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Bruno DEMANGE

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UNCERTAINTY PRINCIPLES ASSOCIATED TO NON-DEGENERATE QUADRATIC FORMS

Bruno Demange

Abstract. – This volume is devoted to several generalisations of the classical Hardy uncertainty principle on Euclidian spaces. Instead of comparing functions and their Fourier transforms a Gaussian, we compare them to the exponential of general non-degenerate quadratic forms, like for example the Lorentz form. Using the Bargmann transform, we translate the problem into the description of several classes of analytic functions of several variables, and at the same time simplify and unify proofs of results presented in several previous papers.

Résumé **(Principes d'incertitude associés à des formes quadratiques non dégénérées)**
Ce volume est consacré a des géneralisations du principe d'incertitude classique de Hardy dans les espaces Euclidiens. Au lieu de comparer les fonctions à des gaussiennes, nous les comparons a l'exponentielle de formes quadratiques non dégénérées, par exemple à la forme de Lorentz. Nous transformons ces problèmes à l'aide de la transformée de Bargmann, en des problèmes de description de certaines classes de fonctions entières de plusieurs variables. Ces méthode améliorent et simplifient des résultats publiés dans des travaux précédents.

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INTRODUCTION

This volume concerns certain forms of the uncertainty principle in harmonic analysis. The uncertainty principle is a general term for theorems that show that if a function f on $\mathbb{R}^d$ and its Fourier transform $\widehat{f}$ approximate g and $\widehat{g}$, then they must be equal.

The history of the uncertainty principle goes back to Heisenberg inequality of quantum mechanics, namely

$$\int |x|^2 |\widehat{f}(x)|^2 \, dx \times \int |\xi|^2 |\widehat{f}(\xi)|^2 \, d\xi \geq \frac{d^2}{16\pi^2} \|f\|_{L^2}^4,$$

where d is the dimension, and $\widehat{f}(y) = \int f(x) \exp(-2i\pi xy) \, dx$. This inequality is well known as the fact that the product of uncertainties of the position and the momentum is bounded below by an explicit constant, that involves the Planck constant. Equality occurs only for the Gaussian functions $f(x) = C \exp(-t|x|^2)$, $t > 0$.

The Hardy uncertainty principle **[13]** precised this unique property of Gaussian functions: if γ is a Gaussian function, there is no function f such that $|f| \leq \gamma$ and $|\widehat{f}| \leq \widehat{\gamma}$, except for the function γ itself (or its multiples). Variants of this results were proved by Morgan **[22]**, Cowling-Price **[10]**, not to mention the work that has been done on Lie Groups. This was illustrated more recently by a lost result of Beurling **[16]**:

$$\int_{\mathbb{R}} |f(x)| |\widehat{f}(y)| \exp(2\pi |xy|) \, dx \, dy < \infty$$

implies that $f = 0$, while Gaussian functions make this integral finite when 2π is replaced by a smaller constant. This has been completed in **[7]**, and one actually has, as a corollary, the following version of Hardy uncertainty principle: if

$$|f(x) \widehat{f}(y)| \leq \exp(-2\pi |xy|)$$

then f is a Gaussian function. In this example, we see that we can ask functions to decrease exponentially in some directions, and not in other, and still get an uncertainty principle.

This paper is essentially about the study of functions satisfying estimates of the form

$$|f(x)| \leq \exp(-\pi |q(x)|), \ |\widehat{f}(\xi)| \leq \exp(-\pi |q'(\xi)|), \tag{0.1}$$

where q and q' are two quadratic forms. We ask for an exponential decrease in some directions, but not in regions close to the isotropic sets of the forms, where they vanish. The classical Hardy uncertainty principle corresponds to positive quadratic forms. Take for example as previously the case of quadratic forms on $\mathbb{R}^2$ defined by $q(x,y)=2xy$ and $q'(\xi,\eta)=2\xi\eta$. We ask for

$$|f(x,y)|\leq \exp(-2\pi|xy|),\ |\widehat{f}(\xi,\eta)|\leq \exp(-2\pi|\xi\eta|). \tag{0.2}$$

Here q and q' are not positive, and we cannot expect solutions to be integrable. Take for example

$$f(x,y)=\operatorname{sgn}(x)\exp(-2\pi|xy|).$$

It is not in any L^p space except L^∞. However, in the distribution sense, we have

$$\widehat{f}(\xi,\eta)=-i\operatorname{sgn}(\xi)\exp(-2\pi|\xi\eta|),$$

so that (0.2) is satisfied.

We see with this example that studying solutions of (0.1) requires to work on the level of distribution. In this setting, (0.1) can be rewritten in the following: study distributions f in the Schwartz space S' so that

$$f(\cdot)\exp(\pm\pi q(\cdot))\in S',\ \widehat{f}(\cdot)\exp(\pm\pi q'(\cdot))\in S'. \tag{0.3}$$

When q and q' are both positive quadratic forms, this corresponds to the classical Hardy uncertainty principle, except that it is stated in a distributional setting. In the simplest case, the conditions are

$$f(\cdot)\exp(\pi|\cdot|^2)\in S',\ \widehat{f}(\cdot)\exp(\pi|\cdot|^2)\in S'. \tag{0.4}$$

To solve this problem, we had to work with more regular objects than distributions. We do this using the Bargmann transform, which is essentially a convolution with a Gaussian function. If f is a tempered distribution, its Bargmann transform is the entire function defined by

$$\mathscr{B}(f)(z)=\exp\big(\frac{\pi}{2}z^2\big)f\star\gamma(z),$$

where $\gamma(x)=\exp(-\pi|x|^2)$. It has been introduced by Bargmann in [**3, 4**].

We already used the Bargmann transform in [**7**], even if not explicitly. There we studied functions satisfying Beurling type conditions, of the form

$$(1+|x|+|y|)^{-N}f(x)\widehat{f}(y)\exp(2\pi|xy|)\in L^1. \tag{0.5}$$

Even if it was the scheme of Hörmander's proof for Beurling theorem, regularity of f is not a direct consequence of (0.5), while Hardy's conditions imply directly that f extends to an entire function of order 2. Our trick was to convolve f with γ. Since f has to be a Hermite function, so does $g=f\star\gamma$. We showed that the new function g satisfies also (0.5). This is the Bargmann transform of f, up to a factor.

We go back to (0.4). We show in the first chapter that f is necessarily a Hermite function, namely $f(x) = P(x)\exp(-\pi|x|^2)$, where P is a polynomial. Equivalently, we prove that the Bargmann transform of f is a polynomial. This is done in two stages. First we show that (0.4) is equivalent to an estimate on the Bargmann transform of f. Then we conclude using a version of Phragmèn-Lindelöf principle. This is a scheme for all our proofs. This distributional version of Hardy uncertainty principle allows to recover known variants, including the result of Cowling-Price [**10**]. We exploit the Bargmann transform to have a distributional version of other uncertainty principles, including the one of Morgan, Beurling (generalizing the results of [**7**]), as well as directional uncertainty principle, mainly in one dimension, where conditions are stated only on the positive numbers axis for example. Let us mention, in this context, a characterization of Bargmann transforms of distributions which are tempered on one side.

When $q(x) = a|x|^2, q'(\xi) = b|\xi|^2$, with $ab > 1$ it follows that there are no solution for $ab > 1$, there are only Gaussian or Hermite solutions when $ab = 1$. The case when $ab < 1$ had partially been studied before (see for example [**17**]). We give here the structure of the distributions satisfying

$$f(\cdot)\exp(\pi a|\cdot|^2) \in S',\ \widehat{f}(\cdot)\exp(\pi b|\cdot|^2) \in S',\ ab < 1,$$

which are actually the members of a space of Gelfand and Shilov. Many Gaussian functions satisfy these estimates, including complex Gaussian functions, and it is easy to characterize them. We show that any other distribution with this property is an average of such Gaussian functions.

This is actually a phenomenon that will happen through the whole paper when considering other pairs of quadratic forms. We will study in general the space $\mathscr{G}(q, q')$ of tempered distributions f satisfying (0.3), given two quadratic forms, that we assume to be non degenerated. As for the case of positive forms, three cases will occur. When there are no Gaussian elements, we call the pair (q, q') a super-critical pair. We expect then $\mathscr{G}(q, q')$ to be small in some sense. We give sufficient conditions so that $\mathscr{G}(q, q') = \{0\}$, and so that it does not contain certain classes of integrable functions. When there are non-real Gaussian elements, we call the pair sub-critical, and critical in the other cases. We give precise characterizations of those pairs in terms of the spectral properties of the matrices of the two quadratic forms.

The case that will be of most interest to us is the critical case. The Gaussian elements of $\mathscr{G}(q, q')$ are then all real, and parameterized by a Group of matrices naturally associated to the quadratic forms. This leads to a natural conjecture on the structure of the elements of $\mathscr{G}(q, q')$: are all off them generated by the Gaussian functions, using averages as above? This conjecture seems even more natural when we have translated the problem on the level of entire functions, using the Bargmann transform. Such a result is established when one of the forms is positive, this is actually deduced from the one dimensional case of Hardy uncertainty principle. However when the two forms have a signature, this is not so simple, and we will not be able to conclude in general. The issue is that they may not be diagonalized in the same basis.

In extreme cases, not only they may not be diagonalized simultaneously, but the group that parametrizes the Gaussian functions contains only one element. This is the case for example when $q(x) = x_1^2 - x_2^2$ and $q'(\xi) = 2\xi_1\xi_2$ on $\mathbb{R}^2$. Then the conjecture is that any distribution f such that

$$f(x)\exp(\pm\pi(x_1^2 - x_2^2)) \in S',\ \widehat{f}(\xi)\exp(\pm 2\pi\xi_1\xi_2) \in S'$$

is a Hermite function $f(x) = P(x)\exp(-\pi|x|^2)$. We could not conclude up to now.

If we take $q(x,y) = 2xy$ and $q'(\xi,\eta) = 2\xi\eta$ on $\mathbb{R}^2$, the space $\mathscr{G}(q,q')$ contains the functions satisfying (0.2). The Gaussian functions in this case are

$$\gamma_t(x,y) = \exp\big(-\pi t x^2 - \frac{\pi}{t}y^2\big).$$

We show that any element of $\mathscr{G}(q,q')$ can be built up using the γ_t. For the example above we have

$$\mathrm{sgn}(x)\exp(-2\pi|xy|) = x\int_0^\infty \gamma_t(x,y)\frac{dt}{\sqrt{\pi t}}.$$

Now take the distribution $f(x,y) = 1(x)\otimes\delta(y)$, where δ is the Dirac mass. It is an element of $\mathscr{G}(q,q')$, since $\widehat{f}(\xi,\eta) = \delta(\xi)1(\eta)$. It is actually the limiting case of γ_t as $t\to 0$. If $\mathscr{F}_2$ is the Fourier transform with respect to the second variable, we have

$$\mathscr{F}_2 f(x,y) = 1 = \exp(-\pi(x^2+y^2)) + \pi(x^2+y^2)\int_0^1 \exp(-\pi t(x^2+y^2))\,dt$$

hence we can express f in terms of the γ_t:

$$f(x,y) = \gamma_1(x,y) + \pi x^2\int_0^1 \gamma_t(x,y)\sqrt{\frac{\pi}{t}}\,dt - \partial_y^2\int_0^1 \gamma_t(x,y)\frac{dt}{4\sqrt{\pi t}}.$$

We prove more generally that any element of $\mathscr{G}(q,q')$ can be decomposed in the following way:

$$f(x,y) = \sum_k P_k(x,y,\partial_x,\partial_y)\int \gamma_t(x,y)\,d\mu_k(t), \tag{0.6}$$

where the sum is finite, P_k are polynomials in x,y and in the partial differential operators ∂_x,∂_y, and μ_k are finite measures on $]0,\infty[$. Since we take derivatives, (0.6) is a distribution in general. However we show that it is regular away from the coordinate axis. Indeed, $f(x,y)$ defined by (0.6) is a real analytic function away from the axis, and satisfies an estimate of the form

$$|f(x,y)| \le C_\varepsilon(1+|x|+|y|)^N\exp(-2\pi|xy|)$$

whenever $|xy| > \varepsilon > 0$, as well as its Fourier transform. As shown by the example above, there are non zero solutions that vanish for $xy \ne 0$. They are exactly linear combinations of distributions of the form

$$\delta^{(k)}(x)\otimes y^l,\ \text{or}\ x^k\otimes\delta^{(l)}(y).$$

Our main results come when considering the analogue of the quadratic forms $2xy$ or x^2-y^2 in higher dimensions. The Lorentz form is defined by $q(x,y) = x_1^2+\cdots+x_d^2-y^2$, $x\in\mathbb{R}^d, y\in\mathbb{R}$. We are able to prove the same structure property as in (0.6) for the elements of $\mathscr{G}(q,q)$, except that the integrals are over the Lorentz group of matrices. The solutions have the property that they are real analytic inside the Lorentz cone, while they can be singular outside. We prove that no element of $\mathscr{G}(q,q)$ is supported in the set $\{q=0\}$, unlike in dimension 1. However we exhibit distributions that vanish inside the cone, as well as their Fourier transforms, without vanishing completely. We prove similar results when considering pairs (q,q') where q is the Lorentz form and q' is any form of the type $q'(\xi,\eta)=a_1\xi_1^2+\cdots+a_d\xi_d^2+a\eta^2$, with $a_i, a\neq 0$.

We do realize that this volume asks more questions than it solves. We organize it as follows. We begin with an history of uncertainty principles of Hardy type, and their different generalizations. We show how the use of the Bargmann transform significantly simplifies their proofs and unifies the results. We mainly focus on results of Hardy, Morgan, Beurling. In the second chapter we go further in details to get richer results, including the aforementioned Hardy uncertainty principle in the sub-critical case. We prove various uncertainty principles where any function or distribution satisfying the conditions is an average of Gaussian functions satisfying the same estimates. In the next chapter we start the study of Hardy uncertainty principle when considering non positive forms. This leads to a classification into critical, sub- and super-critical pairs. The critical pairs will be studied in more details in the fourth chapter. We state there the main conjectures on the structure of the spaces $\mathscr{G}(q,q')$, and the equivalent problems that arise on the level of entire functions. We then prove the main result when we have a Lorentz quadratic form, and variants.

CHAPTER 1

HARDY'S UNCERTAINTY PRINCIPLE AND ITS GENERALIZATIONS

1.1. Hardy's uncertainty principle

Throughout this text we will use the following terminology.

DEFINITION 1.1.1. – *Let A be a real symmetric matrix. It is positive if*

$$\langle Ax, x\rangle > 0$$

whenever $x \neq 0$. It is semi-positive if $\langle Ax, x\rangle \geq 0$ for all x.

A symmetric matrix A is positive if and only if its eigenvalues are positive. It is semi-positive if and only if its eigenvalues are non-negative. We denote by I the identity matrix.

DEFINITION 1.1.2. – *A Gaussian function is a function of the form*

$$f(x) = \exp(-\pi\langle Ax, x\rangle),$$

where A is a positive symmetric matrix. The Fourier transform of f is $\det(A)^{-1/2}$ $\exp(-\pi\big\langle A^{-1}x, x\big\rangle)$. *A Hermite function is a function of the form*

$$f(x) = P(x)\exp(-\pi\langle Ax, x\rangle),$$

where P is a polynomial. The Fourier transform of f has the form

$$Q(x)\exp(-\pi\big\langle A^{-1}x, x\big\rangle),$$

where Q is a polynomial of the same degree as P.

The standard Gaussian function is

$$\gamma(x) = \exp(-\pi|x|^2).$$

We have $\gamma = \widehat{\gamma}$.

Hardy's uncertainty principle is the following, see [**14**].

THEOREM 1.1.3. – *Let A, B be two positive matrices, and $N \in \mathbb{R}$. Let $f \in L^2(\mathbb{R}^d)$ such that, for almost all $x, \xi \in \mathbb{R}^d$,*

$$|f(x)| \leq C(1+|x|)^N \exp(-\pi\langle Ax, x\rangle), \tag{1.1}$$

$$|\widehat{f}(\xi)| \leq C(1+|\xi|)^N \exp(-\pi\langle B\xi, \xi\rangle). \tag{1.2}$$

If $A - B^{-1}$ has a positive eigenvalue, then $f = 0$. If $A = B^{-1}$, then there exists a polynomial P, of degree at most N, such that

$$f(x) = P(x)\exp(-\pi\langle Ax, x\rangle).$$

When $f = 0$ is the only possible conclusion, we will speak of *the weak Hardy's uncertainty principle*, and of *strong Hardy's uncertainty principle* when $A = B^{-1}$ and $N \geq 0$.

Many generalizations of Theorem 1.1.3 in different directions have been given. In [**10**], the following result is obtained:

THEOREM 1.1.4. – *Let $a, b > 0$ with $ab \geq 1$, and $f \in S'(\mathbb{R})$. Let $1 \leq p, q \leq \infty$. Assume that*

$$f(\cdot)\exp(\pi a|\cdot|^2) \in L^p(\mathbb{R}),\ \widehat{f}(\cdot)\exp(\pi b|\cdot|^2) \in L^q.$$

Then $f = 0$ unless $p = q = \infty$ and $ab = 1$.

The condition is optimal since the Gaussian function $f(x) = \exp(-\pi a x^2)$ is a solution for $ab = 1$ and $p = q = \infty$. The corresponding statement for $\mathbb{R}^d$ was first obtained in [**7**]. Functions satisfying close conditions are proved to be Hermite functions:

THEOREM 1.1.5. – *Let $N \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$. Assume that*

$$(1+|x|)^{-N} f(x)\exp(\pi|x|^2) \in L^1(\mathbb{R}^d),\ (1+|\xi|)^{-N}\widehat{f}(\xi)\exp(\pi|\xi|^2) \in L^1(\mathbb{R}^d).$$

Then $f(x) = P(x)\exp(-\pi|x|^2)$, where P is a polynomial of degree less than $N - d$.

Theorem 1.1.4 is a consequence of Theorem 1.1.5. We have stated Theorem 1.1.5 for the standard Gaussian function $\exp(-\pi|x|^2)$. The general formulation, with matrices A and B as in Hardy's uncertainty principle, can be done in the same way (see Theorem 1.3.5).

Morgan gave in [**22**] the following version of the uncertainty principle, where the Gaussian functions have been replaced by a more general family.

THEOREM 1.1.6. – *Let $1 < p < 2$, q be the conjugate exponent, and $a, b > 0$. Let $f \in L^2(\mathbb{R})$ such that for almost all $x, \xi \in \mathbb{R}$,*

$$|f(x)| \leq C\exp(-2\pi p^{-1}a^p|x|^p),\ |\widehat{f}(\xi)| \leq C\exp(-2\pi q^{-1}b^q|\xi|^q).$$

If $ab > |\cos(\frac{p\pi}{2})|^{1/p}$, then $f = 0$.

This is an intermediate result between Paley-Wiener-Schwartz's Theorem, corresponding to $p = 1$, and Hardy's uncertainty principle. Morgan gives a family of solutions when $ab = |\cos(\frac{p\pi}{2})|^{1/p}$. The characterization of all possible solutions may be difficult, since he shows the following: given any $N \in \mathbb{R}$ and $ab = |\cos(\frac{p\pi}{2})|^{1/p}$, one can find a nonzero $f \in L^2(\mathbb{R})$ and $M \in \mathbb{R}$ such that

$$|f(x)| \leq (1+|x|)^N \exp(-2\pi p^{-1} a^p |x|^p), \ |\widehat{f}(\xi)| \leq (1+|\xi|)^M \exp(-2\pi q^{-1} b^q |\xi|^q)$$

Unlike Theorem 1.1.3, N may take negative values. Another version of Theorem 1.1.6 has been given in [**7**]:

Theorem 1.1.7. – *Let $1 < p < 2$, q be the conjugate exponent, and $a, b > 0$. Let $f \in L^2(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} |f(x)| \exp(2\pi p^{-1} a^p |x|^p)\, dx < \infty, \ \int_{\mathbb{R}} |\widehat{f}(\xi)| \exp(2\pi q^{-1} b^q |\xi|^q)\, d\xi < \infty.$$

If $ab > |\cos(\frac{p\pi}{2})|^{1/p}$, then $f = 0$.

The proofs of Theorems 1.1.3, 1.1.4, 1.1.6 and 1.1.7 are very similar and rely on Phragmèn-Lindelöf principle, which can be stated as follows.

Lemma 1.1.8 (Phragmèn-Lindelöf). – *Let $\alpha \geq 1$. Let F be an analytic function of order α in a domain delimited by two lines forming an angle less than $\frac{\pi}{\alpha}$. Assume that F is continuous on the closure of the domain, and has polynomial growth of order N on each line of the boundary. Then it has polynomial growth of order N in the whole domain.*

See [**14**] for details. One can sketch the original proofs of Hardy's and Morgan's uncertainty principles as follows: first observe that the conditions given on f and $\widehat{f}$ enable us to extend them to entire functions. Then one tries to apply Lemma 1.1.8 or its numerous variants (see [**18, 27**]) to f. The proof of Theorem 1.1.5 is slightly different, since we apply Phragmèn-Lindelöf principle to an auxiliary function, obtained by convolution of f with a Gaussian function.

In the next section we will introduce the Bargmann transform, a tool that will be used throughout this paper. We will show that it can be used to unify these proofs and give further generalizations of Theorems 1.1.5 and 1.1.7.

1.2. The Bargmann transform

As mentioned before, the auxiliary function used in the proof of Theorem 1.1.5 is a convolution of f by a Gaussian function. This is almost the classical tool known as the Bargmann transform of f. We still denote by γ the standard Gaussian function $\gamma(x) = \exp(-\pi|x|^2)$.

DEFINITION 1.2.1. – *The Bargmann transform of a tempered distribution $f \in S'(\mathbb{R}^d)$ is defined, for $z \in \mathbb{C}^d$, by*

$$\mathscr{B}(f)(z) = \exp\big(\frac{\pi}{2}z^2\big) f \star \gamma(z). \tag{1.3}$$

Here, $z^2 = z_1^2 + \cdots + z_d^2$.

We denote by $\langle\ \rangle$ the duality bracket between the Schwartz space $S(\mathbb{R}^d)$ and the tempered distributions $S'(\mathbb{R}^d)$. Many properties of $\mathscr{B}$ are shown in [**3, 4**]. For example it is injective. More generally we have the following Lemma, that will be useful later on:

LEMMA 1.2.2. – *Let $t > 0$ and $f \in S'(\mathbb{R}^d)$. Assume that for all polynomial P we have*

$$\big\langle f, P(\cdot)\exp(-t|\cdot|^2)\big\rangle = 0. \tag{1.4}$$

Then $f = 0$.

Proof. – After a change of variables we may assume that $t = \pi$. Relation (1.4) is equivalent to $\mathscr{B}(f) = 0$. Hence $f \star \gamma = 0$, or $\widehat{f}\gamma = 0$, and consequently $f = 0$. □

The Bargmann transform was initially used as an isomorphism from $S'(\mathbb{R}^d)$ into the Fock Space $\mathscr{F}$, that is defined as follows.

DEFINITION 1.2.3. – *The Fock space $\mathscr{F}$ is the space of entire functions F on $\mathbb{C}^d$, such that there exists C and $N > 0$, such that for all $z \in \mathbb{C}^d$,*

$$|F(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{2}|z|^2\big). \tag{1.5}$$

DEFINITION 1.2.4. – *The space $\mathscr{F}_0$ is the space of entire functions F on $\mathbb{C}^d$, such that for all $N > 0$, there exists C_N, such that for all $z \in \mathbb{C}^d$,*

$$|F(z)| \le C_N(1+|z|)^{-N} \exp\big(\frac{\pi}{2}|z|^2\big). \tag{1.6}$$

Consider the topology on $\mathscr{F}_0$ given by the semi-norms

$$Q_N(F) = \sup_z |F(z)|(1+|z|)^N \exp\big(-\frac{\pi}{2}|z|^2\big).$$

Then the dual space of $\mathscr{F}_0$ can be identified with $\mathscr{F}$: any continuous linear form on $\mathscr{F}_0$ can be written as

$$F \longrightarrow \int_{\mathbb{C}^d} F(z)\overline{G(z)}\exp(-\pi|z|^2)\, dV(z),$$

for a uniquely determined $G \in \mathscr{F}$. Here $dV(z)$ is a renormalization of the Lebesgue measure on $\mathbb{C}^d$. Recall that the topology of $S(\mathbb{R}^d)$ is defined by the semi-norms

$$P_N(\phi) = \sup_{|\alpha|\le N, x\in\mathbb{R}^d} (1+|x|)^N |\partial^\alpha \phi(x)|. \tag{1.7}$$

PROPOSITION 1.2.5. – *The Bargmann transform is a homeomorphism from the space $S'(\mathbb{R}^d)$ into $\mathscr{F}$, and from $S(\mathbb{R}^d)$ into $\mathscr{F}_0$.*

The inverse Bargmann transform is given by the following identity, which is the fundamental isometry relation for the Bargmann transform:

PROPOSITION 1.2.6. – *Let $f \in S'(\mathbb{R}^d)$, and $\phi \in S(\mathbb{R}^d)$. Then*

$$\langle \overline{f}, \phi \rangle = \int_{\mathbb{C}^d} \overline{\mathcal{B}(f)(z)}\, \mathcal{B}(\phi)(z) \exp(-\pi |z|^2)\, dV(z). \tag{1.8}$$

Another useful property is the following analogue of Parseval's Identity.

PROPOSITION 1.2.7. – *Let $f \in S'(\mathbb{R}^d)$. For all $z \in \mathbb{C}^d$, we have*

$$\mathcal{B}(f)(z) = \mathcal{B}(\widehat{f})(iz). \tag{1.9}$$

1.3. Hardy's Theorem on S'

1.3.1. Dimension one. – A simple computation shows that the Bargmann transform maps the space of Hermite functions of the form $P(x)\exp(-\pi x^2)$ into the space of polynomials. Thus Theorem 1.1.5 amounts to showing that the Bargmann transform of such a function is a polynomial. We can prove a more general version of Theorem 1.1.5:

THEOREM 1.3.1. – *Let $f \in S'(\mathbb{R})$. Then*

$$f(\cdot)\exp(\pi(\cdot)^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\exp(\pi(\cdot)^2) \in S'(\mathbb{R}) \tag{1.10}$$

if and only if there exists a polynomial P such that for all $x \in \mathbb{R}$, $f(x) = P(x)\exp(-\pi x^2)$.

Proof of Theorem 1.3.1. – Let F be the Bargmann transform of f. Since $f(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}^d)$, there exists N such that for all $\phi \in S(\mathbb{R})$,

$$\big|\big\langle f \exp(\pi(\cdot)^2), \phi(\cdot) \big\rangle\big| \le C P_N(\phi).$$

If we write $F(z)$ as the action of the distribution $f(\cdot)\exp(\pi(\cdot)^2)$ on the test function

$$\phi_z(x) = \exp\big(-2\pi x^2 + 2\pi x z - \frac{\pi}{2} z^2\big),$$

we obtain

$$\begin{aligned} |F(z)| &\le C P_N(\phi_z) \\ &\le C \sup_{x \in \mathbb{R}^d} (1 + |x| + |z|)^{2N} \exp\big(-2\pi |x|^2 + 2\pi |x|\,|\mathcal{R}e(z)| - \frac{\pi}{2}\mathcal{R}e(z^2)\big). \\ &\le C \sup_{r>0} (1 + r + |z|)^{2N} \exp\big(-2\pi(r - \mathcal{R}e(z)/2)^2 + \frac{\pi}{2}|\mathcal{I}m(z)|^2\big), \end{aligned}$$

and thus

$$|F(z)| \le C(1 + |z|)^{2N} \exp\big(\frac{\pi}{2}|\mathcal{I}m(z)|^2\big). \tag{1.11}$$

If we use now the hypothesis on $\widehat{f}$ and Formula (1.9), we obtain

$$|F(z)| \leq C(1+|z|)^{2M} \exp\left(\frac{\pi}{2}|\mathscr{R}e(z)|^2\right) \tag{1.12}$$

for some positive integer M. We conclude that F is a polynomial using the following lemma and Liouville's Theorem. □

LEMMA 1.3.2. – *Let F be a continuous function on $\Omega = \{z; \mathscr{R}e(z) \geq 0,\ \mathscr{I}m(z) \geq 0\}$, holomorphic in the interior. Assume that there exist $C, N > 0$ such that for all $z \in \Omega$,*

$$|F(z)| \leq C(1+|z|)^N \exp\left(|\mathscr{I}m(z)|^2\right).$$

Assume moreover that $|F(ix)| \leq C(1+|x|)^N$ for $x \geq 0$. Then

$$|F(z)| \leq C(1+|z|)^N$$

for all $z \in \Omega$.

Proof. – Consider, for $\varepsilon > 0$, the function $F_\varepsilon(z) = \exp\left(\frac{i}{2}\varepsilon z^2\right)F(z)$. By assumption, it has a polynomial growth of order N on $i\mathbb{R}^+$ and on the half-line $\{(x,y) \in \mathbb{R}^2_+; x = \varepsilon^{-1}y\}$, with constants independent of ε. Lemma 1.1.8 implies that this estimate is true between the lines, and thus $|F(z)| \leq C(1+|z|)^N$ in Ω. □

Theorem 1.3.1 corresponds to the critical case of Theorem 1.1.3. The super critical case is a corollary:

THEOREM 1.3.3. – *Let $f \in S'(\mathbb{R})$. Let $a, b \in \mathbb{R}$. Let $\mathscr{G}(a,b)$ be the space of tempered distributions f such that*

$$f(\cdot)\exp(\pi a(\cdot)^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\exp(\pi b(\cdot)^2) \in S'(\mathbb{R}). \tag{1.13}$$

If $ab > 1$ then $\mathscr{G}(a,b) = \{0\}$. If $ab = 1$ then any element of $\mathscr{G}(a,b)$ can be written as $P(x)\exp(-\pi a x^2)$ for some polynomial P.

1.3.2. Higher dimensions. – We will now give a distributional version of Theorem 1.1.3 in any dimension. For that purpose we need the following result which proves that a too fast Gaussian decay on one direction of $\mathbb{R}^d$ is impossible, except for the zero distribution.

THEOREM 1.3.4. – *Let $a > 1$ and $f \in S'(\mathbb{R}^d)$. Assume that*

$$f(x)\exp(\pi x_1^2) \in S'(\mathbb{R}^d),\ \widehat{f}(\xi)\exp(a\pi\xi_1^2) \in S'(\mathbb{R}^d). \tag{1.14}$$

Then $f = 0$.

Proof. – Let $\psi \in S(\mathbb{R}^{d-1})$. Consider the distribution $T_\psi \in S'(\mathbb{R})$ defined by

$$\langle T_\psi, \phi\rangle = \langle f, \phi \otimes \psi\rangle,$$

where $(\phi \otimes \psi)(x) = \phi(x_1)\psi(x_2, \ldots, x_d)$. Then (1.14) implies that $T_\psi(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R})$ and $\widehat{T_\psi}(\cdot)\exp(a\pi|\cdot|^2) \in S'(\mathbb{R})$. Theorem 1.3.3 implies that $T_\psi = 0$. This is true for all $\psi \in S(\mathbb{R}^{d-1})$, and we conclude that $f = 0$. □

THEOREM 1.3.5. – *Let A, B be two symmetric matrices, with A positive and B invertible. Let $f \in S'(\mathbb{R}^d)$ such that*

$$f(\cdot)\exp(\pi\langle A\cdot,\cdot\rangle) \in S'(\mathbb{R}^d),\ \widehat{f}(\cdot)\exp(\pm\pi\langle B\cdot,\cdot\rangle) \in S'(\mathbb{R}^d). \tag{1.15}$$

If AB has an eigenvalue λ such that $|\lambda| > 1$, then $f = 0$. If all its eigenvalues are 1 *or* -1*, then*

$$f(x) = P(x)\exp(-\pi\langle Ax, x\rangle),$$

where P is a polynomial.

Here I is the identity matrix.

Proof. – Note that AB is conjugated to the symmetric matrix $A^{1/2}BA^{1/2}$, and hence it is diagonalizable. Let $Q \in O_n(\mathbb{R})$ such that ${}^tQA^{1/2}BA^{1/2}Q$ is diagonal, with diagonal coefficients $b_1, \dots, b_n$. Put $P = A^{-1/2}Q$, and $g(x) = f(Px)$.

We are lead to characterize g such that

$$g(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}^d),\ \widehat{g}(\xi)\exp(\pm\pi(b_1\xi_1^2 + \cdot + b_d\xi_d^2)) \in S'(\mathbb{R}^d), \tag{1.16}$$

where $b_i \in \mathbb{R} \setminus \{0\}$ are the eigenvalues of AB.

