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Henry DE THÉLIN & Gabriel VIGNY

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Henry De Thélin

Gabriel Vigny

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Henry De Thélin

Laboratoire Analyse, Géométrie et Applications, UMR 7539, Institut Galilée, Université Paris 13, 99 av. J.-B. Clément, 93430 Villetaneuse, France.

E-mail : dethelin@math.univ-paris13.fr

Gabriel Vigny

LAMFA - UMR 6140, U. P. J. V. 33, rue Saint-Leu, 80039 Amiens, France.

E-mail : gabriel.vigny@u-picardie.fr

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ENTROPY OF MEROMORPHIC MAPS AND DYNAMICS OF BIRATIONAL MAPS

Henry De Thélin, Gabriel Vigny

Abstract. – We study the dynamics of meromorphic maps for a compact Kähler manifold X. More precisely, we give a simple criterion that allows us to produce a measure of maximal entropy. We can apply this result to bound the Lyapunov exponents.

Then, we study the particular case of a family of generic birational maps of $\mathbb{P}^k$ for which we construct the Green currents and the equilibrium measure. We use for that the theory of super-potentials. We show that the measure is mixing and gives no mass to pluripolar sets. Using the criterion we get that the measure is of maximal entropy. It implies finally that the measure is hyperbolic.

Résumé **(Entropie des applications méromorphes et dynamique des applications birationnelles)**

On étudie la dynamique des applications méromorphes sur les variétés kählériennes compactes. Plus précisément, on donne un critère simple qui permet de produire des mesures d'entropie maximale. On peut appliquer ce résultat pour borner les exposants de Lyapounov.

Ensuite, on étudie le cas particulier d'une famille générique d'applications birationnelles de $\mathbb{P}^k$ pour laquelle on construit les courants de Green et la mesure d'équilibre. On utilise pour cela la théorie des super-potentiels. On montre que la mesure est mélangeante et qu'elle n'a pas de masse sur les ensembles pluripolaires. En utilisant le critère on obtient que la mesure est d'entropie maximale. Cela implique finalement que la mesure est hyperbolique.

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CHAPTER 1

INTRODUCTION

Let X be a complex compact Kähler manifold of complex dimension k endowed with a Kähler form ω. We consider $f : X \to X$ a dominant meromorphic map and we denote by I its indeterminacy set. We want to study the dynamics of f in other words the behavior of the sequence of iterates (f^n). We are particularly interested in the study of the ergodic properties of f.

Such study often starts with computing the topological entropy which gives an insight into how complicated the dynamics is. Classical objects in that case are ergodic measures of maximal entropy. Indeed, the support of such measure μ is an invariant set where the complexity of the dynamics is maximal. On this support, the dynamics is well understood in a statistical sense by Birkhoff's ergodic theorem as almost every orbit is equidistributed for the measure μ. One can then study finer properties of the measure: mixing, speed of mixing, dimension of the measure ...

In order to understand more precisely the dynamics near a point x in the support of the measure, one classicaly tries to compute the Lyapunov exponents. Roughly speaking, if no Lyapunov exponent is zero, these are numbers which give a rate of contraction and a rate of expansion in some stable and unstable manifolds. In other words, on the stable manifold the orbit of a point tends to the orbit of x at a speed given by some negative Lyapunov exponent and on the unstable manifold the backward orbit of a point tends to the backward orbit of x at a speed given by some positive Lyapunov exponent. When no Lyapunov exponent is zero, the measure is said to be *hyperbolic*. Finding examples of hyperbolic measures is a central question in dynamics and complex dynamics provides usually many of those (see **[45]** for definitions and results on hyperbolic measures).

In the particular case of complex dynamics, the topological entropy is related to the *dynamical degrees*. For $l = 0 \dots k$, we write:

$$\lambda_l(f) := \int_X f^*(\omega^l) \wedge \omega^{k-l}.$$

The l-th *dynamical degree* of f is defined by (see [**48**] and [**22**]):

$$d_l := \lim_{n\to+\infty} (\lambda_l(f^n))^{1/n}.$$

The degree d_l measures the asymptotic spectral radius of the action of f^* on the cohomology group $H^{l,l}(X)$. The last degree d_k is the *topological degree*. It can be shown that the sequence of degrees is increasing up to a rank s and then it is decreasing (see [**40**]). These quantities are of algebraic nature and there is a bound from above of the topological entropy by $\max_{0\leq s\leq k} \log d_s$ (see [**41**] for the holomorphic case and [**25**], [**22**] for the meromorphic case).

When one of the dynamical degree is strictly higher than the others, f is called *cohomologically hyperbolic*. In that case, it is expected that there exists a measure of maximal entropy $\max_{0\leq s\leq k} \log d_s$ (see [**43**]). Such measure is expected to be hyperbolic (and the saddle points are expected to be equidistributed along the measure). Recall the first author's result on Lyapunov exponents (Corollary 3 in [**11**]) when f is cohomologically hyperbolic: if one can find a measure μ of entropy $\max_{0\leq s\leq k} \log d_s$, then it is hyperbolic with estimates on the Lyapunov exponents provided that $\log dist(x, I\cup C) \in L^1(\mu)$ (I is the indeterminacy set of f and C the critical set). In particular, in order to find hyperbolic measures, it is enough to find measures of maximal entropy.

This will be our aim in the first part of this article (Chapter 2). Following Yomdin's approach ([**55**]), we give a criterion that allows us to produce invariant measures of maximal entropy for a meromorphic map on a compact Kähler manifold X. In a second part (Chapter 3), we study the more precise case of a family of birational maps of $\mathbb{P}^k$ for which we construct the equilibrium measure. We show that it is mixing and using the results of the first part we show that it is of maximal entropy. In particular the measure is hyperbolic. Let us detail our results.

Let d denote the distance in X and recall that I is the indeterminacy set of f. In Chapter 2, we do not assume that f is cohomologically hyperbolic. Chapter 2 is devoted to the proof of the following theorem:

Theorem 1. – *Consider the sequence of measures:*

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} f^i_*\left(\frac{(f^n)^*\omega^l\wedge\omega^{k-l}}{\lambda_l(f^n)}\right).$$

Assume that there exists a converging subsequence $\mu_{\psi(n)} \to \mu$ with:

$$(H) : \lim_{n\to+\infty} \int \log d(x,I)d\mu_{\psi(n)}(x) = \int \log d(x,I)d\mu(x) > -\infty.$$

Then μ is an invariant measure of metric entropy larger than or equal to $\log d_l$.

In the above theorem, when $I = \varnothing$, we define $d(x, I) := 1$ for all $x \in X$ so Hypothesis (H) is automatically satisfied for holomorphic mappings.

Proving the convergence of the sequence μ_n with the hypothesis (H) for $l = s$ such that d_s is the highest dynamical degree gives measures of maximal entropy $\log d_s$. So, as we explained before, Theorem 1 is particularly interesting when f is cohomologically hyperbolic and $l = s$ with d_s the highest dynamical degree. In fact, in that case it is likely that (μ_n) will always converge to a measure of maximal entropy because this is the case in a lot of known cases (Henon maps **[4]** and **[3]**, regular automorphisms of $\mathbb{C}^k$ **[50]**, endomorphisms of $\mathbb{P}^k$ **[37]**, **[36]** and **[38]**, some automorphisms on compact Kähler manifolds **[7]** and **[24]**, ...).

We discuss Hypothesis (H). First $\int \log d(x, I) d\mu(x) > -\infty$ is a natural hypothesis in order to produce hyperbolic measure because it is necessary to define the Lyapunov exponents (**[11]**). It would be a real improvement of Theorem 1 to prove the same result under that weaker hypothesis. Nevertheless, it is not always satisfied by meromorphic maps. Indeed, in **[19]** they are examples modifying Favre's examples (see **[34]**) for which $\log d(x, I)$ is not integrable with respect to the measure μ. On the other hand, semi-continuity of the logarithm implies that

$$\limsup_{n \to +\infty} \int \log d(x, I) d\mu_{\psi(n)}(x) \leq \int \log d(x, I) d\mu(x)$$

so there really is only the other inequality to prove.

We explain the main ideas of the proof of Theorem 1. If f is a Hénon map of $\mathbb{C}^2$, Bedford and Smillie have shown in **[4]** that the Green measure of f is of maximal entropy. Their proof is based on Yomdin's theorem (see **[55]**) and also on the proof of the variational principle. This approach has been used several times since then in dynamics in order to bound from below the entropy of measures (e.g. **[42]**, **[10]** and **[32]**). In all these cases, one can use Yomdin's theorem directly because the application f is either holomorphic or when it is meromorphic everything takes place in a stable open set where f is holomorphic.

The purpose of the first part is to quantify Bedford and Smillie's approach. We will need for that to modify the Bowen ball so that it takes into account the distance to the indeterminacy set I and to quantify Yomdin's theorem. This is the main difficulty. We then use the proof of the variational principle to conclude.

Observe that the criterion can be extended to the case where (X, ω) is a compact Hermitian manifold. In that case, we do not know if the limit:

$$d_l := \lim_{n \to +\infty} (\lambda_l(f^n))^{1/n}$$

exists, but it is sufficient to replace d_l by $\limsup_n (\lambda_l(f^{\psi(n)}))^{1/\psi(n)}$ in the theorem.

In Chapter 3, we work in $\mathbb{P}^k$ endowed with the Fubini Study form ω. We study the dynamics of some birational maps f of $\mathbb{P}^k$, that is maps that are meromorphic and biholomorphic outside some analytic sets. Choose some $1 \leq s \leq k-1$. We assume that $\dim(I^+) = k-s-1$ and $\dim(I^-) = s-1$ where $I^\pm$ are the indeterminacy sets of $f^\pm$ (for $k=2$ the only possibility is $s=1$ and $I^\pm$ are points). Let d and δ be the algebraic degrees of f and f^{-1}.

Let $\mathscr{B}_0$ be the set of these birational maps of $\mathbb{P}^k$ such that:

$$\bigcup_{n\geq 0} f^{-n} I^+ \cap \bigcup_{n\geq 0} f^n I^- = \varnothing.$$

For such a map, we have $\lambda_s(f^n) = d^{sn}$ and we can define $L^n(\omega^s) := d^{-sn}(f^n)^*(\omega^s)$ the sequence of normalized pull-backs and $\Lambda^n(\omega^{k-s}) := d^{-sn}(f^{-n})^*(\omega^{k-s})$ the sequence of normalized push-forwards. The key point of our method is the use of super-potentials introduced by Dinh and Sibony in [**31**]. We sum up the properties of the super-potentials that we need in an appendix. Super-potentials are a generalization of potentials to positive closed currents of bidegree higher than 1. Let $\mathscr{C}_l$ be the set of positive closed current of bidegree (l,l) and mass 1. Then for $S \in \mathscr{C}_l$, the super-potential $\mathscr{U}_S$ of S is a function on $\mathscr{C}_{k-l+1}$ that is uniquely determined by the condition $\mathscr{U}_S(\omega^{k-l+1}) = 0$.

Let $\mathscr{U}_{L^n(\omega^s)}$ and $\mathscr{U}_{\Lambda^n(\omega^{k-s})}$ be the super-potentials of $L^n(\omega^s)$ and $\Lambda^n(\omega^{k-s})$. We study the set $\mathscr{B}$ of map $f \in \mathscr{B}_0$ such that:

(1) $$\lim_n \mathscr{U}_{L^n(\omega^s)}([I^-]_{\mathrm{nor}}) > -\infty \quad \text{and} \quad \lim_n \mathscr{U}_{\Lambda^n(\omega^{k-s})}([I^+]_{\mathrm{nor}}) > -\infty,$$

where $[I^\pm]_{\mathrm{nor}} := \mathrm{vol}(I^\pm)^{-1}[I^\pm]$ is the normalization of the current of integration on $I^\pm$ (we use in fact a slightly different definition that turns out to be equivalent to (1) by Corollary 3.2.5).

We prove in Theorem 3.2.8 below that (1) is implied by the more geometric condition:

$$\sum_{n\geq 0} \left(\frac{1}{d}\right)^n \log \mathrm{dist}(I^+, f^n(I^-)) > -\infty$$

and

$$\sum_{n\geq 0} \left(\frac{1}{\delta}\right)^n \log \mathrm{dist}(I^-, f^{-n}(I^+)) > -\infty.$$

This condition was in fact introduced by Bedford and Diller in [**1**] for the case of projective surface and is equivalent to (1) in this case. Using that hypothesis, the authors define the equilibrium measure and show that the potential of the Green current is integrable for the measure. They proved that the measure is mixing and hyperbolic. Using laminar currents, Dujardin computed the entropy and showed that the measure is of maximal entropy [**33**]. Diller and Guedj extended some parts of these results to a more general case in [**19**]. Note also the generalization of those ideas to the case of meromorphic maps of a surface in the recent articles [**16**], [**17**], [**18**].

When $k \geq 3$, the condition (1) is weaker than the geometric condition. This is due to the fact that distance between supports is a good distance for measure but not for current of higher bidimension. The interest of this condition is that it is generic in the following sense:

THEOREM 2. – *Let E_s be the set of birational maps $f : \mathbb{P}^k \to \mathbb{P}^k$ such that I^+ and I^- satisfy $\dim(I^+) = k - s - 1$ and $\dim(I^-) = s - 1$. Consider the group action:*

$$\Phi : \mathrm{PGL}(k+1, \mathbb{C}) \times E_s \to E_s$$
$$(A, f) \mapsto A \circ f.$$

Then outside a pluripolar set of the orbit $Orb(f)$ of $f \in E_s$, the maps of $Orb(f)$ are in $\mathscr{B}$.

Section 3.3 is devoted to the proof of that statement. It is interesting to note that the condition (1) is not generic in the set of birational maps (assuming one can give a sense to that statement). Indeed, the condition $\dim(I^+) = k-s-1$ and $\dim(I^-) = s-1$ is true for $k = 2$ but does not need to be satisfied in higher dimension (see [**44**] and [**42**] for examples).

We sum up our results. We will construct the Green current of order s and $k - s$ of f and f^{-1} for $f \in \mathscr{B}$. More precisely, we have (see Theorems 3.2.1, 3.2.9 and 3.2.19):

THEOREM 3. – *Let $f \in \mathscr{B}$, then the sequence $L^n(\omega^s)$ is a well defined sequence of currents which converges in the sense of currents to a positive closed current T_s^+ of bidegree (s, s) and of mass 1.*

The current T_s^+ satisfies $f^*(T_s^+) = d^s T_s^+$ and is extremal in the set of positive closed currents.

Then we define the intersection $T_s^+ \wedge T_{k-s}^-$ and we prove (Theorem 3.4.1, Proposition 3.4.4 and Theorem 3.4.15):

THEOREM 4. – *The wedge-product $\mu := T_s^+ \wedge T_{k-s}^-$ is a well defined invariant probability measure for which the potential of the Green current of order 1 is integrable. The measure μ is mixing for f.*

Using a space of test functions introduced by Dinh and Sibony in [**26**] and studied by the second author [**53**], we prove that the measure gives no mass to pluripolar sets. In particular, the measure gives no mass to analytic sets.

Then we use the results of Chapter 2 to prove that (Theorem 3.4.19, Theorem 3.4.21):

THEOREM 5. – *The measure μ is of maximal entropy $\log d^s$ and is hyperbolic. More precisely, the Lyapunov exponents $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_k$ of μ are well defined and we have the estimates:*

$$\chi_1 \geq \cdots \geq \chi_s \geq \frac{1}{2}\log\frac{d_s}{d_{s-1}} = \frac{1}{2}\log d > 0$$

$$0 > -\frac{1}{2}\log\delta = \frac{1}{2}\log\frac{d_{s+1}}{d_s} \geq \chi_{s+1} \geq \cdots \geq \chi_k.$$

REMARK 1.0.1. – In these settings, it is natural to ask whether we have the equidistribution of the saddle points for μ ([**43**]). In the case of $\mathbb{P}^2$, the author of [**1**] prove the (weaker) result that the support of μ is contained in the closure of the saddle periodic points and Dujardin proved the equidistribution in [**33**]. We do not know how to prove such results in our case.

The main difficulty and novelty of that study is that in order to prove the convergences, we deal directly with positive closed currents of bidegree (s, s). When $s > 1$, the potentials U of a positive closed current S of bidegree (s, s) are no longer quasi-plurisubharmonic (qpsh for short) functions but currents satisfying $dd^cU + \omega^s = S$. Two such potentials U and U' differ by a dd^c closed current. Such object can be singular. So we use the new theory of super-potential ([**31**] and also [**32**] for the Kähler case). It provides a calculus on (s, s) positive closed currents. We sum up the properties of super-potentials that we used in an appendix.

The two parts are fairly independent as we only use the results of Chapter 2 at the end of Chapter 3. So they can be read in any order.

Acknowledgements. – We thank the referee for his remarks and in particular for the simplification of the first point of the proof of proposition 2.3.8.

We also thank T.-C. Dinh for helping us constructing an example stable by perturbations in the proof of Theorem 3.1.11.

CHAPTER 2

ENTROPY OF MEROMORPHIC MAPS

2.1. Push-forward of measures by meromorphic maps

Let (X, ω) be a compact Kähler manifold of dimension k. Modulo a normalization of the distance, we can assume that the diameter of X is less than 1. Let f be a dominating meromorphic map and let I be the indeterminacy set of f. Recall that for $l = 0 \ldots k$, we write:

$$\lambda_l(f) := \int_X f^*(\omega^l) \wedge \omega^{k-l}.$$

We recall first how to define the push-forward by f of a measure that gives no mass to I. In all that follows, a measure will mean a finite positive Radon measure.

Let ν be such a measure. On $X \setminus I$, f is a measurable map. So we can define $f_*\nu$ by the formula:

$$(f_*\nu)(A) := \nu(\{x \in X \setminus I \text{ with } f(x) \in A\}) = \nu(f^{-1}(A) \cap (X \setminus I)).$$

When a measure ν gives no mass to the indeterminacy set, we have:

$$\int \varphi \circ f d\nu = \int \varphi d(f_*\nu)$$

for all $\varphi \in L^1(f_*\nu)$. It is implicitly assumed that the integral is on $X \setminus I$. The equality follows from the approximation of function in L^1 by characteristic functions.

The operator f_* has the good property of continuity. Indeed, we have:

Lemma 2.1.1. – *Let ν_n be a sequence of measures that give no mass to I. Then if (ν_n) converges to ν and $\nu(I) = 0$ then $(f_*(\nu_n))$ converges to $f_*\nu$.*

Proof. – Let φ be a continuous function and let $0 \leq \chi_\varepsilon \leq 1$ be a smooth function equal to 0 in an ε-neighborhood I_ε of I and equal to 1 outside a 2ε-neighborhood $I_{2\varepsilon}$ of I. Then, we have:

$$\int \varphi d(f_*\nu_n) = \int \varphi \circ f d\nu_n = \int (1 - \chi_\varepsilon)\varphi \circ f d\nu_n + \int \chi_\varepsilon \varphi \circ f d\nu_n.$$

The first term is bounded in absolute value by $\|\varphi\|_\infty \nu_n(I_{2\varepsilon})$ which can be taken arbitrarily small by taking ε small then n large (because ν gives no mass to I). The second term converges to $\int \chi_\varepsilon \varphi \circ f d\nu$ since $\chi_\varepsilon \varphi \circ f$ is a continuous function. Finally, if ε is small enough, $\int \chi_\varepsilon \varphi \circ f d\nu$ is as close as we want from $\int \varphi \circ f d\nu$ since ν gives no mass to I. □

In this section, we consider in particular the push-forward of the measures

$$\nu_n := \frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)}.$$

The ν_n are well defined probability measures. Indeed, $(f^n)^*\omega^l$ is a form with coefficients in L^1 so it gives no mass to analytic sets of dimension $< k$. This implies that

$$\frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)}$$

is a probability that gives no mass to $\cup_{i\in\mathbb{N}} f^{-i}(I)$ (because f is dominating). So we can push-forward this probability by f^i and we get again a probability. We also make the observation:

$$(f^i)_*(f^j)_* \frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)} = (f^{i+j})_* \frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)},$$

since $(f^j)^{-1}(I)$ is at most a hypersurface. In particular, we have $f^i_*\nu_n = (f^i)_*\nu_n$.

Now, we say that a measure μ that gives no mass to I is *invariant* (or f_*-invariant) if $f_*(\mu) = \mu$. One has the following easy lemma:

Lemma 2.1.2. – *Let μ be a measure that gives no mass to I. Then the following properties are equivalent:*

— *μ is invariant.*
— *For any continuous function φ, we have:*

$$\int \varphi \circ f d\mu = \int \varphi d\mu$$

where the left-hand side integral is taken over $X \setminus I$.

When these properties are true, we even have:

$$\int \varphi \circ f d\mu = \int \varphi d\mu$$

for any φ in $L^1(\mu)$ (with the same abuse of notation for the left-hand side integral that we will do in the whole section).

Let us give some properties of meromorphic maps that will be useful in the proof of Theorem 1. First recall that we denote:

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} f_*^i\left(\frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)}\right).$$

We have seen that it is a well defined sequence of probabilities. Since f is dominating, these measures give no mass to analytic sets of dimension $< k$ so we can define their push-forward.

We need an invariant measure to consider the metric entropy. So we will need the following lemma:

LEMMA 2.1.3. – *If $(\mu_{\psi(n)})$ converges to a measure μ that gives no mass to I, then μ is f_*-invariant.*

Proof. – To simplify the notations, assume that (μ_n) converges to μ.

We can write $f_*(\mu_n) = \mu_n + \alpha_n$ with α_n going to zero. Using Lemma 2.1.1, $f_*(\mu_n)$ converges to $f_*\mu$ and the lemma follows. □

Now since we have an invariant probability measure that gives no mass to I, its mass is 1 on $\Omega = X \setminus \cup_{i\in\mathbb{N}} f^{-i}(I)$. Since $f(\Omega) \subset \Omega$, we can define the metric entropy of μ using partitions (see **[43]** and **[45]**).

We recall the following estimate that we use later:

LEMMA 2.1.4. – *(see* **[20]** *Lemma 2.1)*
There exist constants K and p such that:

$$\|Df(x)\| \leq K d(x, I)^{-p}.$$

2.2. Yomdin's theorem

In this paragraph, we recall some facts on Yomdin's theorem (see **[55]**) using Gromov's version (see **[39]** and **[6]**).

Let l be an integer between 1 and $2k$. If Y is a subset of $\mathbb{C}^k$ (for example a submanifold of real dimension l), we call *C^r-size* (with $r \in \mathbb{N}^*$) of Y, the lower bound of the numbers $t \geq 0$ for which there exists a C^r-map of the unit l-cube into $\mathbb{C}^k$, $h : [0,1]^l \mapsto \mathbb{C}^k$, with $Y \subset h([0,1]^l)$ and $\|D_r h\| \leq t$. Here $D_r h$ is the vector assembled of (the components of) the partial derivatives of h of order $1, \dots, r$. The norm refers to supremum over $x \in [0,1]^l$:

$$\|D_r h\| = \sup_x \|D_r h(x)\|.$$

We make some comments on C^r-size first.

First, the C^1-size bounds the (real) l-dimensional volume of Y and its diameter. More precisely

$$C^1 - \text{size of } Y \geq \max((\text{l-dimensional volume } (Y))^{1/l}, l^{-1/2}\text{Diameter}(Y)).$$

A process that we will use in what follows is the division of a set of C^r-size. If Y is a set of C^r-size smaller than t, we can divide Y in j^l pieces of C^r-size smaller than t/j. For that it is sufficient to divide the l-cube $[0,1]^l$ in j^l equal pieces and then to scale: for example $R : [0,1]^l \mapsto [0,j^{-1}]^l$ and similarly for the $j^l - 1$ other cubes. The composition of $h : [0,1]^l \mapsto \mathbb{C}^k$ which covers Y with the scaling R satisfies $\|D_r(h \circ R)\| \leq t/j$ and the union of the images of these j^l maps covers Y.

Here is now the principal result of Gromov-Yomdin that we will need (see Lemma 3.4 in [**39**]).

THEOREM 2.2.1 ([**39**]). – *Let Y be an arbitrary subset in the graph $\Gamma_g \subset [0,1]^l \times \mathbb{C}^k$ of a C^r-map $g : [0,1]^l \mapsto \mathbb{C}^k$ and take some positive number $\epsilon \leq 1$. Then Y can be divided into $N \leq C(k,l,r)\epsilon^{-l}(1 + \|\partial_r g\|)^{l/r}$ sets of C^r-size $\leq C(k,l,r)\epsilon$Diameter(Y), where $\partial_r g$ denotes the vector assembled of the partial derivatives of g of order exactly r and $C(k,l,r)$ is a universal constant.*

Here is the application of the above theorem that we will use: it is a small variation of Corollary 3.5 in [**39**].

PROPOSITION 2.2.2. – *Let V be an open set of $\mathbb{C}^k$ and $f : V \to \mathbb{C}^k$ a map of class C^r. Let $Y_0 \subset V$ be a set of C^r-size smaller than 1 such that $d(Y_0, \partial V) \geq \sqrt{l}$. Then the intersection of $f(Y_0)$ with a ball of $\mathbb{C}^k$ of radius β can be divided into $N \leq C(k,l,r)\left(1 + \frac{\|D_r f\|}{\beta}\right)^{l/r}$ pieces of C^r-size less than β.*

Proof. – We want to divide $f(Y_0) \cap B(a,\beta)$ into pieces of C^r-size $\leq \beta$. If $H(a, 1/\beta)$ denotes the homothety of center a and ratio $1/\beta$ in $\mathbb{C}^k$, it is equivalent to divide

$$H(a,1/\beta)(f(Y_0) \cap B(a,\beta)) = H(a,1/\beta)(f(Y_0)) \cap B(a,1)$$

into sets of C^r-size less than 1.

By hypothesis, there exists a map $h : [0,1]^l \to \mathbb{C}^k$ of class C^r with $\|D_r h\| \leq 1$ and $Y_0 \subset h([0,1]^l)$. Define $g := H(a,1/\beta) \circ f \circ h$. By the chain rule, we have

$$\|D_r g\| \leq \frac{C'(k,l,r)}{\beta}\|D_r f\|.$$

We apply now the previous theorem to Y the graph of g intersected with $[0,1]^l \times B(a,1)$. So we have that we can cover Y by a number:

$$N \leq C(k,l,r)\left(1 + \frac{C'(k,l,r)}{\beta}\|D_r f\|\right)^{l/r} \leq C(k,l,r)\left(1 + \frac{\|D_r f\|}{\beta}\right)^{l/r}$$

sets of C^r-size ≤ 1 (changing the constant $C(k,l,r)$ if necessary). Since the image of Y by the projection $[0,1]^l \times \mathbb{C}^k \mapsto \mathbb{C}^k$ covers $H(a,1/\beta)(f(Y_0)) \cap B(a,1)$, the proposition follows. □

2.3. Proof of Theorem 1

Let d denote the distance in X normalized such that the diameter of X is less than 1 and recall that I is the indeterminacy set of f. In this paragraph, we prove the following theorem:

THEOREM. – *Consider the sequence of measures:*

$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \left(\frac{(f^n)^* \omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)} \right).$$

Assume that there exists a converging subsequence $\mu_{\psi(n)} \to \mu$ with:

$$(H) : \lim_{n \to +\infty} \int \log d(x,I) d\mu_{\psi(n)}(x) = \int \log d(x,I) d\mu(x) > -\infty.$$

Then μ is an invariant measure of metric entropy larger than or equal to $\log d_l$.

In the above theorem, when $I = \varnothing$, we define $d(x,I) := 1$ for all $x \in X$ so Hypothesis (H) is automatically satisfied for holomorphic mappings.

Remark that the criterion can be extended to the case where (X,ω) is a compact Hermitian manifold. In that case, we do not know if the limit:

$$d_l := \lim_{n \to +\infty} (\lambda_l(f^n))^{1/n}$$

exists, but it is sufficient to replace d_l by $\limsup_n (\lambda_l(f^{\psi(n)}))^{1/\psi(n)}$ in the theorem.

We can also generalize the theorem in the following way: in the definition of μ_n, we can put $\Omega_{1,l}$ instead of ω^l, $\Omega_{2,k-l}$ instead of ω^{k-l} and $\Lambda_l(f^n) = \int (f^n)^* \Omega_{1,l} \wedge \Omega_{2,k-l}$ instead of $\lambda_l(f^n)$ with $\Omega_{1,l}$ a (l,l) smooth form, $\Omega_{2,k-l}$ a $(k-l,k-l)$ smooth form such that one of them is weakly positive, the other is strongly positive and $\Lambda_l(f^n) > 0$. Then, we have the same theorem by replacing d_l by $\limsup_n (\Lambda_l(f^{\psi(n)}))^{1/\psi(n)}$.

In the previous theorem, it is possible to have that the metric entropy of μ is $>$ $\log d_l$. Indeed, if we take for f a Lattès of $\mathbb{P}^1$ of degree $d \geq 2$ and $l = 0$ in the theorem, we obtain on one hand that $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \omega$ converges to the equilibrium measure μ (by using the Birkhoff's theorem since μ is equivalent to ω) which has metric entropy equal to $\log d$ and on the other hand we have that that $\log d_0 = \log 1 = 0 < \log d$.

We begin by giving the ideas of the proof of the theorem.

2.3.1. Ideas of the proof. – Recall that we consider

$$\nu_n = \frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)} \text{ and so } \mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f^i_* \left(\frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)}\right) = \frac{1}{n}\sum_{i=0}^{n-1} f^i_*\nu_n.$$

For the statement of the ideas, assume to simplify that μ_n converges to a measure μ.

First, consider the case where f is holomorphic. As in the article of Bedford and Smillie (see [**4**]), the proof is based in one hand on Yomdin's theorem and in the other hand on the Misiurewicz's proof of the variational principle. We denote by $B_n(x,\epsilon)$ the dynamical ball (or Bowen ball):

$$B_n(x,\epsilon) = \{y \in X \ , \ \max_{0\leq i\leq n-1} d(f^i(x), f^i(y)) \leq \epsilon\}.$$

For $\delta > 0$, by using Yomdin's theorem (see [**55**]), we can prove that $\nu_n(B_n(x,\epsilon)) \leq e^{n\delta}/\lambda_l(f^n)$ by taking ϵ small and then n large enough. This gives an estimation from below for the metric entropy $H_{\nu_n}(\mathscr{P}_{-n})$ where $\mathscr{P}_{-n} = \mathscr{P} \vee f^{-1}\mathscr{P} \vee \cdots \vee f^{-n+1}\mathscr{P}$ with $\mathscr{P}$ a finite partition of X with atoms of diameter less than ϵ. Now, by using Misiurewicz's proof of the variational principle, it implies that $\frac{1}{q}H_{\mu_n}(\mathscr{P}_{-q})$ is bounded from below by something similar to $-2\delta + \frac{1}{n}\log\lambda_l(f^n)$ for n large enough. Now, since the partition $\mathscr{P}$ is finite and if μ puts no mass on its boundary, we obtain

$$\frac{1}{q}H_\mu(\mathscr{P}_{-q}) \geq -2\delta + \log d_l$$

and by taking $q \to \infty$, we have $h_\mu(f) \geq \log d_l$, which is the inequality that we want.

Now, suppose that f is meromorphic. We want to apply the same strategy but the problem is the indeterminacy set.

Namely, the previous estimate on $\nu_n(B_n(x,\epsilon))$ uses Yomdin's theorem and only works for C^∞ maps. The first idea to solve this problem is to use dynamical balls which avoid the indeterminacy set, i.e. of the type

$$B_n(x) = \{y \in X \ , \ d(f^i(x), f^i(y)) \leq \rho(f^i(x)) \text{ for } i \in [0, n-1]\}$$

with $\rho(x) << d(x,I)$. On these balls, the derivatives of f may be large, but we have some estimates on them, and so, we will prove using Yomdin's approach, that $\nu_n(B_n(x)) \leq e^{n\delta}/\lambda_l(f^n)$ for n large enough on a set of *good points* of mass almost 1 for ν_n.

This inequality implies a lower bound on $H_{\nu_n}(\mathscr{P}_{-n})$ but for partitions $\mathscr{P}$ such that the atom $\mathscr{P}(x)$ which contains x has diameter smaller than $\rho(x)$. In particular these partitions are *countable*. Using Misiurewicz's method, we obtain more or less that $\frac{1}{q}H_{\mu_n}(\mathscr{P}_{-q})$ is bounded by below by something similar to $-2\delta + \frac{1}{n}\log\lambda_l(f^n)$ for n

large enough. To conclude, we have to let n goes to ∞: the problem is that $\mathscr{P}_{-q}$ is countable but by using the hypothesis

$$(H): \lim_{n\to+\infty} \int \log d(x,I)d\mu_n(x) = \int \log d(x,I)d\mu(x) > -\infty,$$

we can overcome this difficulty and we obtain the lower bound

$$\frac{1}{q}H_\mu(\mathscr{P}_{-q}) \geq -2\delta + \log d_l.$$

Hypothesis (H) implies that the partition $\mathscr{P}$ has finite entropy, and so if we take the limit $q \to \infty$, we obtain $h_\mu(f) \geq \log d_l$, which is again the inequality that we want.

2.3.2. Proof of Theorem 1. – The hypothesis we made assure us that there exists a subsequence $(\mu_{\psi(n)})$ which converges to a measure μ with:

$$(H): \lim_{n\to+\infty} \int \log d(x,I)d\mu_{\psi(n)}(x) = \int \log d(x,I)d\mu(x) > -\infty.$$

In order to clarify the exposition, we shall write $\psi(n) = n$. We start with the definition of the dynamical ball $B_n(x)$ that we will use.

If $s(x)$ is a function on X with values in $\mathbb{R}^+$, we define (see [**47**]):

$$B(x,s,n,f) := \left\{y,\ d(f^i(x), f^i(y)) \leq s(f^i(x)) \text{ for } i \in [0, n-1]\right\}.$$

We shall use these dynamical balls using for $s(x)$ the particular functions $\rho(x)$ and $\eta(x)$ where :

$$\rho(x) = \left(\frac{d(x,I) \times \cdots \times d(f^{m-1}(x), I)}{K^m}\right)^p$$

(here K and p are the numbers defined at the end of Section 2.1 and $m \in \mathbb{N}^*$ will be chosen later) and:

$$\eta(x) = \left(\frac{d(x,I)}{K}\right)^p.$$

When f is holomorphic (i.e. $I = \varnothing$), take $d(x,I) := 1$ and $p = 2$ in these expressions.

If $n \in \mathbb{N}$ is fixed, by the Euclidean algorithm, we write $n = \phi(n)m + r(n)$ with $0 \leq r(n) < m$. In what follows, we will consider the following dynamical balls:

$$B_n(x) := B(x, \rho, \phi(n), f^m) \cap f^{-\phi(n)m+m}(B(f^{\phi(n)m-m}(x), \eta, r(n)+m, f)).$$

Now, as explained in the paragraph 2.3.1, we will prove that $\nu_n(B_n(x)) \leq e^{n\delta}/\lambda_l(f^n)$ for n large enough on a set of *good points* of mass almost 1 for ν_n. In particular, we will have a lot of dynamical balls. In a second paragraph, we will use this fact to bound from below the entropy of μ.

2.3.3. Upper bound of $\nu_n(B_n(x))$. – We give some notations first. First of all we can put on X a family of chart $(\tau_x)_{x\in X}$ such that $\tau_x(0)=x$, τ_x is defined on $B(0,\epsilon_0)\subset\mathbb{C}^k$ with $\epsilon_0>0$ independent of x and such that the norm of the derivatives of order 1 of the τ_x is bounded from above by a constant independent of x. These charts are obtained from a finite family (U_i,ψ_i) of charts of X by composing them with translations. In $\mathbb{C}^k$, we also consider $\pi_1,\ \ldots,\ \pi_i$ the projections from $\mathbb{C}^k$ onto the vectorial subspaces of dimension $k-l$. In what follows, the choice of these coordinates is supposed to be generic and β_j denotes the standard volume form on $\pi_j(\mathbb{C}^k)$.

Fix $x\in X$ and:

$$\Omega:=(\tau_x)_*(\pi_1^*\beta_1+\cdots+\pi_i^*\beta_i).$$

We want to compute :

$$\nu_n(B_n(x))=\frac{(f^n)^*\omega^l\wedge\omega^{k-l}}{\lambda_l(f^n)}(B_n(x)).$$

Taking K large enough, we can assume that $B_n(x)\subset\tau_x(B(0,\epsilon_0))$ so the previous quantity is less than:

$$C(X)\frac{(f^n)^*\omega^l\wedge\Omega}{\lambda_l(f^n)}(B_n(x))=C(X)\sum_{j=1}^{i}\int\int_{B_n(x)\cap\tau_x(Y_j(t))}\frac{(f^n)^*\omega^l}{\lambda_l(f^n)}dt$$

where $Y_j(t)$ is equal to $\pi_j^{-1}(t)$ for t in the j-th subspace of dimension $k-l$ and dt stands for the Lebesgue measure on that space (we used Fubini theorem: see [**8**] p. 334). Remark that t lives in a ball $B(0,\epsilon_0)$.

So we have a upper bound of $\nu_n(B_n(x))$ by:

$$\frac{C(X)}{\lambda_l(f^n)}\sum_{j=1}^{i}\int\int_{f^n(B_n(x)\cap\tau_x(Y_j(t)))}\omega^l dt.$$

To control this integral, we have to bound from above the $2l$-dimensional volume of $f^n(B_n(x)\cap\tau_x(Y_j(t)))$ for some good points x of ν_n. In order to do that, we explain first what are the good points for ν_n then we will bound the volume using Yomdin's approach and finally we will finish the bound of $\nu_n(B_n(x))$.

Good points for the measure ν_n. – In what follows, we consider a constant $L>0$ and an integer n_0 such that:

$$\int\log d(x,I)d\mu_n(x)\geq -L,$$

for $n\geq n_0$. The existence of these constants follows easily from Hypothesis (H).

Let $\delta>0$. Our goal is to show that the entropy of μ is greater than $\log d_l-\delta$. We choose a constant C_0 large enough ($1/C_0\ll\delta$).

