Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Numéro 123
Nouvelle série

PROJECTIONS IN SEVERAL COMPLEX VARIABLES

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre National de la Recherche Scientifique

Diffusion

Maison de la SMF
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13288 Marseille Cedex 9
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Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

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P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 28 € ($ 42)
Abonnement Europe : 255 €, hors Europe : 290 € ($ 435)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0249-633-X

ISBN 978-85629-304-1

Directrice de la publication : Aline BONAMI

MÉMOIRES DE LA SMF 123

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Société Mathématique de France 2010
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* – 32A25, 32V05, 32V20, 32W30, 58A14.

Key words and phrases. – CR manifold, Szegő kernel, Bergman kernel, heat equation, microlocal analysis.

PROJECTIONS IN SEVERAL COMPLEX VARIABLES

Chin-Yu Hsiao

Abstract. – This work consists two parts. In the first part, we completely study the heat equation method of Menikoff-Sjöstrand and apply it to the Kohn Laplacian defined on a compact orientable connected CR manifold. We then get the full asymptotic expansion of the Szegő projection for $(0, q)$ forms when the Levi form is non-degenerate. This generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. Our main tools are Fourier integral operators with complex valued phase Melin and Sjöstrand functions.

In the second part, we obtain the full asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate. This also generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. We introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the heat equation method of Menikoff and Sjöstrand to this operator. We obtain a description of a new Szegő projection up to smoothing operators. Finally, we get our main result by using the Poisson operator.

***Résumé* (Projecteurs en plusieurs variables complexes).** – Ce travail comporte deux parties. Dans la première, nous appliquons la méthode de Menikoff-Sjöstrand au laplacien de Kohn, défini sur une varieté CR compacte orientée connexe et nous obtenons un développement asymptotique complet du projecteur de Szegő pour les $(0, q)$ formes quand la forme de Levi est non-dégénérée. Cela généralise un résultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous utilisons des opérateurs intégraux de Fourier à phases complexes de Melin et Sjöstrand.

Dans la deuxième partie, nous obtenons un développement asymptotique de la singularité du noyau de Bergman pour les $(0, q)$ formes quand la forme de Levi est non-dégénérée. Cela généralise un résultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous introduisons un nouvel opérateur analogue au laplacien de Kohn défini sur le bord du domaine, et nous y appliquons la méthode de Menikoff-Sjöstrand. Cela donne une description modulo les opérateurs régularisants d'un nouvel projecteur de Szegő. Enfin, nous obtenons notre résultat principal en utilisant l'opérateur de Poisson.

CONTENTS

INTRODUCTION

The Bergman and Szegő projections are classical subjects in several complex variables and complex geometry. By Kohn's regularity theorem for the $\overline{\partial}$-Neumann problem (1963, [**23**]), the boundary behavior of the Bergman kernel is highly dependent on the Levi curvature of the boundary. The study of the boundary behavior of the Bergman kernel on domains with positive Levi curvature (strictly pseudoconvex domains) became an important topic in the field then. In 1965, L. Hörmander [**13**] determined the boundary behavior of the Bergman kernel. C. Fefferman (1974, [**9**]) established an asymptotic expansion at the diagonal of the Bergman kernel. More complete asymptotics of the Bergman kernel was obtained by Boutet de Monvel and Sjöstrand (1976, [**34**]). They also established an asymptotic expansion of the Szegő kernel on strongly pseudoconvex boundaries. All these developments concerned pseudoconvex domains. For the nonpseudoconvex domain, there are few results. R. Beals and P. Greiner (1988, [**1**]) proved that the Szegő projection is a Heisenberg pseudodifferential operator, under certain Levi curvature assumptions. Hörmander (2004, [**19**]) determined the boundary behavior of the Bergman kernel when the Levi form is negative definite by computing the leading term of the Bergman kernel on a spherical shell in $\mathbb{C}^n$.

Other developments recently concerned the Bergman kernel for a high power of a holomorphic line bundle. D. Catlin (1997, [**5**]) and S. Zelditch (1998, [**38**]) adapted a result of Boutet de Monvel-Sjöstrand for the asymptotics of the Szegő kernel on a strictly pseudoconvex boundary to establish the complete asymptotic expansion of the Bergman kernel for a high power of a holomorphic line bundle with positive curvature. X. Dai, K. Liu and X. Ma (2004, [**7**]) obtained the full off-diagonal asymptotic expansion and Agmon estimates of the Bergman kernel for a high power of positive line bundle by using the heat kernel method. Recently, a new proof of the existence of the complete asymptotic expansion was obtained by B. Berndtsson, R. Berman and J. Sjöstrand (2004, [**2**]) and by X. Ma, G. Marinescu (2004, [**27**]). Without the positive curvature assumption, R. Berman and J. Sjöstrand (2005, [**3**]) obtained a full asymptotic expansion of the Bergman kernel for a high power of a line bundle when the curvature is non-degenerate. The approach of Berman and Sjöstrand builds on the heat equation method of Menikoff-Sjöstrand (1978, [**29**]). The expansion was obtained independently by X. Ma and G. Marinescu (2006, [**25**], without a phase function) by using a spectral gap estimate for the Hodge Laplacian. The main analytic tool of

X. Ma and G. Marinescu is the analytic localization technique in local index theory developed by Bismut-Lebeau. We refer the readers to the very nice book ([**26**]) by X. Ma and G. Marinescu for this approach.

Recently, Hörmander (2004, [**19**]) studied the Bergman projection for $(0, q)$ forms. In that paper (page 1306), Hörmander suggested: "A carefull microlocal analysis along the lines of Boutet de Monvel-Sjöstrand should give the asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate."

The main goal for this work is to achieve Hörmander's wish-more precisely, to obtain an asymptotic expansion of the Bergman projection for $(0, q)$ forms. The first step of my research is to establish an asymptotic expansion of the Szegő projection for $(0, q)$ forms. Then, find a suitable operator defined on the boundary of domain which plays the same role as the Kohn Laplacian in the approach of Boutet de Monvel-Sjöstrand.

This work consists two parts. In the first paper, we completely study the heat equation method of Menikoff-Sjöstrand and apply it to the Kohn Laplacian defined on a compact orientable connected CR manifold. We then get the full asymptotic expansion of the Szegő projection for $(0, q)$ forms when the Levi form is non-degenerate. We also compute the leading term of the Szegő projection.

In the second paper, we introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the method of Menikoff-Sjöstrand to this operator. We obtain a description of a new Szegő projection up to smoothing operators. Finally, by using the Poisson operator, we get the full asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate.

These two papers can be read independently. We hope that this work can serve as an introduction to certain microlocal techniques with applications to complex geometry and CR geometry.

Acknowledgements

This work is part of a Ph.D thesis at École polytechnique in 2008. The author would like to thank his advisor Johannes Sjöstrand for his patience, guidance and inspiration. Furthermore, the auther is grateful to Bo Berndtsson, Louis Boutet de Monvel, Xiaonan Ma and George Marinescu for comments and useful suggestions on an early draft of the manuscript. Thanks are due to the referee who comments contributed to the improvement of the paper.

PART I

ON THE SINGULARITIES OF THE SZEGŐ PROJECTION FOR $(0, q)$ FORMS

CHAPTER 1

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, and take a smooth Hermitian metric $(\,|\,)$ on $\mathbb{C}T(X)$ so that $\Lambda^{1,0}T(X)$ is orthogonal to $\Lambda^{0,1}T(X)$ and $(u \mid v)$ is real if u, v are real tangent vectors, where $\Lambda^{0,1}T(X) = \overline{\Lambda^{1,0}T(X)}$ and $\mathbb{C}T(X)$ is the complexified tangent bundle. For $p \in X$, let L_p be the Levi form of X at p (see (1.4)). Given q, $0 \leq q \leq n-1$, the Levi form is said to satisfty condition $Y(q)$ at $p \in X$ if for any $|J| = q$, $J = (j_1, j_2, \ldots, j_q)$, $1 \leq j_1 < j_2 < \cdots < j_q \leq n-1$, we have

$$\left|\sum_{j \notin J} \lambda_j - \sum_{j \in J} \lambda_j\right| < \sum_{j=1}^{n-1} |\lambda_j|, \tag{1.1}$$

where λ_j, $j = 1, \ldots, (n-1)$, are the eigenvalues of L_p (for the precise meaning of the eigenvalues of the Levi form, see the discussion after (1.4)). If the Levi form is non-degenerate at p, then $Y(q)$ holds at p if and only if $q \neq n_-, n_+$, where (n_-, n_+) is the signature of L_p, i.e. the number of negative eigenvalues of L_p is n_- and $n_+ + n_- = n-1$. Let $\Box_b$ be the Kohn Laplacian on X (see Chen-Shaw [**6**] or Chapter 2) and let $\Box_b^{(q)}$ denote the restriction to $(0, q)$ forms. When condition $Y(q)$ holds, Kohn's L^2 estimates give the hypoellipicity with loss of one dervative for the solutions of $\Box_b^{(q)} u = f$ (see Folland-Kohn [**10**], [**6**] and Chapter 2). The Szegő projection is the orthogonal projection onto the kernel of $\Box_b^{(q)}$ in the L^2 space. When condition $Y(q)$ fails, one is interested in the Szegő projection on the level of $(0, q)$ forms. Beals and Greiner (see [**1**]) proved that the Szegő projection is a Heisenberg pseudodifferential operator. Boutet de Monvel and Sjöstrand (see [**34**]) obtained the full asymptotic expansion for the Szegő projection in the case of functions. We have been influenced by these works. The main inspiration for the present paper comes from Berman and Sjöstrand [**3**].

We now start to formulate the main results. First, we introduce some standard notations. Let Ω be a C^∞ paracompact manifold equipped with a smooth density of integration. We let $T(\Omega)$ and $T^*(\Omega)$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T(\Omega)$ and $\mathbb{C}T^*(\Omega)$ respectively. We write $\langle\ ,\rangle$ to denote the pointwise duality between $T(\Omega)$ and $T^*(\Omega)$. We extend $\langle\ ,\rangle$ bilinearly to $\mathbb{C}T(\Omega)\times\mathbb{C}T^*(\Omega)$. Let E be a C^∞ vector bundle over Ω. The fiber of E at $x\in\Omega$ will be denoted by E_x. Let $Y\subset\subset\Omega$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y;\,E)$ and $\mathscr{D}'(Y;\,E)$ respectively. Let $\mathscr{E}'(Y;\,E)$ be the subspace of $\mathscr{D}'(Y;\,E)$ whose elements have compact support in Y. For $s\in\mathbb{R}$, we let $H^s(Y;\,E)$ denote the Sobolev space of order s of sections of E over Y.

The Hermitian metric $(\ |\ )$ on $\mathbb{C}T(X)$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(X)$ that we shall also denote by $(\ |\ )$. Let $\Lambda^{0,q}T^*(X)$ be the bundle of $(0,q)$ forms of X. The Hermitian metric $(\ |\ )$ on $\mathbb{C}T^*(X)$ induces a Hermitian metric on $\Lambda^{0,q}T^*(X)$ also denoted by $(\ |\ )$.

We take (dm) as the induced volume form on X. Let $(\ |\ )$ be the inner product on $C^\infty(X;\,\Lambda^{0,q}T^*(X))$ defined by

$$(f\mid g)=\int_X(f(z)\mid g(z))(dm),\ \ f,g\in C^\infty(X;\,\Lambda^{0,q}T^*(X)). \tag{1.2}$$

Let

$$\pi^{(q)}:L^2(X;\,\Lambda^{0,q}T^*(X))\to\operatorname{Ker}\Box_b^{(q)}$$

be the Szegő projection, i.e. the orthogonal projection onto the kernel of $\Box_b^{(q)}$. Let

$$K_{\pi^{(q)}}(x,y)\in\mathscr{D}'\big(X\times X;\,\mathscr{L}(\Lambda^{0,q}T_y^*(X),\Lambda^{0,q}T_x^*(X))\big)$$

be the distribution kernel of $\pi^{(q)}$ with respect to the induced volume form (dm). Here $\mathscr{L}\big(\Lambda^{0,q}T_y^*(X),\Lambda^{0,q}T_x^*(X)\big)$ is the vector bundle with fiber over (x,y) consisting of the linear maps from $\Lambda^{0,q}T_y^*(X)$ to $\Lambda^{0,q}T_x^*(X)$.

We pause and recall a general fact of distribution theory. Let E and F be C^∞ vector bundles over a paracompact C^∞ manifold Ω equipped with a smooth density of integration. If $A:C_0^\infty(\Omega;\,E)\to\mathscr{D}'(\Omega;\,F)$ is continuous, we write $K_A(x,y)$ or $A(x,y)$ to denote the distribution kernel of A. The following two statements are equivalent

(a) A is continuous: $\mathscr{E}'(\Omega;\,E)\to C^\infty(\Omega;\,F)$,
(b) $K_A\in C^\infty(\Omega\times\Omega;\,\mathscr{L}(E_y,F_x))$.

If A satisfies (a) or (b), we say that A is smoothing. Let $B:C_0^\infty(\Omega;\,E)\to\mathscr{D}'(\Omega;\,F)$. We write $A\equiv B$ if $A-B$ is a smoothing operator. A is smoothing if and only if $A:H^s_{\rm comp}(\Omega;\,E)\to H^{s+N}_{\rm loc}(\Omega;\,F)$ is continuous, for all $N\geq 0$, $s\in\mathbb{R}$, where

$$H^s_{\rm loc}(\Omega;\,F)=\{u\in\mathscr{D}'(\Omega;\,F);\ \varphi u\in H^s(\Omega;\,F);\ \forall\varphi\in C_0^\infty(\Omega)\}$$

and $H^s_{\rm comp}(\Omega;\,E)=H^s_{\rm loc}\big(\Omega;\,E\big)\bigcap\mathscr{E}'(\Omega;\,E\big)$ (see Hörmander **[18]**).

Let $\Lambda^{1,0}T^*(X)$ denote the bundle with fiber $\Lambda^{1,0}T^*_z(X):=\overline{\Lambda^{0,1}T^*_z(X)}$ at $z\in X$. Locally we can choose an orthonormal frame $\omega_1(z),\ldots,\omega_{n-1}(z)$ for the bundle $\Lambda^{1,0}T^*_z(X)$, then $\overline{\omega}_1(z),\ldots,\overline{\omega}_{n-1}(z)$ is an orthonormal frame for the bundle $\Lambda^{0,1}T^*_z(X)$. The real $(2n-2)$ form $\omega=i^{n-1}\omega_1\wedge\overline{\omega}_1\wedge\cdots\wedge\omega_{n-1}\wedge\overline{\omega}_{n-1}$ is independent of the choice of the orthonormal frame. Thus ω is globally defined. Locally there is a real 1 form $\omega_0(z)$ of length one which is orthogonal to $\Lambda^{1,0}T^*_z(X)\oplus\Lambda^{0,1}T^*_z(X)$. $\omega_0(z)$ is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$ form Q on X. Thus, ω_0 can be specified uniquely by requiring that

$$\omega\wedge\omega_0=fQ, \tag{1.3}$$

where f is a positive function. Therefore ω_0, so chosen, is globally defined. We call ω_0 the uniquely determined global real 1 form.

We recall that the Levi form L_p, $p\in X$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(X)$ defined as follows:

$$\begin{split}&\text{For any } Z,\ W\in\Lambda^{1,0}T_p(X),\ \text{pick } \widetilde{Z},\ \widetilde{W}\in C^\infty(X;\,\Lambda^{1,0}T(X)) \text{ that satisfy}\\ &\widetilde{Z}(p)=Z,\ \widetilde{W}(p)=W. \text{ Then } L_p(Z,\overline{W})=\frac{1}{2i}\Big\langle[\widetilde{Z}\ ,\overline{\widetilde{W}}](p)\ ,\omega_0(p)\Big\rangle.\end{split} \tag{1.4}$$

The eigenvalues of the Levi form at $p\in X$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\ |\ )$ on $\Lambda^{1,0}T_p(X)$.

In this work, we assume that

Assumption 1.1. – *The Levi form is non-degenerate at each point of X.*

The characteristic manifold of $\Box^{(q)}_b$ is given by $\Sigma=\Sigma^+\bigcup\Sigma^-$, where

$$\begin{split}\Sigma^+&=\{(x,\xi)\in T^*(X)\setminus 0;\ \xi=\lambda\omega_0(x),\lambda>0\}\,,\\ \Sigma^-&=\{(x,\xi)\in T^*(X)\setminus 0;\ \xi=\lambda\omega_0(x),\lambda<0\}\,.\end{split} \tag{1.5}$$

Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of the Levi form. We define

$$\begin{aligned}\hat\Sigma&=\Sigma^+ &&\text{if } n_+=q\neq n_-,\\ \hat\Sigma&=\Sigma^- &&\text{if } n_-=q\neq n_+,\\ \hat\Sigma&=\Sigma^+\bigcup\Sigma^- &&\text{if } n_+=q=n_-.\end{aligned}$$

Recall that (see **[10]**, **[6]**) if $q\notin\{n_-,n_+\}$, then

$$K_{\pi^{(q)}}(x,y)\in C^\infty\big(X\times X;\,\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))\big).$$

The main result of this work is the following

THEOREM 1.2. – *Let $(X, \Lambda^{1,0}T(X))$ be a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, with a Hermitian metric $(\ |\ )$. Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form and let $q = n_-$ or n_+. Suppose $\Box_b^{(q)}$ has closed range. Then,*

$$\pi^{(q)} : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^s(X;\, \Lambda^{0,q}T^*(X))$$

is continuous, for all $s \in \mathbb{R}$, and $\mathrm{WF}'(K_{\pi^{(q)}}) = \mathrm{diag}\,(\hat{\Sigma} \times \hat{\Sigma})$, where $\mathrm{WF}'(K_{\pi^{(q)}}) = \{(x, \xi, y, \eta) \in T^*(X) \times T^*(X);\ (x, \xi, y, -\eta) \in \mathrm{WF}\,(K_{\pi^{(q)}})\}$. Here $\mathrm{WF}\,(K_{\pi^{(q)}})$ is the wave front set of $K_{\pi^{(q)}}$ in the sense of Hörmander (see **[14]** *or chapter 8 of* **[18]***). Moreover, we have*

$$\begin{aligned} K_{\pi^{(q)}} &= K_{\pi_+^{(q)}} && \text{if } n_+ = q \neq n_-,\\ K_{\pi^{(q)}} &= K_{\pi_-^{(q)}} && \text{if } n_- = q \neq n_+,\\ K_{\pi^{(q)}} &= K_{\pi_+^{(q)}} + K_{\pi_-^{(q)}} && \text{if } n_+ = q = n_-, \end{aligned}$$

where $K_{\pi_+^{(q)}}(x,y)$ satisfies

$$K_{\pi_+^{(q)}}(x,y) \equiv \int_0^\infty e^{i\phi_+(x,y)t} s_+(x,y,t)dt$$

with

$$s_+(x,y,t) \in S^{n-1}_{1,0}\big(X \times X \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big),$$
$$s_+(x,y,t) \sim \sum_{j=0}^\infty s^j_+(x,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(X \times X \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big)$, where $S^m_{1,0}$, $m \in \mathbb{R}$, is the Hörmander symbol space **[14]***,*

$$s^j_+(x,y) \in C^\infty(X \times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))),\ \ j = 0, 1, \ldots,$$

and

$$\phi_+(x,y) \in C^\infty(X \times X), \quad \mathrm{Im}\,\phi_+(x,y) \geq 0, \tag{1.6}$$
$$\phi_+(x,x) = 0, \quad \phi_+(x,y) \neq 0 \ \ \text{if} \ \ x \neq y, \tag{1.7}$$
$$d_x\phi_+ \neq 0, \quad d_y\phi_+ \neq 0 \ \ \text{where} \ \ \mathrm{Im}\,\phi_+ = 0, \tag{1.8}$$
$$d_x\phi_+(x,y)|_{x=y} = \omega_0(x), \quad d_y\phi_+(x,y)|_{x=y} = -\omega_0(x), \tag{1.9}$$
$$\phi_+(x,y) = -\overline{\phi}_+(y,x). \tag{1.10}$$

Similarly,

$$K_{\pi_-^{(q)}}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} s_-(x,y,t)dt$$

with

$$s_-(x,y,t) \sim \sum_{j=0}^{\infty} s_-^j(x,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(X \times X \times]0,\infty[; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big)$, *where*

$$s_-^j(x,y) \in C^\infty\big(X \times X; \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big), \quad j = 0,1,\ldots,$$

and when $q = n_- = n_+$, $\phi_-(x,y) = -\overline{\phi}_+(x,y)$.

A formula for $s_+^0(x,x)$ *will be given in Proposition 1.7. More properties of the phase* $\phi_+(x,y)$ *will be given in Theorem 1.4 and Remark 1.5 below.*

Remark 1.3. – If $Y(q-1)$ and $Y(q+1)$ hold then $\Box_b^{(q)}$ has closed range (see Chapter 6 for a review and references).