Assume that $\max_i |b_i| > 1$. Suppose for simplicity that $|b_1| > 1$. Let ψ be a fixed and compactly supported function on $\mathbb{R}^{d-1}$. Let T_ψ be the element of $S'(\mathbb{R})$ defined by

$$\langle T_\psi, \phi\rangle = \left\langle g, \phi \otimes \widehat{\psi}\right\rangle.$$

Its Fourier transform is defined by

$$\left\langle \widehat{T_\psi}, \phi\right\rangle = \langle \widehat{g}, \phi \otimes \psi\rangle.$$

We first use the fact that $g(x)\exp(\pi x_1^2) \in S'(\mathbb{R}^d)$, and we obtain

$$T_\psi(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}). \tag{1.17}$$

Next we use the inequality $|b_1\xi_1^2 + \cdot + b_d\xi_d^2| \geq |b_1|\xi_1^2 - |\sum_{i>1} b_i\xi_i^2|$, and the fact that ψ is compactly supported. We obtain

$$\widehat{T_\psi}(\cdot)\exp(\pi|b_1|\xi_1^2) \in S'(\mathbb{R}). \tag{1.18}$$

Since $|b_1| > 1$, Theorem 1.3.4 implies that $T_\psi = 0$. Since ψ is arbitrary, we conclude that $g = 0$.

Assume now that $|b_i| = 1$ for all i. Equations (1.17), (1.18), and Theorem 1.3.1 imply that

$$T_\psi(x_1) = P_\psi(x_1)\exp(-\pi x_1^2),$$

where P is a polynomial. The degree of P depends only on the orders of g and $\widehat{g}$, not on the choice of ψ. Hence one can write

$$g(x) = \sum_{k=0}^{N} x_1^k \exp(-\pi x_1^2) \otimes g_k(x_2, \dots, x_d),$$

where the g_k are tempered distributions, and N depends only on the orders of g and $\widehat{g}$. Now (1.16) implies that for all k,

$$f_k(x_2,\dots,x_d)\exp(\pi(x_2^2+\cdots+x_d^2))\in S'(\mathbb{R}^{d-1}),$$
$$\widehat{f_k}(\xi_2,\dots,\xi_d)\exp(\pm\pi(b_2\xi_2^2+\cdots+b_d\xi_d^2))\in S'(\mathbb{R}^{d-1}).$$

The result follows by induction. □

REMARK 1.3.6. – Theorems 1.1.3, 1.1.4 and 1.1.5 are direct corollaries of Theorem 1.3.5. Our proof simplifies all their classical proofs.

1.4. Morgan's uncertainty principle on $S'(\mathbb{R})$

In order to sate Morgan's Theorem in the distribution setting, we need cutoff functions. Throughout this paper, the letter χ will denote a smooth function on $\mathbb{R}$, vanishing in a neighborhood of the origin, and equal to 1 outside a compact set. Similarly, the letter χ_+ will denote a a smooth function equal to 1 in a neighborhood of $+\infty$, and vanishing on $]-\infty,1]$.

THEOREM 1.4.1. – *Let $1<p<2$, q be the conjugate exponent, and $a,b>0$. Let $f\in S'(\mathbb{R})$. Assume that*

$$f(\cdot)\chi(\cdot)\exp(2\pi p^{-1}a^p|\cdot|^p)\in S'(\mathbb{R}),\ \widehat{f}(\cdot)\chi(\cdot)\exp(2\pi q^{-1}b^q|\cdot|^q)\in S'(\mathbb{R}),\tag{1.19}$$

and that $ab>|\cos(\frac{p\pi}{2})|^{1/p}$. Then $f=0$.

Note that condition (1.19) is independent of the particular choice of χ. We have to formulate the hypotheses in this way, since the functions $|\cdot|^p$ and $|\cdot|^q$ are not smooth on $\mathbb{R}$. Recall that the constant $|\cos(\frac{p\pi}{2})|^{1/p}$ is optimal as shown by Morgan's examples in **[22]**.

Proof of Theorem 1.4.1. – We consider the function $F(z)=f\star\gamma(z)$ rather than the Bargmann transform itself. We have, by Proposition 1.2.7,

$$F(z)=\left\langle\widehat{f},\psi_z\right\rangle,$$

with $\psi_z(\xi)=\exp(-\pi\xi^2+2i\pi\xi z)$. We argue then as in the proof of Theorem 1.3.1, and estimate the semi-norms of the test functions involved.

Take the cut-off function χ such that $\chi(r)=1$ for $|r|>2$, and $\chi(r)=0$ for $|r|<1$. First use the fact that $\widehat{f}\in S'(\mathbb{R})$, so that there exists n, such that

$$\begin{aligned}\left|\left\langle\widehat{f},(1-\chi)\psi_z\right\rangle\right|&\le CP_n((1-\chi)\psi_z)\\&\le C\sup_{0<r<2}(1+r+|z|)^{2n}\exp(-\pi r^2+2\pi r|\mathscr{Im}(z)|).\end{aligned}\tag{1.20}$$

Now we use the fact that $\widehat{f}(\cdot)\chi(\cdot)\exp(2\pi q^{-1}b^q|\cdot|^q) \in S'(\mathbb{R})$. One can thus find some $m > 0$, such that

$$\begin{aligned} \left|\left\langle \widehat{f}, \chi\psi_z \right\rangle\right| &\leq CP_m\big(\chi\psi_z \exp(-2\pi q^{-1}b^q|\cdot|^q)\big) \\ &\leq C\sup_{r>1}(1+r+|z|)^{2m}\exp(-\pi r^2 + 2\pi r|\mathscr{Im}(z)| - 2\pi q^{-1}b^q r^q). \end{aligned} \tag{1.21}$$

Combining (1.20) and (1.21), we finally find that there exist $C, N > 0$ such that

$$|F(z)| \leq C\sup_{r>0}(1+r+|z|)^N \exp(-\pi r^2 + 2\pi r|\mathscr{Im}(z)| - 2\pi q^{-1}b^q r^q).$$

Then we use the identity $r|\mathscr{Im}(z)| \leq p^{-1}b^{-p}|\mathscr{Im}(z)|^p + q^{-1}b^q r^q$, and obtain

$$|F(z)| \leq C(1+|z|)^N \exp(2\pi p^{-1}b^{-p}|\mathscr{Im}(z)|^p). \tag{1.22}$$

We will show that for ε small enough, and $\xi \in \mathbb{R}$,

$$|F(\xi)| \leq C(\varepsilon)\exp(-2\pi p^{-1}(a-\varepsilon)^p|\xi|^p). \tag{1.23}$$

We will choose ε so that $(a-\varepsilon)b > |\cos(\frac{p\pi}{2})|^{1/p}$. Then, by a standard argument already used in **[7, 22, 23]**, it will follow from (1.22) and (1.23) that $F = 0$.

So we now prove Inequality (1.23). We argue as in the proof of Inequality 1.22. Writing $F(\xi)$ as the action of f on the test function $x \to \gamma(x-\xi)$, we can prove as well that

$$|F(\xi)| \leq C\sup_{r>0}(1+r+|\xi|)^N \exp(-\pi(r-|\xi|)^2 - 2\pi p^{-1}a^p r^p).$$

In order to estimate the right hand side, we use the following identity: whenever $0 < s < t$ and $\eta > 0$, there exists a constant $C(a,\eta)$ depending only on a and η, such that

$$2p^{-1}a^p(t^p - s^p) \leq C(a,\eta) + (t-s)^2 + \eta s^p.$$

Indeed, the left hand side is bounded by $2a^p(t-s)t^{p-1} \leq (t-s)^2 + a^{2p}t^{2p-2}$, which allows to conclude for $s > t/2$ (remember that $p < 2$); otherwise, we write $2p^{-1}a^p t^p \leq C(a) + \frac{1}{4}t^2 \leq C(a) + (t-s)^2$.

Hence, for $r \leq |\xi|$,

$$\begin{aligned} &(1+r+|\xi|)^N \exp(-\pi(r-|\xi|)^2 - 2\pi p^{-1}a^p r^p) \\ &\qquad\leq C(a,\varepsilon)(1+|\xi|)^N \exp(-2\pi p^{-1}(a-\varepsilon)^p|\xi|^p). \end{aligned}$$

For $r \geq |\xi|$, we write

$$\begin{aligned} &(1+r+|\xi|)^N \exp(-\pi(r-|\xi|)^2 - 2\pi p^{-1}a^p r^p) \\ &\qquad\leq (1+r+|\xi|)^N \exp(-\pi(r-|\xi|)^2)\exp(-2\pi p^{-1}a^p|\xi|^p) \\ &\qquad\leq C(1+|\xi|)^N \exp(-2\pi p^{-1}a^p|\xi|^p), \end{aligned}$$

and (1.23) follows. This proves that $F = 0$, and hence $f = 0$. □

1.5. Beurling's uncertainty principle

A particularly elegant generalization of Theorem 1.1.3 has been given by Beurling. The proof was first forgotten, and then Hörmander published one in [**13**]. The original statement is the following:

THEOREM 1.5.1. – *Let $f \in L^2(\mathbb{R})$. Then*

$$\iint_{\mathbb{R}^2} |f(x)\widehat{f}(y)| \exp(2\pi|xy|)\, dxy < \infty \tag{1.24}$$

if and only if $f = 0$.

This implies Hardy's uncertainty principle (on $\mathbb{R}$, when $A = B^{-1}$ and $N < -1$). in [**7**], we obtained a complete analogue of Beurling's Theorem, with a characterization of Hermite functions, in any dimension. We found then a bilinear version of this result in [**11**], which can be stated as follows.

THEOREM 1.5.2. – *Let $f, g \in L^2(\mathbb{R}^d)$, and $N \in \mathbb{R}$. Assume that*

$$\iint_{\mathbb{R}^{2d}} \frac{|f(x)\widehat{g}(y)| + |\widehat{f}(y)||g(x)|}{(1+|x|+|y|)^N} \exp(2\pi|\langle x, y\rangle|)\, dxy < \infty. \tag{1.25}$$

Then either $f = 0$, or $g = 0$, or f and g are Hermite functions,

$$f(x) = P(x)\exp(-\pi\langle Ax, x\rangle),\ g(x) = Q(x)\exp(-\pi\langle Ax, x\rangle),$$

where A is a positive matrix and P, Q are polynomials such that $deg(P) + deg(Q) < N - d$.

Here we encountered a difficulty: the quadratic form in the exponential is not positive or negative definite. Hence f and $\widehat{f}$ are not automatically entire functions, so we cannot apply a Phragmèn-Lindelöf principle to f or $\widehat{f}$. Let us remark that in dimension 1, Hörmander could do it in [**13**], using a specificity of Formula (1.24), and a tedious version of Phragmèn-Lindelöf principle.

We could overcome this difficulty in [**7**] by considering a convolution of f with a Gaussian function, instead of f itself. This is a natural choice since this new function has still to be a Hermite function. We showed that it also satisfies (1.25). It seems that this is the first use of the Bargmann transform in uncertainty principles.

Here we will show that the Bargmann transform can be used to get a generalization of Theorem 1.5.2 to the setting of distributions. The conditions are given on the tensor products $f \otimes \widehat{g}$ and $g \otimes \widehat{f}$:

THEOREM 1.5.3. – *Let $f, g \in S'(\mathbb{R}^d)$. Assume that*

$$f \otimes \widehat{g} \exp(\pm 2\pi\langle x, y\rangle) \in S'(\mathbb{R}^{2d}),\ \widehat{f} \otimes g \exp(\pm 2\pi\langle x, y\rangle) \in S'(\mathbb{R}^{2d}). \tag{1.26}$$

Then either $f = 0$, $g = 0$, or there exists an orthogonal decomposition of $\mathbb{R}^d$, that is $\mathbb{R}^d = E' \oplus E''$, such that the distributions f and g may be written as

$$f(x) = P(x', \partial_{x''})\exp(-\pi\langle Ax', x'\rangle),\ g(x) = Q(x', \partial_{x''})\exp(-\pi\langle Ax', x'\rangle), \tag{1.27}$$

where A is a real semi-positive symmetric matrix and P and Q are polynomials. Here x' and x'' are the orthogonal projections of x on E' and E''.

Proof. – Let us emphasize that now, in a distribution context, degenerate matrices A are allowed, as well as derivatives of Dirac masses. We may assume that $f \neq 0$ and $g \neq 0$. Denote by F and G the Bargmann transforms of f and g. We can write $F(z_1)G(-iz_2)$ as

$$\exp(\frac{\pi}{2}(z_1^2+z_2^2))\big\langle f(x)\otimes\widehat{g}(y), \exp(-\pi(x-z_1)^2-\pi(y-z_2))\big\rangle. \tag{1.28}$$

We will show as in **[7, 11]** that $F(z)G(-iz)$ is a polynomial. We use the same trick as in the proofs of Theorems 1.3.1 and 1.4.1, writing (1.28) as the action of the distribution $f(x)\otimes\widehat{g}(y)\exp(2\pi|\langle x,y\rangle|)$ against some test function. One actually distinguishes the cases $|\langle x,y\rangle|\leq 1$ and $|\langle x,y\rangle|\geq 1$ to avoid differentiability issues. Let $z_1, z_2 \in \mathbb{C}$. We find

$$\begin{aligned}|F(z_1)G(-iz_2)| \leq C \sup_{x,y\in\mathbb{R}^d} &(1+|x|+|y|+|z_1|+|z_2|)^N\\ &\times \exp(-\pi(|x|^2+|y|^2+2|\langle x,y\rangle|))\\ &\times \exp\big(2\pi\langle x, \mathcal{R}e(z_1)\rangle + 2\pi\langle y, \mathcal{R}e(z_2)\rangle - \frac{\pi}{2}\,\mathcal{R}e(z_1^2+z_2^2)\big).\end{aligned}$$

Put $R^2 = |x|^2+|y|^2+2|\langle x,y\rangle| = \max(|x+y|^2, |x-y|^2)$. Then

$$\begin{aligned}|F(z_1)G(-iz_2)| &\leq C \sup_{R>0}(1+R+|z_1|+|z_2|)^N\exp(-\pi R^2)\\ &\qquad\times \exp\big(\pi R(|\,\mathcal{R}e(z_1+z_2)|+|\,\mathcal{R}e(z_1-z_2)|) - \frac{\pi}{2}\,\mathcal{R}e(z_1^2+z_2^2)\big)\\ &\leq C(1+|z_1|+|z_2|)^N\\ &\qquad\times \exp\big(\frac{\pi}{2}|\,\mathcal{R}e(z_1-z_2)||\,\mathcal{R}e(z_1+z_2)| + \frac{\pi}{2}|\,\mathcal{I}m(z_1,z_2)|^2\big).\end{aligned} \tag{1.29}$$

Using the hypothesis on $\widehat{f\otimes g}$, we can prove as well that

$$\begin{aligned}|F(z_1)G(-iz_2)| &\leq C(1+|z_1|+|z_2|)^N\\ &\qquad\times \exp\big(\frac{\pi}{2}|\,\mathcal{I}m(z_1+z_2)||\,\mathcal{I}m(z_1-z_2)| + \frac{\pi}{2}|\,\mathcal{R}e(z_1,z_2)|^2\big).\end{aligned} \tag{1.30}$$

Next, apply Lemma 1.3.2 to the function $F(z)G(-iz)$. We have

$$|F(z)G(-iz)| \leq C(1+|z|)^N\exp(\pi\min(|\,\mathcal{R}e(z)|^2, |\,\mathcal{I}m(z)|^2)),$$

and hence $F(z)G(-iz)$ is a polynomial in z.

We conclude as is **[7, 11]**, using a standard argument for entire functions of order 2, that F and G have the form

$$F(z) = P(z)\exp(\frac{\pi}{2}\langle Bz,z\rangle),\ G(z) = Q(z)\exp(\frac{\pi}{2}\langle Bz,z\rangle), \tag{1.31}$$

where B is a symmetric complex matrix, and P, Q are polynomials.

It follows from homogeneity and (1.29) that for all $z, \zeta \in \mathbb{C}^d$,

$$|\mathscr{Re}\langle Bz, \zeta\rangle| \leq |\mathscr{Re}(z)||\mathscr{Re}(\zeta)| + \frac{1}{2}(|\mathscr{Im}(z)|^2 + |\mathscr{Im}(\zeta)|^2).$$

Taking z real and ζ imaginary yields $\mathscr{Im}(B) = 0$. While if we take both real, we get that $I - B$ $I + B$ are semi-positive. Put $E'' = \mathrm{Ker}(I + B)$, $E' = E''^{\perp}$, and let B'' be the restriction of B to E''. The inverse Bargmann transform gives (1.27), with $A = (I + B'')^{-1}(I - B'')$. □

As a corollary, one can give a more precise result than Theorem 1.3.1, in view of the degrees of the polynomials involved.

THEOREM 1.5.4. – *Let $f, g \in S'(\mathbb{R}^d)$, and $N > 0$. Assume that*

$$f \otimes \widehat{g} \exp(\pm 2\pi\langle x, y\rangle) \in S'(\mathbb{R}^{2d})$$

and

$$(1 + |x| + |y|)^{-N}|g(x)\widehat{f}(y)| \exp(2\pi|\langle x, y\rangle|) \in L^1(\mathbb{R}^{2d}).$$

Then either $f = 0$, $g = 0$, or f and g can be written as

$$f(x) = P(x)\exp(-\pi\langle Ax, x\rangle),\ g(x) = Q(x)\exp(-\pi\langle Ax, x\rangle),$$

where A is a real positive symmetric matrix, and P and Q are polynomials such that $deg(P) + deg(Q) < N - d$. In particular $f = 0$ or $g = 0$ as soon as $N \leq d$.

The difference with Theorem 1.5.2 is that only one condition of integrability is sufficient to characterize Hermite functions.

1.6. One-directional conditions

In this section we discuss other versions of Theorems 1.3.1, 1.4.1 and 1.5.3. Either the proofs can be done as in the previous section, or they are just corollaries of those theorems.

We can state one-directional versions of Hardy's uncertainty principle.

THEOREM 1.6.1. – *Let $f \in S'(\mathbb{R}^d)$. Assume that*

$$f(x)\exp(\pi x_1^2) \in S'(\mathbb{R}^d) \text{ and } \widehat{f}(\xi)\exp(\pi\xi_1^2) \in S'(\mathbb{R}^d).$$

Then there exists an integer $N \geq 0$ and distributions $f_k \in S'(\mathbb{R}^{d-1})$ such that f may be written as

$$f(x) = \sum_{k=0}^{N} x_1^k \exp(-\pi x_1^2) \otimes f_k(x_2, \ldots, x_d).$$

Proof. – We proceed as in the proof of Theorem 1.3.4. Let $\psi \in S'(\mathbb{R}^{d-1})$ and define the distribution T_ψ on $S(\mathbb{R})$ by

$$\langle T_\psi, \phi\rangle = \langle f, \phi \otimes \psi\rangle.$$

We have $T_\psi(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R})$ and $\widehat{T_\psi}(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R})$. It follows from Theorem 1.3.1 that T_ψ is a Hermite function. Since the order of T_ψ depends only on the order of f, this polynomial has a degree N independent of ψ. Hence we can write

$$T(x) = \sum_{k=0}^{N} x^k \exp(-\pi|x|^2) a_k(\psi).$$

We immediately see that the a_k are tempered distributions of $\mathbb{R}^{d-1}$, and the result follows. □

In particular we have the following.

COROLLARY 1.6.2. – *Let $f \in S'(\mathbb{R}^d)$. Assume that*

$$f(x)\exp(\pi x_i^2) \in S'(\mathbb{R}^d),\ \widehat{f}(\xi)\exp(\pi\xi_i^2) \in S'(\mathbb{R}^d),$$

for all $i = 1, \dots, d$. Then $f(x) = P(x)\exp(-\pi|x|^2)$ for some polynomial P.

We obtain an analogue for Morgan's Theorem:

THEOREM 1.6.3. – *Let $1 < p < 2$ and q be the conjugate exponent. Let $a, b > 0$. Assume that*

$$f(x)\chi(x_1)\exp(2\pi p^{-1}a^p|x_1|^p) \in S'(\mathbb{R}^d).\ \widehat{f}(\xi)\chi(\xi_1)\exp(2\pi q^{-1}b^q|\xi_1|^q) \in S'(\mathbb{R}^d).$$

If $ab > |\cos(\frac{p\pi}{2})|^{1/p}$, then $f = 0$.

REMARK 1.6.4. – The conclusions of Theorems 1.6.1 and 1.6.3 are false if the conditions given do not hold on the same coordinate for the function and the Fourier transform. A counter-example is given the function $\psi(x_1)\widehat{\psi}(x_2)$ on $\mathbb{R}^2$, where ψ is compactly supported.

Nazarov gave in **[23]** an interesting analogue of Theorem 1.1.6, which can be called a one-sided uncertainty principle. It only asks for Morgan's conditions on one half-line. We can generalize this to the setting of tempered distributions. Recall that χ_+ is a smooth function vanishing on $]-\infty, 1]$ and equal to 1 on a neighborhood of $+\infty$.

THEOREM 1.6.5. – *Let $1 < p < 2$ and q be the conjugate exponent. Let $a, b > 0$ and $f \in S'(\mathbb{R}^d)$. Assume that*

$$\begin{aligned} &f(x)\exp(2\pi p^{-1}a^p|x_1|^p)\chi_+(x_1) \in S'(\mathbb{R}^d), \\ &\widehat{f}(\xi)\exp(2\pi q^{-1}b^q|x_1|^q)\chi_+(\xi_1) \in S'(\mathbb{R}^d). \end{aligned} \tag{1.32}$$

If $ab > \sin(\frac{\pi}{p})$, then $f = 0$.

Proof. – We can assume that $d = 1$. We consider the entire function $F(z) = f \star \gamma(z)$. We have, by Proposition 1.2.7, $F(z) = \left\langle \widehat{f}, \phi_z \right\rangle$, with $\phi_z(\xi) = \exp(-\pi\xi^2 + 2i\pi\xi z)$. Write

$$F(z) = \left\langle \widehat{f}, (1-\chi_+)\phi_z \right\rangle + \left\langle \widehat{f}, \chi_+\phi_z \right\rangle.$$

We will show that for $\mathcal{I}m(z) < 0$,

$$|F(z)| \leq C(1+|z|)^N \exp(2\pi p^{-1} b^{-p} |\mathcal{I}m(z)|^p). \tag{1.33}$$

Indeed, since $f \in S'(\mathbb{R})$, we can find M such that $\left|\left\langle \widehat{f}, \phi \right\rangle\right| \leq C P_M(\phi)$ for all Schwartz function ϕ. Hence

$$\begin{aligned} \left|\left\langle \widehat{f}, (1-\chi_+)\phi_z \right\rangle\right| &\leq C P_M((1-\chi)\phi_z) \\ &\leq C \sup_{r\leq 2} (1+|r|+|z|)^{2M} \exp(-\pi r^2 - 2\pi r\, \mathcal{I}m(z)) \\ &\leq C(1+|z|)^{2M} \exp(4\pi |\mathcal{I}m(z)|). \end{aligned}$$

This is smaller than (1.33) since $p > 1$. As in the proof of Theorem 1.4.1, we have as well

$$\left|\left\langle \widehat{f}, \chi_+\phi_z \right\rangle\right| \leq C(1+|z|)^N \exp(2\pi p^{-1} b^{-p} |\mathcal{I}m(z)|^p).$$

We show now that for $\xi > 0$,

$$|F(\xi)| \leq C(\varepsilon) \exp(-2\pi p^{-1} (a-\varepsilon)^p |\xi|^p), \tag{1.34}$$

for arbitrary small ε. Let $\gamma_\xi(x) = \exp(-\pi(x-\xi)^2)$. In the same way, one can prove that

$$|\langle f, (1-\chi_+)\gamma_\xi \rangle| \leq C(1+|\xi|)^M \exp(-\pi(|\xi|-2)^2),$$

which is smaller than (1.34), since $p < 2$. The estimate on $\langle f, \chi\gamma_\xi \rangle$ is done in the same way as in the proof of Theorem 1.4.1.

Finally, we use Phragmèn-Lindelöf principle to show that the estimates (1.33) and (1.34) imply that $f = 0$, as long as we choose ε so that $(a-\varepsilon)b > \sin(\frac{\pi}{p})$. Details on this last point may be found in **[23]**, but let us repeat briefly the argument. Choose $\varepsilon > 0$ and $A > 0$ such that $(a-\varepsilon)b > A > \sin(\pi/p)$. Consider the function $G(z) = F(z^{1/p}) \exp(2\pi p^{-1} b^{-p} A^p z)$. It is analytic for $\mathcal{I}m(z) < 0$ and continuous to the boundary. Moreover, (1.33) and (1.34) imply that $G(Re^{i\theta})$ is exponentially decreasing for $\theta = 0$ and $\theta = -\pi + \eta$, with $\eta > 0$ small enough. By Lemma 1.1.8, G is in particular bounded for $\mathcal{I}m(z) \leq 0$. But the exponential decay on the boundary implies that

$$\int_{\mathbb{R}} \frac{\log|G(x)|}{1+x^2}\, dx = -\infty.$$

Hence Jensen's condition is not satisfied, unless $G = 0$ (see **[8, 18]**). $\square$

Unlike the case of Morgan's Theorem, we do not know examples of solutions for $ab = \sin(\pi/p)$. Nazarov gives in **[23]** an entire function f on $\mathbb{C}$, for which there exist constants $\delta, \alpha > 0$, such that

$$|f(z)| \leq \exp(2\pi p^{-1} |\mathcal{I}m(z)|^p + o(|\mathcal{I}m(z)|^p))$$

for all $z \in \mathbb{C}$ (not only $\mathscr{I}m(z) < 0$), $|f(x)| \leq \exp(-2\pi p^{-1} \sin(\pi/p)^p |x|^p + o(|x|^p))$ for $x \geq 0$, and $f(x+iy) = O(\exp(-\delta|x|^p))$ for $|x| \geq \alpha|y|$. The following lemma, which links the growth of f on the imaginary axis with decay of the Fourier transform, proves that this example gives a solution for any $ab < \sin(\pi/p)$.

LEMMA 1.6.6. – *Let f be an entire function on $\mathbb{C}$ such that, for all $z \in \mathbb{C}$,*

$$|f(z)| \leq \exp(2\pi p^{-1} |\mathscr{I}m(z)|^p + o(|\mathscr{I}m(z)|^p)) \tag{1.35}$$

Assume that there exist $\delta, \alpha > 0$ such that $f(x+iy) = O(\exp(-\delta|x|^p))$ for $|x| \geq \alpha|y|$. Then

$$|\widehat{f}(\xi)| \leq \exp(-2\pi q^{-1}|\xi|^q + o(|\xi|^q))$$

as $\xi \to \pm\infty$, where $p^{-1} + q^{-1} = 1$.

Proof. – We first show that for any $y \in \mathbb{R}$, $R > 0$ and $n \in \mathbb{N}$,

$$|\widehat{f}(\xi)| \leq \frac{n!}{(2\pi|\xi|)^n} R^{-n} \exp\left(2\pi p^{-1} R^p + o(R^p)\right). \tag{1.36}$$

Indeed, by Cauchy formula,

$$|f^{(n)}(x)| \leq n!\, R^{-n} \sup_{|z-x|=R} |f(z)|.$$

When $|x| > (1+\alpha)R$, we use the exponential decay of f to find $C, \delta' > 0$ such that

$$|f^{(n)}(x)| \leq Cn!\, R^{-n} \exp(-\delta'|x|^p).$$

For $|x| \leq (1+\alpha)R$ we use (1.35) and find

$$|f^{(n)}(x)| \leq n!\, R^{-n} \exp(2\pi p^{-1} R^p + o(R^p)).$$

Both inequalities yield

$$\int_{\mathbb{R}} |f^{(n)}(x)|\, dx \leq n!\, R^{-n} \exp(2\pi p^{-1} R^p + o(R^p)),$$

which also gives (1.36). Now we just have to minimize (1.36) with respect to R and n. This is done taking $R^p = \dfrac{n}{2\pi}$ and n of the order $2\pi|\xi|^q$, where q is the conjugate exponent of p. This gives the required decay for $\widehat{f}$. □

We show now that we have analogues of Theorem 1.6.5 for Hardy's uncertainty principle. This corresponds to Theorem 1.6.5 for $p = 2$.

THEOREM 1.6.7. – *Let $f \in S'(\mathbb{R}^d)$ and $a, b > 0$ such that $ab > 1$. Assume that*

$$f(x)\chi_+(x_1) \exp(\pi a|x_1|^2) \in S'(\mathbb{R}^d), \quad \widehat{f}(\xi_1)\chi_+(\xi_1) \exp(\pi b|\xi_1|^2) \in S'(\mathbb{R}^d). \tag{1.37}$$

Then $f = 0$.

Proof. – We may assume that $a = b > 1$, and $d = 1$. Let F be the Bargmann transform of f. We proceed as in the proof of Theorem 1.6.5. The hypothesis on $\widehat{f}$ implies that

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi(1-a)}{2(1+a)}|\mathscr{I}m(z)|^2 + \frac{\pi}{2}|\mathscr{R}e(z)|^2\big) \tag{1.38}$$

for $\mathscr{I}m(z) < 0$, while the hypothesis on f implies that

$$|F(\xi)| \leq C\exp\big(\frac{\pi(1-a)}{2(1+a)}\xi^2\big) \tag{1.39}$$

for $\xi > 0$. Since $a > 1$, we have $|F(z)| \leq C\exp(-\delta|z|^2)$ for $z \in \mathbb{R}_+$ or $z \in i\mathbb{R}_-$, for some $\delta > 0$. The function $H(z) = F(\sqrt{z})$ is analytic on the lower half-plane, continuous on the boundary, and satisfies

$$\int_{\mathbb{R}} \frac{\log|H(x)|dx}{1+x^2} = -\infty.$$

We conclude as before that $f = 0$. □

The condition $ab > 1$ is sharp since the standard Gaussian function satisfies these conditions for $ab = 1$. However the same is valid for

$$f_{\alpha,\beta}(x) = \exp(-\pi(x+\alpha)^2 - 2i\pi\beta x), \tag{1.40}$$

where α, β are non-negative parameters. Its Fourier transform is given by

$$\widehat{f_{\alpha,\beta}}(\xi) = \exp(-\pi(\xi+\beta)^2 + 2i\pi\alpha(\xi+\beta)).$$

Hence we see that $f_{\alpha,\beta}(x)\exp(\pi x^2)$ and $\widehat{f_{\alpha,\beta}}(\xi)\exp(\pi\xi^2)$ are bounded for $x > 0$ and $\xi > 0$.

We can give a precise result in the critical case, when the condition on the Fourier space is two-sided, and when it is one-sided on f.

Theorem 1.6.8. – *Let $f \in S'(\mathbb{R})$. Assume that*

$$f(\cdot)\chi_+(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}). \tag{1.41}$$

Then there exists a tempered distribution μ with support in $]-\infty, 0]$, such that $f = \mu\star\gamma$. Conversely, every such function satisfies (1.41).

Proof. – Define μ by $\widehat{\mu} = \widehat{f}(\cdot)\exp(\pi|\cdot|^2)$. By assumption, $\mu \in S'(\mathbb{R})$, and $f = \mu\star\gamma$. We have to show that μ is supported by the negative axis. The distribution f, which is a function, extends to an entire function of order 2, since $\mu\star\gamma$ does. However we will not be able to exploit the condition given on f directly . Consider instead F, the Bargmann transform of f. We have $F(z) = \exp\big(\frac{\pi}{2}z^2\big)\mu\star\gamma\star\gamma(z)$, hence

$$F(z) = \langle\mu, \phi_z\rangle,$$

where $\phi_z(t) = 1/\sqrt{2}\exp\big(-\frac{\pi}{2}t^2 + \pi tz\big)$. The function F is the Laplace transform of the distribution $\nu = 1/\sqrt{2}\mu(\cdot)\exp\big(-\frac{\pi}{2}|\cdot|^2\big)$.

As in the proof of Theorem 1.6.7, we have the estimate

$$|F(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{2}|\,\mathscr{Im}\, z|^2\big)$$

for $\mathscr{Re}\, z > 0$. The assumption on $\widehat{f}$ implies that

$$|F(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{2}|\,\mathscr{Re}(z)|^2\big)$$

for all $z \in \mathbb{C}$. Now we use Lemma 1.3.2 to find that

$$|F(z)| \le C(1+|z|)^N$$

for $\mathscr{Re}(z) > 0$. Classical results on the Laplace transform (see [**15**], p 191) imply then that ν is supported in the negative axis. This proves that μ is supported by $]-\infty, 0]$.

