codimension r on $(\mathbb{C}^{nr}, 0)$ are denoted by $\mathcal{W}_k(n, r)$.

In **[16]** bounds for the dimension of the space of degree r abelian relations for webs $\mathcal{W}_k(n, r)$ are obtained. These bounds are attained by webs $\mathcal{W}_V$ where V is an extremal subvariety of $\mathbb{P}^{n+r-1}$ in the sense that the dimension of $\mathrm{H}^0(V, \omega_V)$ is maximal among the non-degenerated varieties of same degree and codimension. Recently Hénaut provided sharp bounds for the ℓ-rank of webs $\mathcal{W}_k(n, r)$ for every $\ell \leq r$, cf. **[31]**.

In view of the algebraization results for codimension one webs one is naturally led to wonder if the $\mathcal{W}_k(n, r)$ of maximal rank are algebraizable when k is sufficiently

(5) In a similar sense to the one used for curves, cf. §1.3.

large. Algebraization results for the $\mathcal{W}_k(2,r)$ of maximal r-rank have been obtained by [**21**] ($r=2$) and [**29**] (every $r\geq 2$). For $\ell<r$ or $r\geq 2$ and $n\geq 3$, the characterization of the $\mathcal{W}_k(n,r)$ of maximal ℓ-rank seems to be open.

The study of webs which have codimension not dividing the dimension of the ambient space also leads to beautiful geometry. A prototypal result in this direction is Blaschke-Walberer Theorem [**6**, §35–36] for 3-webs by curves on $(\mathbb{C}^3,0)$ of maximum 1-rank (proven by Blaschke to be 5). It says that these 3-webs can be obtained from cubic hypersurfaces on $\mathbb{P}^4$ by means of an algebraic correspondence.

Concerning webs by curves there are also some interesting results by Damiano. He provided a bound for the $(n-1)$-rank of a web by curves on $(\mathbb{C}^n,0)$ [**19**, Proposition 2.4], found generalizations of Bol's exceptional web $\mathcal{B}_5$ to non-linearizable $(n+3)$-webs by curves on $\mathbb{C}^n$ of maximum $(n-1)$-rank [**19**, Theorem 5.5] and linked the abelian relations of these webs to the Gabrielov-Gelfand-Losik work on the first Pontrjagin class of a manifold, cf. [**35**].

I cannot find a better way to close this survey than recalling a few more words of Chern [**12**] about web geometry:

> (...) *the subject is a wide generalization of the geometry of projective algebraic varieties. Just as intrinsic algebraic varieties are generalized to Kähler manifolds and complex manifolds, such a generalization to web geometry seems justifiable.*

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Jorge Vitório PEREIRA

IMPA
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110 – Jardim Botânico
22460-320 Rio de Janeiro RJ Brasil
E-mail : jvp@impa.br