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MEASURED QUANTUM GROUPOIDS

Franck LESIEUR

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F. Lesieur
LMNO, Université de Caen, BP 5186, F-14032 Caen Cedex.
E-mail : franck.lesieur@math.unicaen.fr
Url : http://www.math.unicaen.fr/~lesieur

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MEASURED QUANTUM GROUPOIDS

Franck Lesieur

Abstract. – In this volume, we give a definition for measured quantum groupoids. We want to get objects with duality extending both quantum groups and groupoids. We base ourselves on J. Kustermans and S. Vaes' works about locally compact quantum groups that we generalize thanks to formalism introduced by M. Enock and J.M. Vallin in the case of inclusion of von Neumann algebras. From a structure of Hopf-bimodule with left and right invariant operator-valued weights, we define a fundamental pseudo-multiplicative unitary. To get a satisfying duality in the general case, we assume the existence of an antipode given by its polar decomposition. This theory is illustrated with many examples among others inclusion of von Neumann algebras (M. Enock) and a sub family of measured quantum groupoids with easier axiomatic.

***Résumé* (Groupoïdes quantiques mesurés).** – Dans cet volume, on définit une notion de groupoïdes quantiques mesurés. On cherche à obtenir des objets munis d'une dualité qui étend celle des groupoïdes et des groupes quantiques. On s'appuie sur les travaux de J. Kustermans et S. Vaes concernant les groupes quantiques localement compacts qu'on généralise grâce au formalisme introduit par M. Enock et J.M. Vallin à propos des inclusions d'algèbres de von Neumann. À partir d'un bimodule de Hopf muni de poids opératoriels invariants à gauche et à droite, on définit un unitaire pseudo-multiplicatif fondamental. Pour obtenir une dualité satisfaisante dans le cas général, on suppose l'existence d'une antipode définie par sa décomposition polaire. Cette théorie est illustrée dans une dernière partie par de nombreux exemples notamment les inclusions d'algèbres de von Neumann (M. Enock) et une sous famille de groupoïdes quantiques mesurés à l'axiomatique plus simple.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Historic

Theory of quantum groups has lot of developments in operator algebras setting. Many contributions are given by [**KaV74, Wor88, ES89, MN91, BS93, Wor95, Wor96, VDa98, KV00**]. In particular, J. Kustermans and S. Vaes' work is crucial: in [**KV00**], they propose a simple definition for locally compact quantum groups which gathers all known examples (locally compact groups, quantum compacts group [**Wor95**], quantum group $ax+b$ [**Wor01, WZ02**], Woronowicz' algebra [**MN91**]...) and they find a general framework for duality of theses objects. The very few number of axioms gives the theory a high manageability which is proved with recent developments in many directions (actions of locally compact quantum groups [**Vae01b**], induced co-representations [**Kus02**], cocycle bi-crossed products [**VV03**]). They complete their work with a theory of locally compact quantum groups in the von Neumann setting [**KV03**].

In geometry, groups are rather defined by their actions. Groupoids category contains groups, group actions and equivalence relation. It is used by G.W. Mackey and P. Hahn ([**Mac66, Hah78a, Hah78b**]), in a measured version, to link theory of groups and ergodic theory. Locally compact groupoids and the operator theory point of view are introduced and studied by J. Renault in [**Ren80, Ren97**]. It covers many interesting examples in differential geometry [**Co94**] *e.g.*, holonomy groupoid of a foliation.

In [**Val96**], J.M. Vallin introduces the notion of Hopf bimodule from which he is able to prove a duality for groupoids. Then, a natural question is to construct a category, containing quantum groups and groupoids, with a duality theory.

In the quantum group case, duality is essentially based on a multiplicative unitary [**BS93**]. To generalize the notion up to the groupoid case, J.M. Vallin introduces pseudo-multiplicative unitaries. In [**Val00**], he exhibits such an object coming from

Hopf bimodule structures for groupoids. Technically speaking, Connes-Sauvageot's theory of relative tensor products is intensively used.

In the case of depth 2 inclusions of von Neumann algebras, M. Enock and J.M. Vallin, and then, M. Enock underline two "quantum groupoids" in duality. They also use Hopf bimodules and pseudo-multiplicative unitaries. At this stage, a non trivial modular theory on the basis (the equivalent for units of a groupoid) is revealed to be necessary and a simple generalization of axioms quantum groups is not sufficient to construct quantum groupoids category: we have to add an axiom on the basis [**Eno00**] *i.e.*, we use a special weight to do the construction. The results are improved in [**Eno04**].

In [**Eno02**], M. Enock studies in detail pseudo-multiplicative unitaries and introduces an analogous notion of S. Baaj and G. Skandalis' regularity. In quantum groups, the fundamental multiplicative unitary is weakly regular and manageable in the sense of Woronowicz. Such properties have to be satisfied in quantum groupoids. Moreover, M. Enock defines and studies compact (resp. discrete) quantum groupoids which have to enter into the general theory.

Lot of works have been led about quantum groupoids but essentially in finite dimension. We have to quote weak Hopf C*-algebras introduced by G. Böhm, F. Nill and K. Szlachányi [**BNS99**], [**BSz96**], and then studied by F. Nill and L. Vaĭnerman [**Nik02, Nil98, NV00, NV02**]. J.M. Vallin develops a quantum groupoids theory in finite dimension thanks to multiplicative partial isometries [**Val01, Val02**]. He proves that his theory coincides exactly with weak Hopf C*-algebras.

1.2. Aims and Methods

In this article, we propose a definition for measured quantum groupoids in any dimensions. "Measured" means we are in the von Neumann setting and we assume existence of the analogous of a measure. We use a similar approach as J. Kustermans and S. Vaes' theory with the formalism of Hopf bimodules and pseudo-multiplicative unitaries. The notion has to recover all known examples and shall extend their duality if already existing.

In our setting, we assume the existence of a scaling group and a co-involution so that we are much more closer to [**MNW03**]. Then, we are able to construct a dual structure for these objects and we prove a duality theorem. We also get uniqueness of the equivalent of Haar measure.

We want to give many examples. First of all, we present a family of measured quantum groupoids of a particular interest: their axiomatic is easier than the general measured quantum groupoids and very similar to J. Kustermans and S. Vaes axiomatic of locally compact quantum groups because we can construct the antipode. However, this new category is not self dual but we can characterize their dual objects. Then we are interested in depth 2 inclusions of von Neumann algebras of Enock's type which are included in our theory and for which we can compute the dual structure. In a forthcoming article, we will study an example of the type $G = G_1G_2$ where G_1 and G_2 are two groupoids such that $G_1 \cap G_2 = G^{(0)}$.

We are inspired by technics developed by J. Kustermans and S. Vaes about locally compact quantum groups in the von Neumann setting [**KV00**], by M. Enock [**Eno04**] as far as the density theorem is concerned which is a key tool for duality and by author's thesis [**Les03**].

1.3. Study plan

After brief recalls about tools and technical points, we define objects we will use. We start by associating a fundamental pseudo-multiplicative unitary to every Hopf bimodule with invariant operator-valued weights. In fact, we shall define several isometries depending on which operator-valued weight we use. Each of them are useful, especially as far as the proof of unitarity of the fundamental isometry is concerned. This point can be also noticed in the crucial paper of S. Baaj and G. Skandalis [**BS93**] where they need a notion of irreducible unitary that means there exists another unitary. Also, in [**KV00**], they need to introduce several unitaries. The fundamental unitary gathers all information on the structure so that we can re-construct von Neumann algebra and co-product.

In the first part, we give axioms of measured quantum groupoids. In this setting, we construct a modulus, which corresponds to modulus of groupoids and a scaling operator which corresponds to scaling factor in locally compact quantum groups. They come from Radon-Nikodym's cocycle of right invariant operator-valued weight with respect to left invariant one thanks to proposition 5.2 of [**Vae01a**]. Then, we prove uniqueness of invariant operator-valued weight up to an element of basis center.

Also, we prove a "manageability" property of the fundamental pseudo-multiplicative unitary. A density result concerning bounded elements can be handled. These are sections 6 and 7. They give interesting results on the structure and a necessary preparation step for duality.

Then, we can proceed to the construction of the dual structure and get a duality theorem.

The second part is devoted to examples. We have a lot of examples for locally compact quantum groups thanks to Woronowicz [**Wor91, Wor01, WZ02, Wor87**] and the cocycle bi-crossed product due to S. Vaes and L. Vaĭnerman [**VV03**]. Theory of measured quantum groupoids has also a lot of examples.

First, we lay stress on Hopf bimodule with invariant operator-valued weights which are “adapted" in a certain sense. This hypothesis corresponds to the choice of a special weight on the basis to do the constructions (like in the groupoid case with a quasi-invariant measure on $G^{\{0\}}$). For them, we are able to construct the antipode S, the polar decomposition of which is given by a co-involution R and a one-parameter group of automorphisms called scaling group τ. In particular, we show that S, R and τ are independent of operator-valued weights.

Then we explain how these so-called adapted measured quantum groupoids fit into our measured quantum groupoids. In this setting, we develop information about modulus, scaling operator and uniqueness. We also characterize them and their dual in the general theory. Groupoids, weak Hopf C*-algebras, quantum groups, quantum groupoids of compact (resp. discrete) type... are of this type.

Depth 2 inclusions of von Neumann algebras also enter into our general setting (but not in adapted measured quantum groupoids unless the basis is semi-finite) and we compute their dual.

Finally, we state stability of the category by direct sum (which reflects the stability of groupoids under disjoint unions), finite tensor product and direct integrals. Then, we are able to construct new examples: in particular we can exhibit quantum groupoids with non scalar scaling operator.

CHAPTER 2

RECALLS

2.1. Weights and operator-valued weights [Str81], [Tak03]

Let N be a von Neumann and ψ a normal, semi-finite faithful (n.s.f.) weight on N; we denote by $\mathcal{N}_\psi$, $\mathcal{M}_\psi$, H_ψ, π_ψ, Λ_ψ, J_ψ, $\Delta_\psi \ldots$ canonical objects of Tomita's theory with respect to (w.r.t.) ψ.

Definition 2.1.1. – Let denote by $\mathcal{T}_\psi$ **Tomita's algebra** w.r.t. ψ defined by:

$$\{x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \mid x \text{ analytic w.r.t. } \sigma^\psi \text{ such that } \sigma_z^\psi(x) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for all } z \in \mathbb{C}\}$$

By [**Str81**, 2.12], we have the following approximating result:

Lemma 2.1.2. – *For all $x \in \mathcal{N}_\psi$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{T}_\psi$ such that:*

i) $||x_n|| \leq ||x||$ *for all* $n \in \mathbb{N}$*;*

ii) $(x_n)_{n\in\mathbb{N}}$ *converges to x in the strong topology;*

iii) $(\Lambda_\psi(x_n))_{n\in\mathbb{N}}$ *converges to $\Lambda_\psi(x)$ in the norm topology of H_ψ.*

Moreover, if $x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$, then we have:

iv) $(x_n)_{n\in\mathbb{N}}$ *converges to x in the *-strong topology;*

iiv) $(\Lambda_\psi(x_n^*))_{n\in\mathbb{N}}$ *converges to $\Lambda_\psi(x^*)$ in the norm topology of H_ψ.*

Let $N \subset M$ be an inclusion of von Neumann algebras and T a normal, semi-finite, faithful (n.s.f.) operator-valued weight from M to N. We put:

$$\mathcal{N}_T = \left\{x \in M \,/\, T(x^*x) \in N^+\right\} \text{ and } \mathcal{M}_T = \mathcal{N}_T^*\mathcal{N}_T$$

We can define a n.s.f. weight $\psi \circ T$ on M in a natural way. Let us recall theorem 10.6 of [**EN96**]:

Proposition 2.1.3. – *Let $N \subset M$ be an inclusion of von Neumann algebras and T be a normal, semi-finite, faithful (n.s.f.) operator-valued weight from M to N and ψ a n.s.f. weight on N. Then we have:*

i) *for all* $x \in \mathcal{N}_T$ *and* $a \in \mathcal{N}_\psi$, xa *belongs to* $\mathcal{N}_T \cap \mathcal{N}_{\psi\circ T}$, *there exists* $\Lambda_T(x) \in Hom_{N^\circ}(H_\psi, H_{\psi\circ T})$ *such that:*

$$\Lambda_T(x)\Lambda_\psi(a) = \Lambda_{\psi\circ T}(xa)$$

and Λ_T *is a morphism of* $M - N$*-bimodules from* $\mathcal{N}_T$ *to* $Hom_{N^\circ}(H_\psi, H_{\psi\circ T})$*;*

ii) $\mathcal{N}_T \cap \mathcal{N}_{\psi\circ T}$ *is a weakly dense ideal of* M *and* $\Lambda_{\psi\circ T}(\mathcal{N}_T \cap \mathcal{N}_{\psi\circ T})$ *is dense in* $H_{\psi\circ T}$, $\Lambda_{\psi\circ T}(\mathcal{N}_T \cap \mathcal{N}_{\psi\circ T} \cap \mathcal{N}_T^* \cap \mathcal{N}_{\psi\circ T}^*)$ *is a core for* $\Delta_{\psi\circ T}^{1/2}$ *and* $\Lambda_T(\mathcal{N}_T)$ *is dense in* $Hom_{N^\circ}(H_\psi, H_{\psi\circ T})$ *for the s-topology defined by* [**BDH88**, 1.3]*;*

iii) *for all* $x \in \mathcal{N}_T$ *and* $z \in \mathcal{N}_T \cap \mathcal{N}_{\psi\circ T}$, $T(x^*z)$ *belongs to* $\mathcal{N}_\psi$ *and:*

$$\Lambda_T(x)^*\Lambda_{\psi\circ T}(z) = \Lambda_\psi(T(x^*z))$$

iv) *for all* $x, y \in \mathcal{N}_T$*:*

$$\Lambda_T(y)^*\Lambda_T(x) = \pi_\psi(T(x^*y)) \text{ and } ||\Lambda_T(x)|| = ||T(x^*x)||^{1/2}$$

and Λ_T *is injective.*

Let us also recall lemma 10.12 of [**EN96**]:

Proposition 2.1.4. – *Let* $N \subseteq M$ *be an inclusion of von Neumann algebras,* T *a n.s.f. operator-valued weight from* M *to* N*,* ψ *a n.s.f. weight on* N *and* $x \in \mathcal{M}_T \cap \mathcal{M}_{\psi\circ T}$*. If we put:*

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t^{\psi\circ T}(x)\ dt$$

then x_n *belongs to* $\mathcal{M}_T \cap \mathcal{M}_{\psi\circ T}$ *and is analytic w.r.t.* $\psi \circ T$*. The sequence converges to* x *and is bounded by* $||x||$*. Moreover,* $(\Lambda_{\psi\circ T}(x_n))_{n\in\mathbb{N}}$ *converges to* $\Lambda_{\psi\circ T}(x)$ *and* $\sigma_z^{\psi\circ T}(x_n) \in \mathcal{M}_T \cap \mathcal{M}_{\psi\circ T}$ *for all* $z \in \mathbb{C}$*.*

Definition 2.1.5. – The set of $x \in \mathcal{N}_\Phi \cap \mathcal{N}_\Phi^* \cap \mathcal{N}_T \cap \mathcal{N}_T^*$, analytic w.r.t. σ^Φ such that $\sigma_z^\Phi(x) \in \mathcal{N}_\Phi \cap \mathcal{N}_\Phi^* \cap \mathcal{N}_T \cap \mathcal{N}_T^*$ for all $z \in \mathbb{C}$ is denoted by $\mathcal{T}_\Phi$ and is called **Tomita's algebra** w.r.t. $\psi \circ T = \Phi$ and T.

Lemma 2.1.2 is still satisfied with Tomita's algebra w.r.t. Φ and T.

2.2. Spatial theory [Co80, Sau83b, Tak03]

Let α be a normal, non-degenerated representation of N on a Hilbert space H. So, H becomes a left N-module and we write ${}_\alpha H$.

Definition 2.2.1 ([**Co80**]). – An element ξ of ${}_\alpha H$ is said to be bounded w.r.t. ψ if there exists $C \in \mathbb{R}^+$ such that, for all $y \in \mathcal{N}_\psi$, we have $||\alpha(y)\xi|| \leq C||\Lambda_\psi(y)||$. The set of **bounded elements** w.r.t. ψ is denoted by $D({}_\alpha H, \psi)$.

By [**Co80**, lem. 2], $D(_\alpha H, \psi)$ is dense in H and $\alpha(N)'$-stable. An element ξ of $D(_\alpha H, \psi)$ gives rise to a bounded operator $R^{\alpha,\psi}(\xi)$ of $Hom_N(H_\psi, H)$ such that, for all $y \in \mathcal{N}_\psi$:

$$R^{\alpha,\psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi$$

For all $\xi, \eta \in D(_\alpha H, \psi)$, we put:

$$\theta^{\alpha,\psi}(\xi,\eta) = R^{\alpha,\psi}(\xi)R^{\alpha,\psi}(\eta)^* \text{ and } \langle \xi,\eta\rangle_{\alpha,\psi} = R^{\alpha,\psi}(\eta)^*R^{\alpha,\psi}(\xi)^*$$

By [**Co80**, lem. 2], the linear span of $\theta^{\alpha,\psi}(\xi,\eta)$ is a weakly dense ideal of $\alpha(N)'$. $\langle \xi,\eta\rangle_{\alpha,\psi}$ belongs to $\pi_\psi(N)' = J_\psi \pi_\psi(N) J_\psi$ which is identified with the opposite von Neumann algebra N^o. The linear span of $\langle \xi,\eta\rangle_{\alpha,\psi}$ is weakly dense in N^o.
By [**Co80**, prop. 3], there exists a net $(\eta_i)_{i\in I}$ of $D(_\alpha H, \psi)$ such that:

$$\sum_{i\in I} \theta^{\alpha,\psi}(\eta_i,\eta_i) = 1$$

Such a net is called a (N,ψ)-**basis** of $_\alpha H$. By [**EN96**, prop. 2.2], we can choose η_i such that $R^{\alpha,\psi}(\eta_i)$ is a partial isometry with two-by-two orthogonal final supports and such that $\langle \eta_i,\eta_j\rangle_{\alpha,\psi} = 0$ unless $i = j$. In the following, we assume these properties hold for all (N,ψ)-basis of $_\alpha H$.

Now, let β be a normal, non-degenerated anti-representation from N on H. So H becomes a right N-module and we write H_β. But β is also a representation of N^o. If ψ^o is the n.s.f. weight on N^o coming from ψ then $\mathcal{N}_{\psi^o} = \mathcal{N}_\psi^*$ and we identify H_{ψ^o} with H_ψ thanks to:

$$(\Lambda_{\psi^o}(x^*) \longmapsto J_\psi \Lambda_\psi(x))$$

Definition 2.2.2 ([**Co80**]). – An element ξ of H_β is said to be bounded w.r.t. ψ^o if there exists $C \in \mathbb{R}^+$ such that, for all $y \in \mathcal{N}_\psi$, we have $||\beta(y^*)\xi|| \leq C||\Lambda_\psi(y)||$. The set of **bounded elements** w.r.t. ψ^o is denoted by $D(H_\beta, \psi^o)$.

$D(_\alpha H, \psi)$ is dense in H and $\beta(N)'$-stable. An element ξ of $D(H_\beta, \psi^o)$ gives rise to a bounded operator $R^{\beta,\psi^o}(\xi)$ of $Hom_{N^o}(H_\psi, H)$ such that, for all $y \in \mathcal{N}_\psi$:

$$R^{\beta,\psi^o}(\xi)\Lambda_\psi(y) = \beta(y^*)\xi$$

For all $\xi, \eta \in D(H_\beta, \psi^o)$, we put:

$$\theta^{\beta,\psi^o}(\xi,\eta) = R^{\beta,\psi^o}(\xi)R^{\beta,\psi^o}(\eta)^* \text{ and } \langle \xi,\eta\rangle_{\beta,\psi^o} = R^{\beta,\psi^o}(\eta)^*R^{\beta,\psi^o}(\xi)^*$$

The linear span of $\theta^{\beta,\psi^o}(\xi,\eta)$ is a weakly dense ideal of $\beta(N)'$. $\langle \xi,\eta\rangle_{\beta,\psi^o}$ belongs to $\pi_\psi(N)$ which is identified with N. The linear span of $\langle \xi,\eta\rangle_{\beta,\psi^o}$ is weakly dense in N. In fact, we know that $\langle \xi,\eta\rangle_{\beta,\psi^o} \in \mathcal{M}_\psi$ by [**Co80**, lem. 4] and by [**Sau83b**, lem. 1.5], we have

$$\Lambda_\psi(\langle \xi,\eta\rangle_{\beta,\psi^o}) = R^{\beta,\psi^o}(\eta)^*\xi$$

A net $(\xi_i)_{i\in I}$ of ψ^o-bounded elements of is said to be a (N^o,ψ^o)-basis of H_β if:

$$\sum_{i\in I}\theta^{\beta,\psi^o}(\xi_i,\xi_i)=1$$

and if ξ_i such that $R^{\beta,\psi^o}(\xi_i)$ is a partial isometry with two-by-two orthogonal final supports and such that $\langle\xi_i,\xi_j\rangle_{\alpha,\psi}=0$ unless $i=j$. Therefore, we have:

$$R^{\beta,\psi^o}(\xi_i)=\theta^{\beta,\psi^o}(\xi_i,\xi_i)R^{\beta,\psi^o}(\xi_i)=R^{\beta,\psi^o}(\xi_i)\langle\xi_i,\xi_i\rangle_{\beta,\psi^o}$$

and, for all $\xi\in D(H_\beta,\psi^o)$:

$$\xi=\sum_{i\in I}R^{\beta,\psi^o}(\xi_i)\Lambda_\psi(\langle\xi,\xi_i\rangle_{\beta,\psi^o})$$

PROPOSITION 2.2.3 ([**Eno02**, prop. 2.10]). – *Let $N\subseteq M$ be an inclusion of von Neumann algebras and T be a n.s.f. operator-valued weight from M to N. There exists a net $(e_i)_{i\in I}$ of $\mathcal{N}_T\cap\mathcal{N}_T^*\cap\mathcal{N}_{\psi\circ T}\cap\mathcal{N}_{\psi\circ T}^*$ such that $\Lambda_T(e_i)$ is a partial isometry, $T(e_j^*e_i)=0$ unless $i=j$ and with orthogonal final supports of sum 1. Moreover, we have $e_i=e_iT(e_i^*e_i)$ for all $i\in I$, and, for all $x\in\mathcal{N}_T$:*

$$\Lambda_T(x)=\sum_{i\in I}\Lambda_T(e_i)T(e_i^*x)\quad\text{and}\quad x=\sum_{i\in I}e_iT(e_i^*x)$$

in the weak topology. Such a net is called a basis for (T,ψ^o). Finally, the net $(\Lambda_{\psi\circ T}(e_i))_{i\in I}$ is a (N^o,ψ^o)-basis of $(H_{\psi\circ T})_s$ where s is the anti-representation which sends $y\in N$ to $J_{\psi\circ T}y^*J_{\psi\circ T}$.

2.3. Relative tensor product [Co80], [Sau83b], [Tak03]

Let H and K be Hilbert space. Let α (resp. β) be a normal and non-degenerated (resp. anti-) representation of N on K (resp. H). Let ψ be a n.s.f. weight on N. Following [**Sau83b**], we put on $D(H_\beta,\psi^o)\odot K$ a scalar product defined by:

$$(\xi_1\odot\eta_1|\xi_2\odot\eta_2)=(\alpha(\langle\xi_1,\xi_2\rangle_{\beta,\psi^o})\eta_1|\eta_2)$$

for all $\xi_1,\xi_2\in D(H_\beta,\psi^o)$ and $\eta_1,\eta_2\in K$. We have identified $\pi_\psi(N)$ with N.

DEFINITION 2.3.1. – The completion of $D(H_\beta,\psi^o)\odot K$ is called **relative tensor product** and is denoted by $H{}_\beta\underset{\psi}{\otimes}{}_\alpha K$.

The image of $\xi\odot\eta$ in $H{}_\beta\underset{\psi}{\otimes}{}_\alpha K$ is denoted by $\xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta$. One should bear in mind that, if we start from another n.s.f. weight ψ' on N, we get another Hilbert space which is canonically isomorphic to $H{}_\beta\underset{\psi}{\otimes}{}_\alpha K$ by [**Sau83b**, prop. 2.6]. However this isomorphism does not send $\xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta$ on $\xi{}_\beta\underset{\psi'}{\otimes}{}_\alpha\eta$.

By [**Sau83b**, def. 2.1], relative tensor product can be defined from the scalar product:

$$(\xi_1 \odot \eta_1|\xi_2 \odot \eta_2) = (\beta(\langle \eta_1, \eta_2\rangle_{\alpha,\psi})\xi_1|\xi_2)$$

for all $\xi_1, \xi_2 \in H$ and $\eta_1, \eta_2 \in D(_\alpha K, \psi)$ that's why we can define a one-to-one flip from $H{}_\beta\underset{\psi}{\otimes}{}_\alpha K$ onto $K{}_\alpha\underset{\psi^o}{\otimes}{}_\beta H$ such that:

$$\sigma_\psi(\xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta) = \eta{}_\alpha\underset{\psi^o}{\otimes}{}_\beta\xi$$

for all $\xi \in D(H_\beta, \psi)$ (resp. $\xi \in H$) and $\eta \in K$ (resp. $\eta \in D(_\alpha K, \psi)$). The flip gives rise at the operator level to ς_ψ from $\mathcal{L}(H{}_\beta\underset{\psi}{\otimes}{}_\alpha K)$ onto $\mathcal{L}(K{}_\alpha\underset{\psi^o}{\otimes}{}_\beta H)$ such that:

$$\varsigma_\psi(X) = \sigma_\psi X \sigma_\psi^*$$

Canonical isomorphisms of change of weights send ς_ψ on $\varsigma_{\psi'}$ so that we write ς_N without any reference to the weight on N.

For all $\xi \in D(H_\beta, \psi^o)$ and $\eta \in D(_\alpha K, \psi)$, we define bounded operators:

$$\begin{aligned} \lambda_\xi^{\beta,\alpha} : K &\longrightarrow H{}_\beta\underset{\psi}{\otimes}{}_\alpha K \\ \eta &\longmapsto \xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho_\eta^{\beta,\alpha} : H &\longrightarrow H{}_\beta\underset{\psi}{\otimes}{}_\alpha K \\ \xi &\longmapsto \xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta \end{aligned}$$

Then, we have:

$$(\lambda_\xi^{\beta,\alpha})^* \lambda_\xi^{\beta,\alpha} = \alpha(\langle \xi, \xi\rangle_{\beta,\psi^o}) \text{ and } (\rho_\eta^{\beta,\alpha})^* \rho_\eta^{\beta,\alpha} = \beta(\langle \eta, \eta\rangle_{\alpha,\psi})$$

By [**Sau83b**, rem. 2.2], we know that $D(_\alpha K, \psi)$ is $\alpha(\sigma^\psi_{-i/2}(\mathcal{D}(\sigma^\psi_{-i/2})))$-stable and for all $\xi \in H$, $\eta \in D(_\alpha K, \psi)$ and $y \in \mathcal{D}(\sigma^\psi_{-i/2})$, we have:

$$\beta(y)\xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta = \xi{}_\beta\underset{\psi}{\otimes}{}_\alpha\alpha(\sigma^\psi_{-i/2}(y))\eta$$

LEMMA 2.3.2. – *If $\xi'{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta = 0$ for all $\xi' \in D(H_\beta, \psi^o)$ then $\eta = 0$.*

Proof. – For all $\xi, \xi' \in D(H_\beta, \psi^o)$, we have:

$$\alpha(\langle \xi', \xi\rangle_{\beta,\psi^o})\eta = (\lambda_\xi^{\beta,\alpha})^* \lambda_{\xi'}^{\beta,\alpha}\eta = (\lambda_\xi^{\beta,\alpha})^*(\xi'{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta) = 0$$

Since the linear span of $\langle \xi', \xi\rangle_{\beta,\psi^o}$ is dense in N, we get $\eta = 0$. □

PROPOSITION 2.3.3. – *Assume $H \neq \{0\}$. Let K' be a closed subspace of K such that $\alpha(N)K' \subseteq K'$. Then:*

$$H{}_\beta\underset{\psi}{\otimes}{}_\alpha K = H{}_\beta\underset{\psi}{\otimes}{}_\alpha K' \quad \Rightarrow \quad K = K'$$

Proof. – Let $\eta \in K'^{\perp}$. For all $\xi, \xi' \in D(H_\beta, \psi^o)$ and $k \in K'$, we have:

$$(\xi {}_\beta \underset{\psi}{\otimes}{}_\alpha k | \xi' {}_\beta \underset{\psi}{\otimes}{}_\alpha \eta) = (\alpha(\langle \xi, \xi' \rangle_{\beta, \nu^o}) k | \eta) = 0$$

Therefore, for all $\xi' \in D(H_\beta, \psi^o)$, we have:

$$\xi' {}_\beta \underset{\psi}{\otimes}{}_\alpha \eta \in (H {}_\beta \underset{\psi}{\otimes}{}_\alpha K')^\perp = (H {}_\beta \underset{\psi}{\otimes}{}_\alpha K)^\perp = \{0\}$$

By the previous lemma, we get $\eta = 0$ and $K = K'$. □

Let H', K', α' and β' like H, K, α and β. Let $A \in \mathcal{L}(H, H')$ and $B \in \mathcal{L}(K, K')$ such that:

$$\forall n \in N, \quad A\beta(n) = \beta'(n)A \quad \text{and} \quad B\alpha(n) = \alpha'(n)B$$

Then we can define an operator $A {}_\beta \underset{\psi}{\otimes}{}_\alpha B \in \mathcal{L}(H {}_\beta \underset{\psi}{\otimes}{}_\alpha K, H' {}_\beta \underset{\psi}{\otimes}{}_\alpha K')$ which naturally acts on elementary tensor products. In particular, if $x \in \beta(N)' \cap \mathcal{L}(H)$ and $y \in \alpha(N)' \cap \mathcal{L}(K)$, we get an operator $x {}_\beta \underset{\psi}{\otimes}{}_\alpha y$ on $H {}_\beta \underset{\psi}{\otimes}{}_\alpha K$. Canonical isomorphism of change of weights sends $x {}_\beta \underset{\psi}{\otimes}{}_\alpha y$ on $x {}_\beta \underset{\psi'}{\otimes}{}_\alpha y$ so that we write $x {}_\beta \underset{N}{\otimes}{}_\alpha y$ without any reference to the weight.

Let P be a von Neumann algebra and ϵ a normal and non-degenerated anti-representation of P on K such that $\epsilon(P)' \subseteq \alpha(N)$. K is equipped with a $N - P$-bimodule structure denoted by ${}_\alpha K_\epsilon$. For all $y \in P$, $1_H {}_\beta \underset{\psi}{\otimes}{}_\alpha \epsilon(y)$ is an operator on $H {}_\beta \underset{\psi}{\otimes}{}_\alpha K$ so that we define a representation of P on $H {}_\beta \underset{\psi}{\otimes}{}_\alpha K$ still denoted by ϵ. If H is a $Q - N$-bimodule, then $H {}_\beta \underset{\psi}{\otimes}{}_\alpha K$ becomes a $Q - P$-bimodule (Connes' fusion of bimodules). If ν is a n.s.f. weight on P and ${}_\zeta L$ a left P-module. It is possible to define two Hilbert spaces $(H {}_\beta \underset{\psi}{\otimes}{}_\alpha K) {}_\epsilon \underset{\nu}{\otimes}{}_\zeta L$ and $H {}_\beta \underset{\psi}{\otimes}{}_\alpha (K {}_\epsilon \underset{\nu}{\otimes}{}_\zeta L)$. These two $\beta(N)' - \zeta(P)'^o$-bimodules are isomorphic. (The proof of [**Val96**, lem. 2.1.3], in the case of commutative $N = P$ is still valid). We speak about associativity of relative tensor product and we write $H {}_\beta \underset{\psi}{\otimes}{}_\alpha K {}_\epsilon \underset{\nu}{\otimes}{}_\zeta L$ without parenthesis.

We identify $H_\psi {}_\beta \underset{\psi}{\otimes}{}_\alpha K$ and K as left N-modules by $\Lambda_\psi(y) {}_\beta \underset{\psi}{\otimes}{}_\alpha \eta \mapsto \alpha(y)\eta$ for all $y \in \mathcal{N}_\psi$. By [**EN96**, 3.10], we have:

$$\lambda_\xi^{\beta,\alpha} = R^{\beta,\psi^o}(\xi) {}_\beta \underset{\psi}{\otimes}{}_\alpha 1_K$$

We recall proposition 2.3 of [**Eno02**]:

PROPOSITION 2.3.4. – *Let $(\xi_i)_{i \in I}$ be a (N^o, ψ^o)-basis of H_β. Then:*

i) *for all $\xi \in D(H_\beta, \psi^o)$ and $\eta \in K$, we have:*

$$\xi {}_\beta \underset{\psi}{\otimes}{}_\alpha \eta = \sum_{i \in I} \xi_i {}_\beta \underset{\psi}{\otimes}{}_\alpha \alpha(\langle \xi, \xi_i \rangle_{\beta, \psi^o}) \eta$$

ii) *we have the following decomposition:*

$$H_{\beta}\underset{\psi}{\otimes}{}_{\alpha}K = \bigoplus_{i\in I}(\xi_{i\beta}\underset{\psi}{\otimes}{}_{\alpha}\alpha(\langle\xi_i,\xi_i\rangle_{\beta,\psi^o})K)$$

We here add a proposition we will use several times.

PROPOSITION 2.3.5. – *Let γ a *-automorphism from N such that $\psi\circ\nu=\psi$. Then:*

$$H_{\beta\circ\gamma}\underset{\psi}{\otimes}{}_{\alpha\circ\gamma}K = H_{\beta}\underset{\psi}{\otimes}{}_{\alpha}K$$

Proof. – Because of invariance of ψ with respect to γ, we have a unitary I from H_ψ such that $I\Lambda_\psi(y)=\Lambda_\psi(\gamma(y))$ for all $y\in\mathscr{N}_\psi$. Moreover $IJ_\psi=J_\psi I$ and $I^*nI=\gamma^{-1}(n)$ for all $n\in N$. For all $\xi\in D(H_\beta,\psi^o)$ and $y\in\mathscr{N}_\psi$, we compute:

$$\begin{aligned}\beta\circ\gamma(y^*))\xi=\beta(\gamma(y)^*)\xi &= R^{\beta,\psi^o}(\xi)J_\psi\Lambda_\psi(\gamma(y))\\ &= R^{\beta,\psi^o}(\xi)J_\psi I\Lambda_\psi(y)=R^{\beta,\nu^o}(\xi)IJ_\psi\Lambda_\psi(y)\end{aligned}$$

that's why we get:

$$D(H_{\beta\circ\gamma},\psi^o)=D(H_\beta,\psi^o) \text{ and } \forall\xi\in D(H_\beta,\psi^o),\ R^{\beta\circ\gamma,\psi^o}(\xi)=R^{\beta,\psi^o}(\xi)I$$

To conclude, we show that scalar products on $D(H_{\beta,\psi^o})\odot K$ used to define $H_{\beta}\underset{\psi}{\otimes}{}_{\alpha}K$ and $H_{\beta\circ\gamma}\underset{\psi}{\otimes}{}_{\alpha\circ\gamma}K$ are equal. If $\xi,\xi'\in D(H_\beta,\nu^o)$ and $\eta,\eta'\in K$, we have:

$$\begin{aligned}(\xi_{\beta\circ\gamma}\underset{\psi}{\otimes}{}_{\alpha\circ\gamma}\eta|\xi'{}_{\beta\circ\gamma}\underset{\psi}{\otimes}{}_{\alpha\circ\gamma}\eta') &= (\alpha(\gamma(\langle\xi,\xi'\rangle_{\beta\circ\gamma,\psi^o}))\eta|\eta')\\ &= (\alpha(\gamma(I^*\langle\xi,\xi'\rangle_{\beta,\psi^o}I))\eta|\eta')\\ &= (\alpha(\langle\xi,\xi'\rangle_{\beta,\psi^o})\eta|\eta')=(\xi_{\beta}\underset{\psi}{\otimes}{}_{\alpha}\xi'|\eta_{\beta}\underset{\psi}{\otimes}{}_{\alpha}\eta')\end{aligned}$$ □

To end the paragraph, we detail finite dimension case. We assume that N, H and K are of finite dimensions. $H_{\beta}\underset{\psi}{\otimes}{}_{\alpha}K$ can be identified with a subspace of $H\otimes K$. We denote by Tr the normalized canonical trace on K and $\tau=\mathrm{Tr}\circ\alpha$. There exist a projection $e_{\beta,\alpha}\in\beta(N)\otimes\alpha(N)$ and $n_o\in Z(N)^+$ such that $(\mathrm{id}\otimes\mathrm{Tr})(e_{\beta,\alpha})=\beta(n_o)$. Let d be the Radon-Nikodym derivative of ψ w.r.t. τ. By [**EV00**, 2.4], and proposition 2.7 of [**Sau83b**], for all $\xi,\eta\in H$:

$$I^{\psi}_{\beta,\alpha}:\xi_{\beta}\underset{\psi}{\otimes}{}_{\alpha}\eta\longmapsto\xi_{\beta}\underset{\tau}{\otimes}{}_{\alpha}\alpha(d)^{1/2}\eta\longmapsto e_{\beta,\alpha}\big(\beta(n_o)^{-1/2}\xi\otimes\alpha(d)^{1/2}\eta\big)$$

defines an isometric isomorphism of $\beta(N)'-\alpha(N)'^o$-bimodules from $H_{\beta}\underset{\psi}{\otimes}{}_{\alpha}K$ onto a subspace of $H\otimes K$, the final support of which is $e_{\beta,\alpha}$.

2.4. Fiber product [Val96], [EV00]

We use previous notations. Let M_1 (resp. M_2) be a von Neumann algebra on H (resp. K) such that $\beta(N) \subseteq M_1$ (resp. $\alpha(N) \subseteq M_2$). We denote by $M_1' {}_\beta\underset{N}{\otimes}{}_\alpha M_2'$ the von Neumann algebra generated by $x {}_\beta\underset{N}{\otimes}{}_\alpha y$ with $x \in M_1'$ and $y \in M_2'$.

Definition 2.4.1. – The commutant of $M_1' {}_\beta\underset{N}{\otimes}{}_\alpha M_2'$ in $\mathcal{L}(H {}_\beta\underset{\psi}{\otimes}{}_\alpha K)$ is denoted by $M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$ and is called **fiber product**.

If P_1 and P_2 are von Neumann algebras like M_1 and M_2, we have:

i) $(M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2) \cap (P_1 {}_\beta\underset{N}{\star}{}_\alpha P_2) = (M_1 \cap P_1) {}_\beta\underset{N}{\star}{}_\alpha (M_2 \cap P_2)$

ii) $\varsigma_N(M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2) = M_2 {}_\alpha\underset{N^o}{\star}{}_\beta M_1$

iii) $(M_1 \cap \beta(N)') {}_\beta\underset{N}{\otimes}{}_\alpha (M_2 \cap \alpha(N)') \subseteq M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$

iv) $M_1 {}_\beta\underset{N}{\star}{}_\alpha \alpha(N) = (M_1 \cap \beta(N)') {}_\beta\underset{N}{\otimes}{}_\alpha 1$

More generally, if β (resp. α) is a normal, non-degenerated *-anti-homomorphism (resp. homomorphism) from N to a von Neumann algebra M_1 (resp. M_2), it is possible to define a von Neumann algebra $M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$ without any reference to a specific Hilbert space. If P_1, P_2, α' and β' are like M_1, M_2, α and β and if Φ (resp. Ψ) is a normal *-homomorphism from M_1 (resp. M_2) to P_1 (resp. P_2) such that $\Phi \circ \beta = \beta'$ (resp. $\Psi \circ \alpha = \alpha'$), then we define a normal *-homomorphism by [**Sau83a**, 1.2.4]:

$$\Phi {}_\beta\underset{N}{\star}{}_\alpha \Psi : M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2 \longrightarrow P_1 {}_{\beta'}\underset{N}{\star}{}_{\alpha'} P_2$$

Assume ${}_\alpha K_\epsilon$ is a $N - P^o$-bimodule and ${}_\zeta L$ a left P-module. If $\alpha(N) \subseteq M_2$, $\epsilon(P) \subseteq M_2$ and if $\zeta(P) \subseteq M_3$ where M_3 is a von Neumann algebra on L, then we can construct $M_1 {}_\beta\underset{N}{\star}{}_\alpha (M_2 {}_\epsilon\underset{N}{\star}{}_\zeta M_3)$ and $(M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2) {}_\epsilon\underset{N}{\star}{}_\zeta M_3$. Associativity of relative tensor product induces an isomorphism between these fiber products and we write $M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2 {}_\epsilon\underset{N}{\star}{}_\zeta M_3$ without parenthesis.

Finally, if M_1 and M_2 are of finite dimensions, then we have:

$$M_1' {}_\beta\underset{N}{\otimes}{}_\alpha M_2' = (I^\psi_{\beta,\alpha})^*(M_1' \otimes M_2')I^\psi_{\beta,\alpha} \text{ and } M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2 = (I^\psi_{\beta,\alpha})^*(M_1 \otimes M_2)I^\psi_{\beta,\alpha}$$

Therefore the fiber product can be identified with a reduction of $M_1 \otimes M_2$ by $e_{\beta,\alpha}$ by [**EV00**, 2.4].

2.5. Slice map [Eno00]

2.5.1. For normal forms Let $A \in M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$ and $\xi_1, \xi_2 \in D(H_\beta, \psi^o)$. We define an element of M_2 by:

$$(\omega_{\xi_1,\xi_2}{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A) = (\lambda^{\beta,\alpha}_{\xi_2})^* A \lambda^{\beta,\alpha}_{\xi_1}$$

so that we have $((\omega_{\xi_1,\xi_2}{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)\eta_1|\eta_2) = (A(\xi_1{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta_1)|\xi_2{}_\beta\underset{\psi}{\otimes}{}_\alpha\eta_2)$ for all $\eta_1, \eta_2 \in K$. Also, we define an operator of M_1 by:

$$(\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \omega_{\eta_1,\eta_2})(A) = (\rho^{\beta,\alpha}_{\eta_2})^* A \rho^{\beta,\alpha}_{\eta_1}$$

for all $\eta_1, \eta_2 \in D({}_\alpha K, \psi)$. We have a Fubini's formula:

$$\omega_{\eta_1,\eta_2}((\omega_{\xi_1,\xi_2}{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)) = \omega_{\xi_1,\xi_2}((\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \omega_{\eta_1,\eta_2})(A))$$

for all $\xi_1, \xi_2 \in D(H_\beta, \psi^o)$ and $\eta_1, \eta_2 \in D({}_\alpha K, \psi)$.

Equivalently, by [**Eno00**, prop. 3.3], for all $\omega_1 \in M_{1*}^+$ and $k_1 \in \mathbb{R}^+$ such that $\omega_1 \circ \beta \leq k_1\psi$ and for all $\omega_2 \in M_{2*}^+$ and $k_2 \in \mathbb{R}^+$ such $\omega_2 \circ \alpha \leq k_2\psi$, we have:

$$\omega_2((\omega_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)) = \omega_1((\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \omega_2)(A))$$

2.5.2. For conditional expectations. – If P_2 is a von Neumann algebra such that $\alpha(N) \subseteq P_2 \subseteq M_2$ and if E is a normal, faithful conditional expectation from M_2 onto P_2, we can define a normal, faithful conditional expectation $(\mathrm{id}_\beta\underset{N}{\star}{}_\alpha E)$ from $M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$ onto $M_1 {}_\beta\underset{N}{\star}{}_\alpha P_2$ such that:

$$(\omega_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(\mathrm{id}_\beta\underset{N}{\star}{}_\alpha E)(A) = E((\omega_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A))$$

for all $A \in M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$, $\omega \in M_{1*}^+$ and $k_1 \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k_1\psi$.

2.5.3. For weights. – If ϕ_1 is n.s.f. weight on M_1 and if A is a positive element of $M_1 {}_\beta\underset{N}{\star}{}_\alpha M_2$, we can define an element of the extended positive part of M_2, denoted by $(\phi_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)$, such that, for all $\eta \in D({}_\alpha L^2(M_2), \psi)$, we have:

$$||((\phi_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A))^{1/2}\eta||^2 = \phi_1((\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \omega_\eta)(A))$$

Moreover, if ϕ_2 is a n.s.f. weight on M_2, we have:

$$\phi_2((\phi_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)) = \phi_1((\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \phi_2)(A))$$

Let $(\omega_i)_{i\in I}$ be an increasing net of normal forms such that $\phi_1 = \sup_{i\in I}\omega_i$. Then we have $(\phi_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A) = \sup_i(\omega_i{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(A)$.

2.5.4. For operator-valued weights. – Let P_1 be a von Neumann algebra such that $\beta(N) \subseteq P_1 \subseteq M_1$ and Φ_i $(i = 1, 2)$ be operator-valued n.s.f. weights from M_i to P_i. By [**Eno00**], for all positive operator $A \in M_1{}_\beta\underset{N}{\star}{}_\alpha M_2$, there exists an element $(\Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(A)$ belonging to $P_1{}_\beta\underset{N}{\star}{}_\alpha M_2$ such that, for all $\xi \in L^2(P_1)$ and $\eta \in D(_\alpha K, \psi)$, we have:

$$||((\Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(A))^{1/2}(\xi{}_\beta\underset{\psi}{\otimes}{}_\alpha \eta)||^2 = ||[\Phi_1((\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \omega_{\eta,\eta})(A))]^{1/2}\xi||^2$$

If ϕ_1 is a n.s.f. weight on P_1, we have:

$$(\phi_1 \circ \Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(A) = (\phi_1{}_\beta\underset{\psi}{\star}{}_\alpha \mathrm{id})(\Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(A)$$

Also, we define an element $(\mathrm{id}_\beta\underset{N}{\star}{}_\alpha \Phi_2)(A)$ of the extended positive part of $M_1{}_\beta\underset{N}{\star}{}_\alpha P_2$ and we have:

$$(\mathrm{id}_\beta\underset{N}{\star}{}_\alpha \Phi_2)((\Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(A)) = (\Phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})((\mathrm{id}_\beta\underset{N}{\star}{}_\alpha \Phi_2)(A))$$

Remark 2.5.1. – We have seen that we can identify $M_1{}_\beta\underset{N}{\star}{}_\alpha \alpha(N)$ with $M_1 \cap \beta(N)'$. Then, it is easy to check that the slice map $\mathrm{id}_\beta\underset{\psi}{\star}{}_\alpha \psi \circ \alpha^{-1}$ (if α is injective) is just the injection of $M_1{}_\beta\underset{N}{\star}{}_\alpha \alpha(N)$ into M_1. Also we see on that example that, if ϕ_1 is a n.s.f. weight on M_1, then $\phi_1{}_\beta\underset{N}{\star}{}_\alpha \mathrm{id}$ (which is equal to $\phi_{1|M_1\cap\beta(N)'}$) does not need to be semi-finite.

CHAPTER 3

FUNDAMENTAL PSEUDO-MULTIPLICATIVE UNITARY

In this section, we construct a fundamental pseudo-multiplicative unitary from a Hopf bimodule with a left invariant operator-valued weight and a right invariant operator-valued weight. Let N and M be von Neumann algebras, α (resp. β) be a faithful, non-degenerate, normal (resp. anti-) representation from N to M. We suppose that $\alpha(N) \subseteq \beta(N)'$.

3.1. Definitions

Definition 3.1.1

A quintuplet $(N, M, \alpha, \beta, \Gamma)$ is said to be a **Hopf bimodule** of basis N if Γ is a normal *-homomorphism from M into $M {}_\beta\underset{N}{\star}{}_\alpha M$ such that, for all $n, m \in N$, we have:

i) $\Gamma(\alpha(n)\beta(m)) = \alpha(n) {}_\beta\underset{N}{\otimes}{}_\alpha \beta(m)$

ii) Γ is co-associative: $(\Gamma {}_\beta\underset{N}{\star}{}_\alpha \mathrm{id}) \circ \Gamma = (\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha \Gamma) \circ \Gamma$

One should notice that property i) is necessary in order to write down the formula given in ii). $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma)$ is a Hopf bimodule called opposite Hopf bimodule. If N is commutative, $\alpha = \beta$ and $\Gamma = \varsigma_N \circ \Gamma$, then $(N, M, \alpha, \alpha, \Gamma)$ is equal to its opposite: we shall speak about a symmetric Hopf bimodule.

Definition 3.1.2. – Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. A normal, semi-finite, faithful operator-valued weight from M to $\alpha(N)$ is said to be **left invariant** if:

$$(\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha T_L)\Gamma(x) = T_L(x) {}_\beta\underset{N}{\otimes}{}_\alpha 1 \qquad \text{for all } x \in \mathcal{M}_{T_L}^+$$

In the same way, a normal, semi-finite, faithful operator-valued weight from M to $\beta(N)$ is said to be **right invariant** if:

$$(T_R {}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})\Gamma(x) = 1 {}_\beta\underset{N}{\otimes}{}_\alpha T_R(x) \qquad \text{for all } x \in \mathcal{M}_{T_R}^+$$

We give several examples in the last section. In this section, $(N, M, \alpha, \beta, \Gamma)$ is a Hopf bimodule with a left operator-valued weight T_L and a right operator-valued weight T_R.

DEFINITION 3.1.3. – A *-anti-automorphism R of M is said to be a **co-involution** if $R \circ \alpha = \beta$, $R^2 = \mathrm{id}$ and $\varsigma_{N^o} \circ (R {}_{\beta}\star_{\alpha}^{}{}_{N} R) \circ \Gamma = \Gamma \circ R$.

REMARK 3.1.4. – With the previous notations, let us notice that $R \circ T_L \circ R$ is a right invariant operator-valued weight from M to $\beta(N)$. Also, let us say that R is an anti-isomorphism of Hopf bimodule from the bimodule and its symmetric.

Let μ be a normal, semi-finite, faithful weight of N. We put:

$$\Phi = \mu \circ \alpha^{-1} \circ T_L \text{ and } \Psi = \mu \circ \beta^{-1} \circ T_R$$

so that, for all $x \in M^+$, we have:

$$(\mathrm{id}_{\beta} \underset{\mu}{\star}_{\alpha} \Phi)\Gamma(x) = T_L(x) \text{ and } (\Psi_{\beta} \underset{\mu}{\star}_{\alpha} \mathrm{id})\Gamma(x) = T_R(x)$$

If H denote a Hilbert space on which M acts, then N acts on H, also, by way of α and β. We shall denote again α (resp. β) for (resp. anti-) the representation of N on H.

3.2. Construction of the fundamental isometry

DEFINITION 3.2.1

Let define $\hat{\beta}$ and $\hat{\alpha}$ by:

$$\begin{aligned} \hat{\beta} : N &\longrightarrow \mathcal{L}(H_\Phi) \\ x &\longmapsto J_\Phi \alpha(x^*) J_\Phi \end{aligned} \qquad \text{and} \qquad \begin{aligned} \hat{\alpha} : N &\longrightarrow \mathcal{L}(H_\Psi) \\ x &\longmapsto J_\Psi \beta(x^*) J_\Psi \end{aligned}$$

Then $\hat{\beta}$ (resp. $\hat{\alpha}$) is a normal, non-degenerate and faithful anti-representation (resp. representation) from N to $\mathcal{L}(H_\Phi)$ (resp. $\mathcal{L}(H_\Psi)$).

PROPOSITION 3.2.2. – *We have* $\Lambda_\Phi(\mathcal{N}_{T_L} \cap \mathcal{N}_\Phi) \subseteq D((H_\Phi)_{\hat{\beta}}, \mu^o)$ *and for all* $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$*, we have:*

$$R^{\hat{\beta},\mu^o}(\Lambda_\Phi(a)) = \Lambda_{T_L}(a)$$

Also, we have $\Lambda_\Psi(\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi) \subseteq D({}_{\hat{\alpha}}(H_\Psi), \mu)$ *and for all* $b \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$*, then:*

$$R^{\hat{\alpha},\mu}(\Lambda_\Psi(b)) = \Lambda_{T_R}(b)$$

REMARK 3.2.3. – We identify H_μ with $H_{\mu\circ\alpha^{-1}}$ and H_μ with $H_{\mu\circ\beta^{-1}}$.

Proof. – Let $y \in \mathcal{N}_\mu$ analytic w.r.t. μ. We have:

$$\begin{aligned}\hat{\beta}(y^*)\Lambda_\Phi(a) &= \Lambda_\Phi(a\sigma^\Phi_{-i/2}(\alpha(y^*))) = \Lambda_\Phi(a\sigma^{\mu\circ\alpha^{-1}}_{-i/2}(\alpha(y^*)))\\ &= \Lambda_\Phi(a\alpha(\sigma^\mu_{-i/2}(y^*))) = \Lambda_{T_L}(a)\Lambda_\mu(\sigma^\mu_{-i/2}(y^*)) = \Lambda_{T_L}(a)J_\mu\Lambda_\mu(y)\end{aligned}$$

Thanks to lemma 2.1.2, we get $\hat{\beta}(y^*)\Lambda_\Phi(a) = \Lambda_{T_L}(a)J_\mu\Lambda_\mu(y)$, for all $y \in \mathcal{N}_\mu$, which gives the first part of the proposition. The end of the proof is very similar. □

PROPOSITION 3.2.4. – *We have* $J_\Phi D((H_\Phi)_{\hat{\beta}}, \mu^o) = D({}_\alpha(H_\Phi), \mu)$ *and for all* $\eta \in D((H_\Phi)_{\hat{\beta}}, \mu^o)$*, we have:*

$$R^{\alpha,\mu}(J_\Phi\eta) = J_\Phi R^{\hat{\beta},\mu^o}(\eta)J_\mu$$

Also, we have $J_\Psi D({}_{\hat{\alpha}}(H_\Psi), \mu) = D((H_\Phi)_\beta, \mu^o)$ *and for all* $\xi \in D((H_\Phi)_\beta, \mu^o)$*, we have:*

$$R^{\beta,\mu^o}(J_\Psi\xi) = J_\Psi R^{\hat{\alpha},\mu}(\xi)J_\mu$$

Proof. – Straightforward. □

COROLLARY 3.2.5. – *We have* $\Lambda_\Phi(\mathcal{T}_{\Phi,T_L}) \subseteq D((H_\Phi)_{\hat{\beta}}, \mu^o) \cap D({}_\alpha(H_\Phi), \mu)$ *and* $\Lambda_\Psi(\mathcal{T}_{\Psi,T_R}) \subseteq D({}_{\hat{\alpha}}(H_\Psi), \mu) \cap D((H_\Psi)_\beta, \mu^o)$.

Proof. – This is a corollary of the two previous propositions. □

REMARK 3.2.6. – The invariance of operator-valued weights does not play a part in the previous propositions.

PROPOSITION 3.2.7. – *We have* $(\omega_{v,\xi} \underset{\mu}{{}_\beta\star_\alpha} \mathrm{id})(\Gamma(a)) \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ *for all elements* $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ *and* $v, \xi \in D(H_\beta, \mu^o)$.

Proof. – By definition of the slice maps, we have:

$$\begin{aligned}(\omega_{v,\xi} \underset{\mu}{{}_\beta\star_\alpha} \mathrm{id})(\Gamma(a))^*(\omega_{v,\xi} \underset{\mu}{{}_\beta\star_\alpha} \mathrm{id})(\Gamma(a)) &= (\lambda^{\beta,\alpha}_v)^*\Gamma(a^*)\lambda^{\beta,\alpha}_\xi(\lambda^{\beta,\alpha}_\xi)^*\Gamma(a)\lambda^{\beta,\alpha}_v\\ &\leq \|\lambda^{\beta,\alpha}_\xi\|^2(\omega_{v,v} \underset{\mu}{{}_\beta\star_\alpha} \mathrm{id})(\Gamma(a^*a))\\ &\leq \|R^{\beta,\mu^o}(\xi)\|^2(\omega_{v,v} \underset{\mu}{{}_\beta\star_\alpha} \mathrm{id})(\Gamma(a^*a))\end{aligned}$$

Then, on one hand, we get, thanks to left invariance of T_L:

$$\begin{aligned}&T_L((\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a))^*(\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a)))\\ \leq\ &\|R^{\beta,\mu^o}(\xi)\|^2T_L((\omega_{v,v}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a^*a)))\\ =\ &\|R^{\beta,\mu^o}(\xi)\|^2(\omega_{v,v}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha T_L)(\Gamma(a^*a))\\ \leq\ &\|R^{\beta,\mu^o}(\xi)\|^2(\lambda_v^{\beta,\alpha})^*(T_L(a^*a)\,{}_\beta\underset{\mu}{\otimes}{}_\alpha 1)\lambda_v^{\beta,\alpha}\\ \leq\ &\|R^{\beta,\mu^o}(\xi)\|^2||T_L(a^*a)||\|\alpha(\langle v,v\rangle_{\beta,\mu^o})\|1\\ \leq\ &\|R^{\beta,\mu^o}(\xi)\|^2\|T_L(a^*a)\|\|R^{\beta,\mu^o}(v)\|^21\end{aligned}$$

So, we get that $(\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a))\in\mathcal{N}_{T_L}$. On the other hand, thanks to left invariance of T_L, we know that:

$$\Phi(((\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a)))^*(\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a)))$$

is less or equal to:

$$\begin{aligned}&\|R^{\beta,\mu^o}(\xi)\|^2\Phi((\omega_{v,v}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a^*a)))\\ =\ &\|R^{\beta,\mu^o}(\xi)\|^2\omega_{v,v}((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\Phi)(\Gamma(a^*a)))\\ =\ &\|R^{\beta,\mu^o}(\xi)\|^2(T_L(a^*a)v|v)\leq\|R^{\beta,\mu^o}(\xi)\|^2\|T_L(a^*a)\|\|v\|^2<+\infty\end{aligned}$$

So, we get that $(\omega_{v,\xi}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(a))\in\mathcal{N}_\Phi$. □

Proposition 3.2.8. – *For all $v,w\in H$ and $a,b\in\mathcal{N}_\Phi\cap\mathcal{N}_{T_L}$, we have:*

$$(v\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a)|w\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(b))=(T_L(b^*a)v|w)$$

For all $v,w\in H$ and $c,d\in\mathcal{N}_\Psi\cap\mathcal{N}_{T_R}$, we have:

$$(\Lambda_\Psi(c)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v|\Lambda_\Psi(d)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta w)=(T_R(d^*c)v|w)$$

Proof. – Using 3.2.2 and 2.1.3, we get that:

$$\begin{aligned}(v\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a)|w\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(b))&=(\alpha(\langle\Lambda_\Phi(a),\Lambda_\Phi(b)\rangle_{\hat\beta,\mu^o})v|w)\\ &=(\alpha(\Lambda_{T_L}(b)^*\Lambda_{T_L}(a))v|w)\\ &=(\alpha(\pi_\mu(\alpha^{-1}(T_L(b^*a))))v|w)\end{aligned}$$

which gives the result after the identification of $\pi_\mu(N)$ with N. The second point is very similar. □

LEMMA 3.2.9. – *Let $a \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $v \in D(H_\beta, \mu^o)$. The following sum:*

$$\sum_{i\in I} \xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))$$

converges in $H_\beta \underset{\mu}{\otimes}{}_\alpha H_\Phi$ for all (N^o, μ^o)-basis $(\xi_i)_{i\in I}$ of H_β and it does not depend on the (N^o, μ^o)-basis of H_β.

Proof. – By 3.2.7, we have $(\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)) \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ for all $i \in I$, and the vectors $\xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))$ are two-by-two orthogonal. Normality and left invariance of Φ imply:

$$\begin{aligned}
&\sum_{i\in I} ||\xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))||^2 \\
&= \sum_{i\in I} (\alpha(\langle \xi_i, \xi_i\rangle_{\beta,\mu^o})\Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))|\Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))) \\
&= \Phi\Big((\lambda_v^{\beta,\alpha})^*\Gamma(a^*)\Big[\sum_{i\in I} \lambda_{\xi_i}^{\beta,\alpha}(\lambda_{\xi_i}^{\beta,\alpha})^*\lambda_{\xi_i}^{\beta,\alpha}(\lambda_{\xi_i}^{\beta,\alpha})^*\Big]\Gamma(a)\lambda_v^{\beta,\alpha}\Big) \\
&= \Phi((\omega_{v,v} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a^*a))) = ((\mathrm{id}_\beta\underset{\mu}{\star}{}_\alpha \Phi)(\Gamma(a^*a))v|v) = (T_L(a^*a)v|v) < \infty
\end{aligned}$$

We deduce that the sum $\sum_{i\in I} \xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))$ converges in $H_\beta \underset{\mu}{\otimes}{}_\alpha H_\Phi$. To prove that the sum does not depend on the (N^o, μ^o)-basis, we compute for all $b \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $w \in D(H_\beta, \mu^o)$:

$$\begin{aligned}
&\Big(\sum_{i\in I} \xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))|w {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi(b)\Big) \\
&= \sum_{i\in I} (\alpha(\langle \xi_i, w\rangle_{\beta,\mu^o})\Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))|\Lambda_\Phi(b)) \\
&= \sum_{i\in I} \Phi(b^*\alpha(\langle \xi_i, w\rangle_{\beta,\mu^o})(\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a))) \\
&= \Phi\Big(b^*\lambda_w^{\beta,\alpha}\Big[\sum_{i\in I} \lambda_{\xi_i}^{\beta,\alpha}(\lambda_{\xi_i}^{\beta,\alpha})^*\Big]\Gamma(a)\lambda_v^{\beta,\alpha}\Big) = \Phi(b^*(\omega_{v,w} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a))).
\end{aligned}$$

As $D(H_\beta, \mu^o) \odot \Lambda_\Phi(\mathcal{N}_{T_L} \cap \mathcal{N}_\Phi)$ is dense in $H_\beta \underset{\mu}{\otimes}{}_\alpha H_\Phi$ and the last expression is independent of the (N^o, μ^o)-basis, we can conclude. □

THEOREM 3.2.10. – *Let H be a Hilbert space on which M acts. There exists a unique isometry U_H, called (left)* ***fundamental isometry****, from $H_\alpha \underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi$ to $H_\beta \underset{\mu}{\otimes}{}_\alpha H_\Phi$ such that, for all (N^o, μ^o)-basis $(\xi_i)_{i\in I}$ of H_β, $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $v \in D(H_\beta, \mu^o)$:*

$$U_H(v {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi(a)) = \sum_{i\in I} \xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i} {}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a))))$$

Proof. – By 3.2.9, we can define the following application:

$$\tilde{U}: D(H_\beta, \mu^o) \times \Lambda_\Phi(\mathcal{N}_T \cap \mathcal{N}_\Phi) \longrightarrow H_\beta \underset{\mu}{\otimes}_\alpha H_\Phi$$
$$(v, \Lambda_\Phi(a)) \longmapsto \sum_{i\in I} \xi_i\, {}_\beta\underset{\mu}{\otimes}_\alpha \Lambda_\Phi((\omega_{v,\xi_i}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(a))))$$

Let $b \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and $w \in D(H_\beta, \mu^o)$. Then, by normality and left invariance of Φ, we have:

$$\begin{aligned}
&(\tilde{U}(v, \Lambda_\Phi(a))|\tilde{U}(w, \Lambda_\Phi(b)))\\
&= \sum_{i,j\in I} (\alpha(\langle \xi_i, \xi_j\rangle_{\beta,\mu^o})\Lambda_\Phi((\omega_{v,\xi_i}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(a)))|\Lambda_\Phi((\omega_{w,\xi_i}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(b))))\\
&= \sum_{i\in I} (\Lambda_\Phi(\alpha(\langle \xi_i, \xi_i\rangle_{\beta,\mu^o})(\omega_{v,\xi_i}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(a)))|\Lambda_\Phi((\omega_{w,\xi_i}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(b))))\\
&= \sum_{i\in I} \Phi((\lambda_w^{\beta,\alpha})^*\Gamma(b^*)\lambda_{\xi_i}^{\beta,\alpha}\alpha(\langle \xi_i, \xi_i\rangle_{\beta,\mu^o})(\lambda_{\xi_i}^{\beta,\alpha})^*\Gamma(a)\lambda_v^{\beta,\alpha})\\
&= \Phi((\lambda_w^{\beta,\alpha})^*\Gamma(b^*)\Big[\sum_{i\in I} \lambda_{\xi_i}^{\beta,\alpha}(\lambda_{\xi_i}^{\beta,\alpha})^*\lambda_{\xi_i}^{\beta,\alpha}(\lambda_{\xi_i}^{\beta,\alpha})^*\Big]\Gamma(a)\lambda_v^{\beta,\alpha})
\end{aligned}$$

Then, properties of (N^o, μ^o)-basis $(\xi_i)_{i\in I}$ of H_β imply that:

$$\begin{aligned}
\Phi((\omega_{v,w}\, {}_\beta\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(b^*a))) &= \omega_{v,w}((\mathrm{id}\, {}_\beta\underset{\mu}{\star}_\alpha \Phi)(\Gamma(b^*a)))\\
&= \omega_{v,w}(T_L(b^*a)) = (T_L(b^*a)v|w)
\end{aligned}$$

By 3.2.8, we get:

$$(\tilde{U}((v, \Lambda_\Phi(a))|\tilde{U}((w, \Lambda_\Phi(b)))) = (v\, {}_\alpha\underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(a)|w\, {}_\alpha\underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(b))$$

so that, from $\tilde{U}$, we can easily define a suitable application U_H which is independent of the (N^o, μ^o)-basis by 3.2.9. □

One can define a right version of U_H from the right invariant weight:

THEOREM 3.2.11. – *Let H be a Hilbert space on which M acts. There exists a unique isometry U'_H, called* ***right fundamental isometry****, from $H_\Psi\, {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}_\beta H$ to $H_\Psi\, {}_\beta\underset{\mu}{\otimes}_\alpha H$ such that, for all (N, μ)-basis $(\eta_i)_{i\in I}$ of ${}_\alpha H$, $a \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ and $v \in D({}_\alpha H, \mu)$:*

$$U'_H(\Lambda_\Psi(a)\, {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}_\beta v) = \sum_{i\in I} \Lambda_\Psi((\mathrm{id}\, {}_\beta\underset{\mu}{\star}_\alpha \omega_{v,\eta_i})(\Gamma(a)))\, {}_\beta\underset{\mu}{\otimes}_\alpha \eta_i$$

3.3. Fundamental isometry and co-product

In this paragraph, we establish several links between fundamental isometry and co-product. In fact, many of the following relations are more or less equivalent to definition of fundamental unitary and, depending of the situation, we will give priority to one or the other relations in our demonstrations.