□

Remark 1.6.9. – All functions of the form $f = \gamma \star \mu$ are entire functions of order 2 on $\mathbb{C}$, and if μ is supported in $]-\infty, 0]$, we have

$$|f(x)| \le C(1+x)^N \exp(-\pi x^2)$$

when $x \ge 0$, and

$$|f(x)| \le C(1+|x|)^N$$

when $x \le 0$. So the conditions on f can be restated as in Hardy's Theorem as a polynomial growth property. But we can not do the same on the Fourier transform side, since $\widehat{\mu}$ may not be a function (take for example the function μ equal to 1 on $\mathbb{R}^-$ and 0 on $\mathbb{R}^+$). If we take $\mu \in L^1$ supported in $]-\infty, -1]$, we have non zero $f \in S'(\mathbb{R})$ such that

$$\widehat{f}(\cdot)\exp(\pi|\cdot|^2) \in L^\infty(\mathbb{R}),\ f(\cdot)\exp(\pi|\cdot|^2) \in L^p(0,\infty),$$

for any value of $p \in [1, \infty]$. Compare this to Theorem 1.1.4.

We end with remarks on distributions satisfying

$$f(x)\chi_+(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\chi_+(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}). \tag{1.42}$$

All linear combinations of the form

$$f_\mu(x) = \int f_{\alpha,\beta}(x)\mu(\alpha,\beta)\, d\alpha\, d\beta \tag{1.43}$$

satisfy(1.42), as long as μ is, for example, a compactly supported distribution, with support inside the set $\Gamma = \{\alpha \ge 0, \beta \ge 0\}$.

We will use a very interesting property of the Bargmann transform, which can be called a one-sided estimate for the Bargmann transform. Let f be a general distribution, and assume that its Bargmann transform is well defined. This is the case for example when $f(\cdot)\exp(-\pi\delta|\cdot|^2) \in S'(\mathbb{R}^d)$, for some $0 \le \delta < 1$. Proposition 1.2.5 states that f is tempered if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,

$$|\mathscr{B}(f)(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{2}|z|^2\big).$$

We will show that the Bargmann transform characterizes the distributions f whose restriction to a half-line is tempered.

LEMMA 1.6.10 (One-sided estimates for the Bargmann transform). – *Let $f \in \mathscr{D}'(\mathbb{R})$ be a distribution. Assume that $f(\cdot)\exp\big(-\frac{\pi}{2}|\cdot|^2\big) \in S'(\mathbb{R})$, so that the Bargmann transform of f is well defined. Then $f\chi_+(\cdot)$ is a tempered distribution if and only if*

$$|\mathscr{B}(f)(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}|z|^2\big),$$

whenever $\mathscr{Re}(z) \geq 0$.

Proof. – The necessity of the condition is immediate, using semi-norms like previously. We want to show that the relation

$$\langle \overline{f}, \phi\rangle = C_0 \int_{\mathbb{C}^d} \overline{\mathscr{B}(f)(z)}\mathscr{B}(\phi)(z)\exp(-\pi|z|^2)\,dV(z) \tag{1.44}$$

is true whenever ϕ is a smooth function compactly supported in an interval $[a,b]$, with $a>0$. Note that this is true for any tempered distribution and any Schwartz function ϕ, by Proposition 1.2.6.

Let G be the function defined by

$$G(\delta) = \frac{C_0}{1+\delta}\int_{\mathbb{C}} \overline{\mathscr{B}(f_\delta)(z)}\mathscr{B}(\phi_\delta)(z)\exp(-\pi|z|^2)\,dV(z),$$

where $f_\delta(x) = f(x\sqrt{1+\delta})\exp(-\pi\delta x^2)$ and $\phi_\delta(x) = \phi(x\sqrt{1+\delta})\exp(+\pi\delta x^2)$. By assumption, for $\delta \geq 1$, $f_\delta \in S'(\mathbb{R})$, hence

$$G(\delta) = \frac{\langle \overline{f_\delta}, \phi_\delta\rangle}{1+\delta} = \frac{\langle \overline{f}, \phi\rangle}{(1+\delta)^{3/2}}.$$

We will show that G is real-analytic on $]0,2]$, continuous at 0, and (1.44) will follow.

Consider the function

$$G(\delta,z) = \int_a^b \phi(x)\exp\big(-\pi\frac{1-\delta}{1+\delta}x^2 + 2\pi x\frac{z}{\sqrt{1+\delta}} - \frac{\pi}{2}z^2\big)\,dx.$$

For $\delta \geq 1$, we have, after a change of variable,

$$G(\delta) = \int_{\mathbb{C}} \overline{\mathscr{B}(f)\Big(\frac{z}{\sqrt{1+\delta}}\Big)\exp\big(-\frac{\pi\delta}{2(1+\delta)}z^2\big)}G(\delta,z)\exp(-\pi|z|^2)\,dV(z). \tag{1.45}$$

We first show that this is a well defined expression when $0 \leq \delta \leq 2$. When $\mathscr{Re}(z) \geq 0$, we have by assumption

$$\begin{aligned}&\Big|\mathscr{B}(f)\Big(\frac{z}{\sqrt{1+\delta}}\Big)\exp\big(-\frac{\pi\delta}{2(1+\delta)}z^2\big)\Big| \\ &\qquad \leq C(1+|z|)^N\exp\big(\frac{\pi(1-\delta)}{2(1+\delta)}|\mathscr{Re}(z)|^2 + \frac{\pi}{2}|\mathscr{Im}(z)|^2\big).\end{aligned} \tag{1.46}$$

Using integrations by parts, and the fact that ϕ is compactly supported, for any M, there exists a constant C_M such that for all z,

$$|G(\delta,z)| \leq C_M(1+|z|)^{-M}\exp\Big(\frac{2\pi b|\mathscr{Re}(z)|}{\sqrt{1+\delta}} - \frac{\pi}{2}|\mathscr{Re}(z)|^2 + \frac{\pi}{2}|\mathscr{Im}(z)|^2\Big). \tag{1.47}$$

Combining this with (1.46), we get

$$\begin{aligned}&\Big|\mathscr{B}(f)\Big(\frac{z}{\sqrt{1+\delta}}\Big)\exp\Big(-\frac{\pi\delta}{2(1+\delta)}z^2\Big)\Big|\exp(-\pi|z|^2)|G(\delta,z)|\\&\quad\le C(1+|z|)^{N-M}\exp\Big(-\frac{\pi(1+2\delta)}{(1+\delta)}|\mathscr{R}e(z)|^2+2\pi b|\mathscr{R}e(z)|\Big)\\&\quad\le C(1+|z|)^{N-M}.\end{aligned}$$

Here C depends on b. Hence, choosing M big enough, we see that (1.45) is an absolutely convergent integral for $\mathscr{R}e(z)\ge 0$, and defines a real-analytic and continuous function for $0\le\delta\le 2$.

We now use the hypothesis $f\exp(-\pi/2|\cdot|^2)\in S'$. We write the expression $\mathscr{B}(f)(\frac{z}{\sqrt{1+\delta}})\exp(-\frac{\pi\delta}{2(1+\delta)}z^2)$ as $\langle f\exp\big(-\pi/2|\cdot|^2\big),\psi(\cdot,z)\rangle$, where

$$\psi(x,z)=\exp\Big(-\pi/2x^2+2\pi x\frac{z}{\sqrt{1+\delta}}-\frac{\pi}{2}z^2\Big).$$

We find the estimate

$$\begin{aligned}&\Big|\mathscr{B}(f)\Big(\frac{z}{\sqrt{1+\delta}}\Big)\exp\Big(-\frac{\pi\delta}{2(1+\delta)}z^2\Big)\Big|\\&\quad\le C(1+|z|)^{N'}\exp\Big(\frac{\pi(3-\delta)}{2(1+\delta)}|\mathscr{R}e(z)|^2+\frac{\pi}{2}|\mathscr{I}m(z)|^2\Big),\end{aligned}\tag{1.48}$$

that we will use for $\mathscr{R}e(z)\le 0$. For $G(\delta,z)$, we use the fact that when $a\le x\le b$ and $\mathscr{R}e(z)\le 0$, we have $x\,\mathscr{R}e(z)\le 0$. We obtain, for every $M>0$, a constant C_M such that for all z with $\mathscr{R}e(z)\le 0$,

$$|G(\delta,z)|\le C_M(1+|z|)^{-M}\exp\Big(-\frac{\pi}{2}|\mathscr{R}e(z)|^2+\frac{\pi}{2}|\mathscr{I}m(z)|^2\Big).\tag{1.49}$$

If we combine this with (1.48), we get

$$\begin{aligned}&\Big|\mathscr{B}(f)\Big(\frac{z}{\sqrt{1+\delta}}\Big)\exp\Big(-\frac{\pi\delta}{2(1+\delta)}z^2\Big)\Big|\exp(-\pi|z|^2)|G(\delta,z)|\\&\quad\le C(1+|z|)^{N'-M}\exp\Big(-\frac{2\pi\delta}{(1+\delta)}|\mathscr{R}e(z)|^2\Big).\\&\quad\le C(1+|z|)^{N'-M}.\end{aligned}$$

We conclude that $G(\delta)$ defined by (1.45) is continuous on $[0,2]$, and real-analytic on $]0,2[$.

It follows that

$$G(0)=\langle\overline{f},\phi\rangle.$$

We now look more carefully at the estimate (1.47) when $\delta=0$. Using integrations by parts, for any M, there exists a seminorm $P_{M'}$ on $S(\mathbb{R})$, such that for $\mathscr{R}e(z)\ge 0$,

$$|G(0,z)|\le CP_{M'}(\phi)(1+|z|)^{-M}\exp\Big(\frac{\pi}{2}|z|^2\Big).$$

Here C does not depend on b and a. Recall that the constant appearing in (1.49) depends only on a seminorm of ϕ. This proves that there exists M' and C, such that

$$\big|\langle \overline{f}, \phi\rangle\big| = |G(0)| \leq CP_{M'}(\phi),$$

and hence $f\chi_+$ is a tempered distribution. □

THEOREM 1.6.11. – *Let $f \in S'(\mathbb{R})$, and F be the Bargmann transform of f. Then*

$$f(\cdot)\chi_+(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\chi_+(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R}). \tag{1.50}$$

if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}$ with $\mathscr{R}e(z) \geq 0$,

$$|F(z)| \leq C(1+|z|)^N \exp(\frac{\pi}{2}|\mathscr{I}m(z)|^2),$$

and for all $z \in \mathbb{C}$ with $\mathscr{I}m(z) \leq 0$,

$$|F(z)| \leq C(1+|z|)^N \exp(\frac{\pi}{2}|\mathscr{R}e(z)|^2).$$

Proof. – The necessity of the estimates can be established as in the proof of Theorem 1.6.7. Assume now that the Bargmann transform of F has these properties.

Consider the distribution g defined by $g(x) = 1/\sqrt{2}f(x/\sqrt{2})\exp\big(\frac{\pi}{2}x^2\big)$. We have $\mathscr{B}(g)(z) = F(\sqrt{2}z)\exp\big(\frac{\pi}{2}z^2\big)$. Hence

$$|\mathscr{B}(g)(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}|z|^2\big)$$

whenever $\mathscr{R}e(z) \geq 0$. We also have $g(\cdot)\exp\big(-\frac{\pi}{2}|\cdot|^2\big) = f(\cdot/\sqrt{2}) \in S'(\mathbb{R})$. By Lemma 1.6.10, we have $g\chi \in S'(\mathbb{R})$, and hence $f(\cdot)\chi(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R})$. We apply the same method for $\widehat{f}$. □

REMARK 1.6.12. – It still remains open to characterize the entire functions F that satisfy both estimates (1.50).

CHAPTER 2

FURTHER RESULTS

This chapter is devoted to further extensions of theorems stated in the previous chapter. Hardy's uncertainty principle, as stated in the distribution case, does not give information on the structure of the solutions in the the case that we will call sub-critical, when there are a lot of solutions, including non Gaussian. This is when $ab < 1$ in Theorem 1.3.3. We show that in dimension 1, the solutions are linear and continuous combinations of the Gaussian solutions. We will encounter this situation a lot in the next chapters, where we study more general versions of Hardy Theorem. We cannot obtain such a precise result in higher dimensions, but we still prove that the solutions can also be built with Gaussian functions. All that is proved with the use of the Bargmann transform introduced in the previous chapter. This tool allows us to state and solve an equivalent problem on entire functions of order 2. At the end of the chapter we study also the one-sided Hardy conditions with the Bargmann transform, and state the conjecture on the form of the solutions.

2.1. Hardy's uncertainty principle in the sub-critical case, dimension 1

We consider here the case $ab < 1$ of Theorem 1.3.3. This amounts in this case to characterize the space $\mathscr{G}(a,b)$ of distributions f such that

$$f(\cdot)\exp(a\pi(\cdot)^2) \in S'(\mathbb{R}),\ \widehat{f}(\cdot)\exp(b\pi(\cdot)^2) \in S'(\mathbb{R}).$$

By Fourier inversion, $\mathscr{G}(a,b)$ is made of entire functions of order 2. We can actually prove the following:

Proposition 2.1.1. – *Let $f \in \mathscr{G}(a,b)$. Then f and $\widehat{f}$ satisfy pointwise estimates of the form*

$$|f(x)| \leq C(1+|x|)^N \exp(-\pi a x^2),\ |\widehat{f}(\xi)| \leq C(1+|\xi|)^N \exp(-\pi b \xi^2), \tag{2.1}$$

where C and N are constants depending only on f.

We see that we do not get any new elements in $\mathscr{G}(a,b)$ by giving conditions in $S'(\mathbb{R})$ instead of L^∞ conditions as in (2.1). However we will see that it is not true in higher dimensions. We need a lemma before proceeding to the proof of Proposition 2.1.1.

LEMMA 2.1.2. – *Let $f \in \mathscr{G}(a,b)$. Then f and $\widehat{f}$ extend to entire functions such that*

$$(2.2)\quad |f(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{b}|\mathscr{I}m(z)|^2\big),\ |\widehat{f}(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{a}|\mathscr{I}m(z)|^2\big),$$

where C and N depend only on f. Moreover, there exist $\varepsilon, \delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|y| \le \varepsilon|x|$,

$$(2.3)\qquad |f(x+iy)| \le C\exp(-\delta x^2),\ |\widehat{f}(x+iy)| \le C\exp(-\delta x^2).$$

Proof of Proposition 2.1.1, assuming Lemma 2.1.2. – We have

$$|\widehat{f}(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{a}|\mathscr{I}m(z)|^2\big),\ |\widehat{f}(x+iy)| \le C\exp(-\delta x^2)$$

when $|y| \le \varepsilon|x|$. As in the proof of Lemma 1.6.6, we have

$$|f(x)| \le C\frac{n!}{(2\pi|x|R)^n}R^M \exp\big(\frac{\pi}{a}R^2\big),$$

where M is an integer depending on N. Minimizing over n and R amounts to take $2\pi R^2 = an$ and for n the integer part of $2a\pi x^2$. We find

$$|f(x)| \le C(1+|x|)^{M'}\exp(-\pi a x^2)$$

for some $M' > M$. The estimate for $\widehat{f}$ is obtained in the same way. □

Proof of Lemma 2.1.2. – By Fourier inversion, we have

$$f(x) = \Big\langle \widehat{f}\exp(\pi b|\cdot|^2), \phi_x \Big\rangle,$$

where $\phi_x(\xi) = \exp(-\pi b\xi^2 + 2i\pi x\xi)$. The right hand side extends to an entire function (replacing x by any complex number). Let N be the order of $T = f\exp(\pi b|\cdot|^2)$. We have

$$|\langle T, \phi_z\rangle| \le C_N p_N(\phi_z),$$

where the semi-norm p_N is defined in (1.7). Hence

$$\begin{aligned}|f(z)| &\le C\sup_{\xi}(1+|\xi|+|z|)^N \exp(-\pi b\xi^2 + 2\pi|\xi||\mathscr{I}m(z)|)\\ &\le C'(1+|z|)^N\exp(\pi/b|\mathscr{I}m(z)|^2).\end{aligned}$$

The corresponding estimate for $\widehat{f}$ is obtained in the same way.

We now prove (2.3). Let $f_1(z) = f(z)\exp(a\pi z^2)$. The restriction of f_1 to the real axis is in $S'(\mathbb{R})$. Hence there exist a function g and n such that $g^{(n)} = f_1$, and g has polynomial growth. The function g extends, as f, to an entire function of order 2. Apply Phragmèn-Lindelöf principle to $g(z)\exp(iCz^2)$, for large C, in the domain $|y| \le |x|$. We get

$$|g(x+iy)| \le C(1+|x|+|y|)^m\exp(2C|xy|).$$

By Cauchy formula, we obtain for $f(z) = g^{(n)}(z)\exp(-a\pi z^2)$ the following estimate:

$$|f(x+iy)| \le C(1+|x|+|y|)^{m'}\exp(-a\pi x^2 + a\pi y^2 + 2C|xy|)$$

for $|y| \le |x|/2$. Then, if ε is small enough and we take $|y| \le \varepsilon|x|$, we get (2.3). □

Estimates (2.2) do not seem to characterize elements of $\mathscr{G}(a,b)$, since we actually need (2.3) to get (2.1). In order to characterize the elements of $\mathscr{G}(a,b)$, we need an equivalent definition in terms of an entire function. For that purpose, we will use the Bargmann transform introduced in Chapter 1.

LEMMA 2.1.3. – *Let $f \in S'(\mathbb{R})$, and $a \geq 0$. Let F be the Bargmann transform of f. Then $f(\cdot)\exp(a\pi(\cdot)^2) \in S'(\mathbb{R})$ if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}$,*

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\frac{1-a}{1+a}|\,\mathscr{Re}(z)|^2 + \frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big). \tag{2.4}$$

Proof. – Since $f(\cdot)\exp(a\pi|\cdot|^2) \in S'(\mathbb{R})$, there exist $C > 0$, $N \in \mathbb{N}$, such that

$$|\langle f, \phi\rangle| \leq CP_N(\exp(-a\pi(\cdot)^2)\phi(\cdot))$$

for all Schwartz function ϕ. We apply this to $\phi(x) = \exp(-\pi(x-z)^2 + \frac{\pi}{2}z^2)$, and obtain (2.4).

Now assume that F satisfies (2.4). This is equivalent to

$$|F(\sqrt{1+a}z)\exp\big(\frac{\pi}{2}az^2\big)| \leq C(1+|z|)^{2N}\exp\big(\frac{\pi}{2}|z|^2\big), \tag{2.5}$$

and by Proposition 1.2.5, the function $G(z) = F(\sqrt{1+a}z)\exp\big(\frac{\pi}{2}az^2\big)$ is the Bargmann transform of a tempered distribution T. But identifying the Bargmann transforms, we see that $f(x)\exp(\pi a x^2) = (1+a)^{1/2}T((1+a)^{1/2}x)$, and hence $f(\cdot)\exp(a\pi(\cdot)^2) \in S'(\mathbb{R})$. □

Using dilations, it is sufficient to study $\mathscr{G}(a,b)$ for $a = b < 1$. It follows from Lemma 2.1.3 that the elements of $\mathscr{G}(a,a)$ are characterized by two conditions on their Bargmann transform F:

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\frac{1-a}{1+a}|\,\mathscr{Re}(z)|^2 + \frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big). \tag{2.6}$$

and

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\frac{1-a}{1+a}|\,\mathscr{Im}(z)|^2 + \frac{\pi}{2}|\,\mathscr{Re}(z)|^2\big). \tag{2.7}$$

We note here that (2.6) and (2.7) imply that

$$|F(z)| \leq C_\delta \exp(\frac{\pi}{2}\delta|z|^2) \tag{2.8}$$

for some $\delta < 1$. in **[17]**, it was already established that (2.8) characterizes the functions f such that

$$f(x) = O(\exp(-\pi\lambda x^2)),\ \widehat{f}(\xi) = O(\exp(-\pi\lambda\xi^2))$$

for some $\lambda > 0$. Our point here is to show that (2.6) and (2.7) are a lot more precise, and enable a characterization for fixed λ.

Put $w = \frac{\pi}{2(1+a)} z^2$. By considering the odd and even parts of F, we are lead to characterize the entire functions H on $\mathbb{C}$ for which there exist C, N such that for all $w \in \mathbb{C}$,

$$|H(w)| \leq C(1+|w|)^N \exp(|w| - a|\,\mathcal{R}e(w)|). \tag{2.9}$$

Theorem 1.3.3 basically proves that for $a = 1$, only the polynomials satisfy this estimate, and for $a > 1$, only the zero function satisfies it.

Assume now that $a < 1$. There are many entire functions satisfying (2.9), including exponential functions. Namely, $\exp(\alpha z)$ satisfies (2.9) if and only if $\alpha \in K_a$, with

$$K_a = \{\alpha \in \mathbb{C}; |\alpha + a| \leq 1\,, |\alpha - a| \leq 1\}. \tag{2.10}$$

There are many results on the description of the classes of entire function of order 1 satisfying estimate similar to (2.9), where the function in the exponential is 1-homogeneous and convex. Here $|w| - a|\,\mathcal{R}e(w)|$ is not convex in w, since the set $\{w; |w| - a|\,\mathcal{R}e(w)| \leq 1\}$ is the union of two ellipses. The natural convex function associated to our problem is the support function of K_a, defined by

$$\phi(w) = \sup_{\alpha \in K_a} \mathcal{R}e(\alpha w).$$

An explicit formula for ϕ is

$$\phi(w) = \begin{cases} |w| - a|\,\mathcal{R}e(w)| & \text{for } |\,\mathcal{R}e(w)| \geq \frac{a}{\sqrt{1-a^2}}|\,\mathcal{I}m(w)| \\ \sqrt{1-a^2}|\,\mathcal{I}m(w)| & \text{else.} \end{cases}$$

Note that $\phi(w) \leq |w| - a|\,\mathcal{R}e(w)|$.

PROPOSITION 2.1.4. – *Let H be an entire function satisfying* (2.9). *Then there exists $C' > 0$ such that for all $w \in \mathbb{C}$,*

$$|H(w)| \leq C'(1+|w|)^N \exp(\phi(w)). \tag{2.11}$$

Proof. – We need to prove this better estimate for $|\,\mathcal{R}e(w)| \leq \frac{a}{\sqrt{1-a^2}}|\,\mathcal{I}m(w)|$. Consider the entire function $H_1(w) = H(w)\exp(i\sqrt{1-a^2}w)$. By (2.9), we have $|H(w)| \leq C(1+|w|)^N$ on the two half-lines defined by $|\,\mathcal{R}e(w)| = \frac{a}{\sqrt{1-a^2}}|\,\mathcal{I}m(w)|$ and $\mathcal{I}m(w) \geq 0$. By Phragmèn-Lindelöf principle, this bound is valid inside the angle, and we get the required estimate. A similar argument works for $\mathcal{I}m(z) \leq 0$. □

PROPOSITION 2.1.5. – *Let H be an entire function. It satisfies* (2.11) *for some C' and N if and only if there exists a distribution $\mu \in S'(\mathbb{R}^2)$, supported by ∂K_a, such that*

$$H(w) = \int \exp(\alpha w)\, d\mu(\alpha). \tag{2.12}$$

Proof. – We will use Paley-Wiener type results of [**20, 21**]. There the authors characterize the entire functions H that can be written as

$$H(w) = \int_{\partial K_a} \exp(\alpha w)\overline{g(\alpha)}\, d\sigma(\alpha),$$

where g is square integrable on ∂K_a with respect to the arc-length measure $d\sigma$. For that, it is necessary and sufficient that $H(w)\exp(-\phi(w))$ is square integrable with respect to a measure on $\mathbb{C}$ naturally associated to K_a. In particular, any H such that

$$|H(w)| \leq C(1+|w|)^{-M} \exp(\phi(w)) \tag{2.13}$$

can be represented this way, if M is large enough.

Assume that H satisfies (2.11). We can write, for any M,

$$H(w) = P_M(w) + w^M H_M(w),$$

where P_M is a polynomial and H_M satisfies (2.13). By [**20**], $w^M H_M(w)$ can be represented as (2.12), where μ is a M-th order derivative of an element of $L^2(\partial K_a)$. Finally, Cauchy Formula yields

$$w^k = \frac{k!}{2i\pi} \int_{\partial K_a} \exp(\alpha w) \frac{d\alpha}{\alpha^{k+1}}$$

for any $k \geq 0$, so that $P_M(w)$ can also be represented this way. □

We come back to the description of $\mathscr{G}(a,a)$. For any $t \in \mathbb{C}$ with $\mathscr{R}e(t) > 0$, the Bargmann transform of $x \to \exp(-\pi t x^2)$ is $z \to (1+t)^{-1/2} \exp\left(\frac{\pi}{2}\frac{1-t}{1+t} z^2\right)$. The homography $\mathscr{C}(t) = \frac{1-t}{1+t}$ is also called the Cayley transform. It maps the half plane $\{t; \mathscr{R}e(t) > 0\}$ onto the open unit ball $\{\alpha; |\alpha| < 1\}$. Let

$$D(a,b) = \{t \in \mathbb{C}; \mathscr{R}e(t) \geq a,\ \mathscr{R}e(t^{-1}) \geq b\}.$$

For $ab < 1$, it is a compact, convex domain, delimited by a circular arc and a line. For $ab = 1$, $D(a,b) = \{a\}$, and for $ab > 1$, $D(a,b) = \varnothing$. We can now give the following complement to Theorem 1.3.3.

THEOREM 2.1.6. – *Let $f \in S'(\mathbb{R})$ and $a, b > 0$. Then $f \in \mathscr{G}(a,b)$ if and only if there exist distributions ν_1, ν_2 on $\mathbb{R}^2$, supported by $\partial D(a,b)$, such that*

$$f(x) = \int \exp(-\pi t x^2)\, d\nu_1(t) + x \int \exp(-\pi t x^2)\, d\nu_2(t). \tag{2.14}$$

Proof. – The case $ab \geq 1$ is covered by Theorem 1.3.3. Assume that $ab < 1$. After a change of variables, we can always assume that $a = b < 1$. Let $f \in \mathscr{G}(a,a)$, and let F be its Bargmann transform. By (2.6) and (2.7), we can write

$$F(z) = H_1\Big(\frac{\pi}{2(1+a)} z^2\Big) + z H_2\Big(\frac{\pi}{2(1+a)} z^2\Big),$$

where each H_i satisfies (2.9) for some $C, N > 0$. By Proposition 2.1.5, we can write

$$F(z) = \int \exp\Big(\frac{\pi\alpha}{2(1+a)} z^2\Big)\, d\mu_1(\alpha) + z \int \exp\Big(\frac{\pi\alpha}{2(1+a)} z^2\Big)\, d\mu_2(\alpha),$$

where μ_1 and μ_2 are distributions supported by ∂K_a. Note that $t \in \partial D(a,a)$ if and only if $(1+a)\mathscr{C}(t) \in \partial K_a$. Hence we can also write

$$F(z) = \int (1+t)^{-1/2} \exp(\frac{\pi}{2}\mathscr{C}(t)z^2)\, d\nu_1(t) + z \int (1+t)^{-1/2} \exp(\frac{\pi}{2}\mathscr{C}(t)z^2)\, d\nu_2(t),$$

where ν_i are supported by $\partial D(a,a)$. Formula (2.14) follows by taking inverse Bargmann transform. □

REMARK 2.1.7. – Since any function $f(x) = \exp(-\pi t x^2)$, for $t \in D(a,b)$, satisfies $|f(x)| \leq \exp(-\pi a x^2)$ and $|\widehat{f}(\xi)| \leq C \exp(-\pi b x^2)$, (2.14) gives directly Proposition 2.1.1. Formula (2.14) states that any element of $\mathscr{G}(a,b)$ is an average of Hermite functions belonging to $\mathscr{G}(a,b)$. Indeed, any distribution ν supported by $\partial D(a,b)$ can be decomposed as a sum of partial derivatives of finite measures on $\partial D(a,b)$: there exist finite measures $m_1, \ldots, m_N$ on $\partial D(a,b)$ such that for all $x \in \mathbb{R}$,

$$f(x) = \sum_{k=1}^{N} x^k \int_{\partial D(a,b)} \exp(-\pi t x^2)\, dm_k(t).$$

2.2. Hardy's uncertainty principle in the sub-critical case, dimension d

We now discuss the sub-critical case of Theorem 1.3.5. Define by $\mathscr{G}(A,B)$ the space of tempered distributions satisfying (1.15). If both are non positive, we cannot expect in general solutions to be entire functions, as will be shown in the next chapter. Unlike dimension 1, it is not obvious that the elements of $\mathscr{G}(A,B)$ are entire functions of order 2 if A is positive and B non positive. However, when A and B are positive, we can prove the analogue of Proposition 2.1.1:

PROPOSITION 2.2.1. – *Let $f \in \mathscr{G}(A,B)$, where A, B are positive matrices. Then we have a pointwise estimate*

$$|f(x)| \leq C(1+|x|)^N \exp(-\pi\langle Ax, x\rangle),\ |\widehat{f}(\xi)| \leq C(1+|\xi|)^N \exp(-\pi\langle B\xi, \xi\rangle),$$

where C and N depend only on f.

Proof. – Do a change of variables so that A and B are diagonal. First prove estimates as in Lemma 2.1.2, then argue as in the proof of Proposition 2.1.1, proving the estimates for each variable. □

Let A, B corresponding to the sub-critical case of Theorem 1.3.5. We can assume that $I - |B|$ is positive, doing a dilation if necessary. The Bargmann transform gives also a characterization of $\mathscr{G}(A,B)$:

Proposition 2.2.2. – *Let A, B be as above. Let $f \in S'(\mathbb{R}^d)$, and F its Bargmann transform. Then $f \in \mathscr{G}(A,B)$ if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,*

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\langle \mathscr{C}(A)\,\mathscr{Re}(z), \mathscr{Re}(z)\rangle + \frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big) \tag{2.15}$$

and

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\langle \mathscr{C}(\varepsilon B)\,\mathscr{Im}(z), \mathscr{Im}(z)\rangle + \frac{\pi}{2}|\,\mathscr{Re}(z)|^2\big), \tag{2.16}$$

for $\varepsilon = 1, -1$.

Here $\mathscr{C}$ stands for the Cayley transform,

$$\mathscr{C}(M) = (I-M)(I+M)^{-1}.$$

This is proved as for Lemma 2.1.3. We point here that the condition on $I - |B|$ is technical. In the next chapter, where we consider critical pairs, we will see what $\mathscr{C}(B)$ should be replaced by, when $|A| = |B| = I$ (Corollary 3.3.4).

If A and B are both positive, Proposition 2.2.2 is true without assumptions on B, since $\mathscr{C}(B)$ is then well defined. Moreover, there exists $0 < \delta < 1$ such that $\langle \mathscr{C}(A)x, x\rangle \leq (1-\delta)|x|^2$ and $\langle \mathscr{C}(B)\xi, \xi\rangle \leq (1-\delta)|\xi|^2$. Combining (2.15) and (2.16), we see that

$$|F(z)| \leq C(1+|z|)^2 \exp(\frac{\pi}{2}(1-\delta/2)|z|^2). \tag{2.17}$$

The inverse Bargmann transform gives then that f and $\widehat{f}$ are entire functions of order 2, as in Proposition 2.2.1, see **[3, 17]**. We will get here a more precise result, similar to Theorem 2.1.6, but less precise.

Define, for A, B positive,

$$D(A,B) = \{M \in \mathscr{S}_d(\mathbb{C}); \mathscr{Re}(M) \geq A,\ \mathscr{Re}(M^{-1}) \geq B\},$$

where $\mathscr{S}_d(\mathbb{C})$ is the set of complex symmetric matrices with positive real part.