We are going to show that Hypothesis (H) implies that the orbits of generic points of the measure $\nu_n = \frac{(f^n)^*\omega^l\wedge\omega^{k-l}}{\lambda_l(f^n)}$ are not close to the indeterminacy set I. They are going to be the *good points.*

LEMMA 2.3.1. – *For $n \geq n_0$, there exists a set A_n of ν_n-measure greater or equal to $1 - C_0^{-1}$ whose points $x \in A_n$ satisfy:*

$$\prod_{i\in[0,n-1]} d(f^i(x), I) \geq e^{-C_0Ln}.$$

Proof. – We have

$$\frac{1}{n}\int \log\left(\prod_{i\in[0,n-1]} d(f^i(x), I)\right) d\nu_n(x) = \frac{1}{n}\int\sum_{i=0}^{n-1}\log d(f^i(x), I)d\nu_n(x).$$

Since $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}(f^i)_*\nu_n$:

$$\frac{1}{n}\int \log\left(\prod_{i\in[0,n-1]} d(f^i(x), I)\right) d\nu_n(x) = \int \log d(x, I)d\mu_n(x).$$

Thanks to our hypothesis, this last integral is $\geq -L$.

Now, if we denote $h(x) = \frac{1}{n}\log\left(\prod_{i\in[0,n-1]} d(f^i(x), I)\right)$ and $A_n := \{x,\, h(x) \geq -C_0L\}$, we have:

$$\int_{A_n} h(x)d\nu_n(x) + \int_{X\setminus A_n} h(x)d\nu_n(x) \geq -L.$$

But $\int_{A_n} h(x)d\nu_n(x) \leq 0$ and $\int_{X\setminus A_n} h(x)d\nu_n(x) \leq -C_0L\nu_n(X\setminus A_n)$.

This implies that $\nu_n(X\setminus A_n) \leq 1/C_0$.

The set A_n is indeed of measure $\geq 1 - C_0^{-1}$ and if $x \in A_n$ then:

$$\prod_{i\in[0,n-1]} d(f^i(x), I) \geq e^{-C_0Ln},$$

which is what we wanted. □

The orbit of points in A_n are not too close to I. These are the *good points* for the measure ν_n.

We now prove the upper bound of the volume.

Upper bound for the volume of $f^n(B_n(x)\cap\tau_x(Y_j(t)))$ for $x\in A_n$. – Let Y_0 denote one of the $\tau_x(Y_j(t))$ (where $Y_j(t)$ is the fiber of π_j with t in the j-th subspace of dimension $k-l$). Our aim is to prove:

PROPOSITION 2.3.2. – *The $2l$-dimensional volume of $f^n(Y_0\cap B_n(x))$ is less or equal to:*

$$C(X,l,r)^{n/m+2m}\times K^{\frac{2npl}{r}+\frac{4mpl}{r}}\times\prod_{0\le i\le n-1}d(f^i(x),I)^{\frac{-4pl}{r}}.$$

Here $C(X,l,r)$ is a constant that depends only on X, of the complex dimension l of Y_0 and the regularity r that we chose. The constants $K=K(f)$ and $p=p(f)$ are those of paragraph 2.1.

Observe that the upper bound does not depend on the fiber $Y_j(t)$ that we consider.

Before proving the proposition, we give the upper bound of the $2l$-dimensional volume of $f^n(B_n(x)\cap\tau_x(Y_j(t)))$ that follows from the proposition.

Recall that we fixed δ and C_0. Now, let r be such that $\frac{1}{r}\log K<\delta$ and $\frac{C_0L}{r}<\delta$. Then, we choose m so that $\frac{1}{m}\log(C(X,l,r))<\delta$ where $C(X,l,r)$ is the constant from the previous proposition. Reformulating the previous proposition we have that the $2l$-dimensional volume of $f^n(Y_0\cap B_n(x))$ is bounded by:

$$C(X,l,r,m,p,K)e^{\delta n}\times e^{2\delta npl}\times\prod_{0\le i\le n-1}d(f^i(x),I)^{\frac{-4pl}{r}}.$$

Finally, if x is in A_n (i.e. if x is a good point for the measure ν_n), the $2l$-dimensional volume of $f^n(Y_0\cap B_n(x))$ is bounded from above by (see Lemma 2.3.1):

$$e^{4\delta npl}e^{\frac{4plC_0Ln}{r}}\le e^{8\delta npl},$$

if n is large (independently of $x\in A_n$).

It is this upper bound that we use now to finish the upper bound of $\nu_n(B_n(x))$ for $x\in A_n$.

End of the proof of the upper bound of $\nu_n(B_n(x))$ for $x\in A_n$

Recall that we have bounded $\nu_n(B_n(x))$ by:

$$\frac{C(X)}{\lambda_l(f^n)}\sum_{j=1}^{i}\int\int_{f^n(B_n(x)\cap\tau_x(Y_j(t)))}\omega^l dt.$$

Now, if $x\in A_n$, we get:

$$\nu_n(B_n(x))\le\frac{e^{10\delta npl}}{\lambda_l(f^n)},\tag{2}$$

for n large enough which does not depend on $x\in A_n$. This quantity is approximately d_l^{-n} and it stands for $x\in A_n$ which is a set of measure $\ge 1-\frac{1}{C_0}$ for ν_n. This is the upper bound that we wanted and it will allow us to bound the entropy of μ.

It remains to prove Proposition 2.3.2, which is the purpose of rest of this section.

Proof of Proposition 2.3.2. – Consider $g = f^a$ an iterate of f and let $x \in X$. We define $g_x = \tau_{g(x)}^{-1} \circ g \circ \tau_x$. We also define $g_{x,s(x)} = h(0, \frac{1}{s(x)}) \circ g_x \circ h(0, s(x))$ where $h(0,t)$ is the homothety of center 0 and ratio t in $\mathbb{C}^k$. Here, $s(x)$ is defined by:

$$s(x) = s_a(x) = \left(\frac{d(x,I) \times \cdots \times d(f^{a-1}(x), I)}{K^a}\right)^p.$$

We will consider later the particular cases $a = 1$ (i.e. $s(x) = \eta(x)$) and $a = m$ (i.e. $s(x) = \rho(x)$).

In what follows, we are going to consider C^r-sizes associated to $2l$ (i.e. sets that will be cover by some $h([0,1]^{2l})$ with $h \in C^r$). First, we prove the following lemma by induction:

LEMMA 2.3.3. – *Let Z_0 be a set of complex dimension l such that the C^r-size of $\tau_x^{-1}(Z_0 \cap B(x, s(x)))$ is $\leq s(x)$. Let $B(x,s,j,g)$ be the dynamical ball:*

$$B(x,s,j,g) = \{y,\ d(g^i(x), g^i(y)) \leq s(g^i(x)) \text{ for } i \in [0, j-1]\}.$$

Then, for $j \geq 1$, we can cover $g^{j-1}(Z_0 \cap B(x,s,j,g))$ by a union of N_j sets Z for which the C^r-size of $\tau_{g^{j-1}(x)}^{-1}(Z)$ is $\leq s(g^{j-1}(x))$ and N_j is bounded from above by :

$$C(X,l,r)^{j-1} \prod_{0 \leq i \leq j-1} s(g^i(x))^{-2l/r}.$$

Proof. – For $j = 1$, the lemma stands by hypothesis.

Assume now that the induction assumption stands for $j-1$.

Observe that:

$$g^{j-1}(Z_0 \cap B(x,s,j,g)) = g(g^{j-2}(Z_0 \cap B(x,s,j-1,g))) \cap B(g^{j-1}(x), s(g^{j-1}(x))).$$

This is true since $B(x,s,j,g) = B(x,s,j-1,g) \cap g^{-j+1}B(g^{j-1}(x), s(g^{j-1}(x)))$ and $\varphi(A \cap \varphi^{-1}(B)) = \varphi(A) \cap B$ for any map φ and any sets A and B.

Let Z be one of the N_{j-1} sets whose union covers $g^{j-2}(Z_0 \cap B(x,s,j-1,g))$. The C^r-size of $\tau_{g^{j-2}(x)}^{-1}(Z)$ is $\leq s(g^{j-2}(x))$ by the induction assumption. To prove the lemma, we bound from above the numbers of sets Y which cover $g(Z) \cap B(g^{j-1}(x), s(g^{j-1}(x)))$ for which the C^r-size of $\tau_{g^{j-1}(x)}^{-1}(Y)$ is $\leq s(g^{j-1}(x))$.

We consider $\widetilde{Z} = h(0, 1/s(g^{j-2}(x))) \circ \tau_{g^{j-2}(x)}^{-1}(Z)$. The C^r-size of $\widetilde{Z}$ is $\leq s(g^{j-2}(x)) \times \frac{1}{s(g^{j-2}(x))} = 1$. Furthermore, since Z is in the ball $B(g^{j-2}(x), s(g^{j-2}(x)))$ (else we only consider the part of Z that is in the ball and we still denote it Z), $\widetilde{Z}$ is in the ball $B(0, C(X))$ (where $C(X)$ is a constant that depends only on X). Using Proposition

2.2.2 of Section 2.2 with $f = g_{g^{j-2}(x),s(g^{j-2}(x))}$ and $Y_0 = \widetilde{Z}$ we get that we can cover $g_{g^{j-2}(x),s(g^{j-2}(x))}(\widetilde{Z}) \cap B(0,\beta)$ (we take $\beta = C(X)\frac{s(g^{j-1}(x))}{s(g^{j-2}(x))}$) by

$$C(X,l,r)\left(1 + \frac{\|D_r g_{g^{j-2}(x),s(g^{j-2}(x))}\|}{\beta}\right)^{2l/r}$$

sets $\widetilde{Y}$ of C^r-size $\leq C(X)\frac{s(g^{j-1}(x))}{s(g^{j-2}(x))}$. Here the norm $\|.\|$ is taken over the ball $B(0, C(X)+\sqrt{2l})$. The images Y of the $\widetilde{Y}$ by $\tau_{g^{j-1}(x)} \circ h(0, s(g^{j-2}(x)))$ cover

$$\begin{aligned}&\tau_{g^{j-1}(x)} \circ h(0,s(g^{j-2}(x)))(g_{g^{j-2}(x),s(g^{j-2}(x))}(\widetilde{Z}) \cap B(0,\beta))\\ &= g(Z) \cap \tau_{g^{j-1}(x)} \circ h(0,s(g^{j-2}(x)))(B(0,\beta))\end{aligned}$$

which contains

$$g(Z) \cap B(g^{j-1}(x), s(g^{j-1}(x))).$$

This is the set we wanted to cover and $\tau^{-1}_{g^{j-1}(x)}(Y) = h(0,s(g^{j-2}(x)))(\widetilde{Y})$ is of C^r-size $\leq s(g^{j-1}(x))$ up to dividing it into $C(X)^{2l}$ pieces as in Section 2.2 (this multiplies N_j by a universal constant).

To finish the proof, we have to count the number of pieces Y that we constructed for which the C^r-size of $\tau^{-1}_{g^{j-1}(x)}(Y)$ is bounded from above by $s(g^{j-1}(x))$. Indeed, the union of those sets covers $g^{j-1}(Z_0 \cap B(x,s,j,g))$.

To control N_j, we need a control of the norm $\|D_r g_{g^{j-2}(x),s(g^{j-2}(x))}\|$ on the ball $B(0, C(X)+\sqrt{2l})$.

We admit temporarily that this norm is $\leq C(X,l,r)s(g^{j-2}(x))^{-1}$.

Then:

$$N_j \leq N_{j-1}C(X,l,r)\left(1 + \frac{\|D_r g_{g^{j-2}(x),s(g^{j-2}(x))}\| s(g^{j-2}(x))}{s(g^{j-1}(x))}\right)^{2l/r},$$

which is smaller than:

$$N_{j-1}C(X,l,r)\left(\frac{2C(X,l,r)}{s(g^{j-1}(x))}\right)^{2l/r} \leq N_{j-1}C(X,l,r)s(g^{j-1}(x))^{-2l/r}$$

up to changing $C(X,l,r)$. This concludes the proof of the lemma up to the upper bound of the norm of $\|D_r g_{g^{j-2}(x),s(g^{j-2}(x))}\|$ on the ball $B(0, C(X)+\sqrt{2l})$.

Upper bound of the norm $\|D_r g_{g^{j-2}(x),s(g^{j-2}(x))}\|$ *on* $B(0, C(X)+\sqrt{2l})$. – Since

$$g_{g^{j-2}(x),s(g^{j-2}(x))} = h(0, \frac{1}{s(g^{j-2}(x))}) \circ g_{g^{j-2}(x)} \circ h(0, s(g^{j-2}(x))),$$

$\|\partial_r g_{g^{j-2}(x),s(g^{j-2}(x))}\|$ is equal to $s(g^{j-2}(x))^{r-1}\|\partial_r g_{g^{j-2}(x)}\|$ where that last norm is taken over the ball

$$B(0, s(g^{j-2}(x))(C(X)+\sqrt{2l}))$$

(see Section 2.2 for notations).

To prove the upper bound of the norm, we are going to prove that:

$$g_{g^{j-2}(x)}(B(0,2s(g^{j-2}(x))(C(X)+\sqrt{2l})))$$

is contained in the ball $B(0,1)$. We will then deduce the upper bound of $\|\partial_r g_{g^{j-2}(x)}\|$ on $B(0,s(g^{j-2}(x))(C(X)+\sqrt{2l}))$ by $C(X,r)(s(g^{j-2}(x))(C(X)+\sqrt{2l}))^{-r}$ thanks to Cauchy inequalities. This gives exactly the upper bound that we want.

If we let $y=g^{j-2}(x)$, we have:

$$g_{g^{j-2}(x)}(B(0,2s(g^{j-2}(x))(C(X)+\sqrt{2l})))=g_y(B(0,2s(y)(C(X)+\sqrt{2l})))$$

which is equal to:

$$\tau^{-1}_{f^a(y)}\circ f^a\circ\tau_y(B(0,2s(y)(C(X)+\sqrt{2l})))$$

because $g=f^a$. Furthermore:

$$\tau^{-1}_{f^a(y)}\circ f^a\circ\tau_y=f_{f^{a-1}(y)}\circ\cdots\circ f_y,$$

with $f_x:=\tau^{-1}_{f(x)}\circ f\circ\tau_x$.

Now we use Lemma 2.1.4 of Section 2.1 to control the differential of f_y on $B(0,2s(y)(C(X)+\sqrt{2l}))$.

If z is a point of the ball $B(0,2s(y)(C(X)+\sqrt{2l}))$ then the distance between $\tau_y(z)$ and I is $\geq d(y,I)-2s(y)C(X)(C(X)+\sqrt{2l})$. But that last quantity is $\geq\frac{d(y,I)}{2}$ since by definition of $s(y)$, we have $s(y)\leq\frac{d(y,I)}{K}$ and we can assume that K is large compared to the constants that depend only on X and l (recall that l is the complex dimension of Z_0: it is between 0 and k, so in particular they are only a finite number of such quantities). Using Lemma 2.1.4, we get an upper bound of $\|Df_y\|$ on the ball $B(0,2s(y)(C(X)+\sqrt{2l}))$ by $KC(X)2^p d(y,I)^{-p}$. Using the control over the differential, this implies that the image of $B(0,2s(y)(C(X)+\sqrt{2l}))$ by f_y is contained in $B(0,KC(X)2^p d(y,I)^{-p}2s(y)(C(X)+\sqrt{2l}))$. But since:

$$s(y)=\left(\frac{d(y,I)\times\cdots\times d(f^{a-1}(y),I)}{K^a}\right)^p,$$

we have:

$$KC(X)2^p d(y,I)^{-p}2s(y)(C(X)+\sqrt{2l})\leq\left(\frac{d(f(y),I)\times\cdots\times d(f^{a-1}(y),I)}{K^{a-1}}\right)^p,$$

since we can assume that K is large compared to the $C(X)$.

So we have proved that the image of $B(0,2s(y)(C(X)+\sqrt{2l}))$ by f_y is contained in

$$B\left(0,\left(\frac{d(f(y),I)\times\cdots\times d(f^{a-1}(y),I)}{K^{a-1}}\right)^p\right).$$

Now, if we do again what we just did for $f(y)$ instead of y, we get that the image by $f_{f(y)} \circ f_y$ of the ball $B(0, 2s(y)(C(X) + \sqrt{2l}))$ is contained in the ball:

$$B\left(0, \left(\frac{d(f^2(y), I) \times \cdots \times d(f^{a-1}(y), I)}{K^{a-2}}\right)^p\right),$$

and so on. At the end, we have that the image of the ball $B(0, 2s(y)(C(X) + \sqrt{2l}))$ by $f_{f^{a-1}(y)} \circ \cdots \circ f_y = \tau^{-1}_{f^a(y)} \circ f^a \circ \tau_y$ is contained in the ball:

$$B\left(0, KC(X)2^p d(f^{a-1}(y), I)^{-p} \left(\frac{d(f^{a-1}(y), I)}{K}\right)^p\right),$$

which is contained in $B(0,1)$ for K large enough (observe that such K does not depend on a as the only requirement is $K^{-p+1}C(X)2^p \leq 1$ where $p \geq 2$).

This concludes the proof of the upper bound of the norm $\|D_r g_{g^{j-2}(x), s(g^{j-2}(x))}\|$ on the ball $B(0, C(X) + \sqrt{2l})$ and that concludes the proof of the lemma. □

Now we will use that lemma to prove Proposition 2.3.2. Recall some notations first. The set Y_0 is one the fiber $\tau_x(Y_j(t))$, $n = m\phi(n) + r(n)$ with $0 \leq r(n) < m$,

$$\rho(x) = \left(\frac{d(x, I) \times \cdots \times d(f^{m-1}(x), I)}{K^m}\right)^p$$

and

$$\eta(x) = \left(\frac{d(x, I)}{K}\right)^p.$$

Recall that:

$$B_n(x) = B(x, \rho, \phi(n), f^m) \cap f^{-\phi(n)m+m}(B(f^{\phi(n)m-m}(x), \eta, r(n) + m, f)).$$

Applying the previous lemma for $g = f^m$ (and thus $s(x) = \rho(x)$), $j = \phi(n)$ and $Z_0 = Y_0 \cap B(x, \rho(x))$ (whose image by τ_x^{-1} is of C^r-size $\leq \rho(x)$ up to dividing into $C(X)^{2l}$ pieces because $Y_j(t)$ is a linear subspace), we get that we can cover $f^{m(\phi(n)-1)}(Y_0 \cap B(x, \rho, \phi(n), g))$ by a number $N_{\phi(n)}$ of sets Z for which the C^r-size of $\tau^{-1}_{g^{\phi(n)-1}(x)}(Z) = \tau^{-1}_{f^{m(\phi(n)-1)}(x)}(Z)$ is $\leq \rho(g^{\phi(n)-1}(x))$ and $N_{\phi(n)}$ bounded from above by:

$$C(X, l, r)^{\phi(n)} \prod_{0 \leq i \leq \phi(n)-1} \rho(g^i(x))^{-2l/r}.$$

So we went up to $f^{m(\phi(n)-1)}(x)$ and we still have to go to $f^n(x)$.

For that, we use the above lemma again with for Z_0 one of the $N_{\phi(n)}$ pieces Z, $g = f$ (so now $s(x) = \eta(x)$), $j = n - m(\phi(n) - 1) = r(n) + m$ and $x = f^{m(\phi(n)-1)}(x)$. We can do that because the C^r-size of $\tau^{-1}_{f^{m(\phi(n)-1)}(x)}(Z_0)$ is $\leq \rho(f^{m(\phi(n)-1)}(x)) \leq \eta(f^{m(\phi(n)-1)}(x))$. So we get that we can cover $f^{r(n)+m-1}(Z \cap$

$B(f^{m(\phi(n)-1)}(x),\eta,r(n)+m,f))$ by a union of M sets Y for which the C^r-size of $\tau^{-1}_{f^{n-1}(x)}(Y)$ is $\leq \eta(f^{n-1}(x))$ and M is less than:

$$C(X,l,r)^{m+r(n)-1}\prod_{1\leq i\leq m+r(n)}\eta(f^{n-i}(x))^{-2l/r}.$$

The sets Y that we constructed belong to (up to keeping the part that lies in it):

$$B(f^{r(n)+m-1+m(\phi(n)-1)}(x),\eta(f^{r(n)+m-1+m(\phi(n)-1)}(x)))=B(f^{n-1}(x),\eta(f^{n-1}(x))).$$

The C^1-size of these Y is smaller than $C(X)\eta(f^{n-1}(x))$ which implies that the diameter of $h([0,1]^{2l})$ (where h is the map in C^r associated to Y) is smaller than $C(X,l)\eta(f^{n-1}(x))$. So, the set $h([0,1]^{2l})$ is contained in

$$B\left(f^{n-1}(x),\frac{d(f^{n-1}(x),I)}{2}\right).$$

Since the differential of f in this last ball is bounded by

$$K2^pd(f^{n-1}(x),I)^{-p}$$

using Lemma 2.1.4, one gets that the images by f of those Y are of C^1-size bounded by $C(X)\eta(f^{n-1}(x))K2^pd(f^{n-1}(x),I)^{-p}$. So their $2l$-dimensional volume is ≤ 1.

Summing up, we have covered

$$f^{r(n)+m}(f^{m(\phi(n)-1)}(Y_0\cap B(x,\rho,\phi(n),g))\cap B(f^{m(\phi(n)-1)}(x),\eta,r(n)+m,f))$$

which contains $f^n(B_n(x)\cap Y_0)$ by a number N of sets Y of volume ≤ 1 with:

$$N\leq C(X,l,r)^{\phi(n)+2m}\prod_{0\leq i\leq\phi(n)-1}\rho(g^i(x))^{-2l/r}\prod_{1\leq i\leq m+r(n)}\eta(f^{n-i}(x))^{-2l/r}.$$

Using now the fact that:

$$\rho(y)=\left(\frac{d(y,I)\times\cdots\times d(f^{m-1}(y),I)}{K^m}\right)^p,$$

and

$$\eta(y)=\left(\frac{d(y,I)}{K}\right)^p,$$

we have:

$$\prod_{0\leq i\leq\phi(n)-1}\rho(g^i(x))^{-2l/r}\leq K^{\frac{2m\phi(n)pl}{r}}\prod_{0\leq i\leq\phi(n)m-1}d(f^i(x),I)^{\frac{-2pl}{r}},$$

and

$$\prod_{1\leq i\leq m+r(n)}\eta(f^{n-i}(x))^{-2l/r}\leq K^{\frac{4mpl}{r}}\prod_{1\leq i\leq m+r(n)}d(f^{n-i}(x),I)^{\frac{-2pl}{r}}.$$

Finally, we have covered $f^n(B_n(x)\cap Y_0)$ by a number N of sets Y of volume ≤ 1 with:

$$N\leq C(X,l,r)^{n/m+2m}K^{\frac{2npl}{r}+\frac{4mpl}{r}}\prod_{0\leq i\leq n-1}d(f^i(x),I)^{\frac{-4pl}{r}}.$$

That concludes the proof of Proposition 2.3.2. □

2.3.4. Lower bound for the entropy of μ. – Recall that we consider a cluster value μ of the sequence

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f^i_* \left(\frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)} \right)$$

and that in order to simplify the notations we assume that (μ_n) converges to μ. By assumption, μ gives no mass to the indeterminacy set I and it is invariant by Lemma 2.1.3. The aim of this section is to prove that the metric entropy $h_\mu(f)$ is $\geq \log d_l - \delta$. This implies Theorem 1 by letting $\delta \to 0$.

So we have to bound $h_\mu(f)$. Here is the plan of this section: first we will construct partitions of finite entropy for μ that will be used latter with the proof of the variational principle to get the lower bound of the entropy that we want.

2.3.4.1. Construction of the partitions. – The proof is the one of Mañé (see Lemma 2 in [**47**]). We give his proof since we will use it in what follows. We consider a function $s(x)$ comprised between 0 and 1. Later, we will take the values $\rho(x)$ or $\eta(x)$ for $s(x)$.

PROPOSITION 2.3.4. – *(Mañé)*
We can construct a countable partition $\mathscr{P}$ of $X \setminus \{s = 0\}$ such that:

1. *If $x \in X \setminus \{s = 0\}$, then diam $\mathscr{P}(x) < s(y)$ for all $y \in \mathscr{P}(x)$ (here $\mathscr{P}(x)$ denotes the atom of the partition that contains x).*
2. *For any probability measure ν such that $\int \log s(x) d\nu(x) > -\infty$, we have $H_\nu(\mathscr{P}) < +\infty$. Here $H_\nu(\mathscr{P})$ denotes the entropy of the partition $\mathscr{P}$ for the measure ν.*

Before proving the proposition, we give a multi-index version of Mañé's lemma (see Lemma 1 in [**47**]). We thank the referee for explaining to us this lemma which simplifies the proof of Proposition 2.3.8.

LEMMA 2.3.5. – *For all $q \in \mathbb{N}^*$, there exists a constant $C(q)$ such that:*

For all family (x_s) of real numbers $0 \leq x_s \leq 1$ indexed by $s = (s_0, \dots, s_{q-1}) \in \mathbb{N}^q$ and for all $A \in \mathbb{N}$ we have:

$$\sum_{|s|\geq A} x_s \log(1/x_s) \leq \sum_{|s|\geq A} |s| x_s + C(q) e^{-\frac{A}{2q}}$$

with the convention that $x_s \log(1/x_s) = 0$ when $x_s = 0$ and $|s| = s_0 + \cdots + s_{q-1}$.

Proof of Lemma 2.3.5. – The proof is the same that Mañé's one.

Let $\mathcal{S}$ be the set of multi-indexes $s \in \mathbb{N}^q$ for which $x_s \neq 0$ and $\log(1/x_s) \leq |s|$. If $s \notin \mathcal{S}$ then $x_s \leq e^{-|s|}$. Furthermore:

$$\sum_{|s|\geq A} x_s \log(1/x_s) \leq \sum_{|s|\geq A,\ s\in\mathcal{S}} |s|x_s + \sum_{|s|\geq A,\ s\notin\mathcal{S}} (\sqrt{x_s})(\sqrt{x_s}) \log(1/x_s).$$

But since $(\sqrt{t}) \log(1/t) \leq 2e^{-1}$ for all $t \geq 0$, we have:

$$\sum_{|s|\geq A} x_s \log(1/x_s) \leq \sum_{|s|\geq A} |s|x_s + 2e^{-1} \sum_{|s|\geq A,\ s\notin\mathcal{S}} \sqrt{x_s}$$

which is less than:

$$\sum_{|s|\geq A} |s|x_s + 2e^{-1} \sum_{|s|\geq A} e^{-|s|/2}.$$

Now, $\{s,\ |s| \geq A\} \subset \cup_{i=0}^{q-1}\{s,\ s_i \geq A/q\}$, so $\sum_{|s|\geq A} e^{-|s|/2}$ is less than

$$\sum_{i=0}^{q-1} \sum_{\{s,\ s_i\geq A/q\}} e^{-s_0/2-\cdots-s_{q-1}/2} \leq q\left(\frac{1}{1-e^{-1/2}}\right)^q e^{-\frac{A}{2q}}.$$

That gives the lemma. □

Proof of Proposition 2.3.4. – Here is Mañé's proof.

First of all, there are constants $C > 0$ and $r_0 > 0$ such that for $0 < r \leq r_0$, there exists a partition $\mathcal{P}_r$ of X whose elements have a diameter $\leq r$ and such that the number of elements of the partition $|\mathcal{P}_r|$ is $\leq C(1/r)^{2k}$.

Now, we define $V_n := \{x,\ e^{-(n+1)} < s(x) \leq e^{-n}\}$ for $n \geq 0$. Since the function s is less than 1, we have that $X \setminus \{s = 0\} = \cup_{n\geq 0} V_n$.

Let $\mathcal{P}$ be the partition defined as follows: for n fixed, we consider the sets $Q \cap V_n$ for $Q \in \mathcal{P}_{r_n}$ with $r_n = e^{-(n+1)}$. This defines a partition of V_n. Now, we get the partition $\mathcal{P}$ of $X \setminus \{s = 0\}$ by taking all the n between 0 and $+\infty$.

If $x \notin \{s = 0\}$, then $x \in V_n$ for some $n \geq 0$ and then the atom $\mathcal{P}(x)$ of $\mathcal{P}$ containing x, is contained in an atom of $\mathcal{P}_{r_n}$, so we have:

$$\mathrm{diam}\mathcal{P}(x) \leq e^{-(n+1)} < s(y)$$

for all $y \in \mathcal{P}(x) \subset V_n$. This proves the first point of Proposition 2.3.4.

We now consider a measure ν such that $\int \log s(x) d\nu(x) > -\infty$. We want to show that $H_\nu(\mathcal{P}) < +\infty$. We have:

$$H_\nu(\mathcal{P}) = \sum_{n=0}^{+\infty} \left(- \sum_{P\in\mathcal{P},\ P\subset V_n} \nu(P) \log \nu(P) \right).$$

Using the inequality:

$$-\sum_{i=1}^{m_0} x_i \log x_i \leq \left(\sum_{i=1}^{m_0} x_i\right)\left(\log m_0 - \log \sum_{i=1}^{m_0} x_i\right)$$

which comes from the convexity of the function $\phi(x) = x\log(x)$ for $x \geq 0$, we get:

$$H_\nu(\mathscr{P}) \leq \sum_{n=0}^{+\infty} \nu(V_n)(\log|\mathscr{P}_{r_n}| - \log \nu(V_n)).$$

Since the number $|\mathscr{P}_{r_n}|$ of elements of $\mathscr{P}_{r_n}$ is less than $Ce^{2k(n+1)}$, we have:

$$H_\nu(\mathscr{P}) \leq \log C + 2k\sum_{n=0}^{+\infty}(n+1)\nu(V_n) + \sum_{n=0}^{+\infty}\nu(V_n)\log\left(\frac{1}{\nu(V_n)}\right).$$

By assumption:

$$\int \log s(x)d\nu(x) = \int_{\cup_{n\geq 0}V_n} \log s(x)d\nu(x) > -\infty.$$

This implies that:

$$\sum_{n=0}^{+\infty} n\nu(V_n) < +\infty,$$

and the proposition is then deduced from Lemma 2.3.5 with $q = 1$ and $A = 0$. □

2.3.4.2. Lower bound for the entropy of μ. – In what follows, we denote $\mathscr{P}$ (resp. $\mathscr{Q}$) the partition previously constructed for $s(x) = \rho(x)$ (resp. $s(x) = \eta(x)$). Notice that $H_\mu(\mathscr{P})$ and $H_\mu(\mathscr{Q})$ are finite by using the previous Proposition since $\log d(x, I) \in L^1(\mu)$ and μ is invariant. We consider the restriction of $\mathscr{P}$ and $\mathscr{Q}$ to $\Omega = X \setminus \cup_{i\geq 0} f^{-i}(I)$ (that we still denote $\mathscr{P}$ and $\mathscr{Q}$). They are partitions of Ω. The advantage of those partitions over Ω is that the f^i are well-defined on them. In particular, we can define for example the partition $f^{-i}(\mathscr{P})$: its atoms are the $f^{-i}(P) := \{x \in \Omega \text{ with } f^i(x) \in P\}$ where the P are the atoms of $\mathscr{P}$. Since $f(\Omega) \subset \Omega$, we get a partition of Ω. The measures that we consider (ν_n, μ_n or μ) have mass 1 on Ω. The parts of X that we drop are of mass 0 for them. We remark that with our convention, we have: $f^{-a}(f^{-b}(P)) = f^{-a-b}(P) = \{x \in \Omega \text{ with } f^{a+b}(x) \in P\}$. Recall that we denote:

$$\nu_n = \frac{(f^n)^*\omega^l \wedge \omega^{k-l}}{\lambda_l(f^n)}$$

and that $\nu_n(A_n) \geq 1 - \frac{1}{C_0}$ (see Lemma 2.3.1).

In what follows, we denote $\nu'_n := \frac{\nu_{n|A_n}}{\nu_n(A_n)}$ (i.e. $\nu'_n(B) = \frac{\nu_n(B\cap A_n)}{\nu_n(A_n)}$).

Define the joint partition $\mathscr{P}_{-n}$ of the partitions $\mathscr{P}$ and $\mathscr{Q}$ by (recall that $n = \phi(n)m + r(n)$ with $0 \leq r(n) < m$):

$$\mathscr{P}_{-n} := \mathscr{P} \vee f^{-1}(\mathscr{P}) \vee \cdots \vee f^{-\phi(n)m+m}(\mathscr{P}) \vee f^{-\phi(n)m+m-1}(\mathscr{Q}) \vee \cdots \vee f^{-n+1}(\mathscr{Q}).$$

First, we have the lemma:

Lemma 2.3.6. – *If n is large enough, then*

$$\nu_n'(\mathcal{P}_{-n}(x)) \leq \frac{e^{10\delta nlp}}{\lambda_l(f^n)} \frac{1}{1-\frac{1}{C_0}}.$$

for every atom $\mathcal{P}_{-n}(x)$ of $\mathcal{P}_{-n}$.

Proof. – Recall that:

$$B_n(x) = B(x,\rho,\phi(n),f^m) \cap f^{-\phi(n)m+m}(B(f^{\phi(n)m-m}(x),\eta,r(n)+m,f)).$$

We have shown in the previous paragraph that if n is large enough then for every $x \in A_n$ (see (2)):

$$\nu_n(B_n(x)) \leq \frac{e^{10\delta npl}}{\lambda_l(f^n)}.$$

Let n be large enough so that the previous property is satisfied. If $\mathcal{P}_{-n}(x)$ does not contain any points of A_n then $\nu_n'(\mathcal{P}_{-n}(x)) = 0$ and the lemma is true. So we can assume that there exists $y \in \mathcal{P}_{-n}(x) \cap A_n$.

By definition of the joint partition, we have $\mathcal{P}_{-n}(x)$ which is equal to:

$$\mathcal{P}(x) \cap \cdots \cap f^{-\phi(n)m+m}(\mathcal{P}(f^{\phi(n)m-m}(x)))$$
$$\cap f^{-\phi(n)m+m-1}(\mathcal{Q}(f^{\phi(n)m-m+1}(x))) \cap \cdots \cap f^{-n+1}(\mathcal{Q}(f^{n-1}(x))).$$

In particular, $f^i(y) \in \mathcal{P}(f^i(x))$ for $i = 0 \dots \phi(n)m-m$ and then $f^i(y) \in \mathcal{Q}(f^i(x))$ for $i = \phi(n)m-m+1 \dots n-1$. By Proposition 2.3.4, the diameter of $\mathcal{P}(f^i(x))$ is $\leq \rho(f^i(y))$ for $i = 0 \dots \phi(n)m-m$ and the diameter of $\mathcal{Q}(f^i(x))$ is $\leq \eta(f^i(y))$ for $i = \phi(n)m-m+1 \dots n-1$ which means:

$$\mathcal{P}_{-n}(x) \subset B_n(y).$$

The lemma follows then first from the estimation of the previous paragraph since $y \in A_n$ and secondly from the fact that $\nu_n(A_n)$ is $\geq 1-\frac{1}{C_0}$. □

Thanks to this estimation on $\nu_n'(\mathcal{P}_{-n}(x))$, we can bound the entropy of μ using a variation of the proof of the variational principle. We refer the reader to [**54**] p.188-190 for the proof of the principle and to [**4**], [**10**] or [**42**] for its use to bound from below the entropies of measures in holomorphic or meromorphic dynamics.

Let q be an integer $2m < q < n$ (with m from the above paragraph). For $0 \leq j \leq q-1$, we let $a(j) = \left[\frac{n-j}{q}\right]$ and then

$$\{0,1,\dots,n-1\} = \{j+rq+i,\ 0 \leq r \leq a(j)-2,\ 0 \leq i \leq q-1\} \cup S(j)$$

where $S(j) = \{0,1,\dots,j-1,j+(a(j)-1)q,j+(a(j)-1)q+1,\dots,n-1\}$ is of cardinality less than $3q$ since $j+(a(j)-1)q \geq j+\left(\frac{n-j}{q}-2\right)q = n-2q$. We took the indexes r up to $a(j)-2$ so that $S(j)$ contains $n-q \dots n-1$ and so in particular $\phi(n)m-m+1 \dots n-1$

(we take q large with respect to m). We denote $S_1(j)$ the elements of $S(j)$ other than $\phi(n)m-m+1\dots n-1$ and $S_2(j)$ the elements $\phi(n)m-m+1\dots n-1$.

Now, we have (see for example Proposition 4.3.3 of [**45**]):

$$H_{\nu'_n}(\mathscr{P}_{-n}) \geq -\log(\sup_{P\in\mathscr{P}_{-n}} \nu'_n(P)) \geq -10\delta nlp + \log\lambda_l(f^n) + \log\left(1-\frac{1}{C_0}\right),$$

by the previous lemma.