PROPOSITION 3.3.1. – *We have* $(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_{H}\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(x)}=\Gamma(x)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}$ *for all* $e,x\in\mathcal{N}_{\Phi}\cap\mathcal{N}_{T_L}$ *and* $(J_{\Psi}fJ_{\Psi}{}_{\beta}\underset{N}{\otimes}{}_{\alpha}1)U'_{H}\lambda^{\hat{\alpha},\beta}_{\Lambda_{\Psi}(y)}=\Gamma(y)\lambda^{\beta,\alpha}_{J_{\Psi}\Lambda_{\Psi}(f)}$ *for all* $f,y\in\mathcal{N}_{\Psi}\cap\mathcal{N}_{T_R}$.

Proof. – Let $v\in D(H_{\beta},\mu^{o})$ and $(\xi_i)_{i\in I}$ a (N^{o},μ^{o})-basis of H_{β}. We have:

$$\begin{aligned}&(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_{H}(v_{\alpha}\underset{\mu^{o}}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(x))\\&=\sum_{i\in I}\xi_{i\,\beta}\underset{\mu}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi}\Lambda_{\Phi}((\omega_{v,\xi_i\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(x)))\\&=\sum_{i\in I}\xi_{i\,\beta}\underset{\mu}{\otimes}{}_{\alpha}(\omega_{v,\xi_i\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(x))J_{\Phi}\Lambda_{\Phi}(e)=\Gamma(x)(v_{\beta}\underset{\mu}{\otimes}{}_{\alpha}J_{\Phi}\Lambda_{\Phi}(e))\end{aligned}$$

By 3.2.2 and 3.2.4, we have $\Lambda_{\Phi}(x)\in D((H_{\Phi})_{\hat{\beta}},\mu^{o})$ and $J_{\Phi}\Lambda_{\Phi}(e)\in D({}_{\alpha}(H_{\Phi}),\mu)$ so that each term of the previous equality is continuous in v. Density of $D(H_{\beta},\mu^{o})$ in H finishes the proof. The last part is very similar. □

PROPOSITION 3.3.2. – *For all* $v,w\in D(H_{\beta},\mu^{o})$ *and* $a\in\mathcal{N}_{\Phi}\cap\mathcal{N}_{T_L}$, *we have:*

$$(\lambda^{\beta,\alpha}_{w})^{*}U_{H}(v_{\alpha}\underset{\mu^{o}}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(a))=\Lambda_{\Phi}((\omega_{v,w\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a)))$$

Also, for all $v',w'\in D({}_{\alpha}H,\mu)$ *and* $b\in\mathcal{N}_{\Psi}\cap\mathcal{N}_{T_R}$, *we have:*

$$(\rho^{\beta,\alpha}_{w'})^{*}U'_{H}(\Lambda_{\Psi}(b)_{\alpha}\underset{\mu^{o}}{\otimes}{}_{\hat{\beta}}v')=\Lambda_{\Psi}((\mathrm{id}_{\beta}\underset{\mu}{\star}{}_{\alpha}\omega_{v',w'})(\Gamma(b)))$$

Proof. – Let $e\in\mathcal{N}_{\Phi}\cap\mathcal{N}_{T_L}$. By 3.3.1, we can compute:

$$\begin{aligned}J_{\Phi}eJ_{\Phi}(\lambda^{\beta,\alpha}_{w})^{*}U_{H}(v_{\alpha}\underset{\mu^{o}}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(a))&=(\lambda^{\beta,\alpha}_{w})^{*}(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_{H}\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(a)}v\\&=(\lambda^{\beta,\alpha}_{w})^{*}\Gamma(a)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}v\\&=(\omega_{v,w\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a))J_{\Phi}\Lambda_{\Phi}(e)\\&=J_{\Phi}eJ_{\Phi}\Lambda_{\Phi}((\omega_{v,w\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a)))\end{aligned}$$

Density of $\mathcal{N}_{\Phi}\cap\mathcal{N}_{T_L}$ in N finishes the proof. The second part is very similar. □

COROLLARY 3.3.3. – *For all* $a\in\mathcal{N}_{T_L}\cap\mathcal{N}_{\Phi}$, $v\in D({}_{\alpha}H,\mu)\cap D(H_{\beta},\mu^{o})$ *and* $w\in D(H_{\beta},\mu^{o})$, *we have:*

$$(\omega_{v,w}*\mathrm{id})(U_{H})\Lambda_{\Phi}(a)=\Lambda_{\Phi}((\omega_{v,w\,\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a)))$$

where we denote by $(\omega_{v,w}*\mathrm{id})(U_{H})$ *the operator* $(\lambda^{\beta,\alpha}_{w})^{*}U_{H}\lambda^{\alpha,\hat{\beta}}_{v}$ *of* $\mathcal{L}(H_{\Phi})$.

Proof. – Straightforward. □

COROLLARY 3.3.4. – *For all* $e, x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ *and* $\eta \in D({}_\alpha H_\Phi, \mu^o)$*, we have:*

$$(\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{J_\Phi \Lambda_\Phi(e),\eta})(\Gamma(x)) = (\mathrm{id} * \omega_{\Lambda_\Phi(x), J_\Phi e^* J_\Phi \eta})(U_H)$$

Also, for all $f, y \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ *and* $\xi \in D((H_\Psi)_\beta, \mu^o)$*, we have:*

$$(\omega_{J_\Psi \Lambda_\Psi(f),\xi\,\beta} \underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(y)) = (\omega_{\Lambda_\Psi(y), J_\Psi f^* J_\Psi \xi} * \mathrm{id})(U'_H)$$

Proof. – Straightforward by 3.3.1. □

COROLLARY 3.3.5. – *For all* $a, b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi^* \cap \mathcal{N}_{T_R}^*$*, we have:*

$$(\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(b)} * \mathrm{id})(U'_H)^* = (\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^*)} * \mathrm{id})(U'_H)$$

Proof. – By 3.3.4, we have for all $e \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$:

$$\begin{aligned}(\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(e^* b)} * \mathrm{id})(U'_H)^* &= (\omega_{J_\Psi \Lambda_\Psi(e), J_\Psi \Lambda_\Psi(b)\,\beta} \underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a))^* \\ &= (\omega_{J_\Psi \Lambda_\Psi(b), J_\Psi \Lambda_\Psi(e)\,\beta} \underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a^*)) \\ &= (\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^* e)} * \mathrm{id})(U'_H).\end{aligned}$$

Let $(u_k)_{k \in K}$ be a family in $\mathcal{N}_\Psi \cap \mathcal{N}_\Psi^*$ such that $u_k \to 1$ in the *-strong topology. We denote:

$$e_k = \frac{1}{\sqrt{\pi}} \int e^{-t^2} \sigma_t^\Psi(u_k)\, dt$$

For all $k \in K$, e_k and $\sigma_{-i/2}^\Psi(e_k^*)$ are bounded and belong to $\mathcal{N}_\Psi$ and converge to 1 in the *-strong topology so that $J_\Psi \Lambda_\Psi(b^* e_k) = \sigma_{-i/2}^\Psi(e_k^*) J_\Psi \Lambda_\Psi(b^*)$ converge to $J_\Psi \Lambda_\Psi(b^*)$ in norm of H_Ψ. Let $\xi, \eta \in D({}_\alpha H, \mu)$ and we compute:

$$\begin{aligned}((\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(b)} * \mathrm{id})(U'_H)^* \xi | \eta) &= (J_\Psi \Lambda_\Psi(b)\,{}_\beta \underset{\mu}{\otimes}{}_\alpha \xi | U'_H(\Lambda_\Psi(a)\,{}_{\hat{\alpha}} \underset{\mu^o}{\otimes}{}_\beta \eta)) \\ &= \lim_{k \in K} (J_\Psi \Lambda_\Psi(e_k^* b)\,{}_\beta \underset{\mu}{\otimes}{}_\alpha \xi | U'_H(\Lambda_\Psi(a)\,{}_{\hat{\alpha}} \underset{\mu^o}{\otimes}{}_\beta \eta)) \\ &= \lim_{k \in K} ((\omega_{\Lambda_\Psi(a), J_\Psi \Lambda_\Psi(e_k^* b)} * \mathrm{id})(U'_H)^* \xi | \eta)\end{aligned}$$

By the previous computation, this last expression is equal to:

$$\begin{aligned}&\lim_{k \in K} ((\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^* e_k)} * \mathrm{id})(U'_H) \xi | \eta) \\ &= \lim_{k \in K} (U'_H(\Lambda_\Psi(a)\,{}_{\hat{\alpha}} \underset{\mu^o}{\otimes}{}_\beta \xi) | J_\Psi \Lambda_\Psi(b^* e_k)\,{}_\beta \underset{\mu}{\otimes}{}_\alpha \eta) \\ &= (U'_H(\Lambda_\Psi(a^*)\,{}_{\hat{\alpha}} \underset{\mu^o}{\otimes}{}_\beta \xi) | J_\Psi \Lambda_\Psi(b^*)\,{}_\beta \underset{\mu}{\otimes}{}_\alpha \eta) = ((\omega_{\Lambda_\Psi(a^*), J_\Psi \Lambda_\Psi(b^*)} * \mathrm{id})(U'_H) \xi | \eta)\end{aligned}$$

By density of $D({}_\alpha H, \mu)$ in H, the result holds. □

3.4. Commutation relations

In this section, we verify commutation relations which are necessary for U_H to be a pseudo-multiplicative unitary and we establish a link between U_H and Γ. We also have similar formulas for U'_H.

LEMMA 3.4.1. – *Let $\xi \in D(H_\beta, \mu^o)$ and $\eta \in D({}_\alpha H, \mu)$.*

i) *For all $a \in \alpha(N)'$, we have $\lambda_\xi^{\beta,\alpha} \circ a = (1{}_\beta\underset{N}{\otimes}{}_\alpha a)\lambda_\xi^{\beta,\alpha}$.*

ii) *For all $b \in \beta(N)'$, we have $\lambda_{b\xi}^{\beta,\alpha} = (b{}_\beta\underset{N}{\otimes}{}_\alpha 1)\lambda_\xi^{\beta,\alpha}$.*

iii) *For all $x \in \mathscr{D}(\sigma_{-i/2}^\mu)$, we have $\lambda_{\beta(x)\xi}^{\beta,\alpha} = \lambda_\xi^{\beta,\alpha} \circ \alpha(\sigma_{-i/2}^\mu(x))$.*

iv) *For all $x \in \mathscr{D}(\sigma_{i/2}^\mu)$, we have $\rho_{\alpha(x)\eta}^{\beta,\alpha} = \rho_\eta^{\beta,\alpha} \circ \beta(\sigma_{i/2}^\mu(x))$.*

Proof. – Straightforward. □

We recall that $\alpha(N)$ and $\beta(N)$ commute with $\hat\beta(N)'$.

PROPOSITION 3.4.2. – *For all $n \in N$, we have:*

i) $U_H(1{}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta}\alpha(n)) = (\alpha(n){}_\beta\underset{N}{\otimes}{}_\alpha 1)U_H$;

ii) $U_H(1{}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta}\beta(n)) = (1{}_\beta\underset{N}{\otimes}{}_\alpha\beta(n))U_H$;

iii) $U_H(\beta(n){}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta}1) = (1{}_\beta\underset{N}{\otimes}{}_\alpha\hat\beta(n))U_H$.

Proof. – By 3.3.1, we can compute for all $n \in N$ and $e, x \in \mathscr{N}_{T_L} \cap \mathscr{N}_\Phi$:

$$\begin{aligned}(\alpha(n){}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi)U_H\rho_{\Lambda_\Phi(x)}^{\alpha,\hat\beta} &= (\alpha(n){}_\beta\underset{N}{\otimes}{}_\alpha 1)\Gamma(x)\rho_{J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha}\\ &= \Gamma(\alpha(n)x)\rho_{J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha}\\ &= (1{}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi)U_H\rho_{\Lambda_\Phi(\alpha(n)x)}^{\alpha,\hat\beta}\\ &= (1{}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi)U_H(1{}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta}\alpha(n))\rho_{\Lambda_\Phi(x)}^{\alpha,\hat\beta}\end{aligned}$$

Usual arguments of density imply the first equality. The second one can be proved in a very similar way. By 3.3.1 and 3.4.1, we can compute for all $n \in \mathscr{T}_\mu$ and $e, x \in \mathscr{N}_{T_L} \cap \mathscr{N}_\Phi$:

$$\begin{aligned}(1{}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi\hat\beta(n))U_H\rho_{\Lambda_\Phi(x)}^{\alpha,\hat\beta} &= \Gamma(x)\rho_{J_\Phi\Lambda_\Phi(e\alpha(n^*))}^{\beta,\alpha}\\ &= \Gamma(x)\rho_{\alpha(\sigma_{-i/2}^\mu(n))J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha}\\ &= \Gamma(x)\rho_{J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha}\beta(n)\\ &= (1{}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi)U_H\rho_{\Lambda_\Phi(x)}^{\alpha,\hat\beta}\beta(n)\\ &= (1{}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi)U_H(\beta(n){}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta}1)\rho_{\Lambda_\Phi(x)}^{\alpha,\hat\beta}\end{aligned}$$

Density of $\mathscr{T}_\mu$ in N and normality of β and $\hat\beta$ finish the proof. □

Proposition 3.4.3. – *For all $x \in M' \cap \mathcal{L}(H)$, we have:*

$$U_H(x_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1) = (x_{\beta}\underset{N}{\otimes}{}_{\alpha}1)U_H$$

Proof. – For all $e, y \in \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$ and $x \in M' \cap \mathcal{L}(H) \subseteq \alpha(N)' \cap \beta(N)' \cap \mathcal{L}(H)$, we have by 3.3.1:

$$\begin{aligned}(x_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(y)} &= (x_{\beta}\underset{N}{\otimes}{}_{\alpha}1)\Gamma(y)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}\\ &= \Gamma(y)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}x\\ &= (1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(y)}x\\ &= (1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H(x_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(y)}\end{aligned}$$

Usual arguments of density imply the result. □

Corollary 3.4.4. – *For all $n \in N$, we have:*

i) $U_{H_{\Phi}}(\hat{\beta}(n)_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1) = (\hat{\beta}(n)_{\beta}\underset{N}{\otimes}{}_{\alpha}1)U_{H_{\Phi}}$

ii) $U_{H_{\Psi}}(\hat{\alpha}(n)_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1) = (\hat{\alpha}(n)_{\beta}\underset{N}{\otimes}{}_{\alpha}1)U_{H_{\Psi}}$

Proposition 3.4.5. – *We have $\Gamma(m)U_H = U_H(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}m)$ for all $m \in M$.*

Proof. – By 3.3.1, we can compute for all $e, x \in \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$:

$$\begin{aligned}(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})\Gamma(m)U_H\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(x)} &= \Gamma(m)(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(x)}\\ &= \Gamma(mx)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}\\ &= (1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(mx)}\\ &= (1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_H(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}m)\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(x)}\end{aligned}$$

Usual arguments of density imply the result. □

3.5. Unitarity of the fundamental isometry

This is a key part of the theory and certainly one of the most difficult. To prove unitary of U_H (resp. U'_H), we establish a reciprocity law where both left and right operator-valued weights are at stake.

3.5.1. First technical result. – We establish results needed for 3.5.3. In the following proposition, we compute some functions θ defined in section 2.2.

PROPOSITION 3.5.1. – *We have for all* $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $m \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*$ *and* $v \in D(H_\beta, \mu^o)$:

$$\theta^{\beta,\mu^o}(v, J_\Psi \Lambda_\Psi(c))m = (\lambda^{\hat{\alpha},\beta}_{\Lambda_\Psi(m^*)})^* \rho^{\hat{\alpha},\beta}_v J_\Psi c^* J_\Psi$$

Proof. – Let $x \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$. On one hand, we get by 3.2.2 and 3.2.4:

$$\begin{aligned}\theta^{\beta,\mu^o}(v, J_\Psi\Lambda_\Psi(c))m\Lambda_\Psi(x) &= R^{\beta,\mu^o}(v)R^{\beta,\mu^o}(J_\Psi\Lambda_\Psi(c))^*\Lambda_\Psi(mx)\\ &= R^{\beta,\mu^o}(v)J_\mu\Lambda_{T_R}(c)^*J_\Psi\Lambda_\Psi(mx).\end{aligned}$$

On the other hand, if $c \in \mathcal{T}_{\Psi,T_R}$, then we have by 3.2.8:

$$\begin{aligned}(\lambda^{\hat{\alpha},\beta}_{\Lambda_\Psi(m^*)})^*\rho^{\hat{\alpha},\beta}_v J_\Psi c^* J_\Psi\Lambda_\Psi(x) &= (\lambda^{\hat{\alpha},\beta}_{\Lambda_\Psi(m^*)})^*(J_\Psi c^* J_\Psi \Lambda_\Psi(x) {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}{}_\beta v)\\ &= T_R(mx\sigma^\Psi_{-i/2}(c))v\\ &= R^{\beta,\mu^o}(v)J_\mu\Lambda_\mu(\beta^{-1}(T_R(\sigma^\Psi_{i/2}(c^*)x^*m^*)))\\ &= R^{\beta,\mu^o}(v)J_\mu\Lambda_{T_R}(c)^*J_\Psi\Lambda_\Psi(mx)\end{aligned}$$

We obtain:

$$(\lambda^{\hat{\alpha},\beta}_{\Lambda_\Psi(m^*)})^*\rho^{\hat{\alpha},\beta}_v J_\Psi c^* J_\Psi\Lambda_\Psi(x) = R^{\beta,\mu^o}(v)J_\mu\Lambda_{T_R}(c)^*J_\Psi\Lambda_\Psi(mx)$$

for all $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ by normality which finishes the proof. □

COROLLARY 3.5.2. – *Let* $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})$. *If* $c \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ *and* $\xi \in H_\Psi, \eta \in D({}_\alpha(H_\Phi), \mu)$, $u \in H$, $v \in D(H_\beta, \mu^o)$, *then we have:*

$$\begin{aligned}&(v {}_\beta\underset{\mu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^* U_{H_\Psi}(\xi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a)) | u {}_\beta\underset{\mu}{\otimes}{}_\alpha J_\Phi e^* J_\Phi \eta)\\ &= (J_\Psi c^* J_\Psi \xi {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}{}_\beta v | \Lambda_\Psi((\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{\eta, J_\Phi\Lambda_\Phi(e)})(\Gamma(a^*))) {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}{}_\beta u)\end{aligned}$$

Proof. – By 3.3.1 and 3.5.1, we can compute:

$$\begin{aligned}&(v {}_\beta\underset{\mu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^* U_{H_\Psi}(\xi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a)) | u {}_\beta\underset{\mu}{\otimes}{}_\alpha J_\Phi e^* J_\Phi \eta)\\ &= ((\rho^{\beta,\alpha}_\eta)^* \lambda^{\beta,\alpha}_v (\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^* (1 {}_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e^* J_\Phi) U_{H_\Psi}(\xi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a)) | u)\\ &= ((\rho^{\beta,\alpha}_\eta)^* \lambda^{\beta,\alpha}_v (\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^* \Gamma(a) \rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(e)} \xi | u)\\ &= \theta^{\beta,\mu^o}(v, J_\Psi\Lambda_\Psi(c))(\rho^{\beta,\alpha}_\eta)^* \Gamma(a) \rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(e)} \xi | u)\\ &= ((\lambda^{\hat{\alpha},\beta}_{\Lambda_\Psi((\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{\eta, J_\Phi\Lambda_\Phi(e)}))(\Gamma(a^*))})^* \rho^{\hat{\alpha},\beta}_v J_\Psi c^* J_\Psi \xi | u)\\ &= (J_\Psi c^* J_\Psi \xi {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}{}_\beta v | \Lambda_\Psi((\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{\eta, J_\Phi\Lambda_\Phi(e)}))(\Gamma(a^*))) {}_{\hat{\alpha}}\underset{\mu^o}{\otimes}{}_\beta u)\end{aligned}$$ □

3.5.2. Second technical result. – In this section, results only depend on 3.3.1 and co-product relation but not on the previous technical result. Let $\mathscr{H}$ be another Hilbert space on which M acts.

LEMMA 3.5.3. – *Let $a, e \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$, $\xi \in D(\mathscr{H}_\beta, \mu^o)$, $\eta \in D({}_\alpha H, \mu)$, and $\zeta \in \mathscr{H}$. We have:*

$$(1_\beta \underset{N}{\otimes}_\alpha J_\Phi e J_\Phi) U_H(\eta_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\zeta_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(a))])$$

$$= (\lambda_\xi^{\beta,\alpha}{}_\beta \underset{N}{\otimes}_\alpha 1)^* (\mathrm{id}_\beta \underset{N}{\star}_\alpha \Gamma)(\Gamma(a))(\zeta_\beta \underset{\mu}{\otimes}_\alpha \eta_\beta \underset{\mu}{\otimes}_\alpha J_\Phi \Lambda_\Phi(e))$$

Proof. – First let assume $\zeta \in D(\mathscr{H}_\beta, \mu^o)$. By 3.3.2 and 3.3.1, we can compute:

$$\begin{aligned}
&(1_\beta \underset{N}{\otimes}_\alpha J_\Phi e J_\Phi) U_H(\eta_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\zeta_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(a))]) \\
&= (1_\beta \underset{N}{\otimes}_\alpha J_\Phi e J_\Phi) U_H(\eta_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi((\omega_{\zeta,\xi\,\beta} \underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(a))) \\
&= \Gamma((\omega_{\zeta,\xi\,\beta} \underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(a)))(\eta_\beta \underset{\mu}{\otimes}_\alpha J_\Phi \Lambda_\Phi(e)) \\
&= (\lambda_\xi^{\beta,\alpha}{}_\beta \underset{N}{\otimes}_\alpha 1)^* (\mathrm{id}_\beta \underset{N}{\star}_\alpha \Gamma)(\Gamma(a))(\zeta_\beta \underset{\mu}{\otimes}_\alpha \eta_\beta \underset{\mu}{\otimes}_\alpha J_\Phi \Lambda_\Phi(e))
\end{aligned}$$

So, we get the result for all $\zeta \in D(\mathscr{H}_\beta, \mu^o)$. The first term of the equality is continuous in ζ because $\eta \in D({}_\alpha H, \mu)$ and $\Lambda_\Phi(a) \in D((H_\Phi)_{\hat{\beta}}, \mu^o)$. Also, since $\eta \in D({}_\alpha H, \mu)$ and $\Lambda_\Phi(a) \in D((H_\Phi)_{\hat{\beta}}, \mu^o)$, the last term of the equality is continuous in ζ. Density of $D(\mathscr{H}_\beta, \mu^o)$ in $\mathscr{H}$ finishes the proof. $\square$

LEMMA 3.5.4. – *The sum $\sum_{i \in I} \eta_{i\,\alpha} \underset{\mu^o}{\otimes}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(a))]$ converges for all $\xi \in D(\mathscr{H}_\beta, \mu^o)$, $\Xi \in \mathscr{H}_\beta \underset{\mu}{\otimes}_\alpha H$, $a \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$ and (N, μ)-basis $(\eta_i)_{i \in I}$ of ${}_\alpha H$.*

Proof. – First, observe that $\eta_{i\,\alpha} \underset{\mu^o}{\otimes}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_{\hat{\alpha}} \underset{\mu^o}{\otimes}_\beta \Lambda_\Phi(a))]$ are orthogonal. To compute, we put: $\Omega_i = \rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}_{\hat{\beta}} \Lambda_\Phi(a)$. By 3.4.1 and 3.4.2, we have:

$$\begin{aligned}
&||\eta_{i\,\alpha} \underset{\mu^o}{\otimes}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i)]||^2 \\
&= (\hat{\beta}(\langle \eta_i, \eta_i \rangle_{\alpha,\mu})(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i) | (\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i)) \\
&= ((\lambda_\xi^{\beta,\alpha})^* (1_\beta \underset{\mu}{\otimes}_\alpha \hat{\beta}(\langle \eta_i, \eta_i \rangle_{\alpha,\mu})) U_{\mathscr{H}}(\Omega_i) | (\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i)) \\
&= ((\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\beta(\langle \eta_i, \eta_i \rangle_{\alpha,\mu})(\Omega_i) | (\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i)) \\
&= (\lambda_\xi^{\beta,\alpha} (\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}(\Omega_i) | U_{\mathscr{H}}(\Omega_i))
\end{aligned}$$

By 3.2.8, it follows that we have, for all $i \in I$:

$$\begin{aligned}
&||\eta_i {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a))]||^2 \\
&\leq ||R^{\beta,\alpha}(\xi)||^2 ((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a) | (\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a)) \\
&\leq ||R^{\beta,\alpha}(\xi)||^2 (T_L(a^*a)(\rho_{\eta_i}^{\beta,\alpha})^* \Xi | (\rho_{\eta_i}^{\beta,\alpha})^* \Xi) \\
&\leq ||R^{\beta,\alpha}(\xi)||^2 ||T(a^*a)|| ((\rho_{\eta_i}^{\beta,\alpha})^* \Xi | (\rho_{\eta_i}^{\beta,\alpha})^* \Xi)
\end{aligned}$$

So, we can sum over $i \in I$ to get that:

$$\sum_{i\in I} ||\eta_i {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a))]||^2$$

is less or equal to $||R^{\beta,\alpha}(\xi)||^2 ||T(a^*a)|| ||\Xi||^2 < \infty$. That's why the sum converges. □

PROPOSITION 3.5.5. – *Let $a, e \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$, $\Xi \in \mathscr{H}_\beta \underset{\mu}{\otimes}_\alpha H$, $\xi \in D(\mathscr{H}_\beta, \mu^o)$, $\eta \in D({}_\alpha(H_\Phi), \mu)$ and $(\eta_i)_{i\in I}$ a (N, μ)-basis of ${}_\alpha H$. We have:*

$$(\rho_{J_\Phi e J_\Phi \eta}^{\beta,\alpha})^* U_H (\sum_{i\in I} \eta_i {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a))])$$

$$= (\lambda_\xi^{\beta,\alpha})^* \Gamma((\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{J_\Phi \Lambda_\Phi(e),\eta})(\Gamma(a)))\Xi$$

Proof. – The existence of the first term comes from the previous lemma. By 3.5.3 and the co-product relation, we can compute:

$$\begin{aligned}
&\sum_{i\in I} (\rho_\eta^{\beta,\alpha})^* (1_\beta \underset{N}{\otimes}{}_\alpha J_\Phi e J_\Phi) U_H (\eta_i {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat{\beta}} [(\lambda_\xi^{\beta,\alpha})^* U_{\mathscr{H}}((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a))]) \\
&= \sum_{i\in I} (\rho_\eta^{\beta,\alpha})^* (\lambda_\xi^{\beta,\alpha} {}_\beta\underset{N}{\otimes}{}_\alpha 1)^* (\mathrm{id}_\beta \underset{N}{\star}{}_\alpha \Gamma)(\Gamma(a))((\rho_{\eta_i}^{\beta,\alpha})^* \Xi_\beta \underset{\mu}{\otimes}{}_\alpha \eta_i {}_\beta\underset{\mu}{\otimes}{}_\alpha J_\Phi \Lambda_\Phi(e)) \\
&= (\rho_\eta^{\beta,\alpha})^* (\lambda_\xi^{\beta,\alpha} {}_\beta\underset{N}{\otimes}{}_\alpha 1)^* (\Gamma_\beta \underset{N}{\star}{}_\alpha \mathrm{id})(\Gamma(a)) \Big(\Big[\sum_{i\in I} \rho_{\eta_i}^{\beta,\alpha} (\rho_{\eta_i}^{\beta,\alpha})^* \Big] \Xi_\beta \underset{\mu}{\otimes}{}_\alpha J_\Phi \Lambda_\Phi(e) \Big) \\
&= (\rho_\eta^{\beta,\alpha})^* (\lambda_\xi^{\beta,\alpha} {}_\beta\underset{N}{\otimes}{}_\alpha 1)^* (\Gamma_\beta \underset{N}{\star}{}_\alpha \mathrm{id})(\Gamma(a))(\Xi_\beta \underset{\mu}{\otimes}{}_\alpha J_\Phi \Lambda_\Phi(e))) \\
&= (\lambda_\xi^{\beta,\alpha})^* (1_\beta \underset{N}{\otimes}{}_\alpha \rho_\eta^{\beta,\alpha})^* (\Gamma_\beta \underset{N}{\star}{}_\alpha \mathrm{id})(\Gamma(a))(\Xi_\beta \underset{\mu}{\otimes}{}_\alpha J_\Phi \Lambda_\Phi(e))) \\
&= (\lambda_\xi^{\beta,\alpha})^* \Gamma((\mathrm{id}_\beta \underset{\mu}{\star}{}_\alpha \omega_{J_\Phi \Lambda_\Phi(e),\eta})(\Gamma(a)))\Xi
\end{aligned}$$

□

With results of the two last sections in hand, we can prove now a reciprocity law where $\mathscr{H}$ will be equal to H_Ψ.

3.5.3. Reciprocity law. – For all monotone increasing net $(e_k)_{k\in K}$ in $\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ of limit equal to 1, the following $(\omega_{J_\Psi\Lambda_\Psi(e_k)})_{k\in K}$ is monotone increasing and converges to Ψ. So, for all $x \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $(\omega_{J_\Psi\Lambda_\Psi(e_k)}\,{}_\beta\underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(x))$ converges to $(\Psi\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma(x))$ in the weak topology. We denote $\zeta_k = J_\Psi\Lambda_\Psi(e_k^* e_k) \in D((H_\Psi)_\beta, \mu^o)$ for all $k \in K$.

Proposition 3.5.6. – *For all $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L}))$, $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}, b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}, c \in \mathcal{T}_{\Psi,T_R}$, $v \in D(H_\beta, \mu^o), \eta \in D({}_\alpha(H_\Phi), \mu)$ and (N,μ)-basis of ${}_\alpha H$, $(\eta_i)_{i\in I}$, we have that the image of:*

$$\sum_{i\in I} \eta_i\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}[(\lambda^{\beta,\alpha}_{\zeta_k})^* U_{H_\Psi}([(\rho^{\beta,\alpha}_{\eta_i})^* U'_H(J_\Psi c^* J_\Psi \Lambda_\Psi(b)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)]\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))]$$

by $(\rho^{\beta,\alpha}_{J_\Phi e^* J_\Phi \eta})^* U_H$ converges, in the weak topology, to:

$$(\rho^{\beta,\alpha}_{J_\Phi e^* J_\Phi\eta})^*(v\,{}_\beta\underset{\mu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^* U_{H_\Psi}(\Lambda_\Psi(b)\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a)))$$

Proof. – Let $u \in H$. We compute the value of the scalar product of:

$$U_H\Big(\sum_{i\in I}\eta_i\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}[(\lambda^{\beta,\alpha}_{\zeta_k})^* U_{H_\Psi}([(\rho^{\beta,\alpha}_{\eta_i})^* U'_H(\Lambda_\Psi(bc)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)]\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))]\Big)$$

by $u\,{}_\beta\underset{\mu}{\otimes}{}_\alpha J_\Phi e^* J_\Phi\eta$. By 3.5.5, we get that it is equal to:

$$(\Gamma((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a)))U'_H(\Lambda_\Psi(bc)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)|\zeta_k\,{}_\beta\underset{\mu}{\otimes}{}_\alpha u)$$

By the right version of 3.4.5, this is equal to:

$$(U'_H(\Lambda_\Psi((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a))bc)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)|\zeta_k\,{}_\beta\underset{\mu}{\otimes}{}_\alpha u)$$

By 3.3.1, we obtain:

$$((\omega_{J_\Psi\Lambda_\Psi(e_k)}\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a))bc))v|u)$$

which converges to:

$$((\Psi\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a)))bc)v|u)$$

Now, by right invariance of T_R, 3.2.8 and 3.5.2, we can compute this last expression:

$$\begin{aligned}
&((\Psi\,{}_\beta\underset{\mu}{\star}{}_\alpha\mathrm{id})(\Gamma((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a)))bc)v|u)\\
&= (T_R((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(e),\eta})(\Gamma(a))bc)v|u)\\
&= (\Lambda_\Psi(bc)\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v|\Lambda_\Psi((\mathrm{id}\,{}_\beta\underset{\mu}{\star}{}_\alpha\omega_{\eta,J_\Phi\Lambda_\Phi(e)})(\Gamma(a^*))))\,{}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta u)\\
&= (v\,{}_\beta\underset{\mu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(c^*))})^* U_{H_\Psi}(\Lambda_\Psi(b)\,{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))|u\,{}_\beta\underset{\mu}{\otimes}{}_\alpha J_\Phi e^* J_\Phi\eta)
\end{aligned}$$

which finishes the proof. □

Let $(\eta_i)_{i\in I}$ be a (N,μ)-basis of ${}_\alpha H$. For all finite subset J of I, we denote by P_J the projection $\sum_{i\in J}\theta^{\alpha,\mu}(\eta_i,\eta_i)\in\alpha(N)'$ so that:

$$\sum_{i\in J}\rho^{\beta,\alpha}_{\eta_i}(\rho^{\beta,\alpha}_{\eta_i})^*=1_\beta\underset{N}{\otimes}{}_\alpha P_J$$

For all $e\in\mathcal{N}_\Phi\cap\mathcal{N}_{T_L}$, we also denote by P^e_J:

$$1_\beta\underset{N}{\otimes}{}_\alpha J_\Phi e^*J_\Phi P_J J_\Phi e J_\Phi=\sum_{i\in J}\rho^{\beta,\alpha}_{J_\Phi e^*J_\Phi\eta_i}(\rho^{\beta,\alpha}_{J_\Phi e^*J_\Phi\eta_i})^*$$

Corollary 3.5.7. – *For all $a\in(\mathcal{N}_\Psi\cap\mathcal{N}_{T_R})^*(\mathcal{N}_\Phi\cap\mathcal{N}_{T_L})$, $b\in\mathcal{N}_\Psi\cap\mathcal{N}_{T_R}$, and $c\in\mathcal{T}_{\Psi,T_R}$, $v\in D(H_\beta,\mu^o)$, $e\in\mathcal{N}_\Phi\cap\mathcal{N}_{T_L}$ and J finite subset of I, we have:*

$$P^e_J U_H\Big(\sum_{i\in I}\eta_i{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}[(\lambda^{\beta,\alpha}_{\zeta_k})^*U_{H_\Psi}([(\rho^{\beta,\alpha}_{\eta_i})^*U'_H(J_\Psi c^*J_\Psi\Lambda_\Psi(b)_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)]_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))]\Big)$$

converges, in the weak topology, to:

$$P^e_J(v_\beta\underset{\mu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^*U_{H_\Psi}(\Lambda_\Psi(b)_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a)))$$

Proof. – We apply to the reciprocity law $\rho^{\beta,\alpha}_{J_\Phi e^*J_\Phi\eta}$ which is a continuous linear operator of H in $H_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi$, and also a continuous linear operator of H with weak topology in $H_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi$ with weak topology. Then, we take finite sums. □

Until the end of the section, we denote by $\mathcal{H}_\Phi$ the closed linear span in H_Φ of $(\lambda^{\beta,\alpha}_w)^*U_{H_\Psi}(v_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))$ where $v\in H_\Psi$, $w\in J_\Psi\Lambda_\Psi(\mathcal{N}_\Psi\cap\mathcal{N}_{T_R})$, and $a\in(\mathcal{N}_\Psi\cap\mathcal{N}_{T_R})^*\mathcal{N}_\Phi\cap\mathcal{N}_{T_L}$. By the third relation of lemma 3.4.1 (resp. proposition 3.4.2), α (resp. $\hat\beta$) is a non-degenerated (resp. anti-) representation of N on $\mathcal{H}_\Phi$.

Lemma 3.5.8. – *Let $a\in(\mathcal{N}_\Psi\cap\mathcal{N}_{T_R})^*(\mathcal{N}_\Phi\cap\mathcal{N}_T)$, $b\in\mathcal{N}_\Psi\cap\mathcal{N}_{T_R}$, $c\in\mathcal{T}_{\Psi,T_R}$, $v\in D(H_\beta,\mu^o)$ and $(\eta_i)_{i\in I}$ a (N,μ)-basis of ${}_\alpha H$. We put, for all $k\in K$:*

$$\Xi_k=(\sum_{i\in I}\eta_i{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}[(\lambda^{\beta,\alpha}_{\zeta_k})^*U_{H_\Psi}([(\rho^{\beta,\alpha}_{\eta_i})^*U'_H(J_\Psi c^*J_\Psi\Lambda_\Psi(b)_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)]_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))]$$

Then the net $(\Xi_k)_{k\in K}$ is bounded.

Proof. – Let $\Xi=v_\beta\underset{\mu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(c)})^*U_{H_\Psi}(\Lambda_\Psi(b)_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))$. By the previous corollary, we know that $P^e_JU_H\Xi_k$ weakly converges to $P^e_J\Xi$, so that:

$$\lim_{J,||e||\leq 1}\lim_k P^e_JU_H\Xi_k=\Xi$$

Consequently, there exists $C\in\mathbb{R}^+$ such that:

$$\sup_{J,||e||\leq 1}\sup_k||P^e_JU_H\Xi_k||\leq C$$

and, the interversion of the supremum gives:

$$C \geq \sup_k \sup_{J,||e||\leq 1} ||P_J^e U_H \Xi_k|| = \sup_k ||U_H \Xi_k|| = \sup_k ||\Xi_k|| \quad \square$$

COROLLARY 3.5.9. – *For all $a \in (\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})^*(\mathcal{N}_\Phi \cap \mathcal{N}_T)$, $b \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$, $c \in \mathcal{T}_{\Psi,T_R}$, $v \in D(H_\beta, \mu^o)$ and $(\eta_i)_{i\in I}$ a (N,μ)-basis of ${}_\alpha H$, we put:*

$$\Xi_k = (\sum_{i\in I} \eta_i {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} [(\lambda^{\beta,\alpha}_{\zeta_k})^* U_{H_\Psi}([(\rho^{\beta,\alpha}_{\eta_i})^* U'_H(J_\Psi c^* J_\Psi \Lambda_\Psi(b) {}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta v)]{}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi(a))]$$

for all $k \in K$, and:

$$\Xi = v {}_\beta\underset{\mu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{J_\Psi \Lambda_\Psi(c)})^* U_{H_\Psi}(\Lambda_\Psi(b) {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi(a))$$

Then $U_H \Xi_k$ converges to Ξ in the weak topology.

Proof. – Let $\Theta \in H {}_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi$ and $\epsilon > 0$. Then, there exists $e \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ of norm less than equal to 1 and a finite subset J of I such that $||(1 - P_J^e)\Theta|| \leq \epsilon$. By 3.5.7, there also exists k_0 such that $|(P_J^e U_H \Xi_k - P_J^e \Xi|\Theta)| \leq \epsilon$ for all $k \geq k_0$. Then, we get:

$$\begin{aligned} & |(U_H \Xi_k - \Xi|\Theta)| \\ \leq\; & |(U_H \Xi_k - P_J^e U_H \Xi_k|\Theta)| + |(P_J^e U_H \Xi_k - P_J^e \Xi|\Theta)| + |(P_J^e \Xi - \Xi|\Theta)| \\ \leq\; & |(U_H \Xi_k|(1 - P_J^e)\Theta)| + \epsilon + |(\Xi|(1 - P_J^e)\Theta)| \\ \leq\; & |(U_H \Xi_k|(1 - P_J^e)\Theta)| + \epsilon + |(\Xi|(1 - P_J^e)\Theta)| \leq (\sup_{k\in k} ||\Xi_k|| + ||\Xi|| + 1)\epsilon \qquad \square \end{aligned}$$

COROLLARY 3.5.10. – *We have the following inclusion:*

$$H {}_\beta\underset{\mu}{\otimes}{}_\alpha \mathcal{H}_\Phi \subseteq U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi)$$

Proof. – By the previous corollary, we know that Ξ belongs to the weak closure of $U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi)$ which is also the norm closure. Now, U_H is an isometry, that's why $U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi)$ is equal to $U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi)$. $\square$

THEOREM 3.5.11. – *$U_H : H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi \to H {}_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi$ is a unitary.*

Proof. – By the previous corollary, we have:

(1) $$H {}_\beta\underset{\mu}{\otimes}{}_\alpha \mathcal{H}_\Phi \subseteq U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi) \subseteq U_H(H {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi) \subseteq H {}_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi.$$

Also, using a (N^o, μ^o)-basis, we have, for all $v \in H_\Psi$ and $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$:

$$U_{H_\Psi}(v {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\phi(a)) = \sum_i \xi_i {}_\beta\underset{\mu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{\xi_i})^* U_{H_\Psi}(v {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\phi(a))$$

so that $U_{H_\Psi}(H_\Psi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi) \subseteq H_\Psi {}_\beta\underset{\mu}{\otimes}{}_\alpha \mathcal{H}_\Phi$. The reverse inclusion is the relation (1) applied to H_Ψ. Consequently, we get:

$$U_{H_\Psi}(H_\Psi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi) = U_{H_\Psi}(H_\Psi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathcal{H}_\Phi)$$

Since U_{H_Ψ} is an isometry, $H_\Psi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H_\Phi = H_\Psi {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \mathscr{H}_\Phi$ and, so $\mathscr{H}_\Phi = H_\Phi$. Finally, by inclusion (1), we obtain $U_H(H {}_{\hat\alpha}\underset{\mu^o}{\otimes}{}_\beta H_\Phi) = H {}_\beta\underset{\mu}{\otimes}{}_\alpha H_\Phi$. □

DEFINITION 3.5.12. – Fundamental isometry U_H is now called (left) **fundamental unitary**. Right version U'_H is called **right fundamental unitary**.

COROLLARY 3.5.13. – *If $[F]$ denote the linear span of a subset F of a vector space E, we have:*

$$\begin{aligned} H_\Phi &= [\Lambda_\Phi((\omega_{v,w}\, {}_\beta\underset{\mu}{\otimes}{}_\alpha \mathrm{id})(\Gamma(a)))|v, w \in D(H_\beta, \mu^o), a \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}] \\ &= [(\lambda_w^{\beta,\alpha})^* U_H(v\, {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi(a))|v \in H, w \in D(H_\beta, \mu^o), a \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}] \\ &= [(\omega_{v,w} * \mathrm{id})(U_H)\xi | v \in D({}_\alpha H, \mu), w \in D(H_\beta, \mu^o), \xi \in H_\Phi] \end{aligned}$$

Proof. – The second equality comes from 3.3.3. The last one is clear. It's sufficient to prove that the last subspace is equal to H_Φ. Let $\eta \in H_\Phi$ in the orthogonal of:

$$[(\omega_{v,w} * \mathrm{id})(U_H)\xi | v \in D({}_\alpha H, \mu), w \in D(H_\beta, \mu^o), \xi \in H_\Phi]$$

Then, for all $v \in D({}_\alpha H, \mu), w \in D(H_\beta, \mu^o)$ and $\xi \in H_\Phi$, we have:

$$(U_H(v\, {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} \xi) | w\, {}_\beta\underset{\mu}{\otimes}{}_\alpha \eta) = ((\omega_{v,w} * \mathrm{id})(U_H)\xi | \eta) = 0$$

Since U_H is a unitary, $w\, {}_\beta\underset{\mu}{\otimes}{}_\alpha \eta = 0$ for all $w \in D(H_\beta, \mu^o)$ from which we easily deduce that $\eta = 0$ (by 2.3.2 for example). □

COROLLARY 3.5.14. – *We have $\Gamma(m) = U_H(1\, {}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta} m)U_H^*$ for all $m \in M$.*

Proof. – Straightforward thanks to unitary of U_H and 3.4.5. □

3.6. Pseudo-multiplicativity

Let put $W = U_{H_\Phi}^*$. We have already proved commutation relations of section 3.4 and, now the aim is to prove that W is a pseudo-multiplicative unitary in the sense of M. Enock and J.M. Vallin [**EV00**, def. 5.6]:

DEFINITION 3.6.1. – We call **pseudo-multiplicative unitary** over N w.r.t. $\alpha, \hat\beta, \beta$ each unitary V from $H\, {}_\beta\underset{\mu}{\otimes}{}_\alpha H$ onto $H\, {}_\alpha\underset{\mu^o}{\otimes}{}_{\hat\beta} H$ which satisfies the following commutation relations, for all $n, m \in N$:

$$(\beta(n)\, {}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta} \alpha(m))V = V(\alpha(m)\, {}_\beta\underset{N}{\otimes}{}_\alpha \hat\beta(n))$$

and

$$(\hat\beta(n)\, {}_\alpha\underset{N^o}{\otimes}{}_{\hat\beta} \beta(m))V = V(\hat\beta(n)\, {}_\beta\underset{N}{\otimes}{}_\alpha \beta(m))$$

and the formula:

$$(V_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)(\sigma_{\mu^o}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}V)\sigma_{2\mu}(1_{\beta}\underset{N}{\otimes}{}_{\alpha}\sigma_{\mu^o})(1_{\beta}\underset{N}{\otimes}{}_{\alpha}V) =$$

$$(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}V)(V_{\beta}\underset{N}{\otimes}{}_{\alpha}1)$$

where the first σ_{μ^o} is the flip from $H_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}H$ onto $H_{\hat{\beta}}\underset{\mu}{\otimes}{}_{\alpha}H$, the second is the flip from $H_{\alpha}\underset{\mu^o}{\otimes}{}_{\beta}H$ onto $H_{\beta}\underset{\mu}{\otimes}{}_{\alpha}H$ and $\sigma_{2\mu}$ is the flip from $H_{\beta}\underset{\mu}{\otimes}{}_{\alpha}H_{\hat{\beta}}\underset{\mu}{\otimes}{}_{\alpha}H$ onto $H_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}(H_{\beta}\underset{\mu}{\otimes}{}_{\alpha}H)$. This last flip turns around the second tensor product. Moreover, parenthesis underline the fact that the representation acts on the furthest leg.

We recall, following [**Eno02**, 3.5], if we use an other n.s.f. weight for the construction of relative tensor product, then canonical isomorphisms of bimodules change the pseudo-multiplicative unitary into another pseudo-multiplicative unitary. The pentagonal relation is essentially the expression of the co-product relation. So, we compute $(\mathrm{id}_{\beta}\underset{N}{\star}{}_{\alpha}\Gamma)\circ\Gamma$ and $(\Gamma_{\beta}\underset{N}{\star}{}_{\alpha}\mathrm{id})\circ\Gamma$ in terms of U_H with the following propositions 3.6.4 and 3.6.6. Until the end of the section, $\mathcal{H}$ is an other Hilbert space on which M acts.

LEMMA 3.6.2. – *We have, for all $\xi_1\in D({}_{\alpha}\mathcal{H},\mu)$ and $\xi_2'\in D(H_{\beta},\mu^o)$:*

$$\lambda_{\xi_1}^{\alpha,\hat{\beta}}(\lambda_{\xi_2'}^{\beta,\alpha})^* = (\lambda_{\xi_2'}^{\beta,\alpha})^*\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_{\mu})\lambda_{\xi_1}^{\alpha,\hat{\beta}}$$

and:

$$U_{\mathcal{H}}\lambda_{\xi_1}^{\alpha,\hat{\beta}}(\lambda_{\xi_2'}^{\beta,\alpha})^*U_H = (\lambda_{\xi_2'}^{\beta,\alpha})^*(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathcal{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_{\mu})(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)\lambda_{\xi_1}^{\alpha,\beta}$$

Proof. – The first equality is easy to verify and the second one comes from the first one. □

PROPOSITION 3.6.3. – *The two following equations hold:*

i) *for all $\xi_1\in D({}_{\alpha}\mathcal{H},\mu),\xi_1'\in D({}_{\alpha}H,\mu),\xi_2\in D(\mathcal{H}_{\beta},\mu^o),\xi_2'\in D(H_{\beta},\mu^o)$ and $\eta_1,\eta_2\in H_{\Phi}$, the scalar product of:*

$$(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathcal{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_{\mu})(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)(\sigma_{\mu}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)([\xi_1'{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\xi_1]_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\eta_1)$$

by $\xi_2'{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\xi_2{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\eta_2$ is equal to $((\omega_{\xi_1,\xi_2}*\mathrm{id})(U_{\mathcal{H}})(\omega_{\xi_1',\xi_2'}*\mathrm{id})(U_H)\eta_1|\eta_2)$.

ii) *for all $a\in\mathcal{N}_{\Phi}\cap\mathcal{N}_{T_L}$, $\xi_1\in\mathcal{H}$ and $\xi_1',\xi_2'\in D(H_{\beta},\mu^o)$, the value of:*

$$(\lambda_{\xi_2'}^{\beta,\alpha})^*(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathcal{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_{\mu})(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)(\sigma_{\mu}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)$$

on $[\xi_1'{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\xi_1]_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(a)$ is equal to:

$$U_{\mathcal{H}}(\xi_1{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}((\omega_{\xi_1',\xi_2'}{}_{\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a))))$$

Proof. – By the previous lemma, we can compute the scalar product of i) in the following way:

$$((\lambda^{\beta,\alpha}_{\xi'_2})^*(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathscr{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_\mu)(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)\lambda^{\alpha,\beta}_{\xi_1}(\xi'_1{}_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\eta_1)|\xi_2{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\eta_2)$$
$$= (U_{\mathscr{H}}\lambda^{\alpha,\hat{\beta}}_{\xi_1}(\lambda^{\beta,\alpha}_{\xi'_2})^*U_H(\xi'_1{}_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\eta_1)|\xi_2{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\eta_2)$$
$$= ((\lambda^{\beta,\alpha}_{\xi_2})^*U_{\mathscr{H}}(\xi_1{}_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}(\omega_{\xi'_1,\xi'_2} * \mathrm{id})(U_H)\eta_1|\eta_2)$$
$$= ((\omega_{\xi_1,\xi_2} * \mathrm{id})(U_{\mathscr{H}})(\omega_{\xi'_1,\xi'_2} * \mathrm{id})(U_H)\eta_1|\eta_2)$$

Also, the second assertion comes from the previous lemma and 3.3.2. Let's first assume that $\xi_1 \in D({}_\alpha\mathscr{H},\mu)$. Then, we compute the vector in demand:

$$(\lambda^{\beta,\alpha}_{\xi'_2})^*(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathscr{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_\mu)(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)\lambda^{\alpha,\beta}_{\xi_1}(\xi'_1{}_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\Lambda_\Phi(a))$$
$$= U_{\mathscr{H}}\lambda^{\alpha,\hat{\beta}}_{\xi_1}(\lambda^{\beta,\alpha}_{\xi'_2})^*U_H(\xi'_1{}_{\alpha}\underset{\mu^o}{\otimes}{}_{\hat{\beta}}\Lambda_\Phi(a))$$
$$= U_{\mathscr{H}}(\xi_1{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\Lambda_\Phi((\omega_{\xi'_1,\xi'_2}{}_{\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a))))$$

So, we obtain the expected equality for all $\xi_1 \in D({}_\alpha\mathscr{H},\mu)$. Since the two expressions are continuous in ξ_1, density of $D({}_\alpha\mathscr{H},\mu)$ in $\mathscr{H}$ implies that the equality is still true for all $\xi_1 \in \mathscr{H}$. □

PROPOSITION 3.6.4. – *For all $a, b \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$, we have:*

$$(\mathrm{id}_{\beta}\underset{\mu}{\star}{}_{\alpha}\Gamma)(\Gamma(a))\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)}$$
$$= (1_{\beta}\underset{N}{\otimes}{}_{\alpha}(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_\Phi bJ_\Phi)U_{\mathscr{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_\mu)(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)(\sigma_{\mu}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)\rho^{\alpha,\hat{\beta}}_{\Lambda_\Phi(a)}$$

Proof. – Let $\xi_1 \in \mathscr{H}$ and $\xi'_1, \xi'_2 \in D(H_\beta, \mu^o)$. We compose the second term of the equality on the left by $(\lambda^{\beta,\alpha}_{\xi'_2})^*$ and we get:

$$(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_\Phi bJ_\Phi)(\lambda^{\beta,\alpha}_{\xi'_2})^*(1_{\beta}\underset{N}{\otimes}{}_{\alpha}U_{\mathscr{H}})\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\sigma_\mu)(1_{\alpha}\underset{N^o}{\otimes}{}_{\beta}U_H)(\sigma_{\mu}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)\rho^{\alpha,\hat{\beta}}_{\Lambda_\Phi(a)}$$

which we evaluate on $\xi'_1{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\xi_1$, to get, by the previous proposition and 3.3.1:

$$(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_\Phi bJ_\Phi)U_{\mathscr{H}}(\xi_1{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}\Lambda_\Phi((\omega_{\xi'_1,\xi'_2}{}_{\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a))))$$
$$= \Gamma((\omega_{\xi'_1,\xi'_2}{}_{\beta}\underset{\mu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(a)))\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)}\xi_1$$
$$= (\lambda^{\beta,\alpha}_{\xi'_2})^*(\mathrm{id}_{\beta}\underset{\mu}{\star}{}_{\alpha}\Gamma)(\Gamma(a))\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)}(\xi'_1{}_{\beta}\underset{\mu}{\otimes}{}_{\alpha}\xi_1)$$

So, the proposition holds. □

LEMMA 3.6.5. – *For all $X \in M_{\beta}\underset{N}{\star}{}_{\alpha}M \subset (1_{\beta}\underset{N}{\otimes}{}_{\alpha}\hat{\beta}(N))'$, we have:*

$$(\Gamma_{\beta}\underset{N}{\star}{}_{\alpha}\mathrm{id})(X) = (U_H{}_{\beta}\underset{N}{\otimes}{}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}X)(U^*_H{}_{\beta}\underset{N}{\otimes}{}_{\alpha}1)$$

Proof. – By 3.5.14, Γ is implemented by U_H so that we easily deduce the lemma. □

PROPOSITION 3.6.6. – *For all $a, b \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, we have:*

$$(\Gamma_{\beta}\underset{N}{\star}_{\alpha}\mathrm{id})(\Gamma(a))\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)}$$
$$= (1_{\beta}\underset{N}{\otimes}_{\alpha}1_{\beta}\underset{N}{\otimes}_{\alpha}J_\Phi bJ_\Phi)(U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}W^*)(U^*_{H\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)\rho^{\alpha,\hat\beta}_{\Lambda_\Phi(a)}$$

Proof. – By the previous lemma and 3.3.1, we can compute:

$$(1_{\beta}\underset{N}{\otimes}_{\alpha}1_{\beta}\underset{N}{\otimes}_{\alpha}J_\Phi bJ_\Phi)(U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}W^*)(U^*_{H\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)\rho^{\alpha,\hat\beta}_{\Lambda_\Phi(a)}$$
$$= (U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}1_{\beta}\underset{N}{\otimes}_{\alpha}J_\Phi bJ_\Phi)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}W^*)\rho^{\alpha,\hat\beta}_{\Lambda_\Phi(a)}U^*_H$$
$$= (U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}(1_{\beta}\underset{N}{\otimes}_{\alpha}J_\Phi bJ_\Phi)W^*\rho^{\alpha,\hat\beta}_{\Lambda_\Phi(a)})U^*_H$$
$$= (U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}\Gamma(a)\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)})U^*_H$$
$$= (U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}\Gamma(a))(U^*_{H\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)} = (\Gamma_{\beta}\underset{N}{\star}_{\alpha}\mathrm{id})(\Gamma(a))\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(b)}$$ □

COROLLARY 3.6.7. – *The following relation is satisfied:*

$$(U^*_{H\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)(\sigma_{\mu^o\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}U^*_H)\sigma_{2\mu}(1_{\beta}\underset{N}{\otimes}_{\alpha}\sigma_{\mu^o})(1_{\beta}\underset{N}{\otimes}_{\alpha}W)$$
$$= (1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}W)(U^*_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)$$

Proof. – We put together 3.6.4 (with $\mathcal{H} = H_\Phi$) and 3.6.6 thanks to the co-product relation. We get:

$$(1_{\beta}\underset{N}{\otimes}_{\alpha}W^*)\sigma_{2\mu^o}(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}\sigma_\mu)(1_{\alpha}\underset{N^o}{\otimes}_{\beta}U_H)$$
$$= (U_{H\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\alpha}\underset{N^o}{\otimes}_{\hat\beta}W^*)(U^*_{H\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)(\sigma_{\mu^o\,\alpha}\underset{N^o}{\otimes}_{\hat\beta}1)$$

Take adjoint and we are. □

THEOREM 3.6.8. – *W is a pseudo-multiplicative unitary over N w.r.t. $\alpha, \hat\beta, \beta$.*

Proof. – W is a unitary from $H_{\Phi\,\beta}\underset{\mu}{\otimes}_{\alpha}H_\Phi$ onto $H_{\Phi\,\alpha}\underset{\mu^o}{\otimes}_{\hat\beta}H_\Phi$ which satisfies the four required commutation relations. The previous corollary, with $H = H_\Phi$, finishes the proof. □

Similar results hold for the right version:

THEOREM 3.6.9. – *If $W' = U'_{H_\Psi}$, then the following relation makes sense and holds:*

$$(W'_{\beta}\underset{N}{\otimes}_{\alpha}1)(\sigma_{\mu\,\beta}\underset{N}{\otimes}_{\alpha}1)(1_{\beta}\underset{N}{\otimes}_{\alpha}U'_H)\sigma_{2\mu^o}(1_{\hat\alpha}\underset{N^o}{\otimes}_{\beta}\sigma_\mu)(1_{\hat\alpha}\underset{N^o}{\otimes}_{\beta}U'_H)$$
$$= (1_{\beta}\underset{N}{\otimes}_{\alpha}U'_H)(W'_{\hat\alpha}\underset{N^o}{\otimes}_{\beta}1)$$

If $H = H_\Psi$, then W' is a pseudo-multiplicative unitary over N^o w.r.t. $\beta, \alpha, \hat\alpha$.

Proof. – For example, it is sufficient to apply the previous results with the opposite Hopf bimodule. □

3.7. Right leg of the fundamental unitary

In the von Neumann setting of the theory of locally compact quantum groups, it is well-known (see [**KV03**]) that we can recover M from the right leg of the fundamental unitary. In this paragraph, we prove the first result in that direction in our setting.

DEFINITION 3.7.1. – We call $A(U'_H)$ (resp. $\mathcal{A}(U'_H)$) the weak closure in $\mathcal{L}(H)$ of the vector space (resp. von Neumann algebra) generated by $(\omega_{v,w} * \mathrm{id})(U'_H)$ with $v \in D({}_{\hat{\alpha}}(H_\Psi), \mu)$ and $w \in D((H_\Psi)_\beta, \mu^o)$.

PROPOSITION 3.7.2. – *$A(U'_H)$ is a non-degenerate involutive algebra* i.e., $A(U'_H) = \mathcal{A}(U'_H)$ *such that:*

$$\alpha(N) \cup \beta(N) \subseteq A(U'_H) = \mathcal{A}(U'_H) \subseteq M \subseteq \hat{\alpha}(N)'$$

Moreover, we have:

$$x \in \mathcal{A}(U'_H)' \cap \mathcal{L}(H) \Longleftrightarrow U'_H(1_{\hat{\alpha}} \underset{N^o}{\otimes}{}_\beta x) = (1_\beta \underset{N}{\otimes}{}_\alpha x)U'_H$$

In fact, we will see later that $A(U'_H) = \mathcal{A}(U'_H) = M$.

Proof. – The second and third points are obtained in [**EV00**, thm. 6.1]. As far as the first point is concerned, it comes from [**Eno02**, prop. 3.6] and 3.3.5 which proves that $A(U'_H)$ is involutive. □

To summarize the results of this section, we state the following theorem:

THEOREM 3.7.3. – *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule, T_L (resp. T_R) be a left (resp. right) invariant n.s.f. operator-valued weight. Then, for all n.s.f. weight μ on N, if $\Phi = \mu \circ \alpha^{-1} \circ T_L$, then the application:*

$$v_\alpha \underset{\mu^o}{\otimes}{}_{\hat{\beta}} \Lambda_\Phi(a) \longmapsto \sum_{i \in I} \xi_{i\,\beta} \underset{\mu}{\otimes}{}_\alpha \Lambda_\Phi((\omega_{v,\xi_i\,\beta} \underset{\mu}{\star}{}_\alpha \mathrm{id})(\Gamma(a)))$$

for all $v \in D((H_\Phi)_\beta, \mu^o)$, $a \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$, (N^o, μ^o)-basis $(\xi_i)_{i \in I}$ of $(H_\Phi)_\beta$ and where $\hat{\beta}(n) = J_\Phi \alpha(n^*) J_\Phi$, extends to a unitary W, the adjoint of which W^* is a pseudo-multiplicative unitary over N w.r.t. $\alpha, \hat{\beta}, \beta$ from $H_{\Phi\,\alpha} \underset{\mu^o}{\otimes}{}_{\hat{\beta}} H_\Phi$ onto $H_{\Phi\,\beta} \underset{\mu}{\otimes}{}_\alpha H_\Phi$. Moreover, for all $m \in M$, we have:

$$\Gamma(m) = W^*(1_\alpha \underset{N^o}{\otimes}{}_{\hat{\beta}} m)W$$

Also, we have similar results from T_R.

We also add a key relation between Γ and the fundamental unitary proved in corollary 3.3.4:

THEOREM 3.7.4. – *For all $e, x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $\eta \in D({}_\alpha H_\Phi, \mu^o)$, we have:*

$$(\mathrm{id}_\beta \underset{\mu}{\star}_\alpha \omega_{J_\Phi \Lambda_\Phi(e),\eta})(\Gamma(x)) = (\mathrm{id} * \omega_{\Lambda_\Phi(x), J_\Phi e^* J_\Phi \eta})(U_H)$$

Also, for all $f, y \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ and $\xi \in D((H_\Psi)_\beta, \mu^o)$, we have:

$$(\omega_{J_\Psi \Lambda_\Psi(f),\xi\, \beta} \underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(y)) = (\omega_{\Lambda_\Psi(y), J_\Psi f^* J_\Psi \xi} * \mathrm{id})(U'_H)$$

PART I

MEASURED QUANTUM GROUPOIDS

In this part, we propose a definition for measured quantum groupoids from which we can develop a full theory that is we construct all expected natural objects, then we perform a dual structure within the category and we also get a duality theorem which extends duality for locally compact quantum groups. Two main ideas are used in this theory. First of all, we use axioms of Masuda-Nakagami-Woronowicz's type: we assume the existence of the antipode defined by its polar decomposition. On the other hand, we introduce a rather weak condition on the modular group of the invariant operator-valued weight. Then we can proceed and we get all known examples as we will see in the second part.

CHAPTER 4

DEFINITION

In the following, $(N, M, \alpha, \beta, \Gamma)$ denotes a Hopf-bimodule. Like in the quantum group case (for example [**KV00**] or [**MNW03**]), we assume that there exist a normal semi-finite and faithful (nsf) left invariant operator-valued weight T_L. We also assume that we have an antipode. Precisely, like in [**MNW03**], we require the existence of a co-involution R of M and a scaling operator τ (deformation operator) which will lead to polar decomposition of the antipode. Axioms we choose for them are well known properties at the quantum groups level. They are quite symmetric, easy to express and adapted to our developments. They give a link between R, τ and the co-product Γ. They stand for strong invariance and relative invariance of the weight in [**MNW03**]. Finally, we add a modular condition on the basis coming from inclusions of von Neumann algebras. The idea is that we have to choose a weight on the basis N to proceed constructions. That is also the case for usual groupoids (see [**Ren80, Val96**] and also section 10).

Definition 4.0.5. – We call $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ a **measured quantum groupoid** if $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule equipped with a nsf left invariant operator-valued weight T_L from M to $\alpha(N)$, a co-involution R of M, a one-parameter group of automorphisms τ of M and a nsf weight ν on N such that, for all $t \in \mathbb{R}$ and $a, b \in \mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$:

$$R((\mathrm{id}_{\beta} \underset{\nu}{\star}_{\alpha} \omega_{J_\Phi \Lambda_\Phi(a)})\Gamma(b^* b)) = (\mathrm{id}_{\beta} \underset{\nu}{\star}_{\alpha} \omega_{J_\Phi \Lambda_\Phi(b)})\Gamma(a^* a)$$

$$\text{and } \tau_t((\mathrm{id}_{\beta} \underset{\nu}{\star}_{\alpha} \omega_{J_\Phi \Lambda_\Phi(a)})\Gamma(b^* b)) = (\mathrm{id}_{\beta} \underset{\nu}{\star}_{\alpha} \omega_{J_\Phi \Lambda_\Phi(\sigma_t^\Phi(a))})\Gamma(\sigma_t^\Phi(b^* b))$$

where $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and such that:

$$\nu \circ \gamma_t = \nu$$

where γ is the unique one-parameter group of automorphisms γ of N satisfying for all $n \in N, t \in \mathbb{R}$:

$$\sigma_t^{T_L}(\beta(n)) = \beta(\gamma_t(n))$$

We recall that the Hopf-bimodule does also admit a nsf right invariant operator-valued weight $T_R = R \circ T_L \circ R$. The rest of the section is devoted to develop several points of the definition and clarify from where γ comes from. Thanks to relation concerning τ, we easily get that:

$$\tau_t \circ \beta = \beta \circ \sigma_t^\nu \quad \text{and} \quad (\tau_t {}_\beta\underset{N}{\star}{}_\alpha \sigma_t^\Phi) \circ \Gamma = \Gamma \circ \sigma_t^\Phi$$

for all $n \in N$ and $t \in \mathbb{R}$ (For the first one, make b goes to 1). The first equality give the behavior τ should have on the basis. In fact, it is necessary, if we want to give a meaning to $\tau_t {}_\beta\underset{N}{\star}{}_\alpha \sigma_t^\Phi$. The last relation is usual in the theory of locally compact quantum groups. Then, we can explain how to recover M from Γ:

Theorem 4.0.6. – *If $\langle F \rangle^{-\mathrm{w}}$ is the weakly closed linear span of F in M, then we have:*

$$\begin{aligned} M &= \langle (\omega {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(m)) \mid m \in M, \omega \in M_*^+, k \in \mathbb{R}^+ \text{ s.t. } \omega \circ \beta \leq k\nu \rangle^{-\mathrm{w}} \\ &= \langle (\mathrm{id} {}_\beta\underset{\nu}{\star}{}_\alpha \omega)(\Gamma(m)) \mid m \in M, \omega \in M_*^+, k \in \mathbb{R}^+ \text{ s.t. } \omega \circ \alpha \leq k\nu \rangle^{-\mathrm{w}} \end{aligned}$$

Proof. – Let call M_R the first subspace of M and M_L the second one. Since $\tau_t(\beta(n)) = \beta(\sigma_t^\nu(n))$ for all $t \in \mathbb{R}$, we have:

$$M_R = \langle (\omega \circ \tau_t {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(m)) \mid m \in M, \omega \in (M_R)_*^+, k \in \mathbb{R}^+ \text{ s.t. } \omega \circ \beta \leq k\nu \rangle^{-\mathrm{w}}$$

Moreover we have $\sigma_t^\Phi((\omega {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})\Gamma(m)) = (\omega \circ \tau_t {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})\Gamma(\sigma_t^\Phi(m))$ so that $\sigma_t^\Phi(M_R) = M_R$ for all $t \in \mathbb{R}$. On the other hand, by proposition 3.2.7, restriction of Φ to M_R is semi-finite. By Takesaki's theorem [**Str81**, thm. 10.1], there exists a unique normal and faithful conditional expectation E from M to M_R such that $\Phi(m) = \Phi(E(m))$ for all $m \in M^+$. Moreover, if P is the orthogonal projection on the closure of $\Lambda_\Phi(\mathcal{N}_\Phi \cap M_R)$ then $E(m)P = PmP$.