Theorem 2.2.3. – *Let A, B be two positive matrices. Let $f \in \mathscr{G}(A,B)$. There exist finite measures $\mu_1, \cdots, \mu_N$ on $\mathscr{S}_d(\mathbb{C})$, whose support in compact, polynomials $P_1, \ldots, P_N$, such that*

$$f(x) = \sum_{i=1}^{N} P_k(x) \int \exp(-\pi\langle Mx, x\rangle)\, d\mu_i(M). \tag{2.18}$$

Here N depends only on f. Conversely, any entire function defined by (2.18) *belongs to some space $\mathscr{G}(A,B)$, with A, B positive.*

Proof. – Let F be the Bargmann transform of f. Consider, for $0 < \delta < 1$,

$$K_\delta = \{M \in \mathscr{S}_n(\mathbb{C}); |\langle \mathscr{C}(M)z, z\rangle| \leq (1-\delta/2)|z|^2\ \forall z \in \mathbb{C}^d\}.$$

It is a compact subset of $\mathscr{S}_d(\mathbb{C})$. By estimate (2.17), $D(A,B) \subset K_\delta$, for some $\delta > 0$. Taking Bargmann transforms of both sides of (2.18), we are lead to the characterization of entire functions F on $\mathbb{C}^d$, satisfying an estimate like $|F(w)| \leq$

$C(1+|w|)^N \exp(|w|^2)$. We can consider the odd and even parts in each variables, and it suffices to characterize entire functions satisfying

$$|F(w)| \leq C(1+|w|)^N \exp(|w_1| + \cdots + |w_d|)$$

in terms of the exponentials $\exp(\lambda w)$, $\lambda \in \mathbb{C}^d$. We claim that any such function can be represented as

$$F(w) = \int_{[0,1]^d} \exp\left(e^{i\theta_1} w_1 + \cdots e^{i\theta_d} w_d\right) d\nu(\theta), \tag{2.19}$$

where ν is a distribution defined on $[0,1]^d$. Indeed, when we develop both sides into power series, we get

$$\widehat{\nu}(n) = F^{(n)}(0).$$

But the estimate on F implies, by Cauchy Formula, that

$$|F^{(n)}(0)| \leq C n_1! \cdots n_d! R^{N-n_1-\cdots-n_d} \exp(R_1 + \cdots + R_d)$$

for any $R_i > 0$. When $R_i = n_i$, we find

$$|F^{(n)}(0)| \leq C(1+|n|)^{N+1/2}.$$

It follows that the series $\sum_n F^{(n)}(0) e^{in\theta}$ converges to a distribution. This completes the proof of (2.19). Going back to the Bargmann transform of f, this proves that it can be written as

$$\sum_{i=1}^{N} P_i(z) \int_{[0,1]^d} \exp\left(\frac{\pi(1-\delta/2)}{2}(e^{i\theta_1} z_1^2 + \cdots + e^{i\theta_d} z_d^2)\right) d\nu_i(\theta), \tag{2.20}$$

with distributions ν_i defined on $[0,1]^d$. After applying inverse Bargmann transform, we get (2.18), with distributions supported in K_δ. The distributions are actually supported by the set of M such that

$$\mathscr{C}(M) = \mathrm{Diag}(\alpha_1, \ldots, \alpha_d),$$

$|\alpha_i| = 1 - \delta/2$, which is much smaller that ∂K_δ. □

Remark 2.2.4. – We used in the proof of Theorem 2.2.3 a simple Paley-Wiener type result for entire functions of order 1, just like in the proof of Theorem 2.1.6. But this time it can be solved easily with Fourier series. Theorem 2.2.3 basically proves that any element of $\mathscr{G}(A, B)$ can be represented as an average of complex Hermite functions. Remember that when AB has an eigenvalue λ such that $|\lambda| > 1$, then $f = 0$, and that the measures above are Dirac masses at $M = A$ when the eigenvalues of AB are 1 or -1, by Theorem 1.3.5.

This result is not as precise as Theorem 2.1.6, since this time we do not have control over the support of the measures. The issue is that (2.17) is stronger than (2.15) and (2.16). We are interested in the case where all eigenvalues of AB are in $]0,1]$, one of them being in $]0,1[$. Do a change of variables so that A and B are diagonal and equal. This is possible since A and B are positive. The diagonal coefficients a_i are such that

$0 < a_i \leq 1$, and we can assume that $a_1 < 1$. Just as (2.9), equations (2.15) and (2.16) can be rewritten as

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\sum_i \frac{|z_i^2|}{1+a_i} - \frac{\pi}{2}\big|\sum \frac{a_i\,\mathcal{R}e(z_i^2)}{1+a_i}\big|\big). \tag{2.21}$$

Thus we are lead to characterize the entire functions satisfying the estimate

$$|H(w)| \leq C(1+|w|)^N \exp\big(\sum_i |w_i| - \big|\sum a_i\,\mathcal{R}e(w_i)\big|\big). \tag{2.22}$$

An exponential function $\exp(\alpha_1 w_1 + \cdots + \alpha_d w_d)$ satisfies (2.22) if and only if

$$(\alpha_1, \ldots, \alpha_d) \in K_{a_1} \times \cdots \times K_{a_d},$$

where K_a is defined by (2.10). The indicator function of $K_{a_1} \times \cdots \times K_{a_d}$ is $\psi(w) = \phi_{a_1}(w_1) + \cdots + \phi_{a_d}(w_d)$, where ϕ_{a_i} is the indicator of K_{a_i}. We do not know if there is a Paley-Wiener Theorem for functions satisfying estimates (2.22). If they can be represented as an average of the exponential $\exp(\alpha z)$, for $\alpha \in K_{a_1} \times \cdots \times K_{a_d}$, then the measures in Theorem 2.2.3 are necessary supported by $D(A,B)$, and actually by a subset of $\partial D(A,B)$. There is probably a relation with this set and the Shilov boundary of $\partial D(A,B)$.

So first we would like to have the convex, 1-homogeneous function ψ in the exponential in (2.22).

CONJECTURE 2.2.5. – *Let the entire function H satisfy* (2.22). *Then*

$$|H(w)| \leq C'(1+|w|)^{N'} \exp(\psi(w))$$

In dimension 1, it was proved using Phragmèn-Lindelöf principle. The same method does not seem to work. Now observe that $\psi(w) \leq \sum |w_i| - a_i|\,\mathcal{R}e(w_i)|$, so by Phragmèn-Lindelöf principle, Conjecture 2.2.5 is equivalent to:

CONJECTURE 2.2.6. – *Let the entire function H satisfy* (2.22). *Then*

$$|H(w)| \leq C'(1+|w|)^{N'} \exp(\sum |w_i| - a_i|\,\mathcal{R}e(w_i)|).$$

The last issue is that very few is known about Paley-Wiener results for functions satisfying the estimate of Conjecture 2.2.5, when $d \geq 2$. We refer to [**19**] for recent results. Note that [**20, 21**] only consider the case $d = 1$.

The problem is more complicated when considering matrices A, B that are not positive. Assume that A is positive and B invertible, as in Theorem 1.3.5. Then one can assume that A and B are diagonal, with coefficients a_i and b_i, and that $|b_i| < 1$. We could assume that $a_i = |b_i| \leq 1$, but then $I - |B|$ could be non positive. The estimates of Proposition 2.2.2 can be rewritten as

$$|F(z)| \leq C(1+|z|)^N \exp\big(\frac{\pi}{2}\sum_i \frac{1-a_i}{1+a_i}|\,\mathcal{R}e(z_i)|^2 + \frac{\pi}{2}|\,\mathcal{I}m(z)|^2\big), \tag{2.23}$$

and

$$|F(z)| \leq C(1+|z|)^N \exp\left(\frac{\pi}{2}\sum_i \frac{1-\varepsilon b_i}{1+\varepsilon b_i}|\mathscr{Im}(z_i)|^2 + \frac{\pi}{2}|\mathscr{Re}(z)|^2\right), \tag{2.24}$$

for $\varepsilon = 1, -1$. Recall that Proposition 2.2.1 was established only when A and B are positive. In the general case, we do not even know if f extends to an entire function. Since A is positive, $\widehat{f}$ does. This would be the case if we had the following:

Conjecture 2.2.7. – *Let the entire function F satisfy* (2.23) *and* (2.24). *Then there exists* $0 < \delta < 1$ *such that*

$$|F(z)| \leq C'(1+|z|)^{N'} \exp\left(\frac{\pi}{2}(1-\delta)|z|^2\right).$$

Note that this estimate is true for $\delta = 0$.

2.3. One-sided Hardy's uncertainty principle.

We conclude this chapter with a discussion on Theorem 1.6.11. We do not have a description of the distributions satisfying (1.50). We would like a description in terms of averages of simple function satisfying these conditions, like in Theorems 2.1.6 and 2.2.3. The simplest functions we think about are defined by (1.40). We have

$$\mathscr{B}(f_{\alpha,\beta})(z) = \frac{1}{\sqrt{2}}\exp(-\pi z(\alpha+i\beta) + \frac{\pi}{2}(\alpha+i\beta)^2).$$

For simplicity we will rather take the functions defined by

$$\mathscr{B}(g_w)(z) = \exp\left(-\frac{\pi}{2}|w|^2 - \pi w z\right),$$

so that $g_{\alpha+i\beta}$ is proportional to $f_{\alpha,\beta}$.

Proposition 2.3.1. – *Let μ be a tempered distribution on $\mathbb{C} = \mathbb{R}^2$, supported by $\Delta = \{w \in \mathbb{C}; \mathscr{Re}(w) \geq 0, \mathscr{Im}(w) \geq 0\}$. The expression*

$$\mathscr{B}(g_\mu)(z) = \int \exp\left(-\frac{\pi}{2}|w|^2 - \pi z w\right) d\mu(w) \tag{2.25}$$

defines an element of $S'(\mathbb{R})$ satisfying (1.50).

Proof. – Let $\phi_z(w) = \exp(-\pi/2|w|^2 - \pi z w))$. Since $\mu \in S'(\mathbb{R}^2)$, there exist C, N such that

$$\left|\int \exp\left(-\frac{\pi}{2}|w|^2 - \pi z w\right) d\mu(w)\right| \leq C p_N(\phi_z) \leq C(1+|z|)^N \exp\left(\frac{\pi}{2}|z|^2\right).$$

By Proposition 1.2.5, g_μ is a well defined element of $S'(\mathbb{R})$. Note that when μ is a compactly supported measure,

$$g_\mu(x) = \sqrt{2}\int \exp(-\pi(x+a)^2 - i\pi ab - 2i\pi bx)\, d\mu(a+ib),$$

and we see directly that it satisfies (1.50).

When μ is any distribution, we prove that $F(z) = \mathcal{B}(g_\mu)(z)$ satisfies the estimates of Theorem 1.6.11. Let χ be a smooth function, equal to 1 on Δ, and to 0 on $\Delta-(1+i)$. Put $\chi_z(w) = \chi(\eta w)$, with $\eta = (1+|z|)^{-1}$. We have

$$\begin{aligned}|F(z)| &= \left|\int \chi_z(w)\phi_z(w)\,d\mu(w)\right| \le Cp(\chi_z\phi_z)_N\\ &\le C\sup_w(1+|w|+|z|)^N \exp\big(-\frac{\pi}{2}|w|^2 - \pi\,\mathcal{R}e(zw)\big),\end{aligned}$$

where the supremum is taken over w such that $\mathcal{R}e(w) \ge -\eta$, $\mathcal{I}m(w) \ge -\eta$. If $\mathcal{R}e(z) \ge 0$ then $\mathcal{R}e(zw) \ge -\eta\,\mathcal{R}e(w) - \mathcal{I}m(z)\,\mathcal{I}m(w) \ge -|\,\mathcal{I}m(z)||\,\mathcal{I}m(w)| - O(1)$, so that

$$\begin{aligned}|F(z)| &\le C\sup_w(1+|w|+|z|)^N \exp(-\pi/2|w|^2 - |\,\mathcal{I}m(z)||\,\mathcal{I}m(w)|)\\ &\le C(1+|z)^N \exp(\pi/2|\,\mathcal{I}m(z)|^2).\end{aligned}$$

The other estimate is obtained in the same way. □

CONJECTURE 2.3.2. – *Assume that* $f \in S'(\mathbb{R})$ *satisfies* (1.50). *Then there exists a tempered distribution* μ *on* $\mathbb{R}^2$, *supported by* Δ, *such that* $f = g_\mu$.

We note that the distribution μ is not uniquely defined. Indeed, we have the reproducing formula of the Bargmann space,

$$F(z) = \int_{\mathbb{C}} F(w)\exp(-\pi|w|^2 + \pi z\overline{w})\,dV(w),$$

where dV is the normalized Lebesgue measure in $\mathbb{C} = \mathbb{R}^2$. So we have $f = g_\mu$, given any $f \in S'(\mathbb{R})$, taking $d\mu(w) = F(\overline{w})\exp(-\pi/2|w|^2)$. This is indeed a tempered distribution, since

$$|F(w)| \le C(1+|w|)^N \exp(\pi/2|w|^2).$$

In view of (2.25), Conjecture (2.3.2) amounts to prove a Paley-Wiener type theorem for entire functions satisfying the estimates of Theorem 1.6.11, namely that they are Laplace transforms of distributions ν supported by Δ, such that $\exp(\pi/2|\cdot|^2)\,d\nu \in S'$. This is another type of Paley-Wiener result, for entire functions of order 2, and with an unbounded support.

CHAPTER 3

CRITICAL AND NON CRITICAL PAIRS

In this chapter we introduce the space $\mathscr{G}(q,q')$ of the distributions satisfying Hardy conditions, when the quadratic forms are not necessary positive. As in the case of the classical Hardy Theorem, the Gaussian functions play a crucial role. We show that there are three kinds of pairs of quadratic forms: the super-critical, critical and sub-critical pairs. We give practical characterizations of them in terms of spectral properties of their matrices. The most interesting case is the critical one, where we show that the Gaussian elements of $\mathscr{G}(q,q')$ are parameterized by the groups of matrices associated to the two forms. This will help us state the conjecture on the structure of those spaces in the next chapter. In the last part of the chapter we give sufficient conditions so that the space $\mathscr{G}(q,q')$ does not contain any non zero element, like in Hardy Theorem in the super-critical case.

3.1. Introduction and definitions

Definition 3.1.1. – *Let q and q' be two non degenerate quadratic forms on $\mathbb{R}^d$. We call $\mathscr{G}(q,q')$ the space of the distributions $f \in S'(\mathbb{R}^d)$ such that*

$$f(\cdot)\exp(\pm\pi q(\cdot)) \in S'(\mathbb{R}^d),\ \widehat{f}(\cdot)\exp(\pm\pi q'(\cdot)) \in S'(\mathbb{R}^d). \tag{3.1}$$

Proposition 3.1.2. – *The space $\mathscr{G}(q,q')$ is stable by differentiation and multiplication by polynomials.*

This proposition is elementary.

3.1.1. Gaussian solutions.

– In this section we will be interested in Gaussian elements in $\mathscr{G}(q,q')$. We also consider complex Gaussian functions as follows:

Definition 3.1.3. – *A complex Gaussian function is a function of the form*

$$f(x) = \exp(-\pi\langle Ax, x\rangle),$$

$x \in \mathbb{R}^d$, where A is a complex symmetric matrix, whose real part is positive.

If $A = B + iC$, with B, C real, symmetric, and B positive, then A is invertible, and

$$A^{-1} = (B + CB^{-1}C)^{-1} - iB^{-1}C(B + CB^{-1}C)^{-1}.$$

We see that $\mathscr{R}e(A^{-1}) \leq \mathscr{R}e(A)^{-1}$.

DEFINITION 3.1.4. – *Let q, q' be two non degenerate quadratic forms. We will call the pair (q, q') a sub-critical pair if $\mathscr{G}(q, q')$ contains a non real Gaussian element. A super-critical pair will be a pair such that $\mathscr{G}(q, q')$ does not contain Gaussian functions. A critical pair will be any other pair,* i.e., *a pair such that the Gaussian elements of $\mathscr{G}(q, q')$ exist and are all real.*

In the sub-critical case, there is a lot of complex Gaussian functions in $\mathscr{G}(q, q')$. Indeed, if $\exp(-\pi\langle Ax, x\rangle)$ is one of them, then all gaussian functions of the form $\exp(-\pi\langle A'x, x\rangle)$, with $\mathscr{R}e(A) = \mathscr{R}e(A')$, are also elements of $\mathscr{G}(q, q')$.

PROPOSITION 3.1.5. – *Let q, q' be two non degenerate quadratic forms. Then $\mathscr{G}(q, q')$ contains a Gaussian function if and only if there exists an invertible matrix P such that $|q(Px)| \leq |x|^2$, $|q'({}^tP^{-1}\xi)| \leq |\xi|^2$ for all x, ξ.*

Proof. – If $|q(Px)| \leq |x|^2$, $|q'({}^tP^{-1}\xi)| \leq |\xi|^2$ for all x, ξ, then the Gaussian function $\exp(-\pi|P^{-1}x|^2)$ is in $\mathscr{G}(q, q')$. Conversely assume that the Gaussian function $\exp(-\pi\langle Ax, x\rangle)$ is in $\mathscr{G}(q, q')$, with A complex symmetric, and $\mathscr{R}e(A)$ positive. Then

$$|q(x)| \leq \langle \mathscr{R}e(A)x, x\rangle, \ |q'(\xi)| \leq \left\langle \mathscr{R}e(A^{-1})\xi, \xi\right\rangle$$

for all x, ξ. We have $\mathscr{R}e(A^{-1}) \leq \mathscr{R}e(A)^{-1}$, and we conclude taking $P = (\mathscr{R}e(A))^{-1/2}$. □

Proposition 3.1.5 implies that if the space $\mathscr{G}(q, q')$ contains a complex Gaussian function f, then $|f| \in \mathscr{G}(q, q')$.

PROPOSITION 3.1.6. – *Let q, q' be two quadratic forms. Assume that $\mathscr{G}(q, q')$ contains a Gaussian function. The pair (q, q') is critical if and only if $|\det(q)\det(q')| = 1$, and sub-critical if and only if $|\det(q)\det(q')| < 1$.*

Proof. – Using Proposition 3.1.5 and a change of variables, we may assume that

$$|q(x)| \leq |x|^2, \ |q'(\xi)| \leq |\xi|^2,$$

so that $\mathscr{G}(q, q')$ contains the standard Gaussian function $\gamma(x) = \exp(-\pi|x|^2)$.

It follows that $|\det(q)| \leq 1$ and $|\det(q')| \leq 1$. If $|\det(q)\det(q')| < 1$, we may assume for example that q' has an eigenvalue λ such that $|\lambda| < 1$. Let e_λ be an associated eigenvector. Choose $b > 0$ such that $|\lambda| = (1 + b^2)^{-1}$, and define B by $B(e_\lambda) = be_\lambda$, $B(x) = 0$ for $x \in e_\lambda^\perp$. Then the non real Gaussian function

$$f(x) = \exp(-\pi\langle (I + iB)x, x\rangle)$$

belongs to $\mathscr{G}(q, q')$.

Assume now that $\mathscr{G}(q,q')$ contains a non real Gaussian function. We show that $|\det(q)\det(q')|<1$. Let

$$f(x)=\exp(-\pi\langle(\alpha+i\beta)x,x\rangle)$$

belong to $\mathscr{G}(q,q')$, with α positive and β a non zero real symmetric matrix. We have $\mathscr{R}e(\alpha+i\beta)^{-1}=(\alpha+\beta\alpha^{-1}\beta)^{-1}$, and hence

$$|q(x)|\leq\langle\alpha x,x\rangle,\ |q'(\xi)|\leq\big\langle(\alpha+\beta\alpha^{-1}\beta)^{-1}\xi,\xi\big\rangle$$

for all x,ξ. It follows that

$$|\det(q)\det(q')|\leq\det\alpha\det((\alpha+\beta\alpha^{-1}\beta)^{-1})<\det(\alpha)\det(\alpha^{-1})=1.$$

This completes the proof. □

We now give a precise characterization of critical, sub-critical and super-critical pairs. We begin with an algebraic one. A contraction is a matrix M such that ${}^tMM\leq I$, which means that $|Mx|\leq|x|$ for all $x\in\mathbb{R}$, where $|\cdot|$ stands for the Euclidean norm.

THEOREM 3.1.7. – *Let $q(x)=\langle Ax,x\rangle$, $q'(\xi)=\langle A'\xi,\xi\rangle$ be two non degenerate quadratic forms. The pair (q,q') is critical if and only if AA' is conjugated to an orthogonal matrix. It is sub-critical if and only if it is conjugated to a contraction that is not orthogonal. It is super-critical in any other case.*

Proof. – Assume that (q,q') is not super-critical. Let P be given by Proposition 3.1.5. Put $B={}^tPAP$ and $B'=P^{-1}A'{}^tP^{-1}$. The eigenvalues of B and B' are in $[-1,1]$. Hence $|BB'x|\leq|x|$ for all x, where $|\cdot|$ stands for the Euclidean norm. We see that AA' is conjugated to a contraction. Assume moreover that (q,q') is critical. Then $|\det(B)\det(B')|=1$ by Proposition 3.1.6, so we see that B and B' have their eigenvalues of modulus 1. It follows that B and B' are orthogonal and symmetric, and hence BB' is orthogonal. If (q,q') is sub-critical, then one of the eigenvalues of B or B' is in $]-1,1[$, and there exists x such that $|BB'x|<|x|$.

Assume now that there exists Q such that $Q^{-1}AA'Q$ is a contraction. Put $B=Q^{-1}A{}^tQ^{-1}$ and $B'={}^tQA'Q$. By the polar decomposition, $|B|B'$ is a contraction. The symmetric matrix $|B|^{1/2}B'|B|^{1/2}$ is conjugated to a contraction, hence it is itself a contraction. Let $P={}^tQ^{-1}|B|^{-1/2}$. Then ${}^tPAP$ is orthogonal and $P^{-1}A'{}^tP^{-1}$ is a contraction, so that $\mathscr{G}(q,q')$ contains a Gaussian function. Now use Proposition 3.1.6. If $Q^{-1}AA'Q$ is an isometry, then $|\det(AA')|=1$, and the pair (q,q') is critical. Else, we have $|\det(AA')|<1$, so the pair is sub-critical. □

Theorem 3.1.7 characterizes critical and sub-critical pairs in a rather inexplicit way. We give now an explicit description in terms of the spectral properties of AA'. The proof is left to the reader.

THEOREM 3.1.8. – *Let* $q(x) = \langle Ax, x\rangle$, $q'(\xi) = \langle A'\xi, \xi\rangle$ *be two non degenerate quadratic forms. Then the pair* (q, q') *is critical if and only if* AA' *is diagonalizable over* $\mathbb{C}$, *with eigenvalues of modulus* 1. *The pair* (q, q') *is sub-critical if and only if* AA' *has a complex eigenvalue* μ *such that* $|\mu| < 1$, *all its other eigenvalues have modulus less than or equal to* 1, *and the restriction of* AA' *to the direct sum of the characteristic spaces associated to eigenvalues of modulus* 1 *is diagonalizable. The pair* (q, q') *is super-critical in any other case.*

REMARK 3.1.9. – In other words, (q, q') is critical if and only if the minimal polynomial of AA' has the form $\Pi(X) = \prod_{\lambda\in\Lambda}(X - \lambda)$, with $|\lambda| = 1$ for $\lambda \in \Lambda$. It is sub-critical if and only if it has the form $\Pi(X) = \prod_{\lambda\in\Lambda}(X - \lambda)^{n_\lambda}$, with $|\lambda| \leq 1$ for $\lambda \in \Lambda$, $n_\lambda = 1$ for $|\lambda| = 1$, and $|\mu| < 1$ for some $\mu \in \Lambda$. The pair (q, q') is super-critical if and only if $\Pi(X) = \prod_{\lambda\in\Lambda}(X - \lambda)^{n_\lambda}$, with Λ containing λ such that $|\lambda| > 1$, or such that $|\lambda| = 1$ and $n_\lambda \geq 2$.

As a corollary we have:

COROLLARY 3.1.10. – *Let* $q(x) = \langle Ax, x\rangle$ *and* $q'(\xi) = \langle A'\xi, \xi\rangle$ *be two non degenerate quadratic forms. Assume that* $|\det(AA')| > 1$, *or more generally that* AA' *has a complex eigenvalue* λ *such that* $|\lambda| > 1$. *Then* (q, q') *is super-critical.*

Given a non degenerate quadratic form q, define the group

$$O(q) = \{P \in GL_d(\mathbb{R});\ q(Px) = q(x)\ \forall x \in \mathbb{R}^d\}. \tag{3.2}$$

If A is an invertible matrix, we define also

$$O(A) = \{P \in GL_d(\mathbb{R});\ {}^tPAP = A\}. \tag{3.3}$$

When (q, q') is critical, all Gaussian elements of $\mathscr{G}(q, q')$ are real. We will describe them. After a change of variable, we may assume that the associated matrices A and A' are orthogonal and symmetric.

THEOREM 3.1.11. – *Let* $q(x) = \langle Ax, x\rangle$, $q'(\xi) = \langle A'\xi, \xi\rangle$, *with matrices* A, A' *orthogonal and symmetric. The Gaussian elements of* $\mathscr{G}(q, q')$ *are precisely given by the functions*

$$\exp(-\pi|g(x)|^2),$$

where the matrix g *belongs to the group* $O(q) \cap O(q')$.

The Cayley transform of a complex matrix M is defined by

$$\mathscr{C}(M) = (I - M)(I + M)^{-1}.$$

The Cayley transform appeared in the proof of Theorem 1.3.1, as naturally involved in the computation of Bargmann transforms of Gaussian functions. Indeed, if $f(x) = \exp(-\pi\langle Mx, x\rangle)$, with $\mathscr{Re}(M)$ positive, then

$$\mathscr{B}(f)(z) = \det(I + M)^{-1} \exp\big(\frac{\pi}{2}\langle \mathscr{C}(M)z, z\rangle\big). \tag{3.4}$$

We see immediately that positive matrices M are transformed through $\mathscr{C}$ in matrices N such that $I - N^t N$ is positive. We have also $\mathscr{C}^2(M) = M$ and $\mathscr{C}(M^{-1}) = -\mathscr{C}(M)$.

Theorem 3.1.11 relies on pure bilinear algebra. It is a direct consequence of the following.

THEOREM 3.1.12. – *Let A, A' be two orthogonal symmetric matrices. Let M be a positive matrix. Then*

$$|\langle Ax, x\rangle| \leq \langle Mx, x\rangle \ \forall x \in \mathbb{R}^d, \ |\langle A'x, x\rangle| \leq \left\langle M^{-1}\xi, \xi\right\rangle \ \forall \xi \in \mathbb{R}^d \tag{3.5}$$

if and only if $\sqrt{M} \in O(A) \cap O(A')$.

LEMMA 3.1.13. – *Let $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = d$. Denote any $x \in \mathbb{R}^d$ by $x = (x_1, x_2)$, with $x_i \in \mathbb{R}^{k_i}$. Put $q(x) = |x_1|^2 - |x_2|^2$. Let M be a positive matrix. Let $N = \mathscr{C}(M)$. Then $|q(x)| \leq \langle Mx, x\rangle$ for all x if and only if*

$$\langle Nx, x\rangle \leq 2|x_1||x_2| \tag{3.6}$$

for all $x \in \mathbb{R}^d$.

Proof. – We will use the conjugate function of a convex function ϕ, which is given by its Legendre transform $\phi^*(x) = \sup_{x'} 2\langle x, x'\rangle - \phi(x')$. We have

$$\sup_{x'} \left(2\langle x, x'\rangle - \langle Mx', x'\rangle - |x'|^2\right) - |x|^2/2 = \frac{1}{2}\langle Nx, x\rangle.$$

Hence

$$\begin{aligned}\frac{1}{2}\langle Nx, x\rangle &\leq \sup_{x'} \left(2\langle x, x'\rangle - 2\max(|x_1'|^2, |x_2'|^2)\right) - |x|^2/2 \\ &= (|x_1| + |x_2|)^2/2 - |x|^2/2 = |x_1||x_2|.\end{aligned}$$

For the converse, recall that $\mathscr{C}(N) = M$. Hence

$$\begin{aligned}\frac{1}{2}\langle Mx, x\rangle &= \sup_{x'} \left(2\langle x, x'\rangle - \langle Nx', x'\rangle - |x'|^2\right) - |x|^2/2 \\ &\geq \sup_{x'} \left(2\langle x, x'\rangle - 2|x_1'||x_2'| - |x'|^2\right) - |x|^2/2 \\ &= |q(x)|/2,\end{aligned}$$

which completes the proof. □

We will prove Theorems 3.1.12 and 3.1.11 in different steps. Let us take notations. Let M be a positive matrix. Let N be the Cayley transform of M, $E_1 = \mathrm{Ker}(A - I)$, $E_1' = \mathrm{Ker}(A' - I)$, $E_2 = \mathrm{Ker}(A + I)$ and $E_2' = \mathrm{Ker}(A' + I)$. We have the orthogonal decompositions

$$\mathbb{R}^d = E_1 \oplus E_2 = E_1' \oplus E_2'.$$

By Lemma 3.1.13, (3.5) is equivalent to

$$-2|x_1'||x_2'| \leq \langle Nx, x\rangle \leq 2|x_1||x_2|, \tag{3.7}$$

where x_1, x_2, x_1', x_2' are the components of x in the orthogonal decompositions given above. Theorem 3.1.12 amounts to show that under Conditions (3.7), $M^{1/2} \in O(A) \cap O(A')$.

We will first consider the case where A and A' commute.

Proposition 3.1.14. – *Let A, A' be two orthogonal and symmetric matrices. Assume that A and A' commute. Let M be a positive matrix such that $N = \mathscr{C}(M)$ satisfies (3.7). Then $M^{1/2} \in O(A) \cap O(A')$.*

Proof. – A fundamental example is when

$$A = A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$|\langle Nx, x\rangle| \leq 2|x_1||x_2|$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. It follows from homogeneity that

$$N = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

for some $v \in]-1, 1[$ (recall that $I - {}^t\!NN$ is positive). Then we can compute

$$M = \frac{1}{1-v^2} \begin{pmatrix} 1+v^2 & -2v \\ -2v & 1+v^2 \end{pmatrix}, \ M^{1/2} = \frac{1}{2} \begin{pmatrix} t+t^{-1} & t-t^{-1} \\ t-t^{-1} & t+t^{-1} \end{pmatrix},$$

with $t = (\frac{1-v}{1+v})^{1/2}$. We clearly have $M^{1/2} \in O(A)$.

Assume now that $A = A'$ and $d \geq 2$. The matrix N satisfies

$$|\langle Nx, x\rangle| \leq 2|x_1||x_2|,$$

hence, in the orthogonal decomposition $\mathbb{R}^d = E_1 \oplus E_2$, N has a bloc form

$$N = \begin{pmatrix} 0 & v \\ {}^t v & 0 \end{pmatrix},$$

for some matrix v, with d_1 lines and d_2 columns, such that $I - {}^t vv$ is positive. There exist $k_i \in O(d_i)$ such that $k_1^{-1} v k_2$ is a quasidiagonal matrix, with zero entries in last position, if any. Since the matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

belongs to $O(A)$, we can assume that v is such a quasidiagonal matrix. Then N has a bloc decomposition, whose diagonal blocs are either 0, or given by 2-dimensional matrices of the form

$$\begin{pmatrix} 0 & v_i \\ v_i & 0 \end{pmatrix},$$

with $|v_i| < 1$, and the result follows from the first part.

In the general case where A and A' commute, there is a common orthonormal basis of eigenvectors of both A and A'. We may assume that the space $\mathbb{R}^d$ is decomposed so that for $x = (x_1, x_2, x_3, x_4)$, $x_i \in \mathbb{R}^{d_i}$, we have

$$Ax = x_1 + x_2 - x_3 - x_4, \; A'\xi = \xi_1 - \xi_2 + \xi_3 - \xi_4.$$

By assumption,

$$-2|(x_1, x_3)||(x_2, x_4)| \leq \langle Nx, x \rangle \leq 2|(x_1, x_2)||(x_3, x_4)| \tag{3.8}$$

for all $x \in \mathbb{R}^d$. We will show that N has the form

$$N = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & w & 0 \\ 0 & {}^t w & 0 & 0 \\ {}^t v & 0 & 0 & 0 \end{pmatrix},$$

where v is a matrix with d_4 columns and d_1 lines, and w has d_2 lines and d_3 columns. Hence $|\langle Nx, x \rangle| \leq 2|(x_1, x_2)||(x_3, x_4)|$ and $|\langle Nx, x \rangle| \leq 2|(x_1, x_3)||(x_2, x_4)|$, so that we can conclude from the previous case.

Consider the canonical basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$. When we apply (3.8) to $x = e_i$, we obtain

$$\langle Ne_i, e_i \rangle = 0. \tag{3.9}$$

Moreover, taking $x_1 = x_2 = 0$, we get

$$\langle N(0, 0, x_3, x_4), (0, 0, x_3, x_4) \rangle \leq 0 \tag{3.10}$$

for all x_3, x_4. The quadratic form on $\mathbb{R}^{d_3+d_4}$ defined by (3.10) is semi-negative, and the trace of its representative matrix is equal to zero by (3.9). Hence $\langle N(0, 0, x_3, x_4), (0, 0, x_3, x_4) \rangle = 0$. A similar argument shows that

$$\begin{aligned} \langle N(x_1, x_2, 0, 0), (x_1, x_2, 0, 0) \rangle &= \langle N(0, x_2, 0, x_4), (0, x_2, 0, x_4) \rangle \\ &= \langle N(x_1, 0, x_3, 0), (x_1, 0, x_3, 0) \rangle = 0 \end{aligned}$$

for all x_1, x_2, x_3, x_4.

Hence N has the required form, with v and w such that $I - {}^t vv$ and $I -^t ww$ are positive. It follows that $|\langle Nx, x \rangle| \leq 2|(x_1, x_2)||(x_3, x_4)|$ and $|\langle Nx, x \rangle| \leq 2|(x_1, x_3)||(x_2, x_4)|$, for all $x \in \mathbb{R}^d$. We conclude using the case $A = A'$. □

We still take the notations given before Proposition 3.1.14. It is easy to check that A and A' commute if and only if the eigenvalues of AA' are all real (assuming A, A' are both orthogonal and symmetric matrices). We now consider the opposite case, namely the case where AA' has no real eigenvalue. This happens exactly when the spaces E_1, E_2, E'_1, E'_2 do not intersect each other.