The phase $\phi_+(x,y)$ is not unique. We can replace $\phi_+(x,y)$ by

$$\widetilde{\phi}(x,y) = f(x,y)\phi_+(x,y), \tag{1.11}$$

where $f(x,y) \in C^\infty(X \times X)$ is real and $f(x,x) = 1$, $f(x,y) = f(y,x)$. Then $\widetilde{\phi}$ satisfies (1.6)–(1.10). We work with local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. Let $p \in \Omega$. We can check that

$$\begin{aligned}\Big\langle \widetilde{\phi}''(p,p)U, V \Big\rangle &= \langle (\phi_+)''(p,p)U, V\rangle + \langle df(p,p), U\rangle \langle d\phi_+(p,p), V\rangle \\ &\quad + \langle df(p,p), V\rangle \langle d\phi_+(p,p), U\rangle, \quad U, V \in \mathbb{C}T(X),\end{aligned}$$

where $(\phi_+)'' = \begin{bmatrix} (\phi_+)''_{xx} & (\phi_+)''_{xy} \\ (\phi_+)''_{yx} & (\phi_+)''_{yy}\end{bmatrix}$ and similarly for $\widetilde{\phi}''$. The Hessian $(\phi_+)''$ of ϕ_+ at (p,p) is well-defined on the space

$$T_{(p,p)}H_+ := \{W \in \mathbb{C}T_p(X) \times \mathbb{C}T_p(X);\ \langle d\phi_+(p,p), W\rangle = 0\}.$$

In Chapter 7, we will define $T_{(p,p)}H_+$ as the tangent space of the formal hypersurface H_+ (see (7.45)) at $(p,p) \in X \times X$.

We define the tangential Hessian of $\phi_+(x,y)$ at (p,p) as the bilinear map:

$$\begin{aligned}&T_{(p,p)}H_+ \times T_{(p,p)}H_+ \to \mathbb{C},\\ &\qquad (U,V) \to \big\langle (\phi''_+)(p,p)U, V\big\rangle, \quad U, V \in T_{(p,p)}H_+.\end{aligned}$$

In Chapter 8, we compute the tangential Hessian of $\phi_+(x,y)$ at (p,p)

Theorem 1.4. – *For a given point* $p \in X$, *let* $U_1(x), \ldots, U_{n-1}(x)$ *be an orthonormal frame of* $\Lambda^{1,0}T_x(X)$ *varying smoothly with* x *in a neighborhood of* p, *for which the Levi form is diagonalized at* p. *We take local coordinates* $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} +$

ix_{2j}, $j=1,\ldots,n-1$, *defined on some neighborhood of* p *such that* $\omega_0(p)=\sqrt{2}dx_{2n-1}$, $x(p)=0$, $(\frac{\partial}{\partial x_j}(p)\mid\frac{\partial}{\partial x_k}(p))=2\delta_{j,k}$, $j,k=1,\ldots,2n-1$ *and*

$$U_j=\frac{\partial}{\partial z_j}-\frac{1}{\sqrt{2}}i\lambda_j\overline{z}_j\frac{\partial}{\partial x_{2n-1}}-\frac{1}{\sqrt{2}}c_jx_{2n-1}\frac{\partial}{\partial x_{2n-1}}+O(|x|^2),\ j=1,\ldots,n-1,$$

where $c_j\in\mathbb{C}$, $\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}})$, $j=1,\ldots,n-1$, *and* λ_j, $j=1,\ldots,n-1$, *are the eigenvalues of* L_p *(this is always possible. see pages 157–160 of* [**1**]*). We also write* $y=(y_1,\ldots,y_{2n-1})$, $w_j=y_{2j-1}+iy_{2j}$, $j=1,\ldots,n-1$. *Then,*

(1.12)

$$\begin{aligned}\phi_+(x,y)=\sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}|\lambda_j|\,|z_j-w_j|^2\\+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+c_j(z_jx_{2n-1}-w_jy_{2n-1})\\+\overline{c}_j(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)+(x_{2n-1}-y_{2n-1})f(x,y)+O(|(x,y)|^3),\end{aligned}$$

where $f\in C^\infty$, $f(0,0)=0$, $f(x,y)=\overline{f}(y,x)$.

Remark 1.5. – With the notations used in Theorem 1.4, since $\frac{\partial\phi_+}{\partial x_{2n-1}}(0,0)\neq0$, from the Malgrange preparation theorem (see Theorem 7.57 of [**18**]), we have

$$\phi_+(x,y)=g(x,y)(\sqrt{2}x_{2n-1}+h(x',y))$$

in some neighborhood of $(0,0)$, where g, $h\in C^\infty$, $g(0,0)=1$, $h(0,0)=0$ and $x'=(x_1,\ldots,x_{2n-2})$. Put $\hat{\phi}(x,y)=\sqrt{2}x_{2n-1}+h(x',y)$. From the global theory of Fourier integral operators (see Theorem 4.2 of Melin-Sjöstrand [**28**]), we see that ϕ_+ and $\hat{\phi}$ are equivalent at $(p,\omega_0(p))$ in the sense of Melin-Sjöstrand (see page 172 of [**28**]). Since $\phi_+(x,y)=-\overline{\phi}_+(y,x)$, we can replace the phase $\phi_+(x,y)$ by

$$\frac{\hat{\phi}(x,y)-\overline{\hat{\phi}}(y,x)}{2}.$$

Then the new function $\phi_+(x,y)$ satisfies (1.12) with $f=0$.

We have the following corollary of Theorem 1.2 (see Chapter 8).

Corollary 1.6. – *There exist*

$$F_+,G_+,F_-,G_-\in C^\infty\big(X\times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))\big)$$

such that

$$K_{\pi^{(q)}_+}=F_+(-i(\phi_+(x,y)+i0))^{-n}+G_+\log(-i(\phi_+(x,y)+i0)),$$
$$K_{\pi^{(q)}_-}=F_-(-i(\phi_-(x,y)+i0))^{-n}+G_-\log(-i(\phi_-(x,y)+i0)).$$

Moreover, we have

$$
\begin{aligned}
F_+ &= \sum_0^{n-1}(n-1-k)!s_+^k(x,y)(-i\phi_+(x,y))^k + f_+(x,y)(\phi_+(x,y))^n,\\
F_- &= \sum_0^{n-1}(n-1-k)!s_-^k(x,y)(-i\phi_-(x,y))^k + f_-(x,y)(\phi_-(x,y))^n,\\
G_+ &\equiv \sum_0^{\infty}\frac{(-1)^{k+1}}{k!}s_+^{n+k}(x,y)(-i\phi_+(x,y))^k,\\
G_- &\equiv \sum_0^{\infty}\frac{(-1)^{k+1}}{k!}s_-^{n+k}(x,y)(-i\phi_-(x,y))^k,
\end{aligned}
\tag{1.13}
$$

where $f_+(x,y), f_-(x,y) \in C^\infty\big(X\times X;\mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))\big)$.

If $w\in\Lambda^{0,1}T^*_z(X)$, let $w^{\wedge,*}:\Lambda^{0,q+1}T^*_z(X)\to\Lambda^{0,q}T^*_z(X)$, $q\geq 0$, be the adjoint of left exterior multiplication $w^\wedge:\Lambda^{0,q}T^*_z(X)\to\Lambda^{0,q+1}T^*_z(X)$. That is,

$$
(w^\wedge u \mid v) = (u \mid w^{\wedge,*}v),
\tag{1.14}
$$

for all $u\in\Lambda^{0,q}T^*_z(X)$, $v\in\Lambda^{0,q+1}T^*_z(X)$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

In Chapter 8, we compute $F_+(x,x)$.

PROPOSITION 1.7. – *For a given point* $p\in X$, *let* $U_1(x),\dots,U_{n-1}(x)$ *be an orthonormal frame of* $\Lambda^{1,0}T_x(X)$, *for which the Levi form is diagonalized at* p. *Let* $e_j(x)$, $j=1,\dots,n-1$, *denote the basis of* $\Lambda^{0,1}T^*_x(X)$, *which is dual to* $\overline{U}_j(x)$, $j=1,\dots,n-1$. *Let* $\lambda_j(x)$, $j=1,\dots,n-1$, *be the eigenvalues of the Levi form* L_x. *We assume that* $q=n_+$ *and that* $\lambda_j(p)>0$ *if* $1\leq j\leq n_+$. *Then*

$$
F_+(p,p) = (n-1)!\frac{1}{2}\left|\lambda_1(p)\right|\cdots\left|\lambda_{n-1}(p)\right|\pi^{-n}\prod_{j=1}^{j=n_+}e_j(p)^\wedge e_j(p)^{\wedge,*}.
$$

In the rest of this section, we will give the outline of the proof of Theorem 1.2. Let M be an open set in $\mathbb{R}^n$ and let $f,g\in C^\infty(M)$. We write $f\asymp g$ if for every compact set $K\subset M$ there is a constant $c_K>0$ such that $f\leq c_K g$ and $g\leq c_K f$ on K.

We will use the heat equation method. We work with some real local coordinates $x=(x_1,\dots,x_{2n-1})$ defined on an open set $\Omega\subset X$. Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of the Levi form. We will say that $a\in C^\infty(\overline{\mathbb{R}}_+\times\Omega\times\mathbb{R}^{2n-1})$ is quasi-homogeneous of degree j if $a(t,x,\lambda\eta)=\lambda^j a(\lambda t,x,\eta)$ for all $\lambda>0$. We consider the problem

$$
\begin{cases}
(\partial_t+\Box_b^{(q)})u(t,x)=0 & \text{in } \mathbb{R}_+\times\Omega,\\
u(0,x)=v(x).
\end{cases}
\tag{1.15}
$$

We shall start by making only a formal construction. We look for an approximate solution of (1.15) of the form $u(t,x) = A(t)v(x)$,

$$A(t)v(x) = \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta)v(y)dyd\eta \tag{1.16}$$

where formally

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta),$$

$a_j(t,x,\eta)$ is a matrix-valued quasi-homogeneous function of degree $-j$.

We let the full symbol of $\Box_b^{(q)}$ be:

$$\text{full symbol of } \Box_b^{(q)} = \sum_{j=0}^{2} p_j(x,\xi)$$

where $p_j(x,\xi)$ is positively homogeneous of order $2-j$ in the sense that

$$p_j(x,\lambda\eta) = \lambda^{2-j} p_j(x,\eta), \ |\eta| \geq 1, \ \lambda \geq 1.$$

We apply $\partial_t + \Box_b^{(q)}$ formally inside the integral in (1.16) and then introduce the asymptotic expansion of $\Box_b^{(q)}(ae^{i\psi})$. Set $(\partial_t + \Box_b^{(q)})(ae^{i\psi}) \sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase $\psi(t,x,\eta)$ should solve

$$\begin{cases} \dfrac{\partial\psi}{\partial t} - ip_0(x,\psi'_x) = O(|\operatorname{Im}\psi|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = \langle x,\eta\rangle \,. \end{cases} \tag{1.17}$$

This equation can be solved with $\operatorname{Im}\psi(t,x,\eta) \geq 0$ and the phase $\psi(t,x,\eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\psi(t,x,\eta) = \langle x,\eta\rangle \text{ on } \Sigma, \ d_{x,\eta}(\psi - \langle x,\eta\rangle) = 0 \text{ on } \Sigma,$$

$$\operatorname{Im}\psi(t,x,\eta) \asymp \Big(|\eta| \frac{t\,|\eta|}{1+t\,|\eta|}\Big)\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2, \quad |\eta| \geq 1.$$

Furthermore, there exists $\psi(\infty,x,\eta) \in C^\infty(\Omega \times \dot{\mathbb{R}}^{2n-1})$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset \Omega \times \dot{\mathbb{R}}^{2n-1}$ there is a constant $c_K > 0$ such that

$$\operatorname{Im}\psi(\infty,x,\eta) \geq c_K\,|\eta|\,\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2, \quad |\eta| \geq 1.$$

If $\lambda \in C(T^*\Omega \smallsetminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min \lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of $\Box_b^{(q)}$, then the solution $\psi(t,x,\eta)$ of (1.17) can be chosen so that for every compact set $K \subset \Omega \times \dot{\mathbb{R}}^{2n-1}$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}$ such that

$$\big|\partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma(\psi(t,x,\eta) - \psi(\infty,x,\eta))\big| \leq c_{\alpha,\beta,\gamma,K}\, e^{-\lambda(x,\eta)t} \text{ on } \overline{\mathbb{R}}_+ \times K.$$

(for the details, see Menikoff-Sjöstrand **[29]** or Chapter 3).

We obtain the transport equations

$$\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(|\mathrm{Im}\,\psi|^N),\ \forall N, \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(|\mathrm{Im}\,\psi|^N),\ \forall N. \end{cases} \tag{1.18}$$

Following the method of Menikoff-Sjöstrand [**29**], we see that we can solve (1.18). Moreover, a_j decay exponetially fast in t when $q \neq n_-, n_+$, and has subexponetially growth in general (subexponetially growth means that a_j satifies (4.13)). We assume that $q = n_-$ or n_+. To get further, we use a trick from Berman-Sjöstrand [**3**]. We use $\overline{\partial}_b\Box_b^{(q)} = \Box_b^{(q+1)}\overline{\partial}_b$, $\overline{\partial_b}^*\Box_b^{(q)} = \Box_b^{(q-1)}\overline{\partial_b}^*$ and get in a formula asymptotic sense

$$\partial_t(\overline{\partial}_b(e^{i\psi}a)) + \Box_b^{(q+1)}(\overline{\partial}_b(e^{i\psi}a)) \sim 0,$$
$$\partial_t(\overline{\partial_b}^*(e^{i\psi}a)) + \Box_b^{(q-1)}(\overline{\partial_b}^*(e^{i\psi}a)) \sim 0.$$

Put

$$\overline{\partial}_b(e^{i\psi}a) = e^{i\psi}\hat{a},\ \overline{\partial_b}^*(e^{i\psi}a) = e^{i\psi}\widetilde{a}.$$

We have

$$(\partial_t + \Box_b^{(q+1)})(e^{i\psi}\hat{a}) \sim 0,$$
$$(\partial_t + \Box_b^{(q-1)})(e^{i\psi}\widetilde{a}) \sim 0.$$

The corresponding degrees of $\hat{a}$ and $\widetilde{a}$ are $q+1$ and $q-1$. We deduce as above that $\hat{a}$ and $\widetilde{a}$ decay exponetially fast in t. This also applies to

$$\Box_b^{(q)}(ae^{i\psi}) = \overline{\partial}_b(\overline{\partial_b}^* ae^{i\psi}) + \overline{\partial_b}^*(\overline{\partial}_b ae^{i\psi}) = \overline{\partial}_b(e^{i\psi}\widetilde{a}) + \overline{\partial_b}^*(e^{i\psi}\hat{a}).$$

Thus, $\partial_t(ae^{i\psi})$ decay exponetially fast in t. Since $\partial_t\psi$ decay exponetially fast in t so does $\partial_t a$. Hence, there exist

$$a_j(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega)\ ,\Lambda^{0,q}T^*(\Omega))\big),\quad j=0,1,\ldots,$$

positively homogeneous of degree $-j$ such that $a_j(t,x,\eta)$ converges exponentially fast to $a_j(\infty,x,\eta)$, for all $j=0,1,\ldots$.

Choose $\chi \in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta) = 1$ when $|\eta| < 1$ and $\chi(\eta) = 0$ when $|\eta| > 2$. We formally set

$$G = \frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta$$

and

$$S = \frac{1}{(2\pi)^{2n-1}}\int\big(e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)d\eta.$$

In Chapter 5, we will show that G is a pseudodifferential operator of order -1 type $(\frac{1}{2},\frac{1}{2})$. In Chapter 6, we will show that $S + \Box_b^{(q)}\circ G \equiv I$, $\Box_b^{(q)}\circ S \equiv 0$. From this, it is not difficult to see that $\pi^{(q)} \equiv S$ if $\Box_b^{(q)}$ has closed range. We deduce that $\pi^{(q)}$ is a Fourier integral operator if $\Box_b^{(q)}$ has closed range. From the global theory of Fourier integral operators (see **[28]** and Chapter 7), we get Theorem 1.2.

CHAPTER 2

$\overline{\partial}_b$-COMPLEX AND THE HYPOELLIPICITY OF $\Box_b$, A REVIEW

References for the basic elements of CR geometry, see the books [**1**], Boggess [**4**] and [**6**]. From now on, we assume that $(X, \Lambda^{1,0}T(X))$ is a compact orientable connected CR manifold of dimension $2n-1$, $n \geq 2$, and we fix a Hermitian metric $(\ |\ )$ on $\mathbb{C}T(X)$. Then there is a real non-vanishing vector field Y on X which is pointwise orthogonal to $\Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X)$. We take Y so that $\|Y\| = 1$, $\langle Y, \omega_0 \rangle = -1$ (we recall that ω_0 is the uniquely determined global real 1 form, see (1.3)). Therefore Y is uniquely determined. We call Y the uniquely determined global real vector field. We have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*(X) &= \Lambda^{1,0}T^*(X) \oplus \Lambda^{0,1}T^*(X) \oplus \{\lambda\omega_0;\ \lambda \in \mathbb{C}\}, \\ \mathbb{C}T(X) &= \Lambda^{1,0}T(X) \oplus \Lambda^{0,1}T(X) \oplus \{\lambda Y;\ \lambda \in \mathbb{C}\}. \end{aligned} \tag{2.1}$$

For U, $V \in C^\infty(X;\ \Lambda^{1,0}T(X))$, from (1.4), we can check that

$$[U\ ,\overline{V}](p) = -(2i)L_p(U(p)\ ,\overline{V(p)})Y(p) + h, \quad h \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X). \tag{2.2}$$

Next we recall the tangential Cauchy-Riemann operator $\overline{\partial}_b$ and $\overline{\partial_b}^*$. Let $\Lambda^q(\mathbb{C}T^*(X))$, $q \in \mathbb{N}$, be the vector bundle of q forms of X. The Hermitian metric $(\ |\ )$ on $\mathbb{C}T^*(X)$ induces a Hermitian metric on $\Lambda^q(\mathbb{C}T^*(X))$ also denoted by $(\ |\ )$. Let

$$\pi^{0,q} : \Lambda^q(\mathbb{C}T^*(X)) \to \Lambda^{0,q}T^*(X)$$

be the orthogonal projection map.

Definition 2.1. – The tangential Cauchy-Riemann operator $\overline{\partial}_b$ is defined by

$$\overline{\partial}_b = \pi^{0,q+1} \circ d : C^\infty(X;\ \Lambda^{0,q}T^*(X)) \to C^\infty(X;\ \Lambda^{0,q+1}T^*(X)).$$

The following is well-known (see page 152 of [**1**]).

Proposition 2.2. – *We have* $\overline{\partial}_b^2 = 0$.

Let $\overline{\partial_b}^*$ be the formal adjoint of $\overline{\partial}_b$, that is $(\overline{\partial}_b f \mid h) = (f \mid \overline{\partial_b}^* h)$, where $f \in C^\infty(X; \Lambda^{0,q}T^*(X)), h \in C^\infty(X; \Lambda^{0,q+1}T^*(X))$ and $(\mid)$ is given by (1.2). $\overline{\partial_b}^*$ is a first order differential operator and $(\overline{\partial_b}^*)^2 = 0$. The Kohn Laplacian $\Box_b$ is given by

$$\Box_b = \overline{\partial}_b \overline{\partial_b}^* + \overline{\partial_b}^* \overline{\partial}_b.$$

From now on, we write $\Box_b^{(q)}$ to denote the restriction to $(0,q)$ forms.

For $z_0 \in X$, we can choose an orthonormal frame $e_1(z), \ldots, e_{n-1}(z)$ for $\Lambda^{0,1}T_z^*(X)$ varying smoothly with z in a neighborhood of z_0. Let $Z_j(z)$, $j = 1, \ldots, n-1$, denote the basis of $\Lambda^{0,1}T_z(X)$, which is dual to $e_j(z)$, $j = 1, \ldots, n-1$. Let Z_j^* be the formal adjoint of Z_j, $j = 1, \ldots, n-1$. That is, $(Z_j f \mid h) = (f \mid Z_j^* h), f, h \in C^\infty(X)$. We have the following (for a proof, see pages 154–156 of [**1**]).

Proposition 2.3. – *With the notations used before, the Kohn Laplacian $\Box_b^{(q)}$ is given by*

$$\begin{aligned}\Box_b^{(q)} &= \overline{\partial}_b \overline{\partial_b}^* + \overline{\partial_b}^* \overline{\partial}_b \\ &= \sum_{j=1}^{n-1} Z_j^* Z_j + \sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \circ [Z_j \ , Z_k^*] + \varepsilon(Z) + \varepsilon(Z^*) + \textit{zero order terms},\end{aligned} \tag{2.3}$$

where $\varepsilon(Z)$ denotes remainder terms of the form $\sum a_k(z) Z_k$ with a_k smooth, matrix-valued and similarly for $\varepsilon(Z^*)$ and the map

$$e_j^\wedge e_k^{\wedge,*} \circ [Z_j, Z_k^*] : C^\infty(X; \Lambda^{0,q}T^*(X)) \to C^\infty(X; \Lambda^{0,q}T^*(X))$$

is defined by

$$(e_j^\wedge e_k^{\wedge,*} \circ [Z_j, Z_k^*])(f(z) e_{j_1} \wedge \cdots \wedge e_{j_q}) = [Z_j, Z_k^*](f)(e_j^\wedge e_k^{\wedge,*}) \circ e_{j_1} \wedge \cdots \wedge e_{j_q}$$

and we extend the definition by linearity. We recall that $e_k^{\wedge,*}$ is given by (1.14).