So the range of P contains $\Lambda_\Phi((\omega {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})\Gamma(x))$ for all ω and $x \in \mathcal{N}_\Phi$. By proposition 3.5.13 implies that $P = 1$ so that E is the identity and $M = M_R$. Now, it is clear that $R(M_R) = M_L$ thanks to co-involution property what completes the proof. □

The theorem enables us to understand that formulas satisfied by R and τ in the definition are sufficient to determine them. For example, we can be ensured of the commutation between R and τ which can be tested on elements of the form $(\mathrm{id} {}_\beta\underset{\nu}{\star}{}_\alpha \omega_{J_\Phi \Lambda_\Phi(a)})\Gamma(b^*b)$. Also, if we put $\Psi = \nu \circ \beta^{-1} \circ T_R = \Phi \circ R$, we get, for all $t \in \mathbb{R}$:

$$\sigma_t^\Psi = R \circ \sigma_{-t}^\Phi \circ R \quad \text{and} \quad \tau_t \circ \alpha = \alpha \circ \sigma_t^\nu \quad \text{and} \quad (\sigma_t^\Psi {}_\beta\underset{N}{\star}{}_\alpha \tau_{-t}) \circ \Gamma = \Gamma \circ \sigma_t^\Psi$$

Then, we can precise the behavior of τ with respect to the Hopf-bimodule structure:

Proposition 4.0.7. – *We have $\Gamma \circ \tau_t = (\tau_t {}_\beta\underset{N}{\star}{}_\alpha \tau_t) \circ \Gamma$ for all $t \in \mathbb{R}$.*

Proof. – Because of the behavior of τ on the basis, it is possible to define a normal *-automorphism $\tau_t {}_\beta\underset{N}{\star}{}_\alpha \tau_t$ of $M {}_\beta\underset{N}{\star}{}_\alpha M$ which naturally acts for all $t \in \mathbb{R}$. By co-product relation, we have for all $t \in \mathbb{R}$:

$$\begin{aligned}(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\Gamma)(\sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t}) \circ \Gamma &= (\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\Gamma)\Gamma \circ \sigma_t^\Psi \\ &= (\Gamma_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma \circ \sigma_t^\Psi = (\Gamma \circ \sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t})\Gamma \\ &= (\sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t} {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t})(\Gamma_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma \\ &= (\sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha [(\tau_{-t} {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t}) \circ \Gamma]) \circ \Gamma\end{aligned}$$

Consequently, for all $m \in M$, $\omega \in M_*^+$, $k \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k\nu$, we have:

$$\begin{aligned}\Gamma \circ \tau_{-t} \circ ((\omega \circ \sigma_t^\Psi)_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma &= (\omega_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})(\sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha (\Gamma \circ \tau_{-t})) \circ \Gamma \\ &= (\omega_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})(\sigma_t^\Psi {}_\beta\underset{\nu}{\star}{}_\alpha [(\tau_{-t} {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t}) \circ \Gamma]) \\ &= [(\tau_{-t} {}_\beta\underset{\nu}{\star}{}_\alpha \tau_{-t}) \circ \Gamma] \circ ((\omega \circ \sigma_t^\Psi)_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma\end{aligned}$$

The theorem 4.0.6 allows us to conclude. □

Then, we get a nice and useful characterization of elements of the basis thanks to Γ:

PROPOSITION 4.0.8. – *For all $x \in M \cap \alpha(N)'$, we have $\Gamma(x) = 1_\beta\underset{N}{\otimes}{}_\alpha x \Leftrightarrow x \in \beta(N)$. Also we have, for all $x \in M \cap \beta(N)'$, $\Gamma(x) = x_\beta\underset{N}{\otimes}{}_\alpha 1 \Leftrightarrow x \in \alpha(N)$.*

Proof. – Let $x \in M \cap \alpha(N)'$ such that $\Gamma(x) = 1_\beta\underset{N}{\otimes}{}_\alpha x$. For all $n \in \mathbb{N}$, we define in the strong topology:

$$x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\sigma_t^\Psi(x)\, dt \quad \text{analytic w.r.t. } \sigma^\Psi,$$

and:

$$y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\tau_{-t}(x)\, dt \quad \text{belongs to } \alpha(N)'.$$

Then we have $\Gamma(x_n) = 1_\beta\underset{N}{\otimes}{}_\alpha y_n$. If $d \in (\mathcal{M}_\Psi \cap \mathcal{M}_{T_R})^+$, then, for all $n \in \mathbb{N}$, we have $dx_n \in \mathcal{M}_\Psi \cap \mathcal{M}_{T_R}$. Let $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$. By right invariance, we get:

$$\begin{aligned}\omega \circ T_R(dx_n) &= \omega((\Psi_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(dx_n))) \\ &= \Psi((\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega)(\Gamma(dx_n))) = \Psi((\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha(y_n\omega))(\Gamma(d))) \\ &= \omega((\Psi_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(d))y_n) = \omega(T_R(d)y_n)\end{aligned}$$

Take the limit over $n \in \mathbb{N}$ to obtain $T_R(dx) = T_R(d)x$ for all $d \in \mathcal{M}_\Psi \cap \mathcal{M}_{T_R}$ and, by semi-finiteness of T_R, we conclude that x belongs to $\beta(N)$. Reverse inclusion comes from axioms. If we apply this result to the opposite Hopf-bimodule, then we get the second point. □

Finally, we are able to explain existence and uniqueness of γ for the definition:

PROPOSITION 4.0.9. – *There exists a unique one-parameter group of automorphisms γ of N such that:*

$$\sigma_t^{T_L}(\beta(n)) = \beta(\gamma_t(n))$$

for all $n \in N$ and $t \in \mathbb{R}$.

Proof. – For all $n \in N$ and $t \in \mathbb{R}$, we have $\sigma_t^\Phi(\beta(n))$ belongs to $M \cap \alpha(N)'$. Then, we can compute:

$$\begin{aligned}\Gamma \circ \sigma_t^\Phi(\beta(n)) &= (\tau_t {}_\beta\underset{N}{\star}{}_\alpha \sigma_t^\Phi) \circ \Gamma(\beta(n)) \\ &= (\tau_t {}_\beta\underset{N}{\star}{}_\alpha \sigma_t^\Phi)(1 {}_\beta\underset{N}{\otimes}{}_\alpha \beta(n)) = 1 {}_\beta\underset{N}{\otimes}{}_\alpha \sigma_t^\Phi(\beta(n))\end{aligned}$$

By the previous proposition, we deduce that $\sigma_t^\Phi(\beta(n))$ belongs to $\beta(N)$ *i.e.*, there exists a unique element $\gamma_t(n)$ in N such that $\sigma_t^\Phi(\beta(n)) = \beta(\gamma_t(n))$. The rest of the proof is straightforward. □

In our definition, we ask γ to leave invariant ν. Just before investigating the structure of these objects, we re-formulate at the Hilbert level relations for R and τ with U_H (or W) coming from theorem 3.7.3. Depending on the situation, we will use one or the other expression.

PROPOSITION 4.0.10. – *Let I be a unitary anti-linear operator which implements R that is $R(m) = Im^*I$ for all $m \in M$ and P be a strictly positive operator which implements τ that is $\tau_t(m) = P^{-it}mP^{it}$ for all $m \in M$ and $t \in \mathbb{R}$. For all $t \in \mathbb{R}$ and $v, w \in D({}_\alpha H_\Phi, \nu)$, we have:*

$$R((\mathrm{id} * \omega_{J_\Phi v, w})(U_H)) = (\mathrm{id} * \omega_{J_\Phi w, v})(U_H)$$

$$\tau_t((\mathrm{id} * \omega_{J_\Phi v, w})(U_H)) = (\mathrm{id} * \omega_{\Delta_\Phi^{-it} J_\Phi v, \Delta_\Phi^{-it} w})(U_H)$$

$$(I {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} J_\Phi) U_H^* = U_H (I {}_\beta\underset{\nu}{\otimes}{}_\alpha J_\Phi) \text{ and } (P^{it} {}_\beta\underset{\nu}{\otimes}{}_\alpha \Delta_\Phi^{it}) U_H = U_H (P^{it} {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} \Delta_\Phi^{it})$$

$$\varsigma_{N^o} \circ (R {}_\beta\underset{N}{\star}{}_\alpha R) \circ \Gamma = \Gamma \circ R \text{ and } (\tau_t {}_\beta\underset{N}{\star}{}_\alpha \sigma_t^\Phi) \circ \Gamma = \Gamma \circ \sigma_t^\Phi$$

Proof. – By theorem 3.7.4, for all $e, x \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $\eta \in D({}_\alpha H_\Phi, \mu^o)$, we recall that:

$$(\mathrm{id} {}_\beta\underset{\nu}{\star}{}_\alpha \omega_{J_\Phi \Lambda_\Phi(e), \eta})(\Gamma(x)) = (\mathrm{id} * \omega_{\Lambda_\Phi(x), J_\Phi e^* J_\Phi \eta})(U_H)$$

Then the first two equalities are equivalent to formulas of the definition and we get straightforward the equalities at the Hilbert level. The last ones come from definition. □

CHAPTER 5

UNIQUENESS, MODULUS AND SCALING OPERATOR

In this section, we obtain results about the modular theory of the left-invariant operator-valued weight. We construct a scaling operator and a modulus which link the left invariant operator-valued weight T_L and the right invariant operator-valued weight $R \circ T_L \circ R$. We also prove that the modulus is a co-character. We also establish uniqueness of the invariant operator-valued weight.

5.1. Definitions of modulus and scaling operators

PROPOSITION 5.1.1

For all $t \in \mathbb{R}$, we have:

(1) $\Gamma \circ \sigma_t^\Phi \tau_{-t} = (\mathrm{id}_{\beta} \underset{N}{\star}{}_{\alpha} \sigma_t^\Phi \tau_{-t}) \circ \Gamma$

(2) $R \circ T_L \circ R \circ \sigma_t^\Phi \tau_{-t} = \beta \circ \gamma_t \sigma_{-t}^\nu \circ \beta^{-1} \circ R \circ T_L \circ R$

(3) $\Phi \circ R \circ \sigma_t^\Phi \tau_{-t} = \Phi \circ R$

Proof. – For all $n \in N$ and $t \in \mathbb{R}$, we have:

$$\sigma_t^\Phi \tau_{-t}(\alpha(n)) = \sigma_t^\Phi(\alpha(\sigma_{-t}^\nu(n))) = \alpha(n)$$

so that we can define $\mathrm{id}_{\beta} \underset{N}{\star}{}_{\alpha} \sigma_t^\Phi \tau_{-t}$. Then, the first statement comes straightforward from definition property of τ and by proposition 4.0.7.

By right invariance of T_R, we deduce, for all $a \in \mathcal{M}_{T_R}^+$:

$$\begin{aligned} T_R \circ \sigma_t^\Phi \tau_{-t}(a) &= (\Psi_{\beta} \underset{\nu}{\star}{}_{\alpha} \mathrm{id})\Gamma(\sigma_t^\Phi \tau_{-t}(a)) \\ &= \sigma_t^\Phi \tau_{-t}((\Psi_{\beta} \underset{\nu}{\star}{}_{\alpha} \mathrm{id})\Gamma(a)) = \sigma_t^\Phi \tau_{-t} \circ T_R(a) \end{aligned}$$

Then, by hypothesis on τ and T_L, we get:

$$T_R \circ \sigma_t^\Phi \tau_{-t} = \sigma_t^\Phi \tau_{-t} \circ \beta \circ \beta^{-1} \circ T_R = \sigma_t^\Phi \circ \beta \circ \sigma_{-t}^\nu \circ \beta^{-1} \circ T_R = \beta \circ \gamma_t \sigma_{-t}^\nu \circ \beta^{-1} \circ T_R$$

To conclude we just have to take $\nu \circ \beta^{-1}$ on the previous relation and use invariance property of σ^ν and γ w.r.t. ν. □

Proposition 5.1.2. – *The one-parameter groups of automorphisms σ^Φ and τ (resp. σ^Ψ and τ) commute each other.*

Proof. – We put $\kappa_t = \gamma_t \sigma^\nu_{-t}$. Since Ψ is κ-invariant, we have $\sigma^\Psi_s \circ \sigma^\Phi_t \circ \tau_{-t} = \sigma^\Phi_t \circ \tau_{-t} \circ \sigma^\Psi_s$, for all $s, t \in \mathbb{R}$ so that:

$$\begin{aligned}(\mathrm{id}_{\beta}\underset{N}{\star}{}_\alpha \kappa_t)\Gamma = \Gamma \circ \kappa_t = \Gamma \circ \sigma^\Psi_{-s} \circ \kappa_t \circ \sigma^\Psi_s &= (\sigma^\Psi_{-s}{}_\beta\underset{N}{\star}{}_\alpha \tau_s) \circ \Gamma \circ \kappa_t \circ \sigma^\Psi_s \\ &= (\sigma^\Psi_{-s}{}_\beta\underset{N}{\star}{}_\alpha \tau_s \circ \kappa_t) \circ \Gamma \circ \sigma^\Psi_s = (\mathrm{id}_\beta\underset{N}{\star}{}_\alpha \tau_s \circ \kappa_t \circ \tau_{-s}) \circ \Gamma\end{aligned}$$

So, for all $a \in M$, $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k\nu$, we get:

$$\sigma^\Phi_t \circ \tau_{-t}((\omega_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})\Gamma(a)) = \tau_s \circ \sigma^\Phi_t \circ \tau_{-t} \circ \tau_{-s}((\omega_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})\Gamma(a))$$

and by theorem 4.0.6, we easily obtain commutation between σ^Φ and τ. By applying the co-involution R to this commutation relation, we end the proof. □

Corollary 5.1.3. – *The one-parameter groups of automorphisms σ^Φ and σ^Ψ commute each other.*

Proof. – By the previous proposition, we compute, for all $s, t \in \mathbb{R}$:

$$\begin{aligned}\Gamma \circ \sigma^\Phi_s \circ \sigma^\Psi_t = (\tau_s{}_\beta\underset{N}{\star}{}_\alpha \sigma^\Phi_s) \circ \Gamma \circ \sigma^\Psi_t &= (\tau_s \sigma^\Psi_t{}_\beta\underset{N}{\star}{}_\alpha \sigma^\Phi_s \tau_{-t}) \circ \Gamma \\ &= (\sigma^\Psi_t \tau_s{}_\beta\underset{N}{\star}{}_\alpha \tau_{-t} \sigma^\Phi_s) \circ \Gamma \\ &= (\sigma^\Psi_t{}_\beta\underset{N}{\star}{}_\alpha \tau_{-t}) \circ \Gamma \circ \sigma^\Phi_s = \Gamma \circ \sigma^\Psi_t \circ \sigma^\Phi_s\end{aligned}$$

Since Γ is injective, we have done. □

By the previous proposition and by [**Vae01a**, prop. 2.5], there exist a strictly positive operator δ affiliated with M and a strictly positive operator λ affiliated to the center of M such that, for all $t \in \mathbb{R}$, we have $[D\Phi \circ R : D\Phi]_t = \lambda^{\frac{1}{2}it^2}\delta^{it}$. Modular groups of Φ and $\Phi \circ R$ are linked by $\sigma^{\Phi\circ R}_t(m) = \delta^{it}\sigma^\Phi_t(m)\delta^{-it}$ for all $t \in \mathbb{R}$ and $m \in M$.

Definition 5.1.4. – We call **scaling operator** the strictly positive operator λ affiliated to $Z(M)$ and **modulus** the strictly positive operator δ affiliated to M such that, for all $t \in \mathbb{R}$, we have:

$$[D\Phi \circ R : D\Phi]_t = \lambda^{\frac{1}{2}it^2}\delta^{it}$$

The following propositions give the compatibility of λ and δ w.r.t. the structure of Hopf-bimodule.

Lemma 5.1.5. – *For all $s, t \in \mathbb{R}$, we have $[D\Phi \circ \sigma^{\Phi\circ R}_s : D\Phi]_t = \lambda^{ist}$.*

Proof. – The computation of the cocycle is straightforward:

$$\begin{aligned}[D\Phi\circ\sigma_s^{\Phi\circ R}:D\Phi]_t &= [D\Phi\circ\sigma_s^{\Phi\circ R}:D\Phi\circ R\circ\sigma_s^{\Phi\circ R}]_t[D\Phi\circ R:D\Phi]_t\\ &= \sigma_{-s}^{\Phi\circ R}([D\Phi:D\Phi\circ R]_t)[D\Phi\circ R:D\Phi]_t\\ &= \delta^{-is}\sigma_{-s}^{\Phi}(\lambda^{-\frac{it^2}{2}}\delta^{-it})\delta^{is}\lambda^{\frac{it^2}{2}}\delta^{it}\\ &= \delta^{-is}\lambda^{-\frac{it^2}{2}}\lambda^{ist}\delta^{-it}\delta^{is}\lambda^{\frac{it^2}{2}}\delta^{it}=\lambda^{ist}\end{aligned}$$

□

PROPOSITION 5.1.6. – *We have $R(\lambda)=\lambda$, $R(\delta)=\delta^{-1}$ and $\tau_t(\delta)=\delta$, $\tau_t(\lambda)=\lambda$ for all $t\in\mathbb{R}$.*

Proof. – Relations between R, λ and δ come from uniqueness of Radon-Nikodym cocycle decomposition. By proposition 5.1.1, we have $\Phi\circ\tau_{-s}=\Phi\circ\sigma_s^{\Phi\circ R}$ for all $s,t\in\mathbb{R}$, so:

$$\tau_s([D\Phi\circ R:D\Phi]_t)=[D\Phi\circ R\circ\tau_{-s}:D\Phi\circ\tau_{-s}]_t=[D\Phi\circ\sigma_s^{\Phi\circ R}\circ R:D\Phi\circ\sigma_s^{\Phi\circ R}]_t$$

Consequently, by the previous lemma, we get:

$$\begin{aligned}&\tau_s([D\Phi\circ R:D\Phi]_t)\\ &=[D\Phi\circ\sigma_s^{\Phi\circ R}\circ R:D\Phi\circ R]_t[D\Phi\circ R:D\Phi]_t[D\Phi:D\Phi\circ\sigma_s^{\Phi\circ R}]_t\\ &=R([D\Phi\circ\sigma_s^{\Phi\circ R}:D\Phi]_{-t}^*)[D\Phi\circ R:D\Phi]_t[D\Phi\circ\sigma_s^{\Phi\circ R}:D\Phi]_t^*\\ &=R(\lambda^{ist})\lambda^{-\frac{it^2}{2}}\delta^{it}\lambda^{-ist}=\lambda^{-\frac{it^2}{2}}\delta^{it}\end{aligned}$$

□

5.2. First result of uniqueness for invariant operator-valued weight

Next, we want to precise where the scaling operator λ sits. We have to prove, first of all, a first result of uniqueness as far as the invariant operator-valued weight is concerned.

Let T_1 and T_2 be two n.s.f. left invariant operator-valued weights from M to $\alpha(N)$ such that $T_1\le T_2$. For all $i\in\{1,2\}$, we put $\Phi_i=\nu\circ\alpha^{-1}\circ T_i$ and $\hat\beta_i(n)=J_{\Phi_i}\alpha(n^*)J_{\Phi_i}$.

We define, as we have done for U_H, an isometry $(U_2)_H$ by the following formula:

$$(U_2)_H(v_\alpha\otimes_{\nu^o\hat\beta_2}\Lambda_{\Phi_2}(a))=\sum_{i\in I}\xi_{i\beta}\otimes_{\nu}{}_\alpha\Lambda_{\Phi_2}((\omega_{v,\xi_i}{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(a)))$$

for all $v\in D(H_\beta,\nu^o)$ and $a\in\mathcal{N}_{\Phi_2}\cap\mathcal{N}_{T_2}$. Then, we know that $(U_2)_H$ is unitary and $\Gamma(m)=(U_2)_H(1_\alpha\otimes_{N^o}{}_{\hat\beta_2}m)(U_2)_H^*$ for all $m\in M$.

Since $T_1\le T_2$, there exists $F\in\mathcal{L}(H_{\Phi_2},H_{\Phi_1})$ such that, for all $x\in\mathcal{N}_{\Phi_2}\cap\mathcal{N}_{T_2}$, we have $F\Lambda_{\Phi_2}(x)=\Lambda_{\Phi_1}(x)$. It is easy to verify that, for all $n\in N$, we have $F\hat\beta_2(n)=\hat\beta_1(n)F$. If we put $P=F^*F$, then P belongs to $M'\cap\hat\beta_2(N)'$ and $J_{\Phi_2}PJ_{\Phi_2}$ belongs to $M\cap\alpha(N)'$.

LEMMA 5.2.1

We have $\Gamma(J_{\Phi_2}PJ_{\Phi_2}) = 1_{\beta}\underset{N}{\otimes}_{\alpha}J_{\Phi_2}PJ_{\Phi_2}$.

Proof. – We have, for all $v, w \in D(H_\beta, \nu^o)$ and $a, b \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$:

$$\begin{aligned}&((1_{\beta}\underset{N}{\otimes}_{\alpha}P)(U_2)_H(v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(a))|(U_2)_H(w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(b)))\\ &= ((U_1)_H(v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(a))|(U_1)_H(w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(b)))\end{aligned}$$

where $(U_1)_H$ is defined in the same way as $(U_2)_H$. The two expressions are continuous in v and w, so by density of $D(H_\beta, \nu^o)$ in H, we get, for all $v, w \in H$ and $a, b \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$:

$$\begin{aligned}&((1_{\beta}\underset{N}{\otimes}_{\alpha}P)(U_2)_H(v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(a))|(U_2)_H(w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(b)))\\ &= ((U_1)_H(v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(a))|(U_1)_H(w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(b)))\\ &= (v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(a)|w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_1}\Lambda_{\Phi_1}(b))\\ &= ((1_{\alpha}\underset{N^o}{\otimes}_{\hat{\beta}_2}P)(v_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(a))|w_{\alpha}\underset{\nu^o}{\otimes}_{\hat{\beta}_2}\Lambda_{\Phi_2}(b))\end{aligned}$$

so that $(U_2)^*_H(1_{\beta}\underset{N}{\otimes}_{\alpha}P)(U_2)_H = 1_{\alpha}\underset{N^o}{\otimes}_{\hat{\beta}_2}P$. In particular, if we take $H = H_\Phi$, then by 4.0.10 we get $(U_2)_H(1_{\alpha}\underset{N^o}{\otimes}_{\hat{\beta}_2}J_{\Phi_2}PJ_{\Phi_2})(U_2)^*_H = 1_{\beta}\underset{N}{\otimes}_{\alpha}J_{\Phi_2}PJ_{\Phi_2}$. Finally, since $J_{\Phi_2}PJ_{\Phi_2} \in M$, we have $\Gamma(J_{\Phi_2}PJ_{\Phi_2}) = 1_{\beta}\underset{N}{\otimes}_{\alpha}J_{\Phi_2}PJ_{\Phi_2}$. □

PROPOSITION 5.2.2. – *If T_1 and T_2 are n.s.f. left invariant weights from M to $\alpha(N)$ such that $T_1 \leq T_2$, then there exists an injective $p \in N$ such that $0 \leq p \leq 1$ and, for all $t \in \mathbb{R}$:*

$$[D\Phi_1 : D\Phi_2]_t = \beta(p)^{it}$$

Proof. – By the previous lemma and proposition 9.2.25, there exists an injective $p \in N$ such that $0 \leq p \leq 1$ and, for all $x, y \in \mathcal{N}_{\Phi_2} \cap \mathcal{N}_{T_2}$, we have $(\Lambda_{\Phi_1}(x)|\Lambda_{\Phi_1}(y)) = (J_{\Phi_2}\beta(p)J_{\Phi_2}\Lambda_{\Phi_2}(x)|\Lambda_{\Phi_2}(y))$. By [**Str81**, prop. 3.13], we get that $\beta(p)$ coincides with the analytic continuation in $-i$ of the cocycle $[D\Phi_1 : D\Phi_2]$. Then, we have, for all $t \in \mathbb{R}$:

$$[D\Phi_1 : D\Phi_2]_t = \beta(p)^{it} \quad \square$$

PROPOSITION 5.2.3. – *Let T_1 be a n.s.f. left invariant operator-valued weight Φ_1 is σ^Φ-invariant. Then, there exists a strictly positive operator q which is affiliated to N^γ such that $\Phi_1 = (\Phi)_{\beta(q)}$.*

Proof. – We put $T_2 = T_L + T_1$. Since Φ_1 is σ^Φ-invariant, then the left invariant operator-valued weight T_2 is n.s.f. Finally, since $T_1 \leq T_2$ and $T_L \leq T_2$, there exists an injective $p \in N$ between 0 and 1 such that $\Phi_1 = (\Phi_2)_{\beta(p)}$ and $\Phi = (\Phi_2)_{\beta(1-p)}$. By **[Str81]**, we have:

$$[D\Phi_1 : D\Phi_2]_t = \beta(p)^{it} \text{ and } [D\Phi : D\Phi_2]_t = \beta(1-p)^{it}$$

Then, we have, for all $t \in \mathbb{R}$:

$$[D\Phi_1 : D\Phi]_t = [D\Phi_1 : D\Phi_2]_t [D\Phi_2 : D\Phi]_t = \beta\Big(\frac{p}{1-p}\Big)^{it}$$

that's why $q = \frac{p}{1-p}$ is the suitable element. Now, by **[Str81]**, we have:

$$\beta(q) = \sigma_t^{\Phi}(\beta(q)) = \beta(\gamma_t(q))$$

so that, by injectivity of β, we get that q is affiliated to N^{γ}. □

LEMMA 5.2.4. – *For all $t \in \mathbb{R}$, $\tau_{-t} \circ T_L \circ \tau_t$ is a n.s.f. left invariant operator-valued weight from M to $\alpha(N)$. Moreover, $\sigma_s^{\Phi\circ\tau_t}(\beta(n)) = \beta(\gamma_s(n))$ for all $s, t \in \mathbb{R}$ and $n \in N$.*

Proof. – For all $t \in \mathbb{R}$, we have $\nu \circ \alpha^{-1} \circ \tau_{-t} \circ T_L \circ \tau_t = \Phi \circ \tau_t$. Then:

$$\begin{aligned}(\mathrm{id}_{\beta \underset{\nu}{\star} \alpha} \nu \circ \alpha^{-1} \circ \tau_{-t} \circ T_L \circ \tau_t) \circ \Gamma &= (\mathrm{id}_{\beta \underset{\nu}{\star} \alpha} \Phi \circ \tau_t) \circ \Gamma \\ &= \tau_{-t} \circ (\mathrm{id}_{\beta \underset{\nu}{\star} \alpha} \Phi) \circ \Gamma \circ \tau_t = \tau_{-t} \circ T_L \circ \tau_t\end{aligned}$$ □

On the other hand, for all $s, t \in \mathbb{R}$ and $n \in N$, since γ and σ^{ν} commute, we have:

$$\begin{aligned}\sigma_s^{\Phi\circ\tau_t}(\beta(n)) &= \tau_{-t} \circ \sigma_s^{\Phi} \circ \tau_t(\beta(n)) = \tau_{-t} \circ \sigma_s^{\Phi}(\beta(\sigma_t^{\nu}(n))) \\ &= \tau_{-t}(\beta(\gamma_s \sigma_t^{\nu}(n))) = \beta(\sigma_{-t}^{\nu} \gamma_s \sigma_t^{\nu}(n)) = \beta(\gamma_s^{\nu}(n))\end{aligned}$$

PROPOSITION 5.2.5. – *There exists a strictly positive operator q affiliated with $Z(N)$ such that the scaling operator $\lambda = \alpha(q) = \beta(q)$. In particular, λ is affiliated with $Z(M) \cap \alpha(N) \cap \beta(N)$.*

Proof. – By the previous lemma, $\tau_s \circ T_L \circ \tau_{-s}$ is left invariant. Moreover, since σ^{Φ} and τ commute, $\Phi \circ \tau_{-s}$ is σ^{Φ}-invariant. That's why, we are in conditions of proposition 5.2.3 so that we get a strictly positive operator q_s affiliated with N^{γ} such that $[D\Phi \circ \tau_{-s} : D\Phi]_t = \beta(q_s)^{it}$. On the other hand, by lemma 5.1.5, we have $[D\Phi \circ \sigma_s^{\Phi\circ R} : D\Phi]_t = \lambda^{ist}$. Since we have $\Phi \circ \tau_{-s} = \Phi \circ \sigma_s^{\Phi\circ R}$, so we obtain that $\lambda^{ist} = \beta(q_s)^{it}$ for all $s, t \in \mathbb{R}$. We easily deduce that there exists a strictly positive operator q affiliated with $Z(N)$ such that $\lambda = \beta(q)$. Finally, since $R(\lambda) = \lambda$, we also have $\lambda = \alpha(q)$. □

5.3. Properties of the modulus

Now, we prove that the modulus δ is a co-character. This will be a key-result for duality.

Proposition 5.3.1. – *For all $n \in N$ and $t \in \mathbb{R}$, we have:*

$$\delta^{it}\alpha(n)\delta^{-it} = \alpha(\gamma_t\sigma_t^{\nu}(n)) \quad \textit{and} \quad \delta^{it}\beta(n)\delta^{-it} = \beta(\gamma_t\sigma_t^{\nu}(n))$$

Proof. – By definition of γ, we have:

$$\beta(\sigma_{-t}^{\nu}(n)) = \sigma_t^{\Psi}(\beta(n)) = \delta^{it}\sigma_t^{\Phi}(\beta(n))\delta^{-it} = \delta^{it}\beta(\gamma_t(n))\delta^{-it}$$

what gives the first equality (we recall that γ and σ^{ν} commute with each other). Then, apply the co-involution to get the second one. □

Thanks to the commutation relations and by proposition 2.3.5, we can define, for all $t \in \mathbb{R}$, a bounded operator $\delta^{it}{}_{\beta}\underset{N}{\otimes}{}_{\alpha}\delta^{it}$ which naturally acts on elementary tensor products.

Lemma 5.3.2. – *There exists a strictly positive operator P on H_{Φ} implementing τ such that, for all $x \in \mathcal{N}_{\Phi}$ and $t \in \mathbb{R}$, we have $P^{it}\Lambda_{\Phi}(x) = \lambda^{\frac{t}{2}}\Lambda_{\Phi}(\tau_t(x))$.*

Proof. – Since $\Phi \circ R = \Phi_{\delta}$, by [**Vae01a**, 5.3], we have:

$$\Lambda_{\Phi}(\sigma_t^{\Phi\circ R}(x)) = \delta^{it}J_{\Phi}\lambda^{\frac{t}{2}}\delta^{it}J_{\Phi}\Delta_{\Phi}^{it}\Lambda_{\Phi}(x)$$

and since λ is affiliated with $Z(M)$, we get $||\Lambda_{\Phi}(\sigma_t^{\Phi\circ R}(x))|| = ||\lambda^{\frac{t}{2}}\Lambda_{\Phi}(x)||$ for all $x \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$. But, we know that Φ is $\sigma_t^{\Phi\circ R} \circ \tau_t$-invariant, so $||\Lambda_{\Phi}(x)|| = ||\lambda^{\frac{t}{2}}\Lambda_{\Phi}(\tau_t(x))||$. Then, there exists P_t on H_{Φ} such that:

$$P_t\Lambda_{\Phi}(x) = \lambda^{\frac{t}{2}}\Lambda_{\Phi}(\tau_t(x))$$

for all $x \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$. For all $s, t \in \mathbb{R}$, we verify that $P_sP_t = P_{st}$ thanks to relation $\tau_t(\lambda) = \lambda$ and the existence of P follows. The fact that P implements τ is clear. □

Lemma 5.3.3. – *We have, for all $a, b \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and $t \in \mathbb{R}$:*

$$\omega_{J_{\Phi}\Lambda_{\Phi}(\lambda^{\frac{t}{2}}\tau_t(a))} = \omega_{J_{\Phi}\Lambda_{\Phi}(a)} \circ \tau_{-t} \ \textit{and} \ \omega_{J_{\Phi}\Lambda_{\Phi}(b)} \circ \sigma_t^{\Phi\circ R} = \omega_{J_{\Phi}\Lambda_{\Phi}(\lambda^{\frac{t}{2}}\sigma_{-t}^{\Phi\circ R}(b))}$$

Proof. – Since τ is implemented by P, the first relation holds. By [**Vae01a**, prop. 2.4], we know that $\Delta_{\Phi\circ R} = J_\Phi \delta J_\Phi \delta \Delta_\Phi$ so that we can compute, for all $x \in M$ and $b \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$:

$$\begin{aligned}(\sigma_t^{\Phi\circ R}(x)J_\Phi\Lambda_\Phi(b)|J_\Phi\Lambda_\Phi(b)) &= (x\Delta_{\Phi\circ R}^{-it}J_\Phi\Lambda_\Phi(b)|\Delta_{\Phi\circ R}^{-it}J_\Phi\Lambda_\Phi(b))\\ &= (xJ_\Phi\delta^{it}J_\Phi\delta^{-it}\Delta_\Phi^{-it}J_\Phi\Lambda_\Phi(b)|J_\Phi\delta^{it}J_\Phi\delta^{-it}\Delta_\Phi^{-it}J_\Phi\Lambda_\Phi(b))\\ &= (x\delta^{-it}J_\Phi\Lambda_\Phi(\sigma_{-t}^\Phi(b))|\delta^{-it}J_\Phi\Lambda_\Phi(\sigma_{-t}^\Phi(b)))\\ &= (xJ_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\sigma_{-t}^\Phi(b)\delta^{it})|J_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\sigma_{-t}^\Phi(b)\delta^{it}))\\ &= (xJ_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\delta^{-it}\sigma_{-t}^\Phi(b)\delta^{it})|J_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\delta^{-it}\sigma_{-t}^\Phi(b)\delta^{it}))\\ &= (xJ_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\sigma_{-t}^{\Phi\circ R}(b))|J_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\sigma_{-t}^{\Phi\circ R}(b)))\end{aligned}$$ □

Proposition 5.3.4. – *We have* $\Gamma\circ\tau_t = (\sigma_t^\Phi {}_\beta\underset{N}{\star}{}_\alpha \sigma_{-t}^{\Phi\circ R})\circ\Gamma$ *for all* $t\in\mathbb{R}$.

Proof. – For all $a, b \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and $t\in\mathbb{R}$, we compute:

$$\begin{aligned}&(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(b)})[(\sigma_{-t}^\Phi{}_\beta\underset{N}{\star}{}_\alpha\sigma_t^{\Phi\circ R})\circ\Gamma\circ\tau_t(a^*a)]\\ &= \sigma_{-t}^\Phi[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(b)}\circ\sigma_t^{\Phi\circ R})(\Gamma\circ\tau_t(a^*a))]\end{aligned}$$

By the previous lemma, this last expression is equal to:

$$\begin{aligned}&\sigma_{-t}^\Phi[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\sigma_{-t}^{\Phi\circ R}(b))})(\Gamma\circ\tau_t(a^*a))]\\ &= \sigma_{-t}^\Phi\circ R[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(\tau_t(a))})(\Gamma(\lambda^t\sigma_{-t}^{\Phi\circ R}(b^*b)))]\\ &= R\circ\sigma_t^{\Phi\circ R}[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(\lambda^{\frac{t}{2}}\tau_t(a))})(\Gamma\circ\sigma_{-t}^{\Phi\circ R}(b^*b))]\end{aligned}$$

Again, by the previous lemma, this last expression is equal to:

$$\begin{aligned}&R\circ\sigma_t^{\Phi\circ R}[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(a)}\circ\tau_{-t}(\Gamma\circ\sigma_{-t}^{\Phi\circ R}(b^*b))]\\ &= R[(\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(a)})(\Gamma(b^*b))] = (\mathrm{id}_\beta\underset{\nu}{\star}{}_\alpha\omega_{J_\Phi\Lambda_\Phi(b)})(\Gamma(a^*a))\end{aligned}$$

So, we conclude that $(\sigma_{-t}^\Phi{}_\beta\underset{N}{\star}{}_\alpha\sigma_t^{\Phi\circ R})\circ\Gamma\circ\tau_t = \Gamma$ for all $t\in\mathbb{R}$. □

Corollary 5.3.5. – *For all* $t\in\mathbb{R}$ *and* $m\in M$, *we have:*

$$(\delta^{it}{}_\beta\underset{N}{\otimes}{}_\alpha\delta^{it})\Gamma(m)(\delta^{-it}{}_\beta\underset{N}{\otimes}{}_\alpha\delta^{-it}) = \Gamma(\delta^{it}m\delta^{-it})$$

In particular, for all $s,t\in\mathbb{R}$, $\Gamma(\delta^{is})$ *and* $\delta^{it}{}_\beta\underset{N}{\otimes}{}_\alpha\delta^{it}$ *commute each other.*

Proof. – For all $t \in \mathbb{R}$, we have:

$$\begin{aligned}(\sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t} {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t}) \circ \Gamma &= (\sigma^{\Phi}_{-t} {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t} \circ \tau_t) \circ \Gamma \circ \sigma^{\Phi\circ R}_{t} \\ &= (\sigma^{\Phi}_{-t} \circ \tau_t {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi\circ R}_{t} \circ \tau_t) \circ \Gamma \circ \sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t} \\ &= (\sigma^{\Phi}_{-t} {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi\circ R}_{t}) \circ \Gamma \circ \tau_t \sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t} \\ &= \Gamma \circ \sigma^{\Phi}_{-t} \circ \sigma^{\Phi\circ R}_{t}\end{aligned}$$

We know that $\sigma^{\Phi}_{-t}\sigma^{\Phi\circ R}_{t}(m) = \delta^{it} m \delta^{-it}$ for all $m \in M$, that's why we get:

$$(\delta^{it} {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \delta^{it})\Gamma(m)(\delta^{-it} {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \delta^{-it}) = \Gamma(\delta^{it} m \delta^{-it})$$

In particular, for all $s \in \mathbb{R}$, we have:

$$(\delta^{it} {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \delta^{it})\Gamma(\delta^{is})(\delta^{-it} {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \delta^{-it}) = \Gamma(\delta^{it}\delta^{is}\delta^{-it}) = \Gamma(\delta^{is}) \quad \square$$

PROPOSITION 5.3.6. – *Let us denote by $\mathcal{T}^{\Psi}_{\Phi,T_L}$ made of elements $a \in \mathcal{N}_{T_R} \cap \mathcal{N}_{\Phi} \cap \mathcal{N}_{\Psi}$, analytic with respect to both Φ and Ψ such that, for all $z, z' \in \mathcal{C}$, $\sigma^{\Psi}_{z} \circ \sigma^{\Phi}_{z}(a)$ belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_{\Phi} \cap \mathcal{N}_{\Psi}$. This linear space is weakly dense in M and the set of $\Lambda_{\Psi}(a)$ (resp. $\Lambda_{\Phi}(a)$), for all $a \in \mathcal{T}^{\Psi}_{\Phi,T_L}$, is a linear dense subset in H. Moreover, the subset $J_{\Psi}\Lambda_{\Psi}(\mathcal{T}^{\Psi}_{\Phi,T_L})$ is included in the domain of δ^z, for all $z \in \mathcal{C}$ and is an essential domain for δ^z.*

Proof. – Let us take $x \in \mathcal{T}^{+}_{\Phi,T_L}$ and let us write $\lambda = \int_0^\infty t \, de_t$ and define $f_p = \int_{\frac{1}{p}}^{p} de_t$. If we put:

$$x_{p,q} = f_p \sqrt{\frac{q}{\pi}} \int_{-\infty}^{+\infty} e^{-qt^2} \sigma^{\Psi}_{t}(x) \, dt$$

we obtain that $x_{p,q}$ belongs to $\mathcal{T}^{+}_{\Phi,T_L}$, is analytical with respect to Ψ, $\sigma^{\Psi}_{z}(x_{p,q})$ is weakly converging to x and $\Lambda_{\Phi}(x_{p,q})$ is weakly converging to $\Lambda_{\Phi}(x)$. Moreover, $\Lambda_{T_L}(x_{p,q})$is weakly converging to $\Lambda_{T_L}(x)$.

Since, for all $y \in M$ and $t \in \mathbb{R}$, we have:

$$\sigma^{\Psi}_{t}\sigma^{\Phi}_{-t} = \delta^{it} x \delta^{-it}$$

we see that, for all such elements $x_{p,q}$ and $z \in \mathbb{C}$, $\delta^{iz} x_{p,q} \delta^{-iz}$ is bounded and belongs to $\mathcal{T}_{\Phi,T_L}$. In particular, $\delta^{-\frac{1}{2}} x_{p,q} \delta^{\frac{1}{2}}$ belongs to $\mathcal{M}_{T_L} \cap \mathcal{T}_{\Phi}$ and is analytic with respect to both Φ and Ψ. Using then the operator e_n introduced in [**Vae01a**, 1.1], which are analytic to both Φ and Ψ and converging to 1 when n goes to infinity, we get that $e_n x_{p,q}$ belongs to $\mathcal{N}_{T_L} \cap \mathcal{N}_{\Phi}$. On the other hand, since:

$$e_n x_{p,q} \delta^{\frac{1}{2}} = (e_n \delta^{\frac{1}{2}}) \delta^{-\frac{1}{2}} x_{p,q} \delta^{\frac{1}{2}}$$

belongs to $\mathcal{N}_{\Phi}$, we see, by [**Vae01a**, 3.3], that $e_n x_{p,q}$ belongs to $\mathcal{N}_{\Psi}$ and, therefore, to $\mathcal{T}^{\Psi}_{\Phi,T_L}$, from which we then get all the results claimed. $\square$

Let recall proposition 2.4 of [**Eno04**]:

PROPOSITION 5.3.7. – *Let a, b in $\mathcal{N}_{T_L}$. Then $T_L(a^*a)$ and $T_L(b^*b)$ are positive self-adjoint closed operators which verify:*

$$\omega_{J_\Psi\Lambda_\Psi(a)}(T_L(b^*b)) = \omega_{J_\Psi\Lambda_\Psi(b)}(T_L(a^*a))$$

LEMMA 5.3.8. – *Let $b \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Phi \cap \mathcal{N}_\Psi$ and X positive affiliated to M be such that $\delta^{-\frac{1}{2}} X \delta^{-\frac{1}{2}}$ is bounded. Then the element of the extended positive part $(\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)\Gamma(X)$ is such that:*

$$\omega_{J_\Psi\Lambda_\Psi(b)}((\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)\Gamma(X)) = \omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b)}(T_L(\delta^{-\frac{1}{2}} X \delta^{-\frac{1}{2}}))$$

If X is bounded, such that $\delta^{-\frac{1}{2}} X \delta^{-\frac{1}{2}}$ is bounded and in $\mathcal{M}_{T_L}^+$ then $(\omega_{J_\Psi\Lambda_\Psi(b)\beta} \underset{\nu}{\star}_\alpha \mathrm{id})\Gamma(X)$ belongs to $\mathcal{M}_{T_L}^+ \cap \mathcal{M}_\Psi^+$. If Y is in $\mathcal{M}_{T_L}^+$, we have:

$$\delta^{-\frac{1}{2}} T_L(Y) \delta^{\frac{1}{2}} = (\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)\Gamma(\delta^{\frac{1}{2}} X \delta^{\frac{1}{2}})$$

Proof. – Let us assume that $a, b \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Phi \cap \mathcal{N}_\Psi$. By [**Vae01a**], $J_\Phi\Lambda_\Phi(a)$ is in the domain of $\delta^{-\frac{1}{2}}$ and $\delta^{-\frac{1}{2}} J_\Phi\Lambda_\Phi(a) = \lambda^{\frac{i}{4}} \delta^{-\frac{1}{2}} J_\Psi\Lambda_\Psi(a)$. Then, we compute the following:

$$\begin{aligned}
&\omega_{J_\Psi\Lambda_\Psi(b)}((\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)\Gamma(a^*a)) = \Phi \circ R((\omega_{J_\Psi\Lambda_\Psi(b)\beta} \underset{\nu}{\star}_\alpha \mathrm{id})\Gamma(a^*a)) \\
&= \Phi((\omega_{J_\Psi\Lambda_\Psi(a)\beta} \underset{\nu}{\star}_\alpha \mathrm{id})\Gamma(b^*b)) = \omega_{J_\Psi\Lambda_\Psi(a)}(T_L(b^*b)) = \omega_{\delta^{-\frac{1}{2}} J_\Phi\Lambda_\Phi(a)}(T_L(b^*b)) \\
&= \omega_{J_\Phi\Lambda_\Phi(a\delta^{-\frac{1}{2}})}(T_L(b^*b)) = \omega_{J_\Phi\Lambda_\Phi(b)}(T_L(\delta^{-\frac{1}{2}} a^*a \delta^{-\frac{1}{2}})) \\
&= \omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b)}(T_L(\delta^{-\frac{1}{2}} a^*a \delta^{-\frac{1}{2}}))
\end{aligned}$$

If X is positive such that $\delta^{-\frac{1}{2}} X \delta^{-\frac{1}{2}}$ is bounded, we may consider X as the upper limit of elements of the type $a_i^* a_i$ where a_i belongs to the dense left ideal $\mathcal{N}_{T_R} \cap \mathcal{N}_\Phi \cap \mathcal{N}_\Psi$. Then every $a_i \delta^{-\frac{1}{2}}$ is bounded and we get the first formula by increasing limits. The proof of the second one is an easy corollary of the first one because we are in the essential domain of $\delta^{\frac{1}{2}}$. □

THEOREM 5.3.9. – *We have $\Gamma(\delta) = \delta_\beta \underset{N}{\otimes}_\alpha \delta$.*

Proof. – Applying Γ to the second equality of the previous proposition, we get for all $Y \in \mathcal{M}_\Phi^+$:

$$\begin{aligned}
\Gamma(\delta^{\frac{1}{2}})(T_L(Y)_\beta \underset{N}{\otimes}_\alpha 1)\Gamma(\delta^{\frac{1}{2}}) &= \Gamma((\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}})) \\
&= (\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)(\Gamma_\beta \underset{\nu}{\star}_\alpha \mathrm{id})\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}})) \\
&= (\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Psi)(\mathrm{id}_\beta \underset{\nu}{\star}_\alpha \Gamma)\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}}))
\end{aligned}$$

Let now $b \in \mathcal{T}^{\Psi}_{\Phi,T_L}$ and define Z by:

$$Z = (\omega_{J_\Psi \Lambda_\Psi(b)} {}_\beta\star_\alpha^{\nu} \mathrm{id})\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}})$$

By corollary 5.3.5, we have:

$$\delta^{-\frac{1}{2}} Z \delta^{-\frac{1}{2}} = (\omega_{\delta^{\frac{1}{2}} J_\Psi \Lambda_\Psi(b)} {}_\beta\star_\alpha^{\nu} \mathrm{id})\Gamma(Y)$$

which is bounded by proposition 5.3.6. By the previous proposition, we get for all $b' \in \mathcal{T}_{T_R} \cap \mathcal{T}_\Psi$:

$$\begin{aligned}
&\omega_{J_\Psi\Lambda_\Psi(b) {}_\beta\otimes_\alpha^{\nu} J_\Psi\Lambda_\Psi(b')}(\Gamma(\delta^{\frac{1}{2}})(T_L(Y) {}_\beta\otimes_\alpha^{N} 1)\Gamma(\delta^{\frac{1}{2}}))\\
&= \omega_{J_\Psi\Lambda_\Psi(b) {}_\beta\otimes_\alpha^{\nu} J_\Psi\Lambda_\Psi(b')}((\mathrm{id} {}_\beta\star_\alpha^{\nu} \mathrm{id} {}_\beta\star_\alpha^{\nu} \Psi)(\mathrm{id} {}_\beta\star_\alpha^{\nu} \Gamma)\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}}))\\
&= \omega_{J_\Psi\Lambda_\Psi(b')}((\mathrm{id} {}_\beta\star_\alpha^{\nu} \Psi)\Gamma((\omega_{J_\Psi\Lambda_\Psi(b)} {}_\beta\otimes_\alpha^{\nu} \mathrm{id})\Gamma(\delta^{\frac{1}{2}} Y \delta^{\frac{1}{2}})))\\
&= \omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b')}(T_L(\delta^{-\frac{1}{2}} Z \delta^{-\frac{1}{2}})) = \omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b')}(T_L((\omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b)} {}_\beta\star_\alpha^{\nu} \mathrm{id})\Gamma(Y)))\\
&= \omega_{\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b) {}_\beta\otimes_\alpha^{\nu} \delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b')}(T_L(Y) {}_\beta\otimes_\alpha^{N} 1)
\end{aligned}$$

from which we infer, by increasing limits, that:

$$\omega_{J_\Psi\Lambda_\Psi(b) {}_\beta\otimes_\alpha^{\nu} J_\Psi\Lambda_\Psi(b')}(\Gamma(\delta)) = ||\delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b) {}_\beta\otimes_\alpha^{\nu} \delta^{\frac{1}{2}} J_\Psi\Lambda_\Psi(b')||^2$$

which finishes the proof by proposition 5.3.6. □

5.4. Uniqueness of invariant operator-valued weight

THEOREM 5.4.1

If T' a n.s.f. left invariant operator-valued weight such that $(\tau_t {}_\beta\star_\alpha^{N} \sigma_t^{\Phi'}) \circ \Gamma = \Gamma \circ \sigma_t^{\Phi'}$, $\nu \circ \gamma' = \nu$ and $\gamma \circ \gamma' = \gamma' \circ \gamma$, then there exists a strictly positive operator h affiliated with $Z(N)$ such that, for all $t \in \mathbb{R}$, we have:

$$\Phi' = \nu \circ \alpha^{-1} \circ T' = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)} \text{ and } [DT' : DT_L]_t = \beta(h^{it})$$

Proof. – We put $\Phi' = \nu \circ \alpha^{-1} \circ T'$. We have for all $s \in \mathbb{R}$:

$$\Gamma \circ \sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'} = (\tau_{-s} {}_\beta\star_\alpha^{N} \sigma_{-s}^{\Phi}) \circ \Gamma \circ \sigma_s^{\Phi'} = (\mathrm{id} {}_\beta\star_\alpha^{N} \sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'}) \circ \Gamma$$

By right invariance of T_R, we have for all $a \in \mathcal{M}_{T_R}^+$:

$$\begin{aligned}
T_R(\sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'}(a)) &= (\Phi \circ R {}_\beta\star_\alpha^{\nu} \mathrm{id})(\Gamma(\sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'}(a)))\\
&= \sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'}((\Phi \circ R {}_\beta\star_\alpha^{\nu} \mathrm{id})\Gamma(a)) = \sigma_{-s}^{\Phi} \circ \sigma_s^{\Phi'}(T_R(a))
\end{aligned}$$

Since γ and γ' leave ν invariant, we get that $\Phi \circ R$ is $\sigma^{\Phi}_{-s} \circ \sigma^{\Phi'}_s$-invariant and, so $\sigma^{\Phi\circ R}_t$ and $\sigma^{\Phi}_{-s} \circ \sigma^{\Phi'}_s$ commute each other. But $\sigma^{\Phi\circ R}$ and σ^{Φ} commute each other that's why $\sigma^{\Phi\circ R}$ and $\sigma^{\Phi'}$ also commute each other. For all $s, t \in \mathbb{R}$, we have:

$$\Gamma(\sigma^{\Phi'}_t(\delta^{is})\delta^{-is}) = (\tau_t {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi'}_t)(\Gamma(\delta^{is}))(\delta^{-is} {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \delta^{-is}) = 1 {}_{\beta}\underset{N}{\otimes}{}_{\alpha} \sigma^{\Phi'}_t(\delta^{is})\delta^{-is}$$

Consequently $\sigma^{\Phi'}_t(\delta^{is})\delta^{-is}$ belongs to $\beta(N)$. For all $n \in N$ and $s, t \in \mathbb{R}$, we have:

$$\begin{aligned}\sigma^{\Phi'}_t(\delta^{is})\beta(n)\sigma^{\Phi'}_t(\delta^{-is}) &= \sigma^{\Phi'}_t((\delta^{is})\sigma^{\Phi'}_{-t}(\beta(n))\delta^{-is}) = \sigma^{\Phi'}_t((\delta^{is})\beta(\gamma'_{-t}(n))\delta^{-is})\\ = \sigma^{\Phi'}_t(\beta(\gamma_s\sigma^{\nu}_s\gamma'_{-t}(n))) &= \beta(\gamma'_t\gamma_s\sigma^{\nu}_s\gamma'_{-t}(n))) = \beta(\gamma_s\sigma^{\nu}_s(n))) = \delta^{is}\beta(n)\delta^{-is}\end{aligned}$$

So $\sigma^{\Phi'}_t(\delta^{is})\delta^{-is}$ belongs to $\beta(Z(N))$ and we easily get that there exists a strictly positive operator k affiliated with $Z(N)$ such that $\sigma^{\Phi'}_t(\delta^{is}) = \beta(k^{ist})\delta^{is}$. Then, we have:

$$\begin{aligned}\sigma^{\Phi'}_s \circ \sigma^{\Phi}_t(m) = \sigma^{\Phi'}_s(\delta^{-it}\sigma^{\Phi\circ R}_t(m)\delta^{it}) &= \beta(k^{-ist})\delta^{-it}\sigma^{\Phi'}_s \circ \sigma^{\Phi\circ R}_t(m)\delta^{it}\beta(k^{ist})\\ &= \beta(k^{-ist})\sigma^{\Phi}_t \circ \sigma^{\Phi'}_s(m)\beta(k^{ist})\end{aligned}$$

Take $m = \delta^{iu}$ to get k is affiliated to N^{γ}. Apply Φ to the previous formula and get:

$$\begin{aligned}\Phi \circ \sigma^{\Phi'}_s \circ \sigma^{\Phi}_s(m^*m) &= \Phi(\beta(k^{-ist})\sigma^{\Phi}_t \circ \sigma^{\Phi'}_s(m^*m)\beta(k^{ist}))\\ &= \Phi(\sigma^{\Phi}_t \circ \sigma^{\Phi'}_s(m^*m)) = \Phi \circ \sigma^{\Phi'}_s(m^*m)\end{aligned}$$

So, by 5.2.3 and left invariance $\sigma^{\Phi'}_{-s} \circ T_L \circ \sigma^{\Phi'}_s$, there exists a strictly positive operator q_s affiliated with $Z(N)$ such that $\Phi \circ \sigma^{\Phi'}_s = \Phi_{\beta(q_s)}$. By usual arguments, we deduce that there exists a strictly positive q affiliated to $Z(N)$ such that $\Phi \circ \sigma^{\Phi'}_s = \Phi_{\beta(q^{-s})}$ and $[D\Phi \circ \sigma^{\Phi'}_s : D\Phi]_s = \beta(q^{-s})$. Then, again by 5.2.3, there exists a strictly positive operator h affiliated to $Z(N)$ such that $\Phi = \Phi_{\beta(h)}$ with $[DT' : DT_L]_t = \beta(h^{it})$. □

Also, we have a similar result for right invariant operator-valued weight.

Corollary 5.4.2. – *If T_R a n.s.f. right invariant operator-valued weight such that $(\sigma^{\Psi'}_t {}_{\beta}\underset{N}{\star}{}_{\alpha} \tau_{-t}) \circ \Gamma = \Gamma \circ \sigma^{\Psi'}_t$, $\nu \circ \gamma' = \nu$ and $\gamma \circ \gamma' = \gamma' \circ \gamma$, then there exists a strictly positive operator h affiliated with $Z(N)$ such that:*

$$T_R = (R \circ T_L \circ R)_{\alpha(h)}$$

We state results of the section in the following theorems:

Theorem 5.4.3. – *Let $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ be a measured quantum groupoid. If T' a n.s.f. left invariant operator-valued weight such that $(\tau_t {}_{\beta}\underset{N}{\star}{}_{\alpha} \sigma^{\Phi'}_t) \circ \Gamma = \Gamma \circ \sigma^{\Phi'}_t$, $\nu \circ \gamma' = \nu$ and $\gamma \circ \gamma' = \gamma' \circ \gamma$, then there exists a strictly positive operator h affiliated with $Z(N)$ such that, for all $t \in \mathbb{R}$:*

$$\nu \circ \alpha^{-1} \circ T' = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)}$$

We have a similar result for the right invariant operator-valued weights.

THEOREM 5.4.4. – *Let $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ be a measured quantum groupoid. Then there exists a strictly positive operator δ affiliated with M called modulus and then there exists a strictly positive operator λ affiliated with $Z(M) \cap \alpha(N) \cap \beta(N)$ called scaling operator such that $[D\nu \circ \alpha^{-1} \circ T_L \circ R : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{\frac{it^2}{2}} \delta^{it}$ for all $t \in \mathbb{R}$.*

Moreover, we have, for all $s, t \in \mathbb{R}$:

i)
$$[D\nu \circ \alpha^{-1} \circ T_L \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{-ist}$$
$$[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist}$$
$$[D\nu \circ \alpha^{-1} \circ T_L \circ \sigma_s^{\nu \circ \alpha^{-1} \circ T_L \circ R} : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{ist}$$
$$[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \sigma_s^{\nu \circ \alpha^{-1} \circ T_L} : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist}$$

ii) $R(\lambda) = \lambda$, $R(\delta) = \delta^{-1}$, $\tau_t(\delta) = \delta$ *and* $\tau_t(\lambda) = \lambda$;

iii) δ *is a group-like element* i.e., $\Gamma(\delta) = \delta {}_\beta\!\otimes_\alpha\!{}_N \delta$.

CHAPTER 6

A DENSITY THEOREM

In this section, we prove that there are sufficiently enough operators which are both bounded under the left-invariant operator-valued weight and the right-invariant operator-valued weight. This allows, as a corollary, to found bounded elements for both α and β which will be useful for duality. This chapter is mostly inspired by chapter 7 of [**Eno04**].

LEMMA 6.0.5. – *Let $y, z \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ and $\xi \in D((H_\Psi)_\beta, \mu^o)$, then we have:*

$$\begin{aligned}
&[(\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H)]^* (\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H) \\
&\leq ||R^{\beta,\nu^0}(\xi)||^2 (\omega_{J_\Psi \Lambda_\Psi(z)} {}_\beta\!\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(y^* y))
\end{aligned}$$

For all $y \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, $z \in M$ and $\xi \in D((H_\Psi)_\beta, \mu^o)$, then we have:

$$\begin{aligned}
&R\left([(\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H)]^* (\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H)\right) \\
&\leq ||R^{\beta,\nu^0}(\xi)||^2 (\omega_{J_\Psi \Lambda_\Psi(y)} {}_\beta\!\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(z^* z))
\end{aligned}$$

Proof. – The first inequality comes straightforward from theorem 3.7.4. Then, apply R to get for all $z \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$:

$$\begin{aligned}
&R\left([(\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H)]^* (\omega_{\Lambda_\Psi(y), J_\Psi z^* J_\Psi \xi} * \mathrm{id})(U'_H)\right) \\
&\leq ||R^{\beta,\nu^0}(\xi)||^2 R(\omega_{J_\Psi \Lambda_\Psi(z)} {}_\beta\!\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(y^* y)) \\
&= ||R^{\beta,\nu^0}(\xi)||^2 (\omega_{J_\Psi \Lambda_\Psi(y)} {}_\beta\!\underset{\mu}{\star}_\alpha \mathrm{id})(\Gamma(z^* z))
\end{aligned}$$

Let us assume now that $z \in M$. Using Kaplansky' s theorem, there exist a family z_i in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, weakly converging to z, with $||z_i|| \leq ||z||$. Then we infer that $R^{\beta,\nu^0}(J_\Psi z_i^* J_\Psi \xi)$ is weakly converging to $R^{\beta,\nu^0}(J_\Psi z^* J_\Psi \xi)$ with:

$$||R^{\beta,\nu^0}(J_\Psi z^* J_\Psi \xi)|| \leq ||R^{\beta,\nu^0}(J_\Psi z^* J_\Psi \xi)||$$

Therefore $(\omega_{\Lambda_\Psi(y),J_\Psi z_i^* J_\Psi\xi} * \mathrm{id})(U'_H)$ is weakly converging to $(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)$ with:

$$||(\omega_{\Lambda_\Psi(y),J_\Psi z_i^* J_\Psi\xi} * \mathrm{id})(U'_H)|| \leq ||(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)||$$

which finishes the proof. □

PROPOSITION 6.0.6. – *If* $z \in \mathcal{N}_{T_L}$, $y \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ *and* $\xi \in D((H_\Psi)_\beta, \mu^o)$ *then* $(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)$ *belongs to* $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$.

Proof. – By the previous lemma and by right left-invariance of Φ, we have:

$$\begin{aligned}
&\Psi\left((\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)^*(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)\right)\\
&= \Phi\circ R\left([(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)]^*(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)\right)\\
&\leq ||R^{\beta,\nu^0}(\xi)||^2\omega_{J_\Psi\Lambda_\Psi(y)}(\mathrm{id}_\beta \underset{\mu}{\star}_\alpha \Phi)(\Gamma(z^*z)) = ||R^{\beta,\nu^0}(\xi)||^2\omega_{J_\Psi\Lambda_\Psi(y)}(T_L(z^*z))
\end{aligned}$$

Also, we have:

$$\begin{aligned}
&T_R\left((\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)^*(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)\right)\\
&= R\circ T_L\circ R\left([(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)]^*(\omega_{\Lambda_\Psi(y),J_\Psi z^* J_\Psi\xi} * \mathrm{id})(U'_H)\right)\\
&\leq ||R^{\beta,\nu^0}(\xi)||^2(\omega_{J_\Psi\Lambda_\Psi(y)\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(\mathrm{id}_\beta\underset{\mu}{\star}_\alpha T_L)(\Gamma(z^*z))\\
&= ||R^{\beta,\nu^0}(\xi)||^2(\omega_{J_\Psi\Lambda_\Psi(y)\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(T_L(z^*z)_\beta\underset{\mu}{\otimes}_\alpha 1)\\
&\leq ||R^{\beta,\nu^0}(\xi)||^2||T_L(z^*z)||||T_R(y^*y)||1
\end{aligned}$$

□

LEMMA 6.0.7. – *For all* $y, z \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ *and* $\eta \in D((H_\Psi)_\beta, \mu^o)$, *we have:*

$$\begin{aligned}
&R[(\omega_{y^*J_\Psi\eta,J_\Psi\Lambda_\Psi(z)} * \mathrm{id})(U'_H)]R[(\omega_{y^*J_\Psi\eta,J_\Psi\Lambda_\Psi(z)} * \mathrm{id})(U'_H)]^*\\
&\leq ||T_R(y^*y)||^2(\omega_{\eta\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(\Gamma(zz^*))
\end{aligned}$$

Proof. – Let us compute:

$$\begin{aligned}
&R[(\omega_{y^*J_\Psi\eta,J_\Psi\Lambda_\Psi(z)} * \mathrm{id})(U'_H)]R[(\omega_{y^*J_\Psi\eta,J_\Psi\Lambda_\Psi(z)} * \mathrm{id})(U'_H)]^*\\
&= (\omega_{\Lambda_\Psi(z),J_\Psi y^*J_\Psi\eta} * \mathrm{id})(U'_H)(\omega_{\Lambda_\Psi(z),J_\Psi y^*J_\Psi\eta} * \mathrm{id})(U'_H)^*\\
&= (\omega_{J_\Psi\Lambda_\Psi(y),\eta\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(\Gamma(z))(\omega_{J_\Psi\Lambda_\Psi(y),\eta\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(\Gamma(z))^*\\
&\leq ||T_R(y^*y)||^2(\omega_{\eta\,\beta}\underset{\mu}{\star}_\alpha\mathrm{id})(\Gamma(zz^*))
\end{aligned}$$

□

PROPOSITION 6.0.8. – *Let* $y_1, z' \in \mathcal{N}_{T_R}\cap\mathcal{N}_\Psi$, $y_2 \in \mathcal{N}_{T_R}\cap\mathcal{N}_\Psi\cap\mathcal{N}_\Phi$, $z \in R(\mathcal{T}^{\Psi}_{\Phi,T_L})^*$ *defined in proposition 5.3.6 and* e_n *the analytic elements associated to the Radon-Nikodym derivative* δ *defined in* **[Vae01a]**. *Then the operators* $(\omega_{y_1^*\Lambda_\Psi(y_2),J_\Psi z^* e_n^*\Lambda_\Psi(z')} * \mathrm{id})(U'_H)$ *belong to* $\mathcal{N}_{T_R}\cap\mathcal{N}_\Psi\cap\mathcal{N}_{T_L}\cap\mathcal{N}_\Phi$.