PROPOSITION 3.1.15. – *Assume that AA' has no real eigenvalue. If (3.7) is satisfied, then $M^{1/2} \in O(A) \cap O(A')$.*

Proof. – In this case the dimension d is even, and E_1, E_2, E_1', and E_2' have dimension $d/2$. Doing a rotation if necessary, we may assume that $E_1 = \{(x_1, 0); x_1 \in \mathbb{R}^{d/2}\}$, $E_2 = \{(0, x_2); x_2 \in \mathbb{R}^{d/2}\}$. Denote by x_1' and x_2' the orthogonal projections of x on E_1' and E_2' respectively, and write $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}^{d/2}$. Then the spaces E_1', E_2' are given by a graph in the decomposition $\mathbb{R}^d = \mathbb{R}^{d/2} \times \mathbb{R}^{d/2}$: there exists an invertible matrix Δ such that

$$E_1' = \{x \in \mathbb{R}^d; x_2 = \Delta x_1\},\ E_2' = \{x \in \mathbb{R}^d; -{}^t\Delta x_2 = x_1\}.$$

Doing further independent rotations in the x_1 and x_2 variables if necessary, we are lead to the case where Δ is diagonal, with positive eigenvalues. Let $\delta_1, \dots, \delta_{d/2}$ be its diagonal entries.

Put $x_2 = 0$ in Relation (3.7): we obtain $\langle N(x_1, 0), (x_1, 0)\rangle \leq 0$ for all $x_1 \in \mathbb{R}^{d/2}$. We also have $\langle N(0, x_2), (0, x_2)\rangle \leq 0$, $\langle N(x_1, \Delta x_1), (x_1, \Delta x_1)\rangle \geq 0$, and $\langle N(-{}^t\Delta x_2, x_2), (-{}^t\Delta x_2, x_2)\rangle \geq 0$. Let (e_i) be the canonical basis of $\mathbb{R}^{d/2}$, so that $\Delta e_i = \delta_i e_i$. Taking $x_1 = x_2 = e_i$, we obtain $\langle N(e_i, e_i), (e_i, e_i)\rangle \geq 0$ and $\langle N(-e_i, e_i), (-e_i, e_i)\rangle \geq 0$. Summing the two quantities, we see that

$$\begin{aligned} 0 &\leq \langle N(e_i, e_i), (e_i, e_i)\rangle + \langle N(-e_i, e_i), (-e_i, e_i)\rangle \\ &= 2\langle N(e_i, 0), (e_i, 0)\rangle + 2\langle N(0, e_i), (0, e_i)\rangle \leq 0. \end{aligned}$$

This gives $\langle N(e_i, 0), (e_i, 0)\rangle = 0$. But since $\langle N(x_1, 0), (x_1, 0)\rangle \leq 0$ for all x_1, we must have $\langle N(x_1, 0), (x_1, 0)\rangle = 0$. We can prove in the same way that $\langle N(0, x_2), (0, x_2)\rangle = 0$ for all x_2. It follows that there exists a matrix v such that

$$N = \begin{pmatrix} 0 & v \\ {}^t v & 0 \end{pmatrix}.$$

By Proposition 3.1.14, we have $M^{1/2} \in O(A)$. By symmetry, we also have $M^{1/2} \in O(A')$, which concludes the proof. □

We still take the notations given before Proposition 3.1.14.

LEMMA 3.1.16. – *Let A, A' be two orthogonal symmetric matrices, and*

$$F = \operatorname{Ker}(AA' - I) \oplus \operatorname{Ker}(AA' + I).$$

Then F is the space spanned by the common eigenvectors to A and A'.

Proof. – If $x \in \operatorname{Ker}(AA' - I)$, we have $AA'x = x = A'Ax$, since $(AA')^{-1} = A'A$, hence $AA'Ax = Ax$ and $Ax \in \operatorname{Ker}(AA' - I)$. We see that $\operatorname{Ker}(AA' \pm I)$ are stable by A and A'. It follows that F is spanned by eigenvectors of A belonging to either $\operatorname{Ker}(AA' - I)$ or $\operatorname{Ker}(AA' + I)$. Let $x \in F$ be such an element. We have $AA'x = \varepsilon x$ and $Ax = \mu x$, with $\varepsilon, \mu \in \{-1, 1\}$. It follows that $A'x = \varepsilon\mu x$, and we see that F is spanned by common eigenvector to A and A'. Now any common eigenvector to A and A' belongs to F, so the result follows. □

PROPOSITION 3.1.17. – *Let A, A' be any orthogonal symmetric matrices. If (3.7) is satisfied, then $M^{1/2} \in O(A) \cap O(A')$.*

Proof. – Denote by F the space generated by the common eigenvectors to A and A'. The spaces $F, F^\perp$ are stable by A and A'. We show now that they are stable by N, *i.e.*, $\langle Nf, g\rangle = 0$ for all $f \in F$ and $g \in F^\perp$.

Given any $x \in \mathbb{R}^d$, let x_1, x_2 be the projections of x on E_1 and E_2, and x_1', x_2' be the projection of x on E_1', E_2'. By assumption,

$$-2|x_1'||x_2'| \le \langle Nx, x\rangle \le 2|x_1||x_2|. \tag{3.11}$$

Let $f \in F$ be a common eigenvector to A and A'. We assume for example that $f \in E_1 \cap E_1'$.

The spaces $F^\perp \cap E_1$, $F^\perp \cap E_2$, $F^\perp \cap E_1'$ and $F^\perp \cap E_2'$ intersect pairwise on the null space by definition of F. As in the proof of Proposition 3.1.15, we see that they have the same dimension, equal to $\dim(F^\perp)/2$, and in particular

$$F^\perp = F^\perp \cap E_1 \oplus F^\perp \cap E_1'.$$

Take $g \in F^\perp \cap E_1$. Relation (3.11) gives $\langle Nf, f\rangle = 0$ and for all $t \in \mathbb{R}$,

$$\langle N(tf+g), tf+g\rangle = \langle Ng, g\rangle + 2t\langle Nf, g\rangle \le 0.$$

Hence $\langle Nf, g\rangle = 0$. The same is true when $g \in F^\perp \cap E_1'$, since then for all $t \in \mathbb{R}$,

$$\langle N(tf+g), tf+g\rangle = \langle Ng, g\rangle + 2t\langle Nf, g\rangle \ge 0.$$

We showed that N stabilizes F and $F^\perp$. We conclude with Propositions 3.1.14 and 3.1.15, considering the restrictions of A, A', N to F and $F^\perp$, respectively. □

Proposition 3.1.17 and Lemma 3.1.13 imply Theorem 3.1.12. We proved that the Gaussian elements of $\mathscr{G}(q, q')$ are parameterized by the group of matrices $G = O(q) \cap O(q')$. Let $K = O(d) \cap G$. Since $|g(x)| = |kg(x)|$ for all $x \in \mathbb{R}^d, g \in G$ and $k \in K$, those Gaussian elements are actually parameterized by the symmetric space G/K. The proof of Theorem 3.1.11 gives then an interesting description of the Cayley transform of G/K. It can happen that $G = K$. In that case, there is only one Gaussian element in $\mathscr{G}(q, q')$. The condition when this occurs is given by the next Theorem:

THEOREM 3.1.18. – *Let $q(x) = \langle Ax, x\rangle$ and $q'(\xi) = \langle A'\xi, \xi\rangle$ be two non degenerate quadratic forms, with A and A' symmetric and orthogonal matrices. Let $F = \mathrm{Ker}(AA' - I) \oplus \mathrm{Ker}(AA' + I)$. Then $\mathscr{G}(q, q')$ contains only one Gaussian function if and only if the non real eigenvalues of AA' have multiplicity 1 in the characteristic polynomial of AA', and A or A' restricted to F is the identity matrix I or $-I$.*

Proof. – By Lemma 3.1.16, F and $F^\perp$ are stable by A and A'. Moreover the proof of Proposition 3.1.17 shows that any matrix N satisfying (3.7) has a bloc decomposition according to the decomposition $\mathbb{R}^d = F \oplus F^\perp$. Hence is suffices to consider separately the cases $F = \mathbb{R}^d$ and $F = \{0\}$.

Assume that $F = \mathbb{R}^d$. All the eigenvalues of AA' are real. After a change of variables, we can assume that A and A' are diagonal, with diagonal coefficients $a_1, \dots, a_n$

and $a'_1, \dots, a'_n$ equal to 1 or -1. If A and A' are not equal to I or $-I$, there exist $i \neq j$ such that $a_i \neq a_j$ and $a'_i \neq a'_j$, and any Gaussian function of the form

$$\exp\big(-\pi \sum_{k \neq i,j} x_k^2 - \pi |g(x_i, x_j)|^2\big),$$

with $g \in O(1,1)$, belongs to $\mathscr{G}(q, q')$. If A or A' is I or $-I$, then Theorem 1.3.5 shows that $\mathscr{G}(q, q')$ contains only the standard Gaussian function.

Assume now that $F = \{0\}$, so that all eigenvalues of AA' are non real. Take the notations of the proof of Proposition 3.1.15. We can assume that the matrix Δ introduced there is diagonal, with positive coefficients. Let N satisfy (3.7). Then by the proof of Proposition 3.1.15, we have

$$\langle Nx, x\rangle = 0$$

for all x in E_1 or E_2. Hence N has the form

$$N = \begin{pmatrix} 0 & v \\ {}^t v & 0 \end{pmatrix}.$$

But by symmetry we have $\langle Nx, x\rangle = 0$ for x in E'_1 and E'_2. It follows that $v\Delta$ and Δv are antisymmetric. We have the relations $v_{i,j}\delta_j = -\delta_i v_{j,i}$ and $v_{i,j}\delta_i = -\delta_j v_{j,i}$ on the coefficients of v. Hence $v = 0$ is the only choice if and only if all the δ_i are different. To conclude the proof, note that the eigenvalues of AA' are exactly the d numbers

$$\frac{1-\delta_k^2}{1+\delta_k^2} \pm \frac{2i\delta_k}{1+\delta_k^2}. \tag{3.12}$$

Indeed, in the orthonormal basis $e_1, \dots, e_d$ we chose, AA' has a representative matrix given by

$$\begin{pmatrix} \mathscr{C}({}^t\Delta\Delta) & 2(I + {}^t\Delta\Delta)^{-1\,t}\Delta \\ -2(I + \Delta{}^t\Delta)^{-1}\Delta & \mathscr{C}(\Delta{}^t\Delta) \end{pmatrix},$$

where $\mathscr{C}$ is the Cayley transform. In the basis $e_1, e_{d/2+1}, \dots, e_{d/2}, e_d$, AA' will have a representative matrix which is bloc diagonal, with blocs of size 2 equal to

$$\begin{pmatrix} \frac{1-\delta_i^2}{1+\delta_i^2} & \frac{2\delta_i}{1+\delta_i^2} \\ -\frac{2\delta_i}{1+\delta_i^2} & \frac{1-\delta_i^2}{1+\delta_i^2} \end{pmatrix}.$$

Hence the eigenvalues of AA' are given by (3.12). □

Unlike Theorem 3.1.8, the condition does not depend on the matrix AA' itself, so that it is difficult to give a condition when (q, q') is a critical pair, without A and A' being orthogonal symmetries. Nevertheless, we have the following sufficient condition:

Corollary 3.1.19. – *Let $q(x) = \langle Ax, x\rangle$ and $q'(\xi) = \langle A'\xi, \xi\rangle$, where A and A' are symmetric invertible matrices. If AA' has d distinct eigenvalues of modulus 1, then $\mathscr{G}(q, q')$ contains only one Gaussian function.*

Proof. – As before, we can assume that the eigenvalues of AA' are all real or all non real. If they are all non real, Theorem 3.1.18 gives the result. Now assume that the eigenvalues of AA' are equal to 1 or -1. Then $d \leq 2$. The case $d = 1$ follows from Theorem 1.3.3. Assume that $d = 2$, and that 1 and -1 are its eigenvalues. Make a change of variables so that A and A' are diagonal, with eigenvalues a_1, a_2, a_1', a_2' equal to 1 or -1. We have $a_1 = a_1'$ and $a_2 = -a_2'$, or $a_1 = -a_1'$ and $a_2 = a_2'$. It follows that A or A' is equal to I or $-I$, and we conclude with Theorem 1.3.5. □

3.1.2. Remarks. – In the sub-critical case, there are real and non real Gaussian elements in $\mathscr{G}(q, q')$. It still seems difficult to give a precise description of them. However Theorem 2.1.6 gives the answer in dimension one. The general idea of our analysis is to show that in the super-critical case, $\mathscr{G}(q, q')$ does not contain a lot of solutions. We will give singular examples where $\mathscr{G}(q, q')$ contains only singular distributions. We will show that, for many super-critical pairs (q, q'), $\mathscr{G}(q, q')$ does not contain any function. Then we will try in some cases to describe completely $\mathscr{G}(q, q')$ when (q, q') is a critical pair. The conjecture that we formulate after our study is that the Gaussian elements of $\mathscr{G}(q, q')$ generate all its elements, using averages, differentiation and multiplication by polynomials (see Proposition 3.1.2). For example, when (q, q') satisfies the conditions of Theorem 3.1.18, we expect the space $\mathscr{G}(q, q')$ to be exactly the space of Hermite functions associated to its unique Gaussian element. We will not be able to show this fact, unless q or q' is positive. For example we do not know if it is true for $q(x) = 2x_1x_2$ and $q'(\xi) = \xi_1^2 - \xi_2^2$ on $\mathbb{R}^2$.

3.1.3. Annihilating pairs of quadratic forms

Definition 3.1.20. – *The pair (q, q') of non degenerate quadratic forms on $\mathbb{R}^d$ is called an annihilating pair if $\mathscr{G}(q, q') = \{0\}$.*

An annihilating pair is necessarily super-critical. If (q, q') is annihilating, then any $f \in L^2(\mathbb{R}^d)$ such that, for $|x|, |\xi| \to \infty$,

$$f(x) = O(\exp(-\pi|q(x)|)), \ \widehat{f}(\xi) = O(\exp(-\pi|q'(\xi)|)),$$

is equal to 0. Such a property is an analogue of Hardy's uncertainty principle for non degenerate quadratic forms. Theorem 1.3.5 gives the annihilating pairs (q, q'), when q or q' is positive:

Proposition 3.1.21. – *Let A and A' be two symmetric matrices, with A positive. Let q, q' be the quadratic forms associated to A and A'. Then the pair (q, q') is annihilating if and only if the matrix AA' has an eigenvalue λ such that $|\lambda| > 1$.*

We call a pair having this property an annihilating pair by reference to annihilating pairs of sets, as defined in [**14**].

DEFINITION 3.1.22. – *Let $E, F \subset \mathbb{R}^d$ be two measurable sets. The pair (E, F) is called a weakly annihilating pair if any $f \in L^2$ with support in E and spectrum in F, is equal to zero. The pair is strongly annihilating if there exists $0 \leq c < 1$ such that for all $f \in L^2(\mathbb{R}^d)$, with support in E,*

$$\int_F |\widehat{f}(\xi)|^2 \, d\xi \leq c\|f\|^2_{L^2(\mathbb{R}^d)}.$$

The link between Definitions 3.1.22 and 3.1.20 is the following.

THEOREM 3.1.23. – *Assume that the pair (q, q') of non degenerate quadratic forms is annihilating. Let $C, C' > 0$, and define the sets*

$$E = \{x \in \mathbb{R}^d; |q(x)| \leq C\}, \; F = \{\xi \in \mathbb{R}^d; |q'(\xi)| \leq C'\}.$$

Then any tempered distribution f with support in E and spectrum in F is equal to zero. In particular, (E, F) is weakly annihilating for L^2 functions.

Note that the notions of strongly/weakly annihilating pairs of sets was defined for functions in L^2. It can as well be defined for functions in L^p spaces. Classical examples of strongly annihilating pairs are pairs of sets of finite measure [**1**]. It is proved in [**25**] that the pairs (E, F), with

$$E = \{x \in \mathbb{R}^d; |q(x)| \leq C\}, \; F = \{\xi \in \mathbb{R}^d; |q'(\xi)| \leq C'\},$$

are strongly annihilating, provided the product CC' is small enough (q and q' are here any non degenerate quadratic forms). We believe that those pairs (E, F) are weakly and strongly annihilating without restriction on C, C'. There are trivial counter-examples when one of the form is degenerated.

Note, however, that particular cases can be proved using the following, which is a corollary of the classical proof for pairs of finite measure ([**1, 14**]. An elementary proof can be found in [**6**]:

PROPOSITION 3.1.24. – *Assume that the subsets E and F of $\mathbb{R}^d$ have the following property: for almost every $x \in \mathbb{R}^d$, the lattice $x + \mathbb{Z}^d$ intersects E and F on finite sets. Then the pair (E, F) is weakly annihilating.*

COROLLARY 3.1.25. – *The pair of sets (E, F), with*

$$E = \{(x, y) \in \mathbb{R}^2; |xy| \leq C\}, \; F = \{(\xi, \eta) \in \mathbb{R}^2; |\xi\eta| \leq C'\},$$

is weakly annihilating, for any value of C and C'.

Note that we can also translate and take rotations of the sets above. Moreover, we can take finite unions of such sets.

3.1.4. Examples in dimension 2. – Assume that q is positive. After a change of variables, we write

$$q(x,y) = x^2 + y^2,\ q'(\xi,\eta) = a\xi^2 + b\eta^2,$$

with $a, b \in \mathbb{R} \setminus \{0\}$. Then the pair is annihilating if and only if $\max(|a|,|b|) > 1$, by Theorem 1.3.5.

Assume now that neither q nor q' is positive. The issue is that they may not have a common basis of reduction as above (see Proposition 3.2.1 below). So we assume moreover that q and q' can be put, after a change of variable, in the form

$$q(x,y) = x^2 - y^2,\ q'(\xi,\eta) = a\xi^2 - b\eta^2,$$

with $a, b > 0$. The difference with the previous case is that $\mathscr{G}(q,q') \neq \{0\}$ when $a = b > 1$. It does not contain any Gaussian function, since $|\det(q)\det(q')| = a^2 > 1$, but the distribution $\delta(x-y)$ defined by

$$\langle \delta(x-y), \phi \rangle = \int \phi(x,x)\, dx \tag{3.13}$$

belongs to $\mathscr{G}(q,q')$.

THEOREM 3.1.26. – *Let $a, b > 0$, and $q(x,y) = x^2 - y^2$, $q'(\xi,\eta) = a\xi^2 - b\eta^2$. Then $\mathscr{G}(q,q') = \{0\}$ if and only if $\max(a,b) > 1$ and $a \neq b$.*

Proof. – When $\max(a,b) \leq 1$, $\mathscr{G}(q,q')$ contains a Gaussian function, by Theorem 3.1.8. When $a = b \geq 1$, (3.13) gives a non zero element of $\mathscr{G}(q,q')$.

Assume now that $a > b$ and $a > 1$. If we divide q' by a suitable constant, we can assume that $a > 1 > b > 0$. Let $f \in \mathscr{G}(q,q')$. Fix a polynomial P on $\mathbb{R}$ and $b < t < 1$. Consider the tempered distribution T_P defined on $S(\mathbb{R})$ by

$$\langle T_P, \phi \rangle = \langle f, \phi \otimes P\gamma_t \rangle,$$

where $\phi \otimes P\gamma_t(x,y) = \phi(x)P(y)\exp(-\pi/t|y|^2)$. Let Q be the polynomial such that $P\gamma_t$ is the Fourier transform of $Q\gamma_{1/t}$. We have

$$\left\langle \widehat{T_P}, \phi \right\rangle = \left\langle \widehat{f}, \phi \otimes Q\gamma_{1/t} \right\rangle.$$

Using the inequality $x^2 - 1/ty^2 \leq |x^2 - y^2| - (1/t - 1)y^2$, the fact that $t < 1$ and $f\exp(\pm\pi q) \in S'(\mathbb{R}^2)$, we find $T_P(\cdot)\exp(\pi|\cdot|^2) \in S'(\mathbb{R})$. In the same way, using the fact that $t > b$, we get $\widehat{T_P}(\cdot)\exp(\pi a|\cdot|^2) \in S'(\mathbb{R})$. Theorem (1.3.4) gives then $T_P = 0$. Since it is true for any polynomial P, Lemma 1.2.2 gives $f = 0$. □

REMARK 3.1.27. – Theorem 5.3.2 will describe the elements of $\mathscr{G}(q,q')$ when $a = b > 1$, while Theorem 5.1.6 describes $\mathscr{G}(q,q')$ for $a = b = 1$. We do not have any analogue of Theorem 2.1.6 for the case $\max(a,b) \leq 1$.

3.2. Annihilating pairs when $d \geq 2$

Let q and q' be two quadratic forms defined by

$$q(x) = \langle Ax, x\rangle,\ q'(\xi) = \langle A'\xi, \xi\rangle,$$

with A, A' real symmetric and invertible matrices. The nature of the space $\mathscr{G}(q, q')$ is unchanged by a linear change of variable, so $\mathscr{G}(q, q')$ is conjugated to $\mathscr{G}(\tilde{q}, \tilde{q}')$, where $\tilde{q}(x) = q(Px)$, $\tilde{q}'(\xi) = q({}^tP^{-1}\xi)$, and P is an invertible matrix. We will focus our attention to the case where P can be chosen so that $\tilde{q}$ and $\tilde{q}'$ are diagonal:

Proposition 3.2.1. – *Let A and A' be two symmetric matrices. Then there exists an invertible matrix P such that ${}^tPAP$ and $P^{-1}A'{}^tP^{-1}$ are diagonal if and only if AA' is diagonalizable over $\mathbb{R}$.*

Proof. – If P exists, then the matrix ${}^tPAA'{}^tP^{-1}$ is diagonal. Conversely, if ${}^tPAA'{}^tP^{-1}$ is diagonal, then the two matrices ${}^tPAP$ and $P^{-1}A'{}^tP^{-1}$ commute, so that they can be diagonalized by the same orthogonal matrix Q. Put $R = PQ$. Then ${}^tRAR$ and $R^{-1}A'{}^tR^{-1}$ are diagonal. □

Remark 3.2.2. – The matrix AA' is diagonalizable over $\mathbb{R}$ for example when A or A' is positive, or when A and A' commute.

We are reduced to quadratic forms defined by

$$q(x) = \sum_{i=1}^{d} \varepsilon_i x_i^2,\ q'(\xi) = \sum_{i=1}^{d} \lambda_i \xi_i^2, \tag{3.14}$$

where $\varepsilon_i \in \{-1, +1\}$ and $\lambda_i \in \mathbb{R}^*$.

As a consequence of Proposition 3.1.21, the following is true.

Theorem 3.2.3. – *Assume that $\varepsilon_i = 1$ for all i. Then $\mathscr{G}(q, q') = \{0\}$ if and only if* $\max_i |\lambda_i| > 1$.

Without assumption on q, we can establish the following result.

Theorem 3.2.4. – *Let $I = \{i; |\lambda_i| = \max_k |\lambda_k|\}$. Assume that all the λ_i, for $i \in I$, have the same sign, and that $\max_k |\lambda_k| > 1$. Then $\mathscr{G}(q, q') = \{0\}$.*

Proof. – The proof follows the lines of the one of Theorem 3.1.26. We can assume that q' has the form

$$q'(\xi) = a \sum_{i=1}^{d_0} \xi_k^2 + \sum_{i>d_0} \lambda_i \xi_i^2,$$

with $a > 1$ and $|\lambda_i| < 1$ for $i > d_0$. Choose t such that $1 > t > \max_{i>d_0} |\lambda_i|$. Let $f \in \mathscr{G}(q, q')$. Define the distribution T_P on $\mathbb{R}^{d_0}$ by

$$\langle T_P, \phi\rangle = \langle f, \phi \otimes P\gamma_t\rangle,$$

where P is a polynomial and $\gamma_t(\cdot) = \exp(-\pi/t|\cdot|^2)$. Since $f \in \mathscr{G}(q,q')$, we have $T_P \in \mathscr{G}(q_0, q_0')$, with

$$q_0(x) = \sum_{i=1}^{d_0} \varepsilon_i x_i^2,\ q_0'(\xi) = a|\xi|^2.$$

Proposition 3.1.21 implies that $T_P = 0$ (since $a > 1$), for all polynomial P, and Lemma 1.2.2 gives $f = 0$. □

REMARK 3.2.5. – The condition given in Theorem 3.2.4 is not necessary in general, unless $d = 1$ or $d = 2$, see Theorem 3.1.26. When one of the quadratic forms is the Lorentz form, the necessary and sufficient condition will be given in Theorem 5.3.1.

COROLLARY 3.2.6. – *Let $q(x) = \sum_i \varepsilon_i x_i^2$, with $\varepsilon_i = \pm 1$, and $q'(\xi) = \sum_{i=1}^d \lambda_i \xi_i^2$, with $\lambda_i \neq 0$. If there exists i such that $|\lambda_i| > |\lambda_j|$ for all $j \neq i$, and $|\lambda_i| > 1$. Then $\mathscr{G}(q,q') = \{0\}$.*

3.3. Annihilating pairs for functions

The space $\mathscr{G}(q,q')$, with $q(x) = q'(x) = 2ax_1x_2$ on $\mathbb{R}^2$ ($a > 1$), is a singular case, as shown by Theorem 3.1.26. It does not contain any Gaussian function, but still contains a non zero element. We will show here in particular that it does not contain any function. Note that this is a consequence of Theorem 5.3.2.

DEFINITION 3.3.1. – *Let $\mathscr{F}(q,q')$ be the space of distributions $f \in S'(\mathbb{R}^d)$ such that*

$$f(\cdot)\exp(\pm\pi q(\cdot)) \in L^1(\mathbb{R}^d),\ \widehat{f}(\cdot)\exp(\pm\pi q'(\cdot)) \in S'(\mathbb{R}^d).$$

This is made of integrable functions, so that the Fourier transform is taken in the usual sense. We can prove the following.

THEOREM 3.3.2. – *Let $q(x) = \langle Ax, x\rangle$ and $q'(\xi) = \langle A'\xi,\xi\rangle$, where A and A' are two symmetric, invertible matrices. Assume that AA' is diagonalizable over $\mathbb{R}$. Then $\mathscr{F}(q,q') = \{0\}$ if only if AA' has an eigenvalue λ such that $|\lambda| \geq 1$.*

For the proof, we do as usual a change of variable so that q and q' are given by (3.14). We will show that $\mathscr{F}(q,q') = \{0\}$ if if only if $\max_i |\lambda_i| \geq 1$. We will use the following estimate, which is fundamental for the remaining of the text. It is a limiting case of the estimates of Proposition 2.2.2.

LEMMA 3.3.3. – *Let $d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. For $x \in \mathbb{R}^d$, we write $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$. Let q be the quadratic form $q(x) = |x_1|^2 - |x_2|^2$. Let $N > 0$. Then there exists $C > 0$ such that for all $z \in \mathbb{C}^d$,*

$$\sup_{x\in\mathbb{R}^d} (1 + |x| + |z|)^N \exp\big(-\pi(|x|^2 + |q(x)|) + 2\pi\langle x, \mathscr{R}e(z)\rangle - \frac{\pi}{2}\mathscr{R}e(z^2)\big)$$
$$\leq C(1 + |z|)^N \exp\big(\pi|\mathscr{R}e(z_1)||\mathscr{R}e(z_2)| + \frac{\pi}{2}|\mathscr{I}m(z)|^2\big).$$

Proof. – Assume first that $q(x) \geq 0$. Then $|x_1| \leq |x_2|$. We have

$$
\begin{aligned}
&(1+|x|+|z|)^N \exp\big(-\pi(|x|^2+|q(x)|)+2\pi\langle x, \mathscr{Re}(z)\rangle - \frac{\pi}{2}\,\mathscr{Re}(z^2)\big)\\
&\quad\leq C(1+|x_+|+|z|)^N\\
&\qquad \times \exp\big(-2\pi|x_1|^2+2\pi|x_1|(|\,\mathscr{Re}(z_1)|+|\,\mathscr{Re}(z_2)|)-\frac{\pi}{2}\,\mathscr{Re}(z^2)\big)\\
&\quad\leq C(1+|z|)^N \exp\big(\frac{\pi}{2}(|\,\mathscr{Re}(z_1)|+|\,\mathscr{Re}(z_2)|)^2-\frac{\pi}{2}\,\mathscr{Re}(z^2)\big)\\
&\quad= C(1+|z|)^N \exp\big(\pi|\,\mathscr{Re}(z_1)||\,\mathscr{Re}(z_2)|+\frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big).
\end{aligned}
$$

The same estimate holds for $q(x) \leq 0$ by symmetry. □

COROLLARY 3.3.4. – *We keep the notations of Lemma 3.3.3. Let $f \in S'(\mathbb{R}^d)$. Then $f(\cdot)\exp(\pm\pi q(\cdot)) \in S'(\mathbb{R}^d)$ if and only of there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,*

$$|\mathscr{B}(f)(z)| \leq C(1+|z|)^N \exp\big(\pi|\,\mathscr{Re}(z_1)||\,\mathscr{Re}(z_2)|+\frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big).$$

Recall that the Bargmann transform is defined by (1.3). Compare this with Lemma 3.1.13.

Proof. – Let χ be a smooth, compactly supported function on $\mathbb{R}$ such that $\chi(t) = 1$ for $|t| \leq 1$, and $\chi(t) = 0$ for $|t| \geq 2$. Let

$$\phi(z,x) = \exp\big(-\pi|x|^2+2\pi\langle x,z\rangle-\frac{\pi}{2}z^2\big).$$

We have $\mathscr{B}(f)(z) = \langle f, (\chi\circ q)\phi(z,\cdot)\rangle + \langle f, (1-\chi\circ q)\phi(z,\cdot)\rangle$.

Since $f \in S'(\mathbb{R}^d)$, we can find $C, M > 0$ such that

$$|\langle f, \phi\rangle| \leq CP_M(\phi)$$

for all Schwartz function ϕ. The semi-norm P_M was defined by (1.7). Hence

$$
\begin{aligned}
&|\langle f, (\chi\circ q)\phi(z,\cdot)\rangle| \leq CP_M((\chi\circ q)\phi(z,\cdot))\\
&\quad\leq C \sup_{|q(x)|\leq 2}(1+|x|+|z|)^{2M}\exp(-\pi|x|^2+2\pi\langle x,\mathscr{Re}(z)\rangle-\frac{\pi}{2}\,\mathscr{Re}(z^2))\\
&\quad\leq C \sup_{x\in\mathbb{R}^d}(1+|x|+|z|)^{2M}\exp(-\pi(|x|^2+|q(x)|)+2\pi\langle x,\mathscr{Re}(z)\rangle-\frac{\pi}{2}\,\mathscr{Re}(z^2))\\
&\quad\leq C(1+|z|)^{2M}\exp\big(\pi|\,\mathscr{Re}(z_1)||\,\mathscr{Re}(z_2)|+\frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big).
\end{aligned}
$$

We used Lemma 3.3.3 for the last inequality.

Now we use the fact that $f(\cdot)\chi\circ q(\cdot)\exp(\pi|q(\cdot)|) \in S'(\mathbb{R}^d)$. There exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,

$$
\begin{aligned}
&|\langle f, (1-\chi\circ q)\phi(z,\cdot)\rangle| \leq CP_N((1-\chi\circ q)e^{-\pi|q(\cdot)|}\phi(z,\cdot))\\
&\quad\leq C \sup_{x\in\mathbb{R}^d}(1+|x|+|z|)^{2M}\exp(-\pi(|x|^2+|q(x)|)+2\pi\langle x,\mathscr{Re}(z)\rangle-\frac{\pi}{2}\,\mathscr{Re}(z^2))\\
&\quad\leq C(1+|z|)^{2N}\exp\big(\pi|\,\mathscr{Re}(z_1)||\,\mathscr{Re}(z_2)|+\frac{\pi}{2}|\,\mathscr{Im}(z)|^2\big),
\end{aligned}
$$

using Lemma 3.3.3 again.