We work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. We let the full symbol of $\Box_b^{(q)}$ be:

$$\text{full symbol of } \Box_b^{(q)} = \sum_{j=0}^{2} p_j(x, \xi) \tag{2.4}$$

where $p_j(x, \xi)$ is positively homogeneous of order $2 - j$. Let q_j, $j = 1, \ldots, n-1$, be the principal symbols of Z_j, $j = 1, \ldots, n-1$, where Z_j, $j = 1, \ldots, n-1$, are as in Proposition 2.3. Then,

$$p_0 = \sum_{j=1}^{n-1} \overline{q}_j q_j. \tag{2.5}$$

The characteristic manifold Σ of $\Box_b^{(q)}$ is

$$\Sigma = \{(x, \xi) \in T^*(X) \setminus 0; \xi = \lambda \omega_0(x), \lambda \neq 0\}. \tag{2.6}$$

From (2.5), we see that p_0 vanishes to second order at Σ. Thus, Σ is a doubly characteristic manifold of $\Box_b^{(q)}$. The subprincipal symbol of $\Box_b^{(q)}$ at $(x_0, \xi_0) \in \Sigma$ is given by

$$p_0^s(x_0, \xi_0) = p_1(x_0, \xi_0) + \frac{i}{2} \sum_{j=1}^{2n-1} \frac{\partial^2 p_0(x_0, \xi_0)}{\partial x_j \partial \xi_j} \in \mathscr{L}\big(\Lambda_{x_0}^{0,q} T^*(X), \Lambda_{x_0}^{0,q} T^*(X)\big). \tag{2.7}$$

It is well-known (see page 83 of Hörmander [**15**]) that the subprincipal symbol of $\Box_b^{(q)}$ is invariantly defined on Σ.

For an operator of the form $Z_j^* Z_j$ this subprincipal symbol is given by

$$\frac{1}{2i} \left\{\overline{q}_j, q_j\right\}$$

and the contribution from the double sum in (2.3) to the subprincipal symbol of $\Box_b^{(q)}$ is

$$\frac{1}{i} \sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \circ \left\{q_j, \overline{q}_k\right\},$$

where $\{q_j, \overline{q}_k\}$ denotes the Poisson bracket of q_j and $\overline{q}_k$. We recall that $\{q_j, \overline{q}_k\} = \sum_{s=1}^{2n-1} \big(\frac{\partial q_j}{\partial \xi_s} \frac{\partial \overline{q}_k}{\partial x_s} - \frac{\partial q_j}{\partial x_s} \frac{\partial \overline{q}_k}{\partial \xi_s}\big)$. We get the subprincipal symbol of $\Box_b^{(q)}$ on Σ,

$$p_0^s = \Big(\sum_{j=1}^{n-1} -\frac{1}{2i} \left\{q_j, \overline{q}_j\right\} \Big) + \sum_{j,k=1}^{n-1} e_j^\wedge e_k^{\wedge,*} \frac{1}{i} \left\{q_j, \overline{q}_k\right\}. \tag{2.8}$$

From (2.2), we see that $[\overline{Z}_k, Z_j] = -(2i) L(\overline{Z}_k, Z_j) Y + h$, $h \in C^\infty\big(X; \Lambda^{1,0} T(X) \oplus \Lambda^{0,1} T(X)\big)$. Note that the principal symbol of $\overline{Z}_k$ is $-\overline{q}_k$. Hence,

$$\left\{\overline{q}_k, q_j\right\} = (2i) L(\overline{Z}_k, Z_j) \sigma_{iY} \text{ on } \Sigma, \tag{2.9}$$

where σ_{iY} is the principal symbol of iY. Thus,

$$p_0^s = \Big(\sum_{j=1}^{n-1} L(\overline{Z}_j, Z_j) - \sum_{j,k=1}^{n-1} 2 e_j^\wedge e_k^{\wedge,*} L(\overline{Z}_k, Z_j) \Big) \sigma_{iY} \text{ on } \Sigma. \tag{2.10}$$

In the rest of this chapter, we will assume the reader is familiar with some basic notions of symplectic geometry. For basic notions and facts of symplectic geometry, see chapter XVIII of [**15**] or chapter 3 of Duistermaat [**8**].

From now on, for any $f \in C^\infty(T^*(X))$, we write H_f to denote the Hamilton field of f. We need the following

LEMMA 2.4. – *We recall that we work with Assumption 1.1. Σ is a symplectic submanifold of $T^*(X)$.*

Proof. – Let $\rho \in \Sigma$. Note that

$$\Sigma = \left\{(x,\xi) \in T^*(X) \smallsetminus 0;\ q_j(x,\xi) = \overline{q}_j(x,\xi) = 0, \quad j = 1,\ldots,n-1\right\}.$$

Let $\mathbb{C}T_\rho(\Sigma)$ and $\mathbb{C}T_\rho(T^*(X))$ be the complexifications of the spaces $T_\rho(\Sigma)$ and $T_\rho(T^*(X))$ respectively. We can choose the basis $H_{q_1},\ldots,H_{q_{n-1}},H_{\overline{q}_1},\ldots,H_{\overline{q}_{n-1}}$ for $T_\rho(\Sigma)^\perp$, where $T_\rho(\Sigma)^\perp$ is the orthogonal to $\mathbb{C}T_\rho(\Sigma)$ in $\mathbb{C}T_\rho(T^*(X))$ with respect to the canonical two form,

$$\sigma = d\xi \wedge dx. \tag{2.11}$$

In view of (2.9), we have $\sigma(H_{q_j},\ H_{\overline{q}_k}) = \{q_j,\overline{q}_k\} = \frac{2}{i}L(\overline{Z}_k,Z_j)\sigma_{iY}$ on Σ. We notice that $\{q_j,q_k\} = 0$ on Σ. Thus, if the Levi form is non-degenerate at each point of X, then σ is non-degenerate on $T_\rho(\Sigma)^\perp$, hence also on $\mathbb{C}T_\rho(\Sigma)$ and Σ is therefore symplectic. $\square$

The fundamental matrix of p_0 at $\rho = (p,\xi_0) \in \Sigma$ is the linear map F_ρ on $T_\rho(T^*(X))$ defined by

$$\sigma(t,F_\rho s) = \langle t, p_0''(\rho)s\rangle\,,\quad t,s \in T_\rho(T^*(X)), \tag{2.12}$$

where σ is the canonical two form (see (2.11)) and

$$p_0''(\rho) = \begin{pmatrix} \frac{\partial^2 p_0}{\partial x\partial x}(\rho) & \frac{\partial^2 p_0}{\partial \xi\partial x}(\rho) \\ \frac{\partial^2 p_0}{\partial x\partial \xi}(\rho) & \frac{\partial^2 p_0}{\partial \xi\partial \xi}(\rho) \end{pmatrix}.$$

We can choose the basis $H_{q_1},\ldots,H_{q_{n-1}},H_{\overline{q}_1},\ldots,H_{\overline{q}_{n-1}}$ for $T_\rho(\Sigma)^\perp$, where $T_\rho(\Sigma)^\perp$ is the orthogonal to $\mathbb{C}T_\rho(\Sigma)$ in $\mathbb{C}T_\rho(T^*(X))$ with respect to canonical two form. We notice that $H_{p_0} = \sum_j(\overline{q}_jH_{q_j} + q_jH_{\overline{q}_j})$. We compute the linearization of H_{p_0} at ρ

$$H_{p_0}\Big(\rho + \sum(t_kH_{q_k} + s_kH_{\overline{q}_k})\Big) = O(|t,s|^2) + \sum_{j,k} t_k\,\{q_k,\overline{q}_j\}\,H_{q_j} + \sum_{j,k} s_k\,\{\overline{q}_k,q_j\}\,H_{\overline{q}_j}.$$

So F_ρ is expressed in the basis $H_{q_1},\ldots,H_{q_{n-1}},H_{\overline{q}_1},\ldots,H_{\overline{q}_{n-1}}$ by

$$F_\rho = \begin{pmatrix} \{q_k,\overline{q}_j\} & 0 \\ 0 & \{\overline{q}_k,q_j\} \end{pmatrix}. \tag{2.13}$$

Again, from (2.9), we see that the non-vanishing eigenvalues of F_ρ are

$$\pm\, 2i\lambda_j\sigma_{iY}(\rho), \tag{2.14}$$

where λ_j, $j = 1,\ldots,n-1$, are the eigenvalues of L_p.

To compute further, we assume that the Levi form is diagonalized at the given point $p \in X$. Then

$$\sum_{j,k} 2e_j^{\wedge}e_k^{\wedge,*}L_p(\overline{Z}_k,Z_j)\sigma_{iY} = \sum_j 2e_j^{\wedge}e_j^{\wedge,*}L_p(\overline{Z}_j,Z_j)\sigma_{iY}.$$

From this, we see that on Σ and on the space of $(0,q)$ forms, $p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}} F$ has the eigenvalues

$$
\begin{aligned}
&\sum_{j=1}^{n-1} |\lambda_j|\,|\sigma_{iY}| + \sum_{j\notin J} \lambda_j \sigma_{iY} - \sum_{j\in J} \lambda_j \sigma_{iY},\ |J| = q, \\
&J = (j_1, j_2, \ldots, j_q),\ 1 \leq j_1 < j_2 < \cdots < j_q \leq n-1,
\end{aligned}
\tag{2.15}
$$

where $\widetilde{\operatorname{tr}} F$ denotes $\sum |\mu_j|$, $\pm\mu_j$ are the non-vanishing eigenvalues of F_ρ.

Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form. Since $\langle Y, \omega_0\rangle = -1$, we have $\sigma_{iY} > 0$ on Σ^+, $\sigma_{iY} < 0$ on Σ^- (we recall that Σ^+ and Σ^- are given by (1.5)). Let

$$\inf\,(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}} F) = \inf\big\{\lambda;\ \lambda : \text{eigenvalue of } p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}} F\big\}.$$

From (2.15), we see that on Σ^+

$$\inf\,(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}} F)\begin{cases} = 0, & q = n_+, \\ > 0, & q \neq n_+. \end{cases}\tag{2.16}$$

On Σ^-

$$\inf\,(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}} F)\begin{cases} = 0, & q = n_-, \\ > 0, & q \neq n_-. \end{cases}\tag{2.17}$$

Let Ω be an open set in $\mathbb{R}^N$. Let P be a classical pseudodifferential operator on Ω of order $m > 1$. P is said to be hypoelliptic with loss of one derivative if $u \in \mathscr{E}'(\Omega)$ and $Pu \in H^s_{\mathrm{loc}}(\Omega)$ implies $u \in H^{s+m-1}_{\mathrm{comp}}(\Omega)$.

We recall classical works by Boutet de Monvel [**32**] and Sjöstrand [**36**].

Proposition 2.5. – *Let Ω be an open set in $\mathbb{R}^N$. Let P be a classical pseudodifferential operator on Ω of order $m > 1$. The symbol of P takes the form*

$$\sigma_P(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + p_{m-2}(x,\xi) + \cdots,$$

where p_j is positively homogeneous of degree j. We assume that $\Sigma = p_m^{-1}(0)$ is a symplectic submanifold of codimension $2d$, $p_m \geq 0$ and p_m vanishes to precisely second order on Σ. Let F be the fundamental matrix of p_m. Let p_m^s be the subprincipal symbol of P. Then P is hypoelliptic with loss of one derivative if and only if $p_m^s(\rho) + \sum_{j=1}^d (\frac{1}{2} + \alpha_j)\,|\mu_j| \neq 0$ at every point $\rho \in \Sigma$ for all $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$, where $\pm i\mu_j$ are the eigenvalues of F at ρ.

Proposition 2.5 also holds if P is a matrix-valued classical pseudodifferential operator on Ω of order $m > 1$ with scalar principal symbol.

From (2.16), (2.17) and Proposition 2.5, we have the following

Proposition 2.6. – $\Box_b^{(q)}$ *is hypoelliptic with loss of one derivative if and only if* $Y(q)$ *holds at each point of* X *(we recall that the definition of* $Y(q)$ *is given by* (1.1)*).*

CHAPTER 3

THE CHARACTERISTIC EQUATION

In this chapter, we consider the characteristic equation for $\partial_t + \Box_b^{(q)}$.

Let $p_0(x,\xi)$ be the principal symbol of $\Box_b^{(q)}$. We work with some real local coordinates $x = (x_1, x_2, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. We identify Ω with an open set in $\mathbb{R}^{2n-1}$. Let $\Omega^{\mathbb{C}}$ be an almost complexification of Ω. That is, $\Omega^{\mathbb{C}}$ is an open set in $\mathbb{C}^{2n-1}$ with $\Omega^{\mathbb{C}} \bigcap \mathbb{R}^{2n-1} = \Omega$. We identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^{2n-1}$. Similarly, let $T^*(\Omega)_{\mathbb{C}}$ be an open set in $\mathbb{C}^{2n-1} \times \mathbb{C}^{2n-1}$ with $T^*\big(\Omega)_{\mathbb{C}} \bigcap (\mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1}\big) = T^*(\Omega)$.

In this chapter, for any function f, we also write f to denote an almost analytic extension (for the precise meaning of almost analytic extension, see Chapter 1 of [**28**]). We look for solutions $\psi(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega) \setminus 0)$ of the problem

$$(3.1)\qquad \begin{cases} \dfrac{\partial \psi}{\partial t} - ip_0(x, \psi'_x) = O(\left|\operatorname{Im}\psi\right|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = \langle x\ ,\eta\rangle \end{cases}$$

with $\operatorname{Im}\psi(t,x,\eta) \geq 0$. More precisely, we look for solutions $\psi(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega) \setminus 0)$ with $\operatorname{Im}\psi(t,x,\eta) \geq 0$ such that $\psi|_{t=0} = \langle x\ ,\eta\rangle$ and for every compact set $K \subset T^*(\Omega) \setminus 0$, $N \geq 0$, there exists $c_{K,N} \geq 0$, such that

$$\left|\frac{\partial \psi}{\partial t} - ip_0(x, \psi'_x)\right| \leq c_{K,N} \left|\operatorname{Im}\psi\right|^N \text{ on } \overline{\mathbb{R}}_+ \times K.$$

Let $f(x,\xi)$, $g(x,\xi) \in C^\infty(T^*(\Omega)_{\mathbb{C}})$. We write $f = g \mod \left|\operatorname{Im}(x,\xi)\right|^\infty$ if, given any compact subset K of $T^*(\Omega)_{\mathbb{C}}$ and any integer $N > 0$, there is a constant $c > 0$ such that $\left|(f-g)(x,\xi)\right| \leq c\left|\operatorname{Im}(x,\xi)\right|^N$, $\forall (x,\xi) \in K$. Let U and V be C^∞ complex vector fields on $T^*(\Omega)_{\mathbb{C}}$. We write $U = V \mod \left|\operatorname{Im}(x,\xi)\right|^\infty$ if $U(f) = V(f) \mod \left|\operatorname{Im}(x,\xi)\right|^\infty$ and $U(\overline{f}) = V(\overline{f}) \mod \left|\operatorname{Im}(x,\xi)\right|^\infty$, for all almost analytic functions f on $T^*(\Omega)_{\mathbb{C}}$.

In the complex domain, the Hamiltonian field H_{p_0} is given by $H_{p_0} = \frac{\partial p_0}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p_0}{\partial x}\frac{\partial}{\partial \xi}$. We notice that H_{p_0} depends on the choice of almost analytic extension of p_0 but we can give an invariant meaning of the exponential map $\exp(-itH_{p_0})$, $t \geq 0$. Note that H_{p_0} vanishes on Σ. We consider the real vector field $-iH_{p_0} + \overline{-iH_{p_0}}$. Let $\Phi(t,\rho)$ be the $-iH_{p_0} + \overline{-iH_{p_0}}$ flow. We notice that for every $T > 0$ there is an open neighborhood

U of Σ in $T^*(\Omega)_{\mathbb{C}}$ such that for all $0 \leq t \leq T$, $\Phi(t,\rho)$ is well-defined if $\rho \in U$. Since we only need to consider Taylor expansions at Σ, for the convenience, we assume that $\Phi(t,\rho)$ is well-defined for all $t \geq 0$ and $\rho \in T^*(\Omega)_{\mathbb{C}}$. The following is well-known (see Chapter 1 of Menikoff-Sjöstrand **[29]** or Proposition B.13 of paper I in **[20]**).

Proposition 3.1. – *Let $\Phi(t,\rho)$ be as above. Let U be a real vector field on $T^*(\Omega)_{\mathbb{C}}$ such that $U = -iH_{p_0} + \overline{-iH_{p_0}} \mod |\mathrm{Im}\,(x,\xi)|^\infty$. Let $\hat{\Phi}(t,\rho)$ be the U flow. Then, for every compact set $K \subset T^*(\Omega)_{\mathbb{C}}$, $N \geq 0$, there is $c_{N,K}(t) > 0$, such that*

$$\left|\Phi(t,\rho) - \hat{\Phi}(t,\rho)\right| \leq c_{N,K}(t)\mathrm{dist}\,(\rho,\Sigma)^N,\ \ \rho \in K.$$

For $t \geq 0$, let

$$G_t = \{(\rho, \Phi(t,\rho));\ \rho \in T^*(\Omega)_{\mathbb{C}}\}\,, \tag{3.2}$$

where $\Phi(t,\rho)$ is as in Proposistion 3.1. We call G_t the graph of the operator $\exp(-itH_{p_0})$. Since H_{p_0} vanishes on Σ, we have $\Phi(t,\rho) = \rho$ if $\rho \in \Sigma$. G_t depends on the choice of almost analytic extension of p_0. Let $\hat{p}_0$ be another almost analytic extension of p_0. Let $\hat{G}_t$ be the graph of $\exp(-itH_{\hat{p}_0})$. From Proposition 3.1, it follows that G_t coincides to infinite order with $\hat{G}_t$ on $\mathrm{diag}\,(\Sigma\times\Sigma)$, for all $t \geq 0$.

In Menikoff-Sjöstrand **[29]**, it was shown that there exist

$$g(t,x,\eta), h(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}})$$

such that $G_t = \{(x, g(t,x,\eta), h(t,x,\eta), \eta);\ (x,\eta) \in T^*(\Omega)_{\mathbb{C}}\}$. Moreover, there exists $\psi(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}})$ such that

$$g(t,x,\eta) - \psi'_x(t,x,\eta) \text{ and } h(t,x,\eta) - \psi'_\eta(t,x,\eta)$$

vanish to infinite order on Σ, for all $t \geq 0$. Furthermore, when (t,x,η) is real, $\psi(t,x,\eta)$ solves (3.1) and we have,

$$\mathrm{Im}\,\psi(t,x,\eta) \asymp \frac{t}{1+t}\big(\mathrm{dist}\,((x,\eta),\Sigma)\big)^2,\ \ t \geq 0,\ |\eta| = 1. \tag{3.3}$$

For the precise meaning of $\asymp$, see the discussion after Proposition 1.7.

Moreover, we have the following

Proposition 3.2. – *There exists $\psi(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega)\setminus 0)$ such that $\mathrm{Im}\,\psi \geq 0$ with equality precisely on $(\{0\}\times T^*(\Omega)\setminus 0)\bigcup(\mathbb{R}_+\times\Sigma)$ and such that (3.1) holds where the error term is uniform on every set of the form $[0,T]\times K$ with $T > 0$ and $K \subset T^*(\Omega)\setminus 0$ compact. Furthermore, ψ is unique up to a term which is $O(|\mathrm{Im}\,\psi|^N)$ locally uniformly for every N and*

$$\psi(t,x,\eta) = \langle x,\eta\rangle \ \text{ on } \Sigma,\ d_{x,\eta}(\psi - \langle x,\eta\rangle) = 0 \text{ on } \Sigma.$$

Moreover, we have

$$\operatorname{Im}\psi(t,x,\eta)\asymp\Big(|\eta|\,\frac{t\,|\eta|}{1+t\,|\eta|}\Big)\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2,\quad t\geq 0,\ |\eta|\geq 1. \tag{3.4}$$

Proposition 3.3. – *There exists a function $\psi(\infty,x,\eta)\in C^\infty(T^*(\Omega)\setminus 0)$ with a uniquely determined Taylor expansion at each point of Σ such that*

For every compact set $K\subset T^*(\Omega)\setminus 0$ there is a $c_K>0$ such that $\operatorname{Im}\psi(\infty,x,\eta)\ \geq\ c_K\,|\eta|\,\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2$, $d_{x,\eta}(\psi(\infty,x,\eta)-\langle x,\eta\rangle)=0$ on Σ.

If $\lambda\in C(T^*(\Omega)\setminus 0)$, $\lambda>0$ and $\lambda|_\Sigma<\min|\lambda_j|$, where $\pm i\,|\lambda_j|$ are the non-vanishing eigenvalues of the fundamental matrix of $\Box_b^{(q)}$, then the solution $\psi(t,x,\eta)$ of (3.1) can be chosen so that for every compact set $K\subset T^*(\Omega)\setminus 0$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}$ such that

$$\big|\partial_x^\alpha\partial_\eta^\beta\partial_t^\gamma(\psi(t,x,\eta)-\psi(\infty,x,\eta))\big|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(x,\eta)t}\ \text{ on } \overline{\mathbb{R}}_+\times K. \tag{3.5}$$

For the proofs of Proposition 3.2 and Proposition 3.3, we refer the reader to **[29]**. From the positively homogeneity of p_0, it follows that we can choose $\psi(t,x,\eta)$ in Proposition 3.2 to be quasi-homogeneous of degree 1 in the sense that $\psi(t,x,\lambda\eta)=\lambda\psi(\lambda t,x,\eta)$, $\lambda>0$ (see Definition 4.1). This makes $\psi(\infty,x,\eta)$ positively homogeneous of degree 1.