On the other hand, by the proof of the variational principle for $0\leq j\leq q-1$, we have:

$$\mathscr{P}_{-n} = \bigvee_{r=0}^{a(j)-2}\left(f^{-(rq+j)}\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}\right)\vee\bigvee_{t\in S_1(j)} f^{-t}\mathscr{P}\vee\bigvee_{t\in S_2(j)} f^{-t}\mathcal{Q}.$$

So, (again by Proposition 4.3.3 in [**45**]):

$$H_{\nu'_n}(\mathscr{P}_{-n}) \leq \sum_{r=0}^{a(j)-2} H_{\nu'_n}(f^{-(rq+j)}\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}) + \sum_{t\in S_1(j)} H_{\nu'_n}(f^{-t}\mathscr{P}) + \sum_{t\in S_2(j)} H_{\nu'_n}(f^{-t}\mathcal{Q})$$

which is equal to:

$$\sum_{r=0}^{a(j)-2} H_{f_*^{rq+j}\nu'_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}) + \sum_{t\in S_1(j)} H_{\nu'_n}(f^{-t}\mathscr{P}) + \sum_{t\in S_2(j)} H_{\nu'_n}(f^{-t}\mathcal{Q})$$

Summing this relation for $j=0\dots q-1$, we get:

$$q\left(-10\delta nlp + \log\lambda_l(f^n) + \log\left(1-\frac{1}{C_0}\right)\right)$$
$$\leq \sum_{j=0}^{q-1}\sum_{r=0}^{a(j)-2} H_{f_*^{rq+j}\nu'_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}) + \sum_{j=0}^{q-1}\left(\sum_{t\in S_1(j)} H_{\nu'_n}(f^{-t}\mathscr{P}) + \sum_{t\in S_2(j)} H_{\nu'_n}(f^{-t}\mathcal{Q})\right).$$

The integers $j+rq$ for $0\leq j\leq q-1$ and $0\leq r\leq a(j)-2$ are all distinct and $\leq n-2q$. So we have that (using the convexity of the function $\Phi(x)=x\log(x)$ for $x>0$):

$$\text{(I): } \frac{q}{n-2q+1}\left(-10\delta nlp + \log\lambda_l(f^n) + \log\left(1-\frac{1}{C_0}\right)\right)$$

which is less than:

$$H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu'_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}) + \sum_{j=0}^{q-1}\left(\sum_{t\in S_1(j)}\frac{H_{\nu'_n}(f^{-t}\mathscr{P})}{n-2q+1} + \sum_{t\in S_2(j)}\frac{H_{\nu'_n}(f^{-t}\mathcal{Q})}{n-2q+1}\right).$$

Here is the plan of the rest of the proof. In a first time, we deduce from that inequality a lower bound of $\frac{1}{q}H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$. Then we will pass to the limit in that inequality.

1) Lower bound of $\frac{1}{q}H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$. – By definition, $\nu_n' := \frac{\nu_{n|A_n}}{\nu_n(A_n)}$. In particular, $\nu_n' \leq \frac{\nu_n}{1-\frac{1}{C_0}}$ and

$$\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n' \leq \frac{1}{(1-\frac{1}{C_0})(n-2q+1)}\sum_{p=0}^{n-2q} f_*^p\nu_n.$$

In order to control $\frac{1}{q}H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$ with the inequality (I), we are going to use the following lemma:

Lemma 2.3.7. – *Let ν and ν' be two probabilities such that $\nu' \leq \beta\nu$ for some $\beta > 1$. Then for any partition Q, we have:*

$$H_{\nu'}(\mathscr{Q}) \leq \beta(H_\nu(\mathscr{Q}) + 1).$$

We assume temporarily this lemma and we continue the proof.

Now, since:

$$\nu_n' \leq \beta\nu_n$$

with $\beta = \frac{1}{1-\frac{1}{C_0}}$, the previous lemma gives that (I) is less than

$$\frac{1}{1-\frac{1}{C_0}}[H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}) + 1$$
$$+\sum_{j=0}^{q-1}\left(\sum_{t\in S_1(j)}\frac{H_{\nu_n}(f^{-t}\mathscr{P})}{n-2q+1} + \sum_{t\in S_2(j)}\frac{H_{\nu_n}(f^{-t}\mathscr{Q})}{n-2q+1}\right) + \frac{3q^2}{n-2q+1}]$$

(since the cardinality of $S(j)$ is $\leq 3q$).

This implies a lower bound of $\frac{1}{q}H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p\nu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$ by

$$\left(1-\frac{1}{C_0}\right)\left(\frac{1}{n-2q+1}\left(-10\delta nlp + \log\lambda_l(f^n) + \log\left(1-\frac{1}{C_0}\right)\right)\right)$$
$$-\frac{1}{q}-\frac{1}{q}\left(\sum_{j=0}^{q-1}\sum_{t\in S_1(j)}\frac{H_{\nu_n}(f^{-t}\mathscr{P})}{n-2q+1} + \sum_{j=0}^{q-1}\sum_{t\in S_2(j)}\frac{H_{\nu_n}(f^{-t}\mathscr{Q})}{n-2q+1}\right) - \frac{3q}{n-2q+1}.$$

It remains now to take the limit of that inequality when n goes to ∞.

2) Pass to the limit $n \to +\infty$. – First:

$$\frac{1}{n-2q+1}\left(-10\delta nlp + \log\lambda_l(f^n) + \log\left(1-\frac{1}{C_0}\right)\right)$$

goes to $-10\delta lp + \log d_l$ when $n \to \infty$. Now, we need the following proposition.

Proposition 2.3.8. – *We have:*

1. *For all* $q > 2m$,

$$H_{\frac{1}{n-2q+1}\sum_{p=0}^{n-2q} f_*^p \nu_n}\Big(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}\Big)$$

converges to $H_\mu(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$ *when* $n \to \infty$.

2. *For* $q > 2m$*:*

$$\frac{1}{q}\sum_{j=0}^{q-1}\sum_{t\in S_1(j)} \frac{H_{\nu_n}(f^{-t}\mathscr{P})}{n-2q+1}$$

converges to 0 *when* $n \to \infty$.

3. *For* $q > 2m$*:*

$$\frac{1}{q}\sum_{j=0}^{q-1}\sum_{t\in S_2(j)} \frac{H_{\nu_n}(f^{-t}\mathscr{Q})}{n-2q+1}$$

converges to 0 *when* $n \to \infty$.

End of the proof of Theorem 1. – We assume temporarily that the proposition is true. We finish the lower bound of the entropy of μ.

If we pass to the limit in the inequality of the previous paragraph, we get:

$$\frac{1}{q}H_\mu\Big(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}\Big) \geq \left(1-\frac{1}{C_0}\right)(-10\delta lp + \log d_l) - \frac{1}{q}.$$

If we let q go to ∞, we have:

$$h_\mu(f) \geq \left(1-\frac{1}{C_0}\right)(-10\delta lp + \log d_l)$$

because the entropy of $\mathscr{P}$ is finite for μ.

This proves the theorem by letting C_0 go to ∞ then by letting δ go to 0.

Up to the proof of Lemma 2.3.7 and Proposition 2.3.8, we have proved Theorem 1.

Proof of Lemma 2.3.7. – The function $\Phi(x) = -x\log(x)$ is increasing on $[0, e^{-1}]$ and decreasing on $[e^{-1}, 1]$. So we have:

$$\begin{aligned} H_{\nu'}(\mathscr{Q}) &= \sum_{Q\in\mathscr{Q}} -\nu'(Q)\log\nu'(Q) \\ &= \sum_{Q\in\mathscr{Q},\ \nu(Q)\leq\frac{e^{-1}}{\beta}} -\nu'(Q)\log\nu'(Q) + \sum_{Q\in\mathscr{Q},\ \nu(Q)>\frac{e^{-1}}{\beta}} -\nu'(Q)\log\nu'(Q) \end{aligned}$$

which is less than:

$$\sum_{Q\in\mathscr{Q},\ \nu(Q)\leq\frac{e^{-1}}{\beta}} -\beta\nu(Q)\log(\beta\nu(Q)) + \sum_{Q\in\mathscr{Q},\ \nu(Q)>\frac{e^{-1}}{\beta}} -\nu'(Q)\log\nu'(Q).$$

Since they are at most $\frac{\beta}{e^{-1}}$ of $Q \in \mathcal{Q}$ with $\nu(Q) > \frac{e^{-1}}{\beta}$ and because on the interval $[0,1]$, the function $\Phi(x)$ is non negative and bounded by e^{-1}, we have:

$$H_{\nu'}(\mathcal{Q}) \leq \beta H_\nu(\mathcal{Q}) + \beta. \qquad \square$$

Proof of Proposition 2.3.8. – In order to simplify the notations, we denote:

$$\mu'_n = \frac{1}{n-2q+1} \sum_{p=0}^{n-2q} f_*^p \nu_n.$$

For the proof of the three points of the proposition, we will use the following lemma:

LEMMA 2.3.9. – *For $i, j = 0 \dots q-1$, we have*

$$0 \geq \int_{\{\rho \circ f^i \leq \epsilon\}} \log \rho \circ f^j \; d\mu'_n \geq -\delta(\epsilon)$$

if n is large enough. Here $\delta(\epsilon)$ goes to 0 when ϵ goes to 0.

Moreover, we have the same lemma with μ instead of μ'_n.

Proof. – We give the proof for μ'_n. It is the same for μ.

First step. – We show first that for $j = 0 \dots 2q-1$, we have

$$0 \geq \int_{\{d(f^j(x),I) \leq \epsilon\}} \log d(f^j(x), I) \; d\mu'_n(x) \geq -\delta'(\epsilon)$$

for n large enough. Here $\delta'(\epsilon)$ goes to 0 when ϵ goes to 0.

The last integral is equal to $A = \int_{\{d(x,I)\leq\epsilon\}} \log d(x,I) \; d((f^j)_*\mu'_n)(x)$. But $(f^j)_*\mu'_n$ is smaller that $\frac{n}{n-2q+1}\mu_n$ for $j = 0 \dots 2q-1$. So, we have

$$A \geq \frac{n}{n-2q+1} \int_{\{d(x,I)\leq\epsilon\}} \log d(x,I) \; d\mu_n(x).$$

Now, $\int_{\{d(x,I)\leq\epsilon\}} \log d(x,I) \; d\mu_n(x)$ converges to $\int_{\{d(x,I)\leq\epsilon\}} \log d(x,I) \; d\mu(x)$. Indeed, on one hand we have that:

$$\int \log d(x,I) \; d\mu_n(x)$$

converges to $\int \log d(x,I) \; d\mu(x)$ by Hypothesis (H). On the other hand:

$$\int_{\{d(x,I)>\epsilon\}} \log d(x,I) \; d\mu_n(x)$$

converges to $\int_{\{d(x,I)>\epsilon\}} \log d(x,I) \; d\mu(x)$ up to choosing ϵ generic so that μ gives no mass to $\{x,\, d(x,I) = \epsilon\}$.

Finally, since $\int_{\{d(x,I)\leq\epsilon\}} \log d(x,I) \; d\mu(x)$ goes to 0 when ϵ converges to 0 by dominated convergence, the first step follows.

Second step. – Here, we prove that for $i, j = 0 \dots 2q-1$, we have

$$0 \geq \int_{\{d(f^i(x),I)\leq\epsilon\}} \log d(f^j(x), I)\ d\mu'_n(x) \geq -\delta''(\epsilon)$$

for n large enough. Here $\delta''(\epsilon)$ goes to 0 when ϵ goes to 0.

For that, we split this integral into two parts:

$$\int_{\{d(f^i(x),I)\leq\epsilon\}\cap\{d(f^j(x),I)\leq\delta'(\epsilon)\}} \log d(f^j(x), I)\ d\mu'_n(x)$$
$$+ \int_{\{d(f^i(x),I)\leq\epsilon\}\cap\{d(f^j(x),I)>\delta'(\epsilon)\}} \log d(f^j(x), I)\ d\mu'_n(x).$$

The first term is larger than

$$\int_{\{d(f^j(x),I)\leq\delta'(\epsilon)\}} \log d(f^j(x), I)\ d\mu'_n(x)$$

and this term is bounded by below by $-\delta'(\delta'(\epsilon))$ if n is large enough by using the first step. That quantity goes to 0 when ϵ goes to 0.

The second term is larger than (if $\epsilon \leq e^{-1}$ and $\delta'(\epsilon) \leq 1$)

$$\int_{\{d(f^i(x),I)\leq\epsilon\}} \log \delta'(\epsilon)\ d\mu'_n(x)$$
$$\geq -\log \delta'(\epsilon) \int_{\{d(f^i(x),I)\leq\epsilon\}} \log d(f^i(x), I)\ d\mu'_n(x) \geq \delta'(\epsilon) \log \delta'(\epsilon)$$

if n is large enough by using the first step. That quantity goes to 0 when ϵ goes to 0.

Third step. – Now, we prove the lemma. Recall that:

$$\rho(x) = \left(\frac{d(x, I) \times \cdots \times d(f^{m-1}(x), I)}{K^m}\right)^p.$$

In particular, we have:

$$\{x,\ \rho \circ f^i(x) \leq \epsilon\} \subset \{x,\ d(f^i(x), I) \leq \epsilon^{\frac{1}{mp}} K\} \cup \cdots \cup \{x,\ d(f^{i+m-1}(x), I) \leq \epsilon^{\frac{1}{mp}} K\}.$$

So, for $i, j = 0 \dots q-1$, we have

$$0 \geq \int_{\{\rho\circ f^i\leq\epsilon\}} \log \rho \circ f^j\ d\mu'_n$$

which is larger than

$$\sum_{l=0}^{m-1} \int_{\{d(f^{l+i}(x),I)\leq\epsilon^{\frac{1}{mp}} K\}} \log \rho \circ f^j\ d\mu'_n.$$

Now, for l between 0 and $m-1$,

$$\int_{\{d(f^{l+i}(x),I)\leq\epsilon^{\frac{1}{mp}}K\}}\log\rho\circ f^j\ d\mu'_n =$$
$$\sum_{s=0}^{m-1}\Big(p\int_{\{d(f^{l+i}(x),I)\leq\epsilon^{\frac{1}{mp}}K\}}\log d(f^{s+j}(x),I)d\mu'_n(x)$$
$$-p\log K\int_{\{d(f^{l+i}(x),I)\leq\epsilon^{\frac{1}{mp}}K\}}d\mu'_n(x)\Big)$$
$$\geq -mp\delta''(\epsilon^{\frac{1}{mp}}K)+mp\log K\int_{\{d(f^{l+i}(x),I)\leq\epsilon^{\frac{1}{mp}}K\}}\log d(f^{l+i}(x),I)d\mu'_n(x)$$

for ϵ small enough so that $\epsilon^{\frac{1}{mp}}K\leq e^{-1}$ and n large enough by using the second step (recall that $m<q$).

Finally this term is larger than $-2mp\log K\delta''(\epsilon^{\frac{1}{mp}}K)$ for n large enough and it was the inequality that we were looking for. □

First point of the proposition 2.3.8. – We thank the referee for the simplification of the proof of this point.

We recall that in this paragraph, we consider the partition $\mathscr{P}$ which is constructed as in Proposition 2.3.4 with $s(x)=\rho(x)$. In particular, we have $V_j:=\{x,\ e^{-(j+1)}<\rho(x)\leq e^{-j}\}$ for $j\in\mathbb{N}$.

For a multi-index $s=(s_0,\dots,s_{q-1})\in\mathbb{N}^q$, we denote by $W_s=V_{s_0}\cap\cdots\cap f^{-q+1}V_{s_{q-1}}$. When $P\in\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P}$, then P is in some W_s for some multi-index s. Moreover, the number of $P\in\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P}$ which are in W_s is bounded from above by $C^qe^{2kq}e^{2k|s|}=C_0^qe^{2k|s|}$ with $|s|=s_0+\cdots+s_{q-1}$.

Now, for $A\in\mathbb{N}^*$, we have

$$H_{\mu'_n}(\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P})=\sum_{s\in\mathbb{N}^q}\ \sum_{P\in\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P},\ P\subset W_s}-\mu'_n(P)\log\mu'_n(P)$$

that we divide as:

$$\sum_{|s|\geq A}\ \sum_{P\in\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P},\ P\subset W_s}-\mu'_n(P)\log\mu'_n(P)+\sum_{|s|<A}\ \sum_{P\in\bigvee_{i=0}^{q-1}f^{-i}\mathscr{P},\ P\subset W_s}-\mu'_n(P)\log\mu'_n(P).$$

We begin with the first term. We use the notations and the ideas of the proof of Proposition 2.3.4.

$$\begin{aligned}
R_1(A) &= \sum_{|s|\geq A} \sum_{P\in\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P},\ P\subset W_s} -\mu'_n(P)\log\mu'_n(P)\\
&\leq \sum_{|s|\geq A} \mu'_n(W_s)(\log(\#\{P\in \bigvee_{i=0}^{q-1} f^{-i}\mathscr{P}\ ,\ P\subset W_s\}) - \log\mu'_n(W_s))\\
&\leq \log C_0^q \sum_{|s|\geq A}\mu'_n(W_s) + 2k\sum_{|s|\geq A}|s|\mu'_n(W_s) + \sum_{|s|\geq A}\mu'_n(W_s)\log\frac{1}{\mu'_n(W_s)}.
\end{aligned}$$

Now, by using Lemma 2.3.5, this is less than

$$C(q)e^{-\frac{A}{2q}} + (\log C_0^q + 2k + 1)\sum_{|s|\geq A}|s|\mu'_n(W_s).$$

On W_s, we have $-\sum_{j=0}^{q-1}\log\rho\circ f^j \geq |s|$. Moreover, the W_s are pairwise disjoints and

$$\cup_{|s|\geq A} W_s \subset \cup_{i=0}^{q-1}\cup_{\{s\in\mathbb{N}^q\ ,\ s_i\geq\frac{A}{q}\}} W_s \subset \cup_{i=0}^{q-1}\{\rho\circ f^i \leq e^{-\frac{A}{q}}\},$$

so,

$$\begin{aligned}
R_1(A) &\leq C(q)e^{-\frac{A}{2q}} + (\log C_0^q+2k+1)\sum_{j=0}^{q-1}\sum_{|s|\geq A}\int_{W_s} -\log\rho\circ f^j(x)d\mu'_n(x)\\
&\leq C(q)e^{-\frac{A}{2q}} + (\log C_0^q+2k+1)\sum_{j=0}^{q-1}\sum_{i=0}^{q-1}\int_{\{\rho\circ f^i(x)\leq e^{-\frac{A}{q}}\}} -\log\rho\circ f^j(x)d\mu'_n(x)
\end{aligned}$$

which is as small that we want if we take A large and then n large enough by using the previous lemma. The same thing is true if we replace μ'_n by μ in $R_1(A)$: it is as small as we want if we take A large enough by using the previous lemma.

We now consider the second term

$$\sum_{|s|<A}\sum_{P\in\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P},\ P\subset W_s} -\mu'_n(P)\log\mu'_n(P).$$

Up to moving slightly the boundaries of the partition $\mathscr{P}$, we can assume that μ gives no mass to the boundary of its elements. In particular, the above term converges to:

$$\sum_{|s|<A}\sum_{P\in\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P},\ P\subset W_s} -\mu(P)\log\mu(P),$$

when n goes to infinity since we only consider a finite number of elements and μ'_n converges to μ.

In particular, this implies that $H_{\mu'_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$ converges to $H_\mu(\bigvee_{i=0}^{q-1} f^{-i}\mathscr{P})$.

Second point of Proposition 2.3.8. – We show that:

$$\frac{1}{q}\sum_{j=0}^{q-1}\sum_{t\in S_1(j)}\frac{H_{\nu_n}(f^{-t}\mathscr{P})}{n-2q+1}$$

converges to 0 when n goes to ∞.

First, we consider $H_{\nu_n}(f^{-t}\mathscr{P})$. We have

$$H_{\nu_n}(f^{-t}\mathscr{P}) = \sum_{s=0}^{+\infty} - \sum_{P\in\mathscr{P},\ P\subset V_s}((f^t)_*\nu_n)(P)\log((f^t)_*\nu_n)(P)$$

which is less than (see the proof of Proposition 2.3.4)

$$\begin{aligned}
&\sum_{s=0}^{+\infty}((f^t)_*\nu_n)(V_s)(\log|\mathscr{P}_{r_s}| - \log((f^t)_*\nu_n)(V_s))\\
&\leq \log C + 2k\sum_{s=0}^{+\infty}(s+1)((f^t)_*\nu_n)(V_s) + \sum_{s=0}^{+\infty}((f^t)_*\nu_n)(V_s)\log\frac{1}{((f^t)_*\nu_n)(V_s)}\\
&\leq \log C + 2k + (2k+1)\sum_{s=0}^{+\infty}s((f^t)_*\nu_n)(V_s) + C(1)\\
&= \log C + 2k + (2k+1)\sum_{s=0}^{s_0-1}s((f^t)_*\nu_n)(V_s) + (2k+1)\sum_{s=s_0}^{+\infty}s((f^t)_*\nu_n)(V_s) + C(1).
\end{aligned}$$

The last inequality comes from Lemma 2.3.5 with $q=1$ and $A=0$.

So, to prove the second point, it is sufficient to show that

$$\frac{1}{q}\sum_{j=0}^{q-1}\sum_{t\in S_1(j)}\frac{\sum_{s=s_0}^{+\infty}s((f^t)_*\nu_n)(V_s)}{n-2q+1}$$

is as small as we want for s_0 large and then n large enough because the cardinality of $S_1(j)$ is smaller than $3q$.

This last term is less than

$$-\frac{1}{q(n-2q+1)}\sum_{j=0}^{q-1}\sum_{t\in S_1(j)}\int_{\{\rho\leq e^{-s_0}\}}\log\rho\ d(f^t)_*\nu_n.$$

But:

$$\sum_{t\in S_1(j)}(f^t)_*\nu_n \leq (\phi(n)m - m + 1)\mu''_n$$

with $\mu''_n = \frac{1}{\phi(n)m-m+1}\sum_{p=0}^{\phi(n)m-m} f^p_*\nu_n$.

Following Lemma 2.3.9 with μ''_n instead of μ'_n and $i = j = 0$ (this is indeed possible because the indexes p in μ''_n goes to $\phi(n)m - m$ which is $\leq n-1-(m-1)$), we deduce that:

$$\frac{1}{q(n-2q+1)} \sum_{j=0}^{q-1} \sum_{t \in S_1(j)} \sum_{s=s_0}^{+\infty} s((f^t)_* \nu_n)(V_s)$$

is as small as we want by taking s_0 large enough and then n large enough.

This gives the second point of Proposition 2.3.8.

Third point of Proposition 2.3.8. — The proof is the same than for the second point by replacing $S_1(j)$ by $S_2(j)$, $\mathscr{P}$ by $\mathscr{Q}$ and ρ by η.

At the end, we have to bound from above:

$$\frac{1}{q(n-2q+1)} \sum_{j=0}^{q-1} \sum_{t \in S_2(j)} \sum_{s=s_0}^{+\infty} s((f^t)_* \nu_n)(V_s),$$

(here the V_s correspond to the partition $\mathscr{Q}$ and to the function η).

That term is less than

$$-\frac{1}{q(n-2q+1)} \sum_{j=0}^{q-1} \sum_{t \in S_2(j)} \int_{\{\eta \leq e^{-s_0}\}} \log \eta \; d(f^t)_* \nu_n.$$

Finally:

$$\sum_{t \in S_2(j)} (f^t)_* \nu_n \leq n \mu_n$$

and since $\int_{\{\eta \leq e^{-s_0}\}} \log \eta d\mu_n$ converges to $\int_{\{\eta \leq e^{-s_0}\}} \log \eta d\mu$, it is also as small as we want if s_0 is large enough then n is large enough. This gives the third point of Proposition 2.3.8, and the proposition follows. □

CHAPTER 3

DYNAMICS OF BIRATIONAL MAPS OF $\mathbb{P}^k$

3.1. A family of birational maps

The purpose of this section is to introduce the family of birational maps $\mathscr{B}$.

Recall that a meromorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ is holomorphic outside an analytic subset $I(f)$ of codimension ≥ 2 in $\mathbb{P}^k$. It can be written in homogeneous coordinates as $[P_0 : \ldots : P_k]$ where the P_i are homogeneous polynomials of algebraic degree d in the $(z_0, \ldots, z_k)$ variable, with $\gcd_i(P_i) = 1$.

Let Γ denote the closure of the graph of the restriction of f to $\mathbb{P}^k \setminus I(f)$. This is an irreducible analytic set of dimension k in $\mathbb{P}^k \times \mathbb{P}^k$.

Let π_1 and π_2 denote the canonical projections of $\mathbb{P}^k \times \mathbb{P}^k$ on its factors. The *indeterminacy locus* $I(f)$ (or indeterminacy set) of f is the set of points $z \in \mathbb{P}^k$ such that $\dim \pi_1^{-1}(z) \cap \Gamma \geq 1$. It is also the common zeros of the P_i: $I(f) = \{P_i = 0\}$.

We assume that f is *dominant*, that is, $\pi_2(\Gamma) = \mathbb{P}^k$. If A is a subset of $\mathbb{P}^k$, define

$$f(A) := \pi_2(\pi_1^{-1}(A) \cap \Gamma) \quad \text{and} \quad f^{-1}(A) := \pi_1(\pi_2^{-1}(A) \cap \Gamma).$$

Define formally for a current S on $\mathbb{P}^k$, not necessarily positive or closed, the pull-back $f^*(S)$ by

$$f^*(S) := (\pi_1)_*\big(\pi_2^*(S) \wedge [\Gamma]\big) \tag{3}$$

where $[\Gamma]$ is the current of integration on Γ.

Similarly, the operator f_* is formally defined by

$$f_*(R) := (\pi_2)_*\big(\pi_1^*(R) \wedge [\Gamma]\big). \tag{4}$$

Here, we consider a birational map f of algebraic degree $d \geq 2$. That is a map such that $\# f^{-1}(z) = 1$ for z generic. Let δ be the algebraic degree of f^{-1}. We denote $I^+ := I(f)$ and $I^- = I(f^{-1})$ the indeterminacy sets of f and f^{-1}.

We also consider the critical sets $\mathscr{C}^+$ (or $\mathscr{C}(f)$) and $\mathscr{C}^-$ (or $\mathscr{C}(f^{-1})$) defined by:

$$\mathscr{C}^+ := f^{-1}(I^-)$$
$$\mathscr{C}^- := (f^{-1})^{-1}(I^+).$$

Write $f = [P_0 : \ldots : P_k]$ where the P_i are homogeneous polynomials of degree d. Let $F = (P_0, \ldots, P_k)$ be the induced map on $\mathbb{C}^{k+1}$. Similarly, write $f^{-1} = [Q_0 : \ldots : Q_k]$ where the Q_i are homogeneous polynomials of degree δ and let $F^{-1} = (Q_0, \ldots, Q_k)$. There is of course an abuse of notation since $F \circ F^{-1} \neq Id$. Instead, we have that:

$$F \circ F^{-1} = P(z_0, \ldots, z_k) \times (z_0, \ldots, z_k),$$

where P is a homogeneous polynomial of degree $d\delta - 1$ equal to 0 in $\pi^{-1}(\mathscr{C}^- \cup I^-)$ where $\pi : \mathbb{C}^{k+1} \to \mathbb{P}^k$ is the canonical projection. That implies that the critical set $\mathscr{C}^-$ is an analytic set of codimension 1 and that $I^- \subset \mathscr{C}^-$. Similarly, we have that $\mathscr{C}^+$ is of codimension 1 and $I^+ \subset \mathscr{C}^+$ (see also Proposition 3.3 in [**15**] and [**50**]). So, $f : \mathbb{P}^k \backslash \mathscr{C}^+ \to \mathbb{P}^k \backslash \mathscr{C}^-$ is a biholomorphism.

For $0 \leq q \leq k$ and $n > 0$, define $\lambda_q(f^n)$ as the degree of $(f^n)^{-1}(L)$ for L a generic linear subspace of codimension q. The number $\lambda_q(f^n)$ gives a size for the action of f^n on the cohomology group $H^{q,q}(\mathbb{P}^k)$. Since the quantity $\lambda_q(f^n)$ is cohomological, we have:

$$\begin{aligned}\lambda_q(f^n) &= \|(f^n)^*(\omega^q)\| = \int_{\mathbb{P}^k} (f^n)^*(\omega^q) \wedge \omega^{k-q} \\ &= \|(f^n)_*(\omega^{k-q})\| = \int_{\mathbb{P}^k} (f^n)_*(\omega^{k-q}) \wedge \omega^q. \end{aligned} \tag{5}$$

We have in particular that $\lambda_1(f) = d$ is the algebraic degree and that $\lambda_k(f^n)$ *the topological degree* of f^n is equal to $\#(f^n)^{-1}(z)$ for z generic. In particular, $\lambda_k(f^n) = 1$ since f is birational.

We define the *dynamical degree of order* q of f by:

$$d_q := \lim_{n\to\infty} (\lambda_q(f^n))^{\frac{1}{n}} \tag{6}$$

These limits always exist and $d_q \leq d_1^q$ [**25**]. The last degree $\lambda_k(f) = d_k$ is equal to 1.

A result by Gromov [**40**, Theorem 1.6] implies that $q \mapsto \log d_q$ is concave in q. In particular, there exists a q_0 such that:

$$1 = d_0 \leq d_1 \leq \cdots \leq d_{q_0} \geq \cdots \geq d_k = 1.$$

Denote for f^{-1}, $\lambda_q^-(f^n)$ and d_q^- the quantities previously defined for f.

In order to work with the currents in cohomology, we need a hypothesis on the indeterminacy sets so that $(f^n)^* = (f^*)^n$ on the cohomology group $H^{q,q}(\mathbb{P}^k)$. If so, we say that the map is *algebraically q-stable* (see [**50**] and [**31**]). For that purpose, we will consider the following class of mappings.

DEFINITION 3.1.1. – *Let* $\mathscr{B}_0$ *be the set of birational maps of* $\mathbb{P}^k$ *such that there exist an integer* $0 < s < k$ *and two integers* $d, \delta > 1$ *such that* $d^s = \delta^{k-s}$ *and for all* $n \geq 1$*:*

— *for all linear subspaces of codimension* $q \leq s$, $\deg(f^{-n}L) = d^{qn}$*;*
— *for all linear subspaces of codimension* $q \leq k-s$, $\deg(f^n L) = \delta^{qn}$*.*

Of course, the fact that $d^s = \delta^{k-s}$ can be deduced from the rest of the definition but we keep it there for simplicity. If a map f is in $\mathscr{B}_0$, then f is algebraically q-stable for $q \leq s$ and f^{-1} is algebraically q-stable for $q \leq k-s$. The following classical lemmas will help us in producing a more geometric characterisation of $\mathscr{B}_0$.

LEMMA 3.1.2. – *Let* f *be a birational map of algebraic degree* d *such that* $\dim(I^+) = k-s-1$ *then* $\lambda_q(f) = d^q$ *for all* $q \leq s$ *and* $\lambda_{s+1}(f) < d^{s+1}$*.*

Proof. – Let f be as above and let $q \leq s$. Then $\lambda_q(f)$ is the mass of the current $f^*(\omega^q)$ which is a form with L^1 coefficients. In particular, $f^*(\omega^q)$ gives no mass to I^+. On the other hand $f^*(\omega^q) = f^*(\omega)^q$ outside I^+. But the current $f^*(\omega)^q$ is a well defined positive closed current of bidimension $(k-q, k-q)$ of mass d^q (see Corollary 4.11 in [**14**, Chapter III]), and it cannot charge I^+ since $k-s-1 < k-q$. So we have the equality $f^*(\omega^q) = f^*(\omega)^q$ on $\mathbb{P}^k$ and both currents have the same mass d^q.

Now the same argument will give that $\lambda_{s+1}(f) < d^{s+1}$ if $f^*(\omega)^{s+1}$ gives some mass to I^+ (Corollary 4.11 in [**14**, Chapter III] still implies that $f^*(\omega)^{s+1}$ is well defined). Let u be a potential of $f^*(\omega)$ that is $dd^c u + d\omega = f^*(\omega)$. We will see in Lemma 3.2.4 below that:

$$u(x) \leq A' \log \operatorname{dist}(x, I^+) + B'.$$

In particular, since $\log \operatorname{dist}(x, I^+) \leq \log \operatorname{dist}(x, y)$ for $y \in I^+$, Theorem 7.8 in [**14**, Chapter III] shows the current $f^*(\omega)^{s+1}$ has its Lelong numbers $> c > 0$ on I^+ for some c. Siu's theorem [**51**] implies that this current is greater than $c[I^+]$. In particular, it gives some mass to $[I^+]$. □

Applying the previous lemma to a map $f \in \mathscr{B}_0$ shows that necessarily $\dim(I^+) = k-s-1$ and $\dim(I^-) = s-1$. We now prove:

LEMMA 3.1.3. – *If a map* f *is in* $\mathscr{B}_0$ *with* s *as above, then for any* $q \leq s$*, no algebraic set of codimension* q *is sent to* I^+ *by some iterate* f^n *of* f*.*

Proof. – Assume that there exist $f \in \mathscr{B}_0$ and an algebraic set A of codimension $q \leq s$ sent to I^+ by f^n. To simplify the proof, assume that $n = 1$ and take the smallest q such that the above property is satisfied. Let $f = [P_0 : P_1 \ldots : P_k]$ where the P_i are homogeneous polynomials of algebraic degree d with no common factors. That way, $I^+ = \{P_0 = \cdots = P_k = 0\}$. Let $\pi : \mathbb{C}^{k+1} \to \mathbb{P}^k$ be the canonical projection and let $F = (P_0, \ldots, P_k)$.

For an element $Z = (z_0, \dots, z_k) \in \mathbb{C}^{k+1}$, we write $|Z|^2 = |z_0|^2 + \cdots + |z_k|^2$. Then, we have that:

$$f^*(\omega) - d\omega = dd^c(\log |F|^2 - d \log |Z|^2).$$

Applying f^* again gives:

$$f^*(f^*(\omega)) - df^*(\omega) = dd^c(\log |F^2|^2 - d \log |F|^2).$$

So, taking the sum gives:

$$f^*(f^*(\omega)) - d^2\omega = dd^c(\log |F^2|^2 - d^2 \log |Z|^2).$$

The singularities of the qpsh function $\varphi := \log |F^2|^2 - d^2 \log |Z|^2$ are exactly on the set $A \cup I^+$ (assuming A is the largest algebraic set sent to I^+) so we have the inequality $\varphi \leq C \log \operatorname{dist}(x, A \cup I^+) + C$ for some $C > 0$.

Again, Corollary 4.11 in [**14**, Chapter III] implies that $f^*(f^*(\omega))^q$ is well defined and Theorem 7.8 in [**14**, Chapter III] shows that the current $(f^*(f^*(\omega)))^q$ has its Lelong numbers $> c > 0$ on A for some c. Siu's theorem [**51**] implies that it gives positive mass to A. But $f^*(f^*(\omega))^q$ is of mass d^{2q} and $f^*(f^*(\omega))^q = (f^2)^*(\omega)^q$ outside $A \cup I^+$. Since the current $(f^2)^*(\omega^q)$ gives no mass to analytic set we have $\lambda_q(f^2) < \lambda_q(f)^2$, a contradiction. □

This leads to the lemma:

Lemma 3.1.4. – *Let f be a birational map with* $\dim(I^+) = k - s - 1$ *and* $\dim(I^-) = s - 1$. *Then the fact that no algebraic set of codimension $q \leq s$ is sent to I^+ by some iterate f^n of f is equivalent to the condition:*

$$\bigcup_{n \geq 0} f^{-n} I(f) \cap \bigcup_{n \geq 0} f^n I(f^{-1}) = \varnothing. \tag{7}$$

Proof. – Assume that there exists some A sent to I^+ by f^n. Then we have that $A \subset \mathscr{C}(f^n)$ since A is contracted and so $f^n(A) \subset I(f^{-n})$. Since $I(f^{-n}) \subset \bigcup_0^{n-1} f^j I(f^{-1})$ we have that the intersection (7) is not empty.

Now assume that the intersection (7) is not empty. Formal manipulations show that (7) is equivalent to:

$$I^- \cap \bigcup_{n \geq 0} f^{-n} I(f) = \varnothing.$$

Hence we can assume that there exists some point $y \in I^- \cap f^{-n} I(f)$. The map $f : \mathscr{C}^+ \to I^-$ is meromorphic and surjective, in particular, the dimension $\dim(f^{-1}(x))$ is upper semi continuous hence $\geq k - s$ at every point. In particular, the set $f^{-1}(y)$ is of codimension $\leq s$ and it is sent by f^{n+1} to $I(f)$. This concludes the proof. □

This leads to the characterisation of $\mathscr{B}_0$:

PROPOSITION 3.1.5. – *Let f be a birational map with* $\dim(I^+) = k - s - 1$ *and* $\dim(I^-) = s - 1$, *then* $f \in \mathscr{B}_0$ *if and only if* f *satisfies (7).*

Proof. – The previous lemmas already give the first implication.

Take f a birational map with $\dim(I^+) = k - s - 1$ and $\dim(I^-) = s - 1$ which satisfies (7). Lemma 3.1.2 gives that $\lambda_q(f) = d^q$ for $q \leq s$ and $\lambda_q(f^{-1}) = \delta^q$ for $q \leq k - s$ where d is the algebraic degree of f and δ is the algebraic degree of f^{-1}. The proof follows from the following lemma. □

We introduce some notations first. Let $\mathscr{C}_s$ denote the convex compact set of (strongly) positive closed currents S of bidegree (s,s) on $\mathbb{P}^k$ and of mass 1, i.e. $\|S\| := \langle S, \omega^{k-s}\rangle = 1$. For a positive closed current T of mass $m(T) > 0$, we denote by T_{nor} the renormalization of T (that is $T_{\text{nor}} = m(T)^{-1}T$). Denote for simplicity $L := \lambda_q(f)^{-1}f^*$ and $\Lambda := (\lambda_{k-q})^{-1}f_* = (\lambda_q^-(f))^{-1}f_*$ which are well defined operators on the elements of $\mathscr{C}_q$ which are smooth near I^- (resp. I^+). We make an abuse of notations and write L instead of L_q, this is not a problem since in what follows $L(S)$ will always be the current $f^*(S)_{\text{nor}}$. The theory of super-potentials (see the appendix) allows us to extend the operator L (resp. Λ) to the currents in $\mathscr{C}_q$ such that their super-potentials are finite at one point of the form $\Lambda(S)$ for $S \in \mathscr{C}_{k-q+1}$ smooth near I^+ (resp. at one point of the form $L(S)$ for $S \in \mathscr{C}_{k-q+1}$ smooth near I^-).

Now we show that a map which satisfies (7) is in fact algebraically q-stable for all q. That is to say no mass is lost on the indeterminacy set by pull-back. More precisely, we have the proposition that uses the theory of super-potential:

LEMMA 3.1.6. – *Let f be a birational map satisfying (7). Then for all* $0 \leq q \leq k$, $\lambda_q(f^n) = (\lambda_q(f))^n$ *for all n and so* $d_q = \lambda_q(f)$ *for all q.*

Proof. – We have to compute the integral:

$$\lambda_q(f^n) = \|(f^n)^*(\omega^q)\| = \int_{\mathbb{P}^k} (f^n)^*(\omega^q) \wedge \omega^{k-q}.$$

The proof is by induction on n: $(f^{n-1})^*(\omega^q)$ is a form in L^1 smooth near I^- by (7). So we can define its pull-back by f which is of mass $\lambda_q(f)\lambda_q(f^{n-1})$. On the other hand, ω^q is smooth near $I(f^{-n})$ so it is $(f^n)^*$-admissible and the mass of $(f^n)^*(\omega^q)$ is $\lambda_q(f^n)$.