Proof. – Let us write $X = (\omega_{y_1^*\Lambda_\Psi(y_2), J_\Psi z^* e_n^* \Lambda_\Psi(z')} * \mathrm{id})(U'_H)$. Since $y_1^* y_2$ belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ and z belongs to $R(\mathcal{N}_{T_R})^* = \mathcal{N}_{T_L}$ and therefore $e_n z$ belongs to $\mathcal{N}_{T_R}$, we get, using proposition 6.0.6, that X belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$. On the other hand, since $y_1, y_2, z^* e_n^* z'$ belong to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, we can use lemma 6.0.7 to get that:

$$\begin{aligned} R(X)R(X)^* &\leq ||T_R(y_1^* y_1)|| (\omega_{J_\Psi \Lambda_\Psi(y_2)} {}_\beta\star_\alpha^\mu \mathrm{id})(\Gamma(z^* e_n^* z' z'^* e_n z)) \\ &\leq ||T_R(y_1^* y_1)|| \, ||z'||^2 (\omega_{J_\Psi \Lambda_\Psi(y_2)} {}_\beta\star_\alpha^\mu \mathrm{id})(\Gamma(z^* e_n^* e_n z)) \end{aligned}$$

Let us apply T_R to this inequality, we get that:

$$T_R(R(X)R(X)^*) \leq ||T_R(y_1^* y_1)|| \, ||z'||^2 T_R(\omega_{J_\Psi \Lambda_\Psi(y_2)} {}_\beta\star_\alpha^\mu \mathrm{id})(\Gamma(z^* e_n^* e_n z))$$

which is equal, thanks to lemma 5.3.8, to:

$$||T_R(y_1^* y_1)|| \, ||z'||^2 \omega_{\delta^{-\frac{1}{2}} J_\Psi \Lambda_\Psi(y_2)}(T_L(\delta^{-\frac{1}{2}} z^* e_n^* e_n z \delta^{-\frac{1}{2}}))$$

With the hypothesis, we get that $\delta^{\frac{1}{2}} z \delta^{-\frac{1}{2}}$ belongs to $\mathcal{N}_{T_L}$ and therefore $e_n z \delta^{-\frac{1}{2}} = (e_n \delta^{-\frac{1}{2}}) \delta^{\frac{1}{2}} z \delta^{-\frac{1}{2}}$ belongs also to $\mathcal{N}_{T_L}$. We also get that $J_\Psi \Lambda_\Psi(y_2)$ belongs to the domain of $\delta^{-\frac{1}{2}}$ which proves that $R(X)^*$ belongs to $\mathcal{N}_{T_R}$ and therefore X belongs to $\mathcal{N}_{T_L}$. We prove by similar computations that X belongs to $\mathcal{N}_\Phi$. □

THEOREM 6.0.9. – *The left ideal $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ is dense in M and $\Lambda_\Psi(\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi)$ is dense in H.*

Proof. – Let y be in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_\Phi$ and z in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$. Taking, by Kaplansky's theorem, a bounded family e_i in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$ strongly converging to 1, we get that $R^{\hat{\alpha},\mu}(e_i^* \Lambda_\Psi(y))$ is weakly converging to $R^{\hat{\alpha},\mu}(\Lambda_\Psi(y))$. Taking also a bounded family f_k in $R(\mathcal{T}^\Phi_{T_R,\Psi})^*$ strongly converging to 1, we get that $R^{\beta,\mu^0}(J_\Psi f_k^* e_n^* \Lambda_\Psi(z))$ is weakly converging, when n, k go to infinity, to $R^{\beta,\mu^0}(\Lambda_\Psi(z))$. Therefore, using the previous proposition, we get that $(\omega_{\Lambda_\Psi(y), J_\Psi \Lambda_\Psi(z)} * \mathrm{id})(U'_H)$ belongs to the weak closure of $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. By proposition 5.3.6, we get that, for any $x \in \mathcal{T}_{T_R,\Psi}$, there exists y_i in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_\Phi$ such that $\Lambda_{T_R}(y_i)$ is weakly converging to $\Lambda_{T_R}(x)$ or equivalently $R^{\hat{\alpha},\mu}(\Lambda_\Psi(y_i))$ is weakly converging to $R^{\hat{\alpha},\mu}(\Lambda_\Psi(x))$. Therefore, we get that $(\omega_{\Lambda_\Psi(x), J_\Psi \Lambda_\Psi(z)} * \mathrm{id})(U'_H)$ belongs to the weak closure of $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. It remains true for x in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}^*_{T_R} \cap \mathcal{N}^*_\Psi$ by density. If now x belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, and h_i is a bounded family in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, since $\Lambda_{T_R}(h_i^* x) = h_i^* \Lambda_{T_R}(x)$ is weakly converging to $\Lambda_{T_R}(x)$, we finally obtain that, for any x, z in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, the operator $(\omega_{\Lambda_\Psi(y), J_\Psi \Lambda_\Psi(z)} * \mathrm{id})(U'_H)$ belongs to the weak closure of $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. By density, for all $\xi \in D({}_{\hat{\alpha}}H, \mu)$ and $\eta \in D(H_\beta, \mu^0)$, the operator $(\omega_{\xi,\eta} * \mathrm{id})(U'_H)$ belongs to the weak closure of $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. Which proves the density of $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ in M by theorem 4.0.6.

Let g_n an increasing sequence of positive elements of $\mathcal{M}_{T_R} \cap \mathcal{M}_\Psi \cap \mathcal{M}_{T_L} \cap \mathcal{M}_\Phi$ strongly converging to 1. The operators:

$$h_n = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_t^\Psi(g_n)\ dt$$

are in $\mathcal{M}_{T_R} \cap \mathcal{M}_\Psi$, analytic with respect to Ψ, and, for any $z \in \mathbb{C}$, $\sigma_z^\Psi(h_n)$ is a bounded sequence strongly converging to 1. Let now $\lambda = \int_0^{+\infty} t\ de_t$ be the scaling operator. Let us write $h'_n = \big(\int_{\frac{1}{n}}^n de_t\big) h_n$. These operators are in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$, analytic with respect to Ψ, and, for any $z \in \mathbb{C}$, $\sigma_z^\Psi(h'_n)$ is a bounded sequence strongly converging to 1. Moreover the operators h'_n belong also to $\mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ by lemma 5.1.5 and [**Vae01a**]. Let now x be in $\mathcal{N}_\Psi$. We get that xh'_n belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$ and that:

$$\Lambda_\Psi(xh'_n) = J_\Psi \sigma_{-i/2}^\Psi(h'_n) J_\Psi \Lambda_\Psi(x)$$

is converging to $\Lambda_\Psi(x)$ which finishes the proof. □

THEOREM 6.0.10. – *Let $\mathcal{T}_{T_R,\Psi,T_L,\Phi}$ be the subset of elements x in $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$, analytic with respect to both Φ and Ψ, and such that, for all $z, z' \in \mathbb{C}$, $\sigma_z^\Psi \circ \sigma_{z'}^\Phi(x)$ belongs to $\mathcal{N}_{T_R} \cap \mathcal{N}_\Psi \cap \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi$. Then $\mathcal{T}_{T_R,\Psi,T_L,\Phi}$ is dense in M and $\Lambda_\Psi(\mathcal{T}_{T_R,\Psi,T_L,\Phi})$ is dense in H.*

Proof. – Let x be a positive operator in $\mathcal{M}_{T_R} \cap \mathcal{M}_\Psi \cap \mathcal{M}_{T_L} \cap \mathcal{M}_\Phi$. Let now $\lambda = \int_0^\infty t\ de_t$ be the scaling operator and let us define:

$$x_n = \left(\int_{\frac{1}{n}}^n de_t\right) \frac{n}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-n(t^2+s^2)} \sigma_t^\Psi \sigma_s^\Phi(x)\ ds\, dt$$

It is not so difficult to see that x_n is analytic both with respect to Φ and Ψ. By lemma 5.1.5 and thanks to [**Vae01a**] and [**EN96**, 10.12], we see that the operators $\sigma_z^\Psi(x_n)$ and $\sigma_z^\Phi(x_n)$ are linear combinations of positive elements in $\mathcal{M}_{T_R} \cap \mathcal{M}_\Psi \cap \mathcal{M}_{T_L} \cap \mathcal{M}_\Phi$. □

COROLLARY 6.0.11. – *There exist a dense linear subspace E of $\mathcal{N}_\Phi$ such that $\Lambda_\Phi(E)$ is dense in $L^2(M, \Phi) = H$ and:*

$$J_\Phi \Lambda_\Phi(E) \subset D({}_\alpha H, \mu) \cap D(H_\beta, \nu^0)$$

Proof. – Let E be the linear subspace spanned by the elements of the form $e_n x$ where e_n are the analytic elements associated to the Radon-Nikodym derivative δ, defined in [**Vae01a**], and x belongs to $\mathcal{T}_{T_R,\Psi,T_L,\Phi}$. It is clear that E is a subset of $\mathcal{N}_\Phi$, dense in M and that $\Lambda_\Phi(E)$ is dense in H. Since $E \subset \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$, we have:

$$J_\Phi \Lambda_\Phi(E) \subset D({}_\alpha H, \mu)$$

Using [**Vae01a**], we get that:

$$J_\Phi \Lambda_\Phi(e_n x) = \delta^{-\frac{1}{2}} J_\Psi \Lambda_\Psi(e_n x)$$

Since $e_n x\delta^{-\frac{1}{2}} = (e_n\delta^{-\frac{1}{2}})\delta^{\frac{1}{2}}x\delta^{-\frac{1}{2}}$ and, by the previous theorem, that $\delta^{\frac{1}{2}}x\delta^{-\frac{1}{2}}$ is a bounded operator in $\mathcal{N}_{T_R}$, so is $e_n x\delta^{-\frac{1}{2}}$ and therefore, we have:

$$\delta^{-\frac{1}{2}}J_\Psi\Lambda_\Psi(e_n x) = \lambda^{\frac{1}{4}}J_\Psi\Lambda_\Psi(e_n x\delta^{-\frac{1}{2}}) \subset J_\Psi\Lambda_\Psi(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$$

and we get that $J_\Phi\Lambda_\Phi(e_n x)$ belongs to $D(H_\beta, \mu^0)$. By linearity, we get the result. □

CHAPTER 7

MANAGEABILITY OF THE FUNDAMENTAL UNITARY

In this section, we prove that the fundamental unitary satisfies a proposition similar to Woronowicz's manageability of [**Wor96**]. Following [**Eno02**, def. 4.1], we define the notion of weakly regular pseudo-multiplicative unitary which is interesting by itself but it will be useful for us to get easily von Neumann algebra structure on the dual structure.

Definition 7.0.12. – We call **manageable operator** the strictly positive operator P on H_Φ such that $P^{it}\Lambda_\Phi(x) = \lambda^{\frac{t}{2}}\Lambda_\Phi(\tau_t(x))$, for all $x \in \mathcal{N}_\Phi$ and $t \in \mathbb{R}$.

Proposition 7.0.13. – *For all $m \in M$, $n \in N$ and $t \in \mathbb{R}$, we have:*

$$P^{it}mP^{-it} = \tau_t(m) \qquad P^{it}\alpha(n)P^{-it} = \alpha(\sigma_t^\nu(n))$$
$$P^{it}\beta(n)P^{-it} = \beta(\sigma_t^\nu(n)) \qquad P^{it}\hat{\beta}(n)P^{-it} = \hat{\beta}(\sigma_t^\nu(n))$$

Proof. – Straightforward. □

Then, we can define operators $P^{it}{}_\beta\underset{\nu}{\otimes}{}_\alpha P^{it}$ on $H_\Phi{}_\beta\underset{\nu}{\otimes}{}_\alpha H_\Phi$ and $P^{it}{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} P^{it}$ on $H_\Phi{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} H_\Phi$ for all $t \in \mathbb{R}$.

Theorem 7.0.14. – *The unitary W satisfies a manageability relation. More exactly, we have:*

$$(\sigma_\nu W^*\sigma_\nu(q_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha v)|p_\alpha\underset{\nu^o}{\otimes}{}_\beta w) = (\sigma_{\nu^o}W\sigma_{\nu^o}(J_\Phi p_\alpha\underset{\nu^o}{\otimes}{}_\beta P^{-1/2}v)|J_\Phi q_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w)$$

for all $v \in \mathcal{D}(P^{-\frac{1}{2}})$, $w \in \mathcal{D}(P^{\frac{1}{2}})$ and $p, q \in D({}_\alpha H_\Phi, \nu) \cap D((H_\Phi)_{\hat{\beta}}, \nu^o)$. Moreover, for all $t \in \mathbb{R}$, we have $W(P^{it}{}_\beta\underset{\nu}{\otimes}{}_\alpha P^{it}) = (P^{it}{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} P^{it})W$.

Proof. – Let $p,q \in D({}_\alpha H_\Phi,\nu)\cap D((H_\Phi)_{\hat\beta},\nu^o)$. For all $v\in\mathcal{D}(D^{1/2})$ and $w\in\mathcal{D}(D^{-1/2})$, we know that:

$$(I(\mathrm{id}*\omega_{q,p})(W)Iv|w)=((\mathrm{id}*\omega_{p,q})(W)P^{1/2}v|P^{-1/2}w)$$

for all $v\in\mathcal{D}(P^{1/2})$ and $w\in\mathcal{D}(P^{-1/2})$. By 4.0.10, we rewrite the formula:

$$(\sigma_\nu W^*\sigma_\nu(q_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha v)|p_\alpha\underset{\nu^o}{\otimes}{}_\beta w)=(\sigma_{\nu^o}W\sigma_{\nu^o}(J_\Phi p_\alpha\underset{\nu^o}{\otimes}{}_\beta P^{-1/2}v)|J_\Phi q_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w)$$

Now, we have to prove $W^*(P^{it}{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}P^{it})=(P^{it}{}_\beta\underset{\nu}{\otimes}{}_\alpha P^{it})W^*$ for all $t\in\mathbb{R}$. First of all, because of the commutation relation between P and β, $D((H_\Phi)_\beta,\nu^o)$ is P^{it}-invariant and if $(\xi_i)_{i\in I}$ is a (N^o,ν^o)-basis of $(H_\Phi)_\beta$, then $(P^{it}\xi_i)_{i\in I}$ is also. Let $v\in D((H_\Phi)_\beta,\nu^o)$ and $a\in\mathcal{N}_{T_L}\cap\mathcal{N}_\Phi$. We compute:

$$\begin{aligned}&(P^{it}{}_\beta\underset{\nu}{\otimes}{}_\alpha P^{it})W^*(v_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))\\ &=\sum_{i\in I}P^{it}\xi_{i\beta}\underset{\nu}{\otimes}{}_\alpha\lambda^{t/2}\Lambda_\Phi(\tau_t((\omega_{v,\xi_i\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(a))))\\ &=\sum_{i\in I}P^{it}\xi_{i\beta}\underset{\nu}{\otimes}{}_\alpha\Lambda_\Phi((\omega_{P^{it}v,P^{it}\xi_i\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(\lambda^{t/2}\tau_t(a))))\\ &=W^*(P^{it}v_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\lambda^{t/2}\Lambda_\Phi(\tau_t(a)))=W^*(P^{it}{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}P^{it})(v_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(a))\end{aligned}$$

□

DEFINITION 7.0.15. – A pseudo-multiplicative unitary $\mathcal{W}$ w.r.t. $\alpha,\beta,\hat\beta$ is said to be **weakly regular** if the weakly closed linear span of $(\lambda_v^{\alpha,\beta})^*\mathcal{W}\rho_w^{\hat\beta,\alpha}$ where v,w belongs to $D({}_\alpha H,\nu)$ is equal to $\alpha(N)'$.

PROPOSITION 7.0.16. – *The operator $\widehat{W}=\sigma_\nu W^*\sigma_\nu$ from $H_{\Phi\hat\beta}\underset{\nu}{\otimes}{}_\alpha H_\Phi$ onto $H_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_\beta H_\Phi$ is a pseudo-multiplicative unitary over N w.r.t. $\alpha,\beta,\hat\beta$ which is weakly regular in the sense of* [**Eno02**, def. 4.1].

Proof. – By [**EV00**], we know that $\widehat{W}$ is a pseudo-multiplicative unitary. We also know that $\langle(\lambda_v^{\alpha,\beta})^*\widehat{W}\rho_w^{\hat\beta,\alpha}\rangle^{-\mathrm{w}}\subset\alpha(N)'$. For all $v\in\mathcal{D}(P^{-\frac12})$, $w\in\mathcal{D}(P^{\frac12})$ and $p,q\in D({}_\alpha H_\Phi,\nu)\cap D((H_\Phi)_{\hat\beta},\nu^o)$, we have, by theorem 7.0.14:

$$((\lambda_p^{\alpha,\beta})^*\widehat{W}\rho_v^{\hat\beta,\alpha}q|w)=(\sigma_{\nu^o}W\sigma_{\nu^o}(J_\Phi p_\alpha\underset{\nu^o}{\otimes}{}_\beta P^{-1/2}v)|J_\Phi q_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w)$$

and on the other hand:

$$\begin{aligned}
(R^{\alpha,\nu}(v)R^{\alpha,\nu}(p)^*q|w) &= (R^{\alpha,\nu}(v)J_\nu R^{\hat{\beta},\nu^o}(J_\Phi p)^* J_\Phi q|w) \\
&= (R^{\alpha,\nu}(v)J_\nu \Lambda_\nu(\langle J_\Phi q, J_\Phi p\rangle_{\hat{\beta},\nu_L^o})|w) \\
&= (P^{-1/2}R^{\alpha,\nu}(v)J_\nu \Lambda_\nu(\langle J_\Phi q, J_\Phi p\rangle_{\hat{\beta},\nu_L^o})|P^{1/2}w) \\
&= (R^{\alpha,\nu}(P^{-1/2}v)\Delta_\nu^{-1/2}J_\nu \Lambda_\nu(\langle J_\Phi q, J_\Phi p\rangle_{\hat{\beta},\nu_L^o})|P^{1/2}w) \\
&= (R^{\alpha,\nu}(P^{-1/2}v)\Lambda_\nu(\langle J_\Phi p, J_\Phi q\rangle_{\hat{\beta},\nu_L^o})|P^{1/2}w) \\
&= (\alpha(\langle J_\Phi p, J_\Phi q\rangle_{\hat{\beta},\nu_L^o})P^{-1/2}v|P^{1/2}w) \\
&= (J_\Phi p {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{-1/2}v|J_\Phi q {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w)
\end{aligned}$$

There exists $\Xi \in H_\Phi {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha H_\Phi$ such that $\sigma_{\nu^o}W\sigma_{\nu^o}\Xi = J_\Phi p {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{-1/2}v$ since W is onto. By definition, there exists a net $(\sum_{k=1}^{n(i)} J_\Phi p_k^i {}_\alpha\underset{\nu^o}{\otimes}{}_\beta P^{-1/2}v_k^i)_{i\in I}$ which converges to Ξ. Then $((\sum_{k=1}^{n(i)}(\lambda_{p_k^i}^{\alpha,\beta})^*\widehat{W}\rho_{v_k^i}^{\hat{\beta},\alpha}q|w))_{i\in I}$ converges to:

$$\begin{aligned}
(\sigma_{\nu^o}W\sigma_{\nu^o}\Xi|J_\Phi q {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w) &= (J_\Phi p {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{-1/2}v|J_\Phi q {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha P^{1/2}w) \\
&= (R^{\alpha,\nu}(v)R^{\alpha,\nu}(p)^*q|w)
\end{aligned}$$

Then, we obtain $\alpha(N)' = \langle R^{\alpha,\nu}(v)R^{\alpha,\nu}(p)^*\rangle^{-\mathrm{w}} \subset \langle(\omega_{v,p} * \mathrm{id})(\widehat{W}\sigma_{\nu^o})\rangle^{-\mathrm{w}}$. □

CHAPTER 8

DUALITY

In this section, a dual measured quantum groupoid is constructed thanks to modulus and scaling operator. Then, we obtain a bi-duality theorem which generalizes Pontryagin duality, locally compact quantum groups duality and duality for groupoids. Finally, we get Heisenberg's relations.

8.1. Dual structure

Definition 8.1.1

The weak closure of the linear span of $(\omega_{\xi,\eta} * \mathrm{id})(W)$, where $\xi \in D((H_\Phi)_\beta, \nu^o)$ and $\eta \in D({}_\alpha H_\Phi, \nu)$, is denoted by $\widehat{M}$. It's a von Neumann algebra because weak regularity of $\hat{W}$ (prop. 7.0.16) and [**Eno02**, prop. 3.2].

Definition 8.1.2. – We put $\hat{\Gamma}$ the application from $\widehat{M}$ into $\mathcal{L}(H_\Phi {}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha H_\Phi)$ such that, for all $x \in \widehat{M}$, we have:

$$\hat{\Gamma}(x) = \sigma_{\nu^o} W(x {}_\beta\underset{N}{\otimes}{}_\alpha 1)W^* \sigma_\nu$$

Proposition 8.1.3. – *The 5-uple $(N, \widehat{M}, \alpha, \hat{\beta}, \hat{\Gamma})$ is a Hopf-bimodule called dual Hopf-bimodule.*

Proof. – The proposition comes from theorems 6.2 and 6.3 of [**EV00**] applied to $\widehat{W} = \sigma_\nu W^* \sigma_\nu$. □

Lemma 8.1.4. – *Let call $M_*^{\alpha,\beta}$ the subspace of M_* spanned by the positive and normal forms such that there exists $k \in \mathbb{R}^+$ and both $\omega \circ \alpha$ and $\omega \circ \beta$ are dominated by $k\nu$. Then, $M_*^{\alpha,\beta}$ is dense *-subalgebra of M_* such that, for all $m \in M$, we have:*

$$\omega\mu(m) = \mu((\omega {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(m))) \quad \textit{and} \quad \omega^*(m) = \overline{\omega \circ R(m^*)}$$

Proof. – By definition $\omega\mu$ belongs to M_*. There exists $\xi \in D(H_\beta, \nu^o)$ such that $\omega = \omega_\xi$. For all $n \in N$, we have:

$$\omega\mu(\alpha(n^*n)) = \mu((\omega_\xi {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(\alpha(n^*n))))$$
$$= \mu((\lambda_\xi^{\beta,\nu^0})^*(\alpha(n^*n)_\beta\underset{N}{\otimes}{}_\alpha 1)\lambda_\xi^{\beta,\nu^0}) = \mu((\lambda_{\alpha(n)\xi}^{\beta,\nu^0})^*\lambda_{\alpha(n)\xi}^{\beta,\nu^0})$$
$$= \mu\circ\alpha(\langle\alpha(n)\xi,\alpha(n)\xi\rangle_{\beta,\nu^0}) \leq k\nu(\langle\alpha(n)\xi,\alpha(n)\xi\rangle_{\beta,\nu^0}) = k||\alpha(n)\xi||^2$$
$$= k\omega\circ\alpha(n^*n) \leq k^2\nu(n^*n)$$

Also, we can prove that $\omega\mu\circ\beta$ is dominated by $k^2\nu$ so that $\omega\mu$ belongs to $M_*^{\alpha,\beta}$. Since $R\circ\alpha = \beta$, $M_*^{\alpha,\beta}$ is *-stable. We have to prove associativity of product and that $(\omega\mu)^* = \mu^*\omega^*$. The first property comes from co-associativity of co-product and the second one comes from co-involution property. We only check the first one because the second proof is very similar computation. Let $\omega,\mu,\chi \in M_*^{\alpha,\beta}$ and $\xi,\xi',\xi'' \in D(H_\beta,\nu^o)$ the corresponding vectors. Then, for all $m \in M$, it is easy to see that:

$$(\omega\mu)\chi(x) = ((\Gamma {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(x))(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')|\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')$$
$$= ((\mathrm{id} {}_\beta\underset{\nu}{\star}{}_\alpha \Gamma)(\Gamma(x))(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')|\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')$$
$$= (\Gamma((\omega_\xi {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(x)))(\xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')|\xi' {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi'')$$
$$= \mu\chi((\omega_\xi {}_\beta\underset{\nu}{\star}{}_\alpha \mathrm{id})(\Gamma(x))) = \omega(\mu\chi)(x)$$

Density condition comes from corollary 6.0.11 for example. □

COROLLARY 8.1.5. – *The contractive application $\widehat{\pi}$ from $M_*^{\alpha,\beta}$ to $\widehat{M}$ such that $\widehat{\pi}(\omega) = (\omega * \mathrm{id})(W)$ is 1-1 and multiplicative.*

Proof. – The application $\widehat{\pi}$ is injective because of theorem 4.0.6. We prove multiplicativity of $\widehat{\pi}$ for positive linear forms because the general case comes then from linearity. Let $\xi,\eta \in D({}_\alpha H,\nu) \cap D(H_\beta,\nu^o)$, $\zeta_1 \in D({}_\alpha H,\nu)$ and $\zeta_2 \in D(H_{\hat{\beta}},\nu^o)$. By proposition 3.6.3 of the first part, we know that:

$$((\omega_\xi * \mathrm{id})(W)(\omega_\eta * \mathrm{id})(W)\zeta_1|\zeta_2)$$

is equal to the scalar product of

$$(\sigma_{\nu^o} {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} 1)(1 {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} W)\sigma_{2\nu}(1 {}_\beta\underset{N}{\otimes}{}_\alpha \sigma_{\nu^o})(1 {}_\beta\underset{N}{\otimes}{}_\alpha W)(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta {}_\beta\underset{\nu}{\otimes}{}_\alpha \zeta_1)$$

by $[\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta] {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} \zeta_2$. Then, by pseudo-multiplicativity of W, this equal to:

$$((W^* {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} 1)(1 {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} W)(W {}_\beta\underset{N}{\otimes}{}_\alpha 1)(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta {}_\beta\underset{\nu}{\otimes}{}_\alpha \zeta_1)|[\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta] {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} \zeta_2)$$

$$= ((1 {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} W)W(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta) {}_\beta\underset{\nu}{\otimes}{}_\alpha \zeta_1|W(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} \zeta_2)$$
$$= ((1 {}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} (\mathrm{id} * \omega_{\zeta_1,\zeta_2})(W))W(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta)|W(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta))$$

Since Γ is implemented by W, this is equal to:

$$(\Gamma((\mathrm{id}*\omega_{\zeta_1,\zeta_2})(W))(\xi_\beta\underset{\nu}{\otimes}_\alpha\eta)|(\xi_\beta\underset{\nu}{\otimes}_\alpha\eta)) = (\omega_\xi\omega_\eta)((\mathrm{id}*\omega_{\zeta_1,\zeta_2})(W))$$
$$= (((\omega_\xi\omega_\eta)*\mathrm{id})(W)\zeta_1|\zeta_2)$$

By density of $D(_\alpha H,\nu)$ and $D(H_{\hat{\beta}},\nu^o)$ in H, we get that $\hat{\pi}$ is multiplicative. □

To get a measured quantum groupoid from the dual Hopf-bimodule, we have to exhibit, first of all, a co-involution. This is done and the following proposition:

Proposition 8.1.6. – *There exists a unique *-anti-automorphism $\widehat{R}$ of $\widehat{M}$ such that, for all $\omega\in M_*^{\alpha,\beta}$, we have $\widehat{R}(\widehat{\pi}(\omega))=\widehat{\pi}(\omega\circ R)$. Moreover $\widehat{R}(x)=J_\Phi x^* J_\Phi$ for all $x\in\widehat{M}$ and $\widehat{R}$ is a co-involution.*

Proof. – For all $\xi\in D(_\alpha(H_\Phi),\nu)$ and $\eta\in D((H_\Phi)_\beta,\nu^o)$, we have:

$$(J_\Phi\hat{\pi}(\omega\circ R)^*J_\Phi\xi|\eta) = (\hat{\pi}(\omega\circ R)J_\Phi\eta|J_\Phi\xi) = ((\omega\circ R*\mathrm{id})(W)J_\Phi\eta|J_\Phi\xi)$$
$$= \omega\circ R((\mathrm{id}*\omega_{J_\Phi\eta,J_\Phi\xi}(W)) = \omega((\mathrm{id}*\omega_{\xi,\eta})(W)) = (\hat{\pi}(\omega)\xi|\eta)$$

So, if we define $\widehat{R}$ by $\widehat{R}(x)=J_\Phi x^*J_\Phi$ for all $x\in\widehat{M}$, we obtain a *-anti-automorphism of $\widehat{M}$ such that, for all $\omega\in M_*^{\alpha,\beta}$, we have $\widehat{R}(\widehat{\pi}(\omega))=\widehat{\pi}(\omega\circ R)$. Uniqueness comes from density of $\widehat{\pi}(M_*^{\alpha,\beta})$ in $\widehat{M}$. By definition, we have $\widehat{R}\circ\alpha=\hat{\beta}$. So, we have to check co-involution property to finish the proof. For all $\omega\in M_*^{\alpha,\beta}$, we compute:

$$\hat{\Gamma}(\hat{\pi}(\omega)) = \hat{W}^*(1_\alpha\underset{N^o}{\otimes}_\beta(\omega*\mathrm{id})(W))\hat{W} = \sigma_{\nu^o}W((\omega*\mathrm{id})(W)_\beta\underset{N}{\otimes}_\alpha 1)W^*\sigma_\nu$$
$$= \sigma_{\nu^o}(\omega*\mathrm{id}*id)((1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)(W_\beta\underset{N}{\otimes}_\alpha 1)(1_\beta\underset{N}{\otimes}_\alpha W^*))\sigma_\nu$$

By pseudo-multiplicativity of W, this is equal to:

$$\sigma_{\nu^o}(\omega*\mathrm{id}*id)((W_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}1)(\sigma_{\nu^o\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}1)(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)\sigma_{2\nu}(1_\beta\underset{N}{\otimes}_\alpha\sigma_{\nu^o}))\sigma_\nu$$
$$= (\omega*\mathrm{id}*id)((1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}\sigma_{\nu^o})(W_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}1)(\sigma_{\nu^o\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}1)(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)\sigma_{2\nu})$$

Then, we get:

$$\hat{\Gamma}\circ\hat{R}(\hat{\pi}(\omega)) = \hat{\Gamma}(\hat{\pi}(\omega\circ R))$$
$$= (\omega\circ R*\mathrm{id}*id)((1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}\sigma_{\nu^o})(W_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}1)(\sigma_{\nu^o\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}1)(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)\sigma_{2\nu})$$

Now, by proposition 4.0.10, we know that: $W=(I_\beta\underset{N}{\otimes}_\alpha J_\Phi)W^*(I_\beta\underset{N}{\otimes}_\alpha J_\Phi)$ so that:

$$(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}\sigma_{\nu^o})(W_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}1)(\sigma_{\nu^o\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}1)(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)\sigma_{2\nu}$$
$$= (I_\alpha\underset{N^o}{\otimes}_\beta J_{\Phi\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}J_\Phi)[(W_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}1)(\sigma_{\nu^o\alpha}\underset{N^o}{\otimes}_{\hat{\beta}}1)(1_\alpha\underset{N^o}{\otimes}_{\hat{\beta}}W)\sigma_{2\nu}(1_\beta\underset{N}{\otimes}_\alpha\sigma_{\nu^o})]^*(I_\beta\underset{N}{\otimes}_\alpha J_{\Phi\hat{\beta}}\underset{N}{\otimes}_\alpha J_\Phi)$$

Since R is implemented by I and $\widehat{R}$ is implemented by J_Φ, we have:

$$\begin{aligned}&\hat{\Gamma}\circ\hat{R}(\hat{\pi}(\omega))\\ &= (\hat{R}_{\alpha}\underset{N^o}{\star}{}_{\hat{\beta}}\hat{R})((\omega * \mathrm{id})[(W_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)(\sigma_{\nu^o}{}_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}1)(1_{\alpha}\underset{N^o}{\otimes}{}_{\hat{\beta}}W)\sigma_{2\nu}(1_{\beta}\underset{N}{\otimes}{}_{\alpha}\sigma_{\nu^o})])\\ &= (\hat{R}_{\alpha}\underset{N^o}{\star}{}_{\hat{\beta}}\hat{R})\circ\varsigma_N\circ\hat{\Gamma}(\hat{\pi}(\omega)) = \varsigma_{N^o}\circ(\hat{R}_{\hat{\beta}}\underset{N}{\star}{}_{\alpha}\hat{R})\circ\hat{\Gamma}(\hat{\pi}(\omega))\end{aligned}$$

A density argument enables us to conclude. □

Then, we have to construct a left-invariant operator-valued weight $\widehat{T_L}$ from $\widehat{M}$ to $\alpha(N)$. We follow J. Kustermans and S. Vaes' paper **[KV00]**: we define in fact a GNS construction $(H, \iota, \widehat{\Lambda})$ and we give a core for $\widehat{\Lambda}$. Let introduce the space $\mathscr{I}$ of $\omega \in M_*^{\alpha,\beta}$ such that there exists $k \in \mathbb{R}^+$ and $|\omega(x^*)| \leq k||\Lambda_\Phi(x)||$ for all $x \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$. Then, by Riesz' theorem, there exists $\xi(\omega) \in H$ such that:

$$\omega(x^*) = (\xi(\omega)|\Lambda_\Phi(x))$$

Lemma 8.1.7. – *The set $\{\xi(\omega)|\ \omega \in \mathscr{I}\}$ is dense in H.*

Proof. – Let $a, b \in E$ define in corollary 6.0.11. Then $\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)}$ belongs to $M_*^{\alpha,\beta}$ and we have, for all $x \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$:

$$\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)}(x^*) = \Phi(b^*x^*a) = \Phi(x^*a\sigma^\Phi_{-i}(b^*)) = (\Lambda_\Phi(a\sigma^\Phi_{-i}(b^*))|\Lambda_\Phi(x))$$

so that $\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)}$ belongs to $\mathscr{I}$ and we have $\xi(\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)}) = \Lambda_\Phi(a\sigma^\Phi_{-i}(b^*))$ which is dense in H. □

In the following, for all form ω, we denote by $\overline{\omega}$ the form such that $\overline{\omega}(x) = \overline{\omega(x^*)}$. Observe that $\omega \in M_*^{\alpha,\beta}$ implies that $\overline{\omega}$ belongs also to $M_*^{\alpha,\beta}$.

Proposition 8.1.8. – *The space $\mathscr{I}$ is a dense left ideal of $M_*^{\alpha,\beta}$ such that, for all $\omega \in M_*^{\alpha,\beta}$ and $\mu \in \mathscr{I}$, we have:*

$$\xi(\omega\mu) = \hat{\pi}(\omega)\xi(\mu)$$

Proof. – If ξ, η belong to $D(_\alpha H, \nu) \cap D(H_\beta, \nu^0)$, then $\omega_{\xi,\eta}$ belongs to $M_*^{\alpha,\beta}$. Moreover, if η belongs also to $D(_{\mathrm{id}}H_\Phi, \Phi) = J_\Phi\Lambda_\Phi(\mathscr{N}_\Phi)$, then we have:

$$|\omega_{\xi,\eta}(x^*)| = |(\xi|x\eta)| \leq ||\xi||||x\eta|| \leq k||\xi||||\Lambda_\Phi(x)||$$

so that, by corollary 6.0.11, we can deduce that $\mathscr{I}$ is dense in $M_*^{\alpha,\beta}$ and therefore in M_*. Now, for all $x \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$, we have:

$$\begin{aligned}\omega\mu(x^*) &= \mu((\omega_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma(x^*)) = \mu(((\overline{\omega}_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma(x))^*)\\ &= (\xi(\mu)|\Lambda_\Phi((\overline{\omega}_\beta\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma(x))) = (\xi(\mu)|(\overline{\omega} * \mathrm{id})(W^*)\Lambda_\Phi(x))\\ &= ((\omega * \mathrm{id})(W)\xi(\mu)|\Lambda_\Phi(x))\end{aligned}$$

so that the proposition holds. □

Definition 8.1.9. – For all $t \in \mathbb{R}$ and $\omega \in M_*$, we define elements of M_* such that, for all $x \in M$:

$$\tau_t^*(\omega)(w) = \omega \circ \tau_t(x), \quad \delta_t^*(\omega)(x) = \omega(\delta^{it}x), \quad \text{and} \quad \rho_t(\omega)(x) = \omega(\delta^{-it}\tau_{-t}(x))$$

Proposition 8.1.10. – *The applications τ^*, δ^* and ρ define strongly continuous one-parameter groups of *-automorphisms of $M_*^{\alpha,\beta}$. Moreover, they leave $\mathscr{I}$ stable and, for all $t \in \mathbb{R}$ and $\omega \in \mathscr{I}$, we have:*

$$\xi(\tau_t^*(\omega)) = \lambda^{-\frac{t}{2}} P^{-it}\xi(\omega), \quad \xi(\delta_t^*(\omega)) = \lambda^{\frac{t}{2}} J_\Phi \delta^{-it} J_\Phi \xi(\omega),$$
$$\text{and} \quad \xi(\rho_t(\omega)) = P^{it} J_\Phi \delta^{it} J_\Phi \xi(\omega)$$

Proof. – Since $\tau_t(\delta) = \delta$, it is easy to see that τ^* and δ^* commute with each other and, for all $t \in \mathbb{R}$, we have $\rho_t = \tau_{-t}^* \circ \delta_{-t}^*$ so that the last statement comes from the two first one. Since τ is implemented by P, τ^* defines a strongly continuous one-parameter representation of M_*. It is the same for δ^*. If ω belongs to $M_*^{\alpha,\beta}$, then there exists $k \in \mathbb{R}+$ such that, for all $t \in \mathbb{R}$, we have:

$$\tau_t^*(\omega) \circ \alpha = \omega \circ \tau_t \circ \alpha = \omega \circ \alpha \circ \sigma_t^\nu \leq k\nu \circ \sigma_t^\nu = k\nu$$

Moreover, there exists $\xi \in D({}_\alpha H, \nu) \cap D(H_\beta, \nu^0)$ such that $\omega = \omega_\xi$ and, for all $t \in \mathbb{R}$ and $n \in N$, we have:

$$\begin{aligned}\delta_t(\omega)(\alpha(n^*n)) &= (\delta^{it}\alpha(n^*n)\xi|\xi) = (\alpha(n)\xi|\alpha(n)\delta^{-it}\xi)\\ &= (\alpha(n)\xi|\delta^{-it}\alpha(\gamma_t\sigma_t^\nu(n)\xi))\end{aligned}$$

so that we get:

$$|\delta_t^*(\omega)(\alpha(n^*n))| \leq k||\Lambda_\nu(n)||^2 = k\nu(n^*n)$$

A similar proof with β allows us to deduce that τ^*, δ^* and ρ belongs to $M_*^{\alpha,\beta}$ as soon as ω belongs to $M_*^{\alpha,\beta}$. It is also straightforward to check that τ_t^* is a *-automorphism of $M_*^{\alpha,\beta}$ thanks to $\Gamma \circ \tau_t = (\tau_t {}_\beta\star_\alpha^{}{}_N \tau_t) \circ \Gamma$ and the commutation between τ and R. Also, it is also straightforward to check that δ_t^* is a *-automorphism of $M_*^{\alpha,\beta}$ thanks to $\Gamma(\delta) = \delta {}_\beta\otimes_\alpha{}_N \delta$ and $R(\delta) = \delta^{-1}$. Finally, for all $x \in \mathscr{N}_\Phi \cap \mathscr{N}_{T_L}$, we have, on one hand:

$$\begin{aligned}\tau_t(\omega)(x^*) &= \omega(\tau_t(x^*)) = \omega \circ \tau_t(x^*) = (\xi(\omega)|\Lambda_\Phi(\tau_t(x)))\\ &= (\xi(\omega)|\lambda^{\frac{-t}{2}} P^{it}\Lambda_\Phi(x)) = (\lambda^{\frac{-t}{2}} P^{-it}\xi(\omega)|\Lambda_\Phi(x))\end{aligned}$$

and on the other hand:

$$\begin{aligned}\delta_t(\omega)(x^*) &= \omega((x\delta^{-it})^*) = (\xi(\omega)|\Lambda_\Phi(x\delta^{-it}))\\ &= (\xi(\omega)|J_\Phi\delta^{-it}J_\Phi\lambda^{\frac{t}{2}}\Lambda_\Phi(x)) = (\lambda^{\frac{t}{2}} J_\Phi\delta^{it}J_\Phi\xi(\omega)|\Lambda_\Phi(x))\end{aligned}$$

That finishes the proof. □

Proposition 8.1.11. – *There exists unique strongly continuous one-parameter groups $\widehat{\tau}, \widehat{\kappa}$ and $\widehat{\sigma}$ of *-automorphisms of $\widehat{M}$ such that, for all $t \in \mathbb{R}$ and $\omega \in M_*^{\alpha,\beta}$, we have:*

$$\widehat{\tau}_t(\widehat{\pi}(\omega)) = \widehat{\pi}(\tau^*_{-t}(\omega)), \quad \widehat{\kappa}_t(\widehat{\pi}(\omega)) = \widehat{\pi}(\delta^*_{-t}(\omega)) \quad \textit{and} \quad \widehat{\sigma}_t(\widehat{\pi}(\omega)) = \widehat{\pi}(\rho_t(\omega))$$

Moreover, for all $t \in \mathbb{R}$ and $x \in \widehat{M}$, the following properties hold:

– $\widehat{\tau}_t(x) = P^{it}xP^{-it}$, $\widehat{\kappa}_t(x) = J_\Phi\delta^{it}J_\Phi x J_\Phi \delta^{-it}J_\Phi$ *and* $\widehat{\sigma}_t(x) = P^{it}J_\Phi\delta^{it}J_\Phi x J_\Phi\delta^{-it}J_\Phi P^{-it}$

– *$\widehat{\tau}$, $\widehat{\kappa}$ and $\widehat{\sigma}$ commute with each other. Also $\widehat{\tau}$ and $\widehat{R}$ do.*

– $\widehat{\kappa} \circ \alpha = \alpha$ *and* $\widehat{\tau} \circ \alpha = \alpha \circ \sigma^\nu_t = \widehat{\sigma} \circ \alpha$

– $(\widehat{\tau}_t {}_\beta\underset{N}{\star}{}_\alpha \widehat{\tau}_t) \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widehat{\tau}_t$, $(\mathrm{id}_\beta\underset{N}{\star}{}_\alpha \widehat{\kappa}_t) \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widehat{\kappa}_t$ *and* $(\widehat{\tau}_t {}_\beta\underset{N}{\star}{}_\alpha \widehat{\sigma}_t) \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widehat{\sigma}_t$

Proof. – By definition, we have $\widehat{\sigma} = \widehat{\tau} \circ \widehat{\kappa} = \widehat{\kappa} \circ \widehat{\tau}$ so that we just have to do the proof for $\widehat{\tau}$ and $\widehat{\kappa}$. For all $\omega \in M_*^{\alpha,\beta}$ and $t \in \mathbb{R}$, we compute the values of $P^{it}\hat{\pi}(\omega)P^{-it}$ and $J_\Phi\delta^{it}J_\Phi\hat{\pi}(\omega)J_\Phi\delta^{-it}J_\Phi$. Let $\mu \in \mathscr{I}$. Since $\hat{\pi}(\omega)$ belongs to $\beta(N)'$, we have on one hand:

$$\begin{aligned} P^{it}\hat{\pi}(\omega)P^{-it}\xi(\mu) &= P^{it}\hat{\pi}(\omega)\lambda^{\frac{t}{2}}\xi(\tau^*_t(\mu)) = \lambda^{\frac{t}{2}}P^{it}\xi(\omega\tau^*_t(\mu)) \\ &= \xi(\tau^*_{-t}(\omega)\mu) = \hat{\pi}(\tau^*_{-t}(\omega))\xi(\mu) \end{aligned}$$

and on the other hand:

$$\begin{aligned} J_\Phi\delta^{it}J_\Phi\hat{\pi}(\omega)J_\Phi\delta^{-it}J_\Phi\xi(\mu) &= J_\Phi\delta^{it}J_\Phi\hat{\pi}(\omega)\lambda^{\frac{-t}{2}}\xi(\delta^*_t(\mu)) \\ &= \xi(\delta^*_{-t}(\omega)\mu) = \hat{\pi}(\delta^*_{-t}(\omega))\xi(\mu) \end{aligned}$$

So, if we define $\widehat{\tau}_t$ by $\widehat{\tau}_t(x) = P^{it}xP^{-it}$ and $\widehat{\kappa}_t$ by $\widehat{\kappa}_t(x) = J_\Phi\delta^{it}J_\Phi x J_\Phi\delta^{-it}J_\Phi$, then we get strongly continuous *-automorphism of $\widehat{M}$ satisfying the first property. By definition, $\widehat{\tau}$ is implemented by P and $\widehat{R}$ by J_Φ. Since P and J_Φ commute with each other, so $\widehat{\tau}$ and $\widehat{R}$ do. Now, $\widehat{\tau}$ and τ coincide on $\alpha(N) \subset M \cap \widehat{M}$ because they are both i:implemented by P. Also $\widehat{\tau}$ coincide with id on $M \cap \widehat{M}$ by definition. By the way, we can give a meaning for formulas of the fourth point. Thanks to manageability of W, we have, for all $t \in \mathbb{R}$ and $x \in \widehat{M}$:

$$\begin{aligned} \widehat{\Gamma}(\widehat{\tau}_t(x)) &= \sigma_\nu W(P^{it}xP^{-it} {}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*\sigma_\nu \\ &= (P^{it} {}_{\hat{\beta}}\underset{N}{\otimes}{}_\alpha P^{it})\sigma_\nu W(x {}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*\sigma_\nu(P^{-it} {}_{\hat{\beta}}\underset{N}{\otimes}{}_\alpha P^{-it}) \\ &= (\widehat{\tau}_t {}_{\hat{\beta}}\underset{N}{\star}{}_\alpha \widehat{\tau}_t)\widehat{\Gamma}(x) \end{aligned}$$

Finally, since the left leg of W leaves in M, we have:

$$\begin{aligned} \widehat{\Gamma}(\widehat{\kappa}_t(x)) &= \sigma_\nu W(J_\Phi\delta^{it}J_\Phi x J_\Phi\delta^{-it}J_\Phi {}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*\sigma_\nu \\ &= (1 {}_{\hat{\beta}}\underset{N}{\otimes}{}_\alpha J_\Phi\delta^{it}J_\Phi)\sigma_\nu W(x {}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*\sigma_\nu(1 {}_{\hat{\beta}}\underset{N}{\otimes}{}_\alpha J_\Phi\delta^{-it}J_\Phi) \\ &= (\mathrm{id}_{\hat{\beta}}\underset{N}{\star}{}_\alpha \widehat{\kappa}_t)\widehat{\Gamma}(x) \end{aligned}$$

□

LEMMA 8.1.12. – *We have* $(\omega R * \mathrm{id})(W^*) = (\tau^*_{-i/2}(\omega) * \mathrm{id})(W)$ *for all* $\omega \in \mathcal{D}(\tau^*_{-i/2})$.

Proof. – We know that $(\mathrm{id} * \mu)(W)$ belongs to $\mathcal{D}(S)$ and that $S((\mathrm{id} * \mu)(W) = (\mathrm{id} * \mu)(W^*)$. So $(\mathrm{id}*\mu)(W)$ belongs to $\mathcal{D}(\tau_{-i/2})$ and $\tau_{-i/2}((\mathrm{id}*\mu)(W)) = R((\mathrm{id}*\mu)(W^*))$. By applying ω to the previous equation, we easily get the result. □

Since $\Psi = \Phi \circ R$, there exists an anti-unitary $\mathcal{J}$ from H_Ψ onto H_Φ such that $\mathcal{J}\Lambda_\Psi(x) = \Lambda_\Phi(R(x^*))$ for all $x \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$.

PROPOSITION 8.1.13. – *For all* $\omega \in \mathcal{I}$ *and* $\mu \in \mathcal{D}(\rho_{i/2})$, $\omega\mu$ *belongs to* $\mathcal{I}$ *and we have:*

$$\xi(\omega\mu) = \mathcal{J}^* \hat{\pi}(\rho_{i/2}(\mu))^* \mathcal{J}\xi(\omega)$$

Proof. – For all $n \in \mathbb{N}$, we put $e_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\delta^{it}dt$ so that $e_n$is analytic with respect to σ^Φ, $\mathcal{N}_\Phi e_n \subset \mathcal{N}_\Phi$ and $\mathcal{N}_\Phi \delta^{-\frac{1}{2}} e_n \subset \mathcal{N}_\Psi$. It is sufficient to prove the proposition for all $\mu \in \mathcal{D}(\tau^*_{-i/2}\delta_{i/2})$. Then, since δ is a co-character, we can compute, for all $x \in \mathcal{N}_\Phi$:

$$\begin{aligned}\Lambda_\Phi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\mu})\Gamma(xe_n)) &= \Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\mu})\Gamma(xe_n)\delta^{-\frac{1}{2}}) \\ &= \Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\mu})\Gamma(xe_n\delta^{-\frac{1}{2}})(1_{\beta}\underset{N}{\otimes}{}_\alpha\delta^{-\frac{1}{2}}))\end{aligned}$$

The computation goes on as follow:

$$\begin{aligned}\Lambda_\Phi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\mu})\Gamma(xe_n)) &= \Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\delta^*_{-i/2}(\mu)})\Gamma(x\delta^{-\frac{1}{2}}e_n)) \\ &= \mathcal{J}^*\Lambda_\Phi(R((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\delta^*_{-i/2}(\mu))\Gamma((x\delta^{-\frac{1}{2}}e_n)^*))) \\ &= \mathcal{J}^*\Lambda_\Phi((\delta^*_{-i/2}(\mu)\circ R_{\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})\Gamma(R(x\delta^{-\frac{1}{2}}e_n)^*)) \\ &= \mathcal{J}^*(\delta^*_{-i/2}(\mu)\circ R * \mathrm{id})(W^*)\Lambda_\Phi(R(x\delta^{-\frac{1}{2}}e_n)^*)) \\ &= \mathcal{J}^*(\delta^*_{-i/2}(\mu)\circ R * \mathrm{id})(W^*)\mathcal{J}\Lambda_\Psi(x\delta^{-\frac{1}{2}}e_n)) \\ &= \mathcal{J}^*(\tau^*_{-i/2}\delta^*_{-i/2}(\mu) * \mathrm{id})(W)\mathcal{J}\Lambda_\Phi(xe_n)) = \mathcal{J}^*(\rho_{i/2}(\mu) * \mathrm{id})(W)\mathcal{J}\Lambda_\Phi(xe_n))\end{aligned}$$

Now, we have:

$$\begin{aligned}(\omega\mu)((xe_n)^*) &= (\omega_{\beta}\underset{\nu}{\star}{}_\alpha\mu)\Gamma((xe_n)^*) = \omega((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\mu)\Gamma((xe_n)^*)) \\ &= (\xi(\omega)|\Lambda_\Phi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_\alpha\overline{\mu})\Gamma(xe_n))) = (\xi(\omega)|\mathcal{J}^*\hat{\pi}(\rho_{i/2}(\mu))\mathcal{J}\Lambda_\Phi(xe_n)) \\ &= (\mathcal{J}^*\hat{\pi}(\rho_{i/2}(\mu))^*\mathcal{J}\xi(\omega)|\Lambda_\Phi(xe_n))\end{aligned}$$

Since $(xe_n)_{n\in\mathbb{N}}$ is converging to x and $(\Lambda_\Phi(xe_n))_{n\in\mathbb{N}}$ is converging to $\Lambda_\Phi(x)$, we finally have:

$$(\omega\mu)(x^*) = (\mathcal{J}^*\hat{\pi}(\rho_{i/2}(\mu))^*\mathcal{J}\xi(\omega)|\Lambda_\Phi(x))$$

so that $\omega\mu \in \mathcal{I}$ and $\xi(\omega\mu) = \mathcal{J}^*\hat{\pi}(\rho_{i/2}(\mu))^*\mathcal{J}\xi(\omega)$. □

Corollary 8.1.14. – *There exists a unique closed densely defined operator $\widehat{\Lambda}$ from $\mathscr{D}(\hat{\Lambda}) \subset \widehat{M}$ to H_Φ such that $\widehat{\pi}(\mathscr{I})$ is a core for $\widehat{\Lambda}$ and $\hat{\Lambda}(\hat{\pi}(\omega)) = \xi(\omega)$ for all $\omega \in \mathscr{I}$.*

Proof. – Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{I}$ and let $w \in H_\Phi$ such that $(\hat{\pi}(\omega_n))_{n\in\mathbb{N}}$ is converging to 0 and $(\xi_n)_{n\in\mathbb{N}}$ is converging to w. If μ belongs to $\mathscr{D}(\rho_{i/2}) \cap \mathscr{I}$, then we have, by the previous proposition, for all $n \in \mathbb{N}$:

$$\hat{\pi}(\omega_n)\xi(\mu) = \mathscr{J}^* \hat{\pi}(\rho_{i/2}(\mu))^* \mathscr{J}\xi(\omega_n)$$

Take the limit to get that $0 = \mathscr{J}^* \hat{\pi}(\rho_{i/2}(\mu))^* \mathscr{J} w$. Since it is easy to check that $\rho_{i/2}(\mathscr{D}(\rho_{i/2}) \cap \mathscr{I})$ is dense in $\mathscr{I}$ we get that $w = 0$. So the formula of the proposition defines a closable operator and its closure satisfy all expected conditions. □

Theorem 8.1.15. – *There exists a unique normal semi-finite faithful weight $\widehat{T_L}$: $\widehat{M} \to \alpha(N)$ such that the normal semi-finite faithful weight $\widehat{\Phi} = \nu \circ \alpha^{-1} \circ \widehat{T_L}$ admits $(H, \iota, \widehat{\Lambda})$ as GNS construction. Moreover, $\widehat{\sigma}$ is the modular group of $\widehat{\Phi}$, the closure of $PJ_\Phi \delta J_\Phi$ (P and $J_\Phi \delta J_\Phi$ commute with each other) coincide with the modular operator of $\widehat{\Phi}$ and $\sigma_t^{\widehat{T_L}}(\hat{\beta}(n)) = \hat{\beta}(\gamma_{-t}(n))$ for all $t \in \mathbb{R}$ and $n \in N$.*

Proof. – Since $\widehat{\pi}$ is a multiplicative application and since $\mathscr{I}$ is a left ideal of $M_*^{\beta,\alpha}$, xy belongs to $\widehat{\pi}(\mathscr{I})$ for all $x \in \widehat{\pi}(M_*^{\beta,\alpha})$ and $y \in \widehat{\pi}(\mathscr{I})$ so, by definition, we have $\widehat{\Lambda}(xy) = x\widehat{\Lambda}(y)$. Using the closeness of $\widehat{\Lambda}$, we show that $\mathscr{D}(\widehat{\Lambda})$ is a left ideal of $\widehat{M}$ and $\widehat{\Lambda}(xy) = x\widehat{\Lambda}(y)$ for all $x \in \widehat{M}$ and $y \in \mathscr{D}(\widehat{\Lambda})$.

By proposition 8.1.11, $\widehat{\sigma}_t(x)$ belongs to $\mathscr{D}(\widehat{\Lambda})$ for all $x \in \widehat{\pi}(\mathscr{I})$ and $t \in \mathbb{R}$ and $\widehat{\Lambda}(\widehat{\sigma}_t(x)) = P^{it} J_\Phi \delta^{it} J_\Phi \widehat{\Lambda}(x)$. Using again the closeness of $\widehat{\Lambda}$, we get that $\widehat{\sigma}_t(x)$ belongs to $\mathscr{D}(\widehat{\Lambda})$ for all $x \in \mathscr{D}(\widehat{\Lambda})$ and $t \in \mathbb{R}$ and we have:

$$\widehat{\Lambda}(\widehat{\sigma}_t(x)) = P^{it} J_\Phi \delta^{it} J_\Phi \widehat{\Lambda}(x)$$

By proposition 8.1.13, for all $\omega \in \mathscr{D}(\rho_{i/2})$ and $x \in \widehat{\pi}(\mathscr{I})$, $x\widehat{\pi}(\omega)$ belongs to $\mathscr{D}(\widehat{\Lambda})$ and we have $\widehat{\Lambda}(x\widehat{\pi}(\omega)) = \mathscr{J}^* \widehat{\pi}(\rho_{i/2}(\omega))^* \mathscr{J} \widehat{\Lambda}(x) = \mathscr{J}^* \widehat{\sigma}_{i/2}(\widehat{\pi}(\rho_{i/2})))^* \mathscr{J} \widehat{\Lambda}(x)$. Since $\widehat{\pi}(\mathscr{D}(\rho_{i/2}))$ is dense in $\mathscr{D}(\widehat{\sigma}_{i/2})$ and $\widehat{\sigma}$-invariant, $\widehat{\pi}(\mathscr{D}(\rho_{i/2}))$ is a core for $\widehat{\sigma}$. The closeness of $\widehat{\Lambda}$ allows us to conclude that xy belongs to $\mathscr{D}(\widehat{\Lambda})$ for all $x \in \mathscr{D}(\widehat{\Lambda})$ and $y \in \mathscr{D}(\widehat{\sigma}_{i/2})$ and we have:

$$\widehat{\Lambda}(xy) = \mathscr{J}^* \widehat{\sigma}_{i/2}(y)^* \mathscr{J} \widehat{\Lambda}(x)$$

Therefore we know, by proposition 5.14 of [**Kus97**], that there exists a normal semi-finite weight $\widehat{\Phi}$ on $\widehat{M}$ such that $(H, \iota, \widehat{\Lambda})$ is a GNS construction for $\widehat{\Phi}$ and $\widehat{\sigma}$ is the modular group of $\widehat{\Phi}$. Moreover, thanks to the previous equation, we have:

$$\widehat{\Lambda}(xy) = \mathscr{J}^* \widehat{\sigma}_{i/2}(y)^* \mathscr{J} \widehat{\Lambda}(x)$$

for all $x \in \mathscr{N}_{\widehat{\Phi}}$ and $y \in \mathscr{D}(\widehat{\sigma}_{i/2}) \cap \mathscr{N}_{\widehat{\Phi}}$. We easily get faithfulness of $\widehat{\Phi}$ from this last relation. We already know that $\alpha(N) \subseteq M \cap \widehat{M}$ and, by proposition 8.1.11 we have, for all $n \in N$:

$$\sigma_t^{\widehat{\Phi}}(\alpha(n)) = \alpha(\sigma_t^{\nu}(n)) = \sigma_t^{\nu\circ\alpha^{-1}}(\alpha(n))$$

By Haagerup's existence theorem, we get the normal semi-finite faithful weight $\widehat{\Phi}$. Finally, we check the last property. For all $n \in N$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned}\sigma_t^{\widehat{\Phi}}(\hat{\beta}(n)) &= P^{it}J_\Phi\delta^{it}\alpha(n^*)\delta^{-it}J_\Phi P^{-it} = P^{it}J_\Phi\alpha(\gamma_{-t}\sigma^{\nu}_{-t}(n^*))J_\Phi P^{-it}\\ &= P^{it}\hat{\beta}(\gamma_{-t}\sigma^{\nu}_{-t}(n))P^{-it} = \hat{\beta}(\gamma_{-t}(n))\end{aligned}$$

because γ and σ^ν commute with each other. □

Lemma 8.1.16. – *For all $x \in \mathscr{N}_{\widehat{T_L}} \cap \mathscr{N}_{\hat{\Phi}}$, $\widehat{\Lambda}(x)$ belongs to $D(H_\beta, \nu^0)$ and we have $R^{\beta,\nu^o}(\hat{\Lambda}(x)) = \Lambda_{\widehat{T_L}}(x)$.*

Proof. – By definition $J_{\widehat{\Phi}}$ and $\mathscr{J}$ implement the same operator on $\alpha(N) \subset M \cap \widehat{M}$ so that $J_{\widehat{\Phi}}\alpha(n^*)J_{\widehat{\Phi}} = \beta(n)$ for all $n \in N$. Then the lemma is a consequence of proposition 3.2.2. □

Lemma 8.1.17. – *For all $\xi \in D((H_\Phi)_{\hat{\beta}}, \nu^o)$, all $\eta \in D((H_\Phi)_{\hat{\beta}}, \nu^o) \cap D({}_\alpha H_\Phi, \nu^o)$ and all $x \in \mathscr{N}_{\hat{\Phi}} \cap \mathscr{N}_{\widehat{T_L}}$, $(\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}(x))$ belongs to $\mathscr{N}_{\hat{\Phi}} \cap \mathscr{N}_{\widehat{T_L}}$ and we have:*

$$\hat{\Lambda}((\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}(x))) = (\mathrm{id} * \omega_{\eta,\xi})(W)\hat{\Lambda}(x)$$

Proof. – Thanks to the pentagonal relation, we can compute for all $\omega \in \mathscr{I}$:

$$\begin{aligned}&(\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}(\hat{\pi}(\omega))) = (\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}((\omega * \mathrm{id})(W)))\\ &= (\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\sigma_{\nu^o}W((\omega * \mathrm{id})(W)\,{}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*\sigma_\nu)\\ &= (\omega * \mathrm{id} * \omega_{\eta,\xi})((1\,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} W)(W\,{}_\beta\underset{N}{\otimes}{}_\alpha 1)(1\,{}_\beta\underset{N}{\otimes}{}_\alpha W^*))\\ &= (\omega * \omega_{\eta,\xi} * \mathrm{id})((W\,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} 1)(\sigma_{\nu^o}\,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} 1)(1\,{}_\alpha\underset{N^o}{\otimes}{}_{\hat{\beta}} W)\sigma_{2\nu}(1\,{}_\beta\underset{N}{\otimes}{}_\alpha \sigma_{\nu^o}))\\ &= \hat{\pi}((\mathrm{id} * \omega_{\eta,\xi})(W)\omega)\end{aligned}$$

Then, by definition, $(\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}(\hat{\pi}(\omega)))$ belongs to $\mathscr{N}_{\hat{\Phi}} \cap \mathscr{N}_{\widehat{T_L}}$ for all $\omega \in \mathscr{I}$ and we have:

$$\hat{\Lambda}((\omega_{\eta,\xi}\,{}_{\hat{\beta}}\underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\hat{\Gamma}(\hat{\pi}(\omega)))) = (\mathrm{id} * \omega_{\eta,\xi})(W)\hat{\Lambda}(\hat{\pi}(\omega))$$

Closeness of $\widehat{\Lambda}$ finishes the proof. □

Proposition 8.1.18. – *The operator-valued weight $\widehat{T_L}$ is left invariant.*

Proof. – Let $(\xi_i)_{i\in I}$ be a (N^o,ν^o)-basis of $(H_\Phi)_{\hat\beta}$. For all $x\in\mathcal{N}_{\widehat\Phi}\cap\mathcal{N}_{\widehat{T_L}}$ and $\eta\in D((H_\Phi)_{\hat\beta},\nu^o)\cap D({}_\alpha H_\Phi,\nu)$, we have:

$$\widehat\Phi((\omega_{\eta}\,{}_{\hat\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\widehat\Gamma(x^*x)))=\sum_{i\in I}\widehat\Phi((\omega_{\eta,\xi_i}\,{}_{\hat\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\widehat\Gamma(x))^*(\omega_{\eta,\xi_i}\,{}_{\hat\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\widehat\Gamma(x)))$$

$$=\sum_{i\in I}||\widehat\Lambda((\omega_{\eta,\xi_i}\,{}_{\hat\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\widehat\Gamma(x))||^2=\sum_{i\in I}||(\mathrm{id}*\omega_{\eta,\xi_i})(W)\widehat\Lambda(x)||^2$$

$$=((\rho_\eta^{\beta,\alpha})^*\rho_\eta^{\beta,\alpha}\widehat\Lambda(x)|\widehat\Lambda(x))=||\widehat\Lambda(x)\,{}_\beta\underset{\nu}{\otimes}{}_\alpha\eta||^2$$

$$=(\alpha(\langle\widehat\Lambda(x),\widehat\Lambda(x)\rangle_{\beta,\nu^o}\eta|\eta)=(\widehat{T_L}(x^*x)\eta|\eta)$$ □

To have a measured quantum groupoid, we need to check a relation between the co-involution $\widehat R$ and Γ. By the way, it will give a link between the two natural GNS constructions of $\Phi_\delta=\Psi=\Phi\circ R$. We put $S_{\widehat\Phi}, J_{\widehat\Phi}$ and $\Delta_{\widehat\Phi}$ to be the fundamental objects associated to $\widehat\Phi$ by the Tomita's theory in the GNS construction $(H,\iota,\widehat\Lambda)$.

Definition 8.1.19. – We put $\mathcal{I}^\sharp$ the subset of $\mathcal{I}$ consisting of elements of the form $\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)}$ where a,b belong to E.

Lemma 8.1.20. – *We have that $\widehat\pi(\mathcal{I}^\sharp)$ is a core for $\widehat\Lambda$ and $\widehat\Lambda(\widehat\pi(\mathcal{I}^\sharp))$ is a core for $S_{\widehat\Phi}$ and for $\Delta^z_{\widehat\Phi}$ for all $z\in\mathbb{C}$.*

Proof. – This lemma comes from standard arguments and by definition of $\widehat\Lambda$. □

Proposition 8.1.21. – *For all $x\in E$, we have $\Lambda_\Phi(x)$ belongs to $\mathcal{D}(S^*_{\widehat\Phi})$ and we have:*

$$S^*_{\widehat\Phi}\Lambda_\Phi(x)=\Lambda_\Phi(S^{-1}(x)^*)$$

Moreover $\Lambda_\Phi(E)$ is a core for $S^*_{\widehat\Phi}$.