We recall that $p_0=q_1\overline{q}_1+\cdots+q_{n-1}\overline{q}_{n-1}$. We can take an almost analytic extension of p_0 so that

$$p_0(x,\xi)=\overline{p}_0(\overline{x},\overline{\xi}). \tag{3.6}$$

From (3.6), we have

$$-\frac{\partial\overline{\psi}}{\partial t}(t,x,-\eta)-ip_0(x,\overline{\psi}'_x(t,x,-\eta))=O(|\operatorname{Im}\psi|^N),\ t\geq 0,$$

for all $N\geq 0$, (x,η) real. Since $p_0(x,-\xi)=p_0(x,\xi)$, we have

$$-\frac{\partial\overline{\psi}}{\partial t}(t,x,-\eta)-ip_0(x,-\overline{\psi}'_x(t,x,-\eta))=O(|\operatorname{Im}\psi|^N),\ t\geq 0, \tag{3.7}$$

for all $N\geq 0$, (x,η) real. From Proposition 3.2, we can take $\psi(t,x,\eta)$ so that $\psi(t,x,\eta)=-\overline{\psi}(t,x,-\eta)$, (x,η) is real. Hence,

$$\psi(\infty,x,\eta)=-\overline{\psi}(\infty,x,-\eta),\quad (x,\eta)\text{ is real.} \tag{3.8}$$

Put $\widetilde{G}_t=\big\{(\overline{y},\overline{\eta},\overline{x},\overline{\xi});\,(x,\xi,y,\eta)\in G_t\big\}$, where G_t is defined by (3.2). From (3.6), it follows that $\Phi(t,\overline{\rho})=\overline{\Phi}(-t,\rho)$, where $\Phi(t,\rho)$ is as in Proposition 3.1. Thus, for all $t\geq 0$,

$$G_t=\widetilde{G}_t. \tag{3.9}$$

Put

$$C_t = \{(x, \psi'_x(t,x,\eta), \psi'_\eta(t,x,\eta), \eta);\ (x,\eta) \in T^*(\Omega^{\mathbb{C}})\} \tag{3.10}$$

and $\widetilde{C}_t = \{(\overline{y}, \overline{\eta}, \overline{x}, \overline{\xi});\ (x,\xi,y,\eta) \in C_t\}$. Since C_t coincides to infinite order with G_t on $\operatorname{diag}(\Sigma \times \Sigma)$, for all $t \geq 0$, from (3.9), it follows that C_t coincides to infinite order with $\widetilde{C}_t$ on $\operatorname{diag}(\Sigma \times \Sigma)$, for all $t \geq 0$. Letting $t \to \infty$, we get the following

PROPOSITION 3.4. – *Let*

$$C_\infty = \{(x, \psi'_x(\infty,x,\eta), \psi'_\eta(\infty,x,\eta), \eta);\ (x,\eta) \in T^*(\Omega^{\mathbb{C}})\} \tag{3.11}$$

and $\widetilde{C}_\infty = \{(\overline{y}, \overline{\eta}, \overline{x}, \overline{\xi});\ (x,\xi,y,\eta) \in C_\infty\}$. *Then* $\widetilde{C}_\infty$ *coincides to infinite order with* C_∞ *on* $\operatorname{diag}(\Sigma \times \Sigma)$.

From Proposition 3.4 and the global theory of Fourier integral operators (see Theorem 4.2 of [**28**]), we have the following

PROPOSITION 3.5. – *The two phases*

$$\psi(\infty,x,\eta) - \langle y,\eta\rangle\,, -\overline{\psi}(\infty,y,\eta) + \langle x,\eta\rangle \in C^\infty(\Omega \times \Omega \times \dot{\mathbb{R}}^{2n-1})$$

are equivalent in the sense of Melin-Sjöstrand [**28**].

We recall that

$$\Sigma = \{(x,\xi) \in T^*(\Omega) \smallsetminus 0;\ q_j(x,\xi) = \overline{q}_j(x,\xi) = 0,\quad j = 1,\ldots,n-1\}\,.$$

For any function $f \in C^\infty(T^*(\Omega))$, we use $\widetilde{f}$ to denote an almost analytic extension with respect to the weight function $\operatorname{dist}((x,\xi),\Sigma)$. Set

$$\widetilde{\Sigma} = \left\{(x,\xi) \in T^*(\Omega)_{\mathbb{C}} \smallsetminus 0;\ \widetilde{q}_j(x,\xi) = \widetilde{\overline{q}}_j(x,\xi) = 0,\quad j = 1,\ldots,n-1\right\}. \tag{3.12}$$

We say that $\widetilde{\Sigma}$ is an almost analytic extension with respect to the weight function $\operatorname{dist}((x,\xi),\Sigma)$ of Σ. Let $f(x,\xi), g(x,\xi) \in C^\infty(W)$, where W is an open set in $T^*(\Omega)_{\mathbb{C}}$. We write $f = g \mod d^\infty_\Sigma$ if, given any compact subset K of W and any integer $N > 0$, there is a constant $c > 0$ such that $|(f-g)(x,\xi)| \leq c\operatorname{dist}((x,\xi),\Sigma)^N$, $\forall (x,\xi) \in K$. From the global theory of Fourier integral operators (see Theorem 4.2 of [**28**]), we only need to consider Taylor expansions at Σ. We may work with the following coordinates (for the proof, see [**29**]).

PROPOSITION 3.6. – *Let* $\rho \in \Sigma$. *Then in some open neighborhood* Γ *of* ρ *in* $T^*(\Omega)_{\mathbb{C}}$, *there are* C^∞ *functions* $\widetilde{x}_j \in C^\infty(\Gamma)$, $\widetilde{\xi}_j \in C^\infty(\Gamma)$, $j = 1,\ldots,2n-1$, *such that*

(a) $\widetilde{x}_j$, $\widetilde{\xi}_j$, $j = 1,\ldots,2n-1$, *are almost analytic functions with respect to the weight function* $\operatorname{dist}((x,\xi),\Sigma)$.

(b) $\det\left(\frac{\partial(x,\xi)}{\partial(\widetilde{x},\widetilde{\xi})}\right) \neq 0$ on $(\Gamma)_{\mathbb{R}}$, *where* $(\Gamma)_{\mathbb{R}} = \Gamma \bigcap T^*(\Omega)$ *and* $\widetilde{x} = (\widetilde{x}_1,\ldots,\widetilde{x}_{2n-1})$, $\widetilde{\xi} = (\widetilde{\xi}_1,\ldots,\widetilde{\xi}_{2n-1})$.

(c) $\widetilde{x}_j$, $\widetilde{\xi}_j$, $j = 1, \ldots, 2n-1$, *form local coordinates of* Γ.

(d) $(\widetilde{x}, \widetilde{\xi})$ *is symplectic to infinite order on* Σ. *That is,* $\{\widetilde{x}_j, \widetilde{x}_k\} = 0 \mod d_\Sigma^\infty$, $\{\widetilde{\xi}_j, \widetilde{\xi}_k\} = 0 \mod d_\Sigma^\infty$, $\{\widetilde{\xi}_j, \widetilde{x}_k\} = \delta_{j,k} \mod d_\Sigma^\infty$, *where* $j,\ k = 1, \ldots, 2n-1$. *Here* $\{f, g\} = \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial \xi}$, $f,\ g \in C^\infty(\Gamma)$.

(e) *We write* $\widetilde{x}'$, $\widetilde{x}''$, $\widetilde{\xi}'$, $\widetilde{\xi}''$ *to denote* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, $(\widetilde{x}_{n+1}, \ldots, \widetilde{x}_{2n-1})$, $(\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n)$ *and* $(\widetilde{\xi}_{n+1}, \ldots, \widetilde{\xi}_{2n-1})$ *respectively. Then,* $\widetilde{\Sigma} \bigcap \Gamma$ *coincides to infinite order with* $\left\{(\widetilde{x}, \widetilde{\xi});\ \widetilde{x}'' = 0,\ \widetilde{\xi}'' = 0\right\}$ *on* $\Sigma \bigcap (\Gamma)_{\mathbb{R}}$ *and*

$$\Sigma \bigcap (\Gamma)_{\mathbb{R}} = \left\{(\widetilde{x}, \widetilde{\xi});\ \widetilde{x}'' = 0,\ \widetilde{\xi}'' = 0,\ \widetilde{x}' \text{ and } \widetilde{\xi}' \text{ are real}\right\}.$$

Furthermore, there is a $(n-1) \times (n-1)$ *matrix of almost analytic functions* $A(\widetilde{x}, \widetilde{\xi})$ *such that for every compact set* $K \subset \Gamma$ *and* $N \geq 0$, *there is a* $c_{K,N} > 0$, *such that*

$$\left| p_0(\widetilde{x}, \widetilde{\xi}) - i \left\langle A(\widetilde{x}, \widetilde{\xi})\widetilde{x}'', \widetilde{\xi}'' \right\rangle \right| \leq c_{K,N} \left| (\widetilde{x}'', \widetilde{\xi}'') \right|^N \text{ on } K,$$

and when $\widetilde{x}'$ *and* $\widetilde{\xi}'$ *are real,* $A(\widetilde{x}', 0, \widetilde{\xi}', 0)$ *has positive eigenvalues* $|\lambda_1|, \ldots, |\lambda_{n-1}|$, *where* $\pm i\lambda_1, \ldots, \pm i\lambda_{n-1}$ *are the non-vanishing eigenvalues of* $F(\widetilde{x}', 0, \widetilde{\xi}', 0)$, *the fundamental matrix of* $\Box_b^{(q)}$. *In particular,*

$$\frac{1}{2}\operatorname{tr} A(\widetilde{x}', 0, \widetilde{\xi}', 0) = \frac{1}{2}\widetilde{\operatorname{tr}}\, F(\widetilde{x}', 0, \widetilde{\xi}', 0).$$

Formally, we write

$$p_0(\widetilde{x}, \widetilde{\xi}) = i \left\langle A(\widetilde{x}, \widetilde{\xi})\widetilde{x}'', \widetilde{\xi}'' \right\rangle + O(\left|(\widetilde{x}'', \widetilde{\xi}'')\right|^N). \tag{3.13}$$

REMARK 3.7. – Set

$$E = \left\{(t, x, \xi, y, \eta) \in \overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}} \times T^*(\Omega)_{\mathbb{C}};\ (x, \xi, y, \eta) \in C_t\right\},$$

where C_t is defined by (3.10). Let $(\widetilde{x}, \widetilde{\xi})$ be the coordinates of Proposition 3.6. In the work of Menikoff-Sjöstrand [**29**], it was shown that there exists $\widetilde{\psi}(t, \widetilde{x}, \widetilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Gamma)$, where Γ is as in Proposition 3.6, such that

$$\begin{cases} \dfrac{\partial \widetilde{\psi}}{\partial t} - ip_0(\widetilde{x}, \widetilde{\psi}'_{\widetilde{x}}) = O(\left|(\widetilde{x}'', \widetilde{\eta}'')\right|^N), & \text{for all } N > 0, \\ \widetilde{\psi}|_{t=0} = \langle \widetilde{x}\ , \widetilde{\eta} \rangle \end{cases}$$

and $\widetilde{\psi}(t, \widetilde{x}, \widetilde{\eta})$ is of the form

$$\widetilde{\psi}(t, \widetilde{x}, \widetilde{\eta}) = \langle \widetilde{x}', \widetilde{\eta}' \rangle + \left\langle e^{-tA(\widetilde{x}', 0, \widetilde{\eta}', 0)}\widetilde{x}'', \widetilde{\eta}'' \right\rangle + \widetilde{\psi}_2(t, \widetilde{x}, \widetilde{\eta}) + \widetilde{\psi}_3(t, \widetilde{x}, \widetilde{\eta}) + \cdots, \tag{3.14}$$

where A is as in Proposition 3.6 and $\widetilde{\psi}_j(t, \widetilde{x}, \widetilde{\eta})$ is a C^∞ homogeneous polynomial of degree j in $(\widetilde{x}'', \widetilde{\eta}'')$. If $\lambda \in C(T^*(\Omega) \setminus 0)$, $\lambda > 0$ and $\lambda|_\Sigma < \min |\lambda_j|$, where $\pm i\,|\lambda_j|$ are the non-vanishing eigenvalues of the fundamental matrix of $\Box_b^{(q)}$, then for every compact set $K \subset \Sigma \bigcap (\Gamma)_{\mathbb{R}}$ and all indices α, β, γ, j, there is a constant $c_{\alpha,\beta,\gamma,j,K}$ such that

$$\left|\partial_{\widetilde{x}}^{\alpha} \partial_{\widetilde{\eta}}^{\beta} \partial_{t}^{\gamma} (\widetilde{\psi}_j(t, \widetilde{x}, \widetilde{\eta}))\right| \leq c_{\alpha,\beta,\gamma,K} e^{-\lambda(\widetilde{x},\widetilde{\eta})t} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{3.15}$$

Put $\widetilde{E} = \left\{(t, \widetilde{x}, \frac{\partial \widetilde{\psi}}{\partial \widetilde{x}}(t, \widetilde{x}, \widetilde{\eta}), \frac{\partial \widetilde{\psi}}{\partial \widetilde{\eta}}(t, \widetilde{x}, \widetilde{\eta}), \widetilde{\eta});\ t \in \overline{\mathbb{R}}_+,\ \widetilde{x}, \widetilde{\eta} \in C^{\infty}(\Gamma)\right\}$. We notice that $\widetilde{E}$ coincides to infinite order with E on $\overline{\mathbb{R}}_+ \times \operatorname{diag}\left((\Sigma \bigcap (\Gamma)_{\mathbb{R}}) \times (\Sigma \bigcap (\Gamma)_{\mathbb{R}})\right)$ (see **[29]**).

CHAPTER 4

THE HEAT EQUATION, FORMAL CONSTRUCTION

We work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. We identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^{2n-1}$.

Definition 4.1. – We say that $a \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega))$ is quasi-homogeneous of degree j if $a(t, x, \lambda\eta) = \lambda^j a(\lambda t, x, \eta)$ for all $\lambda > 0$.

We can check that if a is quasi-homogeneous of degree j, then $\partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma a$ is quasi-homogeneous of degree $j - |\beta| + \gamma$.

In this chapter, we consider the problem

$$\begin{cases} (\partial_t + \Box_b^{(q)})u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u(0,x) = v(x). \end{cases} \tag{4.1}$$

We shall start by making only a formal construction. We look for an approximate solution of (4.1) of the form $u(t,x) = A(t)v(x)$,

$$A(t)v(x) = \frac{1}{(2\pi)^{2n-1}} \iint e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta) v(y) dy d\eta \tag{4.2}$$

where formally

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta),$$

$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$, $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$.

We let the full symbol of $\Box_b^{(q)}$ be:

$$\text{full symbol of } \Box_b^{(q)} = \sum_{j=0}^{2} p_j(x,\xi),$$

where $p_j(x,\xi)$ is positively homogeneous of order $2-j$. We apply $\partial_t + \Box_b^{(q)}$ formally under the integral in (4.2) and then introduce the asymptotic expansion of $\Box_b^{(q)}(ae^{i\psi})$ (see page 148 of [**28**]). Setting $(\partial_t + \Box_b^{(q)})(ae^{i\psi}) \sim 0$ and regrouping the terms according to the degree of quasi-homogeneity. We obtain the transport equations

$$\tag{4.3}\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(|\operatorname{Im}\psi|^N),\ \forall N, \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\dots,a_{j-1}) = O(|\operatorname{Im}\psi|^N),\ \forall N. \end{cases}$$

Here

$$T(t,x,\eta,\partial_t,\partial_x) = \partial_t - i\sum_{j=1}^{2n-1}\frac{\partial p_0}{\partial\xi_j}(x,\psi'_x)\frac{\partial}{\partial x_j} + q(t,x,\eta)$$

where

$$q(t,x,\eta) = p_1(x,\psi'_x) + \frac{1}{2i}\sum_{j,k=1}^{2n-1}\frac{\partial^2 p_0(x,\psi'_x)}{\partial\xi_j\partial\xi_k}\frac{\partial^2\psi(t,x,\eta)}{\partial x_j\partial x_k}$$

and l_j is a linear differential operator acting on $a_0, a_1, \dots, a_{j-1}$. We note that $q(t,x,\eta) \to q(\infty,x,\eta)$ exponentially fast in the sense of (3.5) and that the same is true for the coefficients of l_j.

Let C_t, E be as in (3.10) and Remark 3.7. We recall that for $t \geq 0$,

$$C_t = \left\{(x,\xi,y,\eta) \in T^*(\Omega)_{\mathbb{C}} \times T^*(\Omega)_{\mathbb{C}};\ \xi = \frac{\partial\psi}{\partial x}(t,x,\eta), y = \frac{\partial\psi}{\partial\eta}(t,x,\eta)\right\},$$
$$E = \left\{(t,x,\xi,y,\eta) \in \overline{\mathbb{R}}_+ \times T^*(\Omega)_{\mathbb{C}} \times T^*(\Omega)_{\mathbb{C}};\ (x,\xi,y,\eta) \in C_t\right\}$$

and for $t > 0$, $(C_t)_{\mathbb{R}} = \operatorname{diag}(\Sigma\times\Sigma) = \{(x,\xi,x,\xi) \in T^*(\Omega)\times T^*(\Omega);\ (x,\xi)\in\Sigma\}$.

If we consider $a_0, a_1, \dots$ as functions on E, then the equations (4.3) involve differentiations along the vector field $\nu = \frac{\partial}{\partial t} - iH_{p_0}$. We can consider only Taylor expansions at Σ. Until further notice, our computations will only be valid to infinite order on Σ.

Consider ν as a vector field on E. In the coordinates (t,x,η) we can express ν:

$$\nu = \frac{\partial}{\partial t} - i\sum_{j=1}^{2n-1}\frac{\partial p_0}{\partial\xi_j}(x,\psi'_x)\frac{\partial}{\partial x_j}.$$

We can compute

$$\tag{4.4}\operatorname{div}(\nu) = \frac{1}{i}\Big(\sum_{j=1}^{2n-1}\frac{\partial^2 p_0(x,\psi'_x)}{\partial x_j\partial\xi_j} + \sum_{j,k=1}^{2n-1}\frac{\partial^2 p_0}{\partial\xi_j\partial\xi_k}(x,\psi'_x)\frac{\partial^2\psi}{\partial x_j\partial x_k}(t,x,\eta)\Big).$$

For a smooth function $a(t,x,\eta)$ we introduce the $\frac{1}{2}$ density on E

$$\alpha = a(t,x,\eta)\sqrt{dtdxd\eta}$$

which is well-defined up to some factor i^μ (see Hörmander [**14**]). The Lie derivative of α along ν is $L_\nu(\alpha) = (\nu(a) + \frac{1}{2}\operatorname{div}(\nu)a)\sqrt{dtdxd\eta}$. We see from the expression for T that

$$\tag{4.5}(Ta)\sqrt{dtdxd\eta} = \big(L_\nu + p_0^s(x,\psi'_x(t,x,\eta))\big)(a\sqrt{dtdxd\eta}),$$

where $p_0^s(x,\xi)=p_1(x,\xi)+\frac{i}{2}\sum_{j=1}^{2n-1}\frac{\partial^2 p_0(x,\xi)}{\partial x_j\partial\xi_j}$ is the subprincipal symbol (see (2.7)). Now let $(\widetilde{x},\widetilde{\xi})$ be the coordinates of Proposition 3.6, in which p_0 takes the form (3.13). In these coordinates we have

$$\begin{aligned}H_{p_0}(\widetilde{x},\widetilde{\xi})&=i\left\langle A(\widetilde{x},\widetilde{\xi})\widetilde{x}''\,,\frac{\partial}{\partial\widetilde{x}''}\right\rangle-i\left\langle {}^tA(\widetilde{x},\widetilde{\xi})\widetilde{\xi}'',\frac{\partial}{\partial\widetilde{\xi}''}\right\rangle\\&\quad+\sum_{|\alpha|=1,|\beta|=1}(\widetilde{x}'')^\alpha(\widetilde{\xi}'')^\beta B_{\alpha\beta}(\widetilde{x},\widetilde{\xi},\frac{\partial}{\partial\widetilde{x}},\frac{\partial}{\partial\widetilde{\xi}})\end{aligned}\tag{4.6}$$

and

$$\nu=\frac{\partial}{\partial t}+\left\langle A(\widetilde{x},\widetilde{\psi}'_{\widetilde{x}})\widetilde{x}'',\frac{\partial}{\partial\widetilde{x}''}\right\rangle+\sum_{|\alpha|=1,|\beta|=1}(\widetilde{x}'')^\alpha(\widetilde{\psi}'_{\widetilde{x}''})^\beta C_{\alpha\beta}(\widetilde{x}',\widetilde{\psi}'_{\widetilde{x}},\frac{\partial}{\partial\widetilde{x}}).\tag{4.7}$$

Here $\widetilde{\psi}(t,\widetilde{x},\widetilde{\eta})$ is as in Remark 3.7.