We will now prove that $f^*((f^{n-1})^*(\omega^q)) = (f^n)^*(\omega^q)$.

Let $\pi_{1|\Gamma}$ and $\pi_{2|\Gamma}$ be the restriction of π_1 and π_2 to the graph Γ of f. That way, $f^*(S) = (\pi_1)_*(\pi_{2|\Gamma})^*(S)$ where $S \in \mathscr{C}_q$ is smooth near I^-. We will take $S = (f^{n-1})^*(\omega^q)_{\text{nor}}$.

Let V be a small neighborhood of I^- such that S is smooth here. Outside ${\pi_2}^{-1}(V) \cap \Gamma$, $\pi_{2|\Gamma}$ is a finite map, hence ${\pi_2}^*(S) \wedge [\Gamma]$ is well defined and depends continuously of S here by Theorem 1.1 in [**29**]. Furthermore, if $S_{|\mathbb{P}^k \setminus V}$ does not give mass to a Borel set A then $(\pi_{2|\Gamma})^*(S)$ does not give mass to $(\pi_{2|\Gamma})^{-1}(A)$ outside $\pi_2^{-1}(V)\cap\Gamma$. Since π_1 is holomorphic, the same is true for $f^*(S_{|\mathbb{P}^k\setminus V})$. On V, S is smooth, hence $f^*(S_{|V})$ is a form in L^1 (see e.g. **[25]**).

We consider $S=(f^{n-1})^*(\omega^q)$: $(f^{n-1})^*(\omega^q)$ is a form in L^1 hence it does not give mass to algebraic sets of dimension $\leq k-1$; so $f^*((f^{n-1})^*(\omega^q))$ is a current that does not give mass to algebraic sets of dimension $\leq k-1$. We get that $f^*((f^{n-1})^*(\omega^q))$ and $(f^n)^*(\omega^q)$ are equal wherever they are smooth that is outside analytic sets of dimension $\leq k-1$. We deduce that they are equal hence they have the same mass. □

The following corollary of the previous proof will be used later:

Corollary 3.1.7. – *Let $R\in\mathscr{C}_q$ be a smooth form and f in $\mathscr{B}_0$. Then for all $i,j\geq 0$, we have that $(f^i)^*(R)$ is $(f^j)^*$-admissible and $(f^j)^*((f^i)^*(R))=(f^{i+j})^*(R)$.*

Let $j\geq 0$ and $q\leq k$. Let f in $\mathscr{B}_0$. For a current $S\in\mathscr{C}_q$ which is $(f^j)^*$-admissible, we can define $L_j(S)$ as $(\lambda_q(f^j))^{-1}(f^j)^*(S)$ (similarly we define Λ_j). By Proposition 3.1.6, we have that $\lambda_q(f^j)=\lambda_q(f)^j$ so we can also write $L_j(S)=\lambda_q(f)^{-j}(f^j)^*(S)$. From Corollary 3.1.7, we have that $L^j(S)=L_j(S)$ on smooth forms, the question is: does it also stand for $(f^j)^*$-admissible currents ? The following lemma answers positively:

Lemma 3.1.8. – *Let $S\in\mathscr{C}_q$ for $q\leq k$. Let $n>0$ such that S is $(f^n)^*$-admissible then for all j with $0\leq j\leq n-1$, $L^j(S)$ is well defined, f^*-admissible and $L^{j+1}(S)=L_{j+1}(S)$. In particular, $L^n(S)=L_n(S)$.*

Proof. – Let S be as above, then a super-potential of S is finite at $\Lambda_n(\omega^{k-q+1})$ by hypothesis. Since f satisfies (7), we have that $\Lambda(\omega^{k-q+1})$ is smooth near $I(f^{n-1})$, the previous corollary implies that

$$\Lambda_n(\omega^{k-q+1})=\Lambda_{n-1}(\Lambda(\omega^{k-q+1})).$$

So the super-potentials of S are finite at the image by Λ_{n-1} of a current smooth near $I(f^{n-1})$: it is $(f^{n-1})^*$-admissible (see the appendix). An immediate induction gives that S is $(f^j)^*$-admissible for $j\leq n$.

Now we prove by induction on j that $L^j(S)$ is f^*-admissible and that $L^{j+1}(S)=L_{j+1}(S)$. In order to do that, we show that both currents have the same super-potential and conclude by Proposition A.1.3.

For $j=0$, it is just the fact that S is f^*-admissible. Now, assume the property holds up to the rank j. A super-potential of $L^j(S)=L_j(S)$ is by Proposition A.1.18:

$$\mathscr{U}_{L_j(S)}=\mathscr{U}_{L_j(\omega^q)}+\frac{\lambda_{q-1}(f^j)}{\lambda_q(f^j)}\mathscr{U}_S\circ\Lambda_j$$

on forms smooth near $I(f^j)$. Taking the value at $\Lambda(\omega^{k-q+1})$ (which is smooth near $I(f^j)$) gives:

$$\mathscr{U}_{L_j(S)}(\Lambda(\omega^{k-q+1})) = \mathscr{U}_{L_j(\omega^q)}(\Lambda(\omega^{k-q+1})) + \frac{\lambda_{q-1}(f^j)}{\lambda_q(f^j)}\mathscr{U}_S(\Lambda_j(\Lambda(\omega^{k-q+1}))).$$

The current $L_j(\omega^q) = L^j(\omega^q)$ is f^*-admissible since it is smooth near I^-, that implies that $\mathscr{U}_{L_j(\omega^q)}(\Lambda(\omega^{k-q+1}))$ is finite. Similarly, applying the previous corollary to f^{-1} gives that $\Lambda_j(\Lambda(\omega^{k-q+1})) = \Lambda^{j+1}(\omega^{k-q+1})$ and since S is $(f^{j+1})^*$-admissible then $\mathscr{U}_S(\Lambda_j(\Lambda(\omega^{k-q+1})))$ is also finite.

That gives that $\mathscr{U}_{L_j(S)}(\Lambda(\omega^{k-q+1}))$ is finite so $L_j(S)$ is f^*-admissible. We can now apply Proposition A.1.18 to $L^j(S)$:

$$\begin{aligned}\mathscr{U}_{L^{j+1}(S)} &= \mathscr{U}_{L(\omega^q)} + \frac{\lambda_{q-1}(f)}{\lambda_q(f)}\mathscr{U}_{L^j(S)} \circ \Lambda \\ &= \mathscr{U}_{L(\omega^q)} + \frac{\lambda_{q-1}(f)}{\lambda_q(f)}(\mathscr{U}_{L_j(\omega^q)} + \frac{\lambda_{q-1}(f^j)}{\lambda_q(f^j)}\mathscr{U}_S \circ \Lambda_j) \circ \Lambda\end{aligned}$$

on smooth forms. Since $\mathscr{U}_{L(\omega^q)} + \frac{\lambda_{q-1}(f)}{\lambda_q(f)}\mathscr{U}_{L_j(\omega^q)} \circ \Lambda = \mathscr{U}_{L^{j+1}(\omega^q)}$ on smooth forms, and since $\Lambda_j \circ \Lambda = \Lambda_{j+1}$ on smooth forms, we deduce from Proposition 3.1.6 that:

$$\mathscr{U}_{L^{j+1}(S)} = \mathscr{U}_{L_{j+1}(S)}$$

on smooth forms, hence $L^{j+1}(S) = L_{j+1}(S)$ by Proposition A.1.3. That gives the lemma. □

Summing up what we proved, we have that for an element $f \in \mathscr{B}_0$ the dynamical degrees are ordered as follows:

$$\begin{array}{ccccccccccc} d_0 < & d_1 < & d_2 < & \dots < & d_s & & > & \dots > & d_{k-2} > & d_{k-1} > & d_k \\ \shortparallel & \shortparallel & \shortparallel & & \shortparallel & & & & \shortparallel & \shortparallel & \shortparallel \\ 1 < & d < & d^2 < & \dots < & d^s = & \delta^{k-s} & > & \dots > & \delta^2 > & \delta > & 1. \end{array}$$

Let $f \in \mathscr{B}_0$. Recall that a quasi-potential of a current $T \in \mathscr{C}_q$ is a current U of bidegree $(q-1, q-1)$ such that $T = \omega^q + dd^cU$. We know from the appendix that it is always possible to take U negative. In what follows, for an irreducible analytic set A, we define $[f(A)]$ as the current of integration over $f(A)$ counting the multiplicity of f at A and if A is not irreducible, we decompose it into irreducible components (A_i) and we define $[f(A)]$ as $\sum_i [f(A_i)]$.

Since f satisfies (7), we have that $I^+ \cap f^j(I^-) = \varnothing$ for $j \leq n$. So, $f^n(I^-)$ is well defined and the form $U_{L(\omega)}L(\omega^{s-1})$ is smooth in $f^n(I^-)$ so the following integral is well defined:

$$\int_{[f^n(I^-)]} U_{L(\omega)}L(\omega^{s-1}).$$

The terms $(\mathrm{vol}(I^-))^{-1}$ and $(\mathrm{vol}(I^+))^{-1}$ in the following hypothesis are just here to normalize the integrals. We are going to restrict the study to the subset $\mathscr{B}$ of $\mathscr{B}_0$ given by the following definition:

DEFINITION 3.1.9. – *Let $U_{L(\omega)}$ be a negative quasi-potential of $L(\omega)$ and let $U_{\Lambda(\omega)}$ be a negative quasi-potential of $\Lambda(\omega)$. The set $\mathscr{B}$ is the set of elements $f \in \mathscr{B}_0$ such that:*

$$\sum_{n=0}^{\infty} \left(\frac{1}{d^s}\right)^n (\mathrm{vol}(I^-))^{-1} \int_{f^n(I^-)} U_{L(\omega)} L(\omega^{s-1}) > -\infty$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{\delta^{k-s}}\right)^n (\mathrm{vol}(I^+))^{-1} \int_{f^{-n}(I^+)} U_{\Lambda(\omega)} \Lambda(\omega^{k-s-1}) > -\infty.$$

As usual, for two sets E and F, we denote $\inf_{x\in E, y\in F} \mathrm{dist}(x,y)$ by $\mathrm{dist}(E,F)$. In [**1**], the authors asked for a quantitative and stronger version of (7) similar to:

HYPOTHESIS 3.1.10. – *The birational mapping f satisfies* $\dim(I^+) = k - s - 1$, $\dim(I^-) = s - 1$ *and:*

$$\sum_{n=0}^{\infty} \left(\frac{1}{d}\right)^n \log \mathrm{dist}(I^+, f^n(I^-)) > -\infty$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{\delta}\right)^n \log \mathrm{dist}(I^-, f^{-n}(I^+)) > -\infty.$$

In $\mathbb{P}^2$, the convergence of the series $\sum_{n=0}^{\infty} d^{-n} \log \mathrm{dist}(I^+, f^n(I^-)) > -\infty$ and $\sum_{n=0}^{\infty} \delta^{-n} \log \mathrm{dist}(I^-, f^{-n}(I^+)) > -\infty$ are equivalent (see [**15**]), it has no reason to be true in higher dimension.

In the case of $\mathbb{P}^2$, a map is in $\mathscr{B}$ if and only if it satisfies Hypothesis 3.1.10 (see [**1**, Theorem 4.3] and Theorem 3.2.1 below). That is because the distance between the supports of the currents is a good distance for Dirac masses but not for currents of higher bidimension. We will see in Theorem 3.2.8 that Hypothesis 3.1.10 implies that a map is in $\mathscr{B}$.

We will see in Theorem 3.2.1 that the convergence of the series in Definition 3.1.9 has a clear interpretation in term of super-potentials : it means exactly that the super-potentials of the Green current of order s are finite at $[I^-]_{\mathrm{nor}}$.

Generalizing [**1**] to the case of higher dimension is the main goal of this chapter. The class $\mathscr{B}$ contains the *regular automorphisms* of $\mathbb{C}^k$ since those satisfy $f^n(I^-) = I^-$, $f^{-n}(I^+) = I^+$ and $I^+ \cap I^- = \varnothing$ (see [**50**]). It also contains the *regular birational* maps of $\mathbb{P}^k$ ([**23**]), in this case the indeterminacy sets of I^+ and I^- are trapped in some stable disjoint open sets. The interest of $\mathscr{B}$ is that it is generic in the following sense:

THEOREM 3.1.11. – *Let E_s be the set of birational maps $f : \mathbb{P}^k \to \mathbb{P}^k$ such that I^+ and I^- satisfy* $\dim(I^+) = k - s - 1$ *and* $\dim(I^-) = s - 1$. *Consider the group action:*

$$\Phi : \mathrm{PGL}(k+1, \mathbb{C}) \times E_s \to E_s$$
$$(A, f) \mapsto A \circ f.$$

Then outside a pluripolar set of the orbit $Orb(f)$ of $f \in E_s$, the maps of $Orb(f)$ are in $\mathscr{B}$.

This provides us with many new examples: start with a regular polynomial automorphism of $\mathbb{C}^k$ or a regular birational map in $\mathbb{P}^k$ then consider its orbit under the action of $\mathrm{PGL}(k+1, \mathbb{C})$. Section 3.3 is devoted to the proof of the previous theorem.

Other families of polynomial automorphisms were studied in **[44]** and **[42]**. In these papers, the authors replace the set I^- by the set $f(L_\infty \backslash I^+)$ in the condition $I^+ \cap I^- = \varnothing$. Those mappings are not always in $\mathscr{B}$ since the conditions $\dim(I^+) = k - s - 1$ and $\dim(I^-) = s - 1$ no longer need to be satisfied.

3.2. Construction of the Green currents

Take $f \in \mathscr{B}$. Let s be such that $\dim(I^+) = k - s - 1$ and $\dim(I^-) = s - 1$. The following theorem is the main result of this section.

THEOREM 3.2.1. – *Let $f \in \mathscr{B}$, then the sequence $(L^m(\omega^s))$ converges in the Hartogs' sense to the* Green current of order s of f *that we denote by T_s^+.*

To prove the theorem, we need the following computation of the super-potential of $L^m(\omega^q)$ for $q \leq s$,

LEMMA 3.2.2. – *A super-potential of $L^m(\omega^q)$ is given on $R \in \mathscr{C}_{k-q+1}$ which is $(f^m)_*$-admissible by:*

$$\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^q)} \circ \Lambda^n(R). \tag{8}$$

Proof. – Recall that for $q \leq s$, we have $\lambda_q(f^n) = (d^q)^n$ for all n and so $d_q = d^q$. Recall that for $S \in \mathscr{C}_q$ which is f^*-admissible, $L(S)$ is the element of $\mathscr{C}_q$ defined as $d^{-q} f^*(S)$. Furthermore, any current smooth in a neighborhood of I^- is f^*-admissible. By Corollary 3.1.7, $L^{n-1}(\omega^q)$ is f^*-admissible since f satisfies (7) so we can define $L^n(\omega^q)$.

Now, let $\mathscr{U}_{L(\omega^q)}$ denote a negative super-potential of $L(\omega^q)$ (it is always possible by Proposition A.1.1). Assume first that $R \in \mathscr{C}_{k-q+1}$ is smooth. So, we have that by Proposition A.1.18 that a super-potential of $L^m(\omega^q) = L(L^{m-1}(\omega^q))$ is given on currents in $\mathscr{C}_{k-q+1}$ smooth in a neighborhood of I^+ by:

$$\mathscr{U}_{L(\omega^q)} + \frac{1}{d}\mathscr{U}_{L^{m-1}(\omega^q)} \circ \Lambda.$$

So, by induction, for a smooth element $R \in \mathscr{C}_{k-q+1}$, we have that a super-potential of $L^m(\omega^q)$ is given on R by:

$$\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^q)} \circ \Lambda^n(R). \tag{9}$$

So we have proved the lemma for $R \in \mathscr{C}_{k-q+1}$ smooth. In particular, for such an R, Corollary 3.1.7 gives that a super-potential of $L^m(\omega^q)$ is given on R by:

$$\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^q)} \circ \Lambda_n(R).$$

Now, assume R is only $(f^m)_*$-admissible. Lemma 3.1.8 implies that R is $(f^n)^*$-admissible then for all n with $0 \leq n \leq m$. We consider a sequence of smooth current (R_j) of $\mathscr{C}_{k-q+1}$ converging to R in the Hartogs' sense. Then $\Lambda^n(R_j)$ is smooth near I^+ for $0 \leq n \leq m-1$ by (7) and every term in the super-potential of $L^m(\omega^q)$ is continuous for the Hartogs' convergence, that gives that:

$$\mathscr{U}_{L^m(\omega^q)}(R) = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^q)} \circ \Lambda_n(R).$$

Again, Lemma 3.1.8 gives that $\Lambda_n(R) = \Lambda^n(R)$ for all $n \leq m-1$ if R is $(f^m)_*$-admissible so the lemma is proved. □

REMARK 3.2.3. – The equality (8) also stands for any current $R \in \mathscr{C}_{k-q+1}$ such that $\Lambda^n(R)$ is smooth near I^+ for $0 \leq n \leq m-1$. Indeed such currents are $(f^m)_*$-admissible: every term of the form $\mathscr{U}_{L(\omega^q)} \circ \Lambda^n(R)$ is finite for $n \leq m-1$ and depends continuously on R for the Hartogs' convergence, so we have by (8) that a super-potential of $L^m(\omega^q)$ is finite at R this means by symmetry of the super-potential that a super-potential of R is finite at $L^m(\omega^q)$ hence R is $(f^m)_*$-admissible.

Since the sequence of super-potential of $(L^m(\omega^s))$ given by (8) is decreasing, it is enough to show that it does not converge uniformly to $-\infty$ to show that it converges in the Hartogs' sense (see Proposition A.1.8). In [**31**], the authors prove that fact in the algebraically q-stable case with an additional assumption on the size of $\mathscr{C}^+$ (that fails in our case) using the fact that the sequence is bounded from below by the super-potential of any weak limit of the Cesarò mean of $(L^m(\omega^q))$. Here the idea is to show that the convergence holds at the point $[I^-]_{\text{nor}}$.

We need the following estimate of $\mathscr{U}_{L(\omega)}$ for that. It is similar to Proposition 1.3 in [**1**] though our proof is simpler taking advantage of the fact that we are in $\mathbb{P}^k$.

LEMMA 3.2.4. – *Let $U_{L(\omega)}$ be a quasi-potential of $L(\omega)$. Then there exist constants $A > 0$, B, $A' > 0$, B' such that:*

$$A \log \operatorname{dist}(x, I^+) - B \leq U_{L(\omega)}(x) \leq A' \log \operatorname{dist}(x, I^+) + B', \tag{10}$$

for all x.

Proof of the lemma. – Let $P_1, \ldots, P_{k+1}$ be homogeneous polynomials of degree d with no common factors such that $f = [P_1 : \ldots : P_{k+1}]$. That way, $I^+ = \{P_1 = \cdots = P_{k+1} = 0\}$. For an element $Z = (z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1}$, we write $|Z|^2 = |z_1|^2 + \cdots + |z_{k+1}|^2$. Let $\pi : \mathbb{C}^{k+1} \to \mathbb{P}^k$ denote the canonical projection and we write $F = (P_1, \ldots, P_{k+1})$. Then, we have that:

$$\pi^*(L(\omega) - \omega) = dd^c(\frac{1}{d} \log |F|^2 - \log |Z|^2).$$

Observe that the qpsh function $d^{-1} \log |F|^2 - \log |Z|^2$ is well defined on $\mathbb{P}^k$ since it does not depend on the choice of coordinates. So we can write that $U_{L(\omega)} = d^{-1} \log |F|^2 - \log |Z|^2$. So in $\mathbb{P}^k$, the singularities of $U_{L(\omega)}$ come from the terms in $\log |F|^2$. In the open set of $\mathbb{C}^{k+1}$ defined by $1 - \varepsilon < |Z| < 1 + \varepsilon$, we have that the map $F(Z)$ is equal to $(0, \ldots, 0) \in \mathbb{C}^{k+1}$ exactly in $\pi^{-1}(I^+)$. Using Łojasiewicz Theorem (Chapter IV Theorem 7 in [**46**]), that provide us with two constants $\alpha > 0$ and $C > 0$ such that on $|Z| = 1$ we have:

$$|F(Z)| \geq C(\operatorname{dist}(Z, \pi^{-1}(I^+)))^\alpha.$$

Now from the fact that the projection π is Lipschitz in $|Z| = 1$ and the above bound, we have constants $A > 0$, B such that:

$$U_{L(\omega)} \geq A \log \operatorname{dist}(\, . \,, I^+) - B.$$

For the other inequality, we work in a chart of $\mathbb{P}^k$ where we let z be the coordinate. Let V be a relatively compact open set in the chart. Observe that it is sufficient to prove the upper bound in V. For $y \in I^+$ in the chart, we have that $|F(z)|^2 = ||F(z)|^2 - |F(y)|^2|$ is less than $C' \operatorname{dist}(z, y)$. Taking the infimum over all such y, we get that $|F(z)|^2$ is less than $C' \operatorname{dist}(z, I^+)$. Taking the logarithm gives the estimate in V and the lemma follows. □

Proof of the theorem. – Since f satisfies (7), $\Lambda^n([I^-]_{\mathrm{nor}}) \in \mathscr{C}_{k-s+1}$ is smooth in a neighborhood of I^+ for all n (hence $[I^-]_{\mathrm{nor}}$ is $(f^n)_*$-admissible for all n). So $\Lambda^n([I^-]_{\mathrm{nor}}) = [f^n(I^-)]_{\mathrm{nor}}$ (counting the multiplicity). Hence a super-potential of $L^m(\omega^s)$ is given on $[I^-]_{\mathrm{nor}}$ by (8):

$$\mathscr{U}_{L^m(\omega^s)}([I^-]_{\mathrm{nor}}) = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)}(\Lambda^n([I^-]_{\mathrm{nor}})).$$

In other words:

$$\mathcal{U}_{L^m(\omega^s)}([I^-]_{\mathrm{nor}}) = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega^s)}([f^n(I^-)]_{\mathrm{nor}}).$$

Recall again that $L(\omega^s) = L(\omega^{s-1}) \wedge L(\omega)$ in the sense of current by Corollary 4.11 in [**14**, Chapter III]. So, in particular by Lemma A.2.1, a super-potential of $\mathcal{U}_{L(\omega^s)}$ is given by:

$$\mathcal{U}_{L(\omega^s)}(R) = \mathcal{U}_{L(\omega)}(L(\omega^{s-1}) \wedge R) + \mathcal{U}_{L(\omega^{s-1})}(\omega \wedge R)$$

on currents $R \in \mathscr{C}_{k-s+1}$ such that $L(\omega^{s-1})$ and R are wedgeable. A straightforward induction gives that a super-potential of $L(\omega^s)$ is given by:

$$\sum_{0 \leq j \leq s-1} \mathcal{U}_{L(\omega)}(\omega^j \wedge L(\omega)^{s-1-j} \wedge R), \tag{11}$$

on currents $R \in \mathscr{C}_{k-s+1}$ such that $L(\omega^{s-1})$ and R are wedgeable (since $\omega^j \wedge L(\omega^{s-1-j})$ is more H-regular than $L(\omega^{s-1})$, we have that $\omega^j \wedge L(\omega)^{s-1-j}$ and R are wedgeable by Lemma A.1.14). In particular, $L(\omega^{s-1})$ and $\Lambda^n([I^-]_{\mathrm{nor}})$ are wedgeable by Definition 3.1.9 ((7) is enough for that since $L(\omega^{s-1})$ is smooth near $f^n(I^-)$) so we can take $R = \Lambda^n([I^-]_{\mathrm{nor}})$ in the previous formula.

We want to show that for $0 \leq j \leq s-1$, the following series is convergent:

$$\sum_{n \geq 0} \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega)}(\omega^j \wedge L(\omega)^{s-1-j} \wedge \Lambda^n([I^-]_{\mathrm{nor}})).$$

By Lemma A.1.16, the term of the series can be rewritten as:

$$a_{j,n} = \mathrm{vol}(I^-)^{-1} \frac{1}{d^{sn}} \int_{f^n(I^-)} U_{L(\omega)} \omega^j \wedge L(\omega)^{s-1-j}, \tag{12}$$

since $\Lambda^n([I^-]_{\mathrm{nor}}) = [f^n(I^-)]_{\mathrm{nor}} = \mathrm{vol}(I^-)^{-1} d^{-(s-1)n} [f^n(I^-)]$ (observe that the form $U_{L(\omega)} \omega^j \wedge L(\omega)^{s-1-j}$ is smooth on $f^n(I^-)$ so this integral makes sense). So in particular, Definition 3.1.9 is equivalent to the fact that the series converges for $j = 0$. We prove that the series converges for $j > 0$ by induction.

So let $j > 0$ be given such that the above series converges for $j - 1$. Using $L(\omega) = dd^c U_{L(\omega)} + \omega$, we write:

$$\omega^{j-1} \wedge L(\omega)^{s-j} = \omega^j \wedge L(\omega)^{s-1-j} + dd^c U_{L(\omega)} \wedge \omega^{j-1} \wedge L(\omega)^{s-1-j}.$$

So replacing in (12), we see that:

$$a_{j-1,n} = a_{j,n} + \mathrm{vol}(I^-)^{-1} \frac{1}{d^{sn}} \int_{f^n(I^-)} U_{L(\omega)} dd^c U_{L(\omega)} \wedge \omega^{j-1} \wedge L(\omega)^{s-1-j}.$$

By Stokes, we have that the last integral is equal to:

$$-\int_{f^n(I^-)} dU_{L(\omega)} \wedge d^c U_{L(\omega)} \wedge \omega^{j-1} \wedge L(\omega)^{s-1-j},$$

which is non positive because $\omega^{j-1} \wedge L(\omega)^{s-1-j}$ is positive. That means that:

$$a_{j-1,n} \leq a_{j,n}.$$

Since $a_{j,n} < 0$ (because $U_{L(\omega)} < 0$), we have the convergence of the series for j. That concludes the induction.

Proposition A.1.8 gives the convergence in the Hartogs' sense to T_s^+. □

The previous proof gives in fact the corollary:

COROLLARY 3.2.5. – *The convergence of the series giving $\mathscr{U}_{T_s^+}([I^-]_{\mathrm{nor}})$ is equivalent to the convergence of the first series in 3.1.9:*

$$\sum_n \left(\frac{1}{d^s}\right)^n (\mathrm{vol}(I^-))^{-1} \int_{f^n(I^-)} U_{L(\omega)} L(\omega^{s-1}) > -\infty.$$

REMARK 3.2.6. – Using the same argument for $f^n(I^-)$ instead of I^- shows that the super-potentials of the current T_s^+ are in fact finite at every $[f^n(I^-)]_{\mathrm{nor}}$.

Observe also that the Green current $T_s^+(f^n)$ of f^n is well defined and equal to T_s^+.

REMARK 3.2.7. – We will see in the next section how to construct *the Green current of order* q for $q \leq s$ as $T_q^+ = \lim L^n(\omega^q)$ (see Proposition 3.4.6). We will need to construct the equilibrium measure for that first.

We now prove that if a map satisfies Hypothesis 3.1.10 then it is in $\mathscr{B}$. In fact, for such a map, we can construct directly the Green currents of order $\leq s$, in the general case, such result will only be proved in Section 3.4.

THEOREM 3.2.8. – *Any map satisfying Hypothesis 3.1.10 is in $\mathscr{B}$. For those maps, we have for $q \leq s$ that $(L^m(\omega^q))$ converges in the Hartogs' sense to the* Green current of order q of f *that we denote by T_q^+.*

Proof. – Take f satisfying Hypothesis 3.1.10 and let $q \leq s$. We consider $R_{I^-} \in \mathscr{C}_{k-q+1}$ any positive closed current with support in I^- (for example $\omega^{s-q} \wedge [I^-]_{\mathrm{nor}}$). Then $\Lambda^j(R)$ is smooth near I^+ for all $j \leq m-1$ (it is zero there), so we can apply (8):

$$\mathscr{U}_{L^m(\omega^q)}(R_{I^-}) = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^q)}(\Lambda^n(R_{I^-})).$$

Using (11), we see that

$$\mathscr{U}_{L(\omega^q)}(\Lambda^n(R_{I^-})) = \langle U_{L(\omega)}, S_q \wedge \Lambda^n(R_{I^-})\rangle,$$

where $S_q = \sum_{0 \leq j \leq q-1} \omega^j \wedge L(\omega)^{q-1-j}$ is smooth near $f^n(I^-)$ and is of mass q. The measure $S_q \wedge \Lambda^n(R_{I^-})$ is of mass q with support in $f^n(I^-)$. By Lemma 3.2.4, the function $U_{L(\omega)}$ is greater than $A \log \mathrm{dist}(I^+, f^n(I^-)) - B$ on $f^n(I^-)$. Hypothesis 3.1.10

implies exactly the convergence of the series giving $\mathcal{U}_{L^m(\omega^q)}(R_{I^-})$. That concludes the proof by Proposition A.1.8. □

THEOREM 3.2.9. – *The current T_s^+ is f^*-invariant, that is $L(T_s^+)$ is well defined and equal to T_s^+.*

Furthermore, T_s^+ is the most H-regular current which is f^*-invariant in $\mathscr{C}_s$. In particular, T_s^+ is extremal in the set of f^*-invariant currents of $\mathscr{C}_s$.

Proof. – Recall from the appendix that a current T is f^*-admissible if there exists a current R_0 which is smooth on a neighborhood of I^+ such that the super-potentials of T are finite at $\Lambda(R_0)$. For such T, $f^*(T)$ is well defined and if (T_n) is a sequence of current converging in the Hartogs' sense to T then T_n is f^*-admissible and $f^*(T_n)$ converges to $f^*(T)$ in the Hartogs' sense.

In our case, we take for R_0 the current $[I^-]_{\mathrm{nor}}$ which is smooth near I^+. Then by Remark 3.2.6, the super-potentials of T_s^+ are finite at $\Lambda([I^-]_{\mathrm{nor}}) = [f(I^-)]_{\mathrm{nor}}$. So the current $L(T_s^+)$ is well defined. Now, we have that $(L^{n+1}(\omega^s))_n = (L(L^n(\omega^s)))_n$ converges in the Hartogs' sense to T_s^+ and to $L(T_s^+)$ so that T_s^+ is indeed invariant.

Let $\mathcal{U}_{T_s^+}$ be the super-potential of T_s^+ defined as:

$$\mathcal{U}_{T_s^+} = \sum_{n=0}^{\infty} \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega^s)} \circ \Lambda^n, \tag{13}$$

on smooth forms in $\mathscr{C}_{k-s+1}$. Then composing (13) with Λ gives that on smooth forms in $\mathscr{C}_{k-s+1}$:

$$\mathcal{U}_{T_s^+} = d^{-1}\mathcal{U}_{T_s^+} \circ \Lambda + \mathcal{U}_{L(\omega^s)}.$$

By iteration, we have that on smooth forms in $\mathscr{C}_{k-s+1}$:

$$\mathcal{U}_{T_s^+} = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega^s)} \circ \Lambda^n + \left(\frac{1}{d}\right)^m \mathcal{U}_{T_s^+} \circ \Lambda^m.$$

By Theorem 3.2.1, that implies by difference that:

$$\left(\frac{1}{d}\right)^m \mathcal{U}_{T_s^+} \circ \Lambda^m$$

goes to zero on smooth forms in $\mathscr{C}_{k-s+1}$.

Now, let S be a f^*-invariant current in $\mathscr{C}_s$ such that there are constants $A > 0$ and B satisfying $A\mathcal{U}_{T_s^+} + B \leq \mathcal{U}_S < 0$ for some super-potential $\mathcal{U}_S$. Then on smooth forms in $\mathscr{C}_{k-s+1}$, a super-potential $\mathcal{U}_{L^m(S)}$ of $L^m(S) = S$ is given by:

$$\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega^s)} \circ \Lambda^n + \left(\frac{1}{d}\right)^m \mathcal{U}_S \circ \Lambda^m.$$

Since $\left(\frac{1}{d}\right)^m \mathscr{U}_{T_s^+} \circ \Lambda^m$ goes to zero on smooth forms in $\mathscr{C}_{k-s+1}$, our hypothesis implies that $\left(\frac{1}{d}\right)^m \mathscr{U}_S \circ \Lambda^m$ also goes to zero on smooth forms in $\mathscr{C}_{k-s+1}$. In particular, a super-potential of S is given on smooth forms in $\mathscr{C}_{k-s+1}$ by:

$$\sum_{n=0}^{\infty} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)} \circ \Lambda^n.$$

Now, using the fact that two currents having the same super-potential on smooth forms are in fact equal we deduce that $T_s^+ = S$.

In particular, for $A = 1$, we obtain that T_s^+ is the most H-regular current which is f^*-invariant. It is extremal in the set of f^*-invariant currents of $\mathscr{C}_s$ since if not we could write $T_s^+ = tS_1 + (1-t)S_2$ with S_1 and S_2 two f^*-invariant currents in $\mathscr{C}_s$. Take M small enough so that the super-potentials $\mathscr{U}_1$, $\mathscr{U}_2$ and $\mathscr{U}_{T_s^+}$ of S_1, S_2 and T_s^+ of same mean M are negative. Observe that $\mathscr{U}_{T_s^+} = t\mathscr{U}_1 + (1-t)\mathscr{U}_2$ so that $t^{-1}\mathscr{U}_{T_s^+} \leq \mathscr{U}_1$. Then we can apply the previous result for $A = t^{-1}$ and it follows that $S_1 = T$ (similarly $S_2 = T$). □

In the previous proof, we have obtained:

COROLLARY 3.2.10. – *Let $\mathscr{U}_{T_s^+}$ be the super-potential of T_s^+ defined on smooth forms by:*

$$\mathscr{U}_{T_s^+} = \sum_{n=0}^{\infty} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)} \circ \Lambda^n.$$

Then we have that:

$$\left(\frac{1}{d}\right)^m \mathscr{U}_{T_s^+} \circ \Lambda^m$$

goes to zero on smooth forms.

The current $L(\omega)^{s+1}$ is a well defined element of $\mathscr{C}_{s+1}$ and we have that

$$dd^c U_{L(\omega)} \wedge L(\omega)^s + \omega \wedge L(\omega)^s = L(\omega)^{s+1}$$

by Corollary 4.11 Chapter III in **[14]**. Observe that it is not true though that $L(\omega^{s+1}) = L(\omega)^{s+1}$. Indeed, $f^*(\omega^{s+1})$ is a well defined form (of mass δ^{k-s-1}) which is in L^1 hence that does not give mass to algebraic sets of dimension $\leq k-1$. But $f^*(\omega)^{s+1}$ is a smooth form outside I^+ which coincides with $f^*(\omega^{s+1})$ there. So we have by Siu's Theorem that:

$$f^*(\omega)^{s+1} = \sum_i a_i[I_i^+] + f^*(\omega^{s+1})$$

where the sum goes over all the irreducible components I_i^+ of I^+ and where the a_i are positive numbers. Observe that this formula is related to King's formula (see **[14]** Chapter III proposition 8.18). In particular, one has that:

$$f^*(\omega)^{s+1} \leq C[I^+] + f^*(\omega^{s+1})$$

for $C > 0$ large enough. Similarly, one has that $f_*(\omega)^{k-s+1}$ is well defined and satisfies:

$$f_*(\omega)^{k-s+1} \leq C[I^-] + f_*(\omega^{k-s+1}).$$

The following proposition is useful in the construction of the equilibrium measure.

Proposition 3.2.11. – *The super-potentials of T_s^+ are finite at $\omega^j \wedge \Lambda(\omega)^{k-s+1-j}$ for all $k - s + 1 \geq j \geq 0$.*

Proof. – If two currents S_1 and S_2 in $\mathscr{C}_r$ satisfies $S_1 \leq cS_2$ for a constant $c > 0$ then the super-potentials of S_1 and S_2 of mean 0 satifies $\mathscr{U}_{S_1} \geq c\mathscr{U}_{S_2} + c'$ for some constant c'. In particular, the super-potentials of S_1 are finite wherever $\mathscr{U}_{S_2}$ is.

Now, we have that the super-potentials of T_s^+ are finite at $[I^-]_{\text{nor}}$. Since T_s^+ is f^*-admissible, its super-potential are finite at every point of the form $\Lambda(S)$ for $S \in \mathscr{C}_{k-s+1}$ smooth near I^+. So they are also finite at $\Lambda(\omega^{k-s+1})$. The affinity of the super-potentials of T_s^+ implies that they are finite at $(C[I^-] + f_*(\omega^{k-s+1}))_{\text{nor}}$. So the super-potential of T_s^+ are finite at $(f_*(\omega)^{k-s+1})_{\text{nor}}$. Since for $j \geq 0$, the current $\omega^j \wedge \Lambda(\omega)^{k-s+1-j}$ is more H-regular than $\Lambda(\omega)^{k-s+1}$, we have that the super-potentials of T_s^+ are finite at $\omega^j \wedge \Lambda(\omega)^{k-s+1-j}$ (we use the symmetry of the super-potential : $\mathscr{U}_{T_s^+}(\omega^j \wedge \Lambda(\omega)^{k-s+1-j}) = \mathscr{U}_{\omega^j \wedge \Lambda(\omega)^{k-s+1-j}}(T_s^+))$. □

Corollary 3.2.12. – *The current T_s^+ gives no mass to I^- (nor I^+ by dimension's argument).*

Proof. – From above, the super-potentials of T_s^+ are finite at $\Lambda(\omega) \wedge \omega^{k-s} \in \mathscr{C}_{k-s+1}$. Observe that for two currents R and S in $\mathscr{C}_r$ and $\mathscr{C}_s$ with $r + s \leq k$, then:

$$\mathscr{U}_R(S \wedge \omega^{k+1-r-s}) = \mathscr{U}_{R\wedge\omega^{k+1-r-s}}(S) = \mathscr{U}_S(R \wedge \omega^{k+1-r-s})$$

where all the super-potentials are of same mean.