Proof. – Let $\omega\in\mathcal{I}^\sharp$. For all $\mu\in M_*^{\alpha,\hat\beta}$, we have:

$$\mu(\widehat\pi(\omega)^*)=\mu((\omega*\mathrm{id})(W)^*)=\overline{\omega}((\mathrm{id}*\mu)(W^*))=\overline{\omega}\circ S((\mathrm{id}*\mu)(W))$$
$$=\omega^*\circ\tau_{-\frac{i}{2}}((\mathrm{id}*\mu)(W))=\mu((\omega^*\circ\tau_{-\frac{i}{2}}*\mathrm{id})(W))=\mu(\widehat\pi(\omega^*\circ\tau_{-\frac{i}{2}}))$$

Then, we have:

$$(S_{\widehat\Phi}\widehat\Lambda(\widehat\pi(\omega))|\Lambda_\Phi)=(\widehat\Lambda(\widehat\pi(\omega)^*)|\Lambda_\Phi)=(\widehat\Lambda(\widehat\pi(\omega^*\circ\tau_{-\frac{i}{2}})|\Lambda_\Phi))=(\xi(\omega^*\circ\tau_{-\frac{i}{2}})|\Lambda_\Phi)$$
$$=\omega^*\circ\tau_{-\frac{i}{2}}(x^*)=\overline{\omega(S^{-1}(x))}=(\Lambda_\Phi(S^{-1}(x)^*)|\xi(\omega))=(\Lambda_\Phi(S^{-1}(x)^*)|\widehat\Lambda(\widehat\pi(\omega)))$$

Thus the previous lemma and the fact that $\xi(\omega_{\Lambda_\Phi(a),\Lambda_\Phi(b)})=\Lambda_\Phi(a\sigma^\Phi_{-i}(b^*))$ implies the proposition. □

Proposition 8.1.22. – *For all $x\in\mathcal{N}_{T_R}\cap\mathcal{N}_\Psi$, we have:*

$$J_{\widehat\Phi}\Lambda_{\Phi_\delta}(x)=\Lambda_\Phi(R(x^*))$$

Proof. – Define the anti-unitary $\mathcal{J}$ of H such that $\mathcal{J}\Lambda_{\Phi_\delta}(x) = \Lambda_\Phi(R(x^*))$ for all $x \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$. Let a belongs to E. For all $n \in \mathbb{N}$, we put $e_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\delta^{it}dt$ so that e_n is analytic with respect to σ^Φ, $\mathcal{N}_\Phi e_n \subset \mathcal{N}_\Phi$ and $\mathcal{N}_\Phi \delta^{-\frac{1}{2}} e_n \subset \mathcal{N}_\Psi$. Since $\tau_s(\delta) = \delta$, we see that $\tau_s(e_n) = e_n$ for all $s \in \mathbb{R}$, hence $e_n \in \mathcal{D}(\tau_{\frac{i}{2}})$ and $\tau_{\frac{i}{2}}(e_n) = e_n$. By assumption a belongs to $\mathcal{D}(\tau_{\frac{i}{2}})$ so that ae_n belongs to $\mathcal{D}(\tau_{\frac{i}{2}})$ and $\tau_{\frac{i}{2}}(ae_n) = \tau_{\frac{i}{2}}(a)e_n$. Hence $\tau_{\frac{i}{2}}(ae_n)\delta^{\frac{1}{2}}$ is a bounded operator and its closure is equal to $\tau_{\frac{i}{2}}(a)(\delta^{\frac{1}{2}}e_n)$. We recall that $\kappa_t(x)$ is equal, by definition, to $\tau_t(m)\delta^{it}$ for all $t \in \mathbb{R}$ and $m \in M$. Then ae_n belongs to $\mathcal{D}(\kappa_{\frac{i}{2}})$ and $\kappa_{\frac{i}{2}}(ae_n) = \tau_{\frac{i}{2}}(a)(\delta^{\frac{1}{2}}e_n)$. By assumption, $\tau_{\frac{i}{2}}$ belongs to $\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$. So we see that $\kappa_{\frac{i}{2}}(ae_n)\delta^{-\frac{1}{2}}$ is bounded and its closure equals $\tau_{\frac{i}{2}}(a)e_n \in \mathcal{N}_\Psi \cap \mathcal{N}_{T_R}$ implying that $\kappa_{\frac{i}{2}}(ae_n) \in \mathcal{N}_\Phi \cap \mathcal{N}_{T_L}$ and:

$$\Lambda_\Phi(\kappa_{\frac{i}{2}}(ae_n)) = \Lambda_{\Phi_\delta}(\kappa_{\frac{i}{2}}(ae_n)\delta^{-\frac{1}{2}}) = \Lambda_{\Phi_\delta}(\tau_{\frac{i}{2}}(a)e_n)$$

By definition, we have $\Delta_{\widehat{\Phi}}^{it}\Lambda_\Phi(x) = \Lambda_\Phi(\kappa_t(x))$, we easily get that $\Lambda_\Phi(ae_n)$ belongs to $\mathcal{D}(\Delta_{\widehat{\Phi}}^{-\frac{1}{2}})$ and:

$$\Delta_{\widehat{\Phi}}^{-\frac{1}{2}}\Lambda_\Phi(ae_n) = \Lambda_\Phi(\kappa_{\frac{i}{2}}(ae_n)) = \Lambda_{\Phi_\delta}(\tau_{\frac{i}{2}}(a)e_n)$$

By closedness of $\Delta_{\widehat{\Phi}}^{-\frac{1}{2}}$, this implies that $\Lambda_\Phi(a)$ belongs to $\mathcal{D}(\Delta_{\widehat{\Phi}}^{-\frac{1}{2}})$ and:

$$\Delta_{\widehat{\Phi}}^{-\frac{1}{2}}\Lambda_\Phi(a) = \Lambda_{\Phi_\delta}(\tau_{\frac{i}{2}}(a))$$

Consequently, we have:

$$\mathcal{J}\Delta_{\widehat{\Phi}}^{-\frac{1}{2}}\Lambda_\Phi(a) = \mathcal{J}\Lambda_{\Phi_\delta}(\tau_{\frac{i}{2}}(a)) = \Lambda_\Phi(S^{-1}(a)^*) = S_{\widehat{\Phi}}^*\Lambda_\Phi(a) = J_{\widehat{\Phi}}\Delta_{\widehat{\Phi}}^{-\frac{1}{2}}\Lambda_\Phi(a)$$

Since $\Lambda_\Phi(E)$ is a core for $\Delta_{\widehat{\Phi}}^{-\frac{1}{2}} = J_{\widehat{\Phi}}S_{\widehat{\Phi}}^*$, we have done. □

Finally, we have to recognize what is $\widehat{W}$.

PROPOSITION 8.1.23. – *The unitary $\sigma_\nu W^* \sigma_\nu$ is the fundamental unitary associated with the dual Hopf-bimodule structure.*

Proof. – The fundamental unitary associated with the dual quantum groupoid is denoted by $\widehat{W}$. By definition of $\widehat{W}$ and lemma 8.1.17, we have for all $\xi \in D({}_\alpha H_\Phi, \nu) \cap D((H_\Phi)_{\hat{\beta}}, \nu^o)$, $\eta \in D((H_\Phi)_{\hat{\beta}}, \nu^o)$ and $x \in \mathcal{N}_{\widehat{\Phi}} \cap \mathcal{N}_{\widehat{T_L}}$:

$$\begin{aligned}(\omega_{\xi,\eta} * \mathrm{id})(\widehat{W}^*)\widehat{\Lambda}(x) &= \widehat{\Lambda}((\omega_{\xi,\eta}\,{}_{\hat{\beta}}\!\underset{\nu}{\star}_\alpha \mathrm{id})(\widehat{\Gamma}(x))) \\ &= (\mathrm{id} * \omega_{\xi,\eta})(W)\widehat{\Lambda}(x) = (\omega_{\xi,\eta} * \mathrm{id})(\sigma_{\nu^o} W \sigma_{\nu^o})\widehat{\Lambda}(x)\end{aligned}$$

from which we easily deduce that $\widehat{W} = \sigma_\nu W^* \sigma_\nu$. □

THEOREM 8.1.24. – *$(N, \widehat{M}, \alpha, \hat{\beta}, \widehat{\Gamma}, \widehat{T_L}, \widehat{R}, \widehat{\tau}, \nu)$ is a measured quantum groupoid called* ***dual quantum groupoid*** *of $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$. Fundamental objects of the dual quantum groupoid $(N, \widehat{M}, \alpha, \hat{\beta}, \widehat{\Gamma}, \widehat{R}, \widehat{T_L}, \widehat{\tau}, \nu)$ are given, for all $x \in \widehat{M}$ and $t \in \mathbb{R}$, by:*

i) *$\widehat{W} = \sigma_\nu W^* \sigma_\nu$ is the fundamental unitary,*
ii) *$\widehat{R}(x) = J_\Phi x^* J_\Phi$ is the unitary antipode and $\widehat{\tau}_t(x) = P^{it} x P^{-it}$ is the scaling group,*
iii) *$\widehat{\lambda} = \lambda^{-1}$ is the scaling operator and the closure of $P^{-1} J_\Phi \delta J_\Phi \delta^{-1} \Delta_\Phi^{-1}$ is the modulus $\widehat{\delta}$,*
iv) *$\widehat{P} = P$ is the manipulation operator,*
v) *in the GNS construction $(H, \iota, \widehat{\Lambda})$, the modular operator $\Delta_{\widehat{\Phi}}$ is the closure of $P J_\Phi \delta^{-1} J_\Phi$ and the modular conjugation satisfies $J_{\widehat{\Phi}} \Lambda_{\Phi_\delta}(x) = \Lambda_\Phi(R(x^*))$ for all $x \in \mathcal{N}_{T_R} \cap \mathcal{N}_\Psi$.*

Proof. – By proposition 8.1.3, $(N, \widehat{M}, \alpha, \hat{\beta}, \widehat{\Gamma})$ is a Hopf-bimodule. By theorem 8.1.15, it admits a normal semi-finite faithful left-invariant operator-valued weight $\widehat{T_L}$. By proposition 8.1.6, $\widehat{R}$ is a co-involution for this structure and, by definition, we have $\widehat{R}(\widehat{\pi}(\omega)) = \widehat{\pi}(\omega \circ R)$ for all $\omega \in M_*^{\alpha,\beta}$. Since $J_{\widehat{\Phi}} = \mathscr{J}$ implement R on M and since $= \widehat{W} = \sigma_\nu W^* \sigma_\nu$, we get $R((\mathrm{id} * \omega_{J_\Phi v, w})(\widehat{W})) = (\mathrm{id} * \omega_{J_\Phi w, v})(\widehat{W})$. By proposition 8.1.11, $\widehat{\tau}$ is a scaling group. We just have to check that the one-parameter group of automorphisms $\widehat{\gamma}$ of N leaves ν invariant. However, we have already noticed, in theorem 8.1.15, that we have $\widehat{\gamma}_t = \gamma_{-t}$ for all $t \in \mathbb{R}$. By hypothesis over γ, we have done.

By proposition 8.1.10 and by definition of $\widehat{\tau}$, $\widehat{\pi}(\mathscr{I})$ is stable under $\widehat{\tau}_t$ $t \in \mathbb{R}$ and we have, for all $\omega \in \mathscr{I}$:

$$\widehat{\Lambda}(\widehat{\tau}_t(\widehat{\pi}(\omega))) = \widehat{\Lambda}(\widehat{\pi}(\omega \circ \tau_{-t})) = \xi(\omega \circ \tau_{-t}) = \lambda^{\frac{t}{2}} P^{it} \widehat{\Lambda}(\widehat{\pi}(\omega))$$

Now, by closeness of $\widehat{\Lambda}$, we get that $P^{it}\widehat{\Lambda}(x) = \lambda^{-\frac{t}{2}} \widehat{\Lambda}(\widehat{\tau}_t(x))$ for all $x \in \mathcal{N}_{\widehat{T_L}} \cap \mathcal{N}_{\widehat{\Phi}}$ and $t \in \mathbb{R}$. From this and from lemma 5.1.5, we get that:

$$\lambda^{-ist} = [D\widehat{\Phi} \circ \widehat{\tau}_{-s} : D\widehat{\Phi}]_t = [D\widehat{\Phi} \circ \widehat{\sigma}_s^{\widehat{\Phi} \circ \widehat{R}} : D\widehat{\Phi}]_t = \widehat{\lambda}^{ist}$$

$$\text{and} \qquad P^{it}\widehat{\Lambda}(x) = \widehat{\lambda}^{\frac{t}{2}} \widehat{\Lambda}(\widehat{\tau}_t(x)) = \widehat{P}^{it}\widehat{\Lambda}(x) \quad \square$$

The whole picture is not completely drawn yet because the value of $\widehat{\delta}$ is missing. For this, we need the bi-duality theorem. The expression will finally be given in 8.2.2.

8.2. Bi-duality theorem

In this section, we compute fundamental objects of the dual structure. Also, we can construct the bi-dual quantum groupoid that is the dual quantum groupoid of the dual quantum groupoid and we establish a bi-duality theorem.

THEOREM 8.2.1. – *The measured quantum groupoid* $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ *and its bi-dual* $(N, \widehat{\widehat{M}}, \alpha, \hat{\hat{\beta}}, \widehat{\widehat{\Gamma}}, \widehat{\widehat{T_L}}, \widehat{\widehat{R}}, \widehat{\widehat{\tau}}, \nu)$ *coincide. Moreover, we have* $\widehat{\widehat{\Lambda}} = \Lambda_\Phi$.

Proof. – We know that $J_{\widehat{\Phi}} = \mathscr{J}$. Then, on $\alpha(N) \subset M \cap \widehat{M}$, we have:

$$\hat{\hat{\beta}}(n) = J_{\widehat{\Phi}}\alpha(n)^* J_{\widehat{\Phi}} = \mathscr{J}\alpha(n)^*\mathscr{J} = R(\alpha(n)) = \beta(n)$$

By proposition 8.1.23, we have:

$$\widehat{\widehat{W}} = \sigma_\nu \widehat{W}^* \sigma_\nu = W$$

so that we deduce that the Hopf-bimodule and its bi-dual coincide. We denote by $\widehat{\widehat{\pi}}(\omega) = (\omega * \mathrm{id})(\widehat{W}) = (\mathrm{id} * \omega)(W^*)$ for all $\omega \in M_*^{\alpha,\hat{\beta}}$. By definition of $\widehat{R}$ and $\widehat{\widehat{R}}$, we have for all $\xi, \eta \in D({}_\alpha H, \nu)$:

$$\begin{aligned}\widehat{\widehat{R}}((\mathrm{id} * \omega_{J_\Phi\xi,\eta})(W^*)) = \widehat{\widehat{R}}(\widehat{\widehat{\pi}}(\omega_{J_\Phi\xi,\eta})) &= \widehat{\widehat{\pi}}(\omega_{J_\Phi\xi,\eta} \circ \widehat{R}) \\ &= \widehat{\widehat{\pi}}(\omega_{J_\Phi\eta,\xi}) = (\mathrm{id} * \omega_{J_\Phi\eta,\xi})(W^*)\end{aligned}$$

so that $\widehat{\widehat{R}} = R$. Let $\omega \in \widehat{\mathscr{I}}$. On note $a = \widehat{\widehat{\pi}}(\omega)$. Then, for all $\Theta \in \mathscr{I}$, we have:

$$\begin{aligned}\omega(\hat{\pi}(\Theta)^*) = \omega((\Theta * \mathrm{id})(W)^*) &= \omega((\overline{\Theta} * \mathrm{id})(W^*)) = \overline{\Theta}((\mathrm{id} * \omega)(W^*)) \\ &= \overline{\Theta(a^*)} = \overline{(\xi(\Theta)|\Lambda_\Phi(a))} = (\Lambda_\Phi(a)|\hat{\Lambda}(\hat{\pi}(\Theta)))\end{aligned}$$

Since $\widehat{\pi}(\mathscr{I})$ is a core for $\widehat{\Lambda}$, this implies $\omega(x^*) = (\Lambda_\Phi(a)|\widehat{\Lambda}(x))$ for all $x \in \mathscr{N}_{\widehat{\Phi}}$. By definition of $\widehat{\widehat{\Lambda}}$, we get $\widehat{\widehat{\Lambda}}(\widehat{\widehat{\pi}}(\omega)) = \Lambda_\Phi(a) = \Lambda_\Phi(\widehat{\widehat{\pi}}(\omega))$. Since $\widehat{\widehat{\pi}}(\widehat{\mathscr{I}})$ is a core for $\widehat{\widehat{\Lambda}}$ and by closeness of Λ_Φ we have $\widehat{\widehat{\Lambda}}(y) = \Lambda_\Phi(y)$ for all $y \in \mathscr{N}_{\widehat{\Phi}}$. In particular $\widehat{\widehat{T_L}} = T_L$. Finally, we have to compute $\widehat{\widehat{\tau}}$. For example, we can use proposition 5.3.4, to get for all $t \in \mathbb{R}$:

$$\Gamma \circ \widehat{\widehat{\tau}}_t = \widehat{\widehat{\Gamma}} \circ \widehat{\widehat{\tau}}_t = (\sigma_t^{\widehat{\widehat{\Phi}}} {}_\beta\!\underset{N}{\star}_\alpha \sigma_{-t}^{\widehat{\widehat{\Phi}} \circ \widehat{\widehat{R}}}) \circ \widehat{\widehat{\Gamma}} = (\sigma_t^\Phi {}_\beta\!\underset{N}{\star}_\alpha \sigma_{-t}^{\Phi \circ R}) \circ \Gamma = \Gamma \circ \tau$$

and we can conclude by injectivity of Γ. □

PROPOSITION 8.2.2. – *For all* $t \in \mathbb{R}$, *we have:*

$$\widehat{\delta}^{it} = P^{-it} J_\Phi \delta^{-it} J_\Phi \delta^{-it} \Delta_\Phi^{-it}$$

Proof. – By theorem 8.1.24, we know that $\Delta_{\widehat{\Phi}}^{it} = P^{it} J_\Phi \delta^{it} J_\Phi$ so that we get, thanks to the bi-duality theorem that:

$$\widehat{\delta}^{it} = \widehat{P}^{-it} J_{\widehat{\Phi}} \Delta^{it} J_{\widehat{\Phi}} = P^{-it} J_{\widehat{\Phi}} \Delta^{it} J_{\widehat{\Phi}}$$

From the previous proposition, it is easy to check on $\Lambda_{\Phi_\delta}(x)$ that $J_{\widehat{\Phi}} \Delta J_{\widehat{\Phi}}$ coincide with the modular operator of Ψ in the GNS construction $(H, \iota, \Lambda_{\Phi_\delta})$. Now, by proposition 2.5 of [**Vae01a**], this last modular operator is equal to the closure of $J_\Phi \delta^{-1} J_\Phi \delta \Delta_\Phi$ so that we get the result. □

REMARK 8.2.3. – From this last expression of $\widehat{\delta}$, we can directly verify the following properties which should be satisfied by duality, for all $x \in \widehat{M}$ and $s, t \in \mathbb{R}$:

$$\sigma_s^{\widehat{\Phi}}(\widehat{\delta}^{it}) = \widehat{\lambda}^{ist} \widehat{\delta}^{it}, \qquad \sigma_t^{\widehat{\Phi} \circ \widehat{R}}(x) = \widehat{\delta}^{it} \sigma_t^{\widehat{\Phi}}(x) \widehat{\delta}^{-it} \qquad \text{and} \qquad \widehat{\Gamma}(\delta^{it}) = \delta^{it} {}_{\hat{\beta}} \underset{\nu}{\otimes}_{\alpha} \delta^{it}$$

THEOREM 8.2.4. – *The following properties and their dual hold:*

- $\tau_t(m) = \Delta_{\widehat{\Phi}}^{it} m \Delta_{\widehat{\Phi}}^{-it}$ *and* $R(m) = J_{\widehat{\Phi}} m^* J_{\widehat{\Phi}}$ *for all* $t \in \mathbb{R}$ *and* $m \in M$
- $W(\Delta_{\widehat{\Phi}\,\beta} \underset{\nu}{\otimes}_\alpha \Delta_\Phi) = (\Delta_{\widehat{\Phi}\,\alpha} \underset{\nu^o}{\otimes}_{\hat{\beta}} \Delta_\Phi) W$

and $W(J_{\widehat{\Phi}\,\alpha} \underset{\nu^o}{\otimes}_{\hat{\beta}} J_\Phi) = (J_{\widehat{\Phi}\,\beta} \underset{\nu}{\otimes}_\alpha J_\Phi) W^*$

- $\Delta_{\widehat{\Phi}}^{it} \Delta_\Phi^{is} = \lambda^{ist} \Delta_\Phi^{is} \Delta_{\widehat{\Phi}}^{it}$, $\Delta_\Phi^{it} \delta^{is} = \lambda^{ist} \delta^{is} \Delta_\Phi^{it}$ *and* $\Delta_{\widehat{\Phi}}^{it} \delta^{is} = \delta^{is} \Delta_{\widehat{\Phi}}^{it}$
- $J_{\widehat{\Phi}} J_\Phi = \lambda^{\frac{i}{4}} J_\Phi J_{\widehat{\Phi}}$, $J_\Phi P J_\Phi = P^{-1}$ *and* $J_{\widehat{\Phi}} \delta J_{\widehat{\Phi}} = \delta^{-1}$
- $P^{is} \Delta_\Phi^{it} = \Delta_\Phi^{it} P^{is}$ *and* $P^{is} \delta_\Phi^{it} = \delta_\Phi^{it} P^{is}$

Proof. – Since δ is affiliated to M, $J_\Phi \delta J_\Phi$ is affiliated to M' so that, for all $t \in \mathbb{R}$ and $m \in M$, we have:

$$\Delta_{\widehat{\Phi}}^{it} m \Delta_{\widehat{\Phi}}^{-it} = P^{it} J_\Phi \delta^{it} J_\Phi m J_\Phi \delta^{-it} J_\Phi P^{-it} = P^{it} m P^{-it} = \tau_t(m)$$

We have already noticed that R is implemented by $J_{\widehat{\Phi}}$ by definition of $\widehat{\Phi}$ but we can recover this point thanks to the bi-duality theorem and the fact that, by definition, $\widehat{R}$ is implemented by J_Φ. Now, since we have $R((\mathrm{id} * \omega_{\xi, J_\Phi \eta})(W)) = (\mathrm{id} * \omega_{\eta, J_\Phi \xi})(W)$ for all $\xi, \eta \in D({}_\alpha H, \nu)$, we easily get the second equality of the second point from the first point. Also, we know that $\tau_t((\mathrm{id} * \omega_{\xi, J_\Phi \eta})(W)) = (\mathrm{id} * \omega_{\Delta_\Phi^{it} \xi, \Delta_\Phi^{it} J_\Phi \eta})(W)$ for all $t \in \mathbb{R}$ from which and from the first point we get the first equality of the second point. Since τ and σ commute each other, it is easy to check on $\Lambda_\Phi(x)$ the first equality of the last point. Since $\tau(\delta) = \delta$, we get the last equality of the last point. The last equality of the third point comes from the fact that τ is implemented by $\Delta_{\widehat{\Phi}}$ and that $\tau(\delta) = \delta$. By proposition 5.2 of [**Vae01a**], we have $\sigma_t^\Phi(\delta^{is}) = \lambda^{ist} \delta^{it}$ so that we get the second equality of the third point. Then, for all $s, t \in \mathbb{R}$, we have:

$$\begin{aligned} \Delta_{\widehat{\Phi}}^{it} \Delta_\Phi^{is} &= P^{it} J_\Phi \delta^{it} J_\Phi \Delta_\Phi^{is} = P^{it} J_\Phi \delta^{it} \Delta_\Phi^{is} J_\Phi \\ &= P^{it} J_\Phi \lambda^{-ist} \Delta_\Phi^{is} \delta^{it} J_\Phi = \lambda^{ist} \Delta_\Phi^{is} P^{it} J_\Phi \delta^{it} J_\Phi = \lambda^{ist} \Delta_\Phi^{is} \Delta_{\widehat{\Phi}}^{it} \end{aligned}$$

As far as the fourth point is concerned, the last equality comes from the fact that R is implemented by $J_{\widehat{\Phi}}$ and $R(\delta) = \delta^{-1}$. The second one can be directly checked on $\Lambda_\Phi(x)$. Let us prove the first equality. Let x belongs to $\mathcal{N}_\Psi \cap \mathcal{D}(\sigma^\Psi_{\frac{i}{2}})$. Then, it is easy to see that $R(x^*)$ belongs to $\mathcal{N}_\Phi \cap \mathcal{D}(\sigma^\Phi_{\frac{i}{2}})$. Remembering that the modular conjugation of $\Psi = \Phi_\delta$ associated with the GNS construction $(H, \iota, \Lambda_{\Phi_\delta})$ is equal to $\lambda^{\frac{i}{4}}$ by proposition 2.5 of [**Vae01a**], we get:

$$\begin{aligned} J_{\widehat{\Phi}} J_\Phi \Lambda_{\Phi_\delta}(x) &= \lambda^{\frac{i}{4}} J_{\widehat{\Phi}} \lambda^{\frac{i}{4}} J_\Phi \Lambda_{\Phi_\delta}(x) = \lambda^{\frac{i}{4}} J_{\widehat{\Phi}} \Lambda_{\Phi_\delta}(\sigma^\Psi_{-\frac{i}{2}}(x^*)) = \lambda^{\frac{i}{4}} \Lambda_\Phi(R \circ \sigma^\Psi_{\frac{i}{2}}(x)) \\ &= \lambda^{\frac{i}{4}} \Lambda_\Phi(\sigma^\Psi_{\frac{-i}{2}}(R(x^*)^*) = \lambda^{\frac{i}{4}} J_\Phi \Lambda_\Phi(R(x^*)) = \lambda^{\frac{i}{4}} J_\Phi J_{\widehat{\Phi}} \Lambda_{\Phi_\delta}(x) \end{aligned}$$ □

8.3. Heisenberg's relations

We recall that $\alpha(N) \cup \beta(N) \subset M \subset \widehat{\beta}(N)'$ and $\alpha(N) \cup \widehat{\beta}(N) \subset \widehat{M} \subset \beta(N)'$ in $\mathcal{L}(H)$.

Proposition 8.3.1. – *For all $x \in M'$ and $y \in \widehat{M}'$, we have:*

$$W(x {}_\beta\!\underset{N^o}{\otimes}{}_\alpha y) = (x {}_\alpha\!\underset{N^o}{\otimes}{}_{\hat{\beta}} y) W$$

Proof. – Straightforward by proposition 3.4.3 and by definition of $\widehat{M}$. □

Proposition 8.3.2. – *The following equalities hold:*

$$\begin{array}{llll} i) & M \cap \widehat{M} = \alpha(N) & ii) & M' \cap \widehat{M} = \widehat{\beta}(N) \\ iii) & M \cap \widehat{M}' = \beta(N) & iv) & M' \cap \widehat{M}' = J_\Phi \beta(N) J_\Phi \end{array}$$

Proof. – We start to prove i). We already know that $M \cap \widehat{M} \supset \beta(N)$. In the other way, let $m \in M \cap \widehat{M}$. Then, we have by the previous proposition and the unitarity of W:

$$\Gamma(m) = W^*(1 {}_\alpha\!\underset{N^o}{\otimes}{}_{\hat{\beta}} m) W = W^* W (1 {}_\beta\!\underset{N}{\otimes}{}_\alpha m) = 1 {}_\beta\!\underset{N}{\otimes}{}_\alpha m$$

so that m belongs to $\beta(N)$by proposition 9.2.25. Apply R to get iii) and then apply $\widehat{R}$ to get iv). Finally apply $\widehat{R}$ to i) to get ii). □

PART II

EXAMPLES

In this part, we present a variety of measured quantum groupoids. First of all, we are interested in the so-called adapted measured quantum groupoids. These are a class of measured quantum groupoids with much less complicated axioms because we are able to construct the antipode. The axiomatic is inspired by J. Kustermans and S. Vaes' locally quantum groups with a weak condition on the basis. That is what we develop first. We also characterize adapted measured quantum groupoids and their dual among measured quantum groupoids. Then, we give different examples of adapted measured quantum groupoids and, in particular, the case of groupoids and quantum groups. In a second time, we investigate inclusions of von Neumann algebras of depth 2 which can be seen as measured quantum groupoids but they are not in general of adapted measured quantum groupoids' type. Finally, we explain how to produce new examples from well known measured quantum groupoids thanks to simple operations.

We want to lay stress on a fact: historically speaking, the notion of adapted measured quantum groupoid was the first one we introduce. The main interest of the structure is the rather quite simple axioms. So it is easier to find examples (see sections 10, 11 12, 13). But we discovered examples of quantum space quantum groupoid (section 14) and pairs quantum groupoid (section 15) duals of which are not adapted measured quantum groupoid anymore that is we have not a dual structure within category of adapted measured quantum groupoid. Moreover this category do not cover all inclusions of von Neumann algebras (section 16). That's why we introduce a larger category the now so-called measured quantum groupoid which answer all the problems.

CHAPTER 9

ADAPTED MEASURED QUANTUM GROUPOIDS

In this section, we introduce a new natural hypothesis which gives a link between the right (resp. left) invariant operator-valued weight and the (resp. anti-) representation of the basis.

9.1. Definitions

Definition 9.1.1

We say that a n.s.f. operator-valued weight T_L from M to $\alpha(N)$ is **β-adapted** if there exists a n.s.f. weight ν_L on N such that:

$$\sigma_t^{T_L}(\beta(n)) = \beta(\sigma_{-t}^{\nu_L}(n))$$

for all $n \in N$ and $t \in \mathbb{R}$. We also say that T_L is β-adapted w.r.t. ν_L.

We say that a n.s.f. operator-valued weight T_R from M to $\beta(N)$ is **α-adapted** if there exists a n.s.f. weight ν_R on N such that:

$$\sigma_t^{T_R}(\alpha(n)) = \alpha(\sigma_t^{\nu_R}(n))$$

for all $n \in N$ and $t \in \mathbb{R}$. We also say that T_R is α-adapted w.r.t. ν_R.

Definition 9.1.2. – A Hopf bimodule $(N, M, \alpha, \beta, \Gamma)$ with left (resp. right) invariant n.s.f. operator-valued weight T_L (resp. T_R) from M to $\alpha(N)$ (resp. $\beta(N)$) is said to be a **adapted measured quantum groupoid** if there exists a n.s.f. weight ν on N such that T_L is β-adapted w.r.t. ν and T_R is α-adapted w.r.t. ν. Then, we denote by $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ the adapted measured quantum groupoid and we say that ν is **quasi-invariant**.

Remark 9.1.3. – If a n.s.f. operator-valued weight T_L from M to $\alpha(N)$ is β-adapted w.r.t. ν and if R is a co-involution of M, then the n.s.f. operator-valued weight $R \circ T_L \circ R$ from M to $\beta(N)$ is α-adapted w.r.t. the same weight ν.

LEMMA 9.1.4. – *If μ is a n.s.f. weight on N and if an operator-valued weight T_L is β-adapted w.r.t. ν, then there exists an operator-valued weight S^μ from M to $\beta(N)$, which is α-adapted w.r.t. μ such that $\mu \circ \alpha^{-1} \circ T_L = \nu \circ \beta^{-1} \circ S^\mu$. Also, if χ is a n.s.f. weight on N and if an operator-valued weight T_R is α-adapted w.r.t. ν, then there exists an operator-valued weight S_χ from M to $\alpha(N)$ normal, which is β-adapted w.r.t. χ such that $\chi \circ \beta^{-1} \circ T_R = \nu \circ \beta^{-1} \circ S_\chi$.*

Proof. – For all $n \in N$ and $t \in \mathbb{R}$, we have $\sigma_t^{\mu\circ\alpha^{-1}\circ T_L}(\beta(n)) = \sigma_t^{\nu\circ\beta^{-1}}(\beta(n))$. By Haagerup's theorem, we obtain the existence of S^μ which is clearly adapted. The second part of the lemma is very similar. □

Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be an adapted measured quantum groupoid. Then the opposite adapted measured quantum groupoid is $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma, \nu^o, T_R, T_L)$. We put:

$$\Phi = \nu \circ \alpha^{-1} \circ T_L \quad \text{and} \quad \Psi = \nu \circ \beta^{-1} \circ T_R$$

We also put $S^\nu = S_L$ and $S_\nu = S_R$. By 3.2.2 and 3.2.4, we have:

$$\Lambda_\Phi(\mathscr{T}_{\Phi,S_L}) \subseteq J_\Phi\Lambda_\Phi(\mathscr{N}_\Phi \cap \mathscr{N}_{S_L}) \subseteq D((H_\Phi)_\beta, \nu^o)$$

and we have $R^{\beta,\nu^o}(J_\Phi\Lambda_\Phi(a)) = J_\Phi\Lambda_{S_L}(a)J_\nu$ for all $a \in \mathscr{N}_\Phi \cap \mathscr{N}_{S_L}$.

9.2. Antipode

Then we construct a closed antipode with polar decomposition which leads to a co-involution and a one-parameter group of automorphisms of M called scaling group.

9.2.1. The operator G. – We construct now an closed unbounded operator on H_Φ with polar decomposition which gives needed elements to construct the antipode. We have the following lemmas:

LEMMA 9.2.1. – *For all $\lambda \in \mathbb{C}$, $x \in \mathscr{D}(\sigma^\nu_{i\lambda})$ and $\xi, \xi' \in \Lambda_\Phi(\mathscr{T}_{\Phi,T_L})$, we have:*

$$\begin{aligned} &\alpha(x)\Delta_\Phi^\lambda \subseteq \Delta_\Phi^\lambda \alpha(\sigma^\nu_{i\lambda}(x)) \\ &R^{\alpha,\nu}(\Delta_\Phi^\lambda\xi)\Delta_\nu^\lambda \subseteq \Delta_\Phi^\lambda R^{\alpha,\nu}(\xi) \\ &\textit{and } \sigma^\nu_{i\lambda}(\langle\Delta_\Phi^\lambda\xi, \xi'\rangle_{\alpha,\nu}) = \langle\xi, \Delta_\Phi^{\overline{\lambda}}\xi'\rangle_{\alpha,\nu} \end{aligned} \tag{2}$$

and:

$$\begin{aligned} &\hat{\beta}(x)\Delta_\Phi^\lambda \subseteq \Delta_\Phi^\lambda \hat{\beta}(\sigma^\nu_{i\lambda}(x)) \\ &R^{\hat{\beta},\nu^o}(\Delta_\Phi^\lambda\xi)\Delta_\nu^\lambda \subseteq \Delta_\Phi^\lambda R^{\hat{\beta},\nu^o}(\xi) \\ &\textit{and } \sigma^\nu_{i\lambda}(\langle\Delta_\Phi^\lambda\xi, \xi'\rangle_{\hat{\beta},\nu^o}) = \langle\xi, \Delta_\Phi^{\overline{\lambda}}\xi'\rangle_{\hat{\beta},\nu^o}. \end{aligned} \tag{3}$$

Proof. – Straightforward. □

Then, by **[Sau86]** and proposition 2.3.5, we can define a closed operator $\Delta_{\Phi}^{\lambda}{}_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}\Delta_{\Phi}^{\lambda}$ which naturally acts on elementary tensor products for all $\lambda \in \mathbb{C}$. Moreover, for all $n \in N$, we have $J_{\Phi}\alpha(n) = \hat{\beta}(n^*)J_{\Phi}$, so that we can define a unitary anti-linear operator:

$$J_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}J_{\Phi} : H_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}H_{\Phi} \longrightarrow H_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}H_{\Phi}$$

such that the adjoint is $J_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}J_{\Phi}$. Also, by composition, it is possible to define a natural closed anti-linear operator:

$$S_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}S_{\Phi} : H_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}H_{\Phi} \longrightarrow H_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}H_{\Phi}$$

In the same way, if $F_{\Phi} = S_{\Phi}^*$, then it is possible to define a natural closed anti-linear operator: $F_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}F_{\Phi} : H_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}H_{\Phi} \to H_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}H_{\Phi}$ and we have:

$$(S_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}S_{\Phi})^* = F_{\Phi\hat{\beta}}\underset{\nu}{\otimes}{}_{\alpha}F_{\Phi}$$

LEMMA 9.2.2. – *For all $c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^*(\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$, $e \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and all net $(e_k)_{k\in K}$ of elements of $\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R}$ weakly converging to 1, then $(\lambda^{\beta,\alpha}_{J_{\Psi}\Lambda_{\Psi}(e_k)})^*$ $(1_{\beta}\underset{N}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_{H_{\Psi}}\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(c^*)}$ converges to $(\lambda^{\hat{\alpha},\beta}_{\Lambda_{\Psi}(c)})^*U'^*_{H_{\Phi}}\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}$ in the weak topology.*

Proof. – By 3.3.1, we have, for all $k \in K$:

$$\begin{aligned}
&(\lambda^{\beta,\alpha}_{J_{\Psi}\Lambda_{\Psi}(e_k)})^*(1_{\beta}\underset{\nu}{\otimes}{}_{\alpha}J_{\Phi}eJ_{\Phi})U_{H_{\Psi}}\rho^{\alpha,\hat{\beta}}_{\Lambda_{\Phi}(c^*)}\\
&= (\lambda^{\beta,\alpha}_{J_{\Psi}\Lambda_{\Psi}(e_k)})^*\Gamma(c^*)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)} = \left(\Gamma(c)\lambda^{\beta,\alpha}_{J_{\Psi}\Lambda_{\Psi}(e_k)}\right)^*\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}\\
&= \left((J_{\Psi}e_kJ_{\Psi\beta}\underset{N}{\otimes}{}_{\alpha}1)U'_{H_{\Phi}}\lambda^{\hat{\alpha},\beta}_{\Lambda_{\Psi}(c)}\right)^*\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}\\
&= (\lambda^{\hat{\alpha},\beta}_{\Lambda_{\Psi}(c)})^*U'^*_{H_{\Phi}}(J_{\Psi}e_k^*J_{\Psi\beta}\underset{N}{\otimes}{}_{\alpha}1)\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)} = (\lambda^{\hat{\alpha},\beta}_{\Lambda_{\Psi}(c)})^*U'^*_{H_{\Phi}}\rho^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e)}J_{\Psi}e_k^*J_{\Psi}
\end{aligned}$$

This computation implies the lemma. □

LEMMA 9.2.3. – *If $c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^*(\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$, $e \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$, $\eta \in H_{\Psi}$, $v \in H_{\Phi}$ and a net $(e_k)_{k\in K}$ of $\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R}$ converges weakly to 1, then the net:*

$$((U_{H_{\Psi}}(\eta_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(c^*))|J_{\Psi}\Lambda_{\Psi}(e_k)_{\beta}\underset{\nu}{\otimes}{}_{\alpha}J_{\Phi}e^*J_{\Phi}v))_{k\in K}$$

converges to $(\eta|(\rho_{J_{\Phi}\Lambda_{\Phi}(e)})^*U'_{H_{\Phi}}(\Lambda_{\Psi}(c)_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}v))$.

Proof. – It's a re-formulation of the previous lemma. □

PROPOSITION 9.2.4. – *Let $(\eta_i)_{i\in I}$ be a (N,ν)-basis of ${}_{\alpha}H$, $\Xi \in H_{\Psi\beta}\underset{\nu}{\otimes}{}_{\alpha}H$, $u \in D({}_{\alpha}H,\nu)$, $c \in (\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L})^*(\mathcal{N}_{\Psi} \cap \mathcal{N}_{T_R})$, $h \in \mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L}$ and e be an element of $\mathcal{N}_{\Phi} \cap \mathcal{N}_{T_L} \cap \mathcal{N}^*_{\Phi} \cap \mathcal{N}^*_{T_L}$. Then, we have:*

$$\lim_k \sum_{i\in I}(\eta_{i\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}h^*(\lambda^{\beta,\alpha}_{J_{\Phi}\Lambda_{\Phi}(e_k)})^*U_{H_{\Psi}}((\rho^{\beta,\alpha}_{\eta_i})^*\Xi_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}\Lambda_{\Phi}(c^*))|u_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}}J_{\Phi}\Lambda_{\Phi}(e^*))$$

exists and is equal to $((\rho_u^{\beta,\alpha})^*\Xi|(\rho_{J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha})^*U'_{H_\Psi}(\Lambda_\Psi(c)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta\Lambda_\Phi(h)))$.

Proof. – By 3.4.1 and 3.4.2, we can compute, for all $i\in I$ and $k\in K$:

$$\begin{aligned}&(\eta_{i\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}h^*(\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*U_{H_\Psi}((\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|u_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_\Phi\Lambda_\Phi(e^*))\\
&=(\hat\beta(\langle\eta_i,u\rangle_{\alpha,\nu})(\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*U_{H_\Psi}((\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|gJ_\Phi\Lambda_\Phi(e^*))\\
&=((\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*(1_\beta\underset{\nu}{\otimes}{}_\alpha\hat\beta(\langle\eta_i,u\rangle_{\alpha,\nu})U_{H_\Psi}((\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|J_\Phi e^*J_\Phi\Lambda_\Phi(h)))\\
&=((\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*U_{H_\Psi}(\beta(\langle\eta_i,u\rangle_{\alpha,\nu})(\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|J_\Phi e^*J_\Phi\Lambda_\Phi(h))\end{aligned}$$

Take the sum over i to obtain:

$$\begin{aligned}&\sum_{i\in I}(\eta_{i\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}h^*(\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*U_{H_\Psi}((\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|u_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_\Phi\Lambda_\Phi(e^*))\\
&\qquad=(U_{H_\Psi}((\rho_u^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|J_\Phi\Lambda_\Phi(e_k)_\beta\underset{\nu}{\otimes}{}_\alpha J_\Phi e^*J_\Phi\Lambda_\Phi(h))\end{aligned}$$

so that lemma 9.2.3 implies:

$$\begin{aligned}&\lim_k\sum_{i\in I}(\eta_{i\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}h^*(\lambda_{J_\Phi\Lambda_\Phi(e_k)}^{\beta,\alpha})^*U_{H_\Psi}((\rho_{\eta_i}^{\beta,\alpha})^*\Xi_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(c^*))|u_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_\Phi\Lambda_\Phi(e^*))\\
&\qquad=((\rho_u^{\beta,\alpha})^*\Xi|(\rho_{J_\Phi\Lambda_\Phi(e)}^{\beta,\alpha})^*U'_{H_\Phi}(\Lambda_\Psi(c)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta\Lambda_\Phi(h)))\quad\square\end{aligned}$$

PROPOSITION 9.2.5. – *For all* $a,c\in(\mathcal{N}_\Phi\cap\mathcal{N}_{T_L})^*(\mathcal{N}_\Psi\cap\mathcal{N}_{T_R})$, $b,d\in\mathcal{T}_{\Psi,T_R}$ *and* $g,h\in\mathcal{T}_{\Phi,S_L}$, *the following vector:*

$$U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h)_\beta\underset{\nu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_{H_\Psi}(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*)))$$

belongs to $\mathcal{D}(S_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}S_\Phi)$ *and the value of* $\sigma_\nu(S_{\Phi\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}S_\Phi)$ *on this vector is equal to:*

$$U^*_{H_\Phi}\Gamma(h^*)(\Lambda_\Phi(g)_\beta\underset{\nu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^*U_{H_\Psi}(\Lambda_\Psi(c)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*)))$$

Proof. – For the proof, let denote by $\Xi_1=U'_{H_\Phi}(\Lambda_\Psi(ab)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta\Lambda_\Phi(h))$ and by $\Xi_2=U'_{H_\Phi}(\Lambda_\Psi(cd)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta\Lambda_\Phi(g))$. Then, for all $e,f\in\mathcal{N}_{T_L}\cap\mathcal{N}_\Phi\cap\mathcal{N}^*_{T_L}\cap\mathcal{N}^*_\Phi$, the scalar product of $F_\Phi J_\Phi\Lambda_\Phi(e^*)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}F_\Phi J_\Phi\Lambda_\Phi(f)$ by:

$$U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h)_\beta\underset{\nu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_{H_\Psi}(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*)))$$

is equal to the scalar product of $J_\Phi\Lambda_\Phi(e)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_\Phi\Lambda_\Phi(f^*)$ by:

$$U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h)_\beta\underset{\nu}{\otimes}{}_\alpha(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_{H_\Psi}(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*)))$$

By 3.5.9, this scalar product is equal to the limit over k of the sum over i of:

$$(J_\Phi\Lambda_\Phi(e)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_\Phi\Lambda_\Phi(f^*)|\eta_{i\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}g^*(\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(e_k)})^*U_{H_\Psi}((\rho^{\beta,\alpha}_{\eta_i})^*\Xi_{1\alpha}\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*)))$$

By the previous proposition applied with $\Xi=\Xi_1$, we get the symmetric expression:

$$((\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(f)})^*\Xi_2|(\rho^{\beta,\alpha}_{J_\Phi\Lambda_\Phi(e)})^*\Xi_1)$$

so that, again by the previous proposition applied, this time, with $\Xi = \Xi_2$ we obtain the limit over k of the sum over i of:

$$(\eta_i {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} h^*(\lambda^{\beta,\alpha}_{J_\Psi\Lambda_\Psi(e_k)})^* U_{H_\Psi}((\rho^{\beta,\alpha}_{\eta_i})^*\Xi_2 {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*))|J_\Phi\Lambda_\Phi(f) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} J_\Phi\Lambda_\Phi(e^*))$$

This last expression is equal to the scalar product of:

$$U^*_{H_\Phi}\Gamma(h^*)(\Lambda_\Phi(g) {}_\beta\underset{\nu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*)))$$

by $J_\Phi\Lambda_\Phi(f) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} J_\Phi\Lambda_\Phi(e^*)$ and to the scalar product of:

$$\sigma_{\nu^o} U^*_{H_\Phi}\Gamma(h^*)(\Lambda_\Phi(g) {}_\beta\underset{\nu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*)))$$

by $J_\Phi\Lambda_\Phi(e^*) {}_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha J_\Phi\Lambda_\Phi(f)$. Since the linear span of $J_\Phi\Lambda_\Phi(e^*) {}_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha J_\Phi\Lambda_\Phi(f)$ where $e, f \in \mathcal{N}_{T_L} \cap \mathcal{N}_\Phi \cap \mathcal{N}^*_{T_L} \cap \mathcal{N}^*_\Phi$ is a core of $F_\Phi {}_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha F_\Phi$, we get that:

$$U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h) {}_\beta\underset{\nu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U^*_{H_\Psi}(\Lambda_\Psi(a) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*)))$$

belongs to $\mathcal{D}(S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi)$ and the value of $S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi$ on this vector is:

$$\sigma_{\nu^o} U^*_{H_\Phi}\Gamma(h^*)(\Lambda_\Phi(g) {}_\beta\underset{\nu}{\otimes}{}_\alpha (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*))) \quad \square$$

PROPOSITION 9.2.6. – *There exists a closed densely defined anti-linear operator G on H_Φ such that the linear span of:*

$$(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))$$

with $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}), b, d \in \mathcal{T}_{\Psi,T_R}$, is a core of G and we have:

$$G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))$$

$$= (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*))$$

Moreover, $G\mathcal{D}(G) = \mathcal{D}(G)$ and $G^2 = \mathrm{id}_{|\mathcal{D}(G)}$.

Proof. – For all $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$, $a(n,l), c(n,l) \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$ and $b(n,l), d(n,l) \in \mathcal{T}_{\Psi,T_R}$ and let $w \in H_\Phi$ such that:

$$v_n = \sum_{l=1}^{k_n}(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b(n,l)^*))})^* U_{H_\Psi}(\Lambda_\Psi(a(n,l)) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((c(n,l)d(n,l))^*)) \longrightarrow 0$$

$$w_n = \sum_{l=1}^{k_n}(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d(n,l)^*))})^* U_{H_\Psi}(\Lambda_\Psi(c(n,l)) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((a(n,l)b(n,l))^*)) \longrightarrow w$$

We have $U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h) {}_\beta\underset{\nu}{\otimes}{}_\alpha v_n) \in \mathcal{D}(S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi)$ for all $g, h \in \mathcal{T}_{\Phi,S_L}$ and $n \in \mathbb{N}$ by the previous proposition. Moreover, we have:

$$\sigma_\nu(S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi)U^*_{H_\Phi}\Gamma(g^*)(\Lambda_\Phi(h) {}_\beta\underset{\nu}{\otimes}{}_\alpha v_n) = U^*_{H_\Phi}\Gamma(h^*)(\Lambda_\Phi(g) {}_\beta\underset{\nu}{\otimes}{}_\alpha w_n)$$

Since $\Lambda_\Phi(g)$ and $\Lambda_\Phi(h)$ belongs to $D((H_\Phi)_\beta, \nu^o)$, we obtain:

$$\sigma_\nu(S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi) U^*_{H_\Phi} \Gamma(g^*) \lambda^{\beta,\alpha}_{\Lambda_\Phi(h)} v_n = U^*_{H_\Phi} \Gamma(h^*) \lambda^{\beta,\alpha}_{\Lambda_\Phi(g)} w_n$$

The closure of $S_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} S_\Phi$ implies that $U^*_{H_\Phi} \Gamma(h^*) \lambda^{\beta,\alpha}_{\Lambda_\Phi(g)} w = 0$. So, apply U_{H_Φ}, to get $\Gamma(h^*) \lambda^{\beta,\alpha}_{\Lambda_\Phi(g)} w = 0$. Now, $\mathcal{T}_{\Phi,S_L}$ is dense in M that's why $\lambda^{\beta,\alpha}_{\Lambda_\Phi(g)} w = 0$ for all $g \in \mathcal{T}_{\Phi,S_L}$. Then, by 3.2.4, we have:

$$||\lambda^{\beta,\alpha}_{\Lambda_\Phi(g)} w||^2 = (\alpha(\langle \Lambda_\Phi(g), \Lambda_\Phi(g)\rangle_{\beta,\nu^o}) w|w) = (S_L(\sigma^\Phi_{i/2}(g)\sigma^\Phi_{-i/2}(g^*))w|w)$$

By density of $\mathcal{T}_{\Phi,S_L}$, we obtain $||w||^2 = 0$ *i.e.*, $w = 0$. Consequently, the formula given in the proposition for G gives rise to a closable densely defined well-defined operator on H_Φ. So the required operator is the closure of the previous one. □

Thanks to polar decomposition of the closed operator G, we can give the following definitions:

Definition 9.2.7. – We denote by D the strictly positive operator G^*G on H_Φ (that means positive, self-adjoint and injective) and by I the anti-unitary operator on H_Φ such that $G = ID^{1/2}$.

Since G is involutive, we have $I = I^*$, $I^2 = 1$ and $IDI = D^{-1}$.

9.2.2. A fundamental commutation relation. – In this section, we establish a commutation relation between G and the elements $(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})$. We recall that $W' = U'_{H_\Psi}$. We begin by two lemmas borrowed from **[Eno02]**.

Lemma 9.2.8. – *Let ξ_i be a (N^o, ν^o)-basis of $(H_\Psi)_\beta$. For all $w' \in D({}_{\hat\alpha}H_\Psi, \nu)$ and $w \in H_\Psi$, we have:*

$$W'(w' {}_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta w) = \sum_i \xi_i {}_\beta\underset{\nu}{\otimes}{}_\alpha (\omega_{w',\xi_i} * \mathrm{id})(W')w$$

If we put $\delta_i = (\omega_{w',\xi_i} * \mathrm{id})(W')w$, then $\alpha(\langle \xi_i, \xi_i\rangle_{\beta,\nu^o})\delta_i = \delta_i$. Moreover, if $w \in D({}_{\hat\alpha}(H_\Psi), \nu)$, then $\delta_i \in D({}_{\hat\alpha}(H_\Psi), \nu)$.

For all $v, v' \in D((H_\Psi)_\beta, \nu^o)$ and $i \in I$, there exists $\zeta_i \in D((H_\Psi)_\beta, \nu^o)$ such that $\alpha(\langle \xi_i, \xi_i\rangle_{\beta,\nu^o})\zeta_i = \zeta_i$ and:

$$W'(v' {}_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta v) = \sum_i \xi_i {}_\beta\underset{\nu}{\otimes}{}_\alpha \zeta_i$$

Proof. – Lemma 3.4 of **[Eno02]**. □

Remark 9.2.9. – If $v, v' \in \Lambda_\Psi(\mathcal{T}_{\Psi,T_R}) \subseteq D({}_{\hat\alpha}H, \nu) \cap D(H_\beta, \nu^o)$, then, with notations of the previous lemma, we have $\zeta_i \in D({}_{\hat\alpha}H, \nu) \cap D(H_\beta, \nu^o)$.

LEMMA 9.2.10. – *Let $v, v' \in D(H_\beta, \nu^o)$ and $w, w' \in D({}_{\hat\alpha}H, \nu)$. With notations of the previous lemma, we have:*

$$(\omega_{v,w} * \mathrm{id})(U'^*_H)(\omega_{v',w'} * \mathrm{id})(U'^*_H) = \sum_i (\omega_{\zeta_i,\delta_i} * \mathrm{id})(U'^*_H)$$

in the norm convergence (and also in the weak convergence).

Proof. – Proposition 3.6 of [**Eno02**]. □

LEMMA 9.2.11. – *Let a, c belonging to $(\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$. For all b, d, a', b', c', $d' \in \mathcal{T}_{\Psi,T_R}$, the value of $(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b'^*))})^* U_{H_\Psi}$ on the sum over i of:*

$$\Lambda_\Psi((\omega_{\Lambda_\Psi(ab),\xi_i} * \mathrm{id})(W')a')_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((c'd')^*(\omega_{\xi_i,\Lambda_\Psi(cd)} * \mathrm{id})(W'^*))$$

is equal to:

$$(\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a)_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((cd)^*))$$

Proof. – First, let's suppose that $a \in \mathcal{T}_{\Psi,T_R}$. By 3.3.2 and 3.3.4, we have:

$$\begin{aligned}
&(\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a)_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((cd)^*)) \\
&= (\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})\Lambda_\Phi((\omega_{\Lambda_\Psi(a),\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))\beta} \underset{\nu}{\otimes}{}_\alpha \mathrm{id})(\Gamma((cd)^*))) \\
&= (\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})\Lambda_\Phi((\omega_{\Lambda_\Psi(ab),\Lambda_\Psi(cd)} * \mathrm{id})(U'^*_{H_\Phi}))
\end{aligned}$$

By 9.2.10 and the closure of Λ_Φ, this expression is equal to the sum over $i \in I$ of:

$$\Lambda_\Phi((\omega_{(\omega_{\Lambda_\Psi(ab),\xi_i} * \mathrm{id})(W')\Lambda_\Psi(a'b'),(\omega_{\Lambda_\Psi(cd),\xi_i} * \mathrm{id})(W')\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi}))$$

Again, 3.3.2 and 3.3.4, we obtain the sum over $i \in I$ of the value of $(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b'^*))})^* U_{H_\Psi}$ on:

$$\Lambda_\Psi((\omega_{\Lambda_\Psi(ab),\xi_i} * \mathrm{id})(W')a')_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((c'd')^*(\omega_{\xi_i,\Lambda_\Psi(cd)} * \mathrm{id})(W'^*))$$

A density argument finishes the proof. □

PROPOSITION 9.2.12. – *If $v, w \in \Lambda_\Psi(\mathcal{T}^2_{\Psi,T_R}) \subseteq D({}_{\hat\alpha}(H_\Psi), \nu) \cap D((H_\Psi)_\beta, \nu^o)$, then we have:*

$$(\omega_{v,w} * \mathrm{id})(U'^*_{H_\Phi})G \subseteq G(\omega_{w,v} * \mathrm{id})(U'^*_{H_\Phi}) \tag{4}$$

$$\textit{and } (\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})G^* \subseteq G^*(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi}) \tag{5}$$

Proof. – Let $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$ and $b, d, a', b', c', d' \in \mathcal{T}_{\Psi,T_R}$. By definition of G, we have:

$$(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c)_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((ab)^*)) \in \mathcal{D}(G)$$

and:

$$(\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c)_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_\Phi((ab)^*))$$

$$= (\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(U'^*_{H_\Phi})(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a)_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((cd)^*))$$

By the previous lemma, this is the sum over $i \in I$ of $G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d'^*))})^* U_{H_\Psi}$ on:

$$\Lambda_\Psi((\omega_{\Lambda_\Psi(cd),\xi_i} * \mathrm{id})(W')c')_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((a'b')^*(\omega_{\xi_i,\Lambda_\Psi(ab)} * \mathrm{id})(W'^*))$$

Now, G is a closed operator, so that the sum over $i \in I$ of $(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d'^*))})^* U_{H_\Psi}$ on:

$$\Lambda_\Psi((\omega_{\Lambda_\Psi(cd),\xi_i} * \mathrm{id})(W')c')_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((a'b')^*(\omega_{\xi_i,\Lambda_\Psi(ab)} * \mathrm{id})(W'^*))$$

belongs to $\mathscr{D}(G)$ and by the previous lemma, we obtain:

$$\begin{aligned}
&(\omega_{\Lambda_\Psi(a'b'),\Lambda_\Psi(c'd')} * \mathrm{id})(W'^*) G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c)_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((ab)^*))\\
&= G(\omega_{\Lambda_\Psi(c'd'),\Lambda_\Psi(a'b')} * \mathrm{id})(U'^*_{H_\Phi})(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^* U_{H_\Psi}(\Lambda_\Psi(c)_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((ab)^*))
\end{aligned}$$

Now the linear span:

$$(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^* U_{H_\Psi}(\Lambda_\Psi(a)_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} \Lambda_\Phi((cd)^*))$$

with $a, c \in (\mathscr{N}_\Phi \cap \mathscr{N}_{T_L})^*(\mathscr{N}_\Psi \cap \mathscr{N}_{T_R}), b, d \in \mathscr{T}_{\Psi,T_R}\}$, is a core for G that's why the first inclusion holds. The second one is the adjoint of the first one. □

Corollary 9.2.13. – *For all $v, w \in \Lambda_\Psi(\mathscr{T}^2_{\Psi,T_R})$, we have:*

$$(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D \subseteq D(\omega_{\Delta_\Psi^{-1}v,\Delta_\Psi w} * \mathrm{id})(U'_{H_\Phi})$$

where $D = G^*G$ is defined in 9.2.7.

Proof. – We have:

$$\begin{aligned}
(\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})G &= (\omega_{S_\Psi w,\Delta_\Psi S_\Psi v} * \mathrm{id})(U'^*_{H_\Phi})G && \text{by lemma 3.3.5}\\
&\subseteq G(\omega_{\Delta_\Psi S_\Psi v,S_\Psi w} \star \mathrm{id})(U'^*_{H_\Phi}) && \text{by inclusion (4)}\\
&= G(\omega_{\Delta_\Psi^{-1}v,\Delta_\Psi w} * \mathrm{id})(U'^*_{H_\Phi}) && \text{by lemma 3.3.5}
\end{aligned}$$

In the same way, we can finish the proof:

$$\begin{aligned}
(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D &= (\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})G^* && \text{by definition 9.2.7}\\
&\subseteq G^*(\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})G && \text{by inclusion (5)}\\
&\subseteq G^*G(\omega_{\Delta_\Psi^{-1}v,\Delta_\Psi w} * \mathrm{id})(U'_{H_\Phi}) &&\\
&= D(\omega_{\Delta_\Psi^{-1}v,\Delta_\Psi w} * \mathrm{id})(U'_{H_\Phi}) && \text{by definition 9.2.7.}
\end{aligned}$$
□

9.2.3. Scaling group. – In this section, we give a sense and we prove the following commutation relation $U'_{H_\Phi}(\Delta_\Psi {}_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta D) = (\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)U'_{H_\Phi}$ so as to construct the scaling group τ.

LEMMA 9.2.14. – *For all $\lambda \in \mathbb{C}$ and x analytic w.r.t. ν, we have:*

$$\alpha(x)D^\lambda \subseteq D^\lambda\alpha(\sigma^\nu_{-i\lambda}(x)) \text{ and } \beta(x)D^\lambda \subseteq D^\lambda\beta(\sigma^\nu_{-i\lambda}(x))$$

Proof. – For all $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R})$, $b, d \in \mathcal{T}_{\Psi,T_R}$ and x analytic w.r.t. ν, we have by 3.4.1 and 3.4.2:

$$\begin{aligned}
&\beta(x)G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_\Psi(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))\\
&= \beta(x)(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^*U_\Psi(\Lambda_\Psi(c)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*))\\
&= (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^*(1_\beta\underset{\nu}{\otimes}{}_\alpha\beta(x))U_\Psi(\Lambda_\Psi(c)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((ab)^*))\\
&= (\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))})^*U_\Psi(\Lambda_\Psi(c)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi(\beta(x)b^*a^*))\\
&= G(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\beta(\sigma^\nu_i(x))\sigma^\Psi_{-i}(b^*))})^*U_\Psi(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))\\
&= G\alpha(\sigma^\nu_{-i/2}(x^*))(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_\Psi(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))
\end{aligned}$$

Now, the linear span of:

$$(\lambda^{\beta,\alpha}_{\Lambda_\Psi(\sigma^\Psi_{-i}(b^*))})^*U_\Psi(\Lambda_\Psi(a)_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}\Lambda_\Phi((cd)^*))$$

where $a, c \in (\mathcal{N}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{N}_\Psi \cap \mathcal{N}_{T_R}), b, d \in \mathcal{T}_{\Psi,T_R}\}$, is a core for G, so that we have:

$$\beta(x)G \subseteq G\alpha(\sigma^\nu_{-i/2}(x^*))$$

Take adjoint to obtain $\alpha(x)G^* \subseteq G^*\beta(\sigma^\nu_{i/2}(x^*))$. So, we conclude by:

$$\alpha(x)D = \alpha(x)G^*G \subseteq G^*\beta(\sigma^\nu_{i/2}(x^*))G \subseteq D\alpha(\sigma^\nu_{-i}(x))$$

The second part of the lemma can be proved in a very similar way. □

We now state two lemmas analogous to relations (2) and (3) for Ψ and we justify the existence of natural operators:

LEMMA 9.2.15. – *For all $\lambda \in \mathbb{C}$, $x \in \mathcal{D}(\sigma^\nu_{-i\lambda})$ and $\xi, \xi' \in \Lambda_\Psi(\mathcal{T}_{\Psi,T_R})$, we have:*

$$\beta(x)\Delta^\lambda_\Psi \subseteq \Delta^\lambda_\Psi\beta(\sigma^\nu_{-i\lambda}(x))$$

$$R^{\beta,\nu^o}(\Delta^\lambda_\Psi\xi)\Delta^{-\lambda}_\nu \subseteq \Delta^\lambda_\Psi R^{\beta,\nu^o}(\xi) \tag{6}$$

$$\text{and } \sigma^\nu_{-i\lambda}(\langle\Delta^\lambda_\Psi\xi, \xi'\rangle_{\beta,\nu^o}) = \langle\xi, \Delta^{\overline{\lambda}}_\Psi\xi'\rangle_{\beta,\nu^o}$$

and:

$$\hat{\alpha}(x)\Delta_\Psi^\lambda \subseteq \Delta_\Psi^\lambda \hat{\alpha}(\sigma^\nu_{-i\lambda}(x))$$

$$R^{\hat{\alpha},\nu^o}(\Delta_\Psi^\lambda \xi)\Delta_\nu^{-\lambda} \subseteq \Delta_\Psi^\lambda R^{\hat{\alpha},\nu^o}(\xi) \tag{7}$$

$$\textit{and } \sigma^\nu_{-i\lambda}(\langle \Delta_\Psi^\lambda \xi, \xi' \rangle_{\hat{\alpha},\nu^o}) = \langle \xi, \Delta_\Psi^{\overline{\lambda}} \xi' \rangle_{\hat{\alpha},\nu^o}$$

Proof. – It is sufficient to apply 9.2.1 to the opposite adapted measured quantum groupoid for example. □

Then, we can define, for all $\lambda \in \mathbb{C}$, a closed linear operator $\Delta_\Psi^\lambda {}_\beta\underset{\nu}{\otimes}{}_\alpha D^\lambda$ which naturally acts on elementary tensor products of $H_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha H_\Phi$. With relations (7) in hand, we also get a closed linear operator $\Delta_\Psi^\lambda {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D^\lambda$ on $H_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta H_\Phi$.

Proposition 9.2.16. – *The following relation holds:*

$$U'_{H_\Phi}(\Delta_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D) = (\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)U'_{H_\Phi} \tag{8}$$

Proof. – By 9.2.13, we have, for all $v, w \in \Lambda_\Psi(\mathcal{T}_{\Psi,T_R})$ and $v', w' \in \mathcal{D}(D)$:

$$\begin{aligned}(U'_{H_\Phi}(v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta v') | \Delta_\Psi w {}_\beta\underset{\nu}{\otimes}{}_\alpha D w') &= ((\omega_{v,\Delta_\Psi w} * \mathrm{id})(U'_{H_\Phi})v' | Dw') \\ &= (D(\omega_{\Delta_\Psi^{-1}(\Delta_\Psi v),\Delta_\Psi w} * \mathrm{id})(U'_{H_\Phi})v' | w') \\ &= ((\omega_{\Delta_\Psi v, w} * \mathrm{id})(U'_{H_\Phi})Dv' | w') \\ &= (U'_{H_\Phi}(\Delta_\Psi v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta Dv') | w {}_\beta\underset{\nu}{\otimes}{}_\alpha w')\end{aligned}$$

By definition, we know that $\Lambda_\Psi(\mathcal{T}_{\Psi,T_R}) \odot \mathcal{D}(D)$ is a core for $\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D$ so, for all $u \in \mathcal{D}(\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)$, we have:

$$(U'_{H_\Phi}(v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta v') | (\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)u) = (U'_{H_\Phi}(\Delta_\Psi v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta Dv') | u)$$

Since $\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D$ is self-adjoint, we get:

$$(\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)U'_{H_\Phi}(v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta v') = U'_{H_\Phi}(\Delta_\Psi v {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta Dv')$$

Finally, since $\Lambda_\Psi(\mathcal{T}_{\Psi,T_R}) \odot \mathcal{D}(D)$ is a core for $\Delta_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D$ and by closeness of $\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D$, we deduce that:

$$U'_{H_\Phi}(\Delta_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D) \subseteq (\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)U'_{H_\Phi}$$

Because of unitarity of U'_{H_Φ}, we get that $(\Delta_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D)U'^*_{H_\Phi} \subseteq U'^*_{H_\Phi}(\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)$ and by taking the adjoint, we get the reverse inclusion:

$$(\Delta_\Psi {}_\beta\underset{\nu}{\otimes}{}_\alpha D)U'_{H_\Phi} \subseteq U'_{H_\Phi}(\Delta_\Psi {}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_\beta D) \quad \square$$

We know begin the construction of the scaling group τ strictly speaking. We also prove a theorem which state that $A(U'_H) = M$ and generalize proposition 1.5 of [**KV03**].