Let $f(t,x,\eta)\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega)_{\mathbb{C}})$, $f(\infty,x,\eta)\in C^\infty(T^*(\Omega)_{\mathbb{C}})$. We say that $f(t,x,\eta)$ converges exponentially fast to $f(\infty,x,\eta)$ if $f(t,x,\eta)-f(\infty,x,\eta)$ satisfies the same kind of estimates as (3.5). Recalling the form of $\widetilde{\psi}$ we obtain

$$\nu=\tilde{\nu}=\frac{\partial}{\partial t}+\left\langle A(\widetilde{x}',0,\widetilde{\eta}',0)\widetilde{x}'',\frac{\partial}{\partial\widetilde{x}''}\right\rangle+\sum_{\substack{|\alpha+\beta|=2\\ \alpha\neq0}}(\widetilde{x}'')^\alpha(\widetilde{\eta}'')^\beta D_{\alpha\beta}(t,\widetilde{x},\widetilde{\eta},\frac{\partial}{\partial\widetilde{x}})\tag{4.8}$$

where $D_{\alpha\beta}$ converges exponentially fast to some limit as $t\to+\infty$. We have on Σ,

$$\frac{1}{2}\mathrm{div}\,(\tilde{\nu})=\frac{1}{2}\mathrm{tr}\,A(\widetilde{x}',0,\widetilde{\eta}',0)=\frac{1}{2}\widetilde{\mathrm{tr}}\,F(\widetilde{x}',0,\widetilde{\eta}',0)\tag{4.9}$$

where $F(\widetilde{x}',0,\widetilde{\eta}',0)$ is the fundamental matrix of $\Box_b^{(q)}$. We define $\widetilde{a}(t,\widetilde{x},\widetilde{\eta})$ by

$$\widetilde{a}(t,\widetilde{x},\widetilde{\eta})\sqrt{dtd\widetilde{x}d\widetilde{\eta}}=a(t,x,\eta)\sqrt{dtdxd\eta}.\tag{4.10}$$

Note that the last equation only defines $\widetilde{a}$ up to i^μ. We have

$$(Ta)\sqrt{dtdxd\eta}=(\widetilde{T}\widetilde{a})\sqrt{dtd\widetilde{x}d\widetilde{\eta}}$$

where

$$\begin{aligned}\widetilde{T}&=\frac{\partial}{\partial t}+\left\langle A(\widetilde{x}',0,\widetilde{\eta}',0)\widetilde{x}'',\frac{\partial}{\partial\widetilde{x}''}\right\rangle+\frac{1}{2}\widetilde{\mathrm{tr}}\,F(\widetilde{x}',0,\widetilde{\eta}',0)\\&\quad+p_0^s(\widetilde{x}',0,\widetilde{\eta}',0)+Q(t,\widetilde{x},\widetilde{\eta},\frac{\partial}{\partial\widetilde{x}}).\end{aligned}\tag{4.11}$$

Here

$$Q(t,\widetilde{x},\widetilde{\eta},\frac{\partial}{\partial\widetilde{x}})=\sum_{\substack{|\alpha+\beta|=2\\ \alpha\neq0}}(\widetilde{x}'')^\alpha(\widetilde{\eta}'')^\beta D_{\alpha\beta}(t,\widetilde{x},\widetilde{\eta},\frac{\partial}{\partial\widetilde{x}})+\sum_{|\alpha+\beta|=1}(\widetilde{x}'')^\alpha(\widetilde{\eta}'')^\beta E_{\alpha\beta}(t,\widetilde{x},\widetilde{\eta}).$$

It is easy to see that $E_{\alpha\beta}$ and $D_{\alpha\beta}$ converge exponentially fast to some limits $E_{\alpha\beta}(\infty,\widetilde{x},\widetilde{\eta})$ and $D_{\alpha\beta}(\infty,\widetilde{x},\widetilde{\eta})$ respectively. We need the following

LEMMA 4.2. – *Let A be a $d\times d$ matrix having only positive eigenvalues and consider the map* $\mathscr{A} : u \mapsto \left\langle A \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix} \right\rangle$ *on the space $P^m(\mathbb{R}^d)$ of homogeneous polynomials of degree m. Then* $\exp(t\mathscr{A})(u) = u \circ (\exp(tA))$ *and the map $\mathscr{A}$ is a bijection except for $m = 0$.*

Proof. – We notice that $U(t) : u \mapsto u \circ \exp(tA)$ form a group of operators and that $(\frac{\partial U(t)}{\partial t})\Big|_{t=0} = \mathscr{A}$. This shows that $U(t) = \exp(t\mathscr{A})$. To prove the second statement, suppose that $u \in P^m$, $m \geq 1$ and $\mathscr{A}(u) = 0$. Then $\exp(t\mathscr{A})(u) = u$ for all t, in other words $u(\exp(tA)(x)) = u(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. Since $\exp(tA)(x) \to 0$ when $t \to -\infty$, we obtain $u(x) = u(0) = 0$, which proves the lemma. □

PROPOSITION 4.3. – *Let*

$$c_j(x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\dots,$$

be positively homogeneous functions of degree $-j$. Then, we can find solutions

$$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\dots,$$

of the system (4.3) *with $a_j(0,x,\eta) = c_j(x,\eta)$, $j = 0,1,\dots,$ where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$ such that $a_j(t,x,\eta)$ has unique Taylor expansions on Σ, for all j. Furthermore, let $\lambda(x,\eta) \in C(T^*(\Omega))$ and $\lambda|_\Sigma < \min \tau_j$, where τ_j are the eigenvalues of $\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s$ on Σ. Then for all indices α, β, γ, j and every compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that*

$$\big|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\big| \leq c e^{-t\lambda(x,\eta)} \text{ on } \overline{\mathbb{R}}_+ \times K. \tag{4.12}$$

Proof. – We only need to study Taylor expansions on Σ. Let $(\widetilde{x}, \widetilde{\xi})$ be the coordinates of Proposition 3.6. We define $\widetilde{a}_j(t,\widetilde{x},\widetilde{\eta})$ from $a_j(t,x,\eta)$ as in (4.10). In order to prove (4.12), it is sufficient to prove the corresponding statement for $\widetilde{a}_j$ (see Chapter 1 of **[29]**). We introduce the Taylor expansion of $\widetilde{a}_0$ with respect to $(\widetilde{x}'', \widetilde{\eta}'')$. $\widetilde{a}_0(t,\widetilde{x},\widetilde{\eta}) = \sum_0^\infty \widetilde{a}_0^j(t,\widetilde{x},\widetilde{\eta})$, where $\widetilde{a}_0^j$ is a homogeneous polynomial of degree j in $(\widetilde{x}'', \widetilde{\eta}'')$. Let $c_0(\widetilde{x},\widetilde{\eta}) = \sum_{j=0} \widetilde{c}_0^j(\widetilde{x},\widetilde{\eta})$, where $\widetilde{c}_0^j$ is a homogeneous polynomial of degree j in $(\widetilde{x}'', \widetilde{\eta}'')$. From $\widetilde{T}\widetilde{a}_0 = 0$, we get $\widetilde{a}_0^0(t,\widetilde{x}',\widetilde{\eta}') = e^{-t(\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s)}\widetilde{c}_0^0(\widetilde{x},\widetilde{\eta})$. It is easy to see that for all indices α, β, γ and every compact set $K \subset \Sigma$ there exists a constant $c > 0$ such that $\Big|\partial_t^\gamma \partial_{\widetilde{x}}^\alpha \partial_{\widetilde{\eta}}^\beta \widetilde{a}_0^0\Big| \leq c e^{-t\lambda(\widetilde{x},\widetilde{\eta})}$ on $\overline{\mathbb{R}}_+ \times K$, where $\lambda(\widetilde{x},\widetilde{\eta}) \in C(T^*(\Omega))$, $\lambda|_\Sigma < \min \tau_j$. Here τ_j are the eigenvalues of $\frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s$ on Σ.

Again, from $\widetilde{T}\widetilde{a}_0 = 0$, we get $(\frac{\partial}{\partial t} + \mathscr{A} + \frac{1}{2}\widetilde{\operatorname{tr}} F + p_0^s)\widetilde{a}_0^{j+1}(t,\widetilde{x},\widetilde{\eta}) = \widetilde{b}_0^{j+1}(t,\widetilde{x},\widetilde{\eta})$ where $\widetilde{b}_0^{j+1}(t,\widetilde{x},\widetilde{\eta})$ satisfies the same kind of estimate as $\widetilde{a}_0^0$. By Lemma 4.2, we see that $\exp(-t\mathscr{A})$ is bounded for $t \geq 0$. We deduce a similar estimate for the function

$\widetilde{a}_0^{j+1}(t,\widetilde{x},\widetilde{\eta})$. Continuing in this way we get all the desired estimates for $\widetilde{a}_0$. The next transport equation takes the form $\widetilde{T}\widetilde{a}_1 = \widetilde{b}$ where $\widetilde{b}$ satisfies the estimates (4.12). We can repeat the procedure above and conclude that $\widetilde{a}_1$ satisfies the estimates (4.12). From above, we see that $\widetilde{a}_0$, $\widetilde{a}_1$ have the unique Taylor expansions on Σ. Continuing in this way we get the proposition. □

From Proposition 4.3, we have the following

PROPOSITION 4.4. – *Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form. Let $c_j(x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$, $j = 0,1,\ldots$, be positively homogeneous functions of degree $-j$. Then, we can find solutions*

$$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots,$$

of the system (4.3) with $a_j(0,x,\eta) = c_j(x,\eta)$, $j = 0,1,\ldots$, where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$ and such that for all indices α,β,γ,j, every $\varepsilon > 0$ and compact set $K \subset \Omega$ there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{\varepsilon t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \ \text{on } \overline{\mathbb{R}}_+ \times \Big((K \times \mathbb{R}^{2n-1}) \bigcap \Sigma\Big). \tag{4.13}$$

Furthermore, there exists $\varepsilon_0 > 0$ such that for all indices α,β,γ,j and every compact set $K \subset \Omega$, there exists a constant $c > 0$ such that

$$\begin{aligned} &\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \\ &\text{on } \overline{\mathbb{R}}_+ \times \Big((K \times \mathbb{R}^{2n-1}) \bigcap \Sigma^+\Big) \ \text{if } q \neq n_+ \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma} \\ &\text{on } \overline{\mathbb{R}}_+ \times \Big((K \times \mathbb{R}^{2n-1}) \bigcap \Sigma^-\Big) \ \text{if } q \neq n_-. \end{aligned} \tag{4.15}$$

We need the following formula

PROPOSITION 4.5. – *Let Q be a C^∞ differential operator on Ω of order $k > 0$ with full symbol $q(x,\xi) \in C^\infty(T^*(\Omega))$. For $0 \leq q, q_1 \leq n-1$, q, $q_1 \in \mathbb{N}$, let*

$$a(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big).$$

Then,

$$Q(x,D_x)(e^{i\psi(t,x,\eta)}a(t,x,\eta)) = e^{i\psi(t,x,\eta)} \sum_{|\alpha| \leq k} \frac{1}{\alpha!} q^{(\alpha)}(x, \psi'_x(t,x,\eta))(R_\alpha(\psi, D_x)a),$$

where $D_x = -i\partial_x$, $R_\alpha(\psi,D_x)a = D_y^\alpha \left\{e^{i\phi_2(t,x,y,\eta)}a(t,y,\eta)\right\}\Big|_{y=x}$, $\phi_2(t,x,y,\eta) = (x-y)\psi'_x(t,x,\eta) - \big(\psi(t,x,\eta) - \psi(t,y,\eta)\big)$.

For $0 \leq q, q_1 \leq n-1$, q, $q_1 \in \mathbb{N}$, let

$$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots,$$

be quasi-homogeneous functions of degree $m-j$, $m \in \mathbb{Z}$. Assume that $a_j(t,x,\eta)$, $j = 0,1,\ldots$, are the solutions of the system (4.3). From the proof of Proposition 4.3, it follows that for all indices α, β, γ, j, every $\varepsilon > 0$ and compact set $K \subset \Omega$ there exists a constant $c > 0$ such that

$$\text{(4.16)} \quad \left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a_j(t,x,\eta)\right| \leq c e^{\varepsilon t|\eta|}(1+|\eta|)^{m-j-|\beta|+\gamma} \text{ on } \overline{\mathbb{R}}_+ \times \Big((K \times \mathbb{R}^{2n-1}) \bigcap \Sigma\Big).$$

Let $a(t,x,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)))$ be the asymptotic sum of $a_j(t,x,\eta)$ (see Definition 5.1 and Remark 5.2 for a precise meaning). We formally write $a(t,x,\eta) \sim \sum_{j=0}^\infty a_j(t,x,\eta)$. Let

$$(\partial_t + \Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta)) = e^{i\psi(t,x,\eta)}b(t,x,\eta),$$

where

$$b(t,x,\eta) \sim \sum_{j=0}^\infty b_j(t,x,\eta),$$

$b_j \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$, b_j is a quasi-homogeneous function of degree $m+2-j$, $j = 0,1,\ldots$.

From Proposition 4.5, we see that for all N, every compact set $K \subset \Omega$, $\varepsilon > 0$, there exists $c > 0$ such that

$$\text{(4.17)} \qquad |b(t,x,\eta)| \leq c e^{\varepsilon t|\eta|}(|\eta|^{-N} + |\eta|^{m+2-N}(\operatorname{Im}\psi(t,x,\eta)^N))$$

on $\overline{\mathbb{R}}_+ \times \Big((K \times \mathbb{R}^{2n-1}) \bigcap \Sigma\Big)$, $|\eta| \geq 1$.

Conversely, if $(\partial_t + \Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta)) = e^{i\psi(t,x,\eta)}b(t,x,\eta)$ and b satisfies the same kind of estimates as (4.17), then $a_j(t,x,\eta)$, $j = 0,1,\ldots$, solve the system (4.3) to infinite order at Σ. From this and the particular structure of the problem, we will next show

Proposition 4.6. – *Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form. Suppose condition $Y(q)$ fails. That is, $q = n_-$ or n_+. Let*

$$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots,$$

be the solutions of the system (4.3) with $a_0(0,x,\eta) = I$, $a_j(0,x,\eta) = 0$ when $j > 0$, where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$. Then we can find

$$a_j(\infty,x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots,$$

where $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$, $\varepsilon_0 > 0$ such that for all indices α, β, γ, j, every compact set $K \subset \Omega$, there exists $c > 0$, such that

$$\text{(4.18)} \qquad \left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta (a_j(t,x,\eta) - a_j(\infty,x,\eta))\right| \leq c e^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma\Big)$, $|\eta|\geq 1$.

Furthermore, for all $j=0,1,\ldots,$

$$\begin{cases} \textit{all derivatives of } a_j(\infty,x,\eta) \textit{ vanish at } \Sigma^+, & \textit{if } q=n_-,\ n_-\neq n_+,\\ \textit{all derivatives of } a_j(\infty,x,\eta) \textit{ vanish at } \Sigma^-, & \textit{if } q=n_+,\ n_-\neq n_+. \end{cases} \tag{4.19}$$

Proof. – We assume that $q=n_-$. Put

$$a(t,x,\eta)\sim\sum_j a_j(t,x,\eta).$$

Since $a_j(t,x,\eta)$, $j=0,1,\ldots$, solve the system (4.3), we have

$$(\partial_t+\Box_b^{(q)})(e^{i\psi(t,x,\eta)}a(t,x,\eta))=e^{i\psi(t,x,\eta)}b(t,x,\eta),$$

where $b(t,x,\eta)$ satisfies (4.17). Note that we have the interwing properties

$$\begin{cases} \overline{\partial}_b\Box_b^{(q)}=\Box_b^{(q+1)}\overline{\partial}_b,\\ \overline{\partial_b}^*\Box_b^{(q)}=\Box_b^{(q-1)}\overline{\partial_b}^*. \end{cases} \tag{4.20}$$

Now,

$$\begin{cases} \overline{\partial_b}^*(e^{i\psi}a)=e^{i\psi}\widetilde{a},\\ \overline{\partial}_b(e^{i\psi}a)=e^{i\psi}\hat{a}, \end{cases}$$

$\widetilde{a}\sim\sum_{j=0}^\infty\widetilde{a}_j(t,x,\eta)$, $\hat{a}\sim\sum_{j=0}^\infty\hat{a}_j(t,x,\eta)$, where

$$\hat{a}_j\in C^\infty\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q+1}T^*(\Omega))\big),\quad j=0,1,\ldots,$$
$$\widetilde{a}_j\in C^\infty\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q-1}T^*(\Omega))\big),\quad j=0,1,\ldots,$$

and $\hat{a}_j$, $\widetilde{a}_j$ are quasi-homogeneous of degree $1-j$. From (4.20), we have

$$(\partial_t+\Box_b^{(q-1)})(e^{i\psi}\widetilde{a})=e^{i\psi}b_1,$$
$$(\partial_t+\Box_b^{(q+1)})(e^{i\psi}\hat{a})=e^{i\psi}b_2,$$

where b_1, b_2 satisfy (4.17). Since b_1, b_2 satisfy (4.17), $\widetilde{a}_j$, $\hat{a}_j$, $j=0,1,\ldots$, solve the system (4.3) to infinite order at Σ. We notice that $q-1\neq n_-$, $q+1\neq n_-$. In view of the proof of Proposition 4.3, we can find $\varepsilon_0>0$, such that for all indices α, β, γ, j, every compact set $K\subset\Omega$, there exists $c>0$ such that

$$\begin{cases} \big|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta\widetilde{a}_j(t,x,\eta)\big|\leq ce^{-\varepsilon_0t|\eta|}(1+|\eta|)^{1-j-|\beta|+\gamma}\\ \big|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta\hat{a}_j(t,x,\eta)\big|\leq ce^{-\varepsilon_0t|\eta|}(1+|\eta|)^{1-j-|\beta|+\gamma} \end{cases} \tag{4.21}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma^-\Big)$, $|\eta|\geq 1$.

Now $\Box_b^{(q)}=\overline{\partial}_b\overline{\partial_b}^*+\overline{\partial_b}^*\overline{\partial}_b$, so $\Box_b^{(q)}(e^{i\psi}a)=e^{i\psi}c$, where c satisfies the same kind of estimates as (4.21). From this we see that $\partial_t(e^{i\psi}a)=e^{i\psi}d$, where d has the same properties as c. Since $d=i(\partial_t\psi)a+\partial_ta$ and $\partial_t\psi$ satisfy the same kind of estimates as (4.21), ∂_ta satisfies the same kind of estimates as (4.21). From this we conclude that we can find $a(\infty,x,\eta)\sim\sum_{j=0}^{\infty}a_j(\infty,x,\eta)$, where $a_j(\infty,x,\eta)$ is a matrix-valued C^∞ positively homogeneous function of degree $-j$, $\varepsilon_0>0$, such that for all indices α, β, γ, j and every compact set $K\subset\Omega$, there exists $c>0$ such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))\right|\leq ce^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma^-\Big)$, $|\eta|\geq 1$.

If $n_-=n_+$, then $q-1\neq n_+$, $q+1\neq n_+$. We can repeat the method above to conclude that we can find $a(\infty,x,\eta)\sim\sum_{j=0}^{\infty}a_j(\infty,x,\eta)$, where $a_j(\infty,x,\eta)$ is a matrix-valued C^∞ positively homogeneous function of degree $-j$, $\varepsilon_0>0$, such that for all indices α, β, γ, j and every compact set $K\subset\Omega$, there exists $c>0$ such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta(a_j(t,x,\eta)-a_j(\infty,x,\eta))\right|\leq ce^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma^+\Big)$, $|\eta|\geq 1$.

Now, we assume that $n_-\neq n_+$. From (4.14), we can find $\varepsilon_0>0$, such that for all indices α, β, γ, j and every compact set $K\subset\Omega$, there exists $c>0$ such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta a_j(t,x,\eta)\right|\leq ce^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma^+\Big)$, $|\eta|\geq 1$. The proposition follows. □

CHAPTER 5

SOME SYMBOL CLASSES

We continue to work with some local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. We identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^{2n-1}$.

Definition 5.1. – Let $r(x,\eta)$ be a non-negative real continuous function on $T^*(\Omega)$. We assume that $r(x,\eta)$ is positively homogeneous of degree 1, that is, $r(x,\lambda\eta) = \lambda r(x,\eta)$, for $\lambda \geq 1$, $|\eta| \geq 1$. For $0 \leq q_1, q_2 \leq n-1$, $q_1, q_2 \in \mathbb{N}$ and $k \in \mathbb{R}$, we say that

$$a \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$$

if $a \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$ and for all indices α, β, γ, every compact set $K \subset \Omega$ and every $\varepsilon > 0$, there exists a constant $c > 0$ such that

$$\left|\partial^\gamma_t \partial^\alpha_x \partial^\beta_\eta a(t,x,\eta)\right| \leq c e^{t(-r(x,\eta)+\varepsilon|\eta|)}(1+|\eta|)^{k+\gamma-|\beta|},\ x \in K,\ |\eta| \geq 1.$$

Remark 5.2. – It is easy to see that we have the following properties:

(a) If $a \in \hat{S}^k_{r_1}$, $b \in \hat{S}^l_{r_2}$ then $ab \in \hat{S}^{k+l}_{r_1+r_2}$, $a + b \in \hat{S}^{\max(k,l)}_{\min(r_1,r_2)}$.