So for $R = T_s^+$ and $S = \Lambda(\omega)$, we get that the super-potentials of $\Lambda(\omega)$ are finite at $T_s^+ \wedge \omega^{k-s}$. This is equivalent to the fact that the quasi-potential $U_{\Lambda(\omega)}$ is integrable with respect to the measure $T_s^+ \wedge \omega^{k-s}$. In other words:

$$\int U_{\Lambda(\omega)}\omega^{k-s} \wedge T_s^+$$

is finite. Applying Lemma 3.2.4 to f^{-1} shows that the singularities of $U_{\Lambda(\omega)}$ are in $\log \operatorname{dist}(x, I^-)$ so T_s^+ gives no mass to I^-. □

Remark 3.2.13. – The quantity $\mathscr{U}_{T_s^+}([I^-]_{\text{nor}})$ is related to a generalized Lelong number ([**13**], see also [**9**] for generalized Lelong numbers in dynamics). Let us explain this point. From the previous proposition, we have that the super-potentials of T_s^+ are finite at $\Lambda(\omega)^{k-s+1}$.

We define the Lelong number of T_s^+ associated to the function $U_{\Lambda(\omega)}$ as:

$$\nu(T_s^+, U_{\Lambda(\omega)}) = \lim_{r\to-\infty} \int_{\{U_{\Lambda(\omega)}<r\}} T_s^+ \wedge \Lambda(\omega)^{k-s}.$$

The previous current is well defined by the theory of super-potentials: the super-potentials of T_s^+ are finite at $\omega \wedge \Lambda(\omega)^{k-s}$ which means that T_s^+ and $\Lambda(\omega)^{k-s}$ are wedgeable and their wedge product is well defined by Definition A.1.13.

As in formula (11), we have that a super-potential of $\Lambda(\omega)^{k-s+1}$ is given by:

$$\sum_{0\leq j\leq k-s} \mathscr{U}_{\Lambda(\omega)}(\omega^j \wedge \Lambda(\omega)^{k-s-j} \wedge R),$$

on currents $R \in \mathscr{C}_s$ such that $\Lambda(\omega)^{k-s}$ and R are wedgeable, so in particular for $R = T_s^+$ by the previous proposition. Since the super-potentials of T_s^+ are finite at $\Lambda(\omega)^{k-s+1}$, that implies that every term in the previous sum is finite. So we have in particular that:

$$\mathscr{U}_{\Lambda(\omega)}(\Lambda(\omega)^{k-s} \wedge T_s^+)$$

is finite. That means that the quasi-potential $U_{\Lambda(\omega)}$ is integrable with respect to the measure $\Lambda(\omega)^{k-s} \wedge T_s^+$. So we can use the bound:

$$\begin{aligned}\int_{\{U_{\Lambda(\omega)}<r\}} T_s^+ \wedge \Lambda(\omega)^{k-s} &\leq \frac{1}{r}\int_{\{U_{\Lambda(\omega)}<r\}} U_{\Lambda(\omega)} T_s^+ \wedge \Lambda(\omega)^{k-s} \\ &\leq \frac{1}{r}\int_{\mathbb{P}^k} U_{\Lambda(\omega)} T_s^+ \wedge \Lambda(\omega)^{k-s}.\end{aligned}$$

So we have that:

$$\nu(T_s^+, U_{\Lambda(\omega)}) = 0.$$

This is a generalization of the fact that a psh function finite at the point x has zero Lelong number at x.

A classical question in complex dynamics is to ask by what ω^s can be replaced. In other words, what are the currents T in $\mathscr{C}_s$ such that $L^m(T) \to T_s^+$? The following proposition and theorem give partial results toward this direction.

PROPOSITION 3.2.14. – *Let $(T_m) \in \mathscr{C}_s$ be a sequence of currents such that a super-potential $\mathscr{U}_{T_m}$ of T_m satisfies $\|\mathscr{U}_{T_m}\|_\infty = o(d^m)$. Then $L^m(T_m) \to T_s^+$ in the Hartogs' sense.*

Proof. – Recall first that if $T \in \mathscr{C}_s$ has bounded super-potential it is $(f^n)^*$-admissible (its super-potential is in particular bounded at the point $\Lambda^n(\omega^{k-s+1})$). So the sequence of pull-back is well defined by Lemma 3.1.8. Using Proposition A.1.18 and (8), a super-potential of $L^m(T_m)$ is given on smooth currents in $\mathscr{C}_{k-s+1}$ by:

$$\mathscr{U}_{L^m(T_m)} = \sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)} \circ \Lambda^n + \left(\frac{1}{d}\right)^m \mathscr{U}_{T_m} \circ \Lambda^m.$$

By Theorem 3.2.1 we know that the series $\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)} \circ \Lambda^n$ converges in the Hartogs' sense to $\mathscr{U}_{T_s^+}$. The hypothesis on (T_m) implies that $\left(\frac{1}{d}\right)^m \mathscr{U}_{T_m} \circ \Lambda^m = o(1)$ goes to 0 uniformly on smooth form. Since the control is uniform we have that $(|\mathscr{U}_{L^m(T_m)} - \mathscr{U}_{L^m(\omega^s)}| \to 0)$, and the convergence is in the Hartogs' sense and we can conclude by Proposition A.1.7. □

We also have the following result. We believe the proof is of interest although the result is essentially already known. We refer the reader to [**50**] for the case $q = 1$ and also [**27**] for the general case. See the Appendix for the notion of super-polarity.

THEOREM 3.2.15. – *There exists a super-polar set P of $\mathscr{C}_s$ such that for $S \in \mathscr{C}_s \backslash P$, $L^m(S)$ is well defined and converges to T_s^+.*

Proof. – The set of currents $S \in \mathscr{C}_s$ such that S is not $(f^m)^*$-admissible is super-polar since it is contained in the set of currents such that $\mathscr{U}_S(\Lambda^m(\omega^{k-s+1})) = -\infty$. Now, since a countable union of super-polar set is super-polar, we have that outside a super-polar set of $\mathscr{C}_s$, S is $(f^m)^*$-admissible and so $L^m(S)$ is well defined by Lemma 3.1.8.

As above, a super-potential $\mathscr{U}_{L^m(S)}$ of $L^m(S)$ is given on smooth forms by:

$$\sum_{n=0}^{m-1} \left(\frac{1}{d}\right)^n \mathscr{U}_{L(\omega^s)} \circ \Lambda^n + \left(\frac{1}{d}\right)^m \mathscr{U}_S \circ \Lambda^m,$$

where $\mathscr{U}_S$ is a super-potential of S. For $\Omega \in \mathscr{C}_{k-s+1}$ smooth, consider the current $R(\Omega) \in \mathscr{C}_{k-s+1}$ defined by $R(\Omega) := (\sum_m \left(\frac{1}{d}\right)^m \Lambda^m(\Omega))_{\text{nor}}$. Let P be the set of currents S in $\mathscr{C}_s$ such that $\mathscr{U}_S(R(\omega^{k-s+1})) = -\infty$, then P is super-polar by definition. Observe that for $\Omega \in \mathscr{C}_{k-s+1}$ smooth, we have a constant $c_\Omega > 0$ such that $R(\Omega) \leq c_\Omega R(\omega^{k-s+1})$. In particular, for $S \notin P$, we have that $\mathscr{U}_S(R(\Omega)) > -\infty$.

That implies that for any Ω smooth and $S \notin P$, the sequence $\mathscr{U}_{L^m(S)}(\Omega)$ converges to the value $\mathscr{U}_{T_s^+}(\Omega)$. Indeed, the fact that $\mathscr{U}_S(R(\Omega))$ is finite gives that:

$$\left(\frac{1}{d}\right)^m \mathscr{U}_S \circ \Lambda^m(\Omega)$$

goes to 0. So $\mathscr{U}_{L^m(S)}(\Omega)$ converges to $\mathscr{U}_{T_s^+}(\Omega)$. Then Proposition A.1.7 gives us that the sequence $L^m(S)$ converges in fact to T_s^+ (maybe not in the Hartogs' sense). □

Of course, the above theorem is not optimal and it is conjectured that for T the current of integration on a (very) generic algebraic set of dimension $k-s$, $L^m(T)$ goes to T_s^+ (see in the endomorphisms case **[5]** and **[21]** for the case of measures and see **[35]** for the case of bidegree $(1,1)$ in $\mathbb{P}^2$ and **[30]** in $\mathbb{P}^k$). That is beyond the scope of this study.

Recall that we consider the critical sets $\mathscr{C}^+$ (or $\mathscr{C}(f)$) and $\mathscr{C}^-$ (or $\mathscr{C}(f^{-1})$) defined by:

$$\mathscr{C}^+ := f^{-1}(I^-)$$
$$\mathscr{C}^- := (f^{-1})^{-1}(I^+).$$

We have the proposition:

PROPOSITION 3.2.16. – *The current T_s^+ does not give mass to the critical sets $\mathscr{C}^-$ and $\mathscr{C}^+$. In particular, the current T_s^+ satisfies the equation:*

$$f_*(T_s^+) = \frac{1}{d_s}T_s^+,$$

in $\mathbb{P}^k \setminus \mathscr{C}^-$.

We first need the following lemma:

LEMMA 3.2.17. – *Let φ be a smooth function. Then $f_*(\varphi)$ is in $L^1(T_s^+ \wedge \Lambda(\omega^{k-s}))$ and we have the identity:*

$$\int \varphi T_s^+ \wedge \omega^{k-s} = \int f_*(\varphi) T_s^+ \wedge \Lambda(\omega^{k-s})$$

Proof of the lemma. – First φ is in $L^1(T_s^+ \wedge \omega^{k-s})$ and the quantity:

$$\int \varphi T_s^+ \wedge \omega^{k-s}$$

depends continuously on T_s^+ in the sense of currents.

On the other hand, $f_*(\varphi)$ is in $L^1(T_s^+ \wedge \Lambda(\omega^{k-s}))$ since it is smooth and uniformly bounded outside I^- which has no mass for $T_s^+ \wedge \Lambda(\omega^{k-s})$ (see Remark 3.2.13). Recall that $T_s^+ \wedge \Lambda(\omega)^{k-s}$ is f^*-admissible by Remark 3.2.13 (hence $\mathscr{U}_{\Lambda(\omega)}(\Lambda(\omega)^{k-s} \wedge T_s^+)$ is finite). So we have that

$$\int f_*(\varphi) T_s^+ \wedge \Lambda(\omega^{k-s}) = \int \varphi L(T_s^+ \wedge \Lambda(\omega^{k-s})),$$

as this stands if $T_s^+ \wedge \Lambda(\omega^{k-s})$ was smooth and we can conclude by Hartogs' convergence. Now, applying Lemma A.2.2 to f^{-1} and the invariance of T_s^+, we have that $L(T_s^+ \wedge \Lambda(\omega^{k-s})) = T_s^+ \wedge \omega^{k-s}$. □

Proof of the proposition. – We consider $\mathscr{C}^+$ first. Let V_ε be a small neighborhood of I^+. Since T_s^+ gives no mass to I^+ we can assume that the mass of V_ε for T_s^+ is arbitrarily small. Let W_α be a small neighborhood of $\mathscr{C}^+$. Let $0 \le \varphi \le 1$ be a smooth function such that $\varphi = 1$ in $W_\alpha \backslash V_\varepsilon$, $\varphi = 0$ in $V_{2^{-1}\varepsilon}$ and $\varphi = 0$ outside $W_{2\alpha}$.

Then by the previous lemma:

$$\begin{aligned} \|W_\alpha \backslash V_\varepsilon\|_{T_s^+} &\le \int \varphi T_s^+ \wedge \omega^{k-s} \\ &\le \int f_*(\varphi) T_s^+ \wedge \Lambda(\omega^{k-s}) \\ &\le \int_{f(W_{2\alpha}\backslash V_{2^{-1}\varepsilon})} T_s^+ \wedge \Lambda(\omega^{k-s}) \end{aligned}$$

Now, $f(W_{2\alpha}\backslash V_{2^{-1}\varepsilon})$ is a neighborhood W of I^- that can be taken arbitrarily small by taking ε and α small enough. We have seen in Remark 3.2.13 that the quantity $\int_W T_s^+ \wedge \Lambda(\omega)^{k-s}$ goes to the Lelong number $\nu(T_s^+, U_{\Lambda(\omega)})$ which is equal to zero. Thus $\mathscr{C}^+$ has no mass for T_s^+.

For $\mathscr{C}^-$, we write $T_s^+ = T_{\mathscr{C}^-} + T'$ where $T_{\mathscr{C}^-}$ is a positive closed current with support in $\mathscr{C}^-$ and T' is a positive closed current with no mass on $\mathscr{C}^-$ ([**52**]). Both currents are f^*-admissible since T_s^+ is. But $f^*(T_{\mathscr{C}^-})$ has support in $\mathscr{C}^+$ which means it is equal to zero since $T_s^+ = d_s^{-1}(f^*(T_{\mathscr{C}^-}) + f^*(T'))$ gives no mass to $\mathscr{C}^+$. That implies that $T_{\mathscr{C}^-} = 0$ hence T_s^+ has no mass on $\mathscr{C}^-$.

Now $f : \mathbb{P}^k \setminus \mathscr{C}^+ \to \mathbb{P}^k \setminus \mathscr{C}^-$ is a proper biholomorphism that we denote by f_1. If Θ is a smooth form in $\mathbb{P}^k \setminus \mathscr{C}^-$ then using the invariance of T_s^+:

$$\begin{aligned} \langle (f_1)_*(T_s^+), \Theta\rangle &= \langle T_s^+, (f_1)^*(\Theta)\rangle \\ &= \langle \frac{1}{d^s} f^*(T_s^+), (f_1)^*(\Theta)\rangle \\ &= \langle \frac{1}{d^s} T_s^+, f_*((f_1)^*(\Theta))\rangle. \end{aligned}$$

But $f_*(f_1)^* = (f_1)_*(f_1)^* = \mathrm{id}$ so $f_*((f_1)^*(\Theta)) = \Theta$ and the result follows. □

Remark 3.2.18. – In order to define $\Lambda(T_s^+)$, we need to add to the equation

$$(f_1)_*(T_s^+) = \frac{1}{d_s} T_s^+$$

a term with mass $d_{k-s} - d_s^{-1}$ and support in $\mathscr{C}^-$ in order to obtain a current of mass d_{k-s}. For example, in the case of Hénon maps, we need to add a multiple of the current of integration on the line at infinity. In general, such choice is not clear as the hypersurface $\mathscr{C}^-$ carries many positive closed currents of bidimension $(k-s, k-s)$ and they might be no way to add a current to the equation $(f_1)_*(T_s^+) = \frac{1}{d_s} T_s^+$ in a continuous way.

The previous corollary implies that the Green current is meaningful. For example, in the case of Hénon maps in $\mathbb{P}^2$, the set $\mathscr{C}^-$ and $\mathscr{C}^+$ are in fact L_∞ (the line at infinity) which is totally invariant and the Green current T^+ gives no mass to L_∞.

We can now prove the following stronger result of extremality which implies strong ergodic properties (see Theorem 3.4.15). The inequalities between currents in $\mathscr{C}_s$ have to be understood in the strong sense.

Theorem 3.2.19. – *The current T_s^+ is extremal in $\mathscr{C}_s$, that is if there exists a $c > 0$ and $S \in \mathscr{C}_s$ such that $S \leq cT_s^+$ then $S = T_s^+$.*

Proof. – Applying the previous results to f^n gives that T_s^+ gives no mass to the indeterminacy sets $I(f^{\pm n})$ and critical sets $\mathscr{C}^{\pm n}$ of f^n and f^{-n} (recall that T_s^+ is also the Green current of f^n). Let S be as above, in particular S gives no mass to the sets $I(f^{\pm n})$ and $\mathscr{C}(f^{\pm n})$ and S is $(f^n)^*$-admissible for all n. By Lemma 3.1.8, $L^n(S)$ is well defined for all n and equal to $L_n(S)$ (L_n and Λ_n are the normalized pull-pack and push-forward associated to f^n).

For $n > 0$, we denote by Λ_n' the push forward operator $(f^n)_*$ from $\mathbb{P}^k \setminus \mathscr{C}(f^n)$ to $\mathbb{P}^k \setminus \mathscr{C}(f^{-n})$.

The operator (Λ_n') is positive. That and the previous proposition applied to f^n imply that $(d_s)^n(\Lambda_n')(S) \leq cT_s^+$ in $\mathbb{P}^k \setminus \mathscr{C}(f^{-n})$. We denote by S_n the trivial extension of $(d_s)^n(\Lambda_n')(S)$ over $\mathbb{P}^k$: it does exist since the current $(d_s)^n(\Lambda_n')(S)$ is of bounded mass. We have $S_n \leq cT_s^+$ in $\mathbb{P}^k$. In particular, S_n is $(f^*)^n$-admissible. Using the same argument as in the previous proposition, we see that:

$$(f^n)^*(S_n) = (d_s)^n S,$$

outside $\mathscr{C}(f^n)$. Now S has no mass on $\mathscr{C}(f^n)$ and $(f^n)^*(S_n)$ is less than $c(d^s)^n T_s^+$ (by positivity of the operator $(f^n)^*$) which implies that $(f^n)^*(S_n)$ also has no mass on $I(f^n) \cup \mathscr{C}(f^n)$. So we have:

$$(f^n)^*(S_n) = (d_s)^n S,$$

on $\mathbb{P}^k$. In particular, S_n has mass 1. We have that $L_n(S_n) = S$ and since S_n is $(f^*)^n$-admissible we have $L_n(S_n) = L^n(S_n)$ by Lemma 3.1.8.

For $R \in C^{k-s+1}$ smooth, let $\mathscr{U}_{S_n}$, $\mathscr{U}_{T_s^+,0}$ and $\mathscr{U}_{\Lambda^j(R)}$ be the super-potential of S_n, T_s^+ and $\Lambda^j(R)$ of mean 0. We have from Proposition A.1.18 and (8) that a super-potential $\mathscr{U}_{L^n(S_n)}$ of $L^n(S_n) = S$ on smooth forms is given by:

$$\sum_{j=0}^{n-1} \left(\frac{1}{d}\right)^j \mathscr{U}_{L(\omega^s)} \circ \Lambda^j + \left(\frac{1}{d}\right)^n \mathscr{U}_{S_n} \circ \Lambda^n.$$

Recall that there exists a $M > 0$ that does not depend on neither R nor n such that $\mathscr{U}_{\Lambda^n(R)} - M$ is negative and $\mathscr{U}_{S_n} \leq M$. More precisely, by Proposition A.1.1, there exists a quasi-potential $U_{\Lambda^n(R)}$ of $\Lambda^n(R)$ such that $U_{\Lambda^n(R)} - M\omega^{k-s}$ is negative ($U_{\Lambda^n(R)}$ is a quasi-potential of $\Lambda^n(R)$ of mean 0). Then, we have that:

$$(\mathcal{U}_{\Lambda^n(R)} - M)(S_n) \geq c(\mathcal{U}_{\Lambda^n(R)} - M)(T_s^+).$$

Indeed if S_n and T_s^+ were smooth, it would follow from the fact that $U_{\Lambda^n(R)} - M\omega^{k-s}$ is negative and that $S_n \leq cT_s^+$. The result follows then by Hartogs' convergence: observe that the regularization is obtained by a mean of the composition over the automorphisms of $\mathbb{P}^k$ thus the approximations S'_n and T'^+_s of S_n and T_s^+ also satisfy $S'_n \leq cT'^+_s$.

So we have the estimates:

$$\begin{aligned}
\mathcal{U}_{S_n}(\Lambda^n(R)) &= \mathcal{U}_{\Lambda^n(R)}(S_n) \\
&= (\mathcal{U}_{\Lambda^n(R)} - M)(S_n) + \langle S_n, M\omega^{k-s}\rangle \\
&\geq c(\mathcal{U}_{\Lambda^n(R)} - M)(T_s^+) + \langle S_n, M\omega^{k-s}\rangle \\
&\geq c\mathcal{U}_{T_s^+,0}(\Lambda^n(R)) + \langle S_n - cT_s^+, M\omega^{k-s}\rangle \\
&\geq c\mathcal{U}_{T_s^+,0}(\Lambda^n(R)) + M(1-c).
\end{aligned}$$

The last term multiplied by d^{-n} goes to zero by Corollary 3.2.10. So the super-potential $\mathcal{U}_{L^n(S_n)}$ converges to a super-potential of T_s^+ on smooth forms. By Proposition A.1.7, that implies that $S = T_s^+$. □

The results of this section remain valid for f^{-1}. So we can construct the Green current of order $k-s$ for f^{-1} that we denote by T_{k-s}^-.

3.3. Proof of the genericity (Theorem 3.1.11)

This section is devoted to the proof of Theorem 3.1.11, it is independent from the rest of the paper. The statement of Theorem 3.1.11 is similar to Proposition 4.5 in [**1**]. In addition, we show here that the genericity stands in any orbit. The idea of the proof is to construct in any orbit an element in $\mathcal{B}$ and then to show that the series in Definition 3.1.9 vary as a difference of psh functions (dsh) along the orbit.

Construction of an example stable by perturbations. – Take $f \in E_s$.

We have that:

$$I(A \circ f) = I(f)$$

and

$$I(f^{-1} \circ A^{-1}) = AI(f^{-1}).$$

In particular, for A generic, we can assume that $I^+ \cap I^- = \varnothing$. Remark also that the dimension of these indeterminacy sets does not depend on A.

Consider a projective linear subspace E of $\mathbb{P}^k$ of dimension s such that $E \cap I(f) = \varnothing$. Let V be a neighborhood of E such that $\overline{V} \cap I(f) = \varnothing$.

Choose the coordinates $[z_0 : \ldots : z_k]$ in $\mathbb{P}^k$ such that E is given by $z_0 = \cdots = z_{k-s-1} = 0$ and so that if $E' = \{z_{k-s} = \cdots = z_k = 0\}$ then $E' \cap I(f^{-1}) = \varnothing$, $E' \cap \overline{V} = \varnothing$ and $E' \cap \overline{f(V)} = \varnothing$ (this is always possible). Consider the element A_0 of $\mathrm{PGL}(k+1, \mathbb{C})$ defined by $A_0([z_0 : \ldots : z_k]) = [\lambda z_0 : \ldots : \lambda z_{k-s-1} : z_{k-s} : \ldots : z_k]$ with $\lambda > 0$. We take λ small enough so that:

— $A_0(I(f^{-1})) \subset V$;
— $A_0(f(V)) \subset V$.

Consider the element f_{A_0} defined by $f_{A_0} = A_0 \circ f$. Now, $I(f_{A_0}) = I(f)$ and $I((f_{A_0})^{-1}) = A_0(I(f^{-1})) \subset V$. We have the inclusion:

$$f_{A_0}(I((f_{A_0})^{-1})) = f_{A_0} A_0(I(f^{-1})) \subset (A_0 \circ f)(V) \subset V.$$

An immediate induction gives that $(f_{A_0})^n(I((f_{A_0})^{-1})) \subset V$. So the element f_{A_0} satisfies the first half of Hypothesis 3.1.10 since

$$\mathrm{dist}(I(f_{A_0}), f_{A_0}^n(I(f_{A_0}^{-1}))) \geq \mathrm{dist}(I(f), V) > 0.$$

For each $n, m \in \mathbb{N}$, the condition $f^n(I(f^{-1})) \cap f^{-m}(I(f)) \neq \varnothing$ is algebraic (and not always satisfied by the above), so (7) is satisfied outside a countable union of subvarieties of $\mathrm{PGL}(k+1, \mathbb{C})$. Wherever all these conditions are satisfied, namely wherever (7) holds the dynamical degrees are given by Proposition 3.1.6 so they are constant.

We want to show that we can perturbate that example. Fix E and $\overline{V}$ as above. If $\Phi(f) := A \circ f$ with A close to A_0, then we still have $I(\Phi(f)^{-1}) = AI(f^{-1}) \subset V$ and $A \circ f(V) \subset V$. Thus $\Phi(f)^n(I(\Phi(f)^{-1})) \subset V$ and $I(\Phi(f)) = I(f)$.

This implies that there exists some $\alpha > 0$ such that for every A in a small neighborhood W_0' of A_0 and every $n \in \mathbb{N}$, we have:

$$\mathrm{dist}(I(\Phi(f)), \Phi(f)^n(I(\Phi(f)^{-1}))) \geq \alpha.$$

Since $\mathscr{B}$ and Hypothesis 3.1.10 are invariant by conjugaison and since taking the inverse is a biholomorphism on $\mathrm{PGL}(k+1, \mathbb{C})$, it is equivalent to prove that the set of linear map $A \in \mathrm{PGL}(k+1, \mathbb{C})$ such that $f \circ A \notin \mathscr{B}$ is pluripolar. By the above, we know that there exists a small open set W_0' in $\mathrm{PGL}(k+1, \mathbb{C})$ where the first series in the definition of $\mathscr{B}$ converges.

Now we prove the genericity. In what follows, C denotes a constant independent of n that may change from line to line. Let $W := \mathrm{PGL}(k+1, \mathbb{C})$. It is a Zariski dense open set in the projective space $\widetilde{W} = \mathbb{P}^l$ where $l = (k+1)^2 - 1$. Let c denote the homogeneous coordinate on $\widetilde{W}$. When $c \in W$, we write f_c instead of $f \circ c$. We can extend this notation for $c \in \widetilde{W}$. Of course, in this case f_c is not a birational map.

Consider the rational map:

$$\widetilde{F}_n : \widetilde{W} \times \mathbb{P}^k \to \widetilde{W} \times \mathbb{P}^k$$
$$(c, z) \mapsto (c, f_c^n(z)).$$

Let Π_i denote the canonical projections of $\widetilde{W} \times \mathbb{P}^k$ to its factor for $i = 1, 2$. In $\widetilde{W} \times \mathbb{P}^k$, let $\omega_i := \Pi_i^*(\omega_{FS})$ be the pull-back of the Fubini-Study form by the projection for $i = 1, 2$. That way, $\omega_1 + \omega_2$ is a Kähler form on $\widetilde{W} \times \mathbb{P}^k$.

Action of $\widetilde{F}_n^*$ on the cohomology. – We study the action of $\widetilde{F}_n^*$ on ω_2. Write $c = [c_1 : \ldots : c_{l+1}]$. First we have $\widetilde{F}_n(c, z) = (c, f_c^n(z))$ where the second coordinate is a polynomial of degree d^n in the z_i, of degree $\leq Cd^n$ in the c_i. We compute the mass of $\widetilde{F}_n^*(\omega_2)$. For that, we test against $(\omega_1 + \omega_2)^{k+l-1}$. We expand $(\omega_1 + \omega_2)^{k+l-1}$:

$$(\omega_1 + \omega_2)^{k+l-1} = \sum_{i=0}^{k+l-1} \binom{k+l-1}{i} \omega_1^i \wedge \omega_2^{k+l-1-i}.$$

We have that $\omega_1^i = 0$ for $i > l$ and $\omega_2^{k+l-1-i} = 0$ for $k + l - 1 - i > k$ thus $i < l - 1$. So there are only two terms in the previous sum: for $i = l - 1$ and for $i = l$:

$$\langle \widetilde{F}_n^*(\omega_2), \omega_1^{l-1} \wedge \omega_2^k \rangle \quad \text{and} \quad \langle \widetilde{F}_n^*(\omega_2), \omega_1^l \wedge \omega_2^{k-1} \rangle.$$

By Bézout's theorem, those two terms are $\leq Cd^n$ (the terms can be computed in cohomology so we replace their factors by analytic sets). Here, we use that $\widetilde{F}_n(c, z)$ is a polynomial of degree d^n in the z_i and of degree $\leq Cd^n$ in the c_i.

As a result, we have that:

$$\|\widetilde{F}_n^*(\omega_2)\| \leq Cd^n.$$

and consequently:

$$\|\widetilde{F}_n^*(\omega_2^s)\| \leq Cd^{sn}.$$

We also remark that:

$$\widetilde{F}_n^*(\omega_1) = \omega_1$$

since $\widetilde{F}_n$ acts as the identity on $\widetilde{W}$.

Recall that a map in $\mathscr{B}$ needs to satisfies the condition:

$$\sum_{n=0}^{\infty} \left(\frac{1}{d^s}\right)^n (\mathrm{vol}(I^-))^{-1} \int_{f^n(I^-)} U_{L(\omega)} L(\omega^{s-1}) > -\infty \tag{14}$$

Construction of a function g that test the convergence of (14). – We can write $\widetilde{F}_1^*(\omega_2^s)$ in cohomology:

$$\widetilde{F}_1^*(\omega_2^s) = \sum_{i_1+i_2=s} a_{i_1,i_2}\omega_1^{i_1}\wedge\omega_2^{i_2} + dd^c\mathscr{U}$$

where $\mathscr{U}$ is a negative $(s-1,s-1)$ current, which is C^1 where $\widetilde{F}_1^*(\omega_2^s)$ is smooth (see Proposition 2.3.1 in [**31**] and observe that $\widetilde{W}\times\mathbb{P}^k$ is homogeneous). We also denote $\widetilde{\Omega} = \sum_{i_1+i_2=s} a_{i_1,i_2}\omega_1^{i_1}\wedge\omega_2^{i_2}$ the smooth form cohomologous to $\widetilde{F}_1^*(\omega_2^s)$. Testing against $\omega_1^a\wedge\omega_2^b$ for $a+b=l+k-s$ gives that $a_{i_1,i_2}\geq 0$. In what follows, we take a particular $\mathscr{U}$. We explain now its construction.

LEMMA 3.3.1. – *The indeterminacy set of $\widetilde{F}_1$ has dimension $l+k-s-1$.*

Proof. – The lemma is clear in $W\times\mathbb{P}^k$. In $(\widetilde{W}\setminus W)\times\mathbb{P}^k$, we use a stratification with the dimension of the kernel of c.

Indeed, let F_j be the set of $A\in\widetilde{W}$ such that $\dim(\mathrm{Ker}(A))=j$ in $\mathbb{C}^{k+1}$. Then, F_j is an analytic set of dimension $(k+1)^2-j^2-1$ (choose $k+1-j$ independant vectors in $\mathbb{C}^{k+1}$ then choose j vectors colinear to the first ones and then project to $\mathbb{P}^l$). The indeterminacy set of $\widetilde{F}_1$ is given by $\mathrm{Ker}(A)\cup A^{-1}(I(f))$.

Now, $\dim(A^{-1}(I(f)))\leq k-s-1+j$ so $\dim(\mathrm{Ker}(A)\cup A^{-1}(I(f)))\leq k-s-1+j$ and the dimension of the indeterminacy set of $\widetilde{F}_1$ restricted to F_j is of dimension $\leq (k+1)^2-j^2-1+k-s-1+j\leq (k+1)^2-1+k-s-1=l+k-s-1$. □

In particular, by Theorem 4.5 in [**14**, Chapter III], we have that

$$\widetilde{F}_1^*(\omega_2^s) = (\widetilde{F}_1^*(\omega_2))^s.$$

Since $\widetilde{W}\times\mathbb{P}^k$ is a product of projective spaces, then the positive form $\widetilde{F}_1^*(\omega_2)$ is cohomologous to a positive form β. Let u be a quasi-potential of $\widetilde{F}_1^*(\omega_2)$. In other words, $\widetilde{F}_1^*(\omega_2)=\beta+dd^cu$. We can write $\mathscr{U}$ as in the proof of Theorem 3.2.1, that is:

$$\mathscr{U} = \sum_{j=0}^{s-1} u\widetilde{F}_1^*(\omega_2)^{s-1-j}\wedge\beta^j.$$

In this case $\widetilde{\Omega}=\beta^s$.

We define $\mathscr{I}^- := \widetilde{W}\times I^-$. It is an analytic set of $\widetilde{W}\times\mathbb{P}^k$ of dimension $l+s-1$ such that for $c\in W$, $\mathscr{I}^-\cap\{c\}\times\mathbb{P}^k = I^-(f_c)=I^-$.

Let $[\mathscr{I}^-]$ denote the current of integration on $\mathscr{I}^-$, it is of bidimension $(l+s-1,l+s-1)$. Consider the set

$$V_n := \{c\in\widetilde{W}, f_c^{-n}(I^+(f_c))\cap I^- = \varnothing\},$$

it is a Zariski dense open set in $\widetilde{W}$.

LEMMA 3.3.2. – *The codimension of $V_n^c=\widetilde{W}\setminus V_n$ is ≥ 2.*

Proof. – Let $E' \subset E$ be two linear projective subspaces of $\mathbb{P}^k$ with $\dim(E') = s-1$ and $\dim(E) = s$. Assume that $E \cap I^+(f) = \varnothing$. Choose some projective coordinates $[z_0 : \ldots : z_k]$ such that:

$$E' = \{z_s = \cdots = z_k = 0\},\ E = \{z_{s+1} = \cdots = z_k = 0\}.$$

We denote:

$$F' = \{z_0 = \cdots = z_{s-1} = 0\},\ F = \{z_0 = \cdots = z_s = 0\}.$$

In particular, $\dim(F') = k-s$ and $\dim(F) = k-s-1$. Changing the coordinate if necessary, we can assume that $F \cap f(E) = \varnothing$ and $F' \cap f(E') = \varnothing$ and $I^- \cap F' = \varnothing$ (thus $I^- \cap F = \varnothing$). Now we consider the following elements of $\widetilde{W}$:

$$A' := [z_0 : \ldots : z_k] \mapsto [z_0 : \ldots : z_{s-1} : 0 : \ldots : 0]$$
$$A := [z_0 : \ldots : z_k] \mapsto [z_0 : \ldots : z_s : 0 : \ldots : 0].$$

Then observe that $f \circ A(\mathbb{P}^k \setminus F) = f(E)$ and that $f \circ A$ is holomorphic on $f(E)$ with $f \circ A(f(E)) \subset f(E)$; in particular $I((f \circ A)^n) = F$. Similarly $f \circ A'(\mathbb{P}^k \setminus F') = f(E')$, $f \circ A'$ is holomorphic on $f(E')$ with $f \circ A'(f(E')) \subset f(E')$; in particular $I((f \circ A')^n) = F'$. We deduce that $A, A' \notin V_n^c$.

Now consider the compactification $\widetilde{\mathrm{PGL}(E, \mathbb{C})} \subset \widetilde{W}$ where the inclusion is given by the map $B \mapsto B \circ A$ (with obvious notations). Then $\widetilde{\mathrm{PGL}(E, \mathbb{C})}$ is an analytic set of $\widetilde{W}$ of dimension $(s+1)^2 - 1$. First, if $B \in \mathrm{PGL}(E, \mathbb{C})$ then $f \circ B \circ A$ has the same indeterminacy set than $f \circ A$ and the same image, so $B \circ A \notin V_n^c$. Now consider $\widetilde{\mathrm{PGL}(E, \mathbb{C})} \setminus \mathrm{PGL}(E, \mathbb{C})$, this set is an algebraic irreducible set of codimension 1. Its intersection with V_n^c is not everything since $A' \notin V_n^c$. In particular, $V_n^c \cap \widetilde{\mathrm{PGL}(E, \mathbb{C})}$ is an analytic set of codimension ≥ 2. This implies that V_n^c is not of codimension 1. □

Now, consider on $\cap_{i=0}^{n+1} V_i$:

$$\varphi_n := (\Pi_1)_* \Big(\sum_{j=0}^{s-1} \widetilde{F}_n^* u \widetilde{F}_{n+1}^* (\omega_2)^{s-1-j} \wedge \widetilde{F}_n^* (\beta)^j \wedge [\mathscr{I}^-]\Big).$$

In other words, $\varphi_n = (\Pi_1)_* (\widetilde{F}_n^* (\mathscr{U}) \wedge [\mathscr{I}^-])$.

On the set $\cap_{i=0}^{n+1} V_i$, φ_n is continuous since Π_1 restricted to $\mathscr{I}^-$ is a submersion so the push-forward of a continuous form is continuous. Finally, we define on $\cap_{n=0}^{N+1} V_n$,

$$g_N := \sum_{n=0}^{N} d^{-sn} \varphi_n.$$

Computation of dd^cg_N. – First, we compute dd^cg_N on $\cap_{n=0}^{N+1}V_n$. We have that:

$$dd^cg_N=(\Pi_1)_*\left(\sum_{n=0}^{N}d^{-sn}\sum_{j=0}^{s-1}dd^c\widetilde{F}_n^*u\wedge\widetilde{F}_{n+1}^*(\omega_2)^{s-1-j}\wedge\widetilde{F}_n^*(\beta)^j\wedge[\mathscr{I}^-]\right).$$

Recall that $dd^cu=\widetilde{F}_1^*(\omega_2)-\beta$. So, near $\mathscr{I}^-$

$$dd^c\widetilde{F}_n^*u=\widetilde{F}_{n+1}^*(\omega_2)-\widetilde{F}_n^*(\beta).$$

We obtain

$$dd^cg_N=$$

$$(\Pi_1)_*\left(\sum_{n=0}^{N}d^{-sn}\sum_{j=0}^{s-1}(\widetilde{F}_{n+1}^*(\omega_2)^{s-j}\wedge\widetilde{F}_n^*(\beta)^j-\widetilde{F}_{n+1}^*(\omega_2)^{s-1-j}\wedge\widetilde{F}_n^*(\beta)^{j+1})\wedge[\mathscr{I}^-]\right)$$

which is equal to

$$dd^cg_N=(\Pi_1)_*\left(\sum_{n=0}^{N}d^{-sn}(\widetilde{F}_{n+1}^*(\omega_2^s)-\widetilde{F}_n^*(\beta^s))\wedge[\mathscr{I}^-]\right).$$

Recall that $\beta^s=\widetilde{\Omega}=\sum_{i_1+i_2=s}a_{i_1,i_2}\omega_1^{i_1}\wedge\omega_2^{i_2}$. We show now that $a_{0,s}=d^s$. First we have that:

$$\langle\widetilde{\Omega},\omega_2^{k-s}\wedge\omega_1^l\rangle=a_{0,s}.$$

Since $\widetilde{\Omega}$ and $\widetilde{F}_1^*(\omega_2^s)$ are cohomologous, we deduce:

$$\langle\widetilde{F}_1^*(\omega_2^s),\omega_2^{k-s}\wedge\omega_1^l\rangle=a_{0,s}.$$

So, we want to compute:

$$\int\widetilde{F}_1^*(\omega_2^s)\wedge\omega_2^{k-s}\wedge\omega_1^l.$$

This can be done in cohomology. If L_{k-s} is a generic analytic subspace of dimension $k-s$ in $\mathbb{P}^k$ and L_s is a generic analytic subspace of dimension s in $\mathbb{P}^k$ and $\{c\}\times\mathbb{P}^k$ is a slice, then the previous quantity is the number of intersections of $f_c^{-1}(L_{k-s})\cap L_s$ on the slice. This is equal to d^s since the degree d_s of f_c is d^s on W which is a Zariski open set in $\widetilde{W}$, so we have indeed that $d_s=a_{0,s}$.