DEFINITION 9.2.17. – We denote by M_R the weakly closed linear span of:

$$\{(\omega_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(x)) \mid x \in M,\ \omega \in M_*^+ \text{ s.t. } \exists k \in \mathbb{R}^+,\ \omega \circ \beta \leq k\nu\}$$

Also, we denote by M_L the weakly closed linear span of:

$$\{(\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega)(\Gamma(x)) \mid x \in M,\ \omega \in M_*^+ \text{ s.t. } \exists k \in \mathbb{R}^+,\ \omega \circ \alpha \leq k\nu\}$$

By 3.3.4 and 3.7.2, M_R is equal to the von Neumann subalgebra $A(U'_H)$ of M. Also, M_L is a von Neumann subalgebra of M. Moreover, we know $\alpha(N) \subseteq M_R$ and $\beta(N) \subseteq M_L$, so that $M_{L\beta}\underset{\nu}{\star}{}_{\alpha}M_R$ makes sense. Also, we have, for all $m \in M$:

$$\Gamma(m) \in M_{L\beta}\underset{N}{\star}{}_{\alpha}M_R \tag{9}$$

LEMMA 9.2.18. – *There exists a unique strongly continuous one-parameter group τ of automorphisms of M_R such that $\tau_t(x) = D^{-it}xD^{it}$ for all $t \in \mathbb{R}$ and $x \in M_R$.*

Proof. – By commutation relation (8), for all $t \in \mathbb{R}$ and $v, w \in \Lambda_\Psi(\mathscr{T}_{\Psi,T_R})$, we get that:

$$D^{-it}(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D^{it} = (\omega_{\Delta_\Psi^{-it}v,\Delta_\Psi^{it}w} * \mathrm{id})(U'_{H_\Phi})$$

Consequently, we obtain $D^{-it}M_RD^{it} = M_R$ which is the only point to show. □

LEMMA 9.2.19. – *We have $\tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n))$ for all $n \in N$ and $t \in \mathbb{R}$.*

Proof. – Straightforward by lemma 9.2.14. □

PROPOSITION 9.2.20. – *We have $(\sigma_t^\Psi{}_{\beta}\underset{N}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma = \Gamma \circ \sigma_t^\Psi$ for all $t \in \mathbb{R}$.*

Proof. – By proposition 2.3.5 and thanks to the previous lemma, it is possible to define a normal *-automorphism $\sigma_t^\Psi{}_{\beta}\underset{N}{\star}{}_{\alpha}\tau_{-t}$ of $M_{\beta}\underset{N}{\star}{}_{\alpha}M_R$. By relation (9), the formula makes sense (τ is just defined on M_R). By relation (8), we can compute for all $m \in M$ and $t \in \mathbb{R}$:

$$\begin{aligned}(\sigma_t^\Psi{}_{\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma(m) &= (\Delta_\Psi^{it}{}_{\beta}\underset{\nu}{\otimes}{}_{\alpha}D^{it})\Gamma(m)(\Delta_\Psi^{-it}{}_{\beta}\underset{\nu}{\otimes}{}_{\alpha}D^{-it})\\ &= (\Delta_\Psi^{it}{}_{\beta}\underset{\nu}{\otimes}{}_{\alpha}D^{it})U'_{H_\Phi}(m_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}1)U'^*_{H_\Phi}(\Delta_\Psi^{-it}{}_{\beta}\underset{\nu}{\otimes}{}_{\alpha}D^{-it})\\ &= U'_{H_\Phi}(\Delta_\Psi^{it}{}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}D^{it})(m_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}1)(\Delta_\Psi^{-it}{}_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}D^{-it})U'^*_{H_\Phi}\\ &= U'_{H_\Phi}(\sigma_t^\Psi(m)_{\hat{\alpha}}\underset{\nu^o}{\otimes}{}_{\beta}1)U'^*_{H_\Phi} = \Gamma(\sigma_t^\Psi(m))\end{aligned}$$

□

We are now able to prove that we can re-construct M thanks to the fundamental unitary.

THEOREM 9.2.21. – *If $\langle F\rangle^{-\mathrm{w}}$ is the weakly closed linear span of F in M, then following vector spaces:*

$$M_R = \langle (\omega_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(m)) \mid m\in M, \omega\in M_*^+, k\in\mathbb{R}^+ \text{ s.t. } \omega\circ\beta\leq k\nu\rangle^{-\mathrm{w}}$$
$$A(U'_H) = \langle (\omega_{v,w} * \mathrm{id})(U'_H) \mid v\in D({}_{\hat{\alpha}}(H_\Psi),\mu), w\in D((H_\Psi)_\beta,\mu^o)\rangle^{-\mathrm{w}}$$
$$M_L = \langle (\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega)(\Gamma(m)) \mid m\in M, \omega\in M_*^+, k\in\mathbb{R}^+ \text{ s.t. } \omega\circ\alpha\leq k\nu\rangle^{-\mathrm{w}}$$
$$A(U_H) = \langle (\mathrm{id} * \omega_{v,w})(U_H) \mid v\in D((H_\Psi),\mu^o)_{\hat{\beta}}, w\in D({}_{\alpha}(H_\Psi),\mu)\rangle^{-\mathrm{w}}$$

are equal to the whole von Neumann algebra M.

Proof. – We have already noticed that $M_R = A(U'_H)$ and $M_L = A(U_H)$. Then, we get inspired by [**KV03**]. By 9.2.19, we have $\tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n))$ so:

$$M_L = \langle (\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega\circ\tau_t)(\Gamma(m)) \mid m\in M, \omega\in (M_R)_*^+, k\in\mathbb{R}^+ \text{ s.t. } \omega\circ\alpha\leq k\nu\rangle^{-\mathrm{w}}$$

By 9.2.20, we have $\sigma_t^\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega)\Gamma(m)) = (\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega\circ\tau_t)\Gamma(\sigma_t^\Psi(m))$ that's why $\sigma_t^\Psi(M_L) = M_L$ for all $t\in\mathbb{R}$. On the other hand, by 3.2.7, restriction of Ψ to M_L is semi-finite. By Takesaki's theorem [**Str81**, thm. 10.1], there exists a unique normal and faithful conditional expectation E from M to M_L such that $\Psi(m) = \Psi(E(m))$ for all $m\in M^+$. Moreover, if P is the orthogonal projection on the closure of $\Lambda_\Psi(\mathcal{N}_\Psi\cap M_L)$ then $E(m)P = PmP$.

So the range of P contains $\Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega)\Gamma(x))$ for all ω and $x\in\mathcal{N}_\Psi$. By right version of 3.5.13 implies that $P = 1$ so that E is the identity and $M = M_L$. If we apply the previous result to the opposite adapted measured quantum groupoid, then we get that $M = M_R$. □

COROLLARY 9.2.22. – *There exists a unique strongly continuous one-parameter group τ of automorphisms of M such that, for all $t\in\mathbb{R}$, $m\in M$ and $n\in N$:*

$$\tau_t(m) = D^{-it}mD^{it},\ \tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n)) \text{ and } \tau_t(\beta(n)) = \beta(\sigma_t^\nu(n))$$

Proof. – Straightforward from the previous theorem and 9.2.18. First property comes from 9.2.19 and the second one from 9.2.14. □

DEFINITION 9.2.23. – The one-parameter group τ is called **scaling group**.

Let us notice that it is possible to define normal *-automorphisms $\tau_{t\beta}\underset{N}{\star}{}_{\alpha}\tau_t$ and $\tau_{t\beta}\underset{N}{\star}{}_{\alpha}\sigma_t^\Phi$ of $M_{\beta}\underset{N}{\otimes}{}_{\alpha}M$ for all $t\in\mathbb{R}$, thanks to the previous commutation relations and recalls about tensor products.

PROPOSITION 9.2.24. – *We have $\Gamma\circ\tau_t = (\tau_{t\beta}\underset{N}{\star}{}_{\alpha}\tau_t)\circ\Gamma$ for all $t\in\mathbb{R}$.*

Proof. – By 9.2.20 and co-product relation, we have for all $t \in \mathbb{R}$:

$$\begin{aligned}(\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\Gamma)(\sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma &= (\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\Gamma)\Gamma \circ \sigma_t^{\Psi} \\ &= (\Gamma_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})\Gamma \circ \sigma_t^{\Psi} = (\Gamma \circ \sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t})\Gamma \\ &= (\sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t\,\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t})(\Gamma_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})\Gamma \\ &= (\sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}[(\tau_{-t\,\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma]) \circ \Gamma\end{aligned}$$

Consequently, for all $m \in M$, $\omega \in M_*^+$, $k \in \mathbb{R}^+$ such that $\omega \circ \beta \leq k\nu$, we have:

$$\begin{aligned}\Gamma \circ \tau_{-t} \circ ((\omega \circ \sigma_t^{\Psi})_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})\Gamma &= (\omega_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}(\Gamma \circ \tau_{-t})) \circ \Gamma \\ &= (\omega_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\sigma_t^{\Psi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}[(\tau_{-t\,\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma]) \\ &= [(\tau_{-t\,\beta}\underset{\nu}{\star}{}_{\alpha}\tau_{-t}) \circ \Gamma] \circ ((\omega \circ \sigma_t^{\Psi})_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})\Gamma\end{aligned}$$

The theorem 9.2.21 allows us to conclude. □

PROPOSITION 9.2.25. – *For all $x \in M \cap \alpha(N)'$, we have $\Gamma(x) = 1_{\beta}\underset{N}{\otimes}{}_{\alpha}x \Leftrightarrow x \in \beta(N)$. Also, for all $x \in M \cap \beta(N)'$, we have $\Gamma(x) = x_{\beta}\underset{N}{\otimes}{}_{\alpha}1 \Leftrightarrow x \in \alpha(N)$.*

Proof. – Let $x \in M \cap \alpha(N)'$ such that $\Gamma(x) = 1_{\beta}\underset{N}{\otimes}{}_{\alpha}x$. For all $n \in \mathbb{N}$, we define in the strong topology:

$$x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\sigma_t^{\Psi}(x)\, dt \quad \text{analytic w.r.t. } \sigma^{\Psi},$$

and:

$$y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\tau_{-t}(x)\, dt \quad \text{belongs to } \alpha(N)'.$$

By 9.2.20, we have $\Gamma(x_n) = 1_{\beta}\underset{N}{\otimes}{}_{\alpha}y_n$. If $d \in (\mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R})^+$, then, for all $n \in \mathbb{N}$, we have $dx_n \in \mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R}$. Let $\omega \in M_*^+$ and $k \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$. By right invariance, we get:

$$\begin{aligned}\omega \circ T_R(dx_n) &= \omega((\Psi_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(dx_n))) \\ &= \Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\omega)(\Gamma(dx_n))) = \Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}(y_n\omega))(\Gamma(d))) \\ &= \omega((\Psi_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\Gamma(d))y_n) = \omega(T_R(d)y_n)\end{aligned}$$

Take the limit over $n \in \mathbb{N}$ to obtain $T_R(dx) = T_R(d)x$ for all $d \in \mathcal{M}_{\Psi} \cap \mathcal{M}_{T_R}$ and, by semi-finiteness of T_R, we conclude that x belongs to $\beta(N)$. Reverse inclusion comes from axioms. If we apply this result to the opposite adapted measured quantum groupoid, then we get the second point. □

9.2.4. The antipode and its polar decomposition. – We now approach definition of the antipode.

Lemma 9.2.26. – *We have* $(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D^\lambda \subset D^\lambda(\omega_{\Delta_\Psi^{-\lambda}v,\Delta_\Psi^{\lambda}w} * \mathrm{id})(U'_{H_\Phi})$ *for all* $\lambda \in \mathbb{C}$ *and* $v, w \in \Lambda_\Psi(\mathcal{T}_{\Psi,T_R})$.

Proof. – Straightforward from relation (8). □

Proposition 9.2.27. – *If* I *is the unitary part of the polar decomposition of* G*, then, for all* $v, w \in D((H_\Psi)_\beta, \nu^o)$*, we have:*

$$I(\omega_{J_\Psi w,v} * \mathrm{id})(U'^{*}_{H_\Phi})I = (\omega_{J_\Psi v,w} * \mathrm{id})(U'_{H_\Phi})$$

Proof. – We have $(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D^{1/2} \subseteq D^{1/2}(\omega_{\Delta_\Psi^{-1/2}v,\Delta_\Psi^{1/2}w} \star \mathrm{id})(U'_{H_\Phi})$ for all $v, w \in \Lambda_\Psi(\mathcal{T}_{\Psi,T_R})$ by the previous lemma. On the other hand, by inclusion (5), we have:

$$(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})D^{1/2} = (\omega_{v,w} \star \mathrm{id})(U'_{H_\Phi})G^*I \subseteq D^{1/2}I(\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})I$$

So $I(\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})I = (\omega_{\Delta_\Psi^{-1/2}v,\Delta_\Psi^{1/2}w} * \mathrm{id})(U'_{H_\Phi})$ and, by 3.3.5, we have:

$$I(\omega_{w,v} * \mathrm{id})(U'^{*}_{H_\Phi})I = (\omega_{\Delta_\Psi^{1/2}w,\Delta_\Psi^{-1/2}v} * \mathrm{id})(U'^{*}_{H_\Phi}) = (\omega_{J_\Psi v,J_\Psi w} * \mathrm{id})(U'_{H_\Phi}) \quad \square$$

Corollary 9.2.28. – *There exists a *-anti-automorphism* R *of* M *defined by* $R(m) = Im^*I$ *such that* $R^2 = \mathrm{id}$*. (We recall that* I *denotes the unitary part of the polar decomposition of* G*).*

Proof. – Straightforward from the previous proposition and theorem 9.2.21. □

Definition 9.2.29. – The unique *-anti-automorphism R of M such that $R(m) = Im^*I$, where I denotes the unitary part of the polar decomposition of G, is called **unitary antipode**.

Definition 9.2.30. – The application $S = R\tau_{-i/2}$ is called **antipode**.

The next proposition states elementary properties of the antipode. Straightforward proofs are omitted.

Proposition 9.2.31. – *The antipode* S *satisfies:*

i) *for all* $t \in \mathbb{R}$*, we have* $\tau_t \circ R = R \circ \tau_t$ *and* $\tau_t \circ S = S \circ \tau_t$
ii) $SR = RS$ *and* $S^2 = \tau_{-i}$
iii) S *is densely defined and has dense range*
iv) S *is injective and* $S^{-1} = R\tau_{i/2} = \tau_{i/2}R$
v) *for all* $x \in \mathcal{D}(S)$*,* $S(x^*) \in \mathcal{D}(S)$ *and* $S(S(x)^*)^* = x$

9.2.5. Characterization of the antipode. – In 9.2.30, we define the antipode by giving its polar decomposition. However, we have to verify that S is what it should be.

9.2.5.1. Usual characterization of the antipode.

PROPOSITION 9.2.32. – *For all $v, w \in \Lambda_\Psi(\mathscr{T}_{\Psi,T_R})$, $(\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})$ belongs to $\mathscr{D}(S)$ and we have:*

$$S((\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})) = (\omega_{w,v} * \mathrm{id})(U'^*_{H_\Phi})$$

Moreover, the linear span of $(\omega_{v,w} * \mathrm{id})(U'_{H_\Phi})$, where $v, w \in \Lambda_\Psi(\mathscr{T}_{\Psi,T_R})$, is a core for S.

Proof. – By 9.2.26, we have $(\omega_{w,v} \star \mathrm{id})(U'_{H_\Phi}) \in \mathscr{D}(\tau_{-i/2}) = \mathscr{D}(S)$ and:

$$\begin{aligned} S((\omega_{w,v} * \mathrm{id})(U'_{H_\Phi})) &= R((\omega_{\Delta_\Psi^{-1/2} w, \Delta_\Psi^{1/2} v} * \mathrm{id})(U'_{H_\Phi})) \\ &= (\omega_{S_\Psi v, \Delta_\Psi S_\Psi w} * \mathrm{id})(U'_{H_\Phi}) && \text{by proposition 9.2.27,} \\ &= (\omega_{w,v} * \mathrm{id})(U'^*_{H_\Phi}) && \text{by lemma 3.3.5.} \end{aligned}$$

The involved subspace of M is included in $\mathscr{D}(\tau_{-i/2})$ by 9.2.26, weakly dense in M by theorem 9.2.21 and τ-invariant by 9.2.18 which finishes the proof. □

COROLLARY 9.2.33. – *For $a, b, c, d \in \mathscr{T}_{\Psi,T_R}$, $(\omega_{\Lambda_\Psi(a),\Lambda_\Psi(b)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(cd))$ belongs to $\mathscr{D}(S)$ and we have:*

$$S((\omega_{\Lambda_\Psi(a),\Lambda_\Psi(b)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(cd))) = (\omega_{\Lambda_\Psi(c),\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(\sigma^\Psi_i(a) b^*))$$

Proof. – By 3.3.4, we know that:

$$(\omega_{\Lambda_\Psi(a),\Lambda_\Psi(b)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(cd)) = (\omega_{\Lambda_\Psi(cd),\Lambda_\Psi(b\sigma^\Psi_{-i}(a^*))} * \mathrm{id})(U'_{H_\Phi})$$

which belongs to $\mathscr{D}(S)$. Then, by 3.3.4 and 3.3.5, we have:

$$\begin{aligned} S((\omega_{\Lambda_\Psi(a),\Lambda_\Psi(b)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(cd))) &= S((\omega_{\Lambda_\Psi(cd),\Lambda_\Psi(b\sigma^\Psi_{-i}(a^*))} * \mathrm{id})(W')) \\ &= (\omega_{\Lambda_\Psi(cd),\Lambda_\Psi(b\sigma^\Psi_{-i}(a^*))} * \mathrm{id})(W'^*) \\ &= (\omega_{\Lambda_\Psi(\sigma^\Psi_i(a)b^*),\Lambda_\Psi(\sigma^\Psi_{-i}(d^*c^*))} * \mathrm{id})(W') \\ &= (\omega_{\Lambda_\Psi(c),\Lambda_\Psi(\sigma^\Psi_{-i}(d^*))} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(\sigma^\Psi_i(a) b^*)) \end{aligned}$$ □

9.2.5.2. The co-involution R. – In this section, we give a new expression of R and we show that it is a co-involution of the adapted measured quantum groupoid.

PROPOSITION 9.2.34. – *For all $a, b \in \mathscr{N}_\Psi \cap \mathscr{N}_{T_R}$, we have:*

$$R((\omega_{J_\Psi \Lambda_\Psi(a)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(b^* b))) = (\omega_{J_\Psi \Lambda_\Psi(b)} \beta \underset{\nu}{\star} \alpha \mathrm{id})(\Gamma(a^* a))$$

Proof. – The proposition comes from the following computation:

$$\begin{aligned}
&R((\omega_{J_\Psi\Lambda_\Psi(a),J_\Psi\Lambda_\Psi(a)}\underset{\nu}{{}_\beta\star_\alpha}\mathrm{id})(\Gamma(b^*b)))\\
&= R((\omega_{\Lambda_\Psi(b^*b),J_\Psi\Lambda_\Psi(a^*a)} * \mathrm{id})(U'_{H_\Phi}) && \text{by corollary 3.3.4,}\\
&= (\omega_{\Lambda_\Psi(a^*a),J_\Psi\Lambda_\Psi(b^*b)} * \mathrm{id})(U'_{H_\Phi}) && \text{by definition of } R,\\
&= (\omega_{J_\Psi\Lambda_\Psi(b),J_\Psi\Lambda_\Psi(b)}\underset{\nu}{{}_\beta\star_\alpha}\mathrm{id})(\Gamma(a^*a)) && \text{by corollary 3.3.4.}
\end{aligned}$$

□

REMARK 9.2.35. – We notice that R is T_L-independent.

PROPOSITION 9.2.36. – *We have* $I\alpha(n^*)=\beta(n)I$ *for all* $n\in N$ *and* $R\circ\alpha=\beta$.

Proof. – By 9.2.14, we have, for all $x\in\mathscr{T}_{\Psi,T_R}$:

$$\beta(x)GD^{-1/2}\subseteq G\alpha(\sigma_{-i/2}((x^*))\subseteq GD^{-1/2}\alpha(x^*)\subseteq I\alpha(x^*)$$

and, on the other hand, $\beta(x)GD^{-1/2}\subseteq\beta(x)I$ so that $I\alpha(x^*)=\beta(x)I$. The result holds by normality of α and β. □

By **[Sau83b]**, there exists a unitary and anti-linear operator $I\underset{\nu}{{}_\beta\otimes_\alpha}I$ from $H\underset{\nu}{{}_\beta\otimes_\alpha}H$ onto $H\underset{\nu^o}{{}_\alpha\otimes_\beta}H$, the adjoint of which is $I\underset{\nu^o}{{}_\alpha\otimes_\beta}I$. Also, there exists an anti-isomorphism $R\underset{N}{{}_\beta\star_\alpha}R$ from $M\underset{N}{{}_\beta\star_\alpha}M$ onto $M\underset{N^o}{{}_\alpha\star_\beta}M$ and, by definition of R, we have, for all $X\in M\underset{N}{{}_\beta\star_\alpha}M$:

$$(R\underset{N}{{}_\beta\star_\alpha}R)(X)=(I\underset{\nu}{{}_\beta\otimes_\alpha}I)X^*(I\underset{\nu^o}{{}_\alpha\otimes_\beta}I)$$

We underline the fact that, if $\omega\in M_*^+$, then $\omega\circ R\in M_*^+$ and, if there exists $k\in\mathbb{R}^+$ such that $\omega\circ\alpha\leq k\nu$, then $\omega\circ R\circ\beta\leq k\nu$. Also, if $\theta\in M_*^+$ and $k'\in\mathbb{R}^+$ are such that $\theta\circ\beta\leq k'\nu$, then $\theta\circ R\circ\alpha\leq k\nu$. Then, we have $\omega R\underset{\nu}{{}_\beta\star_\alpha}\theta R=(\omega\underset{\nu^o}{{}_\alpha\star_\beta}\theta)\circ(R\underset{\nu}{{}_\beta\star_\alpha}R)$.

LEMMA 9.2.37. – *For all* $a,x\in\mathscr{N}_{T_R}\cap\mathscr{N}_\Psi$, $\omega\in M_*^+$ *and* $k\in\mathbb{R}^+$ *such that* $\omega\circ\alpha\leq k\nu$, *we have:*

$$\omega\circ R((\omega_{J_\Psi\Lambda_\Psi(a)}\underset{\nu}{{}_\beta\star_\alpha}\mathrm{id})(\Gamma(x)))=(\Lambda_\Psi((\mathrm{id}\underset{\nu}{{}_\beta\star_\alpha}\omega)(\Gamma(a^*a)))|J_\Psi\Lambda_\Psi(x))$$

Proof. – Let $b\in\mathscr{N}_{T_R}\cap\mathscr{N}_\Psi$. By 9.2.34, we can compute:

$$\begin{aligned}
\omega\circ R((\omega_{J_\Psi\Lambda_\Psi(a)}\underset{\nu}{{}_\beta\star_\alpha}\mathrm{id})(\Gamma(b^*b)))&=\omega((\omega_{J_\Psi\Lambda_\Psi(b)}\underset{\nu}{{}_\beta\star_\alpha}\mathrm{id})(\Gamma(a^*a)))\\
&=((\mathrm{id}\underset{\nu}{{}_\beta\star_\alpha}\omega)(\Gamma(a^*a))J_\Psi\Lambda_\Psi(b)|J_\Psi\Lambda_\Psi(b))\\
&=(J_\Psi bJ_\Psi\Lambda_\Psi((\mathrm{id}\underset{\nu}{{}_\beta\star_\alpha}\omega)(\Gamma(a^*a)))|J_\Psi\Lambda_\Psi(b))\\
&=(\Lambda_\Psi((\mathrm{id}\underset{\nu}{{}_\beta\star_\alpha}\omega)(\Gamma(a^*a)))|J_\Psi\Lambda_\Psi(b^*b))
\end{aligned}$$

Linearity and normality of the expressions imply the lemma. □

PROPOSITION 9.2.38. – *We have* $\varsigma_{N^o}\circ(R\underset{N}{{}_\beta\star_\alpha}R)\circ\Gamma=\Gamma\circ R$.

Proof. – Let $a, b \in \mathscr{N}_{T_R} \cap \mathscr{N}_\Psi$, $\omega, \theta \in M_*^+$ and $k, k' \in \mathbb{R}^+$ such that $\omega \circ \alpha \leq k\nu$ and $\theta \circ \beta \leq k'\nu$. Then, we can compute by 9.2.34 and the previous lemma:

$$
\begin{aligned}
&(\theta_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma \circ R((\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})(\Gamma(b^*b))))\\
&= (\theta_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma((\omega_{J_\Psi\Lambda_\Psi(b)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})(\Gamma(a^*a))))\\
&= (\omega_{J_\Psi\Lambda_\Psi(b)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\theta_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\mathrm{id}_{\beta}\underset{N}{\star}_{\alpha}\Gamma)(\Gamma(a^*a))\\
&= (\omega_{J_\Psi\Lambda_\Psi(b)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\theta_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma_{\beta}\underset{N}{\star}_{\alpha}\mathrm{id})(\Gamma(a^*a))\\
&= (\omega_{J_\Psi\Lambda_\Psi(b)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\theta)[\Gamma((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma(a^*a)))]\\
&= (\Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\Gamma(b^*b)))|J_\Psi\Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma(a^*a))))
\end{aligned}
$$

Observe the symmetry of the last expression and use it to proceed towards the computation:

$$
\begin{aligned}
&(\Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\omega)(\Gamma(a^*a)))|J_\Psi\Lambda_\Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\Gamma(b^*b))))\\
&= (\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\omega \circ R)[\Gamma((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\Gamma(b^*b)))]\\
&= (\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\omega \circ R_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\Gamma_{\beta}\underset{N}{\star}_{\alpha}\mathrm{id})(\Gamma(b^*b))\\
&= (\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\omega \circ R_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\mathrm{id}_{\beta}\underset{N}{\star}_{\alpha}\Gamma)(\Gamma(b^*b))\\
&= (\omega \circ R_{\beta}\underset{\nu}{\star}_{\alpha}\theta \circ R)(\Gamma((\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})(\Gamma(b^*b))))\\
&= (\omega_{\alpha}\underset{\nu^o}{\star}_{\beta}\theta)(R_{\beta}\underset{N}{\star}_{\alpha}R)(\Gamma((\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})(\Gamma(b^*b))))\\
&= (\theta_{\beta}\underset{\nu}{\star}_{\alpha}\omega)\varsigma_{N^o}(R_{\beta}\underset{N}{\star}_{\alpha}R)(\Gamma((\omega_{J_\Psi\Lambda_\Psi(a)}{}_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})(\Gamma(b^*b))))
\end{aligned}
$$

Theorem 9.2.21 easily implies the result. □

9.2.5.3. Left strong invariance w.r.t. the antipode.– In this section, T' denotes a left invariant n.s.f. weight from M to $\alpha(N)$. We put $\Phi' = \nu \circ \alpha^{-1} \circ T'$, $J_{\Phi'}$ the anti-linear operator and $\Delta_{\Phi'}$ the modular operator which come from Tomita's theory of Φ', $\sigma^{\Phi'}$ its modular group and $V = (U_{T'})^*_{H_\Phi}$ *i.e.*, the fundamental unitary associated with T'. The next proposition is the left strong invariance w.r.t. S.

PROPOSITION 9.2.39. – *Elements* $(\mathrm{id} * \omega_{v,w})(V)$ *belong to the domain of* S *for all* $v, w \in \Lambda_{\Phi'}(\mathscr{T}_{\Phi',T'})$ *and we have* $S((\mathrm{id} * \omega_{v,w})(V)) = (\mathrm{id} * \omega_{v,w})(V^*)$.

Proof. – By 3.3.4, we have $(\mathrm{id} * \omega)(V) = (\omega \circ R * \mathrm{id})(U'_{H_\Phi})$ for all ω. If $\overline{\omega}(x) = \overline{\omega(x^*)}$, then, by 9.2.32, we have:

$$
\begin{aligned}
S((\mathrm{id} * \omega)(V)) = S((\omega \circ R * \mathrm{id})(U'_{H_\Phi})) &= (\omega \circ R * \mathrm{id})(U'^*_{H_\Phi})\\
&= [(\overline{\omega} \circ R * \mathrm{id})(U'_{H_\Phi})]^*\\
&= [(\mathrm{id} * \overline{\omega})(V)]^* = (\mathrm{id} * \omega)(V^*)
\end{aligned}
$$
□

LEMMA 9.2.40. – *For all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:*

$$(\omega_{v,w} * \mathrm{id})(V)^* = (\omega_{ID^{-1/2}v, ID^{1/2}w} * \mathrm{id})(V)$$

Proof. – We have $(\mathrm{id} * \omega_{w',v'})(V) \in \mathcal{D}(S) = \mathcal{D}(\tau_{-i/2})$ for all v', w' belonging to $\Lambda_{\Phi'}(\mathcal{T}_{\Phi',T'})$ by 9.2.39 and, since τ is implemented by D^{-1}, we have:

$$\begin{aligned}(\mathrm{id} * \omega_{w',v'})(V)D^{1/2} &\subseteq D^{1/2}\tau_{-i/2}((\mathrm{id} * \omega_{w',v'})(V))\\ &= D^{1/2}R(S((\mathrm{id} * \omega_{w',v'})(V)))\\ &= D^{1/2}I[(\mathrm{id} * \omega_{w',v'})(V^*)]^*I\\ &= D^{1/2}I(\mathrm{id} * \omega_{v',w'})(V)I.\end{aligned}$$

Then, for all $v \in \mathcal{D}(D^{1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:

$$\begin{aligned}((\omega_{ID^{-1/2}v, ID^{1/2}w} * \mathrm{id})(V)w'|v') &= ((\mathrm{id} * \omega_{w',v'})(V)D^{1/2}Iv|D^{-1/2}Iw)\\ &= (D^{1/2}I(\mathrm{id} * \omega_{v',w'})(V)v|D^{-1/2}Iw)\\ &= (w|(\mathrm{id} * \omega_{v',w'})v)\\ &= ((\omega_{v,w} * \mathrm{id})(V)^*w', v')\end{aligned}$$

Then, the proposition holds. □

PROPOSITION 9.2.41. – *The following relations are satisfied:*

i) $(I_{\alpha}\otimes_{\epsilon}_{N^o} J_{\Phi'})V = V^*(I_{\beta}\otimes_{\alpha}_{N} J_{\Phi'})$;

ii) $(D^{-1}{}_{\alpha}\otimes_{\epsilon}{}_{\nu^o}\Delta_{\Phi'})V = V(D^{-1}{}_{\beta}\otimes_{\alpha}{}_{\nu}\Delta_{\Phi'})$;

iii) $(\tau_t {}_{\beta}\star_{\alpha}{}_{N} \sigma_t^{\Phi'}) \circ \Gamma = \Gamma \circ \sigma_t^{\Phi'}$ *for all* $t \in \mathbb{R}$.

where $\epsilon(n) = J_{\Phi'}\alpha(n^*)J_{\Phi'}$ *for all* $n \in N$.

Proof. – We denote by $S_{\Phi'}$ the operator of Tomita's theory associated with Φ' and defined as the closed operator on $H_{\Phi'}$ such that $\Lambda_{\Phi'}(\mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*)$ is a core for $S_{\Phi'}$ and $S_{\Phi'}\Lambda_{\Phi'}(x) = \Lambda_{\Phi'}(x^*)$ for all $x \in \mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*$. Then, by definition, we have $\Delta_{\Phi'} = S_{\Phi'}^*S_{\Phi'}$ and $S_{\Phi'} = J_{\Phi'}\Delta_{\Phi'}^{1/2}$. Moreover, for all $m \in M$ and $t \in \mathbb{R}$, we have $\sigma_t^{\Phi'}(m) = \Delta_{\Phi'}^{it}m\Delta_{\Phi'}^{-it}$.

First of all, we verify these relations make sense. We have to prove some commutation relations. We can write for all $n \in \mathcal{T}_\nu$ and $y \in \mathcal{N}_{\Phi'} \cap \mathcal{N}_{\Phi'}^*$:

$$S_{\Phi'}\alpha(n)\Lambda_{\Phi'}(y) = S_{\Phi'}\Lambda_{\Phi'}(\alpha(n)y)$$

$$= \Lambda_{\Phi'}(y^*\alpha(n^*)) = \hat{\alpha}(\sigma_{-i/2}^\nu(n))S_{\Phi'}\Lambda_{\Phi'}(y)$$

so $\hat{\alpha}(\sigma_{-i/2}^\nu(n))S_{\Phi'} \subseteq S_{\Phi'}\alpha(n)$ and by adjoint $\alpha(n)S_{\Phi'}^* \subseteq S_{\Phi'}^*\hat{\alpha}(\sigma_{i/2}^\nu(n)$. Then:

$$\alpha(n)\Delta_{\Phi'} = \alpha(n)S_{\Phi'}^*S_{\Phi'} \subseteq S_{\Phi'}^*\hat{\alpha}(\sigma_{i/2}^\nu(n)S_{\Phi'} \subseteq \Delta_{\Phi'}\alpha(\sigma_i^\nu(n))$$

Since $\beta(n)D^{-1} \subseteq D^{-1}\beta(\sigma_i^\nu(n))$, the second relation makes sense. On an other hand, we know that $I\beta(n) = \alpha(n^*)I$ and $\mathcal{J}\alpha(n) = \epsilon(n^*)J_{\Phi'}$ to terms of the first relation. Finally, for all $t \in \mathbb{R}$, we have:

$$\tau_t \circ \beta = \beta \circ \sigma_t^\nu \quad \text{ and } \sigma_t^{\Phi'}(\alpha(n)) = \Delta_{\Phi'}^{it}\alpha(n)\Delta_{\Phi'}^{-it} = \alpha(\sigma_t^\nu(n))$$

which finishes verifications.

Let $v, w \in \Lambda_\Phi(\mathcal{T}_{\Phi,S_L})$. By 3.2.2, we know that $(\omega_{v,w\,\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(y))$ belongs to $\mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*$ for all $y \in \mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*$. By 3.3.3, we can write $(\omega_{v,w} \star \mathrm{id})(V^*)\Lambda_{\Phi'}(y) = \Lambda_{\Phi'}((\omega_{v,w\,\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(y)))$ so that $(\omega_{v,w} * \mathrm{id})(V^*)\Lambda_{\Phi'}(y)$ belongs to $\mathcal{D}(S_{\Phi'})$. Then, we compute:

$$\begin{aligned}
S_{\Phi'}(\omega_{v,w} * \mathrm{id})(V^*)\Lambda_{\Phi'}(y) &= S_{\Phi'}\Lambda_{\Phi'}((\omega_{v,w\,\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(y))) \\
&= \Lambda_{\Phi'}((\omega_{w,v\,\beta}\underset{\nu}{\star}{}_\alpha\mathrm{id})(\Gamma(y^*))) \\
&= (\omega_{w,v} * \mathrm{id})(V^*)\Lambda_{\Phi'}(y^*) \\
&= (\omega_{w,v} * \mathrm{id})(V^*)S_{\Phi'}\Lambda_{\Phi'}(y)
\end{aligned}$$

Since $\Lambda_{\Phi'}(\mathcal{N}_{T'} \cap \mathcal{N}_{\Phi'} \cap \mathcal{N}_{T'}^* \cap \mathcal{N}_{\Phi'}^*)$ is a core for $S_{\Phi'}$, this implies:

$$(\omega_{w,v} * \mathrm{id})(V^*)S_{\Phi'} \subseteq S_{\Phi'}(\omega_{v,w} * \mathrm{id})(V^*) \tag{10}$$

Take adjoint so as to get:

$$(\omega_{w,v} * \mathrm{id})(V)S_{\Phi'}^* \subseteq S_{\Phi'}^*(\omega_{v,w} * \mathrm{id})(V) \tag{11}$$

Then, we deduce by the previous lemma:

$$\begin{aligned}
(\omega_{v,w} * \mathrm{id})(V)\Delta_{\Phi'} &= (\omega_{v,w} * \mathrm{id})(V)S_{\Phi'}^*S_{\Phi'} \\
&\subseteq S_{\Phi'}^*(\omega_{v,w} * \mathrm{id})(V)S_{\Phi'} \\
&= S_{\Phi'}^*[(\omega_{ID^{-1/2}w,ID^{1/2}v} * \mathrm{id})(V)]^*S_{\Phi'}
\end{aligned}$$

Then by inclusion (10) and the previous lemma, we have:

$$\begin{aligned}
(\omega_{v,w} * \mathrm{id})(V)\Delta_{\Phi'} &\subseteq S_{\Phi'}^*S_{\Phi'}[(\omega_{ID^{1/2}v,ID^{-1/2}w} * \mathrm{id})(V)]^* \\
&= \Delta_{\Phi'}(\omega_{D^{1/2}IID^{1/2}v,D^{-1/2}IID^{-1/2}w} * \mathrm{id})(V) \\
&= \Delta_{\Phi'}(\omega_{Dv,N^{-1}w} * \mathrm{id})(V)
\end{aligned}$$

Consequently, like relation (8), we easily deduce that:

$$(D^{-1}{}_\alpha\underset{\nu^o}{\otimes}{}_\epsilon\Delta_{\Phi'})V = V(D^{-1}{}_\beta\underset{\nu}{\otimes}{}_\alpha\Delta_{\Phi'})$$

Let's prove the first relation. By inclusion (10), for all $v \in \mathcal{D}(N^{-1/2})$ and $w \in \mathcal{D}(D^{1/2})$, we have:

$$\begin{aligned} J_{\Phi'}(\omega_{w,v} * \mathrm{id})(V^*)J_{\Phi'}\Delta_{\Phi'}^{1/2} &= J_{\Phi'}(\omega_{w,v} * \mathrm{id})(V^*)S_{\Phi'} \\ &\subseteq J_{\Phi'}S_{\Phi'}(\omega_{v,w} * \mathrm{id})(V^*) \\ &= \Delta_{\Phi'}^{1/2}(\omega_{v,w} * \mathrm{id})(V^*) \end{aligned} \tag{12}$$

For all $p, q \in \mathcal{D}(\Delta_{\Phi'}^{1/2})$, we have by ii):

$$\begin{aligned} ((\omega_{v,w} * \mathrm{id})(V^*)p, \Delta_{\Phi'}^{1/2}q) &= (V^*(v\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, p)|w\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\,\Delta_{\Phi'}^{1/2}q) \\ &= (V^*(v\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, p)|D^{-1/2}(D^{1/2}w)\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\,\Delta_{\Phi'}^{1/2}q) \\ &= ((D^{-1/2}\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\,\Delta_{\Phi'}^{1/2})V^*(v\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, p)|D^{1/2}w\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, q) \\ &= (V^*(D^{-1/2}v\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\,\Delta_{\Phi'}^{1/2}p)|D^{1/2}w\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, q) \\ &= ((\omega_{D^{-1/2}v,D^{1/2}w} * \mathrm{id})(V^*)\Delta_{\Phi'}^{1/2}p|q). \end{aligned}$$

Since $\Delta_{\Phi'}^{1/2}$ is self-adjoint, we get:

$$(\omega_{D^{-1/2}v,D^{1/2}w} * \mathrm{id})(V^*)\Delta_{\Phi'}^{1/2} \subseteq \Delta_{\Phi'}^{1/2}(\omega_{v,w} * \mathrm{id})(V^*)$$

Also, by the previous lemma, we have:

$$\begin{aligned} (\omega_{D^{-1/2}v,D^{1/2}w} * \mathrm{id})(V^*) &= (\omega_{D^{1/2}w,D^{-1/2}v} * \mathrm{id})(V)^* \\ &= (\omega_{Iw,Iv} * \mathrm{id})(V) \end{aligned}$$

That's why $(\omega_{Iw,Iv} * \mathrm{id})(V)\Delta_{\Phi'}^{1/2} \subseteq \Delta_{\Phi'}^{1/2}(\omega_{v,w} * \mathrm{id})(V^*)$. Since $\Delta_{\Phi'}^{1/2}$ has dense range, this last inclusion and (12) imply that:

$$(\omega_{Iw,Iv} * \mathrm{id})(V) = J_{\Phi'}(\omega_{v,w} * \mathrm{id})(V^*)J_{\Phi'}$$

Then, we can compute:

$$\begin{aligned} &((I\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, J_{\Phi'})V^*(I\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, J_{\Phi'})(v\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, q)|w\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, q) \\ &= (V(Iw\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, J_{\Phi'}q)|Iv\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, J_{\Phi'}p) \\ &= ((\omega_{Iw,Iv} \star \mathrm{id})(V)J_{\Phi'}q|J_{\Phi'}p) = (J_{\Phi'}(\omega_{w,v} \star \mathrm{id})(V^*)q|J_{\Phi'}p) \\ &= ((\omega_{v,w} * \mathrm{id})(V)p|q) = (V(v\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\, q)|w\,{}_\alpha\!\underset{\nu^o}{\otimes}{}_\epsilon\, q) \end{aligned}$$

so that the first relation is proved. We end the proof by the last equality. We know that Γ is implemented by V, $\sigma^{\Phi'}$ by $\Delta_{\Phi'}$ and τ by D so that the relation comes from $(D^{-1}\,{}_\alpha\!\underset{\nu}{\otimes}{}_\epsilon\,\Delta_{\Phi'})V = V(D^{-1}\,{}_\beta\!\underset{\nu}{\otimes}{}_\alpha\,\Delta_{\Phi'})$ like 9.2.20. □

If we take $T' = T_L$ then $V = W^*$, $J_{\Phi'} = J_\Phi$ and $\Delta_{\Phi'} = \Delta_\Phi$ so that we have the following propositions:

PROPOSITION 9.2.42. – *For all* $v, w \in \Lambda_\Phi(\mathcal{T}_{\Phi,S_L})$, $(\mathrm{id} * \omega_{v,w})(W)$ *belongs to* $\mathcal{D}(S)$ *and:*

$$S((\mathrm{id} * \omega_{v,w})(W)) = (\mathrm{id} * \omega_{v,w})(W^*)$$

PROPOSITION 9.2.43. – *We have* $(\omega_{v,w} * \mathrm{id})(W^*)^* = (\omega_{ID^{-1/2}v, ID^{1/2}w} * \mathrm{id})(W^*)$ *for all* $v \in \mathcal{D}(D^{1/2})$ *and* $w \in \mathcal{D}(D^{1/2})$.

PROPOSITION 9.2.44. – *The following relations are satisfied:*
i) $(I_\alpha \underset{N^o}{\otimes}_{\hat\beta} J_\Phi) W^* = W(I_\beta \underset{N}{\otimes}_\alpha J_\Phi)$;
ii) $(D^{-1}{}_\beta \underset{\nu}{\otimes}_\alpha \Delta_\Phi) W^* = W^*(D^{-1}{}_\beta \underset{\nu}{\otimes}_\alpha \Delta_\Phi)$;
iii) $(\tau_t {}_\beta \underset{N}{\star}_\alpha \sigma_t^\Phi) \circ \Gamma = \Gamma \circ \sigma_t^\Phi$ *for all* $t \in \mathbb{R}$.

We summarize the results of this section in the three following theorems:

THEOREM 9.2.45. – *Let* $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ *be an adapted measured quantum groupoid and* W *the pseudo-multiplicative unitary associated with. Then the closed linear span of* $(\mathrm{id} * \omega_{v,w})(W)$ *for all* $v \in D({}_\alpha H_\Phi, \nu)$ *and* $w \in D((H_\Phi)_{\hat\beta}, \nu^o)$ *is equal to the whole von Neumann algebra* M.

THEOREM 9.2.46. – *Let* $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ *be an adapted measured quantum groupoid and* W *the pseudo-multiplicative associated with. If we put* $\Phi = \nu \circ \alpha^{-1} \circ T_L$, *then there exists an unbounded antipode* S *which satisfies:*
i) *for all* $x \in \mathcal{D}(S)$, $S(x)^* \in \mathcal{D}(S)$ *and* $S(S(x)^*)^* = x$
ii) *for all* $v, w \in \Lambda_\Phi(\mathcal{T}_{\Phi,S_L})$, $(\mathrm{id} * \omega_{v,w})(W)$ *belongs to* $\mathcal{D}(S)$ *and:*

$$S((\mathrm{id} * \omega_{v,w})(W)) = (\mathrm{id} * \omega_{v,w})(W^*)$$

S has the following polar decomposition $S = R\tau_{i/2}$, *where* R *is a co-involution of* M *satisfying* $R^2 = \mathrm{id}$, $R \circ \alpha = \beta$ *and* $\varsigma_{N^o} \circ (R_\beta \underset{N}{\star}_\alpha R) \circ \Gamma = \Gamma \circ R$, *and where* τ, *the so-called scaling group, is a one-parameter group of automorphisms such that* $\tau_t \circ \alpha = \alpha \circ \sigma_t^\nu$, $\tau_t \circ \beta = \beta \circ \sigma_t^\nu$ *satisfying* $\Gamma \circ \tau_t = (\tau_t {}_\beta \underset{N}{\star}_\alpha \tau_t) \circ \Gamma$ *for all* $t \in \mathbb{R}$. S, R *and* τ *are independent of* T_L *and of* T_R.

Moreover, $R \circ T_L \circ R$ *is a n.s.f. operator-valued weight which is right invariant and* α*-adapted w.r.t.* ν.

THEOREM 9.2.47. – *Let* $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ *be an adapted measured quantum groupoid. If* R *is the co-involution and* τ *the scaling group, then* $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ *becomes a measured quantum groupoid.*

Proof. – By hypothesis, we know that $\gamma_t = \sigma_{-t}^\nu$ for all $t \in \mathbb{R}$ so that γ leaves ν invariant. By theorem 9.2.46 and proposition 9.2.34, we can construct a co-involution R and a scaling group τ such that $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ becomes a measured quantum groupoid. □

9.3. Uniqueness, modulus and scaling operator

By the general theory of measured quantum groupoids, theorems 5.4.3 and 5.4.4 can be applied and we get the following two theorems in the adapted measured quantum groupoids case:

THEOREM 9.3.1. – *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be an adapted measured quantum groupoid. If T' is a left invariant operator-valued weight which is β-adapted w.r.t. ν, then there exists a strictly positive operator h affiliated with $Z(N)$ such that, for all $t \in \mathbb{R}$:*

$$\nu \circ \alpha^{-1} \circ T' = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)}$$

We have a similar result for the right invariant operator-valued weights.

THEOREM 9.3.2. – *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, R \circ T_L \circ R)$ be a adapted measured quantum groupoid. Then there exists a strictly positive operator δ affiliated with M called modulus and then there exists a strictly positive operator λ affiliated with $Z(M) \cap \alpha(N) \cap \beta(N)$ called scaling operator such that $[D\nu \circ \alpha^{-1} \circ T_L \circ R : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{\frac{it^2}{2}} \delta^{it}$ for all $t \in \mathbb{R}$.*

Moreover, we have, for all $s, t \in \mathbb{R}$:

i)
$$\begin{aligned} &[D\nu \circ \alpha^{-1} \circ T_L \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{-ist} \\ &[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \tau_s : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist} \\ &[D\nu \circ \alpha^{-1} \circ T_L \circ \sigma_s^{\nu\circ\alpha^{-1}\circ T_L \circ R} : D\nu \circ \alpha^{-1} \circ T_L]_t = \lambda^{ist} \\ &[D\nu \circ \alpha^{-1} \circ T_L \circ R \circ \sigma_s^{\nu\circ\alpha^{-1}\circ T_L} : D\nu \circ \alpha^{-1} \circ T_L \circ R]_t = \lambda^{-ist} \end{aligned}$$

ii) *$R(\lambda) = \lambda$, $R(\delta) = \delta^{-1}$, $\tau_t(\delta) = \delta$ and $\tau_t(\lambda) = \lambda$;*

iii) *δ is a group-like element* i.e., *$\Gamma(\delta) = \delta_{\beta}\underset{N}{\otimes}_{\alpha}\delta$.*

Nevertheless, in the setting of adapted measured quantum groupoids, we can improve the previous results. We want to precise where δ sits and the dependence of fundamental elements with respect to the quasi-invariant weight.

PROPOSITION 9.3.3. – *The scaling operator does not depend on the quasi-invariant weight but just on the modular group associated with. If $\dot{\delta}$ is the class of δ for the equivalent relation $\delta_1 \sim \delta_2$ if, and only if there exists a strictly positive operator h affiliated to $Z(N)$ such that $\delta_2^{it} = \beta(h^{it})\delta_1^{it}\alpha(h^{-it})$, then $\dot{\delta}$ does not depend on the quasi-invariant weight but just on the modular group associated with.*

Proof. – If ν' is a n.s.f. weight on N such that $\sigma^{\nu'} = \sigma^{\nu}$, then there exists a strictly positive h affiliated to $Z(N)$ such that $\nu' = \nu_h$. We just have to compute:

$$\begin{aligned}
&[D\nu' \circ \alpha^{-1} \circ T_L \circ R : D\nu' \circ \alpha^{-1} \circ T_L]_t \\
&= [D\nu_h \circ \alpha^{-1} \circ T_L \circ R : D\Phi \circ R]_t [D\Phi \circ R : D\Phi]_t [D\Phi : D\nu_h \circ \alpha^{-1} \circ T_L]_t \\
&= \beta([D\nu_h : D\nu]^*_{-t}) \lambda^{\frac{1}{2}it^2} \delta^{it} \alpha([D\nu : D\nu_h]_t) = \lambda^{\frac{1}{2}it^2} \beta(h^{it}) \delta^{it} \alpha(h^{-it})
\end{aligned}$$ □

PROPOSITION 9.3.4. – *The modulus δ is affiliated with $M \cap \alpha(N)' \cap \beta(N)'$.*

Proof. – Since $\Phi = \nu \circ \beta^{-1} \circ S_L$, with the notation of section 9.2, we have:

$$\lambda^{\frac{it^2}{2}} \delta^{it} = [D\Phi \circ R : D\Phi]_t = [DR \circ T_L \circ R : DS_L]_t$$

which belongs to $M \cap \beta(N)'$. Since λ is affiliated with $Z(M)$, we get that δ is affiliated with $M \cap \beta(N)'$. Finally, since $R(\delta) = \delta$, we obtain that δ is affiliated with $M \cap \alpha(N)' \cap \beta(N)'$. □

Let ν' be a n.s.f. weight on N such that there exist strictly positive operator h and k affiliated with N strongly commuting and $[D\nu' : D\nu]_t = k^{\frac{it^2}{2}} h^{it}$ for all $t \in \mathbb{R}$. By [**Vae01a**, prop. 5.1], it is equivalent to $\sigma^{\nu}_t(h^{is}) = k^{ist} h^{is}$ for all $s, t \in \mathbb{R}$ and $\nu' = \nu_h$ in the sense of [**Vae01a**]. This hypothesis is satisfied, in particular, if σ^{ν} and $\sigma^{\nu'}$ commute each other. In this cas, k is affiliated with $Z(N)$.

PROPOSITION 9.3.5. – *There exists a n.s.f. operator-valued weight T'_L from M to $\alpha(N)$ which is β-adapted w.r.t. ν' such that, for all $t \in \mathbb{R}$, we have:*

$$[DT'_L : DT_L]_t = \beta(k^{\frac{-it^2}{2}} h^{it})$$

Proof. – By 9.1.4, there exists a n.s.f. operator-valued weight S_L from M to $\beta(N)$ such that $\nu \circ \alpha^{-1} \circ T_L = \nu \circ \beta^{-1} \circ S_L$ so that S_L is α-adapted w.r.t. ν. Then, again by 9.1.4, there exists a n.s.f. operator-valued weight T'_L from M to $\alpha(N)$ such that $\nu' \circ \beta^{-1} \circ S = \nu \circ \alpha^{-1} \circ T'_L$ so that T'_L is β-adapted w.r.t. ν'. Then, we compute the Radon-Nikodym cocycle for all $t \in \mathbb{R}$:

$$\begin{aligned}
[DT'_L : DT_L]_t &= [D\nu \circ \alpha^{-1} \circ T'_L : D\nu \circ \alpha^{-1} \circ T_L]_t \\
&= [D\nu' \circ \beta^{-1} \circ S : D\nu \circ \beta^{-1} \circ S]_t \\
&= \beta([D\nu' : D\nu]^*_{-t}) = \beta(k^{\frac{-it^2}{2}} h^{it})
\end{aligned}$$ □

COROLLARY 9.3.6. – *We have:*

$$\nu \circ \alpha^{-1} \circ T'_L = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)} \quad \textit{and} \quad \nu' \circ \alpha^{-1} \circ T'_L = (\nu \circ \alpha^{-1} \circ T_L)_{\alpha(h)\beta(h)}$$

Proof. – Come from [**Vae01a**, prop. 5.1] and the following equality, for all $t \in \mathbb{R}$, $[D\nu' \circ \alpha^{-1} \circ T'_L : D\nu \circ \alpha^{-1} \circ T_L]_t = \alpha(k^{\frac{it^2}{2}}) \beta(k^{\frac{-it^2}{2}}) \alpha(h^{it}) \beta(h^{it})$. □

PROPOSITION 9.3.7. – *T'_L is left invariant.*

Proof. – Let $a \in \mathcal{M}^+_{T'_L}$. By left invariance of T_L, we have:

$$\begin{aligned}(\mathrm{id}_\beta \underset{\nu'}{\star}{}_\alpha \nu' \circ \alpha^{-1} \circ T'_L)(\Gamma(a)) &= (\mathrm{id}_\beta \underset{\nu}{\star}{}_\alpha \nu \circ \alpha^{-1} \circ T'_L)(\Gamma(a)) \\ &= (\mathrm{id}_\beta \underset{\nu}{\star}{}_\alpha (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)})(\Gamma(a)) \\ &= (\mathrm{id}_\beta \underset{\nu}{\star}{}_\alpha \nu \circ \alpha^{-1} \circ T_L)(\Gamma(\beta(h^{1/2})a\beta(h^{1/2}))) \\ &= T_L(\beta(h^{1/2})a\beta(h^{1/2})) = T'(a)\end{aligned}$$

□

We state the right version of these results:

PROPOSITION 9.3.8. – *There exists a n.s.f. right invariant operator-valued weight T'_R which is α-adapted w.r.t. ν' such that, for all $t \in \mathbb{R}$, we have:*

$$[DT'_R : DT_R]_t = \alpha(k^{\frac{it^2}{2}} h^{it})$$

Moreover, we have:

$$\nu \circ \beta^{-1} \circ T'_R = (\nu \circ \beta^{-1} \circ T_R)_{\alpha(h)} \quad \text{and} \quad \nu' \circ \beta^{-1} \circ T'_R = (\nu \circ \beta^{-1} \circ T_R)_{\alpha(h)\beta(h)}$$

LEMMA 9.3.9. – *The application $I^{\nu'}_\nu$ defined by the following formula:*

$$I^{\nu'}_\nu(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta) = \beta(k^{-i/8})\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \alpha(h^{1/2})\eta$$

for all $\xi \in H$ and $\eta \in D({}_\alpha H, \nu) \cap \mathcal{D}(\alpha(h^{1/2}))$, is an isomorphism of $\beta(N)' - \alpha(N)'^o$-bimodules from $H {}_\beta\underset{\nu}{\otimes}{}_\alpha H$ onto $H {}_\beta\underset{\nu'}{\otimes}{}_\alpha H$.

Proof. – For all $x \in \mathcal{N}_{\nu'}$, we have:

$$\alpha(x)\alpha(h^{1/2})\eta = \alpha(xh^{1/2})\eta = R^{\alpha,\nu}(\eta)\Lambda_\nu(xh^{1/2}) = R^{\alpha,\nu}(\eta)\Lambda_{\nu'}(x)$$

so that $\alpha(h^{1/2})\eta \in D({}_\alpha H, \nu)$ and $R^{\alpha,\nu'}(\alpha(h^{1/2})\eta) = R^{\alpha,\nu}(\eta)$. Also, we recall that $J_{\nu'} = J_\nu k^{-i/8} J_\nu k^{i/8} J_\nu$ by [**Vae01a**, prop. 2.5]. Then, we have:

$$\begin{aligned}&(\beta(k^{-i/8})\xi_1 {}_\beta\underset{\nu}{\otimes}{}_\alpha \alpha(h^{1/2})\eta_1 | \beta(k^{-i/8})\xi_2 {}_\beta\underset{\nu}{\otimes}{}_\alpha \alpha(h^{1/2})\eta_2) \\ &= (\beta(J_{\nu'}\langle \alpha(h^{1/2})\eta_1, \alpha(h^{1/2})\eta_2\rangle^*_{\alpha,\nu'} J_{\nu'})\beta(k^{-i/8})\xi_1 | \beta(k^{-i/8})\xi_2) \\ &= (\beta(k^{-i/8} J_\nu k^{-i/8} J_\nu k^{i/8} J_\nu \langle \eta_1, \eta_2\rangle^*_{\alpha,\nu} J_\nu k^{-i/8} J_\nu k^{i/8} J_\nu k^{i/8})\xi_1 | \xi_2) \\ &= (\beta(J_\nu \langle \eta_1, \eta_2\rangle^*_{\alpha,\nu} J_\nu)\xi_1 | \xi_2) = (\xi_1 {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta_1 | \xi_2 {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta_2)\end{aligned}$$

□

REMARK 9.3.10. – For all $\xi \in D(H_\beta, \nu^o)$ and $\eta \in D({}_\alpha H, \nu)$, we have:

$$\begin{aligned}I^{\nu'}_\nu(\xi {}_\beta\underset{\nu}{\otimes}{}_\alpha \eta) &= \beta(k^{-i/8})\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \alpha(h^{1/2})\eta = \xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \alpha(k^{-i/8}h^{1/2})\eta \\ &= \beta(\sigma^\nu_{i/2}(h^{1/2}))\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \alpha(k^{-i/8})\eta = \beta(\sigma^\nu_{i/2}(k^{-i/8}h^{1/2}))\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \eta \\ &= \beta(k^{i/8})\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \alpha(\sigma^\nu_{-i/2}(h^{1/2}))\eta = \beta(k^{i/8}h^{1/2})\xi {}_\beta\underset{\nu'}{\otimes}{}_\alpha \eta\end{aligned}$$

PROPOSITION 9.3.11. – *Let $(N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)$ be an adapted measured quantum groupoid. There exists an adapted measured quantum groupoid $(N,M,\alpha,\beta,\Gamma,\nu',T'_L,T'_R)$ fundamental objects of which, R', τ', λ', δ' and P', are expressed, for all $t\in\mathbb{R}$, in the following way:*

i) $R'=R$, $\lambda'=\lambda$ *and* $\delta'=\delta$

ii) $\tau'_t=Ad_{\alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})}\circ\tau_t=Ad_{\alpha([D\nu':D\nu]^*_t)\beta([D\nu':D\nu]_t)}\circ\tau_t$

iii) $P'^{it}=\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi P^{it}$

Proof. – The existence of $(N,M,\alpha,\beta,\Gamma,\nu',T'_L,T'_R)$ has been already proved. We put $\Phi'=\nu'\circ\alpha^{-1}\circ T'_L$ and $\Psi'=\nu'\circ\beta^{-1}\circ T'_R$. Let $x,y\in\mathcal{N}_{T'_R}\cap\mathcal{N}_{\Psi'}$. By [**Vae01a**, prop. 2.5], we have:

$$J_{\Psi'}\Lambda_{\Psi'}(x)=J_\Psi\alpha(k^{-i/8})\beta(k^{i/8})J_\Psi\alpha(k^{i/8})\beta(k^{-i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2})\beta(h^{1/2}))$$
$$\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)}=\omega_{\alpha(k^{i/8})\beta(k^{-i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2})\beta(h^{1/2}))}$$

Then, we easily verify

$$\lambda^{\beta,\alpha,\nu'}_{\alpha(k^{i/8})\beta(k^{-i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2})\beta(h^{1/2}))}=I^{\nu'}_\nu\lambda^{\beta,\alpha,\nu}_{\alpha(k^{i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2}))}$$

We compute:

$$\begin{aligned}&(\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)}\,{}_\beta\star_{\nu'}{}_\alpha\mathrm{id})(\Gamma(y^*y))\\
&=(\omega_{\alpha(k^{i/8})\beta(k^{-i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2})\beta(h^{1/2}))}\,{}_\beta\star_{\nu'}{}_\alpha\mathrm{id})(\Gamma(y^*y))\\
&=(\omega_{\alpha(k^{i/8})J_\Psi\Lambda_\Psi(x\alpha(h^{1/2}))}\,{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(y^*y))\\
&=(\omega_{J_\Psi\Lambda_\Psi(x\alpha(k^{-i/8}h^{1/2}))}\,{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(y^*y))\end{aligned}$$

Apply R to get:

$$\begin{aligned}&R[(\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)}\,{}_\beta\star_{\nu'}{}_\alpha\mathrm{id})(\Gamma(y^*y))]\\
&=(\omega_{J_\Psi\Lambda_\Psi(y)}\,{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(\alpha(k^{i/8}h^{1/2})x^*x\alpha(k^{-i/8}h^{1/2})))\\
&=(\omega_{\alpha(k^{-i/8}h^{1/2})J_\Psi\Lambda_\Psi(y)}\,{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(x^*x))\\
&=(\omega_{\alpha(k^{i/8})J_\Psi\Lambda_\Psi(y\alpha(h^{1/2}))}\,{}_\beta\star_{\nu}{}_\alpha\mathrm{id})(\Gamma(x^*x))\\
&=(\omega_{J_{\Psi'}\Lambda_{\Psi'}(y)}\,{}_\beta\star_{\nu'}{}_\alpha\mathrm{id})(\Gamma(x^*x))=R'[(\omega_{J_{\Psi'}\Lambda_{\Psi'}(x)}\,{}_\beta\star_{\nu'}{}_\alpha\mathrm{id})(\Gamma(y^*y))]\end{aligned}$$

so that $R = R'$. For all $a \in M$, $\xi \in D(H_\beta, \nu'^o)$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned}
&\tau_t((\omega_{\xi}\, {}_\beta\!\underset{\nu'}{\star}\!{}_\alpha \mathrm{id})(\Gamma(a)))\\
&= \tau_t(\alpha(k^{-i/8}h^{-1/2})(\omega_{\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(a))\alpha(k^{i/8}h^{-1/2}))\\
&= \alpha(\sigma_t^\nu(k^{-i/8}h^{-1/2}))\tau_t((\omega_{\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(a)))\alpha(\sigma_t^\nu(k^{i/8}h^{-1/2}))\\
&= \alpha(k^{-t/2-i/8}h^{-1/2})(\omega_{\Delta_\Psi^{-it}\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(\sigma_{-t}^\Psi(a)))\alpha(k^{-t/2+i/8}h^{-1/2})
\end{aligned}$$

By [**Vae01a**, prop. 2.4 and coro. 2.6], we know that:

$$\begin{aligned}
&(\omega_{\Delta_\Psi^{-it}\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(\sigma_{-t}^\Psi(a)))\\
&= (\omega_{\alpha(k^{\frac{-it^2}{2}}h^{it})\beta(k^{\frac{it^2}{2}}h^{it})\Delta_{\Psi'}^{-it}\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(Ad_{\alpha(k^{\frac{it^2}{2}}h^{-it})\beta(k^{\frac{-it^2}{2}}h^{-it})} \circ \sigma_{-t}^{\Psi'}(a)))
\end{aligned}$$

so that:

$$\begin{aligned}
&\tau_t((\omega_{\xi}\, {}_\beta\!\underset{\nu'}{\star}\!{}_\alpha \mathrm{id})(\Gamma(a)))\\
&= Ad_{\alpha(k^{-t/2+i/8}h^{-1/2})\beta(k^{\frac{it^2}{2}}h^{it})} \circ (\omega_{\beta(k^{\frac{it^2}{2}}h^{it})\Delta_{\Psi'}^{-it}\xi}\, {}_\beta\!\underset{\nu}{\star}\!{}_\alpha \mathrm{id})(\Gamma(\sigma_{-t}^{\Psi'}(a)))\\
&= \alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})(\omega_{\Delta_{\Psi'}^{-it}\xi}\, {}_\beta\!\underset{\nu'}{\star}\!{}_\alpha \mathrm{id})(\Gamma(\sigma_{-t}^{\Psi'}(a)))\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})\\
&= \alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})\tau'_t((\omega_{\xi}\, {}_\beta\!\underset{\nu'}{\star}\!{}_\alpha \mathrm{id})(\Gamma(a)))\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})
\end{aligned}$$

Consequently, we have:

$$\tau'_t(z) = \alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})\tau_t(z)\alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})$$

for all $z \in M$ and $t \in \mathbb{R}$. Now, we compute the Radon-Nikodym cocycle:

$$\begin{aligned}
&[D\nu' \circ \alpha^{-1} \circ T' \circ R : D\nu' \circ \alpha^{-1} \circ T']_t\\
&= [D\nu'\alpha^{-1}T'R : D\nu\alpha^{-1}TR]_t[D\nu\alpha^{-1}TR : D\nu\alpha^{-1}T]_t[D\nu\alpha^{-1}T : D\nu'\alpha^{-1}T']_t\\
&= \alpha([D\nu' : D\nu]_t)\beta([D\nu' : D\nu]_{-t}^*)\lambda^{\frac{it^2}{2}}\delta^{it}\alpha([D\nu : D\nu']_t)\beta([D\nu : D\nu']_{-t}^*)
\end{aligned}$$

which is equal to $\lambda^{\frac{it^2}{2}}\delta^{it}$. Finally, we express the manageable operator P' in terms of P. We have, for all $x \in \mathcal{N}_{T'_L} \cap \mathcal{N}_{\Phi'}$ and $t \in \mathbb{R}$:

$$\begin{aligned}
&P'^{it}\Lambda_{\Phi'}(x) = \lambda'^{t/2}\Lambda_{\Phi'}(\tau'_t(x))\\
&= \lambda^{t/2}\Lambda_\Phi(\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})\tau_t(x)\alpha(k^{\frac{-it^2}{2}}h^{-it})\beta(k^{\frac{it^2}{2}}h^{it})\alpha(h^{1/2})\beta(h^{1/2}))
\end{aligned}$$

which is equal to the value of:

$$\lambda^{t/2}\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})\alpha(k^{t/2})\beta(k^{t/2})J_\Phi$$

on $\Lambda_\Phi(\tau_t(x)\alpha(h^{1/2})\beta(h^{1/2}))$ and the value of:

$$\lambda^{t/2}\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi$$

on $\Lambda_\Phi(\tau_t(x\alpha(h^{1/2})\beta(h^{1/2})))$ which is:

$$\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi\alpha(k^{\frac{it^2}{2}}h^{it})\beta(k^{\frac{-it^2}{2}}h^{-it})J_\Phi P^{it}\Lambda_{\Phi'}(x) \qquad \square$$

Thanks to these formulas, we verify for example that $\tau'_t(\alpha(n)) = \alpha(\sigma_t^{\nu'}(n))$, $\tau'_t(\beta(n)) = \beta(\sigma_t^{\nu'}(n))$ and τ' is implemented by P'.