(b) If $a \in \hat{S}^k_r$ then $\partial^\gamma_t \partial^\alpha_x \partial^\beta_\eta a \in \hat{S}^{k-|\beta|+\gamma}_r$.

(c) If $a_j \in \hat{S}^{k_j}_r$, $j = 0,1,2,\ldots$ and $k_j \searrow -\infty$ as $j \to \infty$, then there exists $a \in \hat{S}^{k_0}_r$ such that $a - \sum_0^{v-1} a_j \in \hat{S}^{k_v}_r$, for all $v = 1,2,\ldots$. Moreover, if $\hat{S}^{-\infty}_r$ denotes $\bigcap_{k\in\mathbb{R}} \hat{S}^k_r$ then a is unique modulo $\hat{S}^{-\infty}_r$.

If a and a_j have the properties of (c), we write

$$a \sim \sum_0^\infty a_j \text{ in the symbol space } \hat{S}^{k_0}_r.$$

From Proposition 4.4 and the standard Borel construction, we get the following

Proposition 5.3. – *Let*

$$c_j(x,\eta) \in C^\infty\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big),\quad j = 0,1,\ldots$$

be positively homogeneous functions of degree $-j$. We can find solutions $a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$, $j = 0,1,\ldots$ of the system (4.3) *with* *the conditions* $a_j(0,x,\eta)=c_j(x,\eta)$, $j=0,1,\dots$, *where* a_j *is a quasi-homogeneous function of degree* $-j$ *such that*

$$a_j\in\hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),\quad j=0,1,\dots,$$

for some r *with* $r>0$ *if* $Y(q)$ *holds and* $r=0$ *if* $Y(q)$ *fails.*

If the Levi form has signature (n_-,n_+), $n_-+n_+=n-1$, *then we can take* $r>0$,

$$\begin{cases}\text{near } \Sigma^+, & \text{if } q=n_-, n_-\neq n_+,\\ \text{near } \Sigma^-, & \text{if } q=n_+, n_-\neq n_+.\end{cases}$$

Again, from Proposition 4.6 and the standard Borel construction, we get the following

Proposition 5.4. – *Let* (n_-,n_+), $n_-+n_+=n-1$, *be the signature of the Levi form. Suppose condition* $Y(q)$ *fails. That is,* $q=n_-$ *or* n_+. *We can find solutions*

$$a_j(t,x,\eta)\in C^\infty\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),\quad j=0,1,\dots$$

of the system (4.3) *with* $a_0(0,x,\eta)=I$, $a_j(0,x,\eta)=0$ *when* $j>0$, *where* $a_j(t,x,\eta)$ *is a quasi-homogeneous function of degree* $-j$, *such that for some* $r>0$ *as in Definition 5.1,*

$$a_j(t,x,\eta)-a_j(\infty,x,\eta)\in\hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),$$

$j=0,1,\dots$, *where*

$$a_j(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),\quad j=0,1,\dots,$$

and $a_j(\infty,x,\eta)$ *is a positively homogeneous function of degree* $-j$.

Furthermore, for all $j=0,1,\dots$,

$$\begin{cases}a_j(\infty,x,\eta)=0 \text{ in a conic neighborhood of } \Sigma^+, & \text{if } q=n_-, n_-\neq n_+,\\ a_j(\infty,x,\eta)=0 \text{ in a conic neighborhood of } \Sigma^-, & \text{if } q=n_+, n_-\neq n_+.\end{cases}$$

Let $b(t,x,\eta)\in\hat{S}^k_r$, $r>0$. Our next goal is to define the operator

$$B(t,x,y)=\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)d\eta$$

as an oscillatory integral (see the proof of Proposition 5.5 for the precise meaning of the integral $B(t,x,y)$). We have the following

Proposition 5.5. – *Let*

$$b(t,x,\eta)\in\hat{S}^k_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))\big)$$

with $r>0$. *Then we can define*

$$B(t):C^\infty_0(\Omega;\Lambda^{0,q_1}T^*(\Omega))\to C^\infty\big(\overline{\mathbb{R}}_+;C^\infty(\Omega;\Lambda^{0,q_2}T^*(\Omega))\big)$$

with distribution kernel $B(t,x,y) = \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)d\eta$ *and* $B(t)$ *has a unique continuous extension*

$$B(t) : \mathscr{E}'(\Omega;\, \Lambda^{0,q_1}T^*(\Omega)) \to C^\infty\big(\overline{\mathbb{R}}_+;\, \mathscr{D}'(\Omega;\, \Lambda^{0,q_2}T^*(\Omega))\big).$$

We have

$$B(t,x,y) \in C^\infty\Big(\overline{\mathbb{R}}_+;\, C^\infty\big(\Omega\times\Omega \smallsetminus \operatorname{diag}(\Omega\times\Omega);\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)\Big)$$

and $B(t,x,y)|_{t>0} \in C^\infty\big(\mathbb{R}_+\times\Omega\times\Omega;\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$.

Proof. – Let

$$S^*\Omega = \left\{(x,\eta)\in\Omega\times\dot{\mathbb{R}}^{2n-1};\, |\eta| = 1\right\}.$$

Set $S^*\Sigma = \Sigma\bigcap S^*\Omega$. Let $V\subset\overline{\mathbb{R}}_+\times S^*\Omega$ be a neighborhood of $(\mathbb{R}_+\times S^*\Sigma)\bigcup(\{0\}\times S^*\Omega)$ such that $V_t = \{(x,\eta);\, (t,x,\eta)\in V\}$ is independent of t for large t. Set $W = \big\{(t,x,\eta)\in\overline{\mathbb{R}}_+\times\Omega\times\dot{\mathbb{R}}^{2n-1};\, (|\eta|\,t, x, \frac{\eta}{|\eta|})\in V\big\}$. Let $\chi_V\in C^\infty(\overline{\mathbb{R}}_+\times S^*\Omega)$ have its support in V, be equal to 1 in a neighborhood of $(\mathbb{R}_+\times S^*\Sigma)\bigcup(\{0\}\times S^*\Omega)$, and be independent of t, for large t. Set $\chi_W(t,x,\eta) = \chi_V(|\eta|\,t, x, \frac{\eta}{|\eta|})\in C^\infty(\overline{\mathbb{R}}_+\times\Omega\times\dot{\mathbb{R}}^{2n-1})$. We have $\chi_W(t,x,\lambda\eta) = \chi_W(\lambda t,x,\eta)$, $\lambda>0$. We can choose V sufficiently small so that

$$|\psi'_x(t,x,\eta)-\eta| \le \frac{|\eta|}{2} \text{ in } W. \tag{5.1}$$

We formally set

$$\begin{aligned} B(t,x,y) &= \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)d\eta \\ &\quad + \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\eta)b(t,x,\eta)d\eta \\ &= B_1(t,x,y) + B_2(t,x,y) \end{aligned}$$

where in $B_1(t,x,y)$ and $B_2(t,x,y)$ we have introduced the cut-off functions $(1-\chi_W)$ and χ_W respectively. Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $|\eta|<1$ and $\chi(\eta)=0$ when $|\eta|>2$. Since $\operatorname{Im}\psi>0$ outside $(\mathbb{R}_+\times\Sigma)\bigcup(\{0\}\times\dot{\mathbb{R}}^{2n-1})$, we have $\operatorname{Im}\psi(t,x,\eta)\ge c\,|\eta|$ outside W, where $c>0$. The kernel $B_{1,\varepsilon}(t,x,y) = \int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)\chi(\varepsilon\eta)d\eta$ converges in the space $C^\infty\big(\overline{\mathbb{R}}_+\times\Omega\times\Omega;\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$ as $\varepsilon\to 0$. This means that

$$B_1(t,x,y) = \lim_{\varepsilon\to 0} B_{1,\varepsilon}(t,x,y) \in C^\infty\big(\overline{\mathbb{R}}_+\times\Omega\times\Omega;\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big).$$

To study $B_2(t,x,y)$ take a $u(y)\in C_0^\infty(K;\, \Lambda^{0,q_1}T^*(\Omega))$, $K\subset\subset\Omega$ and set $\chi_\nu(\eta) = \chi(2^{-\nu}\eta)-\chi(2^{1-\nu}\eta)$, $\nu>0$, $\chi_0(\eta)=\chi(\eta)$. Then we have $\sum_{\nu=0}^\infty\chi_\nu = 1$ and $2^{\nu-1}\le|\eta|\le 2^{\nu+1}$ when $\eta\in\operatorname{supp}\chi_\nu$, $\nu\ne 0$. We assume that $b(t,x,\eta)=0$ if $|\eta|\le 1$. If $x\in K$, we obtain for all indices α, β and every $\varepsilon>0$, there exists $c_{\varepsilon,\alpha,\beta,K}>0$, such that

$$\big|D_x^\alpha D_\eta^\beta\big(\chi_\nu(\eta)\chi_W(t,x,\eta)b(t,x,\eta)\big)\big| \le c_{\varepsilon,\alpha,\beta,K}e^{t(-r(x,\eta)+\varepsilon|\eta|)}(1+|\eta|)^{k-|\beta|}. \tag{5.2}$$

Note that $|D^\alpha \chi_\nu(\eta)| \leq c_\alpha(1+|\eta|)^{-|\alpha|}$ with a constant independent of ν. We have

$$\begin{aligned} B_{2,\nu+1} &= \iint e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_{\nu+1}(\eta)\chi_W(t,x,\eta)b(t,x,\eta)u(y)dyd\eta \\ &= 2^{(2n-1)\nu}\int e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_1(\eta)\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)u(y)dyd\eta, \end{aligned}$$

where $\lambda = 2^\nu$. Since (5.2) holds, we have $\big|D^\alpha_\eta(\chi_W(t,x,2^\nu\eta)b(t,x,2^\nu\eta))\big| \leq c2^{k\nu}$ if $x \in K$, $1 < |\eta| < 4$, where $c > 0$. Since $d_y(\psi(\lambda t,x,\eta) - \langle y\ ,\eta\rangle) \neq 0$, if $\eta \neq 0$, we can integrate by parts and obtain

$$|B_{2,\nu+1}| \leq c2^{\nu(2n-1+k-m)}\sum_{|\alpha|\leq m}\sup|D^\alpha u|\,.$$

Since m can be chosen arbitrary large, we conclude that $\sum_\nu |B_{2,\nu}|$: converges and that $B(t)$ defines an operator

$$B(t) : C^\infty_0(\Omega_y;\, \Lambda^{0,q_1}T^*(\Omega)) \to C^\infty(\overline{\mathbb{R}}_+ \times \Omega_x;\, \Lambda^{0,q_2}T^*(\Omega)).$$

Let $B^*(t)$ be the formal adjoint of $B(t)$ with respect to $(\ |\ )$. From (5.1), we see that $\psi'_x(t,x,\eta) \neq 0$ on W. We can repeat the procedure above and conclude that $B^*(t)$ defines an operator

$$B^*(t) : C^\infty_0(\Omega_x;\, \Lambda^{0,q_2}T^*(\Omega)) \to C^\infty(\overline{\mathbb{R}}_+ \times \Omega_y;\, \Lambda^{0,q_1}T^*(\Omega)).$$

Hence, we can extend $B(t)$ to $\mathscr{E}'(\Omega;\, \Lambda^{0,q_1}T^*(\Omega)) \to C^\infty(\overline{\mathbb{R}}_+;\, \mathscr{D}'(\Omega;\, \Lambda^{0,q_2}T^*(\Omega)))$ by the formula

$$(B(t)u(y)\mid v(x)) = (u(y)\mid B^*(t)v(x)),$$

where $u \in \mathscr{E}'(\Omega;\, \Lambda^{0,q_1}T^*(\Omega))$, $v \in C^\infty_0(\Omega;\, \Lambda^{0,q_2}T^*(\Omega))$.

When $x \neq y$ and $(x,y) \in \Sigma\times\Sigma$, we have $d_\eta(\psi(t,x,\eta) - \langle y\ ,\eta\rangle) \neq 0$, we can repeat the procedure above and conclude that $B(t,x,y) \in C^\infty\Big(\overline{\mathbb{R}}_+;\, C^\infty\big(\Omega\times\Omega \smallsetminus \operatorname{diag}(\Omega\times\Omega);\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)\Big)$.

Finally, in view of the exponential decrease as $t\to\infty$ of the symbol $b(t,x,\eta)$, we see that the kernel $B(t)|_{t>0}$ is smoothing. □

Let $b(t,x,\eta) \in \hat{S}^k_r$ with $r > 0$. Our next step is to show that we can also define the operator $B(x,y) = \int\big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\big)d\eta$ as an oscillatory integral (see the proof of Proposition 5.6 for the precise meaning of the integral $B(x,y)$). We have the following

Proposition 5.6. – *Let*

$$b(t,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$$

with $r > 0$. Assume that $b(t,x,\eta) = 0$ when $|\eta| \leq 1$. We can define

$$B : C^\infty_0(\Omega;\, \Lambda^{0,q_1}T^*(\Omega)) \to C^\infty(\Omega;\, \Lambda^{0,q_2}T^*(\Omega))$$

with distribution kernel

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta$$

and B has a unique continuous extension

$$B:\mathscr{E}'(\Omega;\,\Lambda^{0,q_1}T^*(\Omega))\to\mathscr{D}'(\Omega;\,\Lambda^{0,q_2}T^*(\Omega)).$$

Moreover, $B(x,y)\in C^\infty(\Omega\times\Omega\setminus\operatorname{diag}(\Omega\times\Omega);\,\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega)))$.

Proof. – Let W and $\chi_W(t,x,\eta)$ be as in Proposition 5.5. We formally set

$$\begin{aligned}B(x,y)&=\frac{1}{(2\pi)^{2n-1}}\iint_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)dtd\eta\\&\quad+\frac{1}{(2\pi)^{2n-1}}\iint_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\eta)b(t,x,\eta)dtd\eta\\&=B_1(x,y)+B_2(x,y)\end{aligned}$$

where in $B_1(x,y)$ and $B_2(x,y)$ we have introduced the cut-off functions $(1-\chi_W)$ and χ_W respectively. Since $\operatorname{Im}\psi(t,x,\eta)\geq c'\,|\eta|$ outside W, where $c'>0$, we have

$$\Big|e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(1-\chi_W(t,x,\eta))b(t,x,\eta)\Big|\leq ce^{-c'|\eta|}e^{-\varepsilon_0t|\eta|}(1+|\eta|)^k,\ \varepsilon_0>0$$

and similar estimates for the derivatives. From this, we see that $B_1(x,y)\in C^\infty\big(\Omega\times\Omega;\,\mathscr{L}(\Lambda^{0,q_1}T^*(\Omega),\Lambda^{0,q_2}T^*(\Omega))\big)$.

Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $|\eta|<1$ and $\chi(\eta)=0$ when $|\eta|>2$. To study $B_2(x,y)$ take a $u(y)\in C_0^\infty(K;\,\Lambda^{0,q_1}T^*(\Omega))$, $K\subset\subset\Omega$ and set

$$B_{2,\lambda}(x)=\frac{1}{(2\pi)^{2n-1}}\int_0^\infty\Big(\iint e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)\chi_W(t,x,\eta)\chi(\frac{\eta}{\lambda})u(y)dyd\eta\Big)dt.$$

We have

$$\begin{aligned}&B_{2,2\lambda}(x)-B_{2,\lambda}(x)\\&=\frac{\lambda^{2n-1}}{(2\pi)^{2n-1}}\int_0^\infty\Big(\iint e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)(\chi(\frac{\eta}{2})-\chi(\eta))u(y)dyd\eta\Big)dt.\end{aligned}$$

Since $d_y(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)\neq0$, $\eta\neq0$, we obtain

$$\begin{aligned}&\Big|\iint e^{i\lambda(\psi(\lambda t,x,\eta)-\langle y,\eta\rangle)}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)(\chi(\frac{\eta}{2})-\chi(\eta))u(y)dyd\eta\Big|\\&\leq c\lambda^{-N}\sum_{|\alpha|\leq N}\sup\Big|D^\alpha_{y,\eta}\chi_W(t,x,\lambda\eta)b(t,x,\lambda\eta)(\chi(\frac{\eta}{2})-\chi(\eta))u(y)\Big|\\&\leq c'\lambda^{-N}e^{-\varepsilon_0t|\eta|}(1+|\lambda|)^k,\end{aligned}$$

where c, c', $\varepsilon_0>0$. Hence $B_2(x)=\lim_{\lambda\to\infty}B_{2,\lambda}(x)$ exists. Thus, $B(x,y)$ defines an operator $B:C_0^\infty(\Omega_y;\,\Lambda^{0,q_1}T^*(\Omega))\to C^\infty(\Omega_x;\,\Lambda^{0,q_2}T^*(\Omega))$. Let B^* be the formal adjoint of B with respect to $(\ |\ )$. Since $\psi'_x(t,x,\eta)\neq0$ on W, we can repeat the procedure above and conclude that B^* defines an operator $B^*:C_0^\infty(\Omega_x;\,\Lambda^{0,q_2}T^*(\Omega))\to$

$C^\infty(\Omega_y;\ \Lambda^{0,q_1}T^*(\Omega))$. We can extend B to $\mathscr{E}'(\Omega;\ \Lambda^{0,q_1}T^*(\Omega)) \to \mathscr{D}'(\Omega;\ \Lambda^{0,q_2}T^*(\Omega))$ by the formula

$$(Bu(y) \mid v(x)) = (u(y) \mid B^*v(x)),$$

where $u \in \mathscr{E}'(\Omega;\ \Lambda^{0,q_1}T^*(\Omega))$, $v \in C_0^\infty(\Omega;\ \Lambda^{0,q_2}T^*(\Omega))$.

Finally, when $x \neq y$ and $(x,y) \in \Sigma \times \Sigma$, we have $d_\eta(\psi(t,x,\eta) - \langle y\ ,\eta\rangle) \neq 0$, we can repeat the procedure above and conclude that $B(x,y) \in C^\infty\big(\Omega \times \Omega \smallsetminus \operatorname{diag}(\Omega \times \Omega);\ \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$. □

Remark 5.7. – Let $a(t,x,\eta) \in \hat{S}_0^k\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$. We assume $a(t,x,\eta) = 0$ if $|\eta| \leq 1$ and

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}_r^k\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$$

with $r > 0$, where $a(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q_1}T^*(\Omega), \Lambda^{0,q_2}T^*(\Omega))\big)$. Then we can also define

$$A(x,y) = \int\Big(\int_0^\infty \big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)dt\Big)d\eta$$

as an oscillatory integral by the following formula:

$$A(x,y) = \int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(-t)\big(i\psi_t'(t,x,\eta)a(t,x,\eta) + a_t'(t,x,\eta)\big)dt\Big)d\eta.$$

We notice that $(-t)\big(i\psi_t'(t,x,\eta)a(t,x,\eta) + a_t'(t,x,\eta)\big) \in \hat{S}_r^{k+1}$, $r > 0$.

Let B be as in the proposition 5.6. We can show that B is a matrix of pseudodifferential operators of order $k-1$ type $(\frac{1}{2},\frac{1}{2})$. We review some facts about pseudodifferential operators of type $(\frac{1}{2},\frac{1}{2})$.

Definition 5.8. – Let $k \in \mathbb{R}$ and $q \in \mathbb{N}$. $S^k_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$ is the space of all $a \in C^\infty\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$ such that for every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there is a constant $c_{\alpha,\beta,K} > 0$ such that $\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq c_{\alpha,\beta,K}(1+|\xi|)^{k-\frac{|\beta|}{2}+\frac{|\alpha|}{2}}$, $(x,\xi) \in T^*(\Omega)$, $x \in K$. $S^k_{\frac{1}{2},\frac{1}{2}}$ is called the space of symbols of order k type $(\frac{1}{2},\frac{1}{2})$. We write $S^{-\infty}_{\frac{1}{2},\frac{1}{2}} = \bigcap_{m\in\mathbb{R}} S^m_{\frac{1}{2},\frac{1}{2}}$, $S^{\infty}_{\frac{1}{2},\frac{1}{2}} = \bigcup_{m\in\mathbb{R}} S^m_{\frac{1}{2},\frac{1}{2}}$.

Let $a(x,\xi) \in S^k_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. We can also define

$$A(x,y) = \frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)d\xi$$

as an oscillatory integral and we can show that

$$A : C_0^\infty(\Omega;\ \Lambda^{0,q}T^*(\Omega)) \to C^\infty(\Omega;\ \Lambda^{0,q}T^*(\Omega))$$

is continuous and has unique continuous extension:

$$A : \mathscr{E}'(\Omega;\ \Lambda^{0,q}T^*(\Omega)) \to \mathscr{D}'(\Omega;\ \Lambda^{0,q}T^*(\Omega)).$$

DEFINITION 5.9. – Let $k \in \mathbb{R}$ and let $0 \leq q \leq n-1$, $q \in \mathbb{N}$. A pseudodifferential operator of order k type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ is a continuous linear map $A : C_0^\infty(\Omega;\, \Lambda^{0,q}T^*(\Omega)) \to \mathscr{D}'(\Omega;\, \Lambda^{0,q}T^*(\Omega))$ such that the distribution kernel of A is

$$K_A = A(x,y) = \frac{1}{(2\pi)^{2n-1}} \int e^{i\langle x-y,\xi\rangle} a(x,\xi) d\xi$$

with $a \in S^k_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. We call $a(x,\xi)$ the symbol of A. We shall write $L^k_{\frac{1}{2},\frac{1}{2}}\big(\Omega;\, \Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)\big)$ to denote the space of pseudodifferential operators of order k type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$. We write $L^{-\infty}_{\frac{1}{2},\frac{1}{2}} = \bigcap_{m\in\mathbb{R}} L^m_{\frac{1}{2},\frac{1}{2}}$, $L^{\infty}_{\frac{1}{2},\frac{1}{2}} = \bigcup_{m\in\mathbb{R}} L^m_{\frac{1}{2},\frac{1}{2}}$.