We have the equality:

$$\widetilde{F}_n^*(\widetilde{\Omega})=d^s\widetilde{F}_n^*(\omega_2^s)+\sum_{i_1+i_2=s,\ i_2\neq s}a_{i_1,i_2}\widetilde{F}_n^*(\omega_1^{i_1}\wedge\omega_2^{i_2}).$$

We denote the second term on the right-hand side by S_n. Since $\widetilde{F}_n^*(\omega_1)=\omega_1$, we can bound the mass of S_n by:

$$\begin{aligned}\|S_n\|&\leq\sum_{i_1+i_2=s,\ i_2\neq s}a_{i_1,i_2}\|\omega_1^{i_1}\wedge\widetilde{F}_n^*(\omega_2^{i_2})\|\\&\leq Cd^{n(s-1)}\end{aligned}$$

since $\|\widetilde{F}_n^*(\omega_2^j)\| \leq Cd^{jn}$ for $j \leq s$. So replacing in $dd^c g_N$, we have:

$$\begin{aligned} dd^c g_N =& (\Pi_1)_* \left(\sum_{n=0}^{N} d^{-sn} (\widetilde{F}_{n+1}^*(\omega_2^s) - d^s \widetilde{F}_n^*(\omega_2^s)) \wedge [\mathscr{I}^-] \right) \\ & - (\Pi_1)_* \left(\sum_{n=0}^{N} d^{-sn} S_n \wedge [\mathscr{I}^-] \right). \end{aligned}$$

The second term in the right-hand side is a positive closed current with mass uniformly bounded in N by the above. We now control the first term. Reorganizing the sum, we see that it is equal to:

$$(\Pi_1)_*((d^{-sN} \widetilde{F}_{N+1}^*(\omega_2^s) - d^s \omega_2^s) \wedge [\mathscr{I}^-]).$$

Using the fact that the mass of the positive closed current $\widetilde{F}_{N+1}^*(\omega_2^s)$ is bounded by $Cd^{s(N+1)}$ gives that, on $\cap_{n=0}^{N+1} V_n$:

$$dd^c g_N = \Omega_{1,N}^+ - \Omega_{2,N}^+$$

where $\Omega_{i,N}^+$ is a positive closed current of bidegree $(1,1)$ on $\cap_{n=0}^{N+1} V_n$ with mass less than a constant C (C is independent of N). So, the current $\Omega_{i,N}^+$ has an extension to a positive closed current $\widetilde{\Omega_{i,N}^+}$ with $\|\widetilde{\Omega_{i,N}^+}\| \leq C$. Now, we prove that the function g_N is in $L^1(\widetilde{W})$ and that $dd^c g_N = \widetilde{\Omega_{1,N}^+} - \widetilde{\Omega_{2,N}^+}$ in the sense of currents of $\widetilde{W}$. Take an open set U in $\widetilde{W}$ biholomorphic to a polydisk. On U, we have $\widetilde{\Omega_{i,N}^+} = dd^c u_{i,N}$ where $u_{i,N}$ is a negative psh function. So, on $U \cap (\cap_{n=0}^{N+1} V_n)$,

$$g_N + u_{2,N} = u_{1,N} + h_N$$

with h_N a harmonic function. The equality stands almost everywhere but since g_N is continuous on $\cap_{n=0}^{N+1} V_n$, the equality is true on $U \cap (\cap_{n=0}^{N+1} V_n)$ (see the proof of the inequality $g - m \geq \psi_1 - \psi_2$ below). Since g_N is non positive, the function $u_{1,N} + h_N$ has a psh extension $\widetilde{u_{1,N} + h_N}$ to U. So g_N has an extension $\widetilde{g_N} = \widetilde{u_{1,N} + h_N} - u_{2,N}$ to U. In particular, g_N is in $L^1(\widetilde{W})$ and $dd^c g_N$ has a sense globally. Moreover, since the codimension of V_n^c is larger than 2, we have on U (in the sense of currents):

$$dd^c g_N = dd^c \widetilde{g_N} = dd^c \widetilde{u_{1,N} + h_N} - dd^c u_{2,N} = \widetilde{\Omega_{1,N}^+} - \widetilde{\Omega_{2,N}^+}.$$

In particular, we have $dd^c g_N = \widetilde{\Omega_{1,N}^+} - \widetilde{\Omega_{2,N}^+}$ globally on $\widetilde{W}$.

Now, we write

$$\widetilde{\Omega_{i,N}^+} = a_{i,N}\omega_1 + dd^c \psi_{i,N}$$

with $a_{i,N}$ smaller than C.

We say that a measure is PLB if the qpsh functions are integrable for that measure. Any measure given by a smooth distribution is PLB. In particular, we can find a PLB

probability measure that we denote ν with support in the W_0' defined previously. We have the following lemma (see Proposition 2.4 in [**27**]):

LEMMA 3.3.3. – *The family of qpsh functions in $\widetilde{W}$ such that $dd^c\psi \geq -\omega_1$ and one of the two following conditions:*

$$\max_{\widetilde{W}} \psi = 0 \text{ or } \int \psi d\nu = 0$$

is bounded in $L^1(\nu)$ and is bounded from above.

In particular, in the writing

$$\widetilde{\Omega_{i,N}^+} = a_{i,N}\omega_1 + dd^c\psi_{i,N}$$

we take the normalization $\int \psi_{i,N} d\nu = 0$.

Link with Definition 3.1.9. – Let $c \in W \cap (\cap_{i\leq n+1} V_i)$, then we want to show that:

$$\varphi_n(c) = \int_{f_c^n(I^-(f_c))} U_c$$

where $(f_c)^*(\omega_2^s) = d^s\omega_2^s + dd^cU_c$. First, when $c \in W \cap (\cap_{i\leq n+1} V_i)$, we have:

$$\varphi_n(c) = \int_{I^-(f_c)} \widetilde{F}_n^*(\mathcal{U})_{|\{c\}\times\mathbb{P}^k}.$$

Here, $\widetilde{F}_n^*(\mathcal{U})_{|\{c\}\times\mathbb{P}^k}$ is the restriction of $\widetilde{F}_n^*(\mathcal{U})$ on $\{c\}\times\mathbb{P}^k$ which is well defined because $\widetilde{F}_n^*(\mathcal{U})$ is continuous near $\{c\}\times I^-(f_c)$. But $\widetilde{F}_n^*(\mathcal{U})_{|\{c\}\times\mathbb{P}^k}$ is equal to $(f_c^n)^*(\mathcal{U}_{|\{c\}\times\mathbb{P}^k})$, so

$$\varphi_n(c) = \int_{f_c^n(I^-(f_c))} \mathcal{U}_{|\{c\}\times\mathbb{P}^k}.$$

Recall that

$$\mathcal{U} = \sum_{j=0}^{s-1} u\widetilde{F}_1^*(\omega_2)^{s-1-j} \wedge \beta^j.$$

In particular, $\mathcal{U}_{|\{c\}\times\mathbb{P}^k}$ near $f_c^n(I^-(f_c))$ can be written

$$U_c := \sum_{j=0}^{s-1} u_{|\{c\}\times\mathbb{P}^k} f_c^*(\omega_2)^{s-1-j} \wedge d^j\omega_2^j$$

because the coefficient of β in ω_2 is d ($\beta = d\omega_2 + \dots$). We show now that $dd^cU_c = (f_c)^*(\omega_2^s) - d^s\omega_2^s$.

The singularities of $u_{|\{c\}\times\mathbb{P}^k}$ are in $I^+(f_c)$, so by Theorem 4.5 in [**14**, Chapter III], we have that U_c and $dd^cU_c = \sum_{j=0}^{s-1} dd^c(u_{|\{c\}\times\mathbb{P}^k}) \wedge f_c^*(\omega_2)^{s-1-j} \wedge d^j\omega_2^j$ are well defined in all $\mathbb{P}^k$. Taking the restriction of the equation $\widetilde{F}_1^*(\omega_2) = \beta + dd^cu$ on $\{c\}\times\mathbb{P}^k$, we obtain

$$f_c^*(\omega_2) = d\omega_2 + (dd^cu)_{|\{c\}\times\mathbb{P}^k} = d\omega_2 + dd^c(u_{|\{c\}\times\mathbb{P}^k})$$

since it is true outside $I^+(f_c)$ and $f_c^*(\omega_2)$ or $dd^c(u_{|\{c\}\times\mathbb{P}^k})$ have no mass on this set of dimension $k-s-1$. Moreover u is a qpsh function, so it takes a value at every point.

Finally,

$$dd^c U_c = \sum_{j=0}^{s-1} (f_c^*(\omega_2) - d\omega_2) \wedge f_c^*(\omega_2)^{s-1-j} \wedge d^j \omega_2^j = (f_c)^*(\omega_2^s) - d^s \omega_2^s.$$

Proof of the genericity. – Recall that φ_n is continuous on $\cap_{i\leq n+1} V_i$. It implies that g_N is continuous on $\cap_{i\leq N+1} V_i$ and it decreases to a function g on $\cap_{i\geq 0} V_i$ with g usc on $\cap_{i\geq 0} V_i$. It means that for every point x in $\Lambda = \cap_{i\geq 0} V_i$, we have $\limsup_{y\to x,\ y\in\Lambda} g(y) \leq g(x)$.

Let $m_N = \int g_N d\nu$. We can write on $\widetilde{W}$, $g_N - m_N = \psi_{1,N} - \psi_{2,N}$. Here the equality is true on a set of full Lebesgue measure in $\widetilde{W}$. But, since g_N is continuous on $\cap_{i\leq N+1} V_i$ and the $\psi_{i,N}$ are qpsh, the equality is true for every point in $\cap_{i\leq N+1} V_i$ (see below the proof of the inequality $g - m \geq \psi_1 - \psi_2$).

We apply the previous lemma to the sequences $\psi_{i,N}$ and we have that these sequences are uniformly bounded from above and bounded in $L^1(\nu)$. So we can extract converging subsequences to some limit points ψ_1 and ψ_2 in L^1. The sequence m_N is bounded thanks to the definitions of W_0' and ν. So m_N converges to m by monotone convergence. In particular, $g - m = \psi_1 - \psi_2$ up to a set of zero Lebesgue measure in $\widetilde{W}$. We want to show now that we have

$$g - m \geq \psi_1 - \psi_2$$

for every point in $\Lambda = \cap_{i\geq 0} V_i$. Indeed, assume there is a point $x \in \Lambda$ such that $(g+\psi_2)(x) < m + \psi_1(x) - \varepsilon$. On a chart which contains x, we can write $\psi_1 = \lambda_1 + \xi_1$ with λ_1 smooth and ξ_1 psh.

Since g and ψ_2 are usc on Λ, so is their sum and so $(g+\psi_2)(y) < m + \psi_1(x) - \varepsilon/2$ on a small ball $B(x,r)$ centered at x and of radius r (for $y \in \Lambda$). For a function h, we denote by $m_{B(x,r)}(h)$ the mean value of h on the ball $B(x,r)$. We have that that $m_{B(x,r)}(g+\psi_2) = m_{B(x,r)}(m + \lambda_1 + \xi_1)$ since both functions are equal a.e. and $m_{B(x,r)}(\xi_1) \geq \xi_1(x)$ since ξ_1 is psh, so

$$m + \psi_1(x) - \varepsilon/2 \geq m_{B(x,r)}(g+\psi_2) \geq m_{B(x,r)}(\lambda_1) + m + \xi_1(x)$$

which is false if we take r small enough to have $m_{B(x,r)}(\lambda_1)$ near $\lambda_1(x)$.

In particular, the set of points where $g = -\infty$ is pluripolar since it is included in the set of points where ψ_1 is $-\infty$. By the proof of Theorem 3.2.1, we see that $g \neq -\infty$ is equivalent to the fact that the first half of Definition 3.1.9 is satisfied.

We do the same thing for the second half of Definition 3.1.9 and we conclude since the intersection of two pluripolar sets is pluripolar. □

3.4. The equilibrium measure

3.4.1. Construction of the measure. – We want to define the equilibrium measure μ as $T_s^+ \wedge T_{k-s}^-$. In [**1**], the authors used an approach based on the energy. More precisely, they show that the potential of the Green current is in the Hilbert space H_{T^-} defined by the closure of the smooth forms for the norm $\sqrt{\int d\varphi \wedge d^c\varphi \wedge T^-}$. They deduced from that fact that the measure $T^+ \wedge T^-$ is well defined and that the potential of the Green current is integrable with respect to that measure.

That approach cannot be adapted here since the super-potential is not a function defined on $\mathbb{P}^k$ but a function defined on $\mathscr{C}_{k-s+1}$. Instead, we will use the formalism of super-potential. See Definition A.1.13 for the definition of wedgeability. We prove the theorem:

THEOREM 3.4.1. – *The current T_s^+ and T_{k-s}^- are wedgeable. So the intersection $T_s^+ \wedge T_{k-s}^-$ is a well defined probability measure μ and the quasi-potential of the Green current of order* 1 *is integrable with respect to this measure.*

Recall that $T^+ := T_1^+$ is a well defined invariant current in $\mathscr{C}_1$ ([**50**]) and that it admits the quasi-potential:

$$G := \sum_n (\frac{1}{d} f^*)^n u \tag{15}$$

where $u < 0$ is a quasi-potential of the current $d^{-1}f^*(\omega)$ (we write u instead of $U_{L(\omega)}$ in order to simplify the notations). We denote as before L_n and Λ_n the normalized pull-pack and push-forward associated to f^n. In what follows, for $q \leq s$, $\mathscr{U}_{L^m(\omega^q)}$ denotes the super-potential of $L^m(\omega^q)$ given by (8) in Lemma 3.2.2, that is:

$$\mathscr{U}_{L^m(\omega^q)} = \sum_{n=0}^{m-1} \frac{1}{d^n} \mathscr{U}_{L(\omega^q)} \circ \Lambda^n,$$

on $(f^m)_*$-admissible currents in $\mathscr{C}_{k-q+1}$.

We need the following lemma to construct the measure.

LEMMA 3.4.2. – *The current $\omega^{s-q} \wedge L^n(\omega^q)$ and T_{k-s}^- are wedgeable for all $n \geq 0$ and $0 \leq q \leq s$. Furthermore, for all integers n and n' and $1 \leq q \leq s+1$ we have that $\mathscr{U}_{L^n(\omega)^q}(L^{n'}(\omega)^{s-q+1} \wedge T_{k-s}^-)$ is finite.*

Proof. – We have seen in Proposition 3.2.11 that $\omega^{s-q} \wedge L(\omega)^q$ and T_{k-s}^- are wedgeable for $q \leq s$ and that the super-potentials of T_{k-s}^- are finite at $L(\omega)^{s+1}$. So applying that to f^n instead of f, we have that $\omega^{s-q} \wedge L^n(\omega)^q$ and T_{k-s}^- are wedgeable for all $n \geq 0$ and that the super-potentials of T_{k-s}^- are finite at $L^n(\omega)^{s+1}$.

The case where $q = s+1$ is already known so we assume $1 \leq q \leq s$. The current $L(\omega)^{s+1-q}$ and $L(\omega)^q$ are wedgeable and their wedge-product is $L(\omega)^{s+1}$ (it follows from Corollary 4.11 Chapter III in **[14]** and Lemma A.1.16). So using Lemma A.2.1 we have that a super-potential of $L(\omega)^{s+1}$ is given by:

$$\mathscr{U}_{L(\omega)^q}(L(\omega)^{s-q+1} \wedge R) + \mathscr{U}_{L(\omega)^{s-q+1}}(\omega^q \wedge R),$$

on current $R \in \mathscr{C}_{k-s}$ such that R and $L(\omega)^{s-q+1}$ are wedgeable. In particular, we can take $R = T^-_{k-s}$ at which point the super-potential of $L(\omega)^{s+1}$ is finite. A super-potential of $L(\omega)^{s-q+1} \wedge \omega^q$ is given by:

$$\mathscr{U}_{L(\omega)^{s-q+1}}(\omega^q \wedge \star).$$

So by difference,

$$\mathscr{U}_{L(\omega)^q}(L(\omega)^{s-q+1} \wedge T^-_{k-s})$$

is well defined in the sense of super-potentials (that is it is continuous for the Hartogs' convergence) and is finite.

So we have proved the lemma for $n = n' = 1$.

Applying the result to f^n gives the lemma for $n = n'$. Now, let $n \leq n'$. Then $L^n(\omega)^q$ is more H-regular than $L^{n'}(\omega)^q$. The super-potentials of $L^{n'}(\omega)^q$ are finite at $L^{n'}(\omega)^{s-q+1} \wedge T^-_{k-s}$ so the super-potentials of $L^n(\omega)^q$ are also finite at $L^{n'}(\omega)^{s-q+1} \wedge T^-_{k-s}$.

Similarly, let $n \geq n'$. Then $L^{n'}(\omega)^{s-q+1}$ is more H-regular than $L^n(\omega)^{s-q+1}$ and so Lemma A.1.14 implies that $L^{n'}(\omega)^{s-q+1} \wedge T^-_{k-s}$ is more H-regular than $L^n(\omega)^{s-q+1} \wedge T^-_{k-s}$. The super-potentials of $L^n(\omega)^q$ are finite at $L^n(\omega)^{s-q+1} \wedge T^-_{k-s}$, which means by symmetry of the super-potentials that the super-potentials of $L^n(\omega)^{s-q+1} \wedge T^-_{k-s}$ are finite at $L^n(\omega)^q$. Hence the super-potentials of $L^{n'}(\omega)^{s-q+1} \wedge T^-_{k-s}$ are finite at $L^n(\omega)^q$ which means that the super-potentials of $L^n(\omega)^q$ are finite at $L^{n'}(\omega)^{s-q+1} \wedge T^-_{k-s}$. That gives the lemma. □

Proof of Theorem 3.4.1. – By the above lemma, we have that $L^n(\omega^s) \wedge T^-_{k-s}$ is $(f^n)_*$-admissible since it is finite at $\mathscr{U}_{L^n(\omega)}$. Hence by Lemma 3.1.8, we have that $\Lambda^n(L^n(\omega^s) \wedge T^-_{k-s})$ is well defined and equal to $\Lambda_n(L_n(\omega^s) \wedge T^-_{k-s})$ (recall that Corollary 3.1.7 gives $L_n(\omega^s) = L^n(\omega^s)$).

We consider:

$$\frac{1}{d^n} \mathscr{U}_{L(\omega)}(\Lambda_n(L_n(\omega^s) \wedge T^-_{k-s})).$$

It is finite since by Lemma A.2.2 applied to f^n and the invariance ot T^-_{k-s}, it is equal to

$$\frac{1}{d^n} \mathscr{U}_{L(\omega)}(\omega^s \wedge T^-_{k-s}),$$

and the previous lemma assures us that this is finite.

We use Lemma A.2.3 for f^n instead of f with $S_1 = L(\omega)$, $S_2 = \omega^s$ and $S_3 = T^-_{k-s}$. So we see that the previous quantity is equal to:

$$\mathcal{U}_{L^n(\omega^s)}(L^{n+1}(\omega)\wedge T^-_{k-s}) - \mathcal{U}_{L^n(\omega^s)}(L^n(\omega)\wedge T^-_{k-s}) + \left(\frac{1}{d}\right)^n \mathcal{U}_{L(\omega)}(\Lambda_n(\omega^s\wedge T^-_{k-s})).$$

We now perform some sort of Abel transform. We sum from 0 to N and we regroup the terms in $L^n(\omega)\wedge T^-_{k-s}$ (observe for the first term that $\mathcal{U}_{\omega^s}=0$):

(16)
$$\begin{aligned}\sum_{n=0}^N \frac{1}{d^n}\mathcal{U}_{L(\omega)}(\omega^s\wedge T^-_{k-s}) &= \sum_{n=1}^N(-\mathcal{U}_{L^n(\omega^s)} + \mathcal{U}_{L^{n-1}(\omega^s)})(L^n(\omega)\wedge T^-_{k-s})\\ &\quad + \mathcal{U}_{L^N(\omega^s)}(L^{N+1}(\omega)\wedge T^-_{k-s}) + \sum_{n=0}^N \frac{1}{d^n}\mathcal{U}_{L(\omega)}(\Lambda_n(\omega^s\wedge T^-_{k-s}))\end{aligned}$$

Now, $\mathcal{U}_{L^{n-1}(\omega^s)} - \mathcal{U}_{L^n(\omega^s)} = -d^{-n+1}\mathcal{U}_{L(\omega^s)}\circ\Lambda^{n-1}$ on $(f^n)_*$-admissible currents by Lemma 3.2.2. In particular, we consider the current $L^n(\omega)\wedge T^-_{k-s}$ which is $(f^n)_*$-admissible by the previous lemma. So using again Lemma A.2.2 for f^{n-1} gives:

$$\begin{aligned}(\mathcal{U}_{L^{n-1}(\omega^s)} - \mathcal{U}_{L^n(\omega^s)})(L^n(\omega)\wedge T^-_{k-s}) &= -d^{-n+1}\mathcal{U}_{L(\omega^s)}(\Lambda_{n-1}(L^n(\omega)\wedge T^-_{k-s}))\\ &= -d^{-n+1}\mathcal{U}_{L(\omega^s)}(\Lambda_{n-1}(L_{n-1}(L(\omega))\wedge T^-_{k-s}))\\ &= -d^{-n+1}\mathcal{U}_{L(\omega^s)}(L(\omega)\wedge T^-_{k-s}).\end{aligned}$$

So the series $\sum_{n=1}^N(-\mathcal{U}_{L^n(\omega^s)} + \mathcal{U}_{L^{n-1}(\omega^s)})(L^n(\omega)\wedge T^-_{k-s})$ is also convergent thanks to the previous lemma. We also have that $\mathcal{U}_{L^N(\omega^s)}(L^{N+1}(\omega)\wedge T^-_{k-s})$ is negative since $\mathcal{U}_{L^N(\omega^s)}$ is negative. Thus, letting N go to ∞:

$$\begin{aligned}\left(\sum_{n\geq 0}\frac{1}{d^n}\right)\mathcal{U}_{L(\omega)}(\omega^s\wedge T^-_{k-s}) + \left(\sum_{n\geq 1} d^{-n+1}\right)\mathcal{U}_{L(\omega^s)}(L(\omega)\wedge T^-_{k-s})&\\ \leq \sum_{n\geq 0}\frac{1}{d^n}\mathcal{U}_{L(\omega)}(\Lambda^n(\omega^s\wedge T^-_{k-s})).&\end{aligned}$$

We recognize by (8) that the right-hand side is in fact $\mathcal{U}_{T^+}(\omega^s\wedge T^-_{k-s})$ which in term of quasi-potential is $\int G\omega^s\wedge T^-_{k-s}$ (recall that T^+ is the Green current of order 1). Thus by Hartogs' convergence, we have that $\mathcal{U}_{T^+}(\omega^s\wedge T^-_{k-s})$ is finite (we could also conclude by monotone convergence that $G\in L^1(\omega^s\wedge T^-_{k-s})$, both properties being equivalent).

Observe now that in (16) every term converge. In particular,

$$(\mathcal{U}_{L^N(\omega^s)}(L^{N+1}(\omega)\wedge T^-_{k-s}))_N$$

converges to a finite value. Using Lemma A.2.4, we have the identity:

$$\begin{aligned}\mathcal{U}_{L^N(\omega^s)}(L^{N+1}(\omega)\wedge T^-_{k-s}) &= \mathcal{U}_{L^{N+1}(\omega)}(L^N(\omega^s)\wedge T^-_{k-s})\\ &\quad - \mathcal{U}_{L^{N+1}(\omega)}(\omega^s\wedge T^-_{k-s}) + \mathcal{U}_{L^N(\omega^s)}(\omega\wedge T^-_{k-s}).\end{aligned}$$

On the right-hand side, the first and third terms are negative, the third term is decreasing and we just proved that the second term converges to $\mathcal{U}_{T^+}(\omega^s\wedge T^-_{k-s})$ which is finite. That implies that every term is in fact convergent.

In particular, we have the convergence of $\mathcal{U}_{L^N(\omega^s)}(\omega\wedge T^-_{k-s})$. Since $L^N(\omega^s)\to T^+_s$ in the Hartogs' sense, that means that $\mathcal{U}_{T^+_s}(\omega\wedge T^-_{k-s})$ is finite. Hence the current T^+_s and T^-_{k-s} are wedgeable and their intersection is a well defined probability measure μ (we could also have deduced that from the convergence of the first term but this is more natural).

Now, recall that the function $(R,S)\to\mathcal{U}(R,S):=\mathcal{U}_R(S)=\mathcal{U}_S(R)$ for R and S in $\mathscr{C}_q$ and $\mathscr{C}_{k-q}$ ($\mathcal{U}_R$ and $\mathcal{U}_S$ being the super-potentials of mean 0) is upper semi-continuous. The convergence of $\mathcal{U}(L^{N+1}(\omega),L^N(\omega^s)\wedge T^-_{k-s})$ implies that $\mathcal{U}(T^+,T^+_s\wedge T^-_{k-s})$ is finite which means exactly that the quasi-potential of the Green current is integrable with respect to μ. □

Of course, the measure μ also integrate the potential of the Green current of order 1 of f^{-1}.

Corollary 3.4.3. – *The measure μ gives no mass to the indeterminacy sets I^+ and I^-. Furthermore, $L(\mu)=f^*(\mu)$ and $\Lambda(\mu)=f_*(\mu)$ are well defined in the sense of super-potentials.*

Proof. – The fact that μ is f_*-admissible follows from Theorem 3.4.1 since its super-potentials are finite at the point $L(T^+)=T^+$ and so they are finite at the point $L(\omega)$ which is more H-regular than $L(T^+)$. Since the potential of T^+ is equal to $-\infty$ on I^+ and is in $L^1(\mu)$ (in fact $\log\operatorname{dist}(x,I^+)\in L^1(\mu)$) we have that μ gives no mass to the indeterminacy set I^+, similarly for I^-. □

Proposition 3.4.4. – *The measure μ is invariant, that is $f^*(\mu)$ and $f_*(\mu)$ are equal to μ.*

Proof. – The currents $L^n(\omega^s)$ and $\Lambda^m(\omega^{k-s})$ are wedgeable for m and n in $\mathbb{N}$ since they are more H-regular than T^+_s and T^-_{k-s}. So let $\mu_n:=L^n(\omega^s)\wedge\Lambda^n(\omega^{k-s})$ (resp. $\mu'_n:=L^{n-1}(\omega^s)\wedge\Lambda^{n+1}(\omega^{k-s})$). Now since $L^n(\omega^s)$ and $\Lambda^n(\omega^{k-s})$ converge in the Hartogs' sense to T^+_s and T^-_{k-s} which are wedgeable, we have that μ_n (resp. μ'_n) converges to μ in Hartogs' sense (Proposition A.1.15).

By Lemma A.2.2, we have that $\mu'_n=\Lambda(L^n(\omega^s)\wedge\Lambda^n(\omega^{k-s}))=\Lambda(\mu_n)$ (observe that $L^n(\omega^s)\wedge\Lambda^n(\omega^{k-s})$ is f_* admissible since it is more H-regular than $T^+_s\wedge T^-_{k-s}$ which is f_*-admissible).

So, since μ is f_*-admissible, we have that μ'_n converges in the Hartogs' sense to $\Lambda(\mu) = \mu$ which is what we wanted. □

COROLLARY 3.4.5. – *The measure μ gives no mass to the indeterminacy sets $I(f^{\pm n})$ and the critical sets $\mathscr{C}(f^{\pm n})$.*

Proof. – We already know that the indeterminacy sets have no mass for μ so using the invariance of μ, we have that $\mu(\mathscr{C}(f)) = \mu(f^{-1}(I^-)) = \mu(I^-) = 0$. □

3.4.2. Green currents of order $1 \le q \le s$. – The purpose of this paragraph is to construct the Green currents of order q for $q \le s$. This will allow us to prove that T_s^+ can be written as $(T^+)^s$. As an application, we show that the equilibrium measure gives no mass to the pluripolar sets.

Using the same arguments than in Theorem 3.4.1, we construct the Green currents T_q^+ of order q for $q \le s$:

PROPOSITION 3.4.6. – *For $1 \le q \le s$, the sequence $(L^n(\omega^q))_n$ converges in the Hartogs' sense to T_q^+* the Green current of order q *and the Green currents T_q^+ and T_{k-s}^- are wedgeable. Furthermore, any super-potential $\mathscr{U}_{T_q^+}$ of T_q^+ satisfies*

$$\mathscr{U}_{T_q^+}(T_{s-q+1}^+ \wedge T_{k-s}^-) > -\infty.$$

Proof. – Observe that the roles of q and $s-q+1$ are symmetric, so anything proved for q stands for $s-q+1$. The current $L^n(\omega^{s-q+1}) \wedge T_{k-s}^-$ is $(f^n)_*$ admissible by Lemma 3.4.2. Lemma 3.1.8 implies that $\Lambda^n(L^n(\omega^{s-q+1}) \wedge T_{k-s}^-)$ is well defined and equal to $\Lambda_n(L^n(\omega^{s-q+1}) \wedge T_{k-s}^-)$. So we consider this time:

$$\frac{1}{d^n}\mathscr{U}_{L(\omega^q)}(\Lambda_n(L^n(\omega^{s-q+1}) \wedge T_{k-s}^-)).$$

By Lemma A.2.2 and the invariance ot T_{k-s}^-, it is equal to

$$\frac{1}{d^n}\mathscr{U}_{L(\omega^q)}(\omega^{s-q+1} \wedge T_{k-s}^-),$$

and Lemma 3.4.2 assures us that this is finite.

Using Lemma A.2.3, performing the same Abel transform and using again Lemma A.2.2, we obtain similarly that:

(17)

$$\begin{aligned}\left(\sum_{n=0}^{N} \frac{1}{d^n}\right) \mathscr{U}_{L(\omega^q)}(\omega^{s-q+1} \wedge T_{k-s}^-) = &\left(\sum_{n=1}^{N} -d^{-n+1}\right) \mathscr{U}_{L(\omega^{s-q+1})}(L(\omega^q) \wedge T_{k-s}^-) \\ &+ \mathscr{U}_{L^N(\omega^{s-q+1})}(L^{N+1}(\omega^q) \wedge T_{k-s}^-) \\ &+ \sum_{n=0}^{N} \frac{1}{d^n}\mathscr{U}_{L(\omega^q)}(\Lambda^n(\omega^{s-q+1} \wedge T_{k-s}^-))\end{aligned}$$

We have again that $\mathscr{U}_{L^N(\omega^{s-q+1})}(L^{N+1}(\omega^q)\wedge T^-_{k-s})$ is negative since $\mathscr{U}_{L^N(\omega^{s-q+1})}$ is negative. Thus, letting N go to ∞:

$$\left(\sum_{n\geq 0} d^{-n}\right)\mathscr{U}_{L(\omega^q)}(\omega^{s-q+1}\wedge T^-_{k-s}) + \left(\sum_{n\geq 1} d^{-n+1}\right)\mathscr{U}_{L(\omega^{s-q+1})}(L(\omega^q)\wedge T^-_{k-s})$$
$$\leq \sum_{n\geq 0}\frac{1}{d^n}\mathscr{U}_{L(\omega^q)}(\Lambda_n(\omega^{s-q+1}\wedge T^-_{k-s})).$$

Again, by Proposition A.1.8, we have that the sequence of super-potential of $L^n(\omega^q)$ is decreasing thus to have the convergence in the Hartogs' sense, it is sufficient to have the convergence at one point. We recognize by (8) that the right-hand side gives in fact the convergence at the point $\omega^{s-q+1}\wedge T^-_{k-s}$ (again $\omega^{s-q+1}\wedge T^-_{k-s}$ is $(f^n)_*$-admissible so $\Lambda_n(\omega^{s-q+1}\wedge T^-_{k-s}) = \Lambda^n(\omega^{s-q+1}\wedge T^-_{k-s})$). So we have that $\mathscr{U}_{T^+_q}(\omega^{s-q+1}\wedge T^-_{k-s})$ is finite and $L^n(\omega^q)$ converges to T^+_q in the Hartogs' sense.

In (17) every term converges. In particular,

$$(\mathscr{U}_{L^N(\omega^{s-q+1})}(L^{N+1}(\omega^q)\wedge T^-_{k-s}))_N$$

converges to a finite value. Using Lemma A.2.4, we have the identity:

$$\begin{aligned}\mathscr{U}_{L^N(\omega^{s-q+1})}(L^{N+1}(\omega^q)\wedge T^-_{k-s}) &= \mathscr{U}_{L^{N+1}(\omega^q)}(L^N(\omega^{s-q+1})\wedge T^-_{k-s})\\ &\quad - \mathscr{U}_{L^{N+1}(\omega^q)}(\omega^{s-q+1}\wedge T^-_{k-s})\\ &\quad + \mathscr{U}_{L^N(\omega^{s-q+1})}(\omega^q\wedge T^-_{k-s}).\end{aligned}$$

As above, every term converges to a finite value. In particular, that means that $\mathscr{U}_{T^+_{s-q+1}}(\omega^q\wedge T^-_{k-s})$ is finite (which is already known by exchanging the role of q and $s-q+1$). Finally the convergence of $\mathscr{U}_{L^{N+1}(\omega^q)}(L^N(\omega^{s-q+1})\wedge T^-_{k-s})$ implies that $\mathscr{U}(T^+_q, T^+_{s-q+1}\wedge T^-_{k-s})$ is finite. □

We prove that T^+_q is invariant.

LEMMA 3.4.7. – *For $1\leq q\leq s$, the Green current T^+_q is f^*-admissible and satisfies $T^+_q = L(T^+_q)$. Furthermore, T^+_q is the most H-regular current which is f^*-invariant in $\mathscr{C}_q$. In particular, T^+_q is extremal in the set of f^*-invariant currents of $\mathscr{C}_q$.*

Proof. – For $q=s$, this is Theorem 3.2.9. So take $q<s$. We have that $L^n(\omega^q)$ converges in the Hartogs' sense to T^+_q. So this means that at the point ω^{k-q+1} we have the convergence of the series:

$$\sum_{n\geq 0} d^{-n}\mathscr{U}_{L(\omega^q)}(\Lambda^n(\omega^{k-q+1})).$$

In particular, dropping the first term and multiplying by d, we have the convergence of the series:

$$\sum_{n\geq 0} d^{-n}\mathscr{U}_{L(\omega^q)}(\Lambda^n(\Lambda(\omega^{k-q+1}))).$$

We recognize $\mathscr{U}_{T_q^+}(\Lambda(\omega^{k-q+1})) > -\infty$ hence T_q^+ is f^*-admissible.

By Theorem A.1.17, we see that $L(L^n(\omega^q))$ converges to $L(T_q^+)$ and to T_q^+. So we have proved the first part of the lemma. The rest is exactly as in Theorem 3.2.9. □

Now, we also want to consider the intersection $T_q^+ \wedge T_{k-s}^-$ for $q < s$. First, we have that these intersections are well defined elements of $\mathscr{C}_{k-s+q}$ from Proposition 3.4.6 (T_q^+ and T_{k-s}^- are wedgeable). Furthermore, it is f_*-admissible since we have by Proposition 3.4.6 that

$$\mathscr{U}_{T_{s-q+1}^+}(T_q^+ \wedge T_{k-s}^-) > -\infty$$

and since $L(\omega^{s-q+1})$ is more H-regular than $T_{s-q+1}^+ = L(T_{s-q+1}^+)$, we see that:

$$\mathscr{U}_{L(\omega^{s-q+1})}(T_q^+ \wedge T_{k-s}^-) > -\infty,$$

which means that $T_q^+ \wedge T_{k-s}^-$ is f_*-admissible since by symmetry of the super-potential, its super-potentials are finite at the point $L(\omega^{s-q+1})$.

Using the same argument than in the proof of the invariance of the measure μ, one has:

Proposition 3.4.8. – *The current $T_q^+ \wedge T_{k-s}^- \in \mathscr{C}_{k-s+q}$ is f_*-invariant, that is:*

$$\Lambda(T_q^+ \wedge T_{k-s}^-) = T_q^+ \wedge T_{k-s}^-.$$

Proof. – This follows from the fact that $L^n(\omega^q)$ and $\Lambda^m(\omega^{k-s})$ converge in the Hartogs' sense to T_q^+ and T_{k-s}^- which are wedgeable and we use Proposition A.2.2. □

Now, we use the same arguments than in the proof of Theorem 3.4.1, but we replace T_{k-s}^- by $T_q^+ \wedge T_{k-s}^-$. Our purpose is to show that $T_s^+ = (T^+)^s$. We need the following lemma first:

Lemma 3.4.9. – *Let $q_1 \geq 1$ and q_2 such that $q_1 + q_2 = s - q + 1$. Then the current $T_{q_2}^+$ and $T_q^+ \wedge T_{k-s}^-$ are wedgeable and we have that a super-potential $\mathscr{U}_{T_{q_1}^+}$ of $T_{q_1}^+$ satisfies:*

$$\mathscr{U}_{T_{q_1}^+}(T_{q_2}^+ \wedge T_q^+ \wedge T_{k-s}^-) > -\infty.$$

The proof is essentially the same as the one of Theorem 3.4.1. We need the equivalent of Lemma 3.4.2 first:

Lemma 3.4.10. – *Let $q_1 \geq 1$ and q_2 such that $q_1 + q_2 = s - q + 1$ and let $n \in \mathbb{N}$. Then the currents $L^n(\omega^{q_2})$ and $T_q^+ \wedge T_{k-s}^-$ are wedgeable. Furthermore, for $n' \in \mathbb{N}$:*

$$\mathscr{U}_{L^{n'}(\omega^{q_1})}(L^n(\omega^{q_2}) \wedge T_q^+ \wedge T_{k-s}^-) > -\infty.$$

Proof. – We can assume that $q_2 \geq 1$ (else it is just Proposition 3.4.6). The super-potentials of the current $T_q^+ \wedge T_{k-s}^-$ are finite at $L(\omega^{q_1+q_2}) = L(\omega^{q_1}) \wedge L(\omega^{q_2})$ which is less H-regular than $\omega^{q_1} \wedge L(\omega^{q_2})$. Hence the super-potentials of the current $T_q^+ \wedge T_{k-s}^-$ are finite at $\omega^{q_1} \wedge L(\omega^{q_2})$. This means that the currents $L(\omega^{q_2})$ and $T_q^+ \wedge T_{k-s}^-$ are wedgeable.