PROPOSITION 9.3.12. – *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be adapted measured quantum groupoid and let $\tilde{T_L}$ be an other n.s.f. left invariant operator-valued weight which is β-adapted w.r.t. ν. Then fundamental objects $\tilde{R}$, $\tilde{\tau}$, $\tilde{\lambda}$, $\tilde{\delta}$ and $\tilde{P}$ of the adapted measured quantum groupoid $(N, M, \alpha, \beta, \Gamma, \nu, \tilde{T_L}, T_R)$ can be expressed in the following way:*

i) $\tilde{R} = R$, $\tilde{\tau} = \tau$, $\tilde{\lambda} = \lambda$ *and* $\tilde{P} = P$
ii) $\tilde{\delta} = \delta\alpha(h)\beta(h^{-1})$ *where h is affiliated with $Z(N)$ s.t.* $\tilde{T_L} = (T_L)_{\beta(h)}$

Proof. – By uniqueness theorem, there exists a strictly positive operator h affiliated with $Z(N)$ such that $\nu \circ \alpha^{-1} \circ \tilde{T_L} = (\nu \circ \alpha^{-1} \circ T_L)_{\beta(h)}$ and, for all $t \in \mathbb{R}$, we have $[D\tilde{T_L} : DT_L]_t = \beta(h^{it})$. We have already noticed that R and τ are independent w.r.t. left invariant operator-valued weight and β-adapted w.r.t. ν. We compute then Radon-Nykodim cocycle:

$$\begin{aligned}
&[D\nu\beta^{-1}R\tilde{T_L}R : D\nu\alpha^{-1}\tilde{T_L}]_t \\
&= [D\nu\beta^{-1}R\tilde{T_L}R : D\nu\beta^{-1}RT_LR]_t[D\Psi : D\Phi]_t[D\nu\alpha^{-1}T_L : D\nu\alpha^{-1}\tilde{T_L}]_t \\
&= R([D\tilde{T_L} : DT_L]^*_{-t})[D\Psi : D\Phi]_t[DT_L : D\tilde{T_L}]_t \\
&= \alpha(h^{it})\lambda^{\frac{it^2}{2}}\delta^{it}\beta(h^{-it}) = \lambda^{\frac{it^2}{2}}\delta^{it}\alpha(h^{it})\beta(h^{-it})
\end{aligned}$$

Then, it remains to compute $\tilde{P}$. If, we put $\tilde{\Phi} = \nu \circ \alpha^{-1} \circ \tilde{T_L}$, we have, for all $t \in \mathbb{R}$ and $x \in \mathcal{N}_{\tilde{T_L}} \cap \mathcal{N}_{\tilde{\Phi}}$:

$$\begin{aligned}
\tilde{P}^{it}\Lambda_{\tilde{\Phi}}(x) = \tilde{\lambda}^{t/2}\Lambda_{\tilde{\Phi}}(\tilde{\tau}_t(x)) = \lambda^{t/2}\Lambda_\Phi(\tau_t(x)\beta(h^{1/2})) &= \lambda^{t/2}\Lambda_\Phi(\tau_t(x\beta(h^{1/2})) \\
&= P^{it}\Lambda_\Phi(x\beta(h^{1/2})) = P^{it}\Lambda_{\tilde{\Phi}}(x) \qquad \square
\end{aligned}$$

THEOREM 9.3.13. – *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ and $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ be adapted measured quantum groupoids such that there exist strictly positive operators* *h and k affiliated with N which strongly commute and* $[D\nu' : D\nu]_t = k^{\frac{it^2}{2}} h^{it}$ *for all* $t \in \mathbb{R}$. *For all* $t \in \mathbb{R}$, *fundamental objects of the two structures are linked by:*

i) $R' = R$

ii) $\tau'_t = Ad_{\alpha(k^{\frac{-it^2}{2}} h^{-it})\beta(k^{\frac{it^2}{2}} h^{it})} \circ \tau_t = Ad_{\alpha([D\nu':D\nu]^*_t)\beta([D\nu':D\nu]_t)} \circ \tau_t$

iii) $\lambda' = \lambda$

iv) $\dot{\delta}' = \dot{\delta}$ *where* $\dot{\delta}$ *and* $\dot{\delta}'$ *have been defined in proposition 9.3.3*

v) $P'^{it} = \alpha(k^{\frac{it^2}{2}} h^{it})\beta(k^{\frac{-it^2}{2}} h^{-it}) J_\Phi \alpha(k^{\frac{it^2}{2}} h^{it})\beta(k^{\frac{-it^2}{2}} h^{-it}) J_\Phi P^{it}$

Proof. – We successively apply the two previous propositions. □

We summarize results concerning the change of quasi-invariant weight in the following theorem:

THEOREM 9.3.14. – *Let* $(N, M, \alpha, \beta, \Gamma, \nu, T_L, R \circ T_L \circ R)$ *be a adapted measured quantum groupoid. If* ν' *is a n.s.f. weight on* N *and* h, k *are strictly positive operators, affiliated with* N, *strongly commuting and satisfying* $[D\nu' : D\nu]_t = k^{\frac{it^2}{2}} h^{it}$ *for all* $t \in \mathbb{R}$, *then there exists a n.s.f. left invariant operator-valued weight* $\tilde{T}_L$ *which is* β-*adapted w.r.t.* ν'. *Moreover, if* $(N, M, \alpha, \beta, \Gamma, \nu', T'_L, T'_R)$ *is an other adapted measured quantum groupoid, then, for all* $t \in \mathbb{R}$, *fundamental objects are linked by:*

i) $R' = R$

ii) $\tau'_t = Ad_{\alpha(k^{\frac{-it^2}{2}} h^{-it})\beta(k^{\frac{it^2}{2}} h^{it})} \circ \tau_t = Ad_{\alpha([D\nu':D\nu]^*_t)\beta([D\nu':D\nu]_t)} \circ \tau_t$

iii) $\lambda' = \lambda$

iv) $\dot{\delta}' = \dot{\delta}$ *where* $\dot{\delta}$ *and* $\dot{\delta}'$ *have been defined in proposition 9.3.3*

v) $P'^{it} = \alpha(k^{\frac{it^2}{2}} h^{it})\beta(k^{\frac{-it^2}{2}} h^{-it}) J_\Phi \alpha(k^{\frac{it^2}{2}} h^{it})\beta(k^{\frac{-it^2}{2}} h^{-it}) J_\Phi P^{it}$

9.4. Characterization

In theorem 9.2.47, we explain how an adapted measured quantum groupoid can be seen as a generalized quantum groupoid. But it is easy to characterize them among measured quantum groupoids.

THEOREM 9.4.1. – *A measured quantum groupoid is an adapted measured quantum groupoid if, and only if* $\gamma = \sigma^\nu$ *if, and only if* δ *is affiliated with* $M \cap \alpha(N)' \cap \beta(N)'$.

Proof. – Straightforward. □

In general, we have not a duality within adapted measured quantum groupoid category that is the dual structure coming from measured quantum groupoid is not an adapted measured quantum groupoid anymore. We can be even more precise by characterizing dual objects of adapted measured quantum groupoids.

THEOREM 9.4.2. – *A measured quantum groupoid is the dual of an adapted measured quantum groupoid if, and only if $\gamma_t = \sigma^{\nu}_{-t}$ for al $t \in \mathbb{R}$.*

Proof. – Let us denote by M a measured quantum groupoid and by $\widehat{M}$ its dual. By the bi-duality theorem and the previous theorem, M is the dual of an adapted measured quantum groupoid if, and only if $\widehat{M}$ is an adapted measured quantum groupoid if, and only if $\gamma_{-t} = \hat{\gamma}_t = \sigma^{\nu}_t$ for all $t \in \mathbb{R}$. □

Also, we can deduce a precise result concerning duality within adapted measured quantum groupoids:

THEOREM 9.4.3. – *For all adapted measured quantum groupoid $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$, the dual measured quantum groupoid is an adapted measured quantum groupoid if, and only if the basis N is semi-finite.*

Proof. – $(N, \widehat{M}, \alpha, \hat{\beta}, \hat{\Gamma})$ equipped with $\widehat{T_L}$ et $\widehat{R} \circ \widehat{T_L} \circ \widehat{R}$ is an adapted measured quantum groupoid if, and only if there exists a nsf weight $\widehat{\nu}$ on N such that, for all $t \in \mathbb{R}$, we have $\sigma^{\nu}_t = \sigma^{\widehat{\nu}}_{-t}$. In this case, $\hat{\nu}$ is σ^{ν} invariant, so there exists a strictly positive operator h affiliated to the centralizer of ν such that $[D\hat{\nu} : D\nu]_t = h^{it}$. Then, for all $x \in N$, we have $\sigma^{\nu}_{-t}(x) = h^{it}\sigma^{\nu}_t(x)h^{-it}$ and $\sigma^{\nu}_{-2t}(x) = h^{it}xh^{-it}$. Then σ^{ν}_t is inner for all $t \in \mathbb{R}$ and N is semi-finite by theorem 3.14 of [**Tak03**]. Conversely, if N is semi-finite, there exists a nsf trace tr on N and a strictly positive operator h such that $\nu = tr(h.)$. So $\hat{\nu} = tr(h^{-1}.)$ satisfies conditions. □

CHAPTER 10

GROUPOIDS

DEFINITION 10.0.4. – A **groupoid** G is a small category in which each morphism $\gamma : x \rightarrow y$ is an isomorphism the inverse of which is γ^{-1}. Let $G^{\{0\}}$ the set of objects of G that we identify with $\{\gamma \in G | \gamma \circ \gamma = \gamma\}$. For all $\gamma \in G$, $\gamma : x \rightarrow y$, we denote $x = \gamma^{-1}\gamma = s(\gamma)$ we call source object and $y = \gamma\gamma^{-1} = r(\gamma)$ we call range object. If $G^{\{2\}}$ is the set of pairs (γ_1, γ_2) of G such that $s(\gamma_1) = r(\gamma_2)$, then composition of morphisms makes sense in $G^{\{2\}}$.

In [**Ren80**], J. Renault defines the structure of locally compact groupoid G with a Haar system $\{\lambda^u, u \in G^{\{0\}}\}$ and a quasi-invariant measure μ on $G^{\{0\}}$. We refer to [**Ren80**] for definitions and notations. We put $\nu = \mu \circ \lambda$. We refer to [**Co79**] and [**ADR00**] for discussions about transversal measures.

If G is σ-compact, J.M. Vallin constructs in [**Val96**] two co-involutive Hopf bimodules on the same basis $N = L^\infty(G^{\{0\}}, \mu)$, following T. Yamanouchi's works in [**Yam93**]. The underlying von Neumann algebras are $L^\infty(G, \nu)$ which acts by multiplication on $H = L^2(G, \nu)$ and $\mathscr{L}(G)$ generated by the left regular representation L of G.

We define a (resp. anti-) representation α (resp. β) from N in $L^\infty(G, \nu)$ such that, for all $f \in N$:

$$\alpha(f) = f \circ r \quad \text{and} \quad \beta(f) = f \circ s$$

For all $i, j \in \{\alpha, \beta\}$, we define $G^{\{2\}}_{i,j} \subset G \times G$ and a measure $\nu^2_{i,j}$ such that:

$$H{}_i\underset{N}{\otimes}{}_j H \text{ is identified with } L^2(G^{\{2\}}_{i,j}, \nu^2_{i,j})$$

For example, $G^{\{2\}}_{\beta,\alpha}$ is equal to $G^{\{2\}}$ and $\nu^2_{\beta,\alpha}$ to ν^2. Then, we construct a unitary W_G from $H{}_\alpha\underset{\mu}{\otimes}{}_\alpha H$ onto $H{}_\beta\underset{\mu}{\otimes}{}_\alpha H$, defined for all $\xi \in L^2(G^{\{2\}}_{\alpha,\alpha}, \nu^2_{\alpha,\alpha})$ by:

$$W_G\xi(s, t) = \xi(s, st)$$

for ν^2-almost all (s, t) in $G^{\{2\}}$.

This leads to define co-products Γ_G and $\widehat{\Gamma_G}$ by formulas:

$$\Gamma_G(f) = W_G(1_\alpha \underset{N}{\otimes}_\alpha f)W_G^* \quad \text{and} \quad \widehat{\Gamma_G}(k) = W_G^*(k_\beta \underset{N}{\otimes}_\alpha 1)W_G$$

for all $f \in L^\infty(G,\nu)$ and $k \in \mathscr{L}(G)$, this explicitly gives:

$$\Gamma_G(f)(s,t) = f(st)$$

for all $f \in L^\infty(G,\nu)$ and ν^2-almost all (s,t) in $G^{\{2\}}$,

$$\widehat{\Gamma_G}(L(h))\xi(x,y) = \int_G h(s)\xi(s^{-1}x, s^{-1}y)d\lambda^{r(x)}(s)$$

for all $\xi \in L^2(G^{\{2\}}_{\alpha,\alpha}, \nu^2_{\alpha,\alpha})$, h a continuous function with compact support on G and $\nu^2_{\alpha,\alpha}$-almost all (x,y) in $G^{\{2\}}_{\alpha,\alpha}$. Moreover, we define two co-involutions j_G and $\widehat{j_G}$ by:

$$j_G(f)(x) = f(x^{-1})$$

for all $f \in L^\infty(G,\nu)$ and almost all x,

$$\widehat{j_G}(g) = Jg^*J$$

for all $g \in \mathscr{L}(G)$ and where J is the involution $J\xi = \overline{\xi}$ for all $\xi \in L^2(G)$. Finally, we define two n.s.f. left invariant operator-valued weights P_G and $\widehat{P_G}$:

$$P_G(f)(y) = \int_G f(x)d\lambda^{r(y)}(x) \quad \text{and} \quad \widehat{P_G}(L(f)) = \alpha(f_{|G^{\{0\}}})$$

for all continuous with compact support f on G ν-almost all y in G.

Theorem 10.0.5. – *Let G be a σ-compact, locally compact groupoid with a Haar system and a quasi-invariant measure μon units. Then:*

$$(L^\infty(G^{\{0\}},\mu), L^\infty(G,\nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G P_G j_G)$$

is a commutative adapted measured quantum groupoid and:

$$(L^\infty(G^{\{0\}},\mu), \mathscr{L}(G), \alpha, \alpha, \widehat{\Gamma_G}, \mu, \widehat{P_G}, \widehat{j_G}\widehat{P_G}\widehat{j_G})$$

is a symmetric adapted measured quantum groupoid. The unitary $V_G = W_G^*$ is the fundamental unitary of the commutative structure.

Proof. – By [**Val96**, thm. 3.2.7 and th. 3.3.7], $(L^\infty(G^{\{0\}},\mu), L^\infty(G,\nu), \alpha, \beta, \Gamma_G)$ and $(L^\infty(G^{\{0\}},\mu), \mathscr{L}(G), \alpha, \alpha, \widehat{\Gamma_G})$ are co-involutive Hopf bimodules with left invariant operator-valued weights; to get right invariants operator-valued weights, we consider $j_G P_G j_G$ and $\widehat{j_G}\widehat{P_G}\widehat{j_G}$.

Since $L^\infty(G,\nu)$ is commutative, P_G is adapted w.r.t. μ by [**Val96**, thm. 3.3.4], $\sigma_t^{\mu\circ\alpha^{-1}\circ\widehat{P_G}}$ fixes point by point $\alpha(N)$ so that $\widehat{P_G}$ is adapted w.r.t. μ.

Finally, for all e, f, g continuous functions with compact support and almost all (s,t) in $G^{\{2\}}$, we have, by 3.3.1:

$$(1_\beta \underset{N}{\otimes}_\alpha JeJ)W_G(f_\alpha \underset{\mu}{\otimes}_\alpha g)(s,t) = \overline{e(t)}f(s)g(st) = \Gamma_G(g)(f_\beta \underset{\mu}{\otimes}_\alpha \overline{e})(s,t)$$
$$= (1_\beta \underset{N}{\otimes}_\alpha JeJ)U_H(f_\alpha \underset{\mu}{\otimes}_\alpha g)(s,t)$$

so that we get $U_H = W_G$. □

REMARK 10.0.6. – In the commutative structure, modular function $\frac{d\nu^{-1}}{d\nu}$ and modulus coincide and the scaling operator is trivial.

We have a similar result for adapted measured quantum groupoids in the sense of Hahn ([**Hah78a**] and [**Hah78b**]):

THEOREM 10.0.7. – *From all measured groupoid G, we construct a commutative adapted measured quantum groupoid $(L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G P_G j_G)$ and a symmetric one $(L^\infty(G^{\{0\}}, \mu), \mathscr{L}(G), \alpha, \alpha, \widehat{\Gamma_G}, \mu, \widehat{P_G}, \widehat{j_G}\widehat{P_G}\widehat{j_G})$. Objects are defined in a similar way as in the locally compact case. The unitary V_G is the fundamental unitary of the commutative structure.*

Proof. – Results come from [**Yam93**] for the symmetric case. It is sufficient to apply in this case, technics of [**Val96**] for the commutative case and invariant operator-valued weights. □

CONJECTURE 10.0.8. – *If $(N, M, \alpha, \beta, \Gamma, \mu, T_L, T_R)$ is an adapted measured quantum groupoid such that M is commutative, then there exists a locally compact groupoid G such that:*

$$(N, M, \alpha, \beta, \Gamma, \mu, T_L, T_R) \simeq (L^\infty(G^{\{0\}}, \mu), L^\infty(G, \nu), \alpha, \beta, \Gamma_G, \mu, P_G, j_G \circ P_G \circ j_G)$$

CHAPTER 11

FINITE QUANTUM GROUPOIDS

Definition 11.0.9 (Weak Hopf C*-algebras [**BSz96**]). – We call **weak Hopf C*-algebra** or finite quantum groupoid all $(M, \Gamma, \kappa, \varepsilon)$ where M is a finite dimensional C*-algebra with a co-product $\Gamma : M \to M \otimes M$, a co-unit ε and an antipode $\kappa : M \to M$ such that, for all $x, y \in M$:

i) Γ is a *-homomorphism (not necessary unital);

ii) Unit and co-unit satisfy the following relation:

$$(\varepsilon \otimes \varepsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \varepsilon(xy)$$

iii) κ is an anti-homomorphism of algebra and co-algebra such that:

- $(\kappa \circ *)^2 = \iota$
- $(m(\kappa \otimes \mathrm{id}) \otimes \mathrm{id})(\Gamma \otimes \mathrm{id})\Gamma(x) = (1 \otimes x)\Gamma(1)$.

where m denote the product on M.

We recall some results [**NV00, NV02**] and [**BNS99**]. If $(M, \Gamma, \kappa, \varepsilon)$ is a weak Hopf C*-algebra. We call co-unit range (resp. source) the application $\varepsilon_t = m(\mathrm{id} \otimes \kappa)\Gamma$ (resp. $\varepsilon_s = m(\kappa \otimes \mathrm{id})\Gamma$). We have $\kappa \circ \varepsilon_t = \varepsilon_s \circ \kappa$. There exists a unique faithful positive linear form h, called normalized Haar measure of $(M, \Gamma, \kappa, \varepsilon)$ which is κ-invariant, such that $(\mathrm{id} \otimes h)(\Gamma(1)) = 1$ and, for all $x, y \in M$, we have:

$$(\mathrm{id} \otimes h)((1 \otimes y)\Gamma(x)) = \kappa((i \otimes h)(\Gamma(y)(1 \otimes x)))$$

Moreover, $E_h^s = (h \otimes \mathrm{id})\Gamma$ (resp. $E_h^t = (\mathrm{id} \otimes h)\Gamma$) is a Haar conditional expectation to the source (resp. range) Cartan subalgebra $\varepsilon_s(M)$ (resp. range $\varepsilon_t(M)$) such that $h \circ E_h^s = h$ (resp. $h \circ E_h^t = h$). Range and source Cartan subalgebras commute.

By [**Da03, Nik02, Val03**], we can always assume that $\kappa^2_{|\varepsilon_t(M)} = \mathrm{id}$ thanks to a deformation. **In the following, we assume that the condition holds.**

Since $h \circ \kappa = h$ and $\kappa\varepsilon_t = \varepsilon_s\kappa$, we have $h \circ \varepsilon_t = h \circ \varepsilon_s$.

THEOREM 11.0.10. – *Let $(M, \Gamma, \kappa, \varepsilon)$ be a weak Hopf C^*-algebra, h its normalized Haar measure, E_h^s (resp. E_h^t) its source (resp. range) Haar conditional expectation and $\varepsilon_t(M)$ its range Cartan subalgebra. We put $N = \varepsilon_t(M)$, $\alpha = \mathrm{id}_{|N}$, $\beta = \kappa_{|N}$, $\tilde{\Gamma}$ the co-product Γ viewed as an operator which takes value in:*

$$M_{\beta}\underset{N}{\star}{}_{\alpha}M \simeq (M \otimes M)_{\Gamma(1)}$$

and $\mu = h \circ \alpha = h \circ \beta$. Then $(N, M, \alpha, \beta, \tilde{\Gamma}, \mu, E_h^t, E_h^s)$ is an adapted measured quantum groupoid.

Proof. – α is a representation from N in M and, since $\kappa^2_{|\varepsilon_t(M)} = \mathrm{id}$, β is an anti-representation from N in M. They commute each other because Cartan subalgebras commute and $\kappa \circ \varepsilon_t = \varepsilon_s \circ \kappa$. For all $n \in N$, there exists $m \in M$ such that $n = \varepsilon_t(m)$. So, we have:

$$\tilde{\Gamma}(\alpha(n)) = \tilde{\Gamma}(\varepsilon_t(m)) = \Gamma(1)(\varepsilon_t(m) \otimes 1)\Gamma(1) = \alpha(n)_{\beta}\underset{N}{\otimes}{}_{\alpha}1$$

Also, we have $\tilde{\Gamma}(\beta(n)) = 1_{\beta}\underset{N}{\otimes}{}_{\alpha}\beta(n)$ and $\tilde{\Gamma}$ is a co-product. Then $(N, M, \alpha, \beta, \Gamma)$ is Hopf bimodule. Moreover, for all $n \in N$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned}\sigma_t^{E_h^t}(\beta(n)) = \sigma_t^{h \circ E_h^t}(\beta(n)) = \sigma_t^{h \circ E_h^s}(\beta(n)) &= \sigma_t^{h_{|\beta(N)}}(\beta(n)) \\ &= \beta(\sigma_{-t}^{h_{|\beta(N)} \circ \beta}(n)) = \beta(\sigma_{-t}^{\mu}(n))\end{aligned}$$

and E_h^t is β-adapted w.r.t. μ. Since $E_h^s = \kappa \circ E_h^t \circ \kappa$, then E_h^s is α-adapted w.r.t. μ. □

THEOREM 11.0.11. – *Let $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ be an adapted measured quantum groupoid such that M is finite dimensional. Then, there exist $\tilde{\Gamma}$, κ and ε such that $(M, \tilde{\Gamma}, \kappa, \varepsilon)$ is a weak Hopf C^*-algebra.*

Proof. – By 2.3, we identify via $I^{\nu}_{\beta,\alpha}$, $L^2(M)_{\beta}\underset{N}{\otimes}{}_{\alpha}L^2(M)$ with a subspace of $L^2(M) \otimes L^2(M)$. We put $\tilde{\Gamma}(x) = I^{\nu}_{\beta,\alpha}\Gamma(x)(I^{\nu}_{\beta,\alpha})^*$. By [**Val01**, def. 2.2.3], the fundamental pseudo-multiplicative unitary becomes a multiplicative partial isometry on $L^2(M) \otimes L^2(M)$ of basis $(N, \alpha, \hat{\beta}, \beta)$ by $I = I^{\nu}_{\alpha,\hat{\beta}}W(I^{\nu}_{\beta,\alpha})^*$. I is regular in the sense of [**Val01**, def. 2.6.3] by 7.0.16. Moreover, if we put $H = L^2(M)$, then $Tr_H(R(m)) = Tr_H(m)$ for all $m \in M$ because R is implemented by an anti-unitary, so $Tr_H \circ \beta = Tr_H \circ \alpha = Tr_H \circ \hat{\beta}$ and we conclude by [**Val01**, prop. 3.1.3]. □

REMARK 11.0.12. – With notations of section 2.3, κ and S are linked by:

$$\kappa(x) = \alpha(n_o^{1/2}d^{1/2})\beta(n_o^{-1/2}d^{-1/2})S(x)\alpha(n_o^{-1/2}d^{-1/2})\beta(n_o^{1/2}d^{1/2})$$

CHAPTER 12

QUANTUM GROUPS

THEOREM 12.0.13. – *Adapted measured quantum groupoids, basis N on which is equal to $\mathbb{C}$ are exactly locally compact quantum groups (in the von Neumann setting) introduced by J. Kustermans and S. Vaes in* [**KV03**].

Proof. – In this case, the notion of relative tensor product is just usual tensor product of Hilbert spaces, the notion of fibered product is just tensor product of von Neumann algebras and the notion of operator-valued weight is just weight. □

CHAPTER 13

COMPACT CASE

In this section, we show that pseudo-multiplicative unitaries of compact type in the sense of [**Eno02**] correspond exactly to adapted measured quantum groupoids with a Haar conditional expectation.

DEFINITION 13.0.14. – Let W be a pseudo-multiplicative unitary over N w.r.t. $\alpha, \beta, \hat{\beta}$. Let ν be a n.s.f. weight on N. We say that W is of **compact type** w.r.t. ν if there exists $\xi \in H$ such that:

i) ξ belongs to $D(H_{\hat{\beta}}, \nu^o) \cap D({}_\alpha H, \nu) \cap D(H_\beta, \nu^o)$;
ii) $\langle \xi, \xi \rangle_{\hat{\beta}, \nu^o} = \langle \xi, \xi \rangle_{\alpha, \nu} = \langle \xi, \xi \rangle_{\beta, \nu^o} = 1$
iii) we have $W(\xi {}_{\hat{\beta}} \underset{\nu}{\otimes} {}_\alpha \eta) = \xi {}_\alpha \underset{\nu^o}{\otimes} {}_\beta \eta$ for all $\eta \in H$.

In this case, ξ is said to be **fixed and bi-normalized**. We also say that W is of **discrete type** w.r.t. ν if $\hat{W}$ is of compact type.

By [**Eno02**, prop. 5.11], we recall that, if W is of compact type w.r.t. ν and ξ is a fixed and bi-normalized vector, then ν shall be a faithful, normal, positive form on N which is equal to $\omega_\xi \circ \alpha = \omega_\xi \circ \beta = \omega_\xi \circ \hat{\beta}$ and it is called **canonical form**.

PROPOSITION 13.0.15. – *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. Assume there exist:*

i) *a n.f. left invariant conditional expectation from E to $\alpha(N)$;*
ii) *a n.f. right invariant conditional expectation from F to $\beta(N)$;*
iii) *a n.f. state ν on N such that $\nu \circ \alpha^{-1} \circ E = \nu \circ \beta^{-1} \circ F$.*

Then $(N, M, \alpha, \beta, \Gamma, \nu, E, F)$ is an adapted measured quantum groupoid. Moreover, if R, τ, λ and δ are fondamental objects of the structure, then we have $F = R \circ E \circ R$ and $\lambda = \delta = 1$. Finally, $\Lambda_{\nu \circ \alpha^{-1} \circ E}(1)$ is co-fixed and bi-normalized, and the fundamental pseudo-multiplicative unitary W is weakly regular and of discrete type in sense of [**Eno02**, §5].

Proof. – For all $t \in \mathbb{R}$ and $n \in N$, we have:

$$\sigma_t^E(\beta(n)) = \sigma_t^{\nu\circ\alpha^{-1}\circ E}(\beta(n)) = \sigma_t^{\nu\circ\beta^{-1}\circ F}(\beta(n)) = \beta(\sigma_{-t}^{\nu}(n))$$

Also, we have:

$$\sigma_t^F(\alpha(n)) = \sigma_t^{\nu\circ\beta^{-1}\circ F}(\alpha(n)) = \sigma_t^{\nu\circ\alpha^{-1}\circ E}(\alpha(n)) = \alpha(\sigma_t^{\nu}(n))$$

so that $(N, M, \alpha, \beta, \Gamma, \nu, E, F)$ is an adapted measured quantum groupoid. By definition, we have:

$$[D\nu\circ\alpha^{-1}\circ E\circ R : D\nu\circ\alpha^{-1}\circ E]_t = \lambda^{\frac{it^2}{2}}\delta^{it}$$

On the other hand, since $\nu\circ\alpha^{-1}\circ E = \nu\circ\beta^{-1}\circ F$ and by uniqueness, there exists a strictly positive element h affiliated with $Z(N)$:

$$[D\nu\circ\alpha^{-1}\circ E\circ R : D\nu\circ\alpha^{-1}\circ E]_t = [DR\circ E\circ R : DF]_t = \alpha(h^{it})$$

We deduce that $\lambda = 1$ and $\delta = \alpha(h)$, so $\alpha(h^{-1}) = \delta^{-1} = R(\delta) = \beta(h)$ and by [**Eno00**, 5.2], we get $h = 1$.

We put $\Phi = \nu\circ\alpha^{-1}\circ E$. If $(\xi_i)_{i\in I}$ is a (N^o, ν^o)-basis of $(H_\Phi)_\beta$ then, for all $v \in D(H_\beta, \nu^o)$:

$$\begin{aligned} U_H(v_\alpha \underset{\nu^o}{\otimes}_{\hat\beta}\Lambda_\Phi(1)) &= \sum_{i\in I}\xi_{i\beta}\underset{\nu}{\otimes}_\alpha\Lambda_\Phi((\omega_{v,\xi_i\beta}\underset{\nu}{\star}_\alpha\mathrm{id})(\Gamma(1))) \\ &= \sum_{i\in I}\xi_{i\beta}\underset{\nu}{\otimes}_\alpha\alpha(\langle v, \xi_i\rangle_{\beta,\nu^o})\Lambda_\Phi(1) = v_\beta\underset{\nu}{\otimes}_\alpha\Lambda_\Phi(1)\end{aligned}$$

It is easy to see that $\Lambda_\Phi(1)$ belongs to $D((H_\Phi)_{\hat\beta}, \nu^o) \cap D({}_\alpha H_\Phi, \nu)$ and satisfies $\langle\Lambda_\Phi(1), \Lambda_\Phi(1)\rangle_{\hat\beta,\nu^o} = \langle\Lambda_\Phi(1), \Lambda_\Phi(1)\rangle_{\alpha,\nu} = 1$ so that, by continuity, we get $U_H(v_\alpha \underset{\nu^o}{\otimes}_{\hat\beta}\Lambda_\Phi(1)) = v_\beta\underset{\nu}{\otimes}_\alpha\Lambda_\Phi(1)$ for all $v \in H$ *i.e.*, $\Lambda_\Phi(1)$ is co-fixed and bi-normalized. Since $\nu\circ\alpha^{-1}\circ E = \Phi = \nu\circ\beta^{-1}\circ F$, we have by 3.2.2, for all $n \in \mathcal{N}_\nu$:

$$\beta(n^*)\Lambda_\Phi(1) = \beta(n^*)J_\Phi\Lambda_\Phi(1) = J_\Phi\Lambda_F(1)\Lambda_\nu(n)$$

so that $\Lambda_\Phi(1)$ is β-bounded w.r.t. ν^o and $R^{\beta,\nu^o}(\Lambda_\Phi(1)) = J_\Phi\Lambda_F(1)J_\nu$. Consequently, $\Lambda_\Phi(1)$ is bi-normalized and W is of discrete type. □

Corollary 13.0.16. – *Let W be a weakly regular pseudo-multiplicative unitary over N w.r.t. $\alpha, \beta, \hat\beta$ of compact type w.r.t. the canonical form ν. If ξ a fixed and bi-normalized vector, we put:*

i) *$\mathcal{A}$ the von Neumann algebra generated by right leg of W;*
ii) *$\Gamma(x) = \sigma_{\nu^o}W(x_\alpha \underset{\nu^o}{\otimes}_\beta 1)W^*\sigma_\nu$ for all $x \in \mathcal{A}$;*
iii) *$E = (\omega_{\xi\beta}\underset{\nu}{\star}_\alpha\mathrm{id})\circ\Gamma$ and $F = (\mathrm{id}_\beta\underset{\nu}{\star}_\alpha\omega_\xi)\circ\Gamma$.*

Then $(N, \mathcal{A}, \alpha, \beta, \Gamma, \nu, E, F)$ is an adapted measured quantum groupoid such that E and F are n.f. conditional expectations. Moreover, if R, τ, λ and δ are the fundamental objects of the structure, we have $F = R\circ E\circ R$, $\lambda = \delta = 1$ and the fundamental unitary is $\hat W$.

Proof. – By [**EV00**, 6.3], we know that $(N, \mathcal{A}, \alpha, \beta, \Gamma)$ is a Hopf bimodule. By [**Eno02**, thm. 6.6], E is a n.f. left invariant conditional expectation from $\mathcal{A}$ to $\alpha(N)$. By [**Eno02**, prop. 6.2 and prop. 6.3], F is a n.f. right invariant conditional expectation from $\mathcal{A}$ to $\beta(N)$. Moreover, we clearly have $\omega_\xi \circ E = \omega_\xi \circ F$ so that $\nu \circ \alpha^{-1} \circ E = \nu \circ \beta^{-1} \circ F$. We are in conditions of the previous proposition an we get that $(N, \mathcal{A}, \alpha, \beta, \Gamma, \nu, E, F)$ is an adapted measured quantum groupoid, $F = R \circ E \circ R$ and $\lambda = \delta = 1$. Finally, by [**Eno02**, coro. 7.7], $\hat{W}$ is the fundamental unitary. (More exactly, it is $\sigma_{\nu^o} W_s^* \sigma_\nu$ where W_s is the standard form of W in the sense of [**Eno02**, §7]). □

The converse is also true and so we characterize the compact case:

Corollary 13.0.17. – *Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf bimodule. We assume there exist:*

i) *a co-involution R;*

ii) *a n.f. left invariant conditional expectation from E to $\alpha(N)$.*

Then there exists a n.f. state ν on N such that $(N, M, \alpha, \beta, \Gamma, \nu, E, R \circ E \circ R)$ is an adapted measured quantum groupoid with trivial modulus and scaling operator and the fundamental unitary of which is of discrete type w.r.t. ν.

Proof. – We put $F = R \circ E \circ R$ which is a n.f. right invariant conditional expectation from M to $\beta(N)$. We also put:

$$\tilde{E} = E_{|\beta(N)} : \beta(N) \longrightarrow \alpha(Z(N)) \text{ and } \tilde{F} = F_{|\alpha(N)} : \alpha(N) \longrightarrow \beta(Z(N))$$

We have, for all $m \in M$:

$$\begin{aligned} \tilde{F}E(m) {}_\beta\underset{N}{\otimes}{}_\alpha 1 &= (F {}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(E(m) {}_\beta\underset{N}{\otimes}{}_\alpha 1) \\ &= (F {}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})(\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha E)\Gamma(m) \\ &= (\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha E)(F {}_\beta\underset{N}{\star}{}_\alpha \mathrm{id})\Gamma(m) \\ &= (\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha E)(1 {}_\beta\underset{N}{\otimes}{}_\alpha F(m)) = 1 {}_\beta\underset{N}{\otimes}{}_\alpha \tilde{E}F(m) \end{aligned}$$

so, if $\tilde{F}E(m) = \beta(n)$ for some $n \in Z(N)$, then $\tilde{E}F(m) = \alpha(n)$. Moreover, we have:

$$\begin{aligned} \tilde{E}F(m) {}_\beta\underset{N}{\otimes}{}_\alpha 1 = EF(m) {}_\beta\underset{N}{\otimes}{}_\alpha 1 &= (\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha E)\Gamma(F(m)) \\ &= (\mathrm{id} {}_\beta\underset{N}{\star}{}_\alpha E)(1 {}_\beta\underset{N}{\otimes}{}_\alpha F(m)) = 1 {}_\beta\underset{N}{\otimes}{}_\alpha \tilde{E}F(m) \end{aligned}$$

so that $\alpha(n) = \beta(n)$. Consequently $\tilde{E}F(m) = \tilde{F}E(m)$ and $EF = FE$ is a n.f. conditional expectation from M to:

$$\tilde{N} = \alpha(\{n \in Z(N), \alpha(n) = \beta(n)\}) = \beta(\{n \in Z(N), \alpha(n) = \beta(n)\})$$

Also, we have $R_{|\tilde{N}} = \mathrm{id}$. So, if ω is a n.f. state on $\tilde{N}$, we have $\omega \circ \tilde{E} \circ \beta = \omega \circ \tilde{F} \circ \alpha$ and $\nu = \omega \circ \tilde{E} \circ \beta = \omega \circ \tilde{F} \circ \alpha$ satisfies hypothesis of 13.0.15: then, corollary holds. □

COROLLARY 13.0.18. – *Let* $(N, M, \alpha, \beta, \Gamma, \nu, T_L, T_R)$ *be an adapted measured quantum groupoid such that* T_L *is a conditional expectation. Then there exists a n.f. state* ν' *on* N *such that* $\sigma^{\nu'} = \sigma^{\nu}$ *and the fundamental unitary is of discrete type w.r.t.* ν'.

Proof. – Let R be the co-involution. By the previous corollary, there exists a n.f. state ν' on N such that $(N, M, \alpha, \beta, \Gamma, \nu', T_L, R \circ T_L \circ R)$ is an adapted measured quantum groupoid. Since T_L is β-adapted w.r.t. ν and ν', we have $\sigma^{\nu'} = \sigma^{\nu}$. We easily verify that the fundamental unitary of the first structure coincides with that of the last one which is of discrete type w.r.t. ν' by the previous corollary. □

CHAPTER 14

QUANTUM SPACE QUANTUM GROUPOID

14.1. Definition

Let M be a von Neumann algebra. M acts on $H = L^2(M) = L^2_\nu(M)$ where ν is a n.s.f. weight on M. We denote by M', (resp. $Z(M)'$) the commutant of M (resp. $Z(M)$) in $\mathcal{L}(L^2(M))$. Let tr be a n.s.f. trace on $Z(M)$. $M' \underset{Z(M)}{\star} M = M' \underset{Z(M)}{\otimes} M$ acts on $L^2(M) \underset{tr}{\otimes} L^2(M)$. There exists a n.s.f. operator-valued weight T from M to $Z(M)$ such that $\nu = tr \circ T$.

Let α (resp. β) be the (resp. anti-) representation of M to $M' \underset{Z(M)}{\otimes} M$ such that $\alpha(m) = 1 \underset{Z(M)}{\otimes} m$ (resp. $\beta(m) = j(m) \underset{Z(M)}{\otimes} 1$) where $j(x) = J_\nu x^* J_\nu$ for all $x \in \mathcal{L}(L^2_\nu(M))$.

Proposition 14.1.1. – *The following formula:*

$$I : [L^2(M) \underset{tr}{\otimes} L^2(M)]_\beta \underset{\nu}{\otimes}_\alpha [L^2(M) \underset{tr}{\otimes} L^2(M)] \longrightarrow L^2(M) \underset{tr}{\otimes} L^2(M) \underset{tr}{\otimes} L^2(M)$$
$$[\Lambda_\nu(y) \underset{tr}{\otimes} \eta]_\beta \underset{\nu}{\otimes}_\alpha \Xi \longmapsto \alpha(y)\Xi \underset{tr}{\otimes} \eta$$

for all $\eta \in L^2(M), \Xi \in L^2(M) \underset{tr}{\otimes} L^2(M)$ and $y \in M$, defines a canonical isomorphism such that we have $I([m \underset{Z(M)}{\otimes} z]_\beta \underset{\nu}{\otimes}_\alpha Z) = (\alpha(M)Z \underset{Z(M)}{\otimes} z)I$, for all $m \in M$, $z \in Z(M)'$ and $Z \in \mathcal{L}(L^2(M)) \underset{Z(M)}{\star} M'$.

Proof. – Straightforward. □

We identify $(M' \underset{Z(M)}{\otimes} M)_\beta \underset{M}{\star}_\alpha (M' \underset{Z(M)}{\otimes} M)$ with $M' \underset{Z(M)}{\otimes} Z(M) \underset{Z(M)}{\otimes} M$ and so with $M' \underset{Z(M)}{\otimes} M$. We define a normal *-homomorphism Γ by:

$$M' \underset{Z(M)}{\otimes} M \longrightarrow (M' \underset{Z(M)}{\otimes} M)_\beta \underset{\nu}{\star}_\alpha (M' \underset{Z(M)}{\otimes} M)$$
$$n \underset{Z(M)}{\otimes} m \longmapsto I^*(n \underset{Z(M)}{\otimes} 1 \underset{Z(M)}{\otimes} m)I = [1 \underset{Z(M)}{\otimes} m]_\beta \underset{\nu}{\otimes}_\alpha [n \underset{Z(M)}{\otimes} 1]$$

Γ is, in fact, the identity trough the previous isomorphism.

THEOREM 14.1.2. – *If we put* $T_R = \mathrm{id} \underset{Z(M)}{\star} T$ *and* $R = \varsigma_{Z(M)} \circ (j \underset{Z(M)}{\otimes} j)$, *then* $(M, M' \underset{Z(M)}{\otimes} M, \alpha, \beta, \Gamma, \nu, R \circ T_R \circ R, T_R)$ *becomes an adapted measured quantum groupoid w.r.t.* ν *called* ***quantum space quantum groupoid****.*

Proof. – By definition, Γ is a morphism of Hopf bimodule. We have to prove co-product relation. For all $m \in M$ and $n \in M'$, we have:

$$\begin{aligned}(\Gamma_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id}) \circ \Gamma(n \underset{Z(M)}{\otimes} m) &= [1 \underset{Z(M)}{\otimes} m]_{\beta}\underset{\nu}{\otimes}_{\alpha}[1 \underset{Z(M)}{\otimes} 1]_{\beta}\underset{\nu}{\otimes}_{\alpha}[n \underset{Z(M)}{\otimes} 1] \\ &= (\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\Gamma) \circ \Gamma(n \underset{Z(M)}{\otimes} m)\end{aligned}$$

Now, we show that T_R is right invariant and α-adapted w.r.t. ν. So, for all $m \in M, n \in M'$ and $\xi \in D({}_{\alpha}(L^2(M) \underset{tr}{\otimes} L^2(M)), \nu^o)$, we put $\Psi = \nu \circ \beta^{-1} \circ T_R$ and we compute:

$$\begin{aligned}\omega_\xi((\Psi_{\beta}\underset{\nu}{\star}_{\alpha}\mathrm{id})\Gamma(n \underset{Z(M)}{\otimes} m)) &= \Psi((\mathrm{id}_{\beta}\underset{\nu}{\star}_{\alpha}\omega_\xi)([1 \underset{Z(M)}{\otimes} m]_{\beta}\underset{\nu}{\otimes}_{\alpha}[n \underset{Z(M)}{\otimes} 1])) \\ &= \Psi([1 \underset{Z(M)}{\otimes} m]\beta(\langle [n \underset{Z(M)}{\otimes} 1]\xi, \xi\rangle_{\alpha,\nu})) \\ &= \nu(\langle [n \underset{Z(M)}{\otimes} T(m)]\xi, \xi\rangle_{\alpha,\nu}) \\ &= \omega_\xi(n \underset{Z(M)}{\otimes} T(m)) = \omega_\xi(T_R(n \underset{Z(M)}{\otimes} m))\end{aligned}$$

Finally, we have for all $t \in \mathbb{R}$:

$$\begin{aligned}\sigma_t^{T_R} &= \sigma_t^{\nu' \underset{Z(M)}{\star} \nu}{}_{|(M' \underset{Z(M)}{\otimes} M) \cap \beta(M)'} = \sigma_t^{\nu' \underset{Z(M)}{\star} \nu}{}_{|(M' \underset{Z(M)}{\otimes} M) \cap (M \underset{Z(M)}{\star} \mathcal{L}(L^2(M)))} \\ &= \sigma_t^{\nu' \underset{Z(M)}{\star} \nu}{}_{|Z(M) \underset{Z(M)}{\otimes} M} = (\mathrm{id} \underset{Z(M)}{\otimes} \sigma_t^\nu)_{|1 \underset{Z(M)}{\otimes} M} = 1 \underset{Z(M)}{\otimes} \sigma_t^\nu\end{aligned}$$

so that $\sigma_t^{T_R} \circ \alpha(m) = 1 \underset{Z(M)}{\otimes} \sigma_t^\nu(m) = \alpha(\sigma_t^\nu(m))$ for all $t \in \mathbb{R}$ and $m \in M$. Since it is easy to see that R is a co-involution, we have done. □

14.2. Fundamental elements

By 3.3.1, we can compute the pseudo-multiplicative unitary. Let first notice that $\Phi = \nu' \underset{Z(M)}{\star} \nu = \Psi$ so that $\lambda = \delta = 1$ and:

$$\alpha = 1 \underset{Z(M)}{\otimes} \mathrm{id}, \hat{\alpha} = \mathrm{id} \underset{Z(M)}{\otimes} 1, \beta = j \underset{Z(M)}{\otimes} 1 \text{ and } \hat{\beta} = 1 \underset{Z(M)}{\otimes} j$$

For example, we have $D((H \underset{tr}{\otimes} H)_{\hat{\beta},\nu^o}) \supset H \underset{tr}{\otimes} D(H_j, \nu^o) = H \underset{tr}{\otimes} \Lambda_\nu(\mathcal{N}_\nu)$ and for all $\eta \in H$ and $y \in \mathcal{N}_\nu$, we have $R^{\hat{\beta},\nu^o}(\eta \underset{tr}{\otimes} \Lambda_\nu(y)) = \lambda_\eta^{tr} R^{j,\nu^o}(\Lambda_\nu(y)) = \lambda_\eta^{tr} y$.

LEMMA 14.2.1. – *We have, for all* $\eta \in H$ *and* $e \in \mathcal{N}_\nu$*:*

$$I\rho^{\beta,\alpha}_{\eta \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} = \lambda^{tr}_\eta J_\nu e J_\nu \underset{Z(M)}{\otimes} 1 \textit{ and } I\lambda^{\beta,\alpha}_{\Lambda_\nu(y)\otimes\eta} = \rho^{tr}_\eta (1 \underset{Z(M)}{\otimes} y)$$

Proof. – Straightforward. □

PROPOSITION 14.2.2. – *We have, for all* $\Xi \in H\underset{tr}{\otimes}H, \eta \in H$ *and* $m \in \mathcal{N}_\nu$*:*

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} (\eta \underset{tr}{\otimes} \Lambda_\nu(m))) = I^*(\eta \underset{tr}{\otimes} (1 \underset{Z(M)}{\otimes} m)\Xi)$$

Proof. – For all $m, e \in \mathcal{N}_\nu$ and $m', e' \in \mathcal{N}_{\nu'}$, we have by the previous lemma:

$$\begin{aligned} I\Gamma(m' \underset{Z(M)}{\otimes} m)\rho^{\beta,\alpha}_{J_{\nu'}\Lambda_{\nu'}(e')\underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} &= (m' \underset{Z(M)}{\otimes} 1 \underset{Z(M)}{\otimes} m) I\rho^{\beta,\alpha}_{J_{\nu'}\Lambda_{\nu'}(e')\underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} \\ &= (m' \otimes 1 \otimes m)\lambda^{tr}_{J_{\nu'}\Lambda_{\nu'}(e')} J_\nu e J_\nu \underset{Z(M)}{\otimes} 1 \\ &= \lambda^{tr}_{J_{\nu'}e'J_{\nu'}\Lambda_{\nu'}(m')} J_\nu e J_\nu \underset{Z(M)}{\otimes} m \end{aligned}$$

On the other hand, we have by 14.1.1:

$$\begin{aligned} & I([1 \underset{Z(M)}{\otimes} 1]_\beta \underset{\nu}{\otimes}_\alpha [J_{\nu'} e' J_{\nu'} \underset{Z(M)}{\otimes} J_\nu e J_\nu]) W^* \rho^{\alpha,\hat\beta}_{\Lambda_{\nu'}(m')\underset{tr}{\otimes}\Lambda_{\nu'}(m')} \\ &= (J_{\nu'} e' J_{\nu'} \underset{Z(M)}{\otimes} J_\nu e J_\nu \underset{Z(M)}{\otimes} 1) I W^* \rho^{\alpha,\hat\beta}_{\Lambda_{\nu'}(m')\underset{tr}{\otimes}\Lambda_{\nu'}(m')} \end{aligned}$$

Then, by 3.3.1 and taking the limit over e and e' which go to 1, we get for all $\Xi \in H\underset{tr}{\otimes}H$:

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} (\Lambda_{\nu'}(m') \underset{tr}{\otimes} \Lambda_\nu(m))) = I^*(\Lambda_{\nu'}(m') \underset{tr}{\otimes} (1 \underset{Z(M)}{\otimes} m)\Xi)$$

Now, if $\Xi \in D({}_\alpha(H\underset{tr}{\otimes}H), \nu)$, by continuity and density of $\Lambda_{\nu'}(\mathcal{N}_{\nu'})$ we have for all $\Xi \in D({}_\alpha(H\underset{tr}{\otimes}H), \nu)$:

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} (\eta \underset{tr}{\otimes} \Lambda_\nu(m))) = I^*(\eta \underset{tr}{\otimes} (1 \underset{Z(M)}{\otimes} m)\Xi)$$

Since $\eta \underset{tr}{\otimes} \Lambda_\nu(m) \in D((H\underset{tr}{\otimes}H)_{\hat\beta,\nu^o})$, the relation holds by continuity for all $\Xi \in H\underset{tr}{\otimes}H$. □

REMARK 14.2.3. – If σ_{tr} is the flip of $L^2(M)\underset{tr}{\otimes}L^2(M)$, then $\sigma_{tr} \circ \hat\beta = \beta \circ \sigma_{tr}$ and if $I' = (1 \underset{Z(M)}{\otimes} \sigma_{tr}) I (\sigma_{tr\,\hat\beta} \underset{\nu}{\otimes}_\alpha [1 \underset{Z(M)}{\otimes} 1])\sigma_{\nu^o}$, then I' is the identification:

$$\begin{aligned} I' : [L^2(M)\underset{tr}{\otimes}L^2(M)]_\alpha \underset{\nu^o}{\otimes}_{\hat\beta} [L^2(M)\underset{tr}{\otimes}L^2(M)] &\longrightarrow L^2(M)\underset{tr}{\otimes}L^2(M)\underset{tr}{\otimes}L^2(M) \\ \Xi_\beta \underset{\nu}{\otimes}_\alpha [\eta \underset{tr}{\otimes} \Lambda_\nu(y)] &\longmapsto \eta \underset{tr}{\otimes} \alpha(y)\Xi \end{aligned}$$

for all $\eta \in L^2(M), \Xi \in L^2(M)\underset{tr}{\otimes}L^2(M)$ and $y \in M$. Consequently, by the previous proposition $W^* = I^* I'$.

COROLLARY 14.2.4. – *We can reconstruct the von Neumann algebra thanks to W:*

$$M' \underset{Z(M)}{\otimes} M = \langle (\mathrm{id} * \omega_{\xi,\eta})(W^*) \,|\, \xi \in D((H \underset{tr}{\otimes} H)_{\hat{\beta}}, \nu^o), \eta \in D({}_{\alpha}(H \underset{tr}{\otimes} H), \nu) \rangle^{-\mathrm{w}}$$

Proof. – By 3.4.3, we know that:

$$\langle (\mathrm{id} * \omega_{\xi,\eta})(W^*) \,|\, \xi \in D((H \underset{tr}{\otimes} H)_{\hat{\beta}}, \nu^o, \eta \in D({}_{\alpha}(H \underset{tr}{\otimes} H), \nu) \rangle^{-\mathrm{w}} \subset M' \underset{Z(M)}{\otimes} M$$

Let $\eta, \xi \in H$ and $m, e \in \mathcal{N}_\nu$. Then, for all $\Xi_1, \Xi_2 \in H \underset{tr}{\otimes} H$, we have by 14.2.1:

$$\begin{aligned}
&((\mathrm{id} * \omega_{\eta \underset{tr}{\otimes} \Lambda_\nu(m), \xi \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)})(W^*)\Xi_1 | \Xi_2) \\
&= (W^*(\Xi_1 {}_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}} [\eta \underset{tr}{\otimes} \Lambda_\nu(m)]) | \Xi_2 {}_{\beta}\underset{\nu}{\otimes}{}_{\alpha} [\xi \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)]) \\
&= (I^*(\eta \underset{tr}{\otimes} (1 \underset{Z(M)}{\otimes} m)\Xi_1) | \Xi_2 {}_{\beta}\underset{\nu}{\otimes}{}_{\alpha} [\xi \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)]) \\
&= (\eta \underset{tr}{\otimes} (1 \underset{Z(M)}{\otimes} m)\Xi_1 | \xi \underset{tr}{\otimes} (J_\nu e J_\nu \underset{Z(M)}{\otimes} 1)\Xi_2) \\
&= (((\langle \eta, \xi \rangle_{tr} J_\nu e^* J_\nu \otimes m)\Xi_1 | \Xi_2)
\end{aligned}$$

Consequently, we get the reverse inclusion thanks to the relation:

$$(\mathrm{id} * \omega_{\eta \underset{tr}{\otimes} \Lambda_\nu(m), \xi \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)})(W^*) = \langle \eta, \xi \rangle_{tr} J_\nu e^* J_\nu \underset{Z(M)}{\otimes} m \quad \square$$

Now, we compute G so as to get the antipode.

PROPOSITION 14.2.5. – *If $F_\nu = S_\nu^*$ comes from Tomita's theory, then we have:*

$$G = \sigma_{tr} \circ (F_\nu \underset{tr}{\otimes} F_\nu)$$

Proof. – Let $a = J_\nu a_1 J_\nu \underset{Z(M)}{\otimes} a_2, b = J_\nu b_1 J_\nu \underset{Z(M)}{\otimes} b_2, c = J_\nu c_1 J_\nu \underset{Z(M)}{\otimes} c_2$ and $d = J_\nu d_1 J_\nu \underset{Z(M)}{\otimes} d_2$ be elements of $M' \underset{Z(M)}{\otimes} M$ analytic w.r.t. $\nu' \underset{Z(M)}{\star} \nu$. Then, by 14.2.1, the value of $(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1)) \underset{tr}{\otimes} \Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* W^*$ on

$$[\Lambda_{\nu'}(J_\nu a_1 J_\nu) \underset{tr}{\otimes} \Lambda_\nu(a_2)] {}_{\alpha}\underset{\nu^o}{\otimes}{}_{\hat{\beta}} [\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \underset{tr}{\otimes} \Lambda_\nu(d_2^* c_2^*)]$$

is equal to:

$$\begin{aligned}
&(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1)) \underset{tr}{\otimes} \Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* I^*(\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \underset{tr}{\otimes} \Lambda_{\nu'}(J_\nu a_1 J_\nu) \underset{tr}{\otimes} \Lambda_\nu(d_2^* c_2^* a_2)) \\
&= \left[\rho^{tr}_{\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))}(1 \underset{Z(M)}{\otimes} \sigma^\nu_{i/2}(b_1))\right]^* (\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \underset{tr}{\otimes} \Lambda_{\nu'}(J_\nu a_1 J_\nu) \underset{tr}{\otimes} \Lambda_\nu(d_2^* c_2^* a_2)) \\
&= \langle d_2^* c_2^* \Lambda_\nu(a_2), \Lambda_\nu(\sigma^\nu_{-i}(b_2^*)) \rangle_{tr}\, \Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \underset{tr}{\otimes} \sigma^\nu_{-i/2}(b_1^*) \Lambda_{\nu'}(J_\nu a_1 J_\nu) \\
&= \langle \Lambda_\nu(a_2 b_2), \Lambda_\nu(c_2 d_2) \rangle_{tr}\, J_\nu \Lambda_\nu(d_1^* c_1^*) \underset{tr}{\otimes} J_\nu \Lambda_\nu(a_1 b_1)
\end{aligned}$$

Consequently, by definition of G:

$$G\left[\langle \Lambda_\nu(a_2 b_2), \Lambda_\nu(c_2 d_2) \rangle_{tr}\, J_\nu \Lambda_\nu(d_1^* c_1^*) \underset{tr}{\otimes} J_\nu \Lambda_\nu(a_1 b_1)\right]$$

is equal to the value of $G(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1))\underset{tr}{\otimes}\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* W^*$ on:

$$[\Lambda_{\nu'}(J_\nu a_1 J_\nu)\underset{tr}{\otimes}\Lambda_\nu(a_2)]_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta}[\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu)\underset{tr}{\otimes}\Lambda_\nu(d_2^* c_2^*)]$$

which is equal to the value of $(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(d_1))\underset{tr}{\otimes}\Lambda_\nu(\sigma^\nu_{-i}(d_2^*))})^* W^*$ on:

$$[\Lambda_{\nu'}(J_\nu c_1 J_\nu)\underset{tr}{\otimes}\Lambda_\nu(c_2)]_\alpha \underset{\nu^o}{\otimes}{}_{\hat\beta}[\Lambda_{\nu'}(J_\nu b_1^* a_1^* J_\nu)\underset{tr}{\otimes}\Lambda_\nu(b_2^* a_2^*)]$$

This last vector is $\langle \Lambda_\nu(c_2 d_2), \Lambda_\nu(a_2 b_2)\rangle_{tr}\, J_\nu\Lambda_\nu(b_1^* a_1^*)\underset{tr}{\otimes} J_\nu\Lambda_\nu(c_1 d_1)$. Since G is closed, we get:

$$G\left[J_\nu\Lambda_\nu(d_1^* c_1^*)\underset{tr}{\otimes} J_\nu\Lambda_\nu(a_1 b_1)\right] = \left[J_\nu\Lambda_\nu(b_1^* a_1^*)\underset{tr}{\otimes} J_\nu\Lambda_\nu(c_1 d_1)\right]$$

so that G coincides with $\sigma_{tr}(F_\nu\underset{tr}{\otimes}F_\nu)$. □

The polar decomposition of $G = ID^{1/2}$ is such that $D = \Delta_\nu^{-1}\underset{tr}{\otimes}\Delta_\nu^{-1}$ and $I = \sigma_{tr}(J_\nu\underset{tr}{\otimes}J_\nu)$ so that the scaling group is $\tau_t = \sigma^{\nu'}_{-t}\underset{Z(M)}{\star}\sigma^\nu_t$ for all $t \in \mathbb{R}$ and the unitary antipode is $R = \varsigma_{Z(M)} \circ (j \underset{Z(M)}{\otimes} j)$. We also notice that $\nu' \underset{Z(M)}{\star} \nu$ is τ-invariant.

Remark 14.2.6. – If M is the commutative von Neumann algebra $L^\infty(X)$, then the structure coincides with the quantum space X.

14.3. Dual structure

Here we compute the dual structure and we observe that this is not of adapted measured quantum groupoid's type.

Proposition 14.3.1. – *For all $e, y \in \mathcal{N}_\nu$ and $\eta, \zeta \in H$, we have:*

$$(\omega_{\Lambda_\nu(y)\underset{tr}{\otimes}\eta,\zeta\underset{tr}{\otimes}J_\nu\Lambda_\nu(e)} * \mathrm{id})(W) = 1\underset{Z(M)}{\otimes} J_\nu e^* J_\nu (\rho^{tr}_\zeta)^* \sigma_{tr}\rho^{tr}_\eta y$$

Proof. – For all $\Xi \in H\underset{tr}{\otimes}H, \xi \in H$ and $m \in \mathcal{N}_\nu$, we have:

$$\begin{aligned}
&((\omega_{\Lambda_\nu(y)\underset{tr}{\otimes}\eta,\zeta\underset{tr}{\otimes}J_\nu\Lambda_\nu(e)} * \mathrm{id})(W)\Xi|\xi\underset{tr}{\otimes}\Lambda_\nu(m))\\
&= ([\Lambda_\nu(y)\underset{tr}{\otimes}\eta]_\beta\underset{\nu}{\otimes}{}_\alpha\Xi|W^*([\zeta\underset{tr}{\otimes}J_\nu\Lambda_\nu(e)]_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}[\xi\underset{tr}{\otimes}\Lambda_\nu(m)]))\\
&= ((1\underset{Z(M)}{\otimes}y)\Xi\underset{tr}{\otimes}\eta|\xi\underset{tr}{\otimes}\zeta\underset{tr}{\otimes}mJ_\nu\Lambda_\nu(e)) = (\Xi\underset{tr}{\otimes}\eta|\xi\underset{tr}{\otimes}y^*\zeta\underset{tr}{\otimes}J_\nu e J_\nu\Lambda_\nu(m))\\
&= ((1\underset{Z(M)}{\otimes}\rho^{tr}_\eta)\Xi|(1\underset{Z(M)}{\otimes}\sigma_{tr}\rho^{tr}_{y^*\zeta}J_\nu e J_\nu)(\xi\underset{tr}{\otimes}\Lambda_\nu(m)))\\
&= ((1\underset{Z(M)}{\otimes}J_\nu e^* J_\nu(\rho^{tr}_\zeta)^*\sigma_{tr}\rho^{tr}_\eta y)\Xi|\xi\underset{tr}{\otimes}\Lambda_\nu(m))
\end{aligned}$$

□

Corollary 14.3.2. – *We have $\widehat{M'\underset{Z(M)}{\otimes}M} = 1\underset{Z(M)}{\otimes}Z(M)'$ which is identified with $Z(M)'$.*

Proof. – We already know that $\alpha(M) \cup \hat{\beta}(M) \subset \widehat{M' \underset{Z(M)}{\otimes} M}$ so that $1 \underset{Z(M)}{\otimes} Z(M)' \subset \widehat{M' \underset{Z(M)}{\otimes} M}$. The reverse inclusion comes from the previous proposition. □

With this identification between $1 \underset{Z(M)}{\otimes} Z(M)'$ and $Z(M)'$, the dual structure admits M for basis, id for representation and j for anti-representation. The dual co-product necessarily satisfies $\widehat{\Gamma}(mn) = m_j \underset{\nu}{\otimes}_{\mathrm{id}} n$ for all $m \in M$ and $n \in M'$. If I_ν is the canonical isomorphism from $L^2(M)_j \underset{\nu}{\otimes}_{\mathrm{id}} L^2(M)$ onto $L^2(M)$ given by $I_\nu(\Lambda_\nu(x)_j \underset{\nu}{\otimes}_{\mathrm{id}} \eta) = \alpha(x)\eta$ for all $x \in \mathcal{N}_\nu$ and $\eta \in L^2(M)$, then we have $I_\nu(m_\beta \underset{\nu}{\otimes}_\alpha n) = mnI_\nu$ and we can identify the von Neumann algebra $M'{}_\beta \underset{M}{\star}{}_\alpha M$ with $Z(M)$ and the von Neumann algebra $Z(M)'{}_\beta \underset{M}{\star}{}_\alpha Z(M)'$ with $Z(M)'$. The dual co-product is then identity through this identification.

Lemma 14.3.3. – *We have* $\hat{\Lambda}((\omega_{\Xi, \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} * \mathrm{id})(W)) = (m^* \underset{Z(M)}{\otimes} J_\nu e^* J_\nu)\Xi$ *for all* $m, e \in \mathcal{N}_\nu$ *et* $\Xi \in D((H \underset{tr}{\otimes} H)_\beta, \nu^o)$.

Proof. – Let $m_1, m_2 \in \mathcal{N}_\nu$. Then, we have:

$$
\begin{aligned}
&(\hat{\Lambda}((\omega_{\Xi, \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} * \mathrm{id})(W)) | \Lambda_{\nu'}(J_\nu m_1 J_\nu) \underset{Z(M)}{\otimes} \Lambda_\nu(m_2))\\
&= \omega_{\Xi, \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)}(J_\nu m_1^* J_\nu \underset{Z(M)}{\otimes} m_2^*)\\
&= ((J_\nu m_1^* J_\nu \underset{Z(M)}{\otimes} m_2^*)\Xi | \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e))\\
&= (\Xi | m J_\nu \Lambda_\nu(m_1) \underset{tr}{\otimes} J_\nu e J_\nu \Lambda_\nu(m_2))\\
&= ((m^* \underset{Z(M)}{\otimes} J_\nu e^* J_\nu)\Xi | \Lambda_{\nu'}(J_\nu m_1 J_\nu) \underset{tr}{\otimes} \Lambda_\nu(m_2))
\end{aligned}
$$
□

Proposition 14.3.4. – *The dual operator-valued weight* $\widehat{T_R}$ *coincide with* T^{-1} *in sense of proposition 12.11 of* **[Str81]**. *Also, the dual operator-valued weight* $\widehat{T_L}$ *coincide with* $j \circ T^{-1} \circ j$.