We recall the following classical proposition of Calderon-Vaillancourt (see chapter XVIII of Hörmander [**15**]).

PROPOSITION 5.10. – *If* $A \in L^k_{\frac{1}{2},\frac{1}{2}}\big(\Omega;\, \Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)\big)$. *Then,*

$$A : H^s_{\rm comp}(\Omega;\, \Lambda^{0,q}T^*(\Omega)) \to H^{s-k}_{\rm loc}(\Omega;\, \Lambda^{0,q}T^*(\Omega))$$

is continuous, for all $s \in \mathbb{R}$. *Moreover, if* A *is properly supported (for the precise meaning of properly supported operators, see page 28 of* [**12**]*), then*

$$A : H^s_{\rm loc}(\Omega;\, \Lambda^{0,q}T^*(\Omega)) \to H^{s-k}_{\rm loc}(\Omega;\, \Lambda^{0,q}T^*(\Omega))$$

is continuous, for all $s \in \mathbb{R}$.

We need the following properties of the phase $\psi(t,x,\eta)$.

LEMMA 5.11. – *For every compact set* $K \subset \Omega$ *and all* $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, $|\alpha| + |\beta| \geq 1$, *there exists a constant* $c_{\alpha,\beta,K} > 0$, *such that*

$$\left|\partial_x^\alpha \partial_\eta^\beta (\psi(t,x,\eta) - \langle x,\eta\rangle)\right| \leq c_{\alpha,\beta,K}(1+|\eta|)^{\frac{|\alpha|-|\beta|}{2}} (1 + \operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}},$$

if $|\alpha| + |\beta| = 1$ *and*

$$\left|\partial_x^\alpha \partial_\eta^\beta (\psi(t,x,\eta) - \langle x,\eta\rangle)\right| \leq c_{\alpha,\beta,K}(1+|\eta|)^{1-|\beta|}, \ \text{if } |\alpha| + |\beta| \geq 2,$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$, $|\eta| \geq 1$.

Proof. – For $|\eta| = 1$, we consider Taylor expansions of $\partial_{x_j}(\psi(t,x,\eta) - \langle x,\eta\rangle)$, $j = 1, \ldots, 2n-1$, at $(x_0,\eta_0) \in \Sigma$,

$$\begin{aligned}\partial_{x_j}(\psi(t,x,\eta) - \langle x,\eta\rangle) &= \sum_k \frac{\partial^2\psi}{\partial x_k \partial x_j}(t,x_0,\eta_0)(x_k - x_0^{(k)}) \\ &\quad + \sum_k \frac{\partial^2\psi}{\partial \eta_k \partial x_j}(t,x_0,\eta_0)(\eta_k - \eta_0^{(k)}) \\ &\quad + O(|(x-x_0)|^2 + |(\eta-\eta_0)|^2),\end{aligned}$$

where $x_0 = (x_0^{(1)}, \ldots, x_0^{(2n-1)})$, $\eta_0 = (\eta_0^{(1)}, \ldots, \eta_0^{(2n-1)})$. Thus, for every compact set $K \subset \Omega$ there exists a constant $c > 0$, such that

$$|\partial_x(\psi(t,x,\eta) - \langle x,\eta\rangle)| \leq c\frac{t}{1+t}\mathrm{dist}\,((x,\eta),\Sigma)),$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$ and $|\eta| = 1$. In view of (3.4), we see that $\mathrm{Im}\,\psi(t,x,\eta) \asymp (\frac{t}{1+t})\mathrm{dist}\,((x,\eta),\Sigma))^2$, $|\eta| = 1$. Hence, $(\frac{t}{1+t})^{\frac{1}{2}}\mathrm{dist}\,((x,\eta),\Sigma)) \asymp (\mathrm{Im}\,\psi(t,x,\eta))^{\frac{1}{2}}$, $|\eta| = 1$. Thus, for every compact set $K \subset \Omega$ there exists a constant $c > 0$, such that $|\partial_x(\psi(t,x,\eta) - \langle x,\eta\rangle)| \leq c(\frac{t}{1+t})^{\frac{1}{2}}(\mathrm{Im}\,\psi(t,x,\eta))^{\frac{1}{2}}$, $|\eta| = 1$, $x \in K$. From above, we get for $|\eta| \geq 1$,

$$\begin{aligned}|\partial_x(\psi(t,x,\eta) - \langle x,\eta\rangle)| &= |\eta|\left|\partial_x\Big(\psi(t\,|\eta|\,,x,\frac{\eta}{|\eta|}) - \Big\langle x,\frac{\eta}{|\eta|}\Big\rangle\Big)\right| \\ &\leq c\,|\eta|^{\frac{1}{2}}\left(\frac{t\,|\eta|}{1+t\,|\eta|}\right)^{\frac{1}{2}}(\mathrm{Im}\,\psi(t,x,\eta))^{\frac{1}{2}} \\ &\leq c'(1+|\eta|)^{\frac{1}{2}}(1+\mathrm{Im}\,\psi(t,x,\eta))^{\frac{1}{2}},\end{aligned}$$

where c, $c' > 0$, $x \in K$, $t \in \overline{\mathbb{R}}_+$. Here K is as above. Similarly, for every compact set $K \subset \Omega$ there exists a constant $c > 0$, such that

$$|\partial_\eta(\psi(t,x,\eta) - \langle x,\eta\rangle)| \leq c(1+|\eta|)^{-\frac{1}{2}}(\mathrm{Im}\,\psi(t,x,\eta))^{\frac{1}{2}},$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$ and $|\eta| \geq 1$.

Note that $\left|\partial_x^\alpha\partial_\eta^\beta(\psi(t,x,\eta) - \langle x,\eta\rangle)\right|$ is quasi-homogeneous of degree $1 - |\beta|$. For $|\alpha| + |\beta| \geq 2$, we have $\left|\partial_x^\alpha\partial_\eta^\beta(\psi(t,x,\eta) - \langle x,\eta\rangle)\right| \leq c(1+|\eta|)^{1-|\beta|}$, where $c > 0$, $x \in K$, $t \in \overline{\mathbb{R}}_+$ and $|\eta| \geq 1$. Here K is as above. The lemma follows. □

We also need the following

Lemma 5.12. – *For every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there exist a constant $c_{\alpha,\beta,K} > 0$ and $\varepsilon > 0$, such that*

$$\left|\partial_x^\alpha\partial_\eta^\beta(t\psi_t'(t,x,\eta))\right| \leq c_{\alpha,\beta,K}(1+|\eta|)^{\frac{|\alpha|-|\beta|}{2}}e^{-t\varepsilon|\eta|}(1+\mathrm{Im}\,\psi(t,x,\eta))^{1+\frac{|\alpha|+|\beta|}{2}}$$

if $|\alpha| + |\beta| \leq 1$ and

$$\left|\partial_x^\alpha\partial_\eta^\beta(t\psi_t'(t,x,\eta))\right| \leq c_{\alpha,\beta,K}(1+|\eta|)^{1-|\beta|}e^{-t\varepsilon|\eta|}$$

if $|\alpha| + |\beta| \geq 2$, where $x \in K$, $t \in \overline{\mathbb{R}}_+$, $|\eta| \geq 1$.

Proof. – The proof is essentially the same as the proof of Lemma 5.11. □

Lemma 5.13. – *For every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there exist a constant $c_{\alpha,\beta,K} > 0$ and $\varepsilon > 0$, such that*

$$\begin{aligned}&\left|\partial_x^\alpha\partial_\eta^\beta(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)})\right| \\ &\quad\leq c_{\alpha,\beta,K}(1+|\eta|)^{\frac{|\alpha|-|\beta|}{2}}e^{-\mathrm{Im}\,\psi(t,x,\eta)}(1+\mathrm{Im}\,\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}}\end{aligned} \tag{5.3}$$

and

$$\left|\partial_x^\alpha\partial_\eta^\beta(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle}t\psi_t'(t,x,\eta))\right| \tag{5.4}$$
$$\leq c_{\alpha,\beta,K}(1+|\eta|)^{\frac{|\alpha|-|\beta|}{2}}e^{-t\varepsilon|\eta|}e^{-\operatorname{Im}\psi(t,x,\eta)}(1+\operatorname{Im}\psi(t,x,\eta))^{1+\frac{|\alpha|+|\beta|}{2}},$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $|\eta|\geq 1$.

Proof. – First, we prove (5.3). We proceed by induction over $|\alpha|+|\beta|$. For $|\alpha|+|\beta|\leq 1$, from Lemma 5.11, we get (5.3). Let $|\alpha|+|\beta|\geq 2$. Then

$$\left|\partial_x^\alpha\partial_\eta^\beta(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\right|$$
$$\leq c\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta\\ (\alpha'',\beta'')\neq 0}}\Big(\left|\partial_x^{\alpha'}\partial_\eta^{\beta'}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\right|\times\left|\partial_x^{\alpha''}\partial_\eta^{\beta''}(i\psi(t,x,\eta)-i\langle x,\eta\rangle)\right|\Big),$$

$c>0$. By the induction assumption, we have for every compact set $K\subset\Omega$, there exists a constant $c>0$, such that

$$\left|\partial_x^{\alpha'}\partial_\eta^{\beta'}(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})\right| \tag{5.5}$$
$$\leq c(1+|\eta|)^{\frac{|\alpha'|-|\beta'|}{2}}e^{-\operatorname{Im}\psi(t,x,\eta)}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha'|+|\beta'|}{2}},$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $|\eta|\geq 1$. From Lemma 5.11, we have

$$\left|\partial_x^{\alpha''}\partial_\eta^{\beta''}(i\psi(t,x,\eta)-i\langle x,\eta\rangle)\right|\leq c(1+|\eta|)^{\frac{|\alpha''|-|\beta''|}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha''|+|\beta''|}{2}}, \tag{5.6}$$

where $x\in K$, $t\in\overline{\mathbb{R}}_+$, $|\eta|\geq 1$. Combining (5.5) with (5.6), we get (5.3).

From Leibniz's formula, Lemma 5.12 and (5.3), we get (5.4). □

LEMMA 5.14. – *Let* $b(t,x,\eta)\in\hat{S}_r^k\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$ *with* $r>0$. *We assume that* $b(t,x,\eta)=0$ *when* $|\eta|\leq 1$. *Then*

$$q(x,\eta)=\int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)dt$$

$\in S^{k-1}_{\frac12,\frac12}\big(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$.

Proof. – From Leibniz's formula, we have

$$\left|\partial_x^\alpha\partial_\eta^\beta(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta))\right|$$
$$\leq c\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}\left|(\partial_x^{\alpha'}\partial_\eta^{\beta'}e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle})(\partial_x^{\alpha''}\partial_\eta^{\eta''}b(t,x,\eta))\right|,\quad c>0.$$

From (5.3) and the definition of $\hat{S}^k_r$, we have for every compact set $K \subset \Omega$, there exist a constant $c > 0$ and $\varepsilon > 0$, such that

$$\begin{aligned}\left|\partial^\alpha_x\partial^\beta_\eta q(x,\eta)\right| &\leq c\int_0^\infty e^{-\operatorname{Im}\psi(t,x,\eta)}(1+|\eta|)^{k+\frac{|\alpha|-|\beta|}{2}}(1+\operatorname{Im}\psi(t,x,\eta))^{\frac{|\alpha|+|\beta|}{2}}e^{-\varepsilon t|\eta|}dt\\ &\leq c'(1+|\eta|)^{k-1+\frac{|\alpha|-|\beta|}{2}},\end{aligned}$$

where $c' > 0$, $x \in K$. The lemma follows. □

We will next show

Proposition 5.15. – *Let*

$$b(t,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$$

with $r > 0$. We assume that $b(t,x,\eta) = 0$ when $|\eta| \leq 1$. Let B be as in Proposition 5.6. Then $B \in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\, \Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))$ with symbol

$$q(x,\eta) = \int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)dt$$

$\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$.

Proof. – Choose $\chi \in C^\infty_0(\mathbb{R}^{2n-1})$ so that $\chi(\eta) = 1$ when $|\eta| < 1$ and $\chi(\eta) = 0$ when $|\eta| > 2$. Take a $u(y) \in C^\infty_0(\Omega;\, \Lambda^{0,q}T^*(\Omega))$, then

$$\begin{aligned}Bu &= \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int_0^\infty\Big(\int e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)u(y)\chi(\varepsilon\eta)d\eta\Big)dt\\ &= \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int_0^\infty e^{i\langle x-y,\eta\rangle}\Big(\int e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}b(t,x,\eta)u(y)\chi(\varepsilon\eta)\Big)d\eta dt\\ &= \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\eta\rangle}q(x,\eta)u(y)\chi(\varepsilon\eta)d\eta.\end{aligned}$$

From Lemma 5.14, we know that $q(x,\eta) \in S^{k-1}_{\frac{1}{2},\frac{1}{2}}$. Thus

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{2n-1}}\int e^{i\langle x-y,\eta\rangle}q(x,\eta)u(y)\chi(\varepsilon\eta)d\eta \in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega;\, \Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega)).$$

□

We need the following

Lemma 5.16. – *Let $a(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$ be a classical symbol of order k, that is*

$$a(\infty,x,\eta) \sim \sum_{j=0}^\infty a_j(\infty,x,\eta)$$

in the Hörmander symbol space $S^k_{1,0}\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$, where

$$a_j(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),\quad j = 0,1,\ldots,$$

$a_j(\infty, x, \lambda\eta) = \lambda^{k-j} a_j(\infty, x, \eta)$, $\lambda \geq 1$, $|\eta| \geq 1$, $j = 0, 1, \ldots$. *Assume that* $a(\infty, x, \eta) = 0$ *when* $|\eta| \leq 1$. *Then*

$$p(x,\eta) = \int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)} - e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}\Big) a(\infty,x,\eta)dt$$

$\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$.

Proof. – Note that

$$p(x,\eta) = \int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}(-t)i\psi'_t(t,x,\eta)a(\infty,x,\eta)dt.$$

From (5.4), we can repeat the procedure in the proof of Lemma 5.14 to get the lemma. □

Remark 5.17. – Let $a(t,x,\eta) \in \hat{S}^k_0\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. We assume $a(t,x,\eta) = 0$, if $|\eta| \leq 1$ and

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$$

with $r > 0$, where $a(\infty,x,\eta)$ is as in Lemma 5.16. By Lemma 5.14 and Lemma 5.16, we have

$$\begin{aligned}
&\int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)} a(\infty,x,\eta)\Big)dt \\
&= \int_0^\infty e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)}(a(t,x,\eta) - a(\infty,x,\eta))dt \\
&\quad + \int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)} - e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)}\Big) a(\infty,x,\eta)dt \\
&\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))).
\end{aligned}$$

Let

$$A(x,y) = \frac{1}{(2\pi)^{2n-1}} \iint_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta)\Big) dtd\eta$$

be as in the Remark 5.7. Then as in Proposition 5.15, we can show that $A \in L^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(\Omega; \Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)\big)$ with symbol

$$q(x,\eta) = \int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)} a(\infty,x,\eta)\Big)dt$$

$\in S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$.

We have the following

Proposition 5.18. – *Let* $a(\infty,x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$ *be a classical symbol of order* k. *Then*

$$a(x,\eta) = e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)} a(\infty,x,\eta) \in S^k_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big).$$

Proof. – In view of Lemma 5.13, we have for every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}^{2n-1}$, $\beta \in \mathbb{N}^{2n-1}$, there exists a constant $c_{\alpha,\beta,K} > 0$, such that

$$\left|\partial_x^\alpha \partial_\eta^\beta (e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)})\right| \leq c_{\alpha,\beta,K}(1+|\eta|)^{\frac{|\alpha|-|\beta|}{2}} e^{-\operatorname{Im}\psi(\infty,x,\eta)}(1+\operatorname{Im}\psi(\infty,x,\eta))^{\frac{|\alpha|+|\beta|}{2}},$$

where $x \in K$, $|\eta| \geq 1$. From this and Leibniz's formula, we get the proposition. □

CHAPTER 6

THE HEAT EQUATION

Until further notice, we work with local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. Let

$$b(t,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$$

with $r > 0$. We assume that $b(t,x,\eta) = 0$ when $|\eta| \leq 1$. From now on, we write $\frac{1}{(2\pi)^{2n-1}} \int \big(\int_0^\infty e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} b(t,x,\eta) dt\big) d\eta$ to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ with symbol

$$\int_0^\infty e^{i(\psi(t,x,\eta) - \langle x,\eta\rangle)} b(t,x,\eta) dt \in S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$$

(see Proposition 5.15).

Let $a(t,x,\eta) \in \hat{S}^k_0\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. Assume that

$$a(t,x,\eta) = 0$$

when $|\eta| \leq 1$ and that

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$$

with $r > 0$, where $a(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$ is a classical symbol of order k. From now on, we write

$$\frac{1}{(2\pi)^{2n-1}} \int \Big(\int_0^\infty \big(e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} a(\infty,x,\eta)\big) dt\Big) d\eta$$

to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Omega)$ to sections of $\Lambda^{0,q}T^*(\Omega)$ with symbol

$$\int_0^\infty \big(e^{i(\psi(t,x,\eta) - \langle x,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta) - \langle x,\eta\rangle)} a(\infty,x,\eta)\big) dt$$

in $S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega);\ \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$ (see Remark 5.17). From Proposition 4.5, we have the following

Proposition 6.1. – *Let Q be a C^∞ differential operator on Ω of order m. Let*

$$b(t,x,\eta)\in\hat{S}^k_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$$

with $r>0$. We assume that $b(t,x,\eta)=0$ when $|\eta|\leq 1$. Set

$$Q(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta))=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta),$$

$c(t,x,\eta)\in\hat{S}^{k+m}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$, $r>0$. Put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta,$$
$$C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)dt\Big)d\eta.$$

We have $Q\circ B\equiv C$.

Proposition 6.2. – *Let Q be a C^∞ differential operator on Ω of order m. Let*

$$b(t,x,\eta)\in\hat{S}^k_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big).$$

We assume that $b(t,x,\eta)=0$ when $|\eta|\leq 1$ and that

$$b(t,x,\eta)-b(\infty,x,\eta)\in\hat{S}^k_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$$

with $r>0$, where $b(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$ is a classical symbol of order k. Set

$$\begin{aligned}Q\big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\big)\\&=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta),\end{aligned}$$

where $c(t,x,\eta)\in\hat{S}^{k+m}_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$,

$$c(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$$

is a classical symbol of order $k+m$. Then

$$c(t,x,\eta)-c(\infty,x,\eta)\in\hat{S}^{k+m}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$$

with $r>0$. Put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\big)dt\Big)d\eta,$$
$$C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta)\big)dt\Big)d\eta.$$

We have $Q\circ B\equiv C$.