Conversely, assume that

$$|\mathscr{B}(f)(z)| \le C(1+|z|)^N \exp\big(\pi|\mathscr{R}e(z_1)||\mathscr{R}e(z_2)| + \frac{\pi}{2}|\mathscr{I}m(z)|^2\big)$$

for all $z \in \mathbb{C}^d$. Put $F(z) = \mathscr{B}(f)(z)$, and consider the entire function

$$G(z) = \int_{\mathbb{R}^{d-}} F(\sqrt{2}z_1, iy)\exp\big(\frac{\pi}{2}z_1^2 - \frac{\pi}{2}y^2 - \pi(y-z_2)^2 + \frac{\pi}{2}z_2^2\big)\,dy.$$

It is the Bargmann transform, with respect to the variable y, of the function $F(\sqrt{2}z_1, iy)\exp\big(\frac{\pi}{2}z_1^2 - \frac{\pi}{2}y^2\big)$. The integral is absolutely convergent and we have

$$|G(z)| \le C(1+|z|)^N \exp\big(\frac{\pi}{2}|z|^2\big).$$

By Proposition 1.2.5, there exists a tempered distribution $g \in S'(\mathbb{R}^d)$ whose Bargmann transform is G. Let $\mathscr{F}_2$ denote the Fourier transform with respect to $\mathbb{R}^{d_2}$. It follows by identification of the Bargmann transforms that

$$(\mathscr{F}_2 g)(\sqrt{2}x_1, x_2) = f(x)\exp(\pi(|x_1|^2 - |x_2|^2)),$$

and we conclude that $f\exp(\pi q) \in S'(\mathbb{R}^d)$. We can as well prove that $f\exp(-\pi q) \in S'(\mathbb{R}^d)$, and the proof is complete. □

We are now in position to prove Theorem 3.3.2.

Proof of Theorem 3.3.2. – If $\max_i |\lambda_i| < 1$, choose t such that $\max_i |\lambda_i| < t < 1$. Then the Gaussian function $\gamma_t(x) = \exp(-\pi/t|x|^2)$ belongs to $\mathscr{F}(q, q')$.

Assume now that $\lambda = \max_i |\lambda_i| \ge 1$. We first divide q' by a constant $\lambda \ge 1$ so that $\max_i |\lambda_i| = 1$. Then we separate the λ_i such that $|\lambda_i| < 1$, and tensorize with Hermite functions as in the proof of Theorem 3.2.4. So we will assume that all the λ_i are equal to 1 or -1. Up to a permutation of the variables, we can decompose the space $\mathbb{R}^d$ as $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3} \times \mathbb{R}^{d_4}$, with $d_1 + \cdots + d_4 = d$, so that

$$q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2,\ q'(\xi) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2.$$

We apply Corollary 3.3.4 and find $C, N > 0$ such that for all $z \in \mathbb{C}^d$,

$$|\mathscr{B}(f)(z)| \le C(1+|z|)^N \exp\big(\pi|\mathscr{I}m(z_1,z_3)||\mathscr{I}m(z_2,z_4)| + \frac{\pi}{2}|\mathscr{R}e(z)|^2\big). \tag{3.15}$$

If we use the hypothesis on f we have

$$|\mathscr{B}(f)(z)| \le \int_{\mathbb{R}^d} |f(x)|\exp(\pi|q(x)|)K(x,z)\,dx,$$

where

$$K(x,z) = \exp\big(-\pi(|x|^2 + |q(x)|) + 2\pi\langle x, \mathscr{R}e(z)\rangle - \frac{\pi}{2}\mathscr{R}e(z^2)\big).$$

Using Lemma 3.3.3 we have

$$|K(x,z)| \le C\exp\big(\pi|\mathscr{R}e(z_1,z_3)||\mathscr{R}e(z_2,z_4)| + \frac{\pi}{2}|\mathscr{I}m(z)|^2\big),$$

and hence

$$|\mathscr{B}(f)(z)| \leq C \exp\left(\pi|\mathscr{R}e(z_1,z_2)||\mathscr{R}e(z_3,z_4)| + \frac{\pi}{2}|\mathscr{I}m(z)|^2\right). \tag{3.16}$$

Put $F(z) = \mathscr{B}(f)(z)$. We fix z_2 real, z_3 imaginary, and $z_4 = 0$. Apply Lemma 1.1.8 to $F(z)$ as a function of $z_1 \in \mathbb{C}^{d_1}$. Estimates (3.15) and (3.16) imply that $F(z)$ is constant, as a function of z_1. Then notice that when z_1 is real, we have

$$K(x,z) \leq C, \lim_{z_1\to\infty} K(x,z) = 0.$$

Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{z_1\to\infty} F(z) = 0,$$

and hence $F(z) = 0$.

We can prove in a similar way that, given any partial differential operator D of order $n \in \mathbb{N}$, $DF(z)$ is a polynomial in z_1, of degree at most $n-1$, provided that $z_4 = 0$, z_3 is imaginary and z_2 real.

We apply this to $D = \partial_{z_4}^{n_4}$, given any $n_4 \in \mathbb{N}^{d_4}$. Then $\partial_{z_1}^{n_1}\partial_{z_4}^{n_4} F(z) = 0$ provided $|n_1| \geq |n_4|$. If we take extra derivatives in z_2, z_3, and put $z = 0$, we obtain

$$\partial_z^n F(0) = 0$$

for all $n \in \mathbb{N}^d$ such that $|n_1| \geq |n_4|$. By symmetry, this is also true when $|n_1| \leq |n_4|$, and hence all derivatives of F at 0 are equal to 0. It follows that F and f are identically zero. □

3.3.1. Other subspaces of $\mathscr{F}(q,q')$. – We show in this paragraph that we can extend Theorem 3.3.2 to another class of functions.

We will consider quadratic forms defined by

$$q(x) = \sum_i \varepsilon_i x_i^2, \; q'(\xi) = \sum_i \mu_i \xi_i^2, \tag{3.17}$$

with $\varepsilon_i, \mu_i \in \{-1,+1\}$.

Theorem 3.3.5. – *Let q, q' be defined by* (3.17). *There exists an integer $N \geq 1$ such that every tempered distribution f satisfying*

$$(1+|x|^2)^{-N/2} f(\cdot)\exp(\pm\pi q(\cdot)) \in L^1(\mathbb{R}^d), \; \widehat{f}(\cdot)\exp(\pm\pi q'(\cdot)) \in S'(\mathbb{R}^d)$$

is identically zero.

Proof. – As in the proof of Theorem 3.3.2, write

$$q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2, \; q'(\xi) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2,$$

according to the decomposition $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3} \times \mathbb{R}^{d_4}$. Let F be the Bargmann transform of f. We can as well prove that for any partial differential operator D of order n, $DF(z)$ is a polynomial in z_1, as soon as $z_4 = 0$, z_3 is imaginary and z_2 real.

Let $D = \partial_{z_4}^{n_4}$. By analyticity, $DF(z)$ it is still a polynomial in z_1 when z_2, z_3 are arbitrary, and $z_4 = 0$. Let α be its degree in z_1. We will show that

$$\alpha < N + |n_4| - d_1. \tag{3.18}$$

Fix now $z_4 = 0$, and z_2, z_3 imaginary. We also take z_1 of the form $z_1 = i\xi_1$, $\xi_1 \in \mathbb{R}^{d_1}$. There exists a polynomial Q of degree $|n_4|$ such that

$$\partial_{z_4}^{n_4}\Big|_{z_4=0} \left(e^{-\pi(x_4-z_4)^2 + \frac{\pi}{2} z_4^2}\right) = Q(x_4) e^{-\pi x_4^2}.$$

The quantity $\exp\left(\frac{\pi}{2}(-\xi_1^2 + z_2^2 + z_3^2)\right) \partial_{z_4}^{n_4}\big|_{z_4=0} F(z)$ is equal to

$$\int_{\mathbb{R}^d} f(x) Q(x_4) \exp(-\pi x_1^2 + 2i\pi x_1 \xi_1 - \pi(x_2 - z_2)^2 - \pi(x_3 - z_3)^2 - \pi x_4^2)\, dx.$$

Taking the inverse Fourier transform in ξ_1, we find

$$\int Q(x_4) \exp(-\pi(x_2 - z_2)^2 - \pi(x_3 - z_3)^2 - \pi x_4^2) f(x)\, dx_2\, dx_3\, dx_4 = P(x_1) e^{-\pi x_1^2},$$

where P is a polynomial of degree α, depending on the fixed z_2, z_3. In order to show (3.18), we prove that

$$\int_{\mathbb{R}^{d_1}} \frac{|P(x_1)| dx_1}{(1 + |x_1|)^{N+|n_4|}} < \infty.$$

Indeed,

$$\int_{\mathbb{R}^{d_1}} \frac{|P(x_1)| dx_1}{(1 + |x_1|)^{N+|n_4|}} \leq C(z_2, z_3) \int_{\mathbb{R}^d} \frac{|f(x) Q(x_4)| e^{\pi(x_1^2 - x_2^2 - x_3^2 - x_4^2)}\, dx}{(1 + |x_1|)^{N+|n_4|}}.$$

When $x_1^2 \geq \frac{1}{2}(x_3^2 + x_4^2)$, we have

$$\begin{aligned} \frac{|Q(x_4)| e^{\pi(x_1^2 - x_2^2 - x_3^2 - x_4^2)}}{(1 + |x_1|)^{N+|n_4|}} &\leq C \frac{\exp(\pi q(x) - 2\pi x_2^2)}{(1 + |x_1| + |x_3| + |x_4|)^N} \\ &\leq C \frac{\exp(\pi |q(x)|)}{(1 + |x|)^N}. \end{aligned}$$

An if $x_1^2 \leq \frac{1}{2}(x_3^2 + x_4^2)$, we have

$$\begin{aligned} \frac{|Q(x_4)| e^{\pi(x_1^2 - x_2^2 - x_3^2 - x_4^2)}}{(1 + |x_1|)^{N+|n_4|}} &\leq C \frac{\exp(-\pi/3(x_2^2 + x_3^2 + x_4^2))}{(1 + |x_1|)^{N+|n_4|}} \\ &\leq C(1 + |x|)^{-N} \leq C \frac{\exp(\pi |q(x)|)}{(1 + |x|)^N}. \end{aligned}$$

Hence we have

$$\int_{\mathbb{R}^{d_1}} \frac{|P(x_1)| dx_1}{(1 + |x_1|)^{N+|n_4|}} \leq C(z_2, z_3) \int_{\mathbb{R}^d} \frac{|f(x)| e^{\pi |q(x)|} dx}{(1 + |x|)^N} < \infty,$$

and (3.18) follows. Thus

$$\partial_z^n F(0) = 0$$

for any $n \in \mathbb{N}^d$ such that $|n_1| \geq N + |n_4| - d_1$. By symmetry, this is true whenever one of the following conditions is satisfied:

$$\begin{aligned} &|n_1| \geq N - d_1 + |n_4|, \quad |n_4| \geq N - d_4 + |n_1| \\ &|n_2| \geq N - d_2 + |n_3|, \quad |n_3| \geq N - d_3 + |n_2|. \end{aligned} \tag{3.19}$$

Take for N the integer part of

$$\max\Big(\frac{d_1 + d_4 + 1}{2}, \frac{d_2 + d_3 + 1}{2}\Big). \tag{3.20}$$

We claim that for any $n \in \mathbb{N}^d$, one of the conditions (3.19) is satisfied. Indeed, if this is not the case, we have

$$2 \leq 2N - (d_1 + d_4),\ 2 \leq 2N - (d_3 + d_4),$$

which is a contradiction. Since all the partial derivatives of F at 0 vanish, we have $f = 0$. □

Remark 3.3.6. – A possible value of N is given by (3.20). Note that we have indeed $N \geq 1$. Assume that $q = q'$. Then N is the smallest integer such that $N \geq d/2$. Theorem 3.3.5 is sharp when $q = q'$, $d = 2k$, and q has signature (k, k) on $\mathbb{R}^{2k}$. Indeed, the standard Gaussian function satisfies

$$(1 + |x| + |y|)^{-N} f(x, y) \exp(\pm\pi(|x|^2 - |y|^2)) \in L^1(\mathbb{R}^{2k})$$

whenever $N > k$. This value is also optimal form a form of signature $(k + 1, k)$ or $(k, k + 1)$ on $\mathbb{R}^{2k+1}$. We think that, in the general case, the sharpest constant is $N = \max(k, l)$, where (k, l) is the signature of q, since a Gaussian function satisfies the conditions if and only if $N > \max(k, l)$.

Corollary 3.3.7. – *Let $f \in S'(\mathbb{R}^2)$. Assume that*

$$f(x, y) = O\big(\exp(-2a\pi|xy|)\big),\ \widehat{f}(\xi, \eta) = O\big(\exp(-2b\pi|\xi\eta|)\big).$$

If $ab > 1$, then $f = 0$.

Proof. – We can assume that $a > 1$ and $b = 1$. Hence

$$\widehat{f}(\xi, \eta) \exp(\pm 2\pi\xi\eta) \in L^\infty$$

and

$$\begin{aligned} \int \frac{|f(x, y)| \exp(2\pi|xy|)}{1 + |x| + |y|}\, dx\, dy &\leq C \int \frac{\exp(-2\pi(a - 1)|xy|)}{1 + |x| + |y|}\, dx\, dy \\ &\leq C \int_{x\in\mathbb{R}} \int_{|y|\leq|x|} \frac{\exp(-2\pi(a - 1)|xy|)}{1 + |x|}\, dy\, dx \\ &\leq C \int_{x\in\mathbb{R}} \frac{dx}{(1 + |x|)^2} < \infty. \end{aligned} \tag{3.21}$$

The value given by (3.20) is $N = 1$, we can use Theorem 3.3.5, and we find $f = 0$. □

REMARK 3.3.8. – Corollary 3.3.7 is an analogue of Hardy's uncertainty principle for the non degenerate quadratic form $2xy$, which is the one appearing in Beurling's uncertainty principle. The condition is sharp, since the standard Gaussian function satisfies the hypotheses when $a = b = 1$.

We give the following corollary concerning annihilating pairs of sets.

COROLLARY 3.3.9. – *We take the notations of Theorem 3.3.5. Let* $1 \leq p, q \leq \infty$, *such that* $p^{-1} + q^{-1} = 1$. *Assume that* $q > \frac{d-2}{N}$, *that* $f \in L^p(\mathbb{R}^d)$ *is supported in* $\{x; |q(x)| \leq C\}$, *and that* $\widehat{f}$ *is supported in* $\{\xi; |q'(\xi)| \leq C'\}$, *where* C *and* C' *are two fixed constants. Then* $f = 0$.

Proof. – Recall that for any $t \geq 0$, the function equal to $(1+|x|)^{-t}$ when $|q(x)| \leq C$, and to 0 when $|q(x)| > C$, is in the space $L^1(\mathbb{R}^d)$ if and only if $t > d-2$. Since f is supported in $\{x; |q(x)| \leq C\}$,

$$\int_{\mathbb{R}^d} \frac{|f(x)| \exp(\pi|q(x)|)}{(1+|x|)^N}\, dx \leq C\|f\|_{L^p} \left(\int_{|q(x)| \leq C} (1+|x|)^{-Nq} \right)^{1/q} < \infty.$$

Since moreover $\widehat{f} \exp(\pm\pi q') \in S'(\mathbb{R}^d)$, Theorem 3.3.5 gives $f = 0$. □

REMARK 3.3.10. – If $d = 2$, Corollary 3.3.9 applies for any values of p, even $p = \infty$. When $d = 3$, it applies for $1 \leq p < \infty$.

CHAPTER 4

CRITICAL PAIRS

We study in this chapter the elements of $\mathscr{G}(q, q')$ when the pair is critical. We give necessary and sufficient condition on their Bargmann transform, and we state three conjectures on the form of the elements of $\mathscr{G}(q, q')$.

4.1. Introduction

Take the two quadratic forms defined by

$$q(x) = \langle Ax, x\rangle,\ q'(\xi) = \langle A'\xi, \xi\rangle,$$

where A, A' are symmetric invertible matrices. We will assume throughout in this chapter that the pair (q, q') is critical, which means that AA' is diagonalizable over $\mathbb{C}$, with eigenvalues of modulus 1. We can always make a change of variables so that A and A' are orthogonal and symmetric matrices. Recall that the Gaussian functions in the space $\mathscr{G}(q, q')$ are all real, and are characterized by Theorem 3.1.11.

We will use the Bargmann transform, introduced in the first chapter. We will show that it characterizes the elements of $\mathscr{G}(q, q')$ by the growth of their Bargmann transform.

We now introduce useful operators linked to the Bargmann transform. The annihilation and creation operators from quantum mechanics (see [**3, 12**]), are defined as follows.

Definition 4.1.1. – *The creation operators are defined on $S'(\mathbb{R}^d)$ by*

$$\mathbf{z}_i(f) = x_i f - \frac{1}{2\pi}\partial_{x_i} f. \tag{4.1}$$

The annihilation operators are

$$\mathbf{z}_i^*(f) = x_i f + \frac{1}{2\pi}\partial_{x_i} f. \tag{4.2}$$

The annihilation operators are the formal adjoints of the creation operators. The creation operators commute, and the same is true for the annihilation operators.

Proposition 4.1.2. – *For all $f \in S'(\mathbb{R}^d)$ and $z \in \mathbb{C}^d$, we have*

$$z_i \mathscr{B}(f)(z) = \mathscr{B}(\mathbf{z}_i f)(z), \tag{4.3}$$

$$\partial_{z_i} \mathscr{B}(f)(z) = \mathscr{B}(\mathbf{z}_i^* f)(z). \tag{4.4}$$

Moreover

$$(-2\pi)^k \mathbf{z}_i^k g(x) = e^{\pi x_i^2} \partial_{x_i}^k [e^{-\pi x_i^2} g(x)], \tag{4.5}$$

$$(-2\pi)^k \mathbf{z}_i^{*k} g(x) = e^{-\pi x_i^2} \partial_{x_i}^k [e^{\pi x_i^2} g(x)], \tag{4.6}$$

for any $g \in S'(\mathbb{R}^d)$.

4.2. Characterization of $\mathscr{G}(q, q')$

Assume for simplicity that we have already made a change of variables, so that A and A' are orthogonal and symmetric. Let

$$E_1 = \mathrm{Ker}(A - I),\ E_2 = \mathrm{Ker}(A + I)$$

and

$$E_1' = \mathrm{Ker}(A' - I),\ E_2' = \mathrm{Ker}(A' + I)$$

be the eigenspaces associated to A and A'. For $x \in \mathbb{R}^d$, let x_1 and x_2 the projections of x on E_1 and E_2, respectively. Let x_1' and x_2' be the projections of x on E_1' and E_2'.

For our analysis, we will use the fundamental estimate of Lemma 3.3.3 and Corollary 3.3.4. The following is an immediate consequence.

Theorem 4.2.1. – *Let $f \in S'(\mathbb{R}^d)$. Then $f \in \mathscr{G}(q, q')$ if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,*

$$\begin{aligned} |\mathscr{B}(f)(z)| &\le C(1 + |z|)^N \exp\left(\pi |\mathscr{Re}(z_1)||\mathscr{Re}(z_2)| + \frac{\pi}{2} |\mathscr{Im}(z)|^2\right) \\ |\mathscr{B}(f)(z)| &\le C(1 + |z|)^N \exp\left(\pi |\mathscr{Im}(z_1')||\mathscr{Im}(z_2')| + \frac{\pi}{2} |\mathscr{Re}(z)|^2\right). \end{aligned} \tag{4.7}$$

So our initial problem has been translated into the characterization of a subspace of the Fock space.

When $A = A'$, we can give a more precise result. We can do a rotation in the variables, so that

$$q(x) = q'(x) = |x_1|^2 - |x_2|^2, \tag{4.8}$$

with $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$ $(d_1 + d_2 = d)$.

Theorem 4.2.2. – *Let q be a quadratic form given by (4.8). Let $f \in S'(\mathbb{R}^d)$. Then $f \in \mathscr{G}(q, q)$ if and only if there exist $C, N > 0$, such that for all $z \in \mathbb{C}^d$,*

$$|\mathscr{B}(f)(z)| \le C(1 + |z|)^N \exp(\pi |\mathscr{Re}(z_1)||\mathscr{Re}(z_2)| + \pi |\mathscr{Im}(z_1)||\mathscr{Im}(z_2)|) \tag{4.9}$$

Proof. – Fix $z_1 \in \mathbb{R}^{d_1}$, and $z_2 \in \mathbb{R}^{d_2}$. Consider the analytic function G defined on $\mathbb{C}^*$ by

$$G(t) = \mathscr{B}(f)(tz_1, t^{-1}z_2).$$

It follows from Theorem 4.2.1 that there exist constants C, A, depending on the fixed z_1, z_2, such that for all $t \in \mathbb{C}^*$,

$$|G(t)| \leq C(1 + |t| + |t|^{-1})^N \exp(A(|\mathscr{Re}(t)|^2 + |\mathscr{Re}(t^{-1})|^2))$$

and

$$|G(t)| \leq C(1 + |t| + |t|^{-1})^N \exp(A(|\mathscr{Im}(t)|^2 + |\mathscr{Im}(t^{-1})|^2)).$$

From Lemma 1.3.2, we conclude that

$$|G(t)| \leq C'(1 + |t| + |t|^{-1})^N$$

for all $t \in \mathbb{C}^*$. Hence G is a polynomial in t and t^{-1}, and we can write

$$\mathscr{B}(f)(tz_1, t^{-1}z_2) = \sum_{k=-N}^{N} t^k F_k(z),$$

where F_k are entire functions on $\mathbb{C}^d$.

We will show that (4.9) holds for each of the F_k, for some constants $C, N > 0$. We have

$$F_N(z) = \partial_t^{2N} \mathscr{B}(f)(tz_1, t^{-1}z_2)$$

for any $t \in \mathbb{C}^*$, $z \in \mathbb{C}^d$. Propositions 3.1.2, 4.1.2 and Theorem 4.2.1 imply that there exist $C, M > 0$ such that for all z, ζ, t,

$$\begin{aligned}|F_N(z)| \leq {} & C(1 + |z| + |t| + |t|^{-1})^M \\ & \times \exp\left(\pi|\mathscr{Re}(z_1)||\mathscr{Re}(z_2)| + \frac{\pi}{2}(|\mathscr{Im}(tz_1)|^2 + |\mathscr{Im}(z_2/t)|^2)\right).\end{aligned}$$

If we minimize this estimate over t, we find $C, M' > 0$ such that for all $z \in \mathbb{C}^d$,

$$|F_N(z)| \leq C(1 + |z| + |\zeta|)^{M'} \exp\left(\pi|\mathscr{Re}(z_1)||\mathscr{Re}(z_2)| + \pi|\mathscr{Im}(z_1)||\mathscr{Im}(z_2)|\right).$$

We obtain by induction similar estimates for all the F_k. □

When A and A' commute, we can make a change of variables so that

$$q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2,\ q'(\xi) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2, \tag{4.10}$$

with $x_i, \xi_i \in \mathbb{R}^{d_i}$ and $d_1 + \cdots + d_4 = d$.

Theorem 4.2.3. – *Let q, q' be defined by* (4.10). *Let $f \in S'(\mathbb{R}^d)$. Then $f \in \mathscr{G}(q,q')$ if and only if there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,*

$$\begin{aligned}|\mathscr{B}(f)(z)| &\leq C(1 + |z|)^N \exp\left(\pi|\mathscr{Re}(z_1, z_2)||\mathscr{Re}(z_3, z_4)| + \frac{\pi}{2}|\mathscr{Im}(z)|^2\right)\\ |\mathscr{B}(f)(z)| &\leq C(1 + |z|)^N \exp\left(\pi|\mathscr{Im}(z_1, z_3)||\mathscr{Im}(z_2, z_4)| + \frac{\pi}{2}|\mathscr{Re}(z)|^2\right).\end{aligned} \tag{4.11}$$

This is a reformulation of Theorem 4.2.1. When $d_4 = 0$, we have a more precise version (which includes Theorem 4.2.2). Here

$$q(x_1, x_2, x_3) = |x_1|^2 + |x_2|^2 - |x_3|^2,\ q'(\xi_1, \xi_2, \xi_3) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2.$$

Put

$$q_0(x_2, x_3) = |x_2|^2 - |x_3|^2.$$

It is a non degenerate quadratic forms on $\mathbb{R}^{d_2+d_3}$. We will give a description of the elements of $\mathscr{G}(q, q')$ in terms of $\mathscr{G}(q_0, q_0)$.

THEOREM 4.2.4. – *Let $f \in S'(\mathbb{R}^d)$. Then $f \in \mathscr{G}(q, q')$ if and only if there exist $N \in \mathbb{N}$, distributions $f_k \in \mathscr{G}(q_0, q_0)$, such that*

$$f(x_1, x_2, x_3) = \sum_{k\in\mathbb{N}^{d_1};|k|\le N} x_1^k \exp(-\pi|x_1|^2) f_k(x_2, x_3).$$

Proof. – Such distributions belong clearly to $\mathscr{G}(q, q')$. Let $f \in \mathscr{G}(q, q')$, and F its Bargmann transform. By Theorem 4.2.3, there exist $C, N > 0$ such that for all $z \in \mathbb{C}^d$,

$$|F(z)| \le C(1 + |z|)^N \exp\big(\pi|\mathscr{Re}(z_1, z_2)||\mathscr{Re}(z_3)| + \frac{\pi}{2}|\mathscr{Im}(z)|^2\big)$$

and

$$|F(z)| \le C(1 + |z|)^N \exp\big(\pi|\mathscr{Im}(z_1, z_3)||\mathscr{Im}(z_2)| + \frac{\pi}{2}|\mathscr{Re}(z)|^2\big).$$

Fix $z_2 \in \mathbb{R}^{d_2}$ and $z_3 \in i\mathbb{R}^{d_3}$. By Lemma 1.3.2, we see that $F(z_1, z_2, z_3)$ is a polynomial in z_1. Hence

$$F(z) = \sum_{|k|\le N} z_1^k F_k(z_2, z_3),$$

where the F_k are entire functions depending only on z_2, z_3. We can express each function $F_k(z_2, z_3)$ as a polynomial in ∂_{z_i} applied to $F(z_1, z_2, z_3)$:

$$F_k(z_2, z_3) = P_k(\partial_{z_i}) F(0, z_2, z_3).$$

It follows from Propositions 3.1.2, 4.1.2 and Theorem 4.2.3 that there exist $C, N > 0$, such that for all $(z_2, z_3) \in \mathbb{C}^{d_2+d_3}$,

$$|F_k(z_2, z_3)| \le C(1 + |z_2| + |z_3|)^N \exp\big(\pi|\mathscr{Re}\, z_2||\mathscr{Re}\, z_3| + \frac{\pi}{2}|\mathscr{Im}(z_2, z_3)|^2\big)$$

and

$$|F_k(z_2, z_3)| \le C(1 + |z_2| + |z_3|)^N \exp\big(\pi|\mathscr{Im}\, z_2||\mathscr{Im}\, z_3| + \frac{\pi}{2}|\mathscr{Re}(z_2, z_3)|^2\big).$$

Using Theorem 4.2.3 again, we see that $F_k = \mathscr{B}(f_k)$, with $f_k \in \mathscr{G}(q_0, q_0)$. Hence

$$\mathscr{B}(f)(z) = \sum_{|k|\le N} z_1^k \mathscr{B}(f_k)(z_2, z_3),$$

which is equivalent to

$$f(x) = \sum_{|k|\le N} \mathbf{z}_1^k \gamma(x_1) f_k(x_2, x_3),$$

with $\gamma(x_1) = \exp(-\pi|x_1|^2)$. This completes the proof. □

4.3. Gaussian solutions revisited

Theorem 4.2.1 can be used to obtain the results of Theorem 3.1.11. Indeed, if

$$f(x) = \exp(-\pi\langle Mx, x\rangle)$$

is an element of $\mathscr{G}(q, q')$, where M is a symmetric complex matrix whose real part is positive, then

$$\mathscr{B}(f)(z) = \det(I + M)^{-1/2} \exp\big(\frac{\pi}{2}\langle \mathscr{C}(M)z, z\rangle\big),$$

where $\mathscr{C}(M)$ is the Cayley transform of M. The Gaussian elements are then characterized by their Bargmann transforms, which has the form

$$\exp\big(\frac{\pi}{2}\langle Nz, z\rangle\big),$$

where N is a real symmetric matrix such that

$$-2|x_1'||x_2'| \leq \langle Nx, x\rangle \leq 2|x_1||x_2| \tag{4.12}$$

for all $x \in \mathbb{R}^d$. Recall that x_1, x_2, x_1', x_2' denote the different orthogonal projections of x on the eigenspaces of the matrices A and A'.

Definition 4.3.1. – *Call by* $\mathbb{B}(q, q')$ *the open convex set made of the symmetric matrices* N *satisfying* (4.12), *and such that* $I - {}^tNN$ *is positive.*

When A and A' commute, N has a simple form. We can assume that

$$q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2,\ q'(\xi) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 \tag{4.13}$$

as above.

Proposition 4.3.2. – *A matrix* N *belongs to* $\mathbb{B}(q, q')$, *with* (q, q') *given by* (4.13), *if and only if there exists a real matrix* v *with* d_1 *lines and* d_4 *columns, a matrix* w *with* d_2 *lines and* d_3 *columns, such that* $I - {}^tvv$ *and* $I -{}^t ww$ *are positive, and such that*

$$N = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & w & 0 \\ 0 & {}^tw & 0 & 0 \\ {}^tv & 0 & 0 & 0 \end{pmatrix}.$$

This follows actually from the proof of Proposition 3.1.14. We could also give a description of $\mathbb{B}(q, q')$ in the cases of Propositions 3.1.15 and 3.1.17, but we will not use it.

The spaces $\mathscr{G}(q, q')$ are linear. Hence averages of solutions are still solutions. This enables us to give non Gaussian elements of $\mathscr{G}(q, q')$.

Definition 4.3.3. – *Let* μ *be a finite measure on* $O(q) \cap O(q')$. *Define*

$$\mathbf{G}_\mu(x) = \int_{O(q)\cap O(q')} \exp(-\pi|g(x)|^2)\, d\mu(g). \tag{4.14}$$

This belongs to $\mathscr{G}(q, q')$ by Theorem 3.1.11 and Proposition 3.1.2.

PROPOSITION 4.3.4. – *The function defined by* (4.14) *is a bounded continuous function. The Fourier transform of* $\mathbf{G}_\mu$ *is given by* $\mathbf{G}_\nu$, *where* ν *is the symmetric measure of* μ, *defined by* $\int \phi(g)\, d\nu(g) = \int \phi({}^t g^{-1})\, d\mu(g)$. *We have* $\mathbf{G}_\mu \in \mathscr{G}(q,q')$.

We can build other elements of $\mathscr{G}(q,q')$ using Proposition 3.1.2.

THEOREM 4.3.5. – *Let* $N \geq 1$, $\mu_1, \dots, \mu_N$ *be finite measures on* $O(q) \cap O(q')$. *Let* $P_1, \dots, P_N$ *be polynomials in* x *and in the partial derivatives with respect to* x. *Then the tempered distribution defined by*

$$f = \sum_{k=1}^{N} P_k(x, \partial_x)\mathbf{G}_{\mu_k} \tag{4.15}$$

belongs to $\mathscr{G}(q,q')$.

REMARK 4.3.6. – Although $\mathbf{G}_\mu$ is a continuous and well defined function, the distributions defined by (4.15) are not functions in general.

When the quadratic forms we consider are not of Lorentz type, or not positive, we have not been able to prove the converse of Theorem 4.3.5, and we state this as a conjecture.

CONJECTURE 4.3.7. – *Let* $q(x) = \langle Ax, x\rangle$, $q'(\xi) = \langle A'\xi, \xi\rangle$, *where* A, A' *are orthogonal and symmetric. Any element of* $\mathscr{G}(q,q')$ *can be written in the form (4.15).*

In the next chapter we will show that this is true for the Lorentz quadratic form. Now this can be stated in a simpler way when $\mathscr{G}(q,q')$ contains only one Gaussian element (see Theorem 3.1.18).

CONJECTURE 4.3.8. – *Let* (q,q') *be a critical pair satisfying the hypotheses of Theorem 3.1.18. Let* γ *be its unique Gaussian element. Any* $f \in \mathscr{G}(q,q')$ *is a Hermite function of the form*

$$f(x) = P(x)\gamma(x),$$

where P *is a polynomial.*

As mentioned earlier, we can take $q(x,y) = x^2 - y^2$ and $q'(\xi,\eta) = 2\xi\eta$ on $\mathbb{R}^2$ as an example. In this case, Conjecture 4.3.8 becomes:

CONJECTURE 4.3.9. – *Le* F *be an entire function on* $\mathbb{C}^2$ *satisfying the estimates*

$$|F(z)| \leq C(1+|z|)^N \exp\left(2|\mathscr{Re}(z_1)||\mathscr{Re}(z_2)| + |\mathscr{Im}(z)|^2\right)$$

and

$$|F(z)| \leq C(1+|z|)^N \exp\left(|(\mathscr{Im}\, z_1)^2 - (\mathscr{Im}\, z_2)^2| + |\mathscr{Re}(z)|^2\right).$$

Then F *a polynomial.*

CHAPTER 5

LORENTZ QUADRATIC FORM

This chapter is devoted to the proof of Conjecture 4.3.7 in some cases. The main result is the description of $\mathscr{G}(q,q')$ when q is the Lorentz form defined on $\mathbb{R}^{d+1}$ by

$$q(x,y) = x_1^2 + \cdots + x_d^2 - y^2,$$

where $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}$, and when q' is any quadratic form defined by

$$q'(\xi,\eta) = \varepsilon_1\xi_1^2 + \cdots + \varepsilon_d\xi_d^2 + \varepsilon\eta^2,$$

where $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}$ and $\varepsilon, \varepsilon_i = \pm 1$.