Proof. – Via the identification between $\widehat{M' \underset{Z(M)}{\otimes} M}$ and $Z(M)'$, we have, by proposition 14.3.1:

$$(\omega_{\Xi, \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} * \mathrm{id})(W) = J_\nu e^* J_\nu [(\rho_\zeta^{tr})^* \sigma_{tr} \rho_\eta^{tr}] y$$

Let $m, e, y \in \mathcal{N}_\nu$ and $\eta \in H$. On one hand, we compute:

$$
\begin{aligned}
&||\hat{\Lambda}((\omega_{\Lambda_\nu(y) \underset{tr}{\otimes} \eta, \Lambda_\nu(m) \underset{tr}{\otimes} J_\nu \Lambda_\nu(e)} * \mathrm{id})(W))||^2 = ||m^* \Lambda_\nu(y) \underset{tr}{\otimes} J_\nu e^* J_\nu \eta||^2\\
&= (\langle J_\nu e^* J_\nu \eta, J_\nu e^* J_\nu \eta \rangle_{tr} \Lambda_\nu(m^* y) | \Lambda_\nu(m^* y))
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
& ||\hat{\Lambda}((\omega_{\Lambda_\nu(y)\underset{tr}{\otimes}\eta,\Lambda_\nu(m\underset{tr}{\otimes}J_\nu\Lambda_\nu(e)} * \mathrm{id})(W)))||^2 \\
&= \hat{\Phi}((\rho^{tr}_{J_\nu e^* J_\nu \eta})^* \sigma_{tr} \rho^{tr}_{\Lambda_\nu(y^*m)} (\rho^{tr}_{\Lambda_\nu(y^*m)})^* \sigma_{tr} \rho^{tr}_{J_\nu e^* J_\nu \eta}) \\
&= \hat{\Phi}((\rho^{tr}_{J_\nu e^* J_\nu \eta})^* [\theta^{tr}(\Lambda_\nu(y^*m), \Lambda_\nu(y^*m)) \underset{Z(M)}{\otimes} 1] \rho^{tr}_{J_\nu e^* J_\nu \eta}) \\
&= \hat{\Phi}(\langle J_\nu e^* J_\nu \eta, J_\nu e^* J_\nu \eta \rangle_{tr} \theta^{tr}(\Lambda_\nu(y^*m), \Lambda_\nu(y^*m)))
\end{aligned}
$$

Then we conclude that, for all $m, y \in \mathcal{N}_\nu$, we have:

$$
\begin{aligned}
& \hat{\Phi}(\theta^{tr}(\Lambda_\nu(y^*m), \Lambda_\nu(y^*m))) = ||\Lambda_\nu(m^*y)||^2 = ||\Delta_\nu^{-1/2} J_\nu \Lambda_\nu(y^*m)||^2 \\
&= \nu'(\theta^\nu(J_\nu\Lambda_\nu(y^*m), J_\nu\Lambda_\nu(y^*m))) = \nu' \circ T^{-1}(\theta^{tr}(J_\nu\Lambda_\nu(y^*m), J_\nu\Lambda_\nu(y^*m))) \\
&= \nu \circ j \circ T^{-1} \circ j(\theta^{tr}(\Lambda_\nu(y^*m)), \Lambda_\nu(y^*m)))
\end{aligned}
$$

Therefore $\widehat{T_L} = j \circ T^{-1} \circ j$ and we get the proposition. □

PROPOSITION 14.3.5. – *The dual quantum space quantum groupoid can be identify with $(M, Z(M)', \mathrm{id}, j, \nu, \mathrm{id}, j \circ T^{-1} \circ j, T^{-1})$ which is a measured quantum groupoid but not an adapted measured quantum groupoid. Moreover, expressions for co-involution and scaling group are given, for all $x \in Z(M)'$ and $t \in \mathbb{R}$:*

$$\hat{R}(x) = J_\nu x^* J_\nu \quad \text{and} \quad \hat{\tau}_t(x) = \Delta_\nu^{it} x \Delta_\nu^{-it}$$

Proof. – The proposition gathers results of the section. Nevertheless we lay stress on the following point. We have, for all $t \in \mathbb{R}$ and $m \in M$:

$$\sigma_t^{T^{-1}}(m) = \sigma_{-t}^{T}(m) = \sigma_{-t}^{\nu}(m)$$

instead of $\sigma_t^\nu(m)$ to have an adapted measured quantum groupoid. □

REMARK 14.3.6. – If M is a factor, $\mathcal{L}(H)$ is the von Neumann algebra underlying the structure of quantum space quantum groupoid whereas $M' \otimes M$ is the underlying von Neumann algebra of the dual structure. In general, they are not isomorphic. Nevertheless, if M is abelian or if M is a type I factor (and henceforth a sum of type I factors cf. paragraph 17.1), the structure is self-dual. In the abelian case $M = L^\infty(X)$, we recover the space groupoid X. This example comes from the inclusion of von Neumann algebras [**Eno00**]:

$$Z(M) \subset M \subset Z(M)' \subset \cdots$$

CHAPTER 15

PAIRS QUANTUM GROUPOID

15.1. Definition

Let M be a von Neumann algebra. M acts on $H = L^2(M) = L^2_\nu(M)$ where ν is a n.s.f. weight on M. We denote by M' the commutant of M in $\mathcal{L}(L^2(M))$. $M' \otimes M$ acts on $L^2(M) \otimes L^2(M)$.

Let α (resp. β) be the (resp. anti-) representation of M to $M' \otimes M$ such that $\alpha(m) = 1 \otimes m$ (resp. $\beta(m) = j(m) \otimes 1$) where $j(x) = J_\nu x^* J_\nu$ for all $x \in \mathcal{L}(L^2_\nu(M))$.

Proposition 15.1.1. – *The following formula:*

$$I : [L^2(M) \otimes L^2(M)]_\beta \underset{\nu}{\otimes}_\alpha [L^2(M) \otimes L^2(M)] \longrightarrow L^2(M) \otimes L^2(M) \otimes L^2(M)$$
$$[\Lambda_\nu(y) \otimes \eta]_\beta \underset{\nu}{\otimes}_\alpha \Xi \longmapsto \alpha(y)\Xi \otimes \eta$$

for all $\eta \in L^2(M), \Xi \in L^2(M) \otimes L^2(M)$ *and* $y \in M$*, defines a canonical isomorphism such that we have* $I([m \otimes x]_\beta \underset{\nu}{\otimes}_\alpha [y \otimes n]) = (y \otimes mn \otimes x)I$*, for all* $m \in M, n \in M'$ *and* $x, y \in \mathcal{L}(L^2(M))$.

Proof. – Straightforward. □

Then, we can identify $(M' \otimes M)_\beta \underset{M}{\star}_\alpha (M' \otimes M)$ with $M' \otimes Z(M) \otimes M$. We define a normal *-homomorphism Γ by:

$$M' \otimes M \longrightarrow (M' \otimes M)_\beta \underset{\nu}{\star}_\alpha (M' \otimes M)$$
$$n \otimes m \longmapsto I^*(n \otimes 1 \otimes m)I = [1 \otimes m]_\beta \underset{\nu}{\otimes}_\alpha [n \otimes 1]$$

Theorem 15.1.2. – $(M, M' \underset{Z(M)}{\otimes} M, \alpha, \beta, \Gamma, \nu, \nu' \otimes \mathrm{id}, id \otimes \nu)$ *is an adapted measured quantum groupoid w.r.t.* ν *called* ***pairs quantum groupoid****.*

Proof. – By definition, Γ is a morphism of Hopf bimodule. We have to prove co-product relation. For all $m \in M$ and $n \in M'$, we have:

$$\begin{aligned}(\Gamma_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id}) \circ \Gamma(n \otimes m) &= [1 \otimes m]_{\beta}\underset{\nu}{\otimes}{}_{\alpha}[1 \otimes 1]_{\beta}\underset{\nu}{\otimes}{}_{\alpha}[n \otimes 1] \\ &= (\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\Gamma) \circ \Gamma(n \otimes m)\end{aligned}$$

$R = \varsigma \circ (\beta_\nu \otimes \beta_\nu)$, where $\varsigma : M' \otimes M \to M \otimes M'$ is the flip, is a co-involution so it is sufficient to show that $T_L = \nu' \otimes \mathrm{id}$ is left invariant and β-adapted w.r.t. ν. Let

$m \in M, n \in M'$ and $\xi \in D((L^2(M) \otimes L^2(M))_{\beta,\nu^o})$. We put $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and we compute:

$$\begin{aligned}\omega_\xi((\mathrm{id}_{\beta}\underset{\nu}{\star}{}_{\alpha}\Phi)\Gamma(n \otimes m)) &= \Phi((\omega_{\xi}{}_{\beta}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})([1 \otimes m]_{\beta}\underset{\nu}{\otimes}{}_{\alpha}[n \otimes 1])) \\ &= \Phi([n \otimes 1]\alpha(\langle [1 \otimes m]\xi, \xi\rangle_{\beta,\nu^o})) \\ &= \nu'(n)\nu(\langle [1 \otimes m]\xi, \xi\rangle_{\beta,\nu^o}) \\ &= \nu'(n)\omega_\xi(1 \otimes m) = \omega_\xi(T_L(n \otimes m))\end{aligned}$$

Finally, we prove that $T_R = R \circ T_L \circ R = \mathrm{id} \otimes \nu$ is α-adapted w.r.t. ν. For all $t \in \mathbb{R}$, we have:

$$\sigma_t^{T_R} = \sigma_t^{\nu' \otimes \nu}{}_{|(M' \otimes M) \cap \beta(M)'} = \sigma_t^{\nu' \otimes \nu}{}_{|Z(M) \otimes M} = \mathrm{id} \otimes \sigma_t^{\nu}{}_{|Z(M) \otimes M}$$

so that we have for all $t \in \mathbb{R}$ and $m \in M$:

$$\sigma_t^{T_R} \circ \alpha(m) = 1 \otimes \sigma_t^\nu(m) = \alpha(\sigma_t^\nu(m)) \quad \square$$

REMARK 15.1.3. – If $M = L^\infty(X)$, we find the structure of pairs groupoid $X \times X$.

15.2. Fundamental elements

By 3.3.1, we can compute the pseudo-multiplicative unitary. Let first notice that $\Phi = \nu' \otimes \nu = \Psi$ so that $\lambda = \delta = 1$ and:

$$\alpha = 1 \otimes \mathrm{id}, \hat{\alpha} = \mathrm{id} \otimes 1, \beta = \beta_\nu \otimes 1 \text{ and } \hat{\beta} = 1 \otimes \beta_\nu$$

For example, we have $D((H \otimes H)_{\hat{\beta},\nu^o}) \supset H \otimes D(H_{\beta_\nu}, \nu^o) = H \otimes \Lambda_\nu(\mathscr{N}_\nu)$ and for all $\eta \in H$ and $y \in \mathscr{N}_\nu$, we have $R^{\hat{\beta},\nu^o}(\eta \otimes \Lambda_\nu(y)) = \lambda_\eta R^{\beta_\nu,\nu^o}(\Lambda_\nu(y)) = \lambda_\eta y$.

LEMMA 15.2.1. – *We have, for all $\eta \in H$ and $e \in \mathscr{N}_\nu$:*

$$I\rho^{\beta,\alpha}_{\eta \otimes J_\nu \Lambda_\nu(e)} = \lambda_\eta J_\nu e J_\nu \otimes 1 \textit{ and } I\lambda^{\beta,\alpha}_{\Lambda_\nu(y) \otimes \eta} = \rho_\eta(1 \otimes y)$$

Proof. – Straightforward. □

PROPOSITION 15.2.2. – *We have, for all* $\Xi \in H \otimes H, \eta \in H$ *and* $m \in \mathcal{N}_\nu$*:*

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat{\beta}} (\eta \otimes \Lambda_\nu(m))) = I^*(\eta \otimes (1 \otimes m)\Xi)$$

Proof. – For all $m, e \in \mathcal{N}_\nu$ and $m', e' \in \mathcal{N}_{\nu'}$, we have by the previous lemma:

$$\begin{aligned} I\Gamma(m' \otimes m)\rho^{\beta,\alpha}_{J_{\nu'}\Lambda_{\nu'}(e')\otimes J_\nu \Lambda_\nu(e)} &= (m' \otimes 1 \otimes m) I \rho^{\beta,\alpha}_{J_{\nu'}\Lambda_{\nu'}(e')\otimes J_\nu\Lambda_\nu(e)} \\ &= (m' \otimes 1 \otimes m)\lambda_{J_{\nu'}\Lambda_{\nu'}(e')} J_\nu e J_\nu \otimes 1 \\ &= \lambda_{J_{\nu'}e'J_{\nu'}\Lambda_{\nu'}(m')} J_\nu e J_\nu \otimes m \end{aligned}$$

On the other hand, we have by 15.1.1:

$$\begin{aligned} &I([1 \otimes 1]_\beta \underset{\nu}{\otimes}_\alpha [J_{\nu'}e'J_{\nu'} \otimes J_\nu e J_\nu]) W^* \rho^{\alpha,\hat{\beta}}_{\Lambda_{\nu'}(m')\otimes\Lambda_{\nu'}(m')} \\ &= (J_{\nu'}e'J_{\nu'} \otimes J_\nu e J_\nu \otimes 1) I W^* \rho^{\alpha,\hat{\beta}}_{\Lambda_{\nu'}(m')\otimes\Lambda_{\nu'}(m')} \end{aligned}$$

Then by 3.3.1 and taking the limit over e and e' which go to 1, we get for all $\Xi \in H \otimes H$:

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat{\beta}} (\Lambda_{\nu'}(m') \otimes \Lambda_\nu(m))) = I^*(\Lambda_{\nu'}(m') \otimes (1 \otimes m)\Xi)$$

Now, if $\Xi \in D(_\alpha(H \otimes H), \nu)$, by continuity and density of $\Lambda_{\nu'}(\mathcal{N}_{\nu'})$, we have for all $\Xi \in D(_\alpha(H \otimes H), \nu)$:

$$W^*(\Xi_\alpha \underset{\nu^o}{\otimes}_{\hat{\beta}} (\eta \otimes \Lambda_\nu(m))) = I^*(\eta \otimes (1 \otimes m)\Xi)$$

Since $\eta \otimes \Lambda_\nu(m) \in D((H \otimes H)_{\hat{\beta},\nu^o})$, the previous relation holds by continuity for all $\Xi \in H \otimes H$. □

REMARK 15.2.3. – If σ denotes the flip of $L^2(M) \otimes L^2(M)$, then $\sigma \circ \hat{\beta} = \beta \circ \sigma$ and if $I' = (1 \otimes \sigma) I (\sigma_{\hat{\beta}} \underset{\nu}{\otimes}_\alpha [1 \otimes 1])\sigma_{\nu^o}$, then I' is the identification:

$$\begin{aligned} I' : [L^2(M) \otimes L^2(M)]_\alpha \underset{\nu^o}{\otimes}_{\hat{\beta}} [L^2(M) \otimes L^2(M)] &\longrightarrow L^2(M) \otimes L^2(M) \otimes L^2(M) \\ \Xi_\beta \underset{\nu}{\otimes}_\alpha [\eta \otimes \Lambda_\nu(y)] &\longmapsto \eta \otimes \alpha(y)\Xi \end{aligned}$$

for all $\eta \in L^2(M), \Xi \in L^2(M) \otimes L^2(M)$ and $y \in M$. Consequently, by the previous proposition $W^* = I^* I'$.

COROLLARY 15.2.4. – *We can re-construct the underlying von Neumann algebra thanks to* W*:*

$$M' \otimes M = \langle (\mathrm{id} * \omega_{\xi,\eta})(W^*) \,|\, \xi \in D((H \otimes H)_{\hat{\beta}}, \nu^o), \eta \in D(_\alpha(H \otimes H), \nu) \rangle^{-\mathrm{w}}$$

Proof. – By 3.4.3, we know that:

$$\langle (\mathrm{id} * \omega_{\xi,\eta})(W^*) \,|\, \xi \in D((H \otimes H)_{\hat{\beta}}, \nu^o, \eta \in D(_\alpha(H \otimes H), \nu) \rangle^{-w} \subset M' \otimes M$$

Let $\eta, \xi \in H$ and $m, e \in \mathcal{N}_\nu$. Then, for all $\Xi_1, \Xi_2 \in H \otimes H$, we have, by 15.2.1:

$$\begin{aligned}
&((\mathrm{id} * \omega_{\eta\otimes\Lambda_\nu(m),\xi\otimes J_\nu\Lambda_\nu(e)})(W^*)\Xi_1|\Xi_2) \\
&= (W^*(\Xi_1 {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} \eta \otimes \Lambda_\nu(m))|\Xi_2 {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi \otimes J_\nu\Lambda_\nu(e)) \\
&= (I^*(\eta \otimes (1\otimes m)\Xi_1)|\Xi_2 {}_\beta\underset{\nu}{\otimes}{}_\alpha \xi \otimes J_\nu\Lambda_\nu(e)) \\
&= (\eta \otimes (1\otimes m)\Xi_1|\xi \otimes (J_\nu e J_\nu \otimes 1)\Xi_2) \\
&= (\eta|\xi)((J_\nu e^* J_\nu \otimes m)\Xi_1|\Xi_2)
\end{aligned}$$

Consequently, we get the reverse inclusion thanks to the relation:

(13) $$(\mathrm{id} * \omega_{\eta\otimes\Lambda_\nu(m),\xi\otimes J_\nu\Lambda_\nu(e)})(W^*) = (\eta|\xi)(J_\nu e^* J_\nu \otimes m)$$ □

Now, we compute G so as to get the antipode.

Proposition 15.2.5. – *If $F_\nu = S_\nu^*$ comes from Tomita's theory, we have:*

$$G = \sigma(F_\nu \otimes F_\nu)$$

Proof. – For all $a = J_\nu a_1 J_\nu \otimes a_2, b = J_\nu b_1 J_\nu \otimes b_2, c = J_\nu c_1 J_\nu \otimes c_2$ and $d = J_\nu d_1 J_\nu \otimes d_2$ be analytic elements of $M' \otimes M$ w.r.t. $\nu' \otimes \nu$. Then, by 15.2.1, we have:

$$\begin{aligned}
&(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1))\otimes\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* W^*(\Lambda_{\nu'\otimes\nu}(a) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_{\nu'\otimes\nu}((J_\nu d_1^* J_\nu \otimes d_2^*)c^*)) \\
&= (\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1))\otimes\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* I^*(\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes (1\otimes d_2^* c_2^*)\Lambda_{\nu'\otimes\nu}(a)) \\
&= \left[\rho_{\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))}(1\otimes\sigma^\nu_{i/2}(b_1))\right]^* (\Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes (1\otimes d_2^* c_2^*)\Lambda_{\nu'\otimes\nu}(a)) \\
&= (d_2^* c_2^* \Lambda_\nu(a_2)|\Lambda_\nu(\sigma^\nu_{-i}(b_2^*)))\, \Lambda_{\nu'}(J_\nu d_1^* c_1^* J_\nu) \otimes \sigma^\nu_{-i/2}(b_1^*)\Lambda_{\nu'}(J_\nu a_1 J_\nu) \\
&= \nu(d_2^* c_2^* a_2 b_2)\, J_\nu\Lambda_\nu(d_1^* c_1^*) \otimes J_\nu\Lambda_\nu(a_1 b_1)
\end{aligned}$$

Consequently, by definition of G, we have:

$$\begin{aligned}
&G\left[\nu(d_2^* c_2^* a_2 b_2)\, J_\nu\Lambda_\nu(d_1^* c_1^*) \otimes J_\nu\Lambda_\nu(a_1 b_1)\right] \\
&= G(\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(b_1))\otimes\Lambda_\nu(\sigma^\nu_{-i}(b_2^*))})^* W^*(\Lambda_{\nu'\otimes\nu}(a) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_{\nu'\otimes\nu}((J_\nu d_1^* J_\nu \otimes d_2^*)c^*)) \\
&= (\lambda^{\beta,\alpha}_{\Lambda_\nu(\sigma^\nu_{i/2}(d_1))\otimes\Lambda_\nu(\sigma^\nu_{-i}(d_2^*))})^* W^*(\Lambda_{\nu'\otimes\nu}(c) {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta} \Lambda_{\nu'\otimes\nu}((J_\nu b_1^* J_\nu \otimes b_2^*)a^*)) \\
&= \nu(b_2^* a_2^* c_2 d_2)\, J_\nu\Lambda_\nu(b_1^* a_1^*) \otimes J_\nu\Lambda_\nu(c_1 d_1)
\end{aligned}$$

Since G is anti-linear, we get:

$$G\left[J_\nu\Lambda_\nu(d_1^* c_1^*) \otimes J_\nu\Lambda_\nu(a_1 b_1)\right] = \left[J_\nu\Lambda_\nu(b_1^* a_1^*) \otimes J_\nu\Lambda_\nu(c_1 d_1)\right]$$

so that G coincides with $\sigma(F_\nu \otimes F_\nu)$. □

The polar decomposition of $G = ID^{1/2}$ is such that $D = \Delta_\nu^{-1} \otimes \Delta_\nu^{-1}$ and $I = \Sigma(J_\nu \otimes J_\nu)$ so that the scaling group is $\tau_t = \sigma^{\nu'}_{-t} \otimes \sigma^\nu_t$ for all $t \in \mathbb{R}$ and the unitary antipode is $R = \varsigma \circ (\beta_\nu \otimes \beta_\nu)$. We also notice that $\nu' \otimes \nu$ is τ-invariant.

Corollary 15.2.6. – *We have* $\mathcal{D}(S)=\mathcal{D}(\sigma^{\nu'}_{i/2})\otimes\mathcal{D}(\sigma^{\nu}_{-i/2})$ *and we have*

$$S(J_\nu eJ_\nu\otimes m^*)=J_\nu\sigma^\nu_{i/2}(m)J_\nu\otimes\sigma^\nu_{-i/2}(e^*)$$

for all $e,m\in\mathcal{D}(\sigma^\nu_{i/2})$. *Moreover* $(\mathrm{id}*\omega_{\xi,\eta})(W)\in\mathcal{D}(S)$ *and:*

$$S((\mathrm{id}*\omega_{\xi,\eta})(W))=(\mathrm{id}*\omega_{\xi,\eta})(W^*)$$

for all $\xi,\eta\in D({}_\alpha(H\otimes H),\nu)\cap D((H\otimes H)_{\hat\beta},\nu^o)$.

Proof. – The first part of the corollary is straightforward by what precedes. Let $\zeta,\eta\in H$ and $e,m\in\mathcal{D}(\sigma^\nu_{i/2})$. By 13, we have:

$$\begin{aligned}S((\mathrm{id}*\omega_{\zeta\otimes J_\nu\Lambda_\nu(e),\eta\otimes\Lambda_\nu(m)})(W))&=S((\zeta|\eta)J_\nu eJ_\nu\otimes m^*)\\&=(\zeta|\eta)J\sigma^\nu_{i/2}(m)J\otimes\sigma^\nu_{-i/2}(e^*)\\&=(\mathrm{id}*\omega_{\zeta\otimes J_\nu\Lambda_\nu(e),\eta\otimes\Lambda_\nu(m)})(W^*)\end{aligned}$$

Since S is closed, we can conclude. □

15.3. Dual structure

We are now computing the dual structure.

Proposition 15.3.1. – *For all* $e,y\in\mathcal{N}_\nu$ *and* $\eta,\zeta\in H$, *we have:*

$$(\omega_{\Lambda_\nu(y)\otimes\eta,\zeta\otimes J_\nu\Lambda_\nu(e)}*\mathrm{id})(W)=1\otimes J_\nu e^*J_\nu\rho^*_\zeta\Sigma\rho_\eta y$$

Proof. – For all $\Xi\in H\otimes H,\xi\in H$ and $m\in\mathcal{N}_\nu$, we have:

$$\begin{aligned}&((\omega_{\Lambda_\nu(y)\otimes\eta,\zeta\otimes J_\nu\Lambda_\nu(e)}*\mathrm{id})(W)\Xi|\xi\otimes\Lambda_\nu(m))\\&=([\Lambda_\nu(y)\otimes\eta]_\beta\underset{\nu}{\otimes}{}_\alpha\Xi|W^*([\zeta\otimes J_\nu\Lambda_\nu(e)]_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}[\xi\otimes\Lambda_\nu(m)]))\\&=((1\otimes y)\Xi\otimes\eta|\xi\otimes\zeta\otimes mJ_\nu\Lambda_\nu(e))=(\Xi\otimes\eta|\xi\otimes y^*\zeta\otimes J_\nu eJ_\nu\Lambda_\nu(m))\\&=((1\otimes\rho_\eta)\Xi|(1\otimes\Sigma\rho_{y^*\zeta}J_\nu eJ_\nu)(\xi\otimes\Lambda_\nu(m)))\\&=((1\otimes J_\nu e^*J_\nu\rho^*_\zeta\Sigma\rho_\eta y)\Xi|\xi\otimes\Lambda_\nu(m))\end{aligned}$$

□

Corollary 15.3.2. – *We have* $\widehat{M'\otimes M}=1\otimes\mathcal{L}(H)$.

Proof. – By definition, we recall that:

$$\widehat{M'\otimes M}=\langle(\omega_{\Xi_1,\Xi_2}*\mathrm{id})(W)\,|\,\Xi_1\in D((H\otimes H)_\beta,\nu^o),\Xi_2\in D({}_\alpha(H\otimes H),\nu)\rangle^{-w}$$

and we notice that $\mathcal{L}(H)\otimes 1 \subset \alpha(M)'\cap\beta(M)'$. First of all, we prove that $\widehat{M'\otimes M} \subset 1\otimes\mathcal{L}(H)$. Let $\Xi\in H, \eta\in H$ and $m\in\mathcal{N}_\nu$. For all $x\in\mathcal{L}(H)$, we have:

$$\begin{aligned}
&([1\otimes 1]_\beta\underset{\nu}{\otimes}_\alpha[x\otimes 1])W^*(\Xi_\alpha\underset{\nu}{\otimes}_{\hat\beta}[\eta\otimes J_\nu\Lambda_\nu(m)])\\
&= ([1\otimes 1]_\beta\underset{\nu}{\otimes}_\alpha[x\otimes 1])I^*(\eta\otimes(1\otimes m)\Xi) = I^*(x\eta\otimes(1\otimes m)\Xi)\\
&= W^*(\Xi_\alpha\underset{\nu}{\otimes}_{\hat\beta}[x\eta\otimes J_\nu\Lambda_\nu(m)])\\
&= W^*([1\otimes 1]_\beta\underset{\nu}{\otimes}_\alpha[x\otimes 1])(\Xi_\alpha\underset{\nu}{\otimes}_{\hat\beta}[\eta\otimes J_\nu\Lambda_\nu(m)])
\end{aligned}$$

Therefore, we get that $([1\otimes 1]_\beta\underset{\nu}{\otimes}_\alpha[x\otimes 1])W^* = W^*([1\otimes 1]_\beta\underset{\nu}{\otimes}_\alpha[x\otimes 1])$ for all $x\in\mathcal{L}(H)$. Thus, we get:

$$\widehat{M'\otimes M} \subset (\mathcal{L}(H)\otimes 1)' = 1\otimes\mathcal{L}(H)$$

Then, we prove the reverse inclusion. By the previous proposition, we state that, for all $v,w\in H$:

$$\begin{aligned}
(\omega_{\Lambda_\nu(y)\otimes\eta,\zeta\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W)(v\otimes w) &= (1\otimes J_\nu e^* J_\nu)\rho^*_{y^*\zeta}(1\otimes\Sigma)\rho_\eta(v\otimes w)\\
&= v\otimes(w|y^*\zeta)J_\nu e^* J_\nu\eta
\end{aligned}$$

and therefore we have:

$$\begin{aligned}
\widehat{M'\otimes M} &\supset \langle\omega_{\Lambda_\nu(y)\otimes\eta,\zeta\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W)\,|\,\eta,\zeta\in H, e,y\in\mathcal{N}_\nu\rangle^{-w}\\
&= \langle 1\otimes p\,|\,p \text{ rank 1 projection }\rangle^{-w} = 1\otimes\mathcal{L}(H)
\end{aligned}$$ □

We verify that $(M'\otimes M)\cap\widehat{M'\otimes M} = 1\otimes M = \alpha(M)$. The dual co-product is given by $\hat\Gamma(\widehat{x}) = \sigma_{\nu^o}W(\widehat{x}_\beta\underset{\nu}{\otimes}_\alpha 1)W^*\sigma_\nu$ for all $\widehat{x}\in\widehat{M'\otimes M}$. We us the following identification:

$$\begin{aligned}
\Phi : 1\otimes\mathcal{L}(H) &\longrightarrow \mathcal{L}(H)\\
1\otimes x &\longmapsto x
\end{aligned}$$

which is implemented by λ_e where $e\in H$ is a normalized vector. Then, we know that $\Phi_{\hat\beta}\underset{\nu}{\star}_\alpha\Phi$ is the identification between $[1\otimes\mathcal{L}(H)]_{\hat\beta}\underset{\nu}{\otimes}_\alpha[1\otimes\mathcal{L}(H)]$ and $\mathcal{L}(H)_{\beta_\nu}\underset{\nu}{\star}_{\mathrm{id}}\mathcal{L}(H) \simeq \mathcal{L}(H)$.

Proposition 15.3.3. – *We have $W^*\sigma_\nu(\lambda_{e\,\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\lambda_e) = I^*(\lambda_e\otimes\lambda_e)I_\nu$ for all vector $e\in H$ of norm 1.*

Proof. – Let $m\in\mathcal{N}_\nu$ and $\eta\in H$. We have:

$$\begin{aligned}
&W^*\sigma_\nu(\lambda_{e\,\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\lambda_e)(\Lambda_\nu(m)_{\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\eta) = W^*\sigma_\nu([e\otimes\Lambda_\nu(m)]_{\hat\beta}\underset{\nu}{\otimes}_\alpha[e\otimes\eta])\\
&= W^*([e\otimes\eta]_\alpha\underset{\nu^o}{\otimes}_{\hat\beta}[e\otimes\Lambda_\nu(m)]) = I^*(e\otimes e\otimes m\eta)\\
&= I^*(\lambda_e\otimes\lambda_e)I_\nu(\Lambda_\nu(m)_{\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\eta)
\end{aligned}$$ □

COROLLARY 15.3.4. – *For all $x \in \mathcal{L}(H)$, we have:*

$$(\Phi_{\hat{\beta}\underset{\nu}{\star}\alpha}\Phi) \circ \hat{\Gamma} \circ \Phi^{-1}(x) = I_\nu^* x I_\nu$$

Proof. – Let $x \in \mathcal{L}(H)$ and $e \in H$ be a vector of norm 1. We have:

$$\begin{aligned}
&(\Phi_{\hat{\beta}\underset{\nu}{\star}\alpha}\Phi) \circ \hat{\Gamma} \circ \Phi^{-1}(x) \\
&= (\lambda^*_{e\,\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\lambda^*_e)\sigma_{\nu^o}W([1\otimes x]_{\hat{\beta}}\underset{\nu}{\otimes}_\alpha[1\otimes 1])W^*\sigma_\nu(\lambda_{e\,\beta_\nu}\underset{\nu}{\otimes}_{\mathrm{id}}\lambda_e) \\
&= I_\nu^*(\lambda_e^*\otimes\lambda_e^*)I([1\otimes x]_{\hat{\beta}}\underset{\nu}{\otimes}_\alpha[1\otimes 1])I^*(\lambda_e\otimes\lambda_e)I_\nu \\
&= I_\nu^*(\lambda_e^*\otimes\lambda_e^*)(1\otimes 1\otimes x)(\lambda_e\otimes\lambda_e)I_\nu = I_\nu^* x I_\nu
\end{aligned}$$

□

Now, we are computing the dual operator-valued weight.

LEMMA 15.3.5. – *We have $\hat{\Lambda}((\omega_{\Xi,\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W)) = (m^*\otimes J_\nu e^* J_\nu)\Xi$ for all $m, e \in \mathcal{N}_\nu$ et $\Xi \in D((H\otimes H)_\beta, \nu^o)$.*

Proof. – Let $m_1, m_2 \in \mathcal{N}_\nu$. We have:

$$\begin{aligned}
&(\hat{\Lambda}((\omega_{\Xi,\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W))|\Lambda_{\nu'\otimes\nu}(J_\nu m_1 J_\nu\otimes m_2)) \\
&= \omega_{\Xi,\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)}(J_\nu m_1^* J_\nu\otimes m_2^*) = ((J_\nu m_1^* J_\nu\otimes m_2^*)\Xi|\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)) \\
&= (\Xi|mJ_\nu\Lambda_\nu(m_1)\otimes J_\nu e J_\nu\Lambda_\nu(m_2)) = ((m^*\otimes J_\nu e^* J_\nu)\Xi|\Lambda_{\nu'\otimes\nu}(J_\nu m_1 J_\nu\otimes m_2))
\end{aligned}$$

□

PROPOSITION 15.3.6. – *We have $\hat{T_L} = \mathrm{id}\otimes E_{\nu'}$ where $E_{\nu'}$ is the operator-valued weight from $\mathcal{L}(H)$ to M obtained from the weight ν'.*

Proof. – Let $m, e, y \in \mathcal{N}_\nu$ and $\eta \in H$. On one hand, we compute:

$$\begin{aligned}
||\hat{\Lambda}((\omega_{\Xi,\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W))||^2 &= ||m^*\Lambda_\nu(y)\otimes J_\nu e^* J_\nu\eta||^2 \\
&= ||J_\nu e^* J_\nu \eta||^2||\Lambda_\nu(m^* y)||^2
\end{aligned}$$

On the other hand, by proposition 15.3.1, we have:

$$\begin{aligned}
&||\hat{\Lambda}((\omega_{\Xi,\Lambda_\nu(m)\otimes J_\nu\Lambda_\nu(e)} * \mathrm{id})(W))||^2 \\
&= \hat{\Phi}(\rho_\eta^*(1\otimes\Sigma)\rho_{y^*\Lambda_\nu(m)}(1\otimes J_\nu e J_\nu)(1\otimes J_\nu e^* J_\nu)\rho^*_{y^*\Lambda_\nu(m)}(1\otimes\Sigma)\rho_\eta) \\
&= ||J_\nu e^* J_\nu\eta||^2\hat{\Phi}(1\otimes(\Lambda_\nu(m^*y)\otimes\Lambda_\nu(m^*y)))
\end{aligned}$$

where $\xi\otimes\xi$ is the operator of $\mathcal{L}(H)$ such that $(\xi\otimes\xi)v = (v|\xi)\xi$. Then, if $\xi \in \mathcal{D}(S_\nu)$, then we have:

$$\begin{aligned}
\hat{\Phi}(1\otimes(\xi\otimes\xi)) = ||S_\nu\xi||^2 = (\Delta_\nu\xi|\xi) = \Big(\frac{\mathrm{d}\nu}{\mathrm{d}\nu'}\xi|\xi\Big) &= \nu(\theta^{\nu'}(\xi,\xi)) \\
&= \nu\circ E_{\nu'}(\xi\otimes\xi) = \nu\circ\alpha^{-1}\circ\hat{T_L}(1\otimes(\xi\otimes\xi))
\end{aligned}$$

□

We also have the following formulas, for all $x \in \mathscr{L}(H)$ and $t \in \mathbb{R}$:

$$\hat{R}(1 \otimes x) = 1 \otimes J_\nu x^* J_\nu \quad \text{and} \quad \hat{\tau}_t(1 \otimes x) = 1 \otimes \Delta_\nu^{it} x \Delta_\nu^{-it}$$

The right invariant operator-valued weight is given by: $\hat{T_R} = \hat{R} \circ \hat{T_L} \circ \hat{R} = (\mathrm{id} \otimes E_\nu)$.

PROPOSITION 15.3.7. – *The dual pairs quantum groupoid can be identify with* $(M, \mathscr{L}(H), \mathrm{id}, j, \mathrm{id}, \nu, E_{\nu'}, E_\nu)$ *which is a measured quantum groupoid but not an adapted measured quantum groupoid. Moreover, expressions for co-involution and scaling group are given, for all* $x \in \mathscr{L}(H)$ *and* $t \in \mathbb{R}$*:*

$$\hat{R}(x) = J_\nu x^* J_\nu \quad \textit{and} \quad \hat{\tau}_t(x) = \Delta_\nu^{it} x \Delta_\nu^{-it}$$

Proof. – The proposition gathers results of the section. Nevertheless we lay stress on the following point. We have, for all $t \in \mathbb{R}$ and $m \in M$:

$$\sigma_t^{E_\nu}(m) = \sigma_{-t}^\nu(m)$$

instead of $\sigma_t^\nu(m)$ to have an adapted measured quantum groupoid. □

REMARK 15.3.8. – If $M = L^\infty(X)$, we find the structure of pairs groupoid $X \times X$. This example comes from the inclusion of von Neumann algebras [**Eno00**]:

$$\mathbb{C} \subset M \subset \mathscr{L}(L^2(M)) \subset \mathscr{L}(L^2(M)) \otimes M \subset \cdots$$

CHAPTER 16

INCLUSIONS OF VON NEUMANN ALGEBRAS

Let $M_0 \subseteq M_1$ be an inclusion of von Neumann algebras. We call **basis construction** the following inclusions:

$$M_0 \subseteq M_1 \subseteq M_2 = J_1 M_0' J_1 = End_{M_0^o}(L^2(M_1))$$

By iteration, we construct Jones' tower $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$

DEFINITION 16.0.9. – If $M_0' \cap M_1 \subseteq M_0' \cap M_2 \subseteq M_0' \cap M_3$ is a basis construction, then the inclusion is said to be **of depth** 2.

Let T_1 be a n.s.f. operator-valued weight from M_1 to M_0. By Haagerup's construction [**Str81**, (12.11)] and [**EN96**, (10.1)], it is possible to define a canonical n.s.f. operator-valued weight T_2 from M_2 to M_1 such that, for all $x, y \in \mathcal{N}_{T_1}$, we have:

$$T_2(\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*) = xy^*$$

By iteration, we define, for all $i \geq 1$, a n.s.f. operator-valued weight T_i from M_i to M_{i-1}. If ψ_0 is n.s.f. weight sur M_0, we put $\psi_i = \psi_{i-1} \circ T_i$.

DEFINITION 16.0.10 ([**EN96**, (11.12)], [**EV00**, (3.6)]). – T_1 is said to be regular if restrictions of T_2 to $M_0' \cap M_2$ and of T_3 to $M_1' \cap M_3$ are semi-finite.

PROPOSITION 16.0.11 ([**EV00**, (3.2, 3.8, 3.10)]). – *If $M_0 \subset M_1$ is an inclusion with a regular n.s.f. operator-valued weight T_1 from M_1 to M_0, then there exists a natural *-representation π of $M_0' \cap M_3$ on $L^2(M_0' \cap M_2)$ whose restriction to $M_0' \cap M_2$ is the standard representation of $M_0' \cap M_2$. Moreover, the inclusion is of depth 2 if, and only if π is faithful.*

The following theorem exhibits a structure of measured quantum groupoid coming from inclusion of von Neumann algebras.

Theorem 16.0.12. – *Let $M_0 \subset M_1$ be a depth 2 inclusion of σ-finite von Neumann algebras, equipped with a regular nsf operator-valued weight T_1 in the sense of* **[Eno00]** *and* **[Eno04]**. *Moreover, assume there exists on $M_0' \cap M_1$ a nsf weight χ invariant under the modular automorphism group T_1. Then, by theorem 8.2 of* **[Eno04]**, *we have:*

(1) *there exists an application $\widetilde{\Gamma}$ from $M_0' \cap M_2$ to*

$$(M_0' \cap M_2)_{j_1} \underset{M_0' \cap M_1}{\star}{}_{\mathrm{id}} (M_0' \cap M_2)$$

such that $(M_0' \cap M_1, M_0' \cap M_2, \mathrm{id}, j_1, \widetilde{\Gamma})$ is a Hopf-bimodule, (where id *means here the injection of $M_0' \cap M_1$ into $M_0' \cap M_2$, and j_1 means here the restriction of j_1 coming from Tomita's theory to $M_0' \cap M_1$, considered then as an anti-representation of $M_0' \cap M_1$ into $M_0' \cap M_2$). Moreover, the anti-automorphism j_1 of $M_0' \cap M_2$ is a co-involution for this Hopf-bimodule structure.*

(2) *the nsf operator-valued weight $\widetilde{T_2}$ from $M_0' \cap M_2$ to $M_0' \cap M_1$, restriction of the second canonical weight construct from Jones' tower and T_1, is left invariant.*

(3) *Let χ_2 be the weight $\chi \circ \widetilde{T_2}$; there exist a one-parameter group of automorphisms $\widetilde{\tau}_t$ of $M_0' \cap M_2$, commuting with the modular automorphism group σ^{χ_2}, such that, for all $t \in \mathbb{R}$, we have:*

$$\widetilde{\Gamma} \circ \sigma_t^{\chi_2} = (\widetilde{\tau}_t {}_{j_1} \underset{\chi}{\star}{}_{\mathrm{id}} \sigma_t^{\chi_2}) \circ \widetilde{\Gamma}$$

Moreover, we have $j_1 \circ \widetilde{\tau}_t = \widetilde{\tau}_t \circ j_1$.

Then, $(M_0' \cap M_1, M_0' \cap M_2, \mathrm{id}, j_1, \widetilde{\Gamma}, \widetilde{T_2}, j_1, \widetilde{\tau}, \chi)$ is a measured quantum groupoid.

Proof. – We have $\Phi = \chi_2$. Then, by proposition 6.6 of **[Eno04]**, we have the relation between R and Γ. Also, we notice that $\widetilde{\tau}$ coincide with σ^{χ} on $M_0' \cap M_1$ by theorem 5.10 of **[Eno04]** and that we have, for all $n \in M_0' \cap M_1$ and $t \in \mathbb{R}$:

$$\sigma_t^{\widetilde{T_2}}(j_1(n)) = \sigma_t^{T_2}(j_1(n)) = j_1(\sigma_t^{T_1}(n))$$

by corollary 4.8 and by 4.1 of **[Eno04]**. So that, $\gamma = \sigma^{T_1}$ leaves χ invariant by hypothesis. □

Then we can show that the dual structure coincide with the natural one on the second relative commutant of Jones' tower.

Theorem 16.0.13. – *Let $M_0 \subset M_1$ be a depth 2 inclusion of σ-finite von Neumann algebras, equipped with a regular nsf operator-valued weight T_1 in the sense of* **[Eno00]** *and* **[Eno04]**. *Moreover, assume there exists on $M_0' \cap M_1$ a nsf weight χ invariant under the modular automorphism group T_1.*

(1) *there exists an application* $\widetilde{\Gamma}'$ *from* $M_1' \cap M_3$ *to*

$$(M_1' \cap M_3) {}_{j_1} \underset{M_0' \cap M_1}{\star}{}_{j_2 \circ j_1} (M_1' \cap M_3)$$

such that $(M_0' \cap M_1, M_1' \cap M_3, j_2 \circ j_1, j_1, \widetilde{\Gamma}')$ *is a Hopf-bimodule, where* j_1, j_2 *come from Tomita's theory. Moreover, the anti-automorphism* j_2 *of* $M_1' \cap M_3$ *is a co-involution for this Hopf-bimodule structure.*

(2) *the nsf operator-valued weight* $\widetilde{T_3}$ *from* $M_1' \cap M_3$ *to* $M_1' \cap M_2 = j_1(M_0' \cap M_1)$*, restriction of the second canonical weight construct from Jones' tower and* T_1*, is left invariant.*

(3) *Let* χ_3 *be the weight* $\chi \circ \widetilde{T_3}$*; there exist a one-parameter group of automorphisms* $\widetilde{\tau}_t$ *of* $M_1' \cap M_3$*, commuting with the modular automorphism group* σ^{χ_3}*, such that, for all* $t \in \mathbb{R}$*, we have:*

$$\widetilde{\Gamma} \circ \sigma_t^{\chi_3} = (\widetilde{\tau}_t {}_{j_1} \underset{\chi}{\star}{}_{j_2 \circ j_1} \sigma_t^{\chi_3}) \circ \widetilde{\Gamma}$$

Moreover, we have $j_2 \circ \widetilde{\tau}_t = \widetilde{\tau}_t \circ j_2$.

Then, $(M_0' \cap M_1, M_1' \cap M_3, j_2 \circ j_1, j_1, \widetilde{\Gamma}, \widetilde{T_3}, j_2, \widetilde{\tau}, \chi)$ *is the dual measured quantum groupoid of* $M_0' \cap M_2$ *(equipped with the structure described on 16.0.12).*

Proof. – All objects are constructed from the fundamental unitary that's why the Hopf-bimodule structure of the dual coincide with the structure on the second relative commutant. The uniqueness theorem implies that the dual operator-valued weight coincide with the restriction of T_3 up to an element of the basis. □

We can't characterize, at this stage, inclusions of von Neumann algebras among measured quantum groupoids. A way to answer the question is to know if each measured quantum groupoid acts on a von Neumann algebra .

CHAPTER 17

OPERATIONS ON ADAPTED MEASURED QUANTUM GROUPOIDS

17.1. Elementary operations

17.1.1. Sum of adapted measured quantum groupoids A union of groupoids is still a groupoid. We establish here a similar result at the quantum level:

PROPOSITION 17.1.1. – *Let* $(N_i, M_i, \alpha_i, \beta_i, \Gamma_i, \nu_i, T_L^i, T_R^i)_{i\in I}$ *be a family of adapted measured quantum groupoids. In the von Neumann algebra level, if we identify* $\bigoplus_{i\in I} M_i {}_{\beta}\underset{N_i}{\star}{}_{\alpha} M_i$ *with* $\left(\bigoplus_{i\in I} M_i\right) {}_{\beta}\underset{\bigoplus_{i\in I} N_i}{\star}{}_{\alpha} \left(\bigoplus_{i\in I} M_i\right)$*, then we get:*

$$\left(\bigoplus_{i\in I} N_i, \bigoplus_{i\in I} M_i, \bigoplus_{i\in I} \alpha_i, \bigoplus_{i\in I} \beta_i, \bigoplus_{i\in I} \Gamma_i, \bigoplus_{i\in I} \nu_i, \bigoplus_{i\in I} T_L^i, \bigoplus_{i\in I} T_R^i\right)$$

an adapted measured quantum groupoid where operators act on the diagonal.

Proof. – Straightforward. □

In particular, the sum of two quantum groups with different scaling constants (**[VV03]** for examples) produce an adapted measured quantum groupoid with non scalar scaling operator.

17.1.2. Tensor product of adapted measured quantum groupoids. – Cartesian product of groups correspond to tensor product of quantum groups. In the same way, we have:

PROPOSITION 17.1.2. – *Let* $(N_i, M_i, \alpha_i, \beta_i, \Gamma_i, \nu_i, T_L^i, T_R^i)$ *be adapted measured quantum groupoids for* $i = 1, 2$*. If we identify* $(M_1 {}_{\beta_1}\underset{N_1}{\star}{}_{\alpha_1} M_1) \otimes (M_2 {}_{\beta_2}\underset{N_2}{\star}{}_{\alpha_2} M_2)$ *with* $(M_1 \otimes M_2) {}_{\beta_1\otimes\beta_2}\underset{N_1\otimes N_2}{\star}{}_{\alpha_1\otimes\alpha_2} (M_1 \otimes M_2)$ *as von Neumann algebras, then we have:*

$$(N_1 \otimes N_2, M_1 \otimes M_2, \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2, \Gamma_1 \otimes \Gamma_2, \nu_1 \otimes \nu_2, T_L^1 \otimes T_L^2, T_R^1 \otimes T_R^2)$$

is an adapted measured quantum groupoid.

Proof. – Straightforward. □

17.1.3. Direct integrals of adapted measured quantum groupoids. – In this section, X denote σ-compact, locally compact space and μ a Borel measure on X. Theory of Hilbertian integrals is described in [**Tak03**].

PROPOSITION 17.1.3. – *Let $(N_p, M_p, \alpha_p, \beta_p, \Gamma_p, \nu_p, T_L^p, T_R^p)_{p\in X}$ be a family of adapted measured quantum groupoids. In the von Neumann algebra level, if we identify $\int_X^\oplus M_p {}_{\beta}\underset{N_p}{\star}{}_{\alpha} M_p\, d\mu(p)$ and $\left(\int_X^\oplus M_p\, d\mu(p)\right) {}_{\beta}\underset{\int_X^\oplus N_p\, d\mu(p)}{\star}{}_{\alpha} \left(\int_X^\oplus M_p\, d\mu(p)\right)$, we have:*

$$\left(\int_X^\oplus N_p\, d\mu(p), \int_X^\oplus M_p\, d\mu(p), \int_X^\oplus \alpha_p\, d\mu(p), \int_X^\oplus \beta_p\, d\mu(p), \cdots\right.$$

$$\left.\cdots \int_X^\oplus \Gamma_p\, d\mu(p), \int_X^\oplus \nu_p\, d\mu(p), \int_X^\oplus T_L^p\, d\mu(p), \int_X^\oplus T_R^p\, d\mu(p)\right)$$

is an adapted measured quantum groupoid.

Proof. – Left to the reader. □

[**Bla96**] gives examples. In this case, the basis is $L^\infty(X)$ and $\alpha = \beta = \hat{\beta}$. The fundamental unitary comes from a space onto the same space and then can be viewed as a field of multiplicative unitaries.

17.2. Opposite and commutant structures

DEFINITION 17.2.1

We call Hopf-bimodule morphism from $(N, M_1, \alpha_1, \beta_1, \Gamma_1)$ to $(N, M_2, \alpha_2, \beta_2, \Gamma_2)$ a morphism π of von Neumann algebras from M_1 to M_2 such that:

i) $\pi \circ \alpha_1 = \alpha_2$ et $\pi \circ \beta_1 = \beta_2$;
ii) $\Gamma_2 \circ \pi = (\pi {}_{\beta_1}\underset{N}{\star}{}_{\alpha_1} \pi) \circ \Gamma$.

Also, we call anti-morphism of Hopf-bimodule from $(N, M_1, \alpha_1, \beta_1, \Gamma_1)$ to $(N, M_2, \alpha_2, \beta_2, \Gamma_2)$ a morphism j of von Neumann algebras from M_1 to M_2 such that:

i) $j \circ \alpha_1 = \beta_2$ et $j \circ \beta_1 = \alpha_2$;
ii) $\Gamma_2 \circ j = (j {}_{\beta_1}\underset{N}{\star}{}_{\alpha_1} j) \circ \Gamma_1$.

DEFINITION 17.2.2. – For all Hopf-bimodule $(N, M_1, \alpha_1, \beta_1, \Gamma_1)$ and all 1-1 morphism of von Neumann algebras π from M_1 onto M_2, M_2, $(N, M_2, \pi\circ\alpha_1, \pi\circ\beta_1, (\pi_{\beta_1}\underset{N}{\star}{}_{\alpha_1}\pi)\circ\Gamma\circ\pi^{-1})$ is a Hopf-bimodule called Hopf-bimodule image by π. Also, if j is a 1-1 anti-morphism of von Neumann algebras from M_1 onto M_2, then $(N^0, M_2, j\circ\alpha_1, j\circ\beta_1, (j_{\beta_1}\underset{N}{\star}{}_{\alpha_1}j)\circ\Gamma\circ j^{-1})$ is a Hopf-bimodule called Hopf-bimodule image by j.

PROPOSITION 17.2.3. – *Let π a 1-1 morphism from $(N, M_1, \alpha_1, \beta_1, \Gamma_1)$ onto $(N, M_2, \alpha_2, \beta_2, \Gamma_2)$. If $(N, M_1, \alpha_1, \beta_1, \Gamma_1, R, T_L, \tau, \nu)$ is a measured quantum groupoid, then $(N, M_2, \alpha_2, \beta_2, \Gamma_2, \pi\circ R\circ\pi^{-1}, \pi\circ T_L\circ\pi^{-1}, \pi\circ\tau\circ\pi^{-1}, \nu)$ is a measured quantum groupoid such that:*

$$\lambda_2 = \pi(\lambda_1) \text{ et } \delta_2 = \pi(\delta_1)$$

We denote by $\Phi^1 = \nu\circ\alpha_1^{-1}\circ T_L$ and $\Phi^2 = \Phi^1\circ\pi^{-1}$. If I is the unitary from H_{Φ^1} onto H_{Φ^2} such that $I\Lambda_{\Phi^1}(a) = \Lambda_{\Phi^2}(\pi(a))$ for all $a\in\mathcal{N}_{\Phi^1}$, then fundamental unitaries are linked by:

$$W_2 = (I_{\alpha_1}\underset{N^o}{\otimes}{}_{\hat{\beta}_1} I) W_1 (I^*{}_{\beta_2}\underset{N}{\otimes}{}_{\alpha_2} I^*)$$

Proof. – It is easy to state that $(N, M_2, \alpha_2, \beta_2, \Gamma_2, \pi\circ R\circ\pi^{-1}, \pi\circ T_L\circ\pi^{-1}, \pi\circ\tau\circ\pi^{-1}, \nu)$ is a measured quantum groupoid. For all $v\in D((H_{\Phi^1})_{\beta_1}, \nu^o)$, $a\in\mathcal{N}_{T_L}\cap\mathcal{N}_{\Phi^1}$ and (N^o,ν^o)-basis $(\xi_i)_{i\in I}$ of $(H_{\Phi^1})_{\beta_1}$, we have:

$$
\begin{aligned}
&W_1^*(I^*{}_{\alpha_2}\underset{N^o}{\otimes}{}_{\hat{\beta}_2} I^*)(Iv_{\alpha_2}\underset{\nu^o}{\otimes}{}_{\hat{\beta}_2}\Lambda_{\Psi^2}(\pi(a)) = W_1^*(v_{\alpha_1}\underset{\nu^o}{\otimes}{}_{\hat{\beta}_1}\Lambda_{\Psi^1}(a))\\
&= \sum_{i\in I}\xi_{i\,\beta_1}\underset{\nu}{\otimes}{}_{\alpha_1}\Lambda_{\Phi^1}((\omega_{v,\xi_i\,\beta_1}\underset{\nu}{\star}{}_{\alpha_1}\mathrm{id})\Gamma_1(a))\\
&= \sum_{i\in I}\xi_{i\,\beta_1}\underset{\nu}{\otimes}{}_{\alpha_1}\Lambda_{\Phi^1}((\omega_{v,\xi_i\,\beta_1}\underset{\nu}{\star}{}_{\alpha}\mathrm{id})(\pi^{-1}{}_{\beta_1}\underset{N}{\star}{}_{\alpha_1}\pi^{-1})\Gamma_2(\pi(a))))\\
&= (I^*{}_{\beta_2}\underset{N}{\otimes}{}_{\alpha_2} I^*)\sum_{i\in I} I\xi_{i\,\beta_2}\underset{\nu}{\otimes}{}_{\alpha_2}\Lambda_{\Phi^2}((\omega_{Iv,I\xi_i\,\beta_2}\underset{\nu}{\star}{}_{\alpha_2}\mathrm{id})\Gamma_2(a))\\
&= (I^*{}_{\beta_2}\underset{N}{\otimes}{}_{\alpha_2} I^*) W_2^*(Iv_{\alpha_2}\underset{\nu^o}{\otimes}{}_{\hat{\beta}_2}\Lambda_{\Psi^2}(\pi(a)))
\end{aligned}
$$

Then, we have proved that $W_2 = (I_{\alpha_1}\underset{N^o}{\otimes}{}_{\hat{\beta}_1} I) W_1 (I^*{}_{\beta_2}\underset{N}{\otimes}{}_{\alpha_2} I^*)$. For all $a, b\in\mathcal{N}_{jT_Lj}\cap\mathcal{N}_{\Phi\circ j}$, we have:

$$
\begin{aligned}
&R_2((\mathrm{id}_{\beta_2}\underset{\nu}{\otimes}{}_{\alpha_2}\omega_{J_{\Phi^2}\Lambda_{\Phi^2}(\pi(a))})\Gamma_2(\pi(b^*)\pi(b)))\\
&= (\mathrm{id}_{\beta_2}\underset{\nu}{\otimes}{}_{\alpha_2}\omega_{J_{\Phi^2}\Lambda_{\Phi^2}(\pi(b))})\Gamma_2(\pi(a^*a))\\
&= (\mathrm{id}_{\beta_2}\underset{\nu}{\otimes}{}_{\alpha_2}\omega_{J_{\Phi^2}\Lambda_{\Phi^2}(\pi(b))})(\pi_{\beta_1}\underset{N}{\star}{}_{\alpha_1}\pi)\Gamma_1(a^*a)\\
&= \pi((\mathrm{id}_{\beta_1}\underset{\nu^o}{\otimes}{}_{\alpha_1}\omega_{J_{\Phi^1}\Lambda_{\Phi^1}(b)})\Gamma_1(a^*a)) = \pi R_1((\mathrm{id}_{\beta_1}\underset{\nu^o}{\otimes}{}_{\alpha_1}\omega_{J_{\Phi^1}\Lambda_{\Phi^1}(a)})\Gamma_1(b^*b))\\
&= \pi R_1\pi^{-1}((\mathrm{id}_{\beta_2}\underset{\nu}{\otimes}{}_{\alpha_2}\omega_{J_{\Phi^2}\Lambda_{\Phi^2}(\pi(a))})\Gamma_2(\pi(b^*)\pi(b)))
\end{aligned}
$$

from which we get that $R_2 = \pi\circ R_1\circ\pi^{-1}$ and then $S_2 = \pi\circ S_1\circ\pi^{-1}$ and $\tau_2 = \pi\circ\tau_1\circ\pi^{-1}$ thanks to fundamental unitaries. Finally, we have for all $t\in\mathbb{R}$:

$$\begin{aligned}[D\Phi^2\circ R_2 : D\Phi^2]_t &= [D\Phi^1\circ R_1\circ\pi^{-1} : D\Phi^1\circ\pi^{-1}]_t\\ &= \pi([D\Phi^1\circ R_1 : D\Phi^1]_t) = \pi(\lambda_1)^{\frac{it^2}{2}}\pi(\delta_1)^{it}\end{aligned}$$

and, so we have $\delta_2 = \pi(\delta_1)$ and $\lambda_2 = \pi(\lambda_1)$. □

PROPOSITION 17.2.4. – *Let j a 1-1 anti-morphism from $(N, M_1, \alpha_1, \beta_1, \Gamma_1)$ onto $(N^0, M_2, \alpha_2, \beta_2, \Gamma_2)$. If $(N, M_1, \alpha_1, \beta_1, \Gamma_1, R, T_L, \tau, \nu)$ is a measured quantum groupoid, then $(N^0, M_2, \beta_2, \alpha_2, \Gamma_2, j\circ R\circ j^{-1}, j\circ T_L\circ j^{-1}, j\circ\tau_{-t}\circ j^{-1}, \nu^0)$ is a measured quantum groupoid such that:*

$$\lambda_2 = j(\lambda_1^{-1}) \textit{ et } \delta_2 = j(\delta_1)$$

We denote by $\Phi^1 = \nu\circ\alpha_1^{-1}\circ T_L$ and $\Phi^2 = \Phi^1\circ j^{-1}$. If J is the unitary from H_{Φ^1} onto H_{Φ^2} such that $I\Lambda_{\Phi^1}(a) = J_{\Phi^2}\Lambda_{\Phi^2}(j(a^*))$ for all $a\in\mathcal{N}_{\Phi^1}$, then fundamental unitaries are linked by:

$$W_2 = (J_{\alpha_1}\underset{N^o}{\otimes}_{\hat\beta_1} J)W_1(J^*{}_{\alpha_2}\underset{N}{\otimes}_{\beta_2} J^*)$$

Proof. – The proof is very similar to the previous one. □

DEFINITION 17.2.5. – We call opposite quantum groupoid the image by the co-involution R of the Hopf-bimodule, denoted by $(N, M, \alpha, \beta, \Gamma, R, T_L, \tau, \nu)^{\mathrm{op}}$. THe Hopf-bimodule is then the symmetrized one $(N^o, M, \beta, \alpha, \varsigma_N\circ\Gamma)$.

REMARK 17.2.6. – If N is abelian, $\alpha = \beta$, $\Gamma = \varsigma_N\circ\Gamma$ then the measured quantum groupoid is equal to its opposite: we speak about symmetric quantum groupoid.

We put j the canonical *-anti-isomorphism from M onto M' coming from Tomita's theory, trivial on the center of M and given by $j(x) = J_\Phi x^* J_\Phi$. Then, we have $j\circ\alpha = \widehat{\beta}$ and we put $\varrho = j\circ\beta$. We can construct a unitary $j_{\varrho}\underset{N^o}{\star}_{\hat\beta} j$ from $M'{}_{\varrho}\underset{N^o}{\star}_{\hat\beta} M'$ onto $M_{\beta}\underset{N}{\star}_{\alpha} M$ the adjoint of which is $j_{\beta}\underset{N}{\star}_{\alpha} j$.

DEFINITION 17.2.7. – We call commutant quantum groupoid the image by j of the Hopf bimodule. It is denoted by $(N, M, \alpha, \beta, \Gamma, R, T_L, \tau, \nu)^c$. The Hopf-bimodule is equal to $(N^o, M', \hat\beta, \varrho, (j_{\beta}\underset{N}{\star}_{\alpha} j)\circ\Gamma\circ j)$. We put $\Gamma^c = (j_{\beta}\underset{N}{\star}_{\alpha} j)\circ\Gamma\circ j$.

We describe fundamental objects of the structures.

PROPOSITION 17.2.8. – *We have the following formulas:*

i) $W^{op} = \sigma_{\nu^o} W'^* \sigma_{\nu^o}$, $R^{op} = R$, $\tau_t^{op} = \tau_{-t}$, $\delta^{op} = \delta^{-1}$ *et* $\lambda^{op} = \lambda^{-1}$;

ii) $W^c = (J_{\Phi\,\beta}\underset{N}{\otimes}_{\alpha} J_\Phi)W(J_{\Phi\,\varrho}\underset{N^o}{\otimes}_{\hat\beta} J_\Phi)$, $R^c = jRj$, $\tau_t^c = j\tau_{-t}j$, $\delta^c = j(\delta)$ *et* $\lambda^c = \lambda^{-1}$.

Proof. – It is an easy consequence of propositions 17.2.3 and 17.2.4 except for the relation between W^{op} and W'. For all $v \in D({}_\alpha H_\Psi, \nu)$, $a \in \mathcal{N}_{T_R} \cap \mathcal{N}_{\Psi_R}$ and (N,ν)-basis $(\eta_i)_{i\in I}$ of ${}_\alpha H_\Psi$, we have:

$$\begin{aligned}(W^{\mathrm{op}})^*\sigma_{\nu^o}(\Lambda_\Psi(a)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta v) &= (W^{\mathrm{op}})^*(v_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta \Lambda_\Psi(a))\\ &= \sum_{i\in I}\eta_{i\,\alpha}\underset{\nu^o}{\otimes}{}_\beta\Lambda_\Psi((\omega_{v,\eta_i}\,{}_\alpha\underset{\nu^o}{\star}{}_\beta\mathrm{id})(\varsigma_N\circ\Gamma(a)))\\ &= \sigma_\nu\sum_{i\in I}\Lambda_\Psi((\mathrm{id}_\alpha\underset{\nu^o}{\star}{}_\beta\omega_{v,\eta_i})\Gamma(a))_\beta\underset{\nu}{\otimes}{}_\alpha\eta_i\\ &= \sigma_\nu W'(\Lambda_\Psi(a)_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta v)\end{aligned}$$

Then, we have proved that $W^{\mathrm{op}} = \sigma_{\nu^o}W'^*\sigma_{\nu^o}$. □

COROLLARY 17.2.9. – *We have* $W' = (J_{\widehat{\Phi}}\,{}_\alpha\underset{\nu^o}{\otimes}{}_\beta J_{\widehat{\Phi}})\sigma_\nu W^*\sigma_\nu(J^*_{\widehat{\Phi}}\,{}_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta J^*_{\widehat{\Phi}})$.

Proof. – It is a consequence of the previous proposition and proposition 17.2.4. □

REMARK 17.2.10. – The application $j\circ R$, implemented by $J_\Phi\hat{J}$, gives an isomorphism between the measured quantum groupoid and the opposite of the commutant one.

PROPOSITION 17.2.11. – *We have the following equalities:*

i) $(N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{op\wedge} = (N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{\wedge c}$
ii) $(N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{c\wedge} = (N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{\wedge op}$
iii) $(N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{c\ op} = (N,M,\alpha,\beta,\Gamma,\nu,T_L,T_R)^{op\ c}$

Proof. – The dual of the opposite and the commutant of the dual have the same basis N^0. The von Neumann algebra of the first one is generated by the operators $(\omega * \mathrm{id})(W^{\mathrm{op}})$ so is equal to $J_{\widehat{\Phi}}\{(\omega*\mathrm{id})(W)\}''J^*_{\widehat{\Phi}} = \widehat{M}'$. The representation and the anti-representation over N^0 are both given by $\widehat{\beta}$ and α. Finally, for all $x\in\widehat{M}'$, we have:

$$\Gamma^{\mathrm{op}\wedge}(x) = \sigma_\nu W^{\mathrm{op}}(x_\alpha\underset{N^o}{\otimes}{}_\beta 1)(W^{\mathrm{op}})^*\sigma_{\nu^o} = W'^*(1_\beta\underset{N}{\otimes}{}_\alpha x)W'$$

By the previous corollary, we have:

$$\Gamma^{\mathrm{op}\wedge}(x) = \sigma_\nu(J_{\widehat{\Phi}}\,{}_\alpha\underset{\nu^o}{\otimes}{}_{\hat\beta}J_{\widehat{\Phi}})W(J_{\widehat{\Phi}}xJ_{\widehat{\Phi}}\,{}_\beta\underset{N}{\otimes}{}_\alpha 1)W^*(\mathcal{J}_\beta\underset{\nu}{\otimes}{}_{\hat\alpha}J_{\widehat{\Phi}})\sigma_{\nu^o}$$

$$= (J_{\widehat{\Phi}}\,{}_{\hat\beta}\underset{\nu}{\otimes}{}_\alpha J_{\widehat{\Phi}})\hat\Gamma(J_{\widehat{\Phi}}xJ_{\widehat{\Phi}})(J_{\widehat{\Phi}}\,{}_{\hat\alpha}\underset{\nu^o}{\otimes}{}_\beta J_{\widehat{\Phi}}) = \hat\Gamma^{\mathrm{c}}(x)$$

So i) is proved. ii) comes from i) and the bi-duality theorem.

The opposite of the commutant and the commutant of the opposite have the same basis N^0 and the same von Neumann algebra M'. The representation and the anti-representation are both given by ϱ and $\widehat{\beta}$. By **[Vae01a]**, we have $J_\Psi = \lambda^{i/4} J_\Phi$. Then we get, for all $x \in M'$:

$$\begin{aligned}\Gamma^{\mathrm{op\ c}}(x) &= (J_\Psi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} J_\Psi)\varsigma_N \Gamma(J_\Psi x J_\Psi)(J_\Psi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} J_\Psi)\\ &= \varsigma_{N^o}(J_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} J_\Phi)\Gamma(J_\Phi x J_\Phi)(J_\Phi {}_\alpha\underset{\nu^o}{\otimes}{}_{\hat{\beta}} J_\Phi)\Gamma^{\mathrm{c\ op}}(x)\end{aligned}$$

because $\lambda^{i/4} \in Z(M) \cap \alpha(N) \cap \beta(N)$ So iii) is proved. □

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