On the other hand, $\mathscr{U}_{L(\omega^{q_1+q_2})}(T_q^+ \wedge T_{k-s}^-)$ is finite. We can use Lemma A.2.1 and we have that:

$$\mathscr{U}_{L(\omega^{q_1+q_2})}(T_q^+ \wedge T_{k-s}^-) = \mathscr{U}_{L(\omega^{q_2})}(\omega^{q_1} \wedge T_q^+ \wedge T_{k-s}^-) + \mathscr{U}_{L(\omega^{q_1})}(L(\omega^{q_2}) \wedge T_q^+ \wedge T_{k-s}^-).$$

Again taking the difference with $\mathscr{U}_{L(\omega^{q_2})}(\omega^{q_1} \wedge T_q^+ \wedge T_{k-s}^-)$, we have that:

$$\mathscr{U}_{L(\omega^{q_1})}(L(\omega^{q_2}) \wedge T_q^+ \wedge T_{k-s}^-)$$

is well defined in the sense of super-potentials and is finite. We have proved the lemma for $n = n' = 1$. The rest follows as in Lemma 3.4.2. □

Proof of Lemma 3.4.9. – We replace T_{k-s}^- by $T_q^+ \wedge T_{k-s}^-$ and we do the same computations. Lemma A.2.2, A.2.3 and A.2.4 still apply for $T_q^+ \wedge T_{k-s}^-$. □

We can now prove the following corollary. Observe that if a sequence S_n converges in the Hartogs' sense to S and a sequence R_n converges in the Hartogs' sense to R with $R_n \wedge S_n$ wedgeable converging in the Hartogs' sense to a current T, we cannot claim a priori that S and R are wedgeable and that $T = R \wedge S$. But if S and R are wedgeable, then we do have $T = R \wedge S$.

Corollary 3.4.11. – *The current T_s^+ satisfies $T_s^+ = (T^+)^s$. Consequently, one has $\mu = (T^+)^s \wedge (T^-)^{k-s}$ where $T^\pm$ is the Green current of order 1 of $f^\pm$.*

Proof. – Applying the previous lemma to $q = 1$, $q_1 = 1$ and $q_2 = s-1$ gives that

$$\mathscr{U}_{T^+}(T_{s-1}^+ \wedge T^+ \wedge T_{k-s}^-) > -\infty.$$

Since ω^{k-s+1} is more H-regular than $T^+ \wedge T_{k-s}^-$ that implies that:

$$\mathscr{U}_{T^+}(T_{s-1}^+ \wedge \omega^{k-s+1}) > -\infty.$$

In particular that T^+ and T_{s-1}^+ are wedgeable. Since $L^n(\omega)$ and $L^n(\omega^{s-1})$ converges in the Hartogs' sense to T^+ and T_{s-1}^+ and $L^n(\omega^s)$ converges in the Hartogs' sense to T_s^+, Proposition A.1.15 implies that $T^+ \wedge T_{s-1}^+ = T_s^+$. An easy induction gives the result for T_s^+ and the result follows for μ. □

Remark 3.4.12. – We do not know how to prove the previous result without constructing T_{k-s}^- first. In the case where f satisfies Hypothesis 3.1.10, the result was proved directly (see Theorem 3.2.8). This illustrates the difference between Hypotheses 3.1.10 and Definition 3.1.9. For a map satisfying Hypotheses 3.1.10, we have that the potential of T^+ is finite at every point of I^-, if it is only in $\mathscr{B}$, we can only say that $\int_{I^-} U_{T^+}\omega^{s-1}$ is finite since $T^+ \wedge \omega^{s-1}$ is more H-regular than $T_s^+ = (T^+)^s$.

Now, we improve the previous results and we show that the measure μ gives no mass to pluripolar sets (hence analytic sets). The proof relies on a space of test functions introduced by Dinh and Sibony in **[26]** and studied by the second author in **[53]**. Recall that the space $W^{1,2}(\mathbb{P}^k)$ is the set of functions in L^2 whose differential in the sense of currents can be represented by a form in L^2. The space $W^*(\mathbb{P}^k)$ is the set of functions φ in $W^{1,2}(\mathbb{P}^k)$ such that there exists a positive closed current S_φ of bidegree $(1,1)$ satisfying:

$$d\varphi \wedge d^c\varphi \leq S_\varphi. \tag{18}$$

For $\varphi \in W^*$, we define the norm:

$$\|\varphi\|_*^2 = \|\varphi\|_{L^2}^2 + \inf\Big\{m(S),\ S \text{ closed, satisfying (18)}\Big\}.$$

Let ψ be a qpsh function in $W^*(\mathbb{P}^k)$. Consider the regularization ψ_n of ψ obtained through an approximation of the identity in $\mathrm{PGL}(k+1,\mathbb{C})$. Let S be minimal in (18) for ψ and let S_n be the smooth regularization of S obtained through the same approximation of the identity. Using Lemma 5 in **[53]**, we have

- ψ_n "decreases" to ψ.
- $d\psi_n \wedge d^c\psi_n \leq S_n$, and $m(S_n) \to m(S)$ thus $\lim \|\psi_n\|_* = \|\psi\|_*$.

If φ is a qpsh function in $\mathbb{P}^k$ with $\varphi \leq -2$, then $\psi := -\log -\varphi$ is in $W^*(\mathbb{P}^k)$, thus for every pluripolar set in $\mathbb{P}^k$ there exists a qpsh function in $W^*(\mathbb{P}^k)$ equal to $-\infty$ on that set (see Example 1 p. 253 in [**53**]). In particular, if the qpsh functions in $W^*(\mathbb{P}^k)$ are integrable for a measure, the measure cannot give mass to the pluripolar sets. We can now state the theorem:

THEOREM 3.4.13. – *The measure μ gives no mass to pluripolar sets (hence analytic sets). More precisely, there exists $C > 0$ such that for $\psi < 0$ a qpsh function in $W^*(\mathbb{P}^k)$, we have that:*

$$|\mu(\psi)| \leq C\|\psi\|_*.$$

Proof. – Let ψ and ψ_n be as above. Recall that G is the potential of T^+. Let T_m^+ and T_m^- be sequence of smooth currents in $\mathscr{C}_1$ converging to T^+ and T^- in the Hartogs' sense. Then $\mu_m = (T_m^+)^s \wedge (T_m^-)^{k-s}$ converges to μ in the Hartogs' sense by Proposition A.1.15. Let G_m be the associated potential of T_m^+. Using Stokes' formula and Cauchy-Schwarz inequality, we have that:

$$\begin{aligned}\left|\int \psi_n d\mu_m\right| &= \left|\int \psi_n(dd^c G_m + \omega)\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|\\ &\leq \left|\int d\psi_n \wedge d^c G_m \wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|\\ &+\left|\int \psi_n\omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|\\ &\leq \left(\int d\psi_n\wedge d^c\psi_n\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right)^{\frac12}\\ &\times\left(\int dG_m\wedge d^c G_m\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right)^{\frac12}\\ &+\left|\int \psi_n\omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|.\end{aligned}$$

Let S_n be the positive closed current of bidegree $(1,1)$ such that $d\psi_n\wedge d^c\psi_n\leq S_n$. Using again Stokes' formula for the second term of the product yields:

$$\begin{aligned}\left|\int \psi_n d\mu_m\right| &\leq \left(\int S_n\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right)^{\frac12}\\ &\times\left(-\int G_m dd^c G_m\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right)^{\frac12}\\ &+\left|\int \psi_n\omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|.\end{aligned}$$

We let m go to ∞, we have that $\left|\int \psi_n d\mu_m\right|$ converges to $\left|\int \psi_n d\mu\right|$,

$$\left(\int S_n\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right)$$

converges to $\left(\int S_n\wedge T_{s-1}^+\wedge T_{k-s}^-\right)$, and $\left|\int \psi_n\omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\right|$ converges to $\left|\int \psi_n\omega\wedge T_{s-1}^+\wedge T_{k-s}^-\right|$. The term:

$$\begin{aligned}\int G_m dd^c G_m\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s} &= \int G_m (T_m^+)^{s}\wedge (T_m^-)^{k-s}\\ &-\int G_m\omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s}\end{aligned}$$

can be rewritten as:

$$\mathscr{U}_1(T_m^+, (T_m^+)^s\wedge (T_m^-)^{k-s}) - \mathscr{U}_1(T_m^+, \omega\wedge (T_m^+)^{s-1}\wedge (T_m^-)^{k-s})$$

which by Hartogs' convergence goes with m to:

$$\mathscr{U}_1(T^+,\mu) - \mathscr{U}_1(T^+, \omega\wedge T_{s-1}^+\wedge T_{k-s}^-)$$

which is finite by Theorem 3.4.1 (observe that $\omega \wedge T^+_{s-1} \wedge T^-_{k-s}$ is more H-regular than μ). So we have that:

$$\left|\int \psi_n d\mu\right| \leq C \left(\int S_n \wedge T^+_{s-1} \wedge T^-_{k-s}\right)^{\frac{1}{2}} + \left|\int \psi_n \omega \wedge T^+_{s-1} \wedge T^-_{k-s}\right|,$$

where $C^2 = |\mathscr{U}_1(T^+, \mu) - \mathscr{U}_1(T^+, \omega \wedge T^+_{s-1} \wedge T^-_{k-s})|$ is a constant.

The term $\left(\int S_n \wedge T^+_{s-1} \wedge T^-_{k-s}\right)^{\frac{1}{2}}$ is controlled by $\|\psi\|_* + \varepsilon$ for n large enough because S_n is smooth so that wedge-product is well defined and the mass can be computed in cohomology.

We use an induction to control in the same way the term $\left|\int \psi_n \omega \wedge T^+_{s-1} \wedge T^-_{k-s}\right|$ (at the last step, we have a term in $\int -\psi_n \omega^k$). Since for n large enough we have $\|\psi_n\|_* \leq \|\psi\|_* + \varepsilon$ ($\varepsilon > 0$), we have proved that:

$$\left|\int \psi_n d\mu\right| \leq C(\|\psi\|_* + \varepsilon).$$

By monotone convergence and letting $\varepsilon \to 0$, we have the theorem. □

3.4.3. Mixing, entropy and hyperbolicity of μ. – We now prove that μ is mixing, that is $\lim_{n\to\infty} \mu(\varphi\psi \circ f^n) = \mu(\varphi)\mu(\psi)$ for φ and ψ smooth functions on $\mathbb{P}^k$. Here the function $\psi \circ f^n$ is not smooth, so by definition $\mu(\varphi\psi \circ f^n)$ is the integral of $\varphi\psi \circ f^n$ on $\mathbb{P}^k \setminus I(f^n)$ for the measure μ which gives no mass to $I(f^n)$. Recall that $I(f^n) \subset \mathscr{C}(f^n)$.

We need the classical lemma ([**50**] and [**42**]):

Lemma 3.4.14. – *Let ψ be smooth function on $\mathbb{P}^k$, then the sequence of currents $(\psi \circ f^n T^+_s)_n$ converges to cT^+_s where $c = \mu(\psi)$. Moreover, we have that $\|d(\psi \circ f^n T^+_s)\|$ and $\|dd^c(\psi \circ f^n T^+_s)\|$ go to zero.*

Proof. – The norm $\|d(\psi \circ f^n T^+_s)\|$ is the operator's norm on the space of smooth forms.

We can assume that $0 \leq \psi \leq 1$. Then, the sequence $(\psi \circ f^n T^+_s)_n$ is bounded so we can extract a subsequence converging in the sense of currents to $S \geq 0$ which satisfies $S \leq T^+_s$. In order to show that S is closed and that $\|d(\psi \circ f^n T^+_s)\| \to 0$, we only need to show that for every smooth $(0,1)$-form θ we have that $|\langle \psi \circ f^n T^+_s, \partial(\theta \wedge \omega^{k-s-1})\rangle|$ goes to 0 uniformly on θ (see [**12**] p. 3 for details). In other words, we want to compute the limit of:

$$\int_{\mathbb{P}^k \setminus I(f^n)} \psi \circ f^n T^+_s \wedge \partial(\theta) \wedge \omega^{k-s-1}.$$

We are going to use the technics of [**49**]. Let $v < 0$ be a qpsh function equal to $-\infty$ on $\mathscr{C}(f^n)$ and smooth outside $\mathscr{C}(f^n)$. Let max′ be a smooth convex increasing function approximating the function $\max^+ := \max(x, 0)$ such that its derivative is less than 1. Let $v_j = \max'(v/j+1)$. Then (v_j) is an increasing sequence of smooth qpsh ($i\partial\bar\partial v_j + \omega \geq 0$) functions with $0 \leq v_j \leq 1$ converging uniformly to 1 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$ and equal to 0 on some neighborhood of $\mathscr{C}(f^n)$. Let $\alpha : [0,1] \to [0,1]$ be a smooth function equal to 0 in $[0,1/3]$ and to 1 in $[2/3,1]$. Then the sequence of functions $v'_j := \alpha \circ v_j$ is equal to 1 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$ for j large enough and is equal to 0 on some neighborhood of $\mathscr{C}(f^n)$.

Since T_s^+ gives no mass to $\mathscr{C}(f^n)$, the previous quantity is the limit when j goes to ∞ of:

$$\langle v'_j \psi \circ f^n T_s^+, \partial(\theta) \wedge \omega^{k-s-1} \rangle.$$

By Stokes' formula, it is equal to:

$$-\langle v'_j \partial(\psi \circ f^n) \wedge T_s^+, \theta \wedge \omega^{k-s-1} \rangle - \langle \psi \circ f^n \partial(v'_j) \wedge T_s^+, \theta \wedge \omega^{k-s-1} \rangle.$$

We apply Cauchy-Schwarz inequality for the first term of the sum, we bound the absolute value of the first term of the previous quantity by:

$$\langle (v'_j)^2 i\partial\psi \circ f^n \wedge \bar\partial\psi \circ f^n \wedge T_s^+, \omega^{k-s-1} \rangle^{\frac{1}{2}} \times \langle i\theta \wedge \overline{\theta} \wedge T_s^+, \omega^{k-s-1} \rangle^{\frac{1}{2}}.$$

The second term of the product is bounded and does not depend on j and n (uniformly in $\|\theta\|$). For the first term, observe that :

$$i\partial\psi \circ f^n \wedge \bar\partial\psi \circ f^n \wedge T_s^+ = d^{-sn}(f^n)^*(i\partial\psi \wedge \bar\partial\psi \wedge T_s^+)$$

in the integral since f^n is smooth on the support of v'_j and one can multiply a positive closed current by a smooth form and take the pull-back by a smooth function. So, assuming that $i\partial\psi \wedge \bar\partial\psi \leq \omega$, we have that the first term is less than:

$$\langle d^{-sn}(f^n)^*(\omega \wedge T_s^+), \omega^{k-s-1} \rangle^{\frac{1}{2}} = (\delta^{-(k-s)n}\delta^{(k-s-1)n})^{\frac{1}{2}} = \delta^{-n/2}$$

which goes to 0 when n goes to ∞ independtly of j.

Now we have to control the term:

$$\langle \partial(v'_j)\psi \circ f^n \wedge T_s^+, \theta \wedge \omega^{k-s-1} \rangle.$$

We have that $\partial(v'_j) = \alpha'(v_j)\partial v_j$ and observe that the sequence of functions $(\alpha'(v_j))$ is bounded and converges uniformly to 0 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$. We apply Cauchy-Schwarz inequality and we get that:

$$\langle \partial(v'_j)\psi \circ f^n \wedge T_s^+, \theta \wedge \omega^{k-s-1} \rangle^2 \leq \langle i\partial(v_j) \wedge \bar\partial(v_j) \wedge T_s^+, \omega^{k-s-1} \rangle \langle i(\alpha'(v_j))^2 \theta \wedge \overline{\theta} \wedge T_s^+, \omega^{k-s-1} \rangle.$$

The first term of the product is equal by Stokes' formula to:

$$\langle -v_j \wedge i\partial\bar\partial(v_j) \wedge T_s^+, \omega^{k-s-1} \rangle$$

Since $0 \leq v_j \leq 1$ and $i\partial\bar\partial v_j + \omega \geq 0$, it is less than:

$$\langle v_j \omega \wedge T_s^+, \omega^{k-s-1} \rangle \leq \langle \omega \wedge T_s^+, \omega^{k-s-1} \rangle$$

which is bounded independently of n and j. The second term of the product goes to 0 when $j \to \infty$ uniformly on θ by dominated convergence since T_s^+ gives no mass to $\mathscr{C}(f^n)$. So letting $j \to \infty$ first, we see that:

$$\langle \psi \circ f^n T_s^+, \partial(\theta) \wedge \omega^{k-s-1} \rangle$$

goes to 0 when $n \to \infty$ uniformly on $\|\theta\|$.

By Theorem 3.2.19, this shows that $S = cT_s^+$. To compute c, consider $\langle \psi \circ f^n T_s^+, \omega^{k-s} \rangle$. It is equal to $\langle T_s^+ \wedge \Lambda^n(\omega^{k-s}), \psi \rangle$: replace T_s^+ by a smooth approximation T_m^+, then $\psi \circ f^n L^n(T_m^+) = d^{-ns}(f^n)^*(\psi T_m^+)$, so

$$\langle \psi \circ f^n L^n(T_m^+), \omega^{k-s} \rangle = \langle T_m^+ \wedge \Lambda^n(\omega^{k-s}), \psi \rangle$$

and let m go to ∞. So we have $\langle \psi \circ f^n T_s^+, \omega^{k-s} \rangle = \langle T_s^+ \wedge \Lambda^n(\omega^{k-s}), \psi \rangle$ because $T_s^+ \wedge \omega^{k-s}$ gives no mass to $I(f^n)$.

By Theorem 3.4.1, we have that $T_s^+ \wedge \Lambda^n(\omega^{k-s})$ converges (in the Hartogs' sense hence in the sense of currents) to μ which means that $c = \mu(\psi)$. In particular, c does not depend on the choice of S and the first part of the lemma follows.

Now we show that $\|dd^c(\psi \circ f^n T_s^+)\|$ goes to zero. Let Θ be a test form of bidegree $(k-s-1, k-s-1)$. Again, we consider a smooth approximation of T_s^+ that we denote T_m^+. Using the fact that $(\psi \circ f^n) L^n(T_m^+) = d^{-sn}(f^n)^*(\psi T_m^+)$, we compute:

$$\begin{aligned} \langle \psi \circ f^n L^n(T_m^+), dd^c\Theta \rangle &= \langle d^{-sn}(f^n)^*(\psi T_m^+), dd^c\Theta) \rangle \\ &= \langle d^{-sn}(f^n)^*(dd^c(\psi) \wedge T_m^+), \Theta \rangle. \end{aligned}$$

Writing $\Theta = \Theta^+ - \Theta^-$ we can assume that Θ is positive (so $\Theta \leq B\omega^{k-s-1}$ with $B > 0$ large enough which depends only on $\|\Theta\|$). Let $A > 0$ be such that $-A\omega \leq dd^c\psi \leq A\omega$. It is sufficient to control:

$$\langle d^{-sn}(f^n)^*(\omega \wedge T_m^+), \omega^{k-s-1} \rangle.$$

We recognize that this is equal by definition to $d_{s+1}^n d^{-sn} = \delta^{-n}$. We let m go to ∞ and we have that $\langle dd^c(\psi \circ f^n \wedge T_s^+), \Theta \rangle$ goes to 0 with n uniformly on $\|\Theta\|$. □

THEOREM 3.4.15. – *The measure μ is mixing.*

Proof. – Let ψ and φ be real smooth functions on $\mathbb{P}^k$. We can assume without loss of generality that $0 \leq \psi, \varphi \leq 1$. Then for S in $\mathscr{C}_{k-s}$ smooth, we have by the above lemma that:

$$\langle (\varphi\psi \circ f^n) T_s^+, S \rangle$$

converges to:

$$\mu(\psi) \langle \varphi T_s^+, S \rangle.$$

We consider a sequence (T_m^-) of smooth currents in $\mathscr{C}_1$ converging in the Hartogs' sense to T^- (the Green current of order 1 of f^{-1}). Then let $m = (m_1, m_2, \ldots, m_{k-s})$

and $m' = (m'_1, m'_2, \dots, m'_{k-s})$ in $\mathbb{N}^{k-s}$. We have that $T^-_{m_1} \wedge \cdots \wedge T^-_{m_{k-s}}$ converges to T^-_{k-s} in the Hartogs' sense when the m_i go to ∞. We decompose:

$$T^-_{m_1} \wedge \cdots \wedge T^-_{m_{k-s}} - T^-_{m'_1} \wedge \cdots \wedge T^-_{m'_{k-s}}$$

as:

$$\begin{aligned} (T^-_{m_1} - T^-_{m'_1}) \wedge T^-_{m_2} \wedge \cdots \wedge T^-_{m_{k-s}} + \\ T^-_{m'_1} \wedge (T^-_{m_2} - T^-_{m'_2}) \wedge \cdots \wedge T^-_{m_{k-s}} + \\ \cdots \\ T^-_{m'_1} \wedge \cdots \wedge T^-_{m'_{k-s-1}} \wedge (T^-_{m_{k-s}} - T^-_{m'_{k-s}}). \end{aligned}$$

As in the previous lemma, let (v_j) be an increasing sequence of smooth qpsh ($i\partial\bar{\partial} v_j + \omega \geq 0$) functions with $0 \leq v_j \leq 1$ converging uniformly to 1 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$ and equal to 0 on some neighborhood of $\mathscr{C}(f^n)$.

We also define $v'_j := \alpha \circ v_j$ with $\alpha : [0,1] \to [0,1]$ a smooth function equal to 0 in $[0, 1/3]$ and to 1 in $[2/3, 1]$ so that the sequence of functions v'_j is equal to 1 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$ for j large enough and is equal to 0 on some neighborhood of $\mathscr{C}(f^n)$.

We consider the quantity $\langle v'_j \varphi\psi \circ f^n T^+_s, (T^-_{m_1} - T^-_{m'_1}) \wedge \cdots \wedge T^-_{m_{k-s}} \rangle$. Write $T^-_i = \omega + dd^c g_i$ where the g_i are decreasing. By Stokes' formula, we have that:

$$\begin{aligned} \langle v'_j \varphi\psi \circ f^n T^+_s, (T^-_{m_1} - T^-_{m'_1}) \wedge \cdots \wedge T^-_{m_{k-s}} \rangle = \\ -\langle (\varphi\psi \circ f^n dv'_j + v'_j \psi \circ f^n d\varphi + v'_j \varphi d\psi \circ f^n) \wedge T^+_s, d^c(g_{m_1} - g_{m'_1}) \wedge \cdots \wedge T^-_{m_{k-s}} \rangle. \end{aligned}$$

Write the last sum $I + II + III$ with obvious notations. Using Cauchy Schwarz inequality for the first term, we have that:

$$\begin{aligned} |I|^2 \leq \langle dv_j \wedge d^c v_j \wedge T^+_s, T^-_{m_2} \wedge \cdots \wedge T^-_{m_{k-s}} \rangle \times \\ \langle (\alpha'(v_j))^2 d(g_{m_1} - g_{m'_1}) \wedge d^c(g_{m_1} - g_{m'_1}) \wedge T^+_s, T^-_{m_2} \wedge \cdots \wedge T^-_{m_{k-s}} \rangle. \end{aligned}$$

As in the proof of the previous lemma, we have that this term goes to zero when $j \to \infty$ since $\alpha'(v_j)$ converges uniformly to 0 on the compact sets of $\mathbb{P}^k \backslash \mathscr{C}(f^n)$.

Now for II, we use Cauchy Schwarz inequality and we have that:

$$\begin{aligned} |II|^2 \leq & \langle d\varphi \wedge d^c\varphi \wedge T^+_s, T^-_{m_2} \wedge \cdots \wedge T^-_{m_{k-s}} \rangle \\ & \langle d(g_{m_1} - g_{m'_1}) \wedge d^c(g_{m_1} - g_{m'_1}) \wedge T^+_s, T^-_{m_2} \wedge \cdots \wedge T^-_{m_{k-s}} \rangle. \end{aligned}$$

The first term of the product is bounded as it converges to $\int d\varphi \wedge d^c\varphi \wedge T_s^+ \wedge (T^-)^{s-1}$. By Stokes, we recognize that the second term is equal to

$$\begin{aligned}\langle -(g_{m_1} - g_{m_1'}) \wedge dd^c(g_{m_1} - g_{m_1'}) \wedge T_s^+, T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle = \\ \langle -(g_{m_1} - g_{m_1'}) \wedge (T_{m_1}^- - T_{m_1'}^-) \wedge T_s^+, T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle = \\ \mathscr{U}_{T_{m_1}^-}(T_s^+ \wedge T_{m_1'}^- \wedge T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^-) - \mathscr{U}_{T_{m_1'}^-}(T_s^+ \wedge T_{m_1'}^- \wedge T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^-) + \\ \mathscr{U}_{T_{m_1'}^-}(T_s^+ \wedge T_{m_1}^- \wedge T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^-) - \mathscr{U}_{T_{m_1}^-}(T_s^+ \wedge T_{m_1}^- \wedge T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^-).\end{aligned}$$

Observe that this term goes to 0 when the m_i, m_i' are large enough. Indeed recall that $\mathscr{U}_1(S,T)$ is continuous for the Hartogs' convergence (Lemma A.1.11), so:

$$\mathscr{U}_{T_{m_1}^-}(T_s^+ \wedge T_{m_1'}^- \wedge T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^-)$$

converges to $\mathscr{U}_{T^-}(\mu)$ which is finite and so does the other terms in the majoration of II (the convergence is uniform else we could extract a subsequence which does not converge).

Now we bound III. Applying Cauchy-Schwarz inequality gives:

$$\begin{aligned}|III|^2 \leq & \langle |v_j'|^2 d\psi \circ f^n \wedge d^c\psi \circ f^n \wedge T_s^+, T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle \\ & \langle d(g_{m_1} - g_{m_1'}) \wedge d^c(g_{m_1} - g_{m_1'}) \wedge T_s^+, T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle\end{aligned}$$

Observe that the second integral is the same than in the bound of II so it goes to zero. For the first term of the product, we use that f^n is smooth in the support of v_j' and thus $d\psi \circ f^n \wedge d^c\psi \circ f^n = (f^n)^*(d\psi \wedge d^c\psi)$ in the integral. We can assume that $d\psi \wedge d^c\psi \leq \omega$. Using the invariance of T_s^+ and the fact that v_j' is equal to 0 near $\mathscr{C}(f^n)$, we have that $(f^n)^*(\omega) \wedge T_s^+ = d^{-sn}(f^n)^*(\omega \wedge T_s^+)$ in the integral, so the first term in the bound of III is less than:

$$\frac{1}{d^{sn}} \langle (f^n)^*(\omega \wedge T_s^+), T_{m_2}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle.$$

That last term can be computed cohomologically and is equal to $\frac{\delta^{n(k-s-1)}}{d^{ns}} < 1$. So as for II, we have that III goes to 0 uniformly in n.

Letting j go to ∞, we have that

$$\langle \varphi\psi \circ f^n T_s^+, T_{m_1}^- \wedge \cdots \wedge T_{m_{k-s}}^- - T_{m_1'}^- \wedge \cdots \wedge T_{m_{k-s}'}^- \rangle$$

converges uniformly to 0. In particular, we can interchange the limit in:

$$\lim_m \lim_n \langle (\varphi\psi \circ f^n) T_s^+, T_{m_1}^- \wedge \cdots \wedge T_{m_{k-s}}^- \rangle$$

which gives $\lim_n \mu(\varphi\psi \circ f^n) = \mu(\varphi)\mu(\psi)$ hence the mixing. □

We now show that the measure μ satisfies the hypothesis of Chapter 2 and we deduce from that a bound of its entropy. Recall that we denote by μ_n the sequence of probabilities:

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} f_*^i\left(\frac{(f^n)^*\omega^s\wedge\omega^{k-s}}{\lambda_s(f^n)}\right).$$

In our case, using Lemma A.2.2 and hypothesis (H), we can write it as:

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} L^{n-i}(\omega^s)\wedge\Lambda^i(\omega^{k-s}).$$

We consider the hypothesis (H): there exists a subsequence $\mu_{\psi(n)}$ of μ_n converging to a measure μ' such that:

$$(H):\ \lim_{n\to+\infty}\int \log d(x,I)d\mu_{\psi(n)}(x) = \int \log d(x,I)d\mu'(x) > -\infty.$$

In here, we do not need to take a subsequence:

Proposition 3.4.16. – *The sequence (μ_n) converges to μ and satisfies the hypothesis (H).*

Proof. – Let φ be a smooth test function. Choose $\varepsilon > 0$. By Theorem 3.4.1, since $L^{n-i}(\omega^s)$ and $\Lambda^i(\omega^{k-s})$ converge in the Hartogs' sense, Proposition A.1.15 assures us that $L^{n-i}(\omega^s)\wedge\Lambda^i(\omega^{k-s})$ converges in the Hartogs' sense to μ. So we have for $\sqrt{n}\leq i\leq n-\sqrt{n}$ and n large enough that$|L^{n-i}(\omega^s)\wedge\Lambda^i(\omega^{k-s})(\varphi)-\mu(\varphi)|\leq\varepsilon$. The fact that (μ_n) goes to μ follows since they are $o(n)$ terms for which the estimation does not stand.

Now, by Lemma A.2.5, we see that there exist constants $A_{i,n}\geq 0$ such that $\mathcal{U}_{L^{n-i}(\omega^s)\wedge\Lambda^i(\omega^{k-s})}\geq\mathcal{U}_\mu - A_{i,n}$ with $A_{i,n}$ uniformly bounded from above by C and arbitrarily close to zero for i and n large enough. We consider super-potentials of mean 0. In particular:

$$\mathcal{U}_{\mu_n}\geq\mathcal{U}_\mu-\frac{1}{n}\sum_{i=0}^{n-1}A_{i,n}.$$

So we have that the sequence μ_n is more H-regular than μ for all n. We also have the convergence in the Hartogs' sense to μ since $\frac{1}{n}\sum_{i=0}^{n-1}A_{i,n}$ goes to 0 when $n\to+\infty$.

Thus $\mu_n(G)\to\mu(G)$ which is finite by Theorem 3.4.1 where G is a negative potential of the Green current of order 1 that we denote T^+. Since T^+ is less H-regular than $L(\omega)$, we have that if $U_{L(\omega)}$ is a quasi-potential of $L(\omega)$ then $\mu_n(U_{L(\omega)})\to\mu(U_{L(\omega)})$ which is also finite. By Lemma 3.2.4, we have that:

$$AU_{L(\omega)}(x) < \log\operatorname{dist}(x,I^+)$$

for $A > 0$ large enough. We denote $\varphi := \log \mathrm{dist}(x, I^+)$. Since μ gives no mass to I^+ that means that $AU_{L(\omega)}(x) \leq \varphi \leq 0$ for μ a.e point, so we have that $\varphi \in L^1(\mu)$. We have the classical lemma:

LEMMA 3.4.17. – *Let ν_n be a sequence of measures converging to ν in the sense of measures. Then for v an upper semi-continuous function, we have that*

$$\limsup \nu_n(v) \leq \nu(v).$$

Proof. – Recall that an usc function can be written as the limit of a decreasing sequence of continuous functions. So for some small $\alpha > 0$ we can take $v' \geq v$ a continuous function such that $\int v' d\nu \leq \int v d\nu + \alpha$ by monotone convergence. In particular:

$$\int v d\nu_n \leq \int v' d\nu_n \to \int v' d\nu \leq \int v d\nu + \alpha.$$

And the result follows by letting $\alpha \to 0$. □

End of the proof of the proposition. – Now, φ is upper semi-continuous, so:

$$\limsup \mu_n(\varphi) \leq \mu(\varphi)$$

We also have that $(A+1)U_{L(\omega)} - \varphi$ is upper semi-continuous (we use the fact that it is equal to $-\infty$ on I^+). That and $\mu_n(U_{L(\omega)}) \to \mu(U_{L(\omega)})$ give:

$$\liminf \mu_n(\varphi) \geq \mu(\varphi).$$

This is exactly the fact that μ satisfies Hypothesis (H). □

We can now apply Theorem 1 to get the proposition:

THEOREM 3.4.18. – *The topological entropy of f is greater than $\log d_s = s \log d$. More precisely, the entropy of μ is greater than $s \log d$.*

On the other hand, the topological entropy is always bounded by $\max_{0 \leq s \leq k} \log d_s$ (see **[25]** for the projective case and **[22]** for the Kähler case). So we have the fundamental result:

THEOREM 3.4.19. – *The topological entropy of f is equal to $\log d_s$. Moreover, the entropy of μ is equal to $s \log d$ so μ is a measure of maximal entropy.*

REMARK 3.4.20. – A very natural question asked by the referee is whether μ is unique. To answer such a question one often use some sort of control on the size of the stable and unstable manifolds that seems very difficult to get here (see for example **[2]** in the case of Hénon maps).

This allows us to use the first author's estimate of the Lyapunov exponents (Corollary 3 in [**11**]). To apply that result, we need to have that $\log(\mathrm{dist}(x, \mathscr{C}^+))$ is integrable with respect to μ. For that observe that the function $U_{L(\omega)}$ is integrable with respect to μ. By invariance, $f_*(U_{L(\omega)})$ is also integrable. Write $U_{L(\omega)}$ as in Lemma 3.2.4:

$$U_{L(\omega)} = d^{-1} \log |F|^2 - \log |Z|^2,$$

where $f = [P_1 : \ldots : P_{k+1}]$ and $F = (P_1, \ldots, P_{k+1})$. Write $f^{-1} = [Q_1 : \ldots : Q_{k+1}]$ where the Q_i are homogeneous polynomials of degree δ and write $F^{-1} = (Q_1, \ldots, Q_{k+1})$. There is of course an abuse of notation since $F \circ F^{-1} \neq Id$ instead, we have that:

$$F \circ F^{-1} = P(z_1, \ldots, z_{k+1}) \times (z_1, \ldots, z_{k+1}),$$

where P is an homogeneous polynomial of degree $d\delta - 1$ equal to 0 in $\pi^{-1}(\mathscr{C}^-)$ and $\pi : \mathbb{C}^{k+1} \to \mathbb{P}^k$ is the canonical projection. Then, we have that:

$$f_*(\frac{1}{d} \log |F|^2 - \log |Z|^2) = \frac{1}{d} \log |F \circ F^{-1}|^2 - \log |F^{-1}(Z)|^2.$$

We recognize $d^{-1} \log |F \circ F^{-1}|^2 - \delta \log |Z|^2 + \delta \log |Z|^2 - \log |F^{-1}(Z)|^2$. But

$$\delta \log |Z|^2 - \log |F^{-1}(Z)|^2 = -\delta U_{\Lambda(\omega)}$$

is in $L^1(\mu)$ and by difference, so is $d^{-1} \log |F \circ F^{-1}|^2 - \delta \log |Z|^2$. As in Lemma 3.2.4, we then have that $\log \mathrm{dist}(., \mathscr{C}^-)$ is in $L^1(\mu)$. Similarly, $\log \mathrm{dist}(., \mathscr{C}^+)$ is in $L^1(\mu)$.

THEOREM 3.4.21. – *The Lyapunov exponents $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_k$ of μ are well defined and we have the estimates:*

$$\chi_1 \geq \cdots \geq \chi_s \geq \frac{1}{2} \log \frac{d_s}{d_{s-1}} = \frac{1}{2} \log d > 0$$

$$0 > -\frac{1}{2} \log \delta = \frac{1}{2} \log \frac{d_{s+1}}{d_s} \geq \chi_{s+1} \geq \cdots \geq \chi_k.$$

In particular, the measure μ is hyperbolic.

APPENDIX A

SUPER-POTENTIALS

A.1. Definitions and properties of super-potentials

We recall here the facts and definitions we use on super-potentials. Everything in this section was taken from **[31]** so we refer the reader to this paper for proofs and details.

Recall that $\mathscr{C}_s$ is the convex compact set of (strongly) positive closed currents S of bidegree (s,s) on $\mathbb{P}^k$ and of mass 1. To develop the calculus, we have to consider $\mathscr{C}_s$ as an infinite dimensional space with special families of currents that we parametrize by the unit disc Δ in $\mathbb{C}$. We call these families *special structural discs of currents.* The notion of structural varieties of $\mathscr{C}_s$ was introduced in **[28]**. In some sense, we consider $\mathscr{C}_s$ as a space of infinite dimension admitting "complex subvarieties" of finite dimension. For S in $\mathscr{C}_s$, it is always possible to construct a special structural variety $\varphi : \Delta \to \mathscr{C}_s$ such that $\varphi(0) = S$ and $\varphi(z)$ is a smooth form for $z \neq 0$. The points of the structural disk centered at R are defined as the regularization of R by an approximation of the identity in $\mathrm{Aut}(\mathbb{P}^k)$; the structural disk can be seen as a positive closed current in $\Delta \times \mathbb{P}^k$ and the points of the disks are the vertical slices of that current.