We first prove Conjecture 4.3.7 when q and q' are equal to the Lorentz form. In this case the elements of $\mathscr{G}(q,q')$ have very interesting properties. We show that they are smooth inside the Lorentz cone, while they can be singular outside. We point out examples that vanish inside the Lorentz cone, as well as their Fourier transforms, without vanishing identically. As a corollary we obtain the main result mentioned above. We will complete Theorem 3.2.4 and give the exact conditions on q and q' so that $\mathscr{G}(q,q') = \{0\}$, when q is the Lorentz form and q' has only diagonal terms.

5.1. The Bargmann transform of $\mathscr{G}(q,q)$

Theorem 4.2.2 characterizes the Bargmann transform of the elements of $\mathscr{G}(q,q)$. In this section we will describe this space, which is the space of entire functions F on $\mathbb{C}^{d+1}$, for which there exist $C, N > 0$, such that for all $(z,\zeta) \in \mathbb{C}^d \times \mathbb{C}$,

$$|F(z,\zeta)| \leq C(1+|z|+|\zeta|)^N \exp(\pi|\mathscr{R}e(z)||\mathscr{R}e(\zeta)| + \pi|\mathscr{I}m(z)||\mathscr{I}m(\zeta)|). \tag{5.1}$$

Recall that in the whole chapter, the letters C and N denote constants that may vary from line to line.

LEMMA 5.1.1. – *Let F be an entire function satisfying* (5.1). *There exists a decomposition $F = \sum_{k=-N}^{N} F_k$, with entire functions F_k satisfying the estimate*

$$|F_k(z,\zeta)| \leq C(1+|z|+|\zeta|)^N \exp(\pi|\mathscr{R}e(\zeta z)|), \tag{5.2}$$

and the homogeneity condition

$$F_k(tz, t^{-1}\zeta) = t^k F_k(z,\zeta),\ (t \in \mathbb{C}^*,\ -N \le k \le N). \tag{5.3}$$

Proof. – We proceed as in the proof of Theorem 4.2.2. We showed that there exist entire functions F_k on $\mathbb{C}^{d+1}$ such that

$$\mathscr{B}(f)(tz, t^{-1}\zeta) = \sum_{k=-N}^{N} t^k F_k(z,\zeta).$$

Each of the F_k satisfies (5.1).

Relations (5.3) are obtained by taking partial derivatives at $t = 0$. We prove now (5.2). Because of (5.3), taking $t = \zeta^{-1}$,

$$F_k(z,\zeta) = \zeta^{-k} F_k(\zeta z, 1).$$

Using (5.1), we find $M, C > 0$ such that for all z, ζ,

$$|F_k(z,\zeta)| \le C|\zeta|^{-k}(1 + |z| + |\zeta|)^M e^{\pi|\,\mathscr{R}e(\zeta z)|}.$$

This gives (5.2) for $k \le 0$, and for $|\zeta| \ge |z|^{-1}$ when $k > 0$. If $k > 0$ and $|\zeta||z| \le 1$, write

$$|F_k(z,\zeta)| = |z|^k |F_k(|z|^{-1}z, |z|\zeta)| \le C|z|^k \tag{5.4}$$

and (5.2) is proved. □

Let $\mathbb{B}$ be the open unit ball of $\mathbb{R}^d$, and $\overline{\mathbb{B}}$ its closure. Let $S'_{\overline{B}}$ be the space of distributions on $\mathbb{R}^d$ supported by $\overline{\mathbb{B}}$.

THEOREM 5.1.2. – *Let F be an entire function on $\mathbb{C}^d$. Then it satisfies* (5.1) *for some C and N, if and only if there exist $M \ge 0$, distributions $\mu_i \in S'_{\overline{B}}$, and polynomials P_i, such that for all $z \in \mathbb{C}^d$ and $\zeta \in \mathbb{C}$,*

$$F(z,\zeta) = \sum_{i=1}^{M} P_i(z,\zeta) \int \exp(\pi\langle v, \zeta z\rangle)\, d\mu_i(v). \tag{5.5}$$

Proof. – Use the decomposition of F as in Lemma 5.1.1. Since

$$|F_k(z,1)| \le C(1+|z|)^N \exp(\pi|\,\mathscr{R}e(z)|),$$

we can apply Paley-Wiener-Schwartz Theorem. Hence $F_k(\cdot, 1)$ is the Laplace transform of a distribution $\nu_k \in S'_{\overline{B}}$. It follows that

$$F_k(z,\zeta) = \zeta^{-k} \int \exp(\pi\langle v, \zeta z\rangle)\, d\nu_k(v).$$

When $k > 0$, all the moments of ν_k of order up to $k-1$ vanish, since F_k is an entire function. It follows that for $k > 0$, ν_k can be written as

$$\nu_k = \sum_{|\alpha|=k} \partial_v^\alpha \nu_{k,\alpha},$$

where $\nu_{k,\alpha} \in S'_{\overline{B}}$. Integrations by parts give then (5.5). Conversely, any entire function defined by (5.5) satisfies (5.1) for some constants C and N. □

5.1.1. Description of $\mathscr{G}(q,q)$. – We now describe the space $\mathscr{G}(q,q)$ itself.

DEFINITION 5.1.3. – *We define for $k \geq 0$ the injective operator $T_k(\mu)$ from $S'_{\overline{\mathbb{B}}}$ into $S'(\mathbb{R}^d)$ by*

$$\mathscr{B}(T_k(\mu))(z,\zeta) = \zeta^k \int \exp(\pi\langle v, \zeta z\rangle)\, d\mu(v).$$

This can also be done for $k < 0$. Define by $S'_{\overline{\mathbb{B}},k}$ be the space of distributions on $\mathbb{R}^d$ supported by $\overline{\mathbb{B}}$, that vanish on all polynomials of degree less that $|k|$.

DEFINITION 5.1.4. – *Let $k < 0$. Define the injective operator T_k from $S'_{\overline{\mathbb{B}},k}$ to $S'(\mathbb{R}^{d+1})$ by*

$$\mathscr{B}(T_k(\mu)) = \zeta^k \int \exp(\pi\langle v, \zeta z\rangle)\, d\mu(v).$$

These operators are well defined by Proposition 1.2.5. Indeed, in each case, the expression

$$\zeta^k \int e^{\pi\zeta v z}\, d\mu(v)$$

defines an element of $\mathscr{F}$. Theorems 5.1.2 and 4.2.2 give actually the following.

PROPOSITION 5.1.5. – *If $k \geq 0$, T_k maps $S'_{\overline{\mathbb{B}}}$ into $\mathscr{G}(q,q)$. If $k < 0$, it maps $S'_{\overline{\mathbb{B}},k}$ into $\mathscr{G}(q,q)$.*

THEOREM 5.1.6. – *Any element of $\mathscr{G}(q,q)$ can be written as a finite sum*

$$f = \sum_k P_k(x, y, \partial_x, \partial_y)\mathbf{G}_{\mu_k},$$

where P_k are polynomials, and μ_k are finite measures on the Lorentz group $O(d,1)$.

The functions $\mathbf{G}_{\mu_k}$ were defined in Definition 4.3.3. We will use the following fact on the structure of the elements of $S'_{\overline{\mathbb{B}}}$:

LEMMA 5.1.7. – *Every $\mu \in S'_{\overline{\mathbb{B}}}$ can be decomposed as a finite sum of derivatives of finite measures μ_k supported by $\overline{\mathbb{B}}$, that satisfy*

$$\int_{\mathbb{B}} \frac{|d\mu_k(v)|}{(1-|v|^2)^{1/2}} < \infty. \tag{5.6}$$

Proof. – It is a standard fact that every distribution on the ball may be written as a finite sum of partial derivatives of finite measures on the closed ball (see **[24]**, chapter III). Hence it is sufficient to decompose a finite measure as in the statement of the lemma.

To do so, we choose local coordinates inside $\mathbb{B}$, around a point v_0. When the point is inside the ball, the measure $(1-|v|^2)^{-1/2}\,d\mu(v)$ is clearly finite in a neighborhood of v_0. So we have only to consider the case $|v_0|=1$. Changing coordinates, we have to show that any finite measure $d\mu(t)$ supported by $[0,1]^d$ is a sum of derivatives of measures $d\nu(t)$ supported by $[0,1]^d$ such that $t_1^{-1/2}\,d\nu(t)$ is finite. Write

$$\begin{aligned}\int \phi(t)\,d\mu(t) &= \int_{[0,1]^d}\left(\phi(1/2,t') + \int_{1/2}^{t_1}\partial_{s_1}\phi(s_1,t')ds_1\right)d\mu(t_1,t')\\ &= \int \phi\,d\mu_1 + \int \partial_{s_1}\phi(s_1,s')\,d\nu(s),\end{aligned}$$

where the measure $d\mu_1$ is supported by the set $\{t_1=1/2\}\cap[0,1]^d$, and thus satisfies the required conclusion, and the measure $d\nu$ satisfies by definition

$$\begin{aligned}\int_{[0,1]^d} t_1^{-1/2}|d\nu(t)| &\leq \int_{[0,1]^d}\left|\int_{1/2}^{t_1} s_1^{-1/2}ds_1\right||d\mu(t_1,t')|\\ &\leq \int_{[0,1]^d} 2(\sqrt{t_1}+\sqrt{1/2})|d\mu(t)| \leq 4|\mu| < \infty,\end{aligned}$$

from which we conclude for the lemma. □

Proof of Theorem 5.1.6. – Theorems 5.1.2 and 4.2.2 imply that any $f\in\mathscr{G}(q,q)$ can be written as a finite sum

$$f = \sum_k T_k(\mu_k),$$

where $\mu_k\in S'_{\overline{B}}$. Hence it suffices to prove that each $T_k(\mu)$ can be put in this form, given any $\mu\in S'_{\overline{B}}$.

Define the Gaussian function γ_v, for $v\in\mathbb{B}$, so that $\mathscr{B}(\gamma_v)(z,\zeta)$ is proportional to $\exp(\pi\langle v,\zeta z\rangle)$. By Proposition 4.3.2, every Gaussian element of $\mathscr{G}(q,q)$ is equal to γ_v, for some v. A simple computation shows that

$$\gamma_v(x,y) = \exp\big(-\pi(x_1^2+\cdots+x_d^2-y^2+\frac{2}{1-|v|^2}(y-\langle v,x\rangle)^2)\big), \tag{5.7}$$

and that

$$\mathscr{B}(\gamma_v)(z,\zeta) = \frac{1}{2}(1-|v|^2)^{1/2}\exp(\pi\langle v,\zeta z\rangle).$$

For any finite measure ν on $\mathbb{B}$, define

$$\tilde{\mathbf{G}}_\mu(x,y) = \int_{\mathbb{B}}\gamma_v(x,y)\,d\nu(v).$$

There exists at least one finite measure $\tilde{\nu}$ on $O(d,1)$ such that $\tilde{\mathbf{G}}_\nu = \mathbf{G}_{\tilde{\nu}}$. We have

$$\mathscr{B}(\tilde{\mathbf{G}}_\nu)(z,\zeta) = \frac{1}{2}\int_{\mathbb{B}} \exp(\pi\langle v, \zeta z\rangle)(1-|v|^2)^{1/2}\, d\nu(v).$$

Let $k \geq 0$ and $\mu \in S'_{\overline{B}}$. By Lemma 5.1.7, there exists a decomposition

$$\mu = \frac{1}{2}\sum_\alpha (-1)^\alpha \partial_v^\alpha\big((1-|v|^2)^{1/2}\nu_k(v)\big),$$

where each ν_k is a finite measure on $\mathbb{B}$. If we use the creation and annihilation operators of Definition 4.1.1, we have

$$\begin{aligned}\mathscr{B}(T_k(\mu))(z,\zeta) &= \frac{1}{2}\sum_\alpha \pi^{|\alpha|}\zeta^k(\zeta z)^\alpha \mathscr{B}(T_0(\sqrt{1-|v|^2}\nu_k))(z,\zeta)\\ &= \sum_\alpha \pi^{|\alpha|}\mathscr{B}\left(\boldsymbol{\zeta}^k(\boldsymbol{\zeta}\mathbf{z})^\alpha \tilde{\mathbf{G}}_{\nu_k}\right)(z,\zeta),\end{aligned}$$

from which it follows that

$$T_k(\mu) = \sum_\alpha \pi^{|\alpha|}\boldsymbol{\zeta}^k(\boldsymbol{\zeta}\mathbf{z})^\alpha \tilde{\mathbf{G}}_{\nu_k}. \tag{5.8}$$

This gives the result in this case.

Assume now that $k < 0$. Here $\mu \in S'_{\overline{B},k}$, hence there exists a decomposition

$$\mu = \sum_{|\alpha|=|k|} \partial_v^\alpha \mu_{\alpha,k},$$

with $\mu_{\alpha,k} \in S'_{\overline{\mathbb{B}}}$. It follows that

$$\mathscr{B}(T_k(\mu))(z,\zeta) = \sum_{|\alpha|=|k|} (-\pi z)^\alpha \mathscr{B}(T_0(\mu_{\alpha,k})),$$

and hence $T_k(\mu) = \sum_{|\alpha|=|k|}(-\pi\mathbf{z})^\alpha T_0(\mu_{\alpha,k})$. By the previous case, $T_k(\mu)$ has also the required form. □

5.1.2. Properties of the elements of $\mathscr{G}(q,q)$. – When $d=1$, the description of $\mathscr{G}(q,q)$ is simpler. After a rotation, we can assume that

$$q(x,y) = q'(x,y) = 2xy,$$

with $(x,y) \in \mathbb{R}^2$. Then the group $SO(q)$ is made of the matrices of the form

$$g_\tau = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1}\end{pmatrix},$$

where $\tau \in \mathbb{R}^*$. Putting $t = \tau^2$, we have:

THEOREM 5.1.8. – *Let $f \in S'(\mathbb{R}^2)$. Then $f \in \mathscr{G}(q,q)$ if and only if there exist $N > 0$, finite measures μ_k on $\mathbb{R}_+^*$ and polynomials P_k such that*

$$f(x,y) = \sum_{k=1}^{N} P_k(x,y,\partial_x,\partial_y) \int_{\mathbb{R}_+^*} \exp(-\pi t x^2 - \pi/t y^2)\, d\mu_k(t).$$

A particular example was mentioned in the introduction. It is the function given by $f(x,y) = \operatorname{sgn}(x)\exp(-2\pi|xy|)$. We actually have

$$f(x,y) = \int_0^\infty x e^{-\pi t x^2 - \frac{\pi}{t} y^2} t^{-1/2}\, dt, \tag{5.9}$$

see Formula (5.14) below. It is not of the required form, because the measure $t^{-1/2}\,dt$ is not finite. But cutting the integral at $t=1$, we write:

$$f(x,y) = x\int_0^1 e^{-\pi t x^2 - \frac{\pi}{t} y^2} t^{-1/2}\, dt - \frac{1}{2\pi}\partial_x \int_1^\infty e^{-\pi t x^2 - \frac{\pi}{t} y^2} t^{-3/2}\, dt.$$

Another example is the distribution $f(x,y) = \delta(x)1(y)$, see Theorem 3.1.26. We can also put it in that form using integrations by parts.

We now give properties of the elements of $\mathscr{G}(q,q)$, when q is the Lorentz form on $\mathbb{R}^{d+1}$.

THEOREM 5.1.9. – *Let $f \in \mathscr{G}(q,q)$. Then f is a real-analytic function when $y^2 > |x|^2$. Moreover, there exist $C, M, m \geq 0$ such that for all (x,y) with $y^2 > |x|^2$,*

$$|f(x,y)| \leq C(1+|x|+|y|)^M \big||x|^2 - y^2\big|^{-m} e^{-\pi||x|^2-y^2|}.$$

If $d=1$, this is true for any (x,y) such that $x^2 \neq y^2$.

REMARK 5.1.10. – Even though the conditions on the elements f of $\mathscr{G}(q,q)$ are given in a distribution sense, f satisfies in the Lorentz cone a pointwise estimate analogous to Hardy's uncertainty principle.

Proof. – When taking formally derivatives of $\mathbf{G}_\mu$ with respect to x and y under the integral, a singularity at $|v|=1$ appears. It is of the form $(1-|v|^2)^{-m}$, with $m \geq 0$. We will prove that the integral is still absolutely convergent provided $|y| > |x|$. Note that we have the estimate

$$(1-|v|^2)^{-m} e^{-2\pi(|x|^2-y^2) - \frac{2\pi(y-\langle v,x\rangle)^2}{1-|v|^2}} \leq C(1+|y|)^{2m}(|y|-|x|)^{-2m} \tag{5.10}$$

for all $v \in \mathbb{B}$. The real part of

$$|x|^2 - y^2 + \frac{1}{1-|v|^2}(y - \langle v,x\rangle)^2$$

is non negative, whenever (x,y) is in a complex neighborhood of some point (x_0,y_0) such that $|y_0|^2 > |x_0|^2$. We conclude with Lebesgue's Theorem that G_μ is real analytic for $y^2 > |x|^2$. So is f by Theorem 5.1.6. □

The following is a corollary of this proof.

Corollary 5.1.11. – *Let $\mu \in S'_{\overline{B}}$, and $k \geq 0$. Then for any (x,y) such that $y^2 > |x|^2$, we have*

$$T_k(\mu)(x,y) = 2\int \boldsymbol{\zeta}^k \gamma_v(x,y)(1-|v|^2)^{-1/2}\, d\mu(v).$$

Here $\boldsymbol{\zeta}$ is the creation operator associated to the variable y.

Proof. – Note that this makes sense since the function

$$\boldsymbol{\zeta}^k \gamma_v(x,y)(1-|v|^2)^{-1/2},$$

extended by 0 for $|v| \geq 1$, is smooth and compactly supported by $\overline{B}$, as soon as $y^2 > |x|^2$. We can decompose μ as a finite sum

$$\mu = \frac{1}{2}\sum_{\alpha}(-1)^{\alpha}\partial_v^{\alpha}\big((1-|v|^2)^{1/2}\mu_{\alpha}(v)\big),$$

where each μ_α is a finite measure on $\mathbb{B}$. We have $\mathscr{B}(\gamma_v)(z,\zeta) = \frac{1}{2}(1-|v|^2)^{1/2} \exp(\pi\langle v, \zeta z\rangle)$. Put $f_v = \frac{1}{2}(1-|v|^2)^{1/2}\gamma_v$. For any α, we have also $\frac{1}{2}(1-|v|^2)^{1/2}\partial_v^{\alpha} f_v = (\pi\boldsymbol{\zeta}\mathbf{z})^{\alpha}\gamma_v$. Indeed, the Bargmann transform of both functions coincide. It follows that

$$\begin{aligned}
\mathscr{B}(T_k(\mu))(z,\zeta) &= \zeta^k \int \exp(\pi\langle v, \zeta z\rangle)\, d\mu(v)\\
&= \frac{1}{2}\zeta^k \sum_{\alpha}\int \partial_v^{\alpha}\big(\exp(\pi\langle v,\zeta z\rangle)\big)(1-|v|^2)^{1/2}\, d\mu_{\alpha}(v)\\
&= \sum_{\alpha}\mathscr{B}(\boldsymbol{\zeta}^k(\pi\boldsymbol{\zeta}\mathbf{z})^{\alpha}\mathbf{G}_{\mu_\alpha})(z,\zeta).
\end{aligned}$$

Hence $T_k(\mu) = \sum_\alpha \boldsymbol{\zeta}^k(\pi\boldsymbol{\zeta}\mathbf{z})^{\alpha}\mathbf{G}_{\mu_\alpha}$. In the proof of Theorem 5.1.9, we showed that we can compute the derivatives under the integral defining $\mathbf{G}_{\mu_\alpha}$, provided $y^2 > |x|^2$. Hence

$$\begin{aligned}
T_k(\mu) &= \sum_{\alpha}\int_{\mathbb{B}} \boldsymbol{\zeta}^k(\pi\boldsymbol{\zeta}\mathbf{z})^{\alpha}\gamma_v\, d\mu_{\alpha}(v)\\
&= \sum_{\alpha}\int_{\mathbb{B}} \boldsymbol{\zeta}^k \frac{1}{2}(1-|v|^2)^{1/2}\partial_v^{\alpha} f_v\, d\mu_{\alpha}(v)\\
&= \int_{\mathbb{B}} \boldsymbol{\zeta}^k f_v\, d\mu(v),
\end{aligned}$$

as required. □

Recall that Theorem 5.1.6 establishes that any $f \in \mathscr{G}(q,q)$ can be decomposed as a finite sum

$$f = \sum_k T_k(\mu_k),$$

where $\mu_k \in S'_{\overline{B}}$ for $k \geq 0$, and $\mu_k \in S'_{\overline{B},k}$ when $k < 0$. The following lemma, that we will use later, proves that this decomposition is unique.

Lemma 5.1.12. – *Write*

$$f = \sum_{k=-N}^{N} T_k(\mu_k).$$

For any k, $T_{k+N}(\mu_k)$ *can be expressed as a polynomial in the annihilation and creation operators, applied to* f.

Proof. – Using Proposition 4.1.2, we obtain

$$\boldsymbol{\zeta}^N f = \sum_{k=-N}^{N} T_{k+N}(\mu_k) = \sum_{k=0}^{2N} T_k(\mu_{k-N}).$$

Taking the Bargmann Transforms, we find that for any $a > 0$,

$$\mathscr{B}(\boldsymbol{\zeta}^N f)(a^{-1}z, a\zeta) = \sum_{k=0}^{2N} a^k \zeta^k \int e^{\pi\langle v, \zeta z\rangle}\, d\mu_{k-N}(v).$$

We conclude by taking derivatives at $a = 1$, and induction on k. □

5.2. Vanishing elements of $\mathscr{G}(q,q)$

In this section, q is still the Lorentz form on $\mathbb{R}^{d+1}$. We want to show that the elements of $\mathscr{G}(q,q)$ cannot vanish on large sets. Let C be the light cone, defined by $C = \{q = 0\}$. The Lorentz cone is $\{(x,y); y^2 > |x|^2\}$.

By Theorem 5.1.9, all distributions vanishing in an open subset of the Lorentz cone vanishes in one connected component of it. We will first exhibit non trivial elements of $\mathscr{G}(q,q)$ that vanish for $y^2 > |x|^2$, and characterize them.

5.2.1. Examples of vanishing solutions.

– For θ in the unit sphere $\mathbb{S}_{d-1}$ of $\mathbb{R}^d$, let

$$\mathbf{E}_\theta(x,y) = \sqrt{2}e^{-\pi(|x|^2-y^2)}\delta_{y=\langle x,\theta\rangle}.$$

It is a measure defined by $\langle \mathbf{E}_\theta, \phi\rangle = \sqrt{2}\int_{\mathbb{R}^d} e^{-\pi|x-\langle x,\theta\rangle\theta|^2}\phi(x,\langle x,\theta\rangle)dx$. The support of $\mathbf{E}_\theta$ is exactly the hyperplane H_θ of equation $y = \langle x,\theta\rangle$, which is tangent to the light cone C, and contained in the complementary of the Lorentz cone. When $d = 1$, its support is the line $y = \theta x$, $\theta \in \{\pm 1\}$, which is on the light cone. Also $\widehat{\mathbf{E}_\theta} = \mathbf{E}_{-\theta}$. In fact $\mathbf{E}_\theta$ can be seen as the weak limit as $r \to 1$ of $f_{r\theta}$ defined in the proof of Corollary 5.1.11. Hence its Bargmann Transform is

$$\mathscr{B}(\mathbf{E}_\theta)(z,\zeta) = e^{\pi\langle\theta,\zeta z\rangle}. \tag{5.11}$$

From this expression we see that $\mathbf{E}_\theta = T_0(\delta_\theta)$, so it is a particular element of $\mathscr{G}(q,q)$, vanishing on the Lorentz cone, as well as its Fourier transform.

The aim of this section is to prove that every element of $\mathscr{G}(q,q)$ vanishing in one connected component the Lorentz cone arises as a (continuous) linear combination of the $\mathbf{E}_\theta$:

$$\sum_{k=1}^{k_0} P_k(x,y,\partial_x,\partial_y)\mathbf{E}_{m_k} \tag{5.12}$$

where P_k are polynomials, m_k distributions on $\mathbb{S}_{d-1}$. Here, if m is a distribution on $\mathbb{S}_{d-1}$, $\mathbf{E}_m$ is defined by

$$\langle \mathbf{E}_m, \phi\rangle = \int_{S_{d-1}}\int_{\mathbb{R}^d} \phi(x,\langle x,\theta\rangle)e^{-\pi|x|^2+\pi\langle x,\theta\rangle^2}\,dx\,dm(\theta),$$

which will be formally denoted by

$$\mathbf{E}_m = \int_{\mathbb{S}_{d-1}} \mathbf{E}_\theta(x,y)\,dm(\theta).$$

In particular, if a solution vanishes in one component of C_+, it vanishes in the whole cone C_+, as well as its Fourier Transform. The idea of the proof is to show that any μ_k arising in the decomposition

$$f = \sum_k T_k(\mu_k)$$

is supported by the unit sphere $\mathbb{S}_{d-1}$ instead of $\overline{\mathbb{B}}$.

5.2.2. Characterization of vanishing solutions. – We define for $|v| < 1$ the function $\phi_v(x,y)$:

$$\phi_v(x,y) = e^{-2\pi\frac{(y-\langle v,x\rangle)^2}{1-|v|^2}}(1-|v|^2)^{-1/2}.$$

Note that $v \to \phi_v(x,y)$, extended by 0 when $|v| \geq 1$, is a smooth function with support equal to $\overline{\mathbb{B}}$, as long as $|y| > |x|$, since then $(y-\langle v,x\rangle)^2 \geq (|y|-|x|)^2$.

We first begin with a proposition of independent interest.

Proposition 5.2.1. – *Let $\mu \in S'_{\overline{\mathbb{B}}}$, and $k \geq 0$. Assume that*

$$T_k(\mu)(x,y) = 0 \tag{5.13}$$

for any $y > |x|$. Then the distribution μ is supported by the unit sphere $\mathbb{S}_{d-1}$.

Proof. – First consider the case $k = 0$. By assumption,

$$\int (1-|v|^2)^{-1/2}e^{-\frac{2\pi(y-\langle v,x\rangle)^2}{1-|v|^2}}\,d\mu(v) = 0$$

for all $y > |x|$ (see Lemma 5.1.11). We want first to replace the integrated term $e^{-\frac{2\pi(y-\langle v,x\rangle)^2}{1-|v|^2}}$ by the more suitable $e^{-\frac{(y-\langle v,x\rangle)}{(1-|v|^2)^{1/2}}}$. We will use a classical formula, which is linked to the principle of subordination ([**26**], p46):

$$e^{-|\beta|} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u}e^{-\frac{\beta^2}{4u}}u^{-1/2}\,du. \tag{5.14}$$

We will show that

$$\int e^{-\frac{y-\langle v,x\rangle}{\sqrt{1-|v|^2}}}(1-|v|^2)^{-1/2}\,d\mu(v)=0 \tag{5.15}$$

for all $y>|x|$, which makes sense since we test the distribution μ on a smooth function in v, as long as $y>|x|$.

Put $\beta(x,y,v)=\frac{y-\langle v,x\rangle}{\sqrt{1-|v|^2}}$. We remark that the double integral

$$\iint (1-|v|^2)^{-N}\frac{e^{-u}}{\sqrt{u}}e^{-\frac{\beta(x,y,v)^2}{4u}}\,du\,d\nu(v)$$

is absolutely convergent whenever $N\geq 0$, $y>|x|$, and for any finite measure $d\nu$. Hence a direct use of Fubini's Theorem yields (5.15) when μ is a measure. Otherwise we write μ as a finite sum of derivatives of finite measures supported by $\overline{\mathbb{B}}$ (see **[24]**, chapter III), integrate by parts, and exchange derivatives and integration in v, and still obtain (5.15).

Now we take derivatives with respect to x in (5.15), and let $x=0$. For any polynomial P on $\mathbb{R}^d$,

$$\int P\left(\frac{v}{(1-|v|^2)^{1/2}}\right)e^{-y(1-|v|^2)^{-1/2}}(1-|v|^2)^{-1/2}\,d\mu(v)=0. \tag{5.16}$$

To conclude it is sufficient to show that $(1-|v|^2)^N d\,\mu(v)=0$, for N big enough, depending on the order of the distribution μ. By density of the polynomials, it is sufficient to show that

$$\int Q(v)(1-|v|^2)^N\,d\mu(v)=0 \tag{5.17}$$

for any homogeneous polynomial Q and N big enough, but fixed. We want to deduce (5.17) from (5.16), with P defined by

$$P\left(\frac{v}{\sqrt{1-|v|^2}}\right)=\frac{Q(v)}{(1-|v|^2)^{k/2}},$$

and $k=deg(Q)$. We use the fact that

$$\int_0^\infty e^{-y(1-|v|^2)^{-1/2}}y^{2N+k}dy=C(1-|v|^2)^{N+(k+1)/2}. \tag{5.18}$$

We remark that the double integral

$$\iint |Q(v)|\sqrt{1-|v|^2}^{\,-N_0-k-1}e^{-y(1-|v|^2)^{-1/2}}y^{2N+k}\,dy\,d\nu(v)$$

is absolutely convergent for any N,N_0 such that $N\geq N_0$, and any finite measure ν.

Hence, when μ is a finite measure, the exchange of integrations in y and v is a consequence of Fubini's Theorem, taking $N=1$ for example, and we get (5.17). For a general distribution, we write μ as a sum of derivatives of order up to N_0 of finite measures $d\nu$ of $S'_{\overline{\mathbb{B}}}$. We conclude that $(1-|v|^2)^N\,d\mu(v)=0$, for any $N\geq N_0$, which proves that $d\mu$ is supported by $\mathbb{S}_{d-1}$.

Consider now the case $k > 0$. By Proposition 4.1.2, we have

$$\begin{aligned}\mathscr{B}(T_0(\partial_{v_1}^k \mu)) &= (-\pi\zeta z_1)^k \int e^{\pi\zeta vz}\, d\mu(v)\\ &= (-\pi z_1)^k \mathscr{B}(T_k(\mu))(z,\zeta)\\ &= \mathscr{B}\left((-\pi \mathbf{z}_1)^k T_k(\mu)\right)(z,\zeta).\end{aligned}$$

Since $T_k(\mu)(x,y) = 0$ for $y > |x|$, the same is true for $(-\pi\mathbf{z}_1)^k T_k(\mu)$. Hence $T_0(\partial_{v_1}^k \mu))(x,y) = 0$ for $y > |x|$. By the previous case, $\partial_{v_1}^k \mu = 0$ in $\mathbb{B}$. This is actually true for any derivative in v of order k, and we conclude that μ is given by a polynomial inside the ball.

We are thus lead to prove that if $d\mu(v) = P(v)\,dv$, where P is a polynomial, and if $T_k(\mu)$ vanishes for $y > |x|$, then $P = 0$. But if this is the case, we have, using Lemma 5.1.11, and Formula (4.5), for all $y > |x|$,

$$\int (1-|v|^2)^{-1/2} \partial_y^k \left[e^{-2\pi \frac{2\pi(y-\langle v,x\rangle)^2}{1-|v|^2}} \right] P(v)\,dv = 0.$$

We conclude as before, using Relation (5.14), that

$$\int (1-|v|^2)^{-(k+1)/2} e^{-\frac{y-\langle v,x\rangle}{\sqrt{1-|v|^2}}} P(v)\,dv = 0.$$

Then we take derivatives in x, let $x = 0$, use (5.18), and find finally that (5.17) holds for $d\mu(v) = P(v)\,dv$. Hence $P = 0$. ☐

THEOREM 5.2.2. – *Suppose that $f \in \mathscr{G}(q,q)$ vanishes on an open subset of the Lorentz cone. Then it is can be written as* (5.12).

Proof. – By real analyticity of the solutions (see Theorem 5.1.9), f vanishes is a connected component of C_+, for example in the set $\{y > |x|\}$. We know that f can be put in the form

$$f = \sum_{k=-k_0}^{k_0} T_k(\mu_k).$$

We want to show that every μ_k is a distribution supported by the unit sphere $\mathbb{S}_{d-1}$.

By Lemma 5.1.12, there exist polynomials P_k such that

$$T_{k+k_0}(\mu_k) = P_k(\mathbf{z}, \mathbf{z}^*, \boldsymbol{\zeta}, \boldsymbol{\zeta}^*) f.$$

Hence $T_{k+k_0}(\mu_k)$ vanishes on the cone, and by Proposition 5.2.1, we obtain that μ_k is supported by $\mathbb{S}_{d-1}$. The structure of distributions supported by $\mathbb{S}_{d-1}$ is known (see **[24]**, chapter III). It follows that $\mathscr{B}(f)$ has the form

$$\mathscr{B}(f)(z,\zeta) = \sum_{\alpha,k} z^\alpha \zeta^k \int_{\mathbb{S}_{d-1}} e^{\pi\zeta z\theta}\, dm_{\alpha,k}(\theta)$$

where each $m_{\alpha,k}$ is a distribution defined on the sphere. But this is the Bargmann Transform of

$$\sum_{\alpha,k} \mathbf{z}^{\alpha}\boldsymbol{\zeta}^{k}\boldsymbol{E}_{m_{\alpha,k}}, \tag{5.19}$$

hence f is equal to (5.19). □

5.2.3. Weak uncertainty principles for Lorentz form. – Let us consider more precisely the case $d = 1$. The unit sphere is reduced to $\{-1, 1\}$. All the distributions of the form $\mathbf{E}_m$ are in this case combinations of $\delta_{y=x}$ and $\delta_{y=-x}$. We do a rotation in the variables so that we consider the form $2xy$ instead of $y^2 - x^2$. The following is true.

THEOREM 5.2.3. – *Let $f \in \mathscr{G}(q, q)$. If f vanishes on an open set, then it is a finite linear combination of the distributions*

$$x^k \otimes \delta^{(l)}(y),\ \delta^{(k)}(x) \otimes y^l.$$

Proof. – It is easy to see that these distributions are those of type (5.12). We can assume, since the four quadrants are equivalent, that f vanishes in a subset of $\{x > 0,\ y > 0\}$. We conclude using Theorem 5.2.2. □

Theorem 5.2.2 is rather restrictive. Nevertheless a lot of solutions can be put in the form (5.12). Some solutions are even locally integrable functions. For example take σ equal to the surface measure on the unit sphere when $d > 1$. Up to a constant we have

$$\mathbf{E}_{\sigma}(x, y) = \int_{\mathbb{S}_{d-1}} e^{-\pi(|x|^2 - y^2)} \delta_{y=\langle x,\theta\rangle}\, d\sigma(\theta). \tag{5.20}$$

As an average of measures, it is a measure. It is actually locally integrable since an easy computation shows that

$$\mathbf{E}_{\sigma}(x, y) = C(d)\frac{1}{|x|}\left(1 - \frac{y^2}{|x|^2}\right)^{\frac{d-3}{2}} e^{-\pi(|x|^2 - y^2)}\chi_{-|x|<y<|x|}. \tag{5.21}$$

It has the following properties, due to (5.20) and (5.21):

PROPOSITION 5.2.4. – *The function $\mathbf{E}_{\sigma}$ defined by (5.21) is a slowly increasing function on $\mathbb{R}^{d+1}$, and is in particular locally integrable. It is its own Fourier Transform, and vanishes exactly when $y^2 > |x|^2$. Moreover, when $d \geq 3$, $\mathbf{E}_{\sigma}$ is in $L^p(\mathbb{R}^{d+1})$ for p in the range $\frac{d-1}{d-2} < p < d + 1$. In particular, when $d \geq 4$, $\mathbf{E}_{\sigma} \in L^2(\mathbb{R}^{d+1})$.*

It is not obvious at first glance that $\widehat{\mathbf{E}_{\sigma}} = \mathbf{E}_{\sigma}$ if we look at the formula (5.21). To prove it one has to use (5.20) and the fact that $\widehat{\mathbf{E}_{\theta}} = \mathbf{E}_{-\theta}$.

Before giving weak uncertainty principles associated to the Lorentz Form, we begin by a lemma which will be useful.

LEMMA 5.2.5. – *Let m be a distribution on the sphere such that*

$$\int_{\mathbb{S}_{d-1}} \psi(\langle \theta', \theta \rangle)\, dm(\theta) = 0$$

for all smooth ψ supported by a subinterval $J \subset]0,1[$ and $\theta' \in \mathbb{S}_{d-1}$. Then $m = 0$.

Proof. – We first prove the lemma when $d = 2$. Using polar coordinates in the complex plane, the hypothesis is rewritten as

$$\int_{\mathbb{T}} \psi(\cos(\theta - \theta'))\, dm(\theta) = 0$$

for all smooth ψ supported in J. Here m is a distribution on the torus $\mathbb{T}$. Using the function cos^{-1} and changing variables, we can as well assume that

$$\int_{\mathbb{T}} [\psi(\theta - \theta') + \psi(\theta' - \theta)]\, dm(\theta) = 0$$

for any smooth function ψ supported by a fixed subinterval $\tilde{J}$ of $]0, \pi/2[$. Take for ψ an approximate identity converging to the Dirac mass at $a \in \tilde{J}$. The first term tends to the translate of m by $-a$, and the second one to the translate by a. The Fourier coefficients of the sum, that is $\cos(ak)\widehat{m}(k)$ vanish for a in a small interval. Hence $\widehat{m}(k) = 0$ for all k. We conclude that $m = 0$.

The general case is done in the same way, using the harmonic analysis on the sphere. The operator L_ψ defined by

$$L_\psi(m)(\theta') = \int_{\mathbb{S}_{d-1}} \psi(\langle \theta', \theta \rangle)\, dm(\theta)$$

maps distributions defined on $\mathbb{S}_{d-1}$ (and hence polynomials) into the space of continuous functions. Moreover it commutes with the action of the orthogonal group $SO(d)$ on $\mathbb{S}_{d-1}$. In fact it is a generalized convolution operator on $\mathbb{S}_{d-1}$. It follows (see [**9**], Chapter II.4) that if a distribution $m(\theta)$ is decomposed as

$$m = \sum_k m_k$$

where m_k is a harmonic polynomial of order k, and the sum converges in the distribution sense, then

$$L_\psi(m) = \sum c_k(\psi) m_k,$$

where the coefficients $c_k(\psi)$ are the Fourier coefficients of the operator L_ψ. Since $L_\psi(m) = 0$ it follows that

$$c_k(\psi) m_k = 0$$

for any k and ψ supported in J. So it suffices to prove that for any k, there exists ψ such that $c_k(\psi) \neq 0$.

The coefficient $c_k(\psi)$ is given by the scalar product of the zonal function $\psi(\langle\theta, e_d\rangle)$ with the zonal polynomial of order k, $Z_k(\langle\theta, e_d\rangle)$, where $e_d = (0, \dots, 0, 1)$. The zonal polynomial is given up to a constant by the Gegenbauer polynomial

$$Z_k(t) = (1-t^2)^{-(d-3)/2}\partial_t^k(1-t^2)^{(d-3+2k)/2}.$$

It follows that

$$c_k(\psi) = \int_{-1}^{1} \psi(t) Z_k(t)(1-t^2)^{(d-3)/2}\, dt.$$

Since Z_k does not vanish, there exists a smooth ψ supported in J such that $C_k(\psi) \neq 0$. □

We now give sufficient conditions so that the elements of $\mathscr{G}(q, q)$ vanish everywhere. Let us insist on the fact that the next theorem is not true for $d = 1$.

THEOREM 5.2.6. – *Let $d > 1$. Let $f \in \mathscr{G}(q, q)$. Suppose that f vanishes on an open subset of the Lorentz cone, and on an open subset of the complementary invariant by rotations in the x variable. Then $f = 0$.*

Proof. – By assumption f vanishes in a connected component of the cone. The distribution f may be written as

$$f = \sum_{k=-k_0}^{k_0} T_k(\mu_k)$$

and we can express each $T_{k+k_0}(\mu_k)$ as a polynomial in the creation and annihilation operators applied to f by Lemma 5.1.12. Hence $T_{k+k_0}(\mu_k)$ vanishes on the same set, and it suffices to consider the case $f = T_k(\mu)$, where $k \geq 0$ and $\mu \in S'_{\mathbb{B}}$. We want to show that $\mu = 0$. As in the proof of Proposition 5.2.1, it suffices to consider the case $k = 0$. By Proposition 5.2.1, μ is supported by $\mathbb{S}_{d-1}$. Write

$$\mathscr{B}(T_0(\mu))(z, \zeta) = \int e^{\pi\zeta v z}\, d\mu(v).$$

The distribution μ is a finite sum of radial derivatives at $r = 1$ of extensions to a neighborhood of $\mathbb{S}_{d-1}$ of distributions defined on $\mathbb{S}_{d-1}$ (see **[24]**, chapter III). Hence

$$\begin{aligned}\mathscr{B}(T_0(\mu))(z, \zeta) &= \sum_{l=0}^{L} \int \pi^l(\zeta\langle\theta, z\rangle)^l e^{\pi\langle v, \zeta z\rangle}\, dm_l(\theta)\\ &= \sum_{l=0}^{L} \zeta^l \partial_\zeta^l \mathscr{B}(\mathbf{E}_{m_l})(z, \zeta),\end{aligned}$$

where each distribution m_l is defined on $\mathbb{S}_{d-1}$. Then

$$T_0(\mu) = \sum_{l=0}^{L} \boldsymbol{\zeta}^l \boldsymbol{\zeta}^{*l} \mathbf{E}_{m_l}.$$

We will prove that $m_L = 0$ and conclude by induction.

What we know is that f vanishes on an open set of the form $\Omega = \{(x,y); |x| \in I\ , y/|x| \in J\}$, where I is a subinterval of $]0,\infty[$ and J a subinterval of $]-1,1[$. We can assume that J does not contain 0. It follows that $\langle T_0(\mu), \phi\rangle = 0$ whenever $\phi \in S'(\mathbb{R}^{d+1})$ is supported in Ω.

We take ϕ of the form $\phi_1(|x|)\phi_2(x/|x|)\psi(y/|x|)$ where ϕ_1 is smooth and supported on I and ψ is smooth supported on J, and ϕ_2 is a smooth function defined on $\mathbb{S}_{d-1}$. If we denote by $\psi_{|x|}$ the function $\psi_{|x|}(y) = \psi(y/|x|)$, we have

$$0 = \sum_{l=0}^{L} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}^d} \phi_1(|x|)\phi_2(x/|x|)e^{-\pi|x|^2+\pi(\langle x,\theta\rangle)^2} \boldsymbol{\zeta}^l \boldsymbol{\zeta}^{*l} \psi_{|x|}(\langle x,\theta\rangle)\, dx\, dm_l(\theta).$$

Remark that

$$\boldsymbol{\zeta}^l \boldsymbol{\zeta}^{*l} \psi_{|x|}(y) = \sum_{k=-2L}^{2L} |x|^k \psi_k(y/|x|)$$

with smooth ψ_k supported in J, and $\psi_{2L}(y) = y^{2L}\psi(y)$. So if we take ϕ_1 such that

$$\int_0^\infty r^{d-1+k}\phi_1(r)e^{-\pi r^2}\, dr = 0 \tag{5.22}$$

for any $k = -2L, \dots, 2L-1$, and

$$\int_{\mathbb{R}^d} r^{d-1+2L}\phi_1(r)e^{-\pi r^2}\, dr = 1, \tag{5.23}$$

we see that the only remaining term is

$$\begin{aligned} 0 &= \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}^d} \phi_1(|x|)\phi_2(x/|x|)e^{-\pi|x|^2+\pi(\langle x,\theta\rangle)^2} (\langle x,\theta\rangle)^{2L} \psi(\langle x/|x|,\theta\rangle)\, dx\, dm_L(\theta) \\ &= \int_{\mathbb{S}_{d-1}} \int_0^\infty \int_{\mathbb{S}_{d-1}} \phi_1(r)\phi_2(\theta') r^{2L+d-1} \langle\theta',\theta\rangle^{2L} \psi(\langle\theta',\theta\rangle)\, dr\, d\sigma(\theta')\, dm_L(\theta) \\ &= \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \phi_2(\theta')\langle\theta',\theta\rangle^{2L}\psi(\langle\theta',\theta\rangle)\, d\sigma(\theta')\, dm_L(\theta). \end{aligned}$$

Replace $\psi(t)$ by $t^{-2L}\psi(t)$ (recall that ψ is supported away from 0). Since ϕ_2 is arbitrary, we get

$$\int_{\mathbb{S}_{d-1}} \psi(\langle\theta',\theta\rangle)\, dm_L(\theta) = 0$$

for all $\theta' \in \mathbb{S}_{d-1}$ and ψ supported in J. Note that the last quantity is a smooth function of θ'. To conclude that $m_L = 0$ we use Lemma 5.2.5. □

REMARK 5.2.7. – The rotation invariance of the set is fundamental. If we use (5.12) with measures m_k supported on small caps of the sphere, then the corresponding solution vanish on an open subset of the complementary of the cone.

COROLLARY 5.2.8. – *Let $d \geq 1$. Assume that an element f of $\mathscr{G}(q,q)$ vanishes on $\{y > a|x|\}$ with $0 < a < 1$. Then $f = 0$.*

Corollary 5.2.9. – *Let $d > 1$. If a distribution f is supported in the set $\{||x|^2 - y^2| < A\}$, and $\widehat{f}$ is supported in $\{||\xi|^2 - \eta^2| < B\}$, for two constants A and B, then $f = 0$. Hence these two sets form an annihilating pair for distributions. In particular, unlike the case $d = 1$, there is no distribution f such that both f and $\widehat{f}$ are supported by the light cone.*

This is an interesting complement of Theorem 3.1.23. Note that the pair (q, q) is not annihilating in the sense of Definition 3.1.20.

5.3. The supercritical case with Lorentz form

We give here a complement to Theorem 3.2.4 in the case of a Lorentz quadratic form. Let q be the Lorentz form, and

$$q'(x, y) = a_1 x_1^2 + \cdots + a_d x_d^2 - by^2, \tag{5.24}$$

where $a_i, b \in \mathbb{R} \setminus \{0\}$. We can assume that $b > 0$, changing q to $-q$ if necessary. Let $a = \max_i |a_i|$. Then according to Theorem 3.2.4, the space of tempered distributions f such that

$$f(\cdot)\exp(\pm\pi q) \in S'(\mathbb{R}^{d+1}),\ \widehat{f}(\cdot)\exp(\pm\pi q') \in S'(\mathbb{R}^{d+1}) \tag{5.25}$$

is reduced to zero whenever $\max(a, b) > 1$, $a \neq b$ and $a_i > 0$ for all i. We will complete Theorem 3.2.4 and characterize the pairs for which this is the case.

Theorem 5.3.1. – *Let q be the Lorentz quadratic form on $\mathbb{R}^{d+1}$, and define q' by* (5.24)*. Let $I_+ = \{i; a_i = a\}$, $I_- = \{i; a_i = -a\}$ and $J = \{j; |a_j| < a\}$. Then the space $\mathscr{G}(q, q')$ of distributions satisfying* (5.25) *is reduced to zero if and only if one of the four following conditions is satisfied:*

1. $\max(a, b) > 1$ *and* $a \neq b$,
2. $a = b > 1$ *and* $I_- \neq \varnothing$
3. $a = b > 1$, $I_- = \varnothing$ *and* $card(I_+) > 1$,
4. $a = b > 1$, $I_- = \varnothing$, $card(I_+) = 1$, *and* $\max_{j \in J} |a_j| > 1$.

As mentioned in the remark following Theorem 3.2.4, the key point to establish such a result is the description of the solutions in the critical case ($a_i = b = 1$), which is done in Theorem 5.1.6. A particular case of Theorem 5.3.1 is when $a_i = a = b > 1$:

Theorem 5.3.2. – *Let $q(x, y) = |x|^2 - y^2$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, and $q'(x, y) = a(|x|^2 - y^2)$, with $a > 1$. If $d \geq 2$, then $\mathscr{G}(q, q') = \{0\}$. If $d = 1$, then $\mathscr{G}(q, q')$ is made of the distributions f of the form*

$$f(x, y) = \sum_k P_k(x)\delta^{(k)}(x - y) + \sum_k Q_k(x)\delta^{(k)}(x + y),$$

where P_k, Q_k are polynomials, and δ is the Dirac measure.

We also mention here without proof an immediate corollary of Theorem 4.2.4:

THEOREM 5.3.3. – *Let $n_1 \geq 1$, $n_2, n_3 \geq 0$, such that $n = n_1 + n_2 + n_3$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^{n_1+n_2+n_3}$, and $a > 1$, put $q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2$ and $q'(\xi) = a(-|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2)$. Then $\mathscr{G}(q, q') = \{0\}$.*

Proof of Theorem 5.3.1 assuming Theorem 5.3.2. – Assume that case (1) is satisfied. When $b > a$, we use Theorem (3.2.4). When $a > b$, the a_i such that $|a_i| = a$ may have different signs. We can give a direct proof in that case. Arguing as is the proof of Theorem (3.2.4), we can eliminate the variables y and x_j, for $j \in J$. We are lead to the case $q = x_1^2 + \cdots + x_d^2$ and $q' = a_1 x_1^2 + \cdots + a_d x_d^2$, with $|a_i| = a > 1$. Theorem 1.3.5 allows to conclude. In cases (2) and (3), we reduce as well to the case where $J = \varnothing$. We conclude with Theorem 5.3.3 in case (2) and Theorem 5.3.2 in case (3).

We consider now case (4). Let $f \in \mathscr{G}(q, q')$. We have for example $I_+ = \{1\}$. Here $J \neq \varnothing$. Choose λ such that $a > \lambda > \max\{|a_j|; j = 2, \ldots, d\} > 1$ and put $q'' = \lambda^{-1} q$. Let t such that $\lambda^{-1} \max\{|a_j|; j = 2, \ldots, d\} < t < 1$. For any polynomial P in the variables x_j, $j = 2, \ldots, d$, consider the tempered distribution T_P on $\mathbb{R}^2$ defined by

$$\langle T_P, \phi \rangle = \big\langle f, \phi(x_1, y) \otimes P \exp(-\pi t^{-1} |\cdot|^2) \big\rangle.$$

Since $f \in \mathscr{G}(q, q'')$, we have $T_P \in \mathscr{G}(x_1^2 - y^2, a/\lambda(x_1^2 - y^2))$. Theorem 5.3.2 gives in particular that there exists n depending only on the order of f such that $(x_1^2 - y^2)^n T_P(x_1, y) = 0$. Take

$$\phi(x, y) = (x_1^2 - y^2)^n Q(x_1, y) \exp(-\pi t^{-1}(x_1^2 + y^2)),$$

where Q is a polynomial, and use Lemma 1.2.2 to conclude that $(x_1^2 - y^2)^n f = 0$. Hence

$$f(x, y) \exp(\pi \sum_{j \in J} x_j^2) \in S'(\mathbb{R}^{d+1}).$$

For the same reason,

$$\widehat{f}(\xi, \eta) \exp(\pm \pi (a_2 \xi_2^2 + \cdots + a_d \xi_d^2)) \in S'(\mathbb{R}^{d+1}).$$

Theorem 1.3.5 gives $f = 0$, since $\max\{|a_j|, j = 2, \ldots, d\} > 1$.

In the remaining cases, there is always a non zero element in $\mathscr{G}(q, q')$. Indeed, when $\max(a, b) \leq 1$, the standard Gaussian function is a solution. And if $a = b > 1$, $I_- = \varnothing$, $I_+ = \{1\}$, and $\max_{j \in J} a_j \leq 1$, we can take f equal to

$$\delta(x_1 - y) \otimes \gamma(x_2, \ldots, x_d),$$

where γ is the standard Gaussian function. □

We prove now Theorem 5.3.2.

Proof. – We first consider the case $d \geq 2$. Let $f \in \mathscr{G}(q, q')$. We will use the fact that for any $1 \leq \alpha \leq a$, $f(\frac{\cdot}{\sqrt{\alpha}}) \in \mathscr{G}(q, q)$. The distribution f itself belongs to $\mathscr{G}(q, q)$. Theorems 5.1.2 and 4.2.2 imply that f can be uniquely written as a finite sum

$$f = \sum_k T_k(\mu_k),$$

where $\mu_k \in S'_{\overline{B}}$. We will prove by induction on k that $\mu_k = 0$. By Lemma 5.1.12, and the formula

$$(-\pi)^k \mathbf{z}_1^k T_k(\mu) = T_0(\partial_1^k \mu).$$

we only have to consider the case $f = T_0(\mu)$.

Redefine the Gaussian function γ_ε, for $\varepsilon > 0$, by

$$\gamma_\varepsilon(x,y) = \varepsilon^{-d/2} \exp(-\frac{\pi}{\varepsilon}(x_1^2 + \cdots + x_d^2 + y^2)).$$

We have $f \star \gamma_1 \star \gamma_\varepsilon = f \star \gamma_{1+\varepsilon}$, or equivalently

$$\int \mathscr{B}(f)(x,y) \exp\big(-\frac{\pi}{2}(|x|^2 + y^2) - \frac{\pi}{\varepsilon}((x-z)^2 + (y-\zeta)^2)\big)\, dx\, dy$$
$$= \mathscr{B}(f_\varepsilon)((1+\varepsilon)^{-1/2}(z,\zeta)) \exp(-\frac{\pi(z^2+\zeta^2)}{2(1+\varepsilon)}),$$

where $f_\varepsilon(\cdot) = \varepsilon^{-d/2} f(\sqrt{1+\varepsilon}\,\cdot)$. By assumption on f, $f_\varepsilon \in \mathscr{G}(q,q)$ for ε small enough. For $|v| < 1$, call $\phi_{v,\varepsilon}(z,\zeta)$ the expression

$$\int \exp(\pi\langle v, xy\rangle) \exp\big(-\frac{\pi}{2}(x^2+y^2) - \frac{\pi}{\varepsilon}((x-z)^2 + (y-\zeta)^2) + \frac{\pi(z^2+\zeta^2)}{2(1+\varepsilon)}\big)\, dx\, dy,$$

so that

$$\mathscr{B}(f_\varepsilon)((1+\varepsilon)^{-1/2}(z,\zeta)) = \int \phi_{v,\varepsilon}(z,\zeta)\, d\mu(v).$$

An straightforward computation shows that

$$\phi_{v,\varepsilon}(0,\zeta) = \frac{C(\varepsilon)}{\sqrt{2+\varepsilon(1-|v|^2)}} \exp(\frac{\pi}{2}\tau(v,\varepsilon)\zeta^2)$$

with $\tau(v,\varepsilon) \approx \varepsilon\frac{1-|v|^2}{4}$ as $\varepsilon \to 0$. Theorem 5.1.2 implies that $\mathscr{B}(f_\varepsilon)(0,\zeta)$ is a polynomial. Take derivatives with respect to ζ, and the limit at $\varepsilon = 0$. We get

$$\int (1-|v|^2)^n\, d\mu(v) = 0$$

for n large enough. Hence μ is supported by $\mathbb{S}_{d-1}$.

In the same way, we have

$$\phi_{v,\varepsilon}(z,0) = \frac{C(\varepsilon)}{\kappa(v,\varepsilon)} \exp(-\pi\langle M(v,\varepsilon)z, z\rangle),$$

with $\kappa(v,\varepsilon) \approx 1$ as $\varepsilon \to 0$. Here $M(v,\varepsilon)$ is a real matrix such that

$$\langle M(v,\varepsilon)z, z\rangle \approx \frac{\varepsilon(z^2 - \langle v,z\rangle^2)}{4}$$

as $\varepsilon \to 0$. Since $\mathscr{B}(f_\varepsilon)(z,0)$ is a polynomial in z, we see that

$$\int (z^2 - \langle v,z\rangle^2)^n\, d\mu(v) = 0$$

for n large enough. This is also true for a partial derivative of μ, since $T_0(\partial_1 v)(\mu)$ can be expressed as a polynomial in the creation and annihilation operators applied to $T_0(\mu)$. Hence

$$\int (z^2 - \langle v, z\rangle^2)^n P(\langle v, z\rangle)\, d\mu(v) = 0$$

for any polynomial P, and n large enough. It follows that

$$\int (z^2 - \langle v, z\rangle^2)^n \exp(\pi\langle v, \zeta z\rangle)\, d\mu(v) = 0 \tag{5.26}$$

for large n, $\zeta \in \mathbb{C}$ and $z \in \mathbb{C}^d$.

We prove now that (5.26) implies that $\mu = 0$. The distribution μ may be written as a finite sum

$$\mu = \sum_{k=0}^{N} \partial_r^k m_k$$

of radial derivatives of distributions m_k on the unit sphere $\mathbb{S}^{d-1}$. Relation (5.26) may be rewritten as

$$0 = \big(z^2 + \frac{1}{4\pi^2}\partial_\zeta^2\big)^n \sum_{k=0}^{N}(-1)^k \zeta^k \partial_\zeta^k \int_{\mathbb{S}^{d-1}} \exp(2i\pi\langle\theta, z\zeta\rangle)\, dm_k(\theta).$$

Take real z, ζ, and take a Fourier transform with respect to ζ. We find

$$0 = (x_1^2 + \cdots + x_d^2 - y^2)^n \sum_{k=0}^{N}(-1)^k \partial_y^k y^k \int_{\mathbb{S}^{d-1}} \delta_{y=\langle x,\theta\rangle}\, dm_k(\theta).$$

Hence the distribution

$$g(x,y) = \exp(-\pi(|x|^2 - y^2)) \sum_{k=0}^{N}(-1)^k \partial_y^k y^k \int_{\mathbb{S}^{d-1}} \delta_{y=\langle x,\theta\rangle}\, dm_k(\theta)$$

vanishes for $|x|^2 - y^2 \neq 0$. Since it belongs to $\mathscr{G}(q,q)$ by (5.12), Theorem 5.2.6 implies that $g = 0$, and hence $\mu = 0$. This conclude the proof when $d \geq 2$.

When $d = 1$, the previous argument may be adapted, but one has to be more careful in the reduction of the problem, since non zero distributions μ are allowed. We give a simpler proof. We can assume that the two quadratic forms are given by

$$q(x,y) = 2xy,\ q'(\xi,\eta) = 2a\xi\eta,$$

with $a > 1$. Then any element of $\mathscr{G}(q,q)$ can be decomposed as in Theorem 5.1.8. Let $f \in \mathscr{G}(q,q')$. We will use the fact that $f_a(x,y) = f(x, ya^{-1})$ belongs to $\mathscr{G}(q,q)$. Let g be the Fourier transform of f with respect to the variable y. Theorem 5.1.8 implies that $g(x,y)$, for $(x,y) \neq 0$, can be decomposed as

$$g(x,y) = \sum_{k,l} x^k y^l g_{k,l}(x^2 + y^2),$$

where the sum is finite, and $g_{k,l}$ are real analytic functions on $\mathbb{R}_*^+$. Since $f_a \in \mathscr{G}(q,q)$, we also have

$$g(x,y) = \sum_{k,l} x^k y^l h_{k,l}(x^2 + y^2/a^2),$$

where the sum is finite, and $h_{k,l}$ are real analytic functions on $\mathbb{R}_*^+$. These two expressions cannot occur simultaneously, unless g is given by a polynomial for $(x,y) \neq 0$. It follows that g is a sum of a polynomial, and a distribution supported by the origin (which is a sum of derivatives of Dirac measures). The result follows. □

5.4. Description of other spaces $\mathscr{G}(q,q')$

THEOREM 5.4.1. – *Let q be the Lorentz form on $\mathbb{R}^d$, and q' given by*

$$q'(\xi) = \sum_{i=1}^{d} \varepsilon_i' \xi_i^2,$$

with $\varepsilon_i' \in \{-1,+1\}$. Then any element of f of $\mathscr{G}(q,q')$ can be written as

$$f(x) = \sum_{k=1}^{N} P_k(x,\partial_x) \int_{O(q)\cap O(q')} \exp(-\pi |g(x)|^2)\, d\mu_k(g),$$

where P_k are polynomials and μ_k are finite measures on the group $O(q)\cap O(q')$.

Proof. – We may write, changing the sign of q' if necessary,

$$q(x) = |x_1|^2 + |x_2|^2 - |x_3|^2, \quad q'(\xi) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2,$$

with $x_i, \xi_i \in \mathbb{R}^{d_i}$, $d_3 = 1$, and $d_1 + d_2 + d_3 = d$. We apply Theorem 4.2.4, and then Theorem 5.1.6 to the form $|x_2|^2 - |x_3|^2$. It gives the required form for f, once we have noticed that any matrix of the form

$$\begin{pmatrix} I & 0 \\ 0 & g_0 \end{pmatrix},$$

with $g_0 \in O(d_2,1)$, belongs to $O(q)\cap O(q')$. □

We now give two generalizations of Theorem 5.1.8. The proofs are very similar to the one of Theorem 5.1.6, and we will skip them.

We will first describe the space of distributions f on $\mathbb{R}^{2d}$ such that

$$f(x,y)\exp(\pi \sum_i \varepsilon_i x_i y_i) \in S'(\mathbb{R}^{2d}),\ \widehat{f}(\xi,\eta)\exp(\pi \sum_i \varepsilon_i' x_i y_i) \in S'(\mathbb{R}^{2d}), \tag{5.27}$$

for all choices of $\varepsilon_i, \varepsilon_i' \in \{-1,+1\}$. Particular examples are the distributions f such that,

$$f(x,y) = O\big(\exp(-2\pi \sum |x_i y_i|)\big),$$
$$\widehat{f}(\xi,\eta) = O\big(\exp(-2\pi \sum |\xi_i \eta_i|)\big).$$

In view of Theorem 5.1.8, every function of the form

$$\mathbf{H}_\mu(x,y) = \int_{(\mathbb{R}_+^*)^d} \exp(-\pi \sum_i [t_i x_i^2 + y_i^2/t_i])\, d\mu(t_1,\ldots,t_d)$$

is a solution, when μ is a finite measure on $(\mathbb{R}_+^*)^d$.

THEOREM 5.4.2. – *Let $f \in S'(\mathbb{R}^{2d})$. Then f satisfies* (5.27) *if and only if there exist polynomials P_k, $k = 1,\ldots,N$, finite measures μ_k on $(\mathbb{R}_+^*)^d$, such that*

$$f(x,y) = \sum_k P_k(x,y,\partial_x,\partial_y)\mathbf{H}_{\mu_k}(x,y).$$

Sketch of the proof. – Let F be the Bargmann transform of f, and

$$G(z,\zeta) = F\left(\frac{z+\zeta}{\sqrt{2}}, \frac{z-\zeta}{\sqrt{2}}\right).$$

We can show as in the proof of Theorem 5.1.6 that

$$|G(z,\zeta)| \le C(1+|z|+|\zeta|)^N \exp(\pi \sum_i |\mathscr{Re}(z_i)||\mathscr{Re}(\zeta_i)| + |\mathscr{Im}(z_i)||\mathscr{Im}(\zeta_i)|.)$$

We conclude with Paley-Wiener-Schwartz's Theorem. □

We can as well prove the following generalization of Theorem 5.1.8. Let χ be a smooth function, equal to 0 in a neighborhood of the origin, and to 1 in the complement of some compact set. We consider the distributions $f \in S'(\mathbb{R}^d)$ such that

$$\left\{\begin{array}{l} \chi(|x||y|)f(x,y)\exp(2\pi|x||y|) \in S'(\mathbb{R}^d), \\ \chi(|\xi||\eta|)\widehat{f}(\xi,\eta)\exp(2\pi|\xi||\eta|) \in S'(\mathbb{R}^d). \end{array}\right. \tag{5.28}$$

Again, we use a cutoff function because the norm is not smooth at the origin. Examples are distributions satisfying

$$f(x,y) = O\big(\exp(-2\pi|x||y|)\big),$$
$$\widehat{f}(\xi,\eta) = O\big(\exp(-2\pi|\xi||\eta|)\big).$$

It is not necessary for x and y to have the same number of components. We choose $(x,y) \in \mathbb{R}^d$, with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$, $k+l = d$.

THEOREM 5.4.3. – *Let $f \in S'(\mathbb{R}^d)$. Then f satisfies* (5.28) *if and only if there exist polynomials P_k, finite measures μ_k on $\mathbb{R}_+^*$, such that for all x, y,*

$$f(x,y) = \sum_k P_k(x,y,\partial_x,\partial_y) \int_{\mathbb{R}_+^*} \exp(-\pi t|x|^2 - \pi|y|^2/t)\, d\mu_k(t).$$

Sketch of the proof. – Let F be the Bargmann transform of f. Then it can be shown, as when $d=1$, that there exist $C, N > 0$ such that for all z, ζ,

$$|F(z,\zeta)| \le C(1+|z|+|\zeta|)^N \exp(\pi|\mathscr{Re}(z^2-\zeta^2)|).$$

This is done as in the proof of Lemma 5.1.1. We conclude with Paley-Wiener-Schwartz's Theorem. □

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