Let S be a current in $\mathscr{C}_s$ with $s \geq 1$. If U is a $(s-1, s-1)$-current such that $dd^c U = S - \omega^s$, we say that U is a *quasi-potential* of S. The integral $\langle U, \omega^{k-s+1}\rangle$ is *the mean* of U. Observe that such quasi-potential is defined up to a dd^c-closed current. For $s = 1$ such functions are constant a.e., but in the general case, they can be singular currents. Nevertheless, we have the proposition:

Proposition A.1.1. – *Let S be a current in $\mathscr{C}_s$. Then, there is a negative quasi-potential U of S depending linearly on S such that for every r with $1 \leq r < k/(k-1)$ and for $1 \leq \rho < 2k/(2k-1)$*

$$\|U\|_{\mathscr{L}^r} \leq c_r \quad \textit{and} \quad \|dU\|_{\mathscr{L}^\rho} \leq c_\rho$$

for some positive constants c_r, c_ρ independent of S. Moreover, U depends continuously on S with respect to the $\mathscr{L}^r$ topology on U and the weak topology on S.

We are going to introduce a *super-potential* associated to S. It is an affine upper semi-continuous function $\mathscr{U}_S$ defined on $\mathscr{C}_{k-s+1}$ with values in $\mathbb{R} \cup \{-\infty\}$. For $R \in \mathscr{C}_{k-s+1}$ smooth, we define the super-potential of mean M of S by $\mathscr{U}_S(R) := \langle S, U_R \rangle$ where U_R is a quasi-potential of R of mean $\langle U_R, \omega^s \rangle = M$. The integral $\langle S, U_R \rangle$ does not depend on the choice of U_R with a fixed mean M. If S is smooth, we have $\mathscr{U}_S(R) = \langle U_S, R \rangle$ where U_S is a quasi-potential of S of mean M. Now assume that R is not smooth. Consider the above special structural variety $\varphi : \Delta \to \mathscr{C}_{k-s+1}$ associated to $R \in \mathscr{C}_{k-s+1}$ and write R_θ for $\varphi(\theta)$. Then the function $u(\theta) := \mathscr{U}_S(R_\theta)$ defined on Δ^* can be extended as a quasi-subharmonic function on Δ. Let (S_θ) and let (R_θ) be special structural disks associated to $S \in \mathscr{C}_s$ and $R \in C_{k-s+1}$. Then we have the proposition:

Proposition A.1.2. – *The function $\mathscr{U}_S$ can be extended in a unique way to an affine upper semi continuous function on $\mathscr{C}_{k-s+1}$ with values in $\mathbb{R} \cup \{-\infty\}$, also denoted by $\mathscr{U}_S$, such that*

$$\mathscr{U}_S(R) = \lim_{\theta \to 0} \mathscr{U}_{S_\theta}(R) = \lim_{\theta \to 0} \mathscr{U}_S(R_\theta).$$

In particular, we have

$$\mathscr{U}_S(R) = \limsup_{R' \to R} \mathscr{U}_S(R') \text{ with } R' \text{ smooth.}$$

Moreover, there is a constant $c \geq 0$ independent of S such that if $\mathscr{U}_S$ is the super-potential of mean m of S, then $\mathscr{U}_S \leq m + c$ everywhere.

Super-potentials determine the current, more precisely, we have the proposition:

Proposition A.1.3. – *Let I be a compact subset in $\mathbb{P}^k$ with $(2k-2s)$-dimensional Hausdorff measure 0. Let S, S' be currents in $\mathscr{C}_s$ and $\mathscr{U}_S$, $\mathscr{U}_{S'}$ be super-potentials of S, S'. If $\mathscr{U}_S = \mathscr{U}_{S'}$ on smooth forms in $\mathscr{C}_{k-s+1}$ with compact support in $\mathbb{P}^k \setminus I$, then $S = S'$.*

For $I = \varnothing$, this tells us that the values of the super-potential on smooth forms determine uniquely the current.

A crucial notion to prove the convergence of currents is the following:

Definition A.1.4. – *Let (S_n) be a sequence in $\mathscr{C}_s$ converging to a current S. Let $\mathscr{U}_{S_n}$ (resp. $\mathscr{U}_S$) be the super-potential of mean M_n (resp. M) of S_n (resp. S). Assume that M_n converge to M. If $\mathscr{U}_{S_n} \geq \mathscr{U}_S$ for every n, we say that (S_n)* converge to S in the Hartogs' sense. *If a current S' in $\mathscr{C}_s$ admits a super-potential $\mathscr{U}_{S'}$ such that $\mathscr{U}_{S'} \geq \mathscr{U}_S$ we say that S' is* more H-regular than S.

Smooth currents are dense for the Hartog's convergence, more precisely:

PROPOSITION A.1.5. – *Let $S \in \mathscr{C}_s$ and let $\mathscr{U}$ be a super-potential of S of mean M. There is a sequence of smooth forms (S_n) in $\mathscr{C}_s$ with super-potentials $\mathscr{U}_n$ of mean M_n such that*

- $\mathrm{supp}(S_n)$ *converge to* $\mathrm{supp}(S)$;
- S_n *converge to* S *and* $M_n \to M$;
- $(\mathscr{U}_n)$ *decreases to* $\mathscr{U}$.

We have the following convergence theorem:

PROPOSITION A.1.6. – *Let (S_n) be a sequence in $\mathscr{C}_s$ converging to a current S. Let $\mathscr{U}_{S_n}$ (resp. $\mathscr{U}_S$) be the super-potential of mean M_n (resp. M) of S_n (resp. S). Assume that M_n converge to M. Let $\mathscr{U}$ be a continuous function on a compact subset K of $\mathscr{C}_{k-s+1}$ such that $\mathscr{U}_S < \mathscr{U}$ on K. Then, for n large enough we have $\mathscr{U}_{S_n} < \mathscr{U}$ on K. In particular, we have $\limsup \mathscr{U}_{S_n} \leq \mathscr{U}_S$ on $\mathscr{C}_{k-s+1}$. Furthermore, if $S_n \to S$ in the Hartogs' sense, then $\mathscr{U}_{S_n} \to \mathscr{U}_S$ pointwise.*

In $\mathscr{C}_{k-s+1}$, they are points which are more "regular" than other, namely smooth forms. This is a difference with psh functions. In particular, it is often easier to obtain the convergence at such points:

PROPOSITION A.1.7. – *Let (S_n) be a sequence in $\mathscr{C}_s$ and $\mathscr{U}_{S_n}$ be super-potentials of mean M_n of S_n. Assume that $(\mathscr{U}_{S_n})$ converges to a finite function $\mathscr{U}$ on smooth forms in $\mathscr{C}_{k-s+1}$. Then, (M_n) converges to a constant M, (S_n) converges to a current S and $\mathscr{U}$ is equal to the super-potential of mean M of S on smooth forms in $\mathscr{C}_{k-s+1}$.*

The following is the main argument to get the convergence of the Green current:

PROPOSITION A.1.8. – *Let $\mathscr{U}_{S_n}$ be super-potentials of mean M_n of S_n. Assume that $\mathscr{U}_{S_n}$ decrease to a function $\mathscr{U} \not\equiv -\infty$. Then, (S_n) converges to a current S, (M_n) converges to a constant M and $\mathscr{U}$ is the super-potential of mean M of S.*

In particular, the convergence at one point of the super-potentials gives the convergence of the currents in the Hartogs' sense in the case of decreasing super-potentials.

An interesting symmetry result is that if $\mathscr{U}_S$ and $\mathscr{U}_R$ are super-potentials of the same mean M of R and S respectively, then $\mathscr{U}_S(R) = \mathscr{U}_R(S)$.

There is a notion of *super-polarity* for Borel subsets E of $\mathscr{C}_{k-s+1}$. This notion does not describe "small" sets E but rather how singular are the currents in E.

DEFINITION A.1.9. – We say that E is *super-polar* in $\mathscr{C}_{k-s+1}$ if there is a super-potential $\mathscr{U}_S$ of a current S in $\mathscr{C}_s$ such that $E \subset \{\mathscr{U}_S = -\infty\}$.

Denote by $\widetilde{E}$ the set of currents $cR + (1-c)R'$ with $R \in \widehat{E}$, $R' \in \mathscr{C}_{k-s+1}$ and $0 < c \leq 1$, and $\widehat{E}$ the *barycentric hull* of E, i.e. the set of currents $\int R d\nu(R)$ where ν is a probability measure on $\mathscr{C}_{k-s+1}$ such that $\nu(E) = 1$. Then, $\widetilde{E}$ and $\widehat{E}$ are convex.

Proposition A.1.10. – *The following properties are equivalent*

1. *E is super-polar in $\mathscr{C}_{k-s+1}$.*
2. *$\widehat{E}$ is super-polar in $\mathscr{C}_{k-s+1}$.*
3. *$\widetilde{E}$ is super-polar in $\mathscr{C}_{k-s+1}$.*

Moreover, a countable union of super-polar sets is super-polar.

One of the purposes of super-potentials is to define the wedge product of current (see Section 4 in [**31**]). We define a universal function $\mathscr{U}_s$ on $\mathscr{C}_s \times \mathscr{C}_{k-s+1}$ by

$$\mathscr{U}_s(S, R) := \mathscr{U}_S(R) = \mathscr{U}_R(S)$$

where $\mathscr{U}_S$ and $\mathscr{U}_R$ are super-potentials of mean 0 of S and R. The function $\mathscr{U}_s$ is is u.s.c. on $\mathscr{C}_s \times \mathscr{C}_{k-s+1}$. It even enjoys a nice continuity for the Hartogs' convergence:

Lemma A.1.11. – *Let $(S_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ be sequences of currents in $\mathscr{C}_s$ and $\mathscr{C}_{k-s+1}$ converging in the Hartogs' sense to S and R respectively. Then, $\mathscr{U}_s(S_n, R_n)$ converge to $\mathscr{U}_s(S, R)$. Moreover, if $\mathscr{U}_s(S, R)$ is finite, then $\mathscr{U}_s(S_n, R_n)$ is finite for every n.*

We have the proposition:

Proposition A.1.12. – *Let $s_1 \in \mathbb{N}^*$ and $s_2 \in \mathbb{N}^*$ with $s_1 + s_2 \leq k$. The following conditions are equivalent and are symmetric on $R_1 \in \mathscr{C}_{s_1}$ and $R_2 \in \mathscr{C}_{s_2}$:*

1. *$\mathscr{U}_{s_1}(R_1, R_2 \wedge \Omega)$ is finite for at least one smooth form Ω in $\mathscr{C}_{k-s_1-s_2+1}$.*
2. *$\mathscr{U}_{s_1}(R_1, R_2 \wedge \Omega)$ is finite for every smooth form Ω in $\mathscr{C}_{k-s_1-s_2+1}$.*
3. *There are sequences $(R_{i,n})_{n\geq 0}$ in $\mathscr{C}_{s_i}$ converging to R_i and a smooth form Ω in $\mathscr{C}_{k-s_1-s_2+1}$ such that $\mathscr{U}_{s_1}(R_{1,n}, R_{2,n} \wedge \Omega)$ is bounded.*

Definition A.1.13. – *We say that R_1 and R_2 are wedgeable if they satisfy the conditions in Proposition A.1.12.*

Assume that $R_1 \in \mathscr{C}_{s_1}$ and $R_2 \in \mathscr{C}_{s_2}$ are wedgeable. For every smooth real form φ of bidegree $(k - s_1 - s_2, k - s_1 - s_2)$, write $dd^c\varphi = c(\Omega^+ - \Omega^-)$ where $\Omega^\pm$ are smooth forms in $\mathscr{C}_{k-s_1-s_2+1}$ and c is a positive constant. We define the wedge-product (or the intersection) $R_1 \wedge R_2$ by its action on the smooth forms by:

$$(19) \qquad \langle R_1 \wedge R_2, \varphi\rangle := \langle R_2, \omega^{s_1} \wedge \varphi\rangle + c\,\mathscr{U}_{s_1}(R_1, R_2 \wedge \Omega^+) - c\,\mathscr{U}_{s_1}(R_1, R_2 \wedge \Omega^-).$$

The right hand side of (19) is independent of the choice of c, $\Omega^\pm$ and depends linearly on φ. Moreover, $R_1 \wedge R_2$ defines a positive closed $(s_1 + s_2, s_1 + s_2)$-current of mass 1 with support in $\mathrm{supp}(R_1) \cap \mathrm{supp}(R_2)$ which depends linearly on each R_i and is symmetric with respect to the variables. The notion of wedgeability behave well with the notion of H-regularity:

LEMMA A.1.14. – *Let R_i and R'_i be currents in $\mathscr{C}_{s_i}$. Assume that R_1 and R_2 are wedgeable. If R'_i is more H-regular than R_i for $i = 1, 2$, then R'_1 and R'_2 are wedgeable and $R'_1 \wedge R'_2$ is more H-regular than $R_1 \wedge R_2$.*

We will use the following proposition in the construction of the equilibrium measure:

PROPOSITION A.1.15. – *Let R_1, R_2 be wedgeable currents as above and $R_{i,n}$ be currents in $\mathscr{C}_{s_i}$ converging to R_i in the Hartogs' sense. Then, $R_{1,n}$, $R_{2,n}$ are wedgeable and $R_{1,n} \wedge R_{2,n}$ converge to $R_1 \wedge R_2$ in the Hartogs' sense.*

For several currents (more than 2), the notion of wedgeability is defined by induction: that is R_1, R_2 and R_3 are wedgeable if R_1 and R_2 are wedgeable and $R_1 \wedge R_2$ and R_3 are wedgeable. One shows that this definition is in fact symmetric in the R_i and we have Proposition A.1.15 for several currents.

An interesting subcase is when we consider currents $R_1, \dots, R_l$ such that R_i is of bidegree $(1,1)$ for $i \geq 2$. For $2 \leq i \leq l$, there is a quasi-psh function u_i on $\mathbb{P}^k$ such that

$$dd^c u_i = R_i - \omega.$$

LEMMA A.1.16. – *The currents $R_1, \dots, R_l$ are wedgeable if and only if for every $2 \leq i \leq l$, u_i is integrable with respect to the trace measure of $R_1 \wedge \cdots \wedge R_{i-1}$. In particular, the last condition is symmetric with respect to $R_2, \dots, R_l$.*

If R_2 has a quasi-potential integrable with respect to R_1, it is classical to define the wedge-product $R_1 \wedge R_2$ by

$$R_1 \wedge R_2 := dd^c(u_2 R_1) + \omega \wedge R_1.$$

One defines $R_1 \wedge \cdots \wedge R_l$ by induction. These two definitions coincide.

The other use of super-potentials is to define pull-back and push-forward of current by meromorphic maps (see section 5.1 in [**31**]). We state the result in the case where f is birational although the results are true in the case where f is just meromorphic. Recall that pull-back and push-forward of a current are defined formally by formulae (3) and (4) of the previous section:

$$f^*(S) := (\pi_1)_*\big(\pi_2^*(S) \wedge [\Gamma]\big)$$
$$f_*(R) := (\pi_2)_*\big(\pi_1^*(R) \wedge [\Gamma]\big),$$

where $[\Gamma]$ is the current of integration of Γ. We denote by $I^+ := I(f)$ and $I^- = I'(f) = I(f^{-1})$ the indeterminacy sets of f and f^{-1}.

In particular, for a current in $R \in \mathscr{C}_{k-s+1}$ smooth near I^+ the push-forward is a well defined positive closed $(k-s+1, k-s+1)$-current and the mass λ_{s-1} of $f_*(R)$ does not depend on R. Similarly, for a current S in C_s smooth near I^- the pull-back is a well defined positive closed (s,s)-current and the mass of $f^*(S)$ is equal to λ_s. So as above we define for these currents $\Lambda(R) = \lambda_{s-1}^{-1} f_*(R)$ and $L(S) = \lambda_s^{-1} f^*(S)$ (the normalized push-forward and pull-back).

Using the theory of super-potentials we can extend these definitions to other currents. Namely, we say that a current $S \in \mathscr{C}_s$ is *f^*-admissible* if there exists a current $R_0 \in \mathscr{C}_{k-s+1}$ which is smooth on a neighborhood of I^+ such that the super-potentials of S are finite at $\Lambda(R_0)$. For such S, if (S_n) is a sequence of currents converging in the Hartogs' sense to S then S_n is f^*-admissible and $(\lambda_s)^{-1} f^*(S_n)$ converges in the Hartogs' sense to a limit independent on the choice of (S_n) that we denote $(\lambda_s)^{-1} f^*(S)$ (in particular $f^*(S)$ is of mass λ_s). In other words, we have the continuity result:

THEOREM A.1.17. – *Let S be an f^* admissible current. Let S_n be a sequence converging to S in the Hartogs' sense, then S_n is f^*-admissible and $L(S_n)$ converges in the Hartogs' sense to $L(S)$.*

We say that S is *invariant under f^** or that S is *f^*-invariant* if S is f^*-admissible and $L(S) = S$.

PROPOSITION A.1.18. – *Let S be an f^*-admissible current in $\mathscr{C}_s$. Let $\mathscr{U}_S$, $\mathscr{U}_{L(\omega^s)}$ be super-potentials of S and $L(\omega^s)$. Then $\lambda_s^{-1}\lambda_{s-1}\mathscr{U}_S \circ \Lambda + \mathscr{U}_{L(\omega^s)}$ is equal to a super-potential of $L(S)$ on $R \in \mathscr{C}_{k-s+1}$, smooth in a neighbourhood of I^+.*

Similarly, one define push-forward of currents. We remark that an element $S \in \mathscr{C}_s$ smooth near I^- is f^*-admissible and that the two available definitions of $L(S)$ coincide.

A.2. Additional properties

We state now some properties of the super-potentials that we need. Recall that $f \in \mathscr{B}$ and that s is such that $\dim(I^+) = k-s-1$ and $\dim(I^-) = s-1$.

LEMMA A.2.1. – *Let $S_1 \in \mathscr{C}_{r_1}$ and $S_2 \in \mathscr{C}_{r_2}$ be wedgeable currents with $r_1 + r_2 \leq k$. There exist super-potentials $\mathscr{U}_{S_1\wedge S_2}$, $\mathscr{U}_{S_1}$ and $\mathscr{U}_{S_2}$ of $S_1 \wedge S_2$, S_1 and S_2 such that:*

$$\mathscr{U}_{S_1\wedge S_2}(R) = \mathscr{U}_{S_1}(R \wedge S_2) + \mathscr{U}_{S_2}(R \wedge \omega^{r_1})$$

for all $R \in \mathscr{C}_{k-r_1-r_2+1}$ such that R and S_2 are wedgeable.

Proof. – Let $S_{1,n}$ and $S_{2,m}$ be sequence of smooth currents in $\mathscr{C}_{r_1}$ and $\mathscr{C}_{r_2}$ converging to S_1 and S_2 in the Hartogs' sense. If $U_{S_{1,n}}$ and $U_{S_{2,m}}$ are smooth quasi-potentials of $S_{1,n}$ and $S_{2,m}$, then:

$$S_{1,n} \wedge S_{2,m} = dd^c(U_{S_{1,n}} \wedge S_{2,m} + U_{S_{2,m}} \wedge \omega^{r_1}) + \omega^{r_1+r_2}.$$

So, if $\mathscr{U}_{S_{1,n} \wedge S_{2,m}}$, $\mathscr{U}_{S_{1,n}}$ and $\mathscr{U}_{S_{2,m}}$ are super-potentials of mean 0 of the currents, we have for any R in $\mathscr{C}_{k-r_1-r_2+1}$:

$$\mathscr{U}_{S_{1,n} \wedge S_{2,m}}(R) = \mathscr{U}_{S_{1,n}}(S_{2,m} \wedge R) + \mathscr{U}_{S_{2,m}}(\omega^{r_1} \wedge R) - \mathscr{U}_{S_{1,n}}(S_{2,m} \wedge \omega^{k-r_1-r_2+1}).$$

Now, we take R such that R and S_2 are wedgeable. We let $n \to \infty$. By Proposition A.1.15, $S_{1,n} \wedge S_{2,m}$ converges in the Hartogs' sense to $S_1 \wedge S_{2,m}$. So by Proposition A.1.6, we have that:

$$\mathscr{U}_{S_1 \wedge S_{2,m}}(R) = \mathscr{U}_{S_1}(S_{2,m} \wedge R) + \mathscr{U}_{S_{2,m}}(\omega^{r_1} \wedge R) - \mathscr{U}_{S_1}(S_{2,m} \wedge \omega^{k-r_1-r_2+1}),$$

where the super-potentials are of mean 0. Similarly, we let $m \to \infty$. Recall that $S_{2,m} \wedge R$ converges to $S_2 \wedge R$ in the Hartogs' sense, hence $\mathscr{U}_{S_1}(S_{2,m} \wedge R) = \mathscr{U}_{S_{2,m} \wedge R}(S_1)$ converges to $\mathscr{U}_{S_2 \wedge R}(S_1)$. So we have indeed:

$$\mathscr{U}_{S_1 \wedge S_2}(R) = \mathscr{U}_{S_1}(S_2 \wedge R) + \mathscr{U}_{S_2}(\omega^{r_1} \wedge R) - \mathscr{U}_{S_1}(S_2 \wedge \omega^{k-r_1-r_2+1}).$$

Since $\mathscr{U}_{S_1}(S_2 \wedge \omega^{k-r_1-r_2+1})$ does not depend on R and is finite because S_1 and S_2 are wedgeable, we can add it to $\mathscr{U}_{S_1 \wedge S_2}$ (this just change the mean of the super-potentials) and we have the lemma. □

LEMMA A.2.2. – *Let $T_1 \in \mathscr{C}_{r_1}$ be an f_*-admissible current with $r_1 \geq k - s$. Let $T_2 \in \mathscr{C}_{r_2}$ be an f^*-admissible current with $r_1 + r_2 \leq k$ such that $L(T_2)$ and T_1 are wedgeable and $L(T_2) \wedge T_1$ is f_*-admissible. Assume also that T_2 and $\Lambda(T_1)$ are wedgeable. Then:*

$$\Lambda(L(T_2) \wedge T_1) = T_2 \wedge \Lambda(T_1).$$

Proof. – Assume first that T_2 is smooth. Let $T_{1,n}$ and $L_{2,m}$ be sequences in $\mathscr{C}_{r_1}$ and $\mathscr{C}_{r_2}$ converging in the Hartogs' sense to T_1 and $L(T_2)$. Let Θ be a smooth current of bidegree $k - r_1 - r_2$. We want to show that:

$$\langle T_2 \wedge \Lambda(T_1), \Theta \rangle = \langle \Lambda(L(T_2) \wedge T_1), \Theta \rangle$$

for all Θ smooth.

First assume that Θ is closed and (strongly) positive. Up to a multiplicative constant, we assume that $\Theta \in \mathscr{C}_{k-r_1-r_2}$. Since everything is smooth:

$$\begin{aligned}\langle \Lambda(L_{2,m} \wedge T_{1,n}), \Theta \rangle &= \langle L_{2,m} \wedge T_{1,n}, L(\Theta) \rangle \\ &= \langle T_{1,n}, L_{2,m} \wedge L(\Theta) \rangle.\end{aligned}$$

Since $L_{2,m}$ converges to $L(T_2)$, we have that $L_{2,m} \wedge L(\Theta)$ converges to $L(T_2 \wedge \Theta)$ in the sense of currents. Indeed, the sequence $(L_{2,m} \wedge L(\Theta))_m$ is of mass 1. We can extract a converging subsequence (in the sense of currents). Observe that its limit is less than $L(T_2) \wedge L(\omega^{k-r_1-r_2})$ which gives no mass to I^+ by dimension's arguments. So its limit gives no mass to I^+ either. But outside I^+, $L_{2,m} \wedge L(\Theta)$ converges to the smooth form $L(T_2) \wedge L(\Theta)$. That implies that $L_{2,m} \wedge L(\Theta)$ converges to the trivial extension of $L(T_2) \wedge L(\Theta)$. This extension is equal to the form $L(T_2 \wedge \Theta)$ since it has coefficients in L^1 (as in Lemma 2.1.1 test the convergence against a smooth form Ψ and write it $\xi\Psi + (1-\xi)\Psi$ where ξ is a cut-off function equal to 1 in a small neighborhood of I^+).

So, letting $m \to \infty$ and using the fact that $T_2 \wedge \Theta$ is smooth give:

$$\lim_{m\to\infty} \langle \Lambda(L_{2,m} \wedge T_{1,n}), \Theta\rangle = \langle T_{1,n}, L(T_2 \wedge \Theta)\rangle \qquad = \langle \Lambda(T_{1,n}), T_2 \wedge \Theta\rangle$$
$$= \langle T_2 \wedge \Lambda(T_{1,n}), \Theta\rangle.$$

Now, we let $n \to \infty$, $\Lambda(T_{1,n})$ converges to $\Lambda(T_1)$ in the Hartogs' sense (Proposition A.1.17) hence Proposition A.1.15 gives that $T_2 \wedge \Lambda(T_{1,n})$ converges to $T_2 \wedge \Lambda(T_1)$ in the sense of currents. On the other hand, when n and m go to ∞, $\Lambda(L_{2,m} \wedge T_{1,n})$ converge to $\Lambda(L(T_2) \wedge T_1)$ in the sense of currents by Propositions A.1.15 and A.1.17.

So we have indeed that:

$$\langle T_2 \wedge \Lambda(T_1), \Theta\rangle = \langle \Lambda(L(T_2) \wedge T_1), \Theta\rangle$$

for Θ closed.

Now, for Θ not necessarily closed, we can assume that Θ is positive and $\Theta \leq C\omega^{k-r_1-r_2}$ for C large enough. Again, we have that

$$\langle \Lambda(L_{2,m} \wedge T_{1,n}), \Theta\rangle = \langle T_{1,n}, L_{2,m} \wedge L(\Theta)\rangle.$$

The positive current $L_{2,m} \wedge L(\Theta)$ is less than $CL_{2,m} \wedge L(\omega^{k-r_1-r_2})$ so it is of mass less than C. We can extract a converging subsequence (in the sense of currents). Observe that its limit is less than $CL(T_2) \wedge L(\omega^{k-r_1-r_2}) = CL(T_2 \wedge \omega^{k-r_1-r_2})$ which gives no mass to I^+ by dimension's arguments. So its limit gives no mass to I^+ either. Again outside I^+, $L_{2,m} \wedge L(\Theta)$ converges to the smooth form $L(T_2) \wedge L(\Theta)$. That implies that $L_{2,m} \wedge L(\Theta)$ converges to the trivial extension of $L(T_2) \wedge L(\Theta)$ which is equal to the form $L(T_2 \wedge \Theta)$ which has coefficients in L^1. We have again that:

$$\langle T_{1,n}, L(T_2 \wedge \Theta)\rangle = \langle T_2 \wedge \Lambda(T_{1,n}), \Theta\rangle.$$

That gives the conclusion as before.

Now, for T_2 not necessarily smooth, we can approximate T_2 by a sequence of smooth currents converging in the Hartogs' sense to T_2. Since both members of the equality:

$$\Lambda(L(T_2) \wedge T_1) = T_2 \wedge \Lambda(T_1),$$

depend continuously on T_2 for the Hartogs' convergence (wedge-product, pull-pack and push-forward are continuous for the Hartogs' convergence) we get the lemma from the smooth case. □

Some of the hypothesis of the following lemma are not necessary, but the following version is enough for our purpose:

LEMMA A.2.3. – *Let S_1, S_2 and S_3 in $\mathscr{C}_{r_1}$, $\mathscr{C}_{r_2}$ and $\mathscr{C}_{r_3}$ with $r_1+r_2+r_3=k+1$. Assume that S_2 is smooth and that $L(S_2)$ and S_3 are wedgeable. Assume that $L(S_2)\wedge S_3$ is f_*-admissible. Assume also that the super-potential $\mathscr{U}_{S_1}$ of S_1 is finite at $\Lambda(L(S_2)\wedge S_3)$. Finally, we also assume that S_1 is f^*-admissible, that S_3 and $L(S_1)$ are wedgeable and that their wedge product is finite at the super-potential $\mathscr{U}_{L(S_2)}$ of $L(S_2)$. Then we have the formula:*

$$\mathscr{U}_{S_1}\left(\Lambda\left(L\left(S_2\right)\wedge S_3\right)\right)=\left(\frac{\lambda_{r_1}}{\lambda_{r_1-1}}\right)\Big(\mathscr{U}_{L(S_2)}\left(S_3\wedge L\left(S_1\right)\right)-\mathscr{U}_{L(S_2)}\left(S_3\wedge L\left(\omega^{r_1}\right)\right)\Big)\\ +\mathscr{U}_{S_1}\left(\Lambda\left(\omega^{r_2}\wedge S_3\right)\right)$$

Proof. – First, observe that ω^{r_1} is more H-regular than S_1 hence $L(\omega^{r_1})$ is more H-regular than $L(S_1)$. So, we have that S_3 and $L(\omega^{r_1})$ are wedgeable and $S_3\wedge L(\omega^{r_1})$ is more H-regular than $S_3\wedge L(S_1)$. In particular, $\mathscr{U}_{L(S_2)}(S_3\wedge L(\omega^{r_1}))$ is finite. Similarly, the expression $\mathscr{U}_{S_1}(\Lambda(\omega^{r_2}\wedge S_3))$ is finite and everything is well defined in it.

Let S_{1,m_1}, L_{2,m_2} and S_{3,m_3} be sequences of smooth currents converging in the Hartogs' sense to S_1, $L(S_2)$ and S_3. Let U_{1,m_1} and U_{2,m_2} be smooth quasi-potential of S_{1,m_1} and L_{2,m_2}. For smooth currents, we have the identity:

$$\begin{aligned}\mathscr{U}_{S_{1,m_1}}(\Lambda(L_{2,m_2}\wedge S_{3,m_3}))&=\langle U_{1,m_1},\Lambda(L_{2,m_2}\wedge S_{3,m_3})\rangle\\&=\langle(\lambda_{r_1-1}(f))^{-1}f^*(U_{1,m_1}),L_{2,m_2}\wedge S_{3,m_3}\rangle.\end{aligned}$$

By Stokes, we recognize:

$$\langle(\lambda_{r_1-1}(f))^{-1}f^*(U_{1,m_1}),\omega^{r_2}\wedge S_{3,m_3}\rangle+\langle dd^c((\lambda_{r_1-1}(f))^{-1}f^*(U_{1,m_1})),U_{2,m_2}\wedge S_{3,m_3}\rangle.$$

Since f^* commutes with dd^c, we have that

$$\begin{aligned}dd^c((\lambda_{r_1-1}(f))^{-1}f^*(U_{1,m_1}))&=(\lambda_{r_1-1}(f))^{-1}f^*(S_{1,m_1}-\omega^{r_1})\\&=\left(\frac{\lambda_{r_1}}{\lambda_{r_1-1}}\right)\left(L(S_{1,m_1})-L(\omega^{r_1})\right).\end{aligned}$$

The lemma follows then by letting m_1 then m_2 then m_3 go to ∞ and using the continuity of the wedge product, the pull-back, push-forward and value at a point for the super-potential for the Hartogs' convergence. □

We also have the following integration by parts lemma:

LEMMA A.2.4. – *Let S_1, S_2 and S_3 in $\mathscr{C}_{r_1}$, $\mathscr{C}_{r_2}$ and $\mathscr{C}_{r_3}$ with $r_1+r_2+r_3=k+1$. Assume that the S_i are two by two wedgeable. Then if $\mathscr{U}_{S_1}$ and $\mathscr{U}_{S_2}$ are super-potentials of S_1 and S_2 finite at $S_2\wedge S_3$ and $S_1\wedge S_3$:*

$$\mathscr{U}_{S_1}(S_2\wedge S_3)-\mathscr{U}_{S_1}(\omega^{r_2}\wedge S_3)=\mathscr{U}_{S_2}(S_1\wedge S_3)-\mathscr{U}_{S_2}(\omega^{r_1}\wedge S_3)$$

Proof. – First observe that $\omega^{r_2}\wedge S_3$ and $\omega^{r_1}\wedge S_3$ are more H-regular than $S_2\wedge S_3$ and $S_1\wedge S_3$ so every term is finite.

Now, if every term is smooth, we write U_{S_1} and U_{S_2} quasi-potentials of S_1 and S_2. By Stokes:

$$\begin{aligned}\mathscr{U}_{S_1}(S_2\wedge S_3)-\mathscr{U}_{S_1}(\omega^{r_2}\wedge S_3)&=\langle U_{S_1},S_2\wedge S_3-\omega^{r_2}\wedge S_3\rangle=\langle U_{S_1},dd^cU_2\wedge S_3\rangle\\&=\langle dd^cU_{S_1},U_2\wedge S_3\rangle=\langle S_1,U_2\wedge S_3\rangle-\langle\omega^{r_1},U_2\wedge S_3\rangle\\&=\mathscr{U}_{S_2}(S_1\wedge S_3)-\mathscr{U}_{S_2}(\omega^{r_1}\wedge S_3).\end{aligned}$$

So the result follows in the general case by Hartogs' convergence. □

We also have the following refinement of Lemma A.1.14 whose proof is similar:

LEMMA A.2.5. – *Let $R_{1,n}$ and $R_{2,m}$ be sequence of currents in $\mathscr{C}_{p_1}$ and $\mathscr{C}_{p_2}$ converging in the Hartogs' sense to R_1 and R_2 which are wedgeable. Then there exists a constant $A_{n,m}>0$ such that*

$$\mathscr{U}_{R_{1,n}\wedge R_{2,m}}\geq\mathscr{U}_{R_1\wedge R_2}-A_{n,m}$$

where the super-potentials are of mean 0 and where $A_{n,m}$ is uniformly bounded from above in n and m and is arbitrarily small for n and m large enough.

Proof. – By Lemma A.1.14, $R_{1,n}$ and $R_{2,m}$ are wedgeable.

The symbols U and $\mathscr{U}$ below denote quasi-potentials and super-potentials of mean 0. Assume first that all the terms are smooth. By hypothesis, there is a constant a such that $\mathscr{U}_{R_{1,n}}+a\geq\mathscr{U}_{R_1}$ and $\mathscr{U}_{R_{2,m}}+a\geq\mathscr{U}_{R_2}$. Write $r=k-p_1-p_2+1$. Consider a smooth form R in $\mathscr{C}_r$ and choose U_R smooth. We have the computation:

$$\begin{aligned}\mathscr{U}_{R_{1,n}\wedge R_{2,m}}(R)&=\langle R_{1,n}\wedge R_{2,m},U_R\rangle=\langle R_{2,m},\omega^{p_1}\wedge U_R\rangle+\langle R_{1,n}-\omega^{p_1},R_{2,m}\wedge U_R\rangle\\&=\langle R_{2,m},\omega^{p_1}\wedge U_R\rangle+\langle dd^cU_{R_{1,n}},R_{2,m}\wedge U_R\rangle\\&=\langle R_{2,m},\omega^{p_1}\wedge U_R\rangle+\langle U_{R_{1,n}},R_{2,m}\wedge dd^cU_R\rangle\\&=\langle R_{2,m},\omega^{p_1}\wedge U_R\rangle+\mathscr{U}_{R_{1,n}}(R_{2,m}\wedge R)-\mathscr{U}_{R_{1,n}}(R_{2,m}\wedge\omega^r).\\&=\mathscr{U}_R(R_{2,m}\wedge\omega^{p_1})+\mathscr{U}_{R_{1,n}}(R_{2,m}\wedge R)-\mathscr{U}_{R_{1,n}}(R_{2,m}\wedge\omega^r).\end{aligned}$$

That identity also holds when the currents are not smooth by Hartogs' convergence. We have the same identity for $R_1 \wedge R_{2,m}$ and $R_1 \wedge R_2$. By difference, we have:

$$\begin{aligned}\mathscr{U}_{R_{1,n}\wedge R_{2,m}}(R) - \mathscr{U}_{R_1\wedge R_{2,m}}(R) + \mathscr{U}_{R_1\wedge R_{2,m}}(R) - \mathscr{U}_{R_1\wedge R_2}(R) = \\ \mathscr{U}_{R_{1,n}}(R_{2,m}\wedge R) - \mathscr{U}_{R_1}(R_{2,m}\wedge R) - \mathscr{U}_{R_{1,n}}(R_{2,m}\wedge \omega^r) + \mathscr{U}_{R_1}(R_{2,m}\wedge \omega^r) \\ +\mathscr{U}_{R_{2,m}}(R_1\wedge R) - \mathscr{U}_{R_2}(R_1\wedge R) - \mathscr{U}_{R_{2,m}}(R_1\wedge \omega^r) + \mathscr{U}_{R_2}(R_1\wedge \omega^r).\end{aligned}$$

So:

$$\begin{aligned}\mathscr{U}_{R_{1,n}\wedge R_{2,m}}(R) - \mathscr{U}_{R_1\wedge R_2}(R) \geq \\ -2a - \mathscr{U}_{R_{1,n}}(R_{2,m}\wedge \omega^r) + \mathscr{U}_{R_1}(R_{2,m}\wedge \omega^r) - \mathscr{U}_{R_{2,m}}(R_1\wedge \omega^r) + \mathscr{U}_{R_2}(R_1\wedge \omega^r).\end{aligned}$$

The last quantity does not depend on R and is uniformly bounded from below: the terms with a minus sign are greater than $-M$ since the super-potentials are of mean 0, and since $R_{2,m}\wedge\omega^r$ converges to $R_2\wedge\omega^r$ in the Hartogs' sense and $\mathscr{U}_{R_1}(R_2\wedge\omega^r)$ is finite, we have that $\mathscr{U}_{R_1}(R_{2,m}\wedge\omega^r)$ and $\mathscr{U}_{R_2}(R_1\wedge\omega^r)$ are uniformly bounded from below.

This gives that the constant $A_{n,m}$ of the lemma is uniformly bounded from above in n and m. Now, we can choose $A_{n,m}$ going to zero by Proposition A.1.15: if not, we can extract subsequences such that $A_{n_i,m_i} \geq \varepsilon > 0$ and it contradicts the fact that $R_{1,n_i}\wedge R_{2,m_i}$ converges in the Hartogs' sense to $R_1 \wedge R_2$. □

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