We return to our problem. From now on, we assume that our operators are properly supported. We assume that $Y(q)$ holds. Let

$$a_j(t,x,\eta)\in\hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big),\quad j=0,1,\ldots,$$

where $r > 0$, be as in Proposition 5.3 with $a_0(0,x,\eta) = I$, $a_j(0,x,\eta) = 0$ when $j > 0$. Let $a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta)$ in $\hat{S}^0_r(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega)))$, where $r > 0$. Let

$$(6.1) \qquad (\partial_t + \Box_b^{(q)})\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\big) = e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta).$$

From Proposition 4.5, we see that for every compact set $K \subset \Omega$, $\varepsilon > 0$ and all indices α, β and $N \in \mathbb{N}$, there exists $c_{\alpha,\beta,N,\varepsilon,K} > 0$ such that

$$(6.2) \qquad \left|\partial_x^\alpha \partial_\eta^\beta b(t,x,\eta)\right| \le c_{\alpha,\beta,N,\varepsilon,K} e^{t(-r(x,\eta)+\varepsilon|\eta|)}\big(|\eta|^{-N} + |\eta|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N\big),$$

where $t \in \overline{\mathbb{R}}_+$, $x \in K$, $|\eta| \ge 1$. Choose $\chi \in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta) = 1$ when $|\eta| < 1$ and $\chi(\eta) = 0$ when $|\eta| > 2$. Set

$$(6.3) \qquad A(x,y) = \frac{1}{(2\pi)^{2n-1}} \int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))dt\Big)d\eta.$$

We have the following proposition

Proposition 6.3. – *Suppose $Y(q)$ holds. Let A be as in* (6.3)*. We have $\Box_b^{(q)}A \equiv I$.*

Proof. – We have

$$\begin{aligned}
&\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))\Big)\\
&\quad= \frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))\\
&\qquad - \frac{1}{(2\pi)^{2n-1}}\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta)),
\end{aligned}$$

where $b(t,x,\eta)$ is as in (6.1), (6.2). From Proposition 6.1, we have

$$\begin{aligned}
\Box_b^{(q)} \circ A \equiv{}& \frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))dt\Big)d\eta\\
&- \frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty \frac{\partial}{\partial t}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta.
\end{aligned}$$

From (6.2), it follows that $\int\big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)(1-\chi(\eta))dt\big)d\eta$ is smoothing. Choose a cut-off function $\chi_1(\eta) \in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi_1(\eta) = 1$ when $|\eta| < 1$ and $\chi_1(\eta) = 0$ when $|\eta| > 2$. Take a $u(y) \in C_0^\infty(\Omega; \Lambda^{0,q}T^*(\Omega))$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint\Big(\int_0^\infty \frac{\partial}{\partial t}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle}a(t,x,\eta)\big)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)dt\Big)d\eta dy\\
&\quad= -\lim_{\varepsilon\to 0}\iint e^{i\langle x-y,\eta\rangle}(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy.
\end{aligned}$$

Hence $\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty \frac{\partial}{\partial t}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta \equiv -I$. Thus $\Box_b^{(q)} \circ A \equiv I$. $\Box$

REMARK 6.4. – We assume that $Y(q)$ holds. From Proposition 6.3, we know that, for every local coordinate patch X_j, there exists a properly supported operator $A_j : \mathscr{D}'(X_j;\, \Lambda^{0,q}T^*(X_j)) \to \mathscr{D}'(X_j;\, \Lambda^{0,q}T^*(X_j))$ such that

$$A_j : H^s_{\rm loc}(X_j;\, \Lambda^{0,q}T^*(X_j)) \to H^{s+1}_{\rm loc}(X_j;\, \Lambda^{0,q}T^*(X_j))$$

and $\Box^{(q)}_b \circ A_j - I : H^s_{\rm loc}(X_j;\, \Lambda^{0,q}T^*(X_j)) \to H^{s+m}_{\rm loc}(X_j;\, \Lambda^{0,q}T^*(X_j))$ for all $s \in \mathbb{R}$ and $m \geq 0$. We assume that $X = \bigcup_{j=1}^k X_j$. Let $\{\chi_j\}$ be a C^∞ partition of unity subordinate to $\{X_j\}$ and set $Au = \sum_j A_j(\chi_j u)$, $u \in \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X))$. A is well-defined as a continuous operator:

$$A : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+1}(X;\, \Lambda^{0,q}T^*(X))$$

for all $s \in \mathbb{R}$. We notice that A is properly supported. We have $\Box^{(q)}_b \circ A - I : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+m}(X;\, \Lambda^{0,q}T^*(X))$ for all $s \in \mathbb{R}$ and $m \geq 0$.

Assume that $Y(q)$ fails. Let

$$a_j(t,x,\eta) \in \hat{S}^{-j}_0\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots,$$

and $a_j(\infty,x,\eta) \in C^\infty\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$, $j = 0,1,\ldots$, be as in Proposition 5.4. We recall that for some $r > 0$,

$$a_j(t,x,\eta) - a_j(\infty,x,\eta) \in \hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big), \quad j = 0,1,\ldots.$$

Let

$$a(\infty,x,\eta) \sim \sum_{j=0}^\infty a_j(\infty,x,\eta) \tag{6.4}$$

in the Hörmander symbol space $S^0_{1,0}\big(T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. Let

$$a(t,x,\eta) \sim \sum_{j=0}^\infty a_j(t,x,\eta) \tag{6.5}$$

in $\hat{S}^0_0\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big)$. We take $a(t,x,\eta)$ so that for every compact set $K \subset \Omega$ and all indices α, β, γ, k, there exists $c > 0$, c is independent of t, such that

$$\left|\partial^\gamma_t \partial^\alpha_x \partial^\beta_\eta \Big(a(t,x,\eta) - \sum_{j=0}^k a_j(t,x,\eta)\Big)\right| \leq c(1+|\eta|)^{-k-1+\gamma-|\beta|}, \tag{6.6}$$

where $t \in \overline{\mathbb{R}}_+$, $x \in K$, $|\eta| \geq 1$, and

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}^0_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega);\, \mathscr{L}(\Lambda^{0,q}T^*(\Omega), \Lambda^{0,q}T^*(\Omega))\big) \text{ with } r > 0.$$

Let

$$(\partial_t + \Box^{(q)}_b)\big(e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta)\big) = e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} b(t,x,\eta). \tag{6.7}$$

Then $b(t,x,\eta)\in\hat S^2_0\big(\overline{\mathbb R}_+\times T^*(\Omega);\mathscr L(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)$ and

$$b(t,x,\eta)-b(\infty,x,\eta)\in\hat S^2_r\big(\overline{\mathbb R}_+\times T^*(\Omega);\mathscr L(\Lambda^{0,q}T^*(\Omega),\Lambda^{0,q}T^*(\Omega))\big)\text{ with } r>0,\tag{6.8}$$

where $b(\infty,x,\eta)$ is a classical symbol of order 2. Moreover, we have

$$\begin{aligned}&(\partial_t+\Box_b^{(q)})\Big(\frac{1}{(2\pi)^{2n-1}}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)\Big)\\&\quad=\frac{1}{(2\pi)^{2n-1}}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\big).\end{aligned}\tag{6.9}$$

From Proposition 4.5, we see that for every compact set $K\subset\Omega$ and all indices α, β and $N\in\mathbb N$, there exists $c_{\alpha,\beta,N,K}>0$ such that

$$\big|\partial_x^\alpha\partial_\eta^\beta b(t,x,\eta)\big|\le c_{\alpha,\beta,N,K}\big(|\eta|^{-N}+|\eta|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N\big),\tag{6.10}$$

where $t\in\overline{\mathbb R}_+$, $x\in K$, $|\eta|\ge1$. Thus,

$$\big|\partial_x^\alpha\partial_\eta^\beta b(\infty,x,\eta)\big|\le c_{\alpha,\beta,N,K}\big(|\eta|^{-N}+|\eta|^{2-N}(\operatorname{Im}\psi(\infty,x,\eta))^N\big).\tag{6.11}$$

From (6.8), (6.10) and (6.11), it follows that for every compact set $K\subset\Omega$, $\varepsilon>0$ and all indices α, β and $N\in\mathbb N$, there exists $c_{\alpha,\beta,N,\varepsilon,K}>0$ such that

$$\begin{aligned}&\big|\partial_x^\alpha\partial_\eta^\beta\big(b(t,x,\eta)-b(\infty,x,\eta)\big)\big|\\&\quad\le c_{\alpha,\beta,N,\varepsilon,K}\Big(e^{t(-r(x,\eta)+\varepsilon|\eta|)}\big(|\eta|^{-N}+|\eta|^{2-N}(\operatorname{Im}\psi(t,x,\eta))^N\big)\Big)^{\frac12},\end{aligned}\tag{6.12}$$

where $t\in\overline{\mathbb R}_+$, $x\in K$, $|\eta|\ge1$, $r>0$.

Choose $\chi\in C_0^\infty(\mathbb R^{2n-1})$ so that $\chi(\eta)=1$ when $|\eta|<1$ and $\chi(\eta)=0$ when $|\eta|>2$. Set

$$\begin{aligned}G(x,y)=\frac{1}{(2\pi)^{2n-1}}&\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\\&-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta.\end{aligned}\tag{6.13}$$

Put

$$S(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)d\eta.\tag{6.14}$$

We have the following

Proposition 6.5. – *We assume that $Y(q)$ fails. Let G and S be as in* (6.13) *and* (6.14) *respectively. Then $S+\Box_b^{(q)}\circ G\equiv I$ and $\Box_b^{(q)}\circ S\equiv0$.*

Proof. – We have

$$\begin{aligned}&\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)\\&\quad=\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\Big(b(t,x,\eta)-i\frac{\partial\psi}{\partial t}a-\frac{\partial a}{\partial t}\Big),\end{aligned}$$

where $b(t,x,\eta)$ is as in (6.7). Letting $t\to\infty$, we get

$$\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)=\frac{1}{(2\pi)^{2n-1}}e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta),$$

where $b(\infty,x,\eta)$ is as in (6.8) and (6.11). From (6.11), we have

$$\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)d\eta$$

is smoothing. Thus $\Box_b^{(q)}\circ S\equiv 0$.

In view of (6.9), we have

$$\begin{aligned}&\Box_b^{(q)}\Big(\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)(1-\chi(\eta))-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)(1-\chi(\eta))\Big)\Big)\\&=\frac{1}{(2\pi)^{2n-1}}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)(1-\chi(\eta))\\&\quad-\frac{1}{(2\pi)^{2n-1}}\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta)).\end{aligned}$$

From Proposition 6.2, we have

$$\begin{aligned}\Box_b^{(q)}\circ G=\Box_b^{(q)}\Bigg(\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\Big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\\&-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\Bigg)\\\equiv\frac{1}{(2\pi)^{2n-1}}\Bigg(\int\Big(\int_0^\infty\Big(&e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)\\&-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\\&-\int\Big(\int_0^\infty\frac{\partial}{\partial t}\Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\Big)(1-\chi(\eta))dt\Big)d\eta\Bigg).\end{aligned}$$

In view of (6.11) and (6.12), we see that

$$\begin{aligned}&\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta\\&=\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}\big)b(\infty,x,\eta)(1-\chi(\eta))dt\Big)d\eta\\&\quad+\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}\big(b(t,x,\eta)-b(\infty,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta\end{aligned}$$

is smoothing.

Choose a cut-off function $\chi_1(\eta) \in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi_1(\eta) = 1$ when $|\eta| < 1$ and $\chi_1(\eta) = 0$ when $|\eta| > 2$. Take a $u \in C_0^\infty(\Omega; \Lambda^{0,q}T^*(\Omega))$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int\Big(\int_0^\infty \frac{\partial}{\partial t}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\big)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)dt\Big)d\eta dy\\
&\quad=\lim_{\varepsilon\to 0}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy\\
&\qquad-\lim_{\varepsilon\to 0}\int e^{i\langle x-y,\eta\rangle}(1-\chi(\eta))\chi_1(\varepsilon\eta)u(y)d\eta dy.
\end{aligned}$$

Hence

$$\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty \frac{\partial}{\partial t}\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta \equiv S - I.$$

Thus $S + \Box_{b,q}\circ G \equiv I$. □

In the rest of this chapter, we recall some facts about Hilbert space theory for $\Box_b^{(q)}$. Our basic reference for these matters is [**1**]. Let A be as in Remark 6.4. A has a formal adjoint $A^* : \mathscr{D}'(X; \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X; \Lambda^{0,q}T^*(X))$, $(A^*u \mid v) = (u \mid Av)$, $u \in C^\infty(X; \Lambda^{0,q}T^*(X))$, $v \in C^\infty(X; \Lambda^{0,q}T^*(X))$.

Lemma 6.6. – *A^* is well-defined as a continuous operator:*

$$A^* : H^s(X; \Lambda^{0,q}T^*(X)) \to H^{s+1}(X; \Lambda^{0,q}T^*(X))$$

for all $s \in \mathbb{R}$. Moreover, we have $A^* \equiv A$.

Proof. – The first statement is a consequence of the theorem of Calderon and Vaillancourt (see Proposition 5.10). In view of Remark 6.4, we see that $\Box_b^{(q)}\circ A \equiv I$. Thus $A^*\circ\Box_b^{(q)} \equiv I$. We have

$$\begin{aligned}
A^* - A &\equiv A^*\circ(\Box_b^{(q)}\circ A) - A\\
&\equiv (A^*\circ\Box_b^{(q)})\circ A - A\\
&\equiv A - A\\
&\equiv 0.
\end{aligned}$$

The lemma follows. □

From this, we get a two-sided parametrix for $\Box_b^{(q)}$.

Proposition 6.7. – *We assume that $Y(q)$ holds. Let A be as in Remark 6.4. Then $\Box_b^{(q)}\circ A \equiv A\circ\Box_b^{(q)} \equiv I$.*

Proof. – In view of Lemma 6.6, we have $A^* \equiv A$. Thus $I \equiv \Box_b^{(q)}\circ A \equiv A^*\circ\Box_b^{(q)} \equiv A\circ\Box_b^{(q)}$. □

REMARK 6.8. – The existence of a two-sided parametrix for $\Box_b^{(q)}$ under condition $Y(q)$ is a classical result. See [**1**].

DEFINITION 6.9. – Suppose Q is a closed densely defined operator

$$Q : H \supset \operatorname{Dom} Q \to \operatorname{Ran} Q \subset H,$$

where H is a Hilbert space. Suppose that Q has closed range. By the partial inverse of Q, we mean the bounded operator $N : H \to H$ such that $Q \circ N = \pi_2$, $N \circ Q = \pi_1$ on $\operatorname{Dom} Q$, where π_1, π_2 are the orthogonal projections in H such that $\operatorname{Ran} \pi_1 = (\operatorname{Ker} Q)^\perp$, $\operatorname{Ran} \pi_2 = \operatorname{Ran} Q$. In other words, for $u \in H$, let $\pi_2 u = Qv$, $v \in (\operatorname{Ker} Q)^\perp \bigcap \operatorname{Dom} Q$. Then, $Nu = v$.

Set $\operatorname{Dom} \Box_b^{(q)} = \Big\{u \in L^2(X;\, \Lambda^{0,q}T^*(X));\, \Box_b^{(q)} u \in L^2(X;\, \Lambda^{0,q}T^*(X))\Big\}$.

LEMMA 6.10. – *We consider $\Box_b^{(q)}$ as an operator*

$$\Box_b^{(q)} : L^2(X;\, \Lambda^{0,q}T^*(X)) \supset \operatorname{Dom} \Box_b^{(q)} \to L^2(X;\, \Lambda^{0,q}T^*(X)).$$

If $Y(q)$ holds, then $\Box_b^{(q)}$ has closed range.

Proof. – Suppose $u_j \in \operatorname{Dom} \Box_b^{(q)}$ and $\Box_b^{(q)} u_j = v_j \to v$ in $L^2(X;\, \Lambda^{0,q}T^*(X))$. We have to show that there exists $u \in \operatorname{Dom} \Box_b^{(q)}$ such that $\Box_b^{(q)} u = v$. From Proposition 6.7, we have $\Box_b^{(q)} A = I - F_1$, $A\Box_b^{(q)} = I - F_2$, where F_j, $j = 1, 2$, are smoothing operators. Now, $A\Box_b^{(q)} u_j = (I - F_2)u_j \to Av$ in $L^2(X;\, \Lambda^{0,q}T^*(X))$. Since F_2 is compact, there exists a subsequence $u_{j_k} \to u$ in $L^2(X;\, \Lambda^{0,q}T^*(X))$, $k \to \infty$. We have $(I - F_2)u = Av$ and $\Box_b^{(q)} u_{j_k} \to \Box_b^{(q)} u$ in $H^{-2}(X;\, \Lambda^{0,q}T^*(X))$, $k \to \infty$. Thus $\Box_b^{(q)} u = v$. Now $v \in L^2(X;\, \Lambda^{0,q}T^*(X))$, so $u \in \operatorname{Dom} \Box_b^{(q)}$. We have proved the lemma. $\Box$

It follows that $\operatorname{Ran} \Box_b^{(q)} = (\operatorname{Ker} \Box_b^{(q)})^\perp$. Notice also that $\Box_b^{(q)}$ is self-adjoint. Now, we can prove the following classical result

PROPOSITION 6.11. – *We assume that $Y(q)$ holds. Then* $\dim \operatorname{Ker} \Box_b^{(q)} < \infty$ *and $\pi^{(q)}$ is a smoothing operator . Let N be the partial inverse. Then $N = A + F$ where A is as in Proposition 6.7 and F is a smoothing operator. Let N^* be the formal adjoint of N,*

$$(N^* u \mid v) = (u \mid Nv),\ u \in C^\infty(X;\, \Lambda^{0,q}T^*(X)),\ v \in C^\infty(X;\, \Lambda^{0,q}T^*(X)).$$

Then, $N^* = N$ on $L^2(X;\, \Lambda^{0,q}T^*(X))$.

Proof. – From Proposition 6.7, we have $A\Box_b^{(q)} = I - F_1$, $\Box_b^{(q)}A = I - F_2$, where F_1, F_2 are smoothing operators. Thus $\operatorname{Ker}\Box_b^{(q)} \subset \operatorname{Ker}(I - F_1)$. Since F_1 is compact, $\operatorname{Ker}(I - F_1)$ is finite dimensional and contained in $C^\infty(X;\, \Lambda^{0,q}T^*(X))$. Thus $\dim \operatorname{Ker}\Box_b^{(q)} < \infty$ and $\operatorname{Ker}\Box_b^{(q)} \subset C^\infty(X;\, \Lambda^{0,q}T^*(X))$.

Let $\{\phi_1, \phi_2, \ldots, \phi_m\}$ be an orthonormal basis for $\operatorname{Ker}\Box_b^{(q)}$. The projection $\pi^{(q)}$ is given by $\pi^{(q)}u = \sum_{j=1}^m (u \mid \phi_j)\phi_j$. Thus $\pi^{(q)}$ is a smoothing operator. Notice that $I - \pi^{(q)}$ is the orthogonal projection onto $\operatorname{Ran}\Box_b^{(q)}$ since $\Box_b^{(q)}$ is formally self-adjoint with closed range.

For $u \in C^\infty(X;\, \Lambda^{0,q}T^*(X))$, we have

$$\begin{aligned}(N - A)u &= (A\Box_b^{(q)} + F_1)Nu - Au\\ &= A(I - \pi^{(q)})u + F_1Nu - Au\\ &= -A\pi^{(q)}u + F_1N(\Box_b^{(q)}A + F_2)u\\ &= -A\pi^{(q)}u + F_1(I - \pi^{(q)})Au + F_1NF_2u.\end{aligned}$$

Here $-A\pi^{(q)}$, $F_1(1 - \pi^{(q)})A : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+m}(X;\, \Lambda^{0,q}T^*(X))$ $\forall s \in \mathbb{R}$ and $m \geq 0$, so $-A\pi^{(q)}$, $F_1(1 - \pi^{(q)})A$ are smoothing operators. Since

$$\begin{aligned}F_1NF_2 : \mathscr{E}'(X;\, \Lambda^{0,q}T^*(X)) &\to C^\infty(X;\, \Lambda^{0,q}T^*(X))\\ &\to L^2(X;\, \Lambda^{0,q}T^*(X)) \to C^\infty(X;\, \Lambda^{0,q}T^*(X)),\end{aligned}$$

we have that F_1NF_2 is a smoothing operator. Thus $N = A + F$, F is a smoothing operator.

Since $N\pi^{(q)} = \pi^{(q)}N = 0 = N^*\pi^{(q)} = \pi^{(q)}N^* = 0$, we have $N^* = (N\Box_b^{(q)} + \pi^{(q)})N^* = N\Box_b^{(q)}N^* = N$. The proposition follows. $\Box$

Now, we assume that $Y(q)$ fails but that $Y(q-1)$, $Y(q+1)$ hold. In view of Lemma 6.10, we see that $\Box_b^{(q-1)}$ and $\Box_b^{(q+1)}$ have closed range. We write $\overline{\partial}_b^{(q)}$ to denote the map: $\overline{\partial}_b : C^\infty(X;\, \Lambda^{0,q}T^*(X)) \to C^\infty(X;\, \Lambda^{0,q+1}T^*(X))$. Let $\overline{\partial}_b^{(q),*}$ denote the formal adjoint of $\overline{\partial}_b$. We have $\overline{\partial}_b^{(q),*} : C^\infty(X;\, \Lambda^{0,q+1}T^*(X)) \to C^\infty(X;\, \Lambda^{0,q}T^*(X))$. Let $N_b^{(q+1)}$ and $N_b^{(q-1)}$ be the partial inverses of $\Box_b^{(q+1)}$ and $\Box_b^{(q-1)}$ respectively. From Proposition 6.11, we have

$$(N_b^{(q+1)})^* = N_b^{(q+1)}, \quad (N_b^{(q-1)})^* = N_b^{(q-1)},$$

where $(N_b^{(q+1)})^*$ and $(N_b^{(q-1)})^*$ are the formal adjoints of $N_b^{(q+1)}$ and $N_b^{(q-1)}$ respectively. Put

$$M = \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*} \tag{6.15}$$

and

$$\pi = I - (\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}). \tag{6.16}$$

In view of Proposition 6.11, we see that M is well-defined as a continuous operator: $M : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^s(X;\, \Lambda^{0,q}T^*(X))$ and π is well-defined as a continuous operator: $\pi : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s-1}(X;\, \Lambda^{0,q}T^*(X))$, for all $s \in \mathbb{R}$.

Let π^* and M^* be the formal adjoints of π and M respectively. We have the following

LEMMA 6.12. – *If we consider π and M as operators:*

$$\pi, M : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)),$$

then

$$\pi^* = \pi,\ \ M^* = M, \tag{6.17}$$

$$\Box_b^{(q)}\pi = 0 = \pi\Box_b^{(q)}, \tag{6.18}$$

$$\pi + \Box_b^{(q)}M = I = \pi + M\Box_b^{(q)}, \tag{6.19}$$

$$\pi M = 0 = M\pi, \tag{6.20}$$

$$\pi^2 = \pi. \tag{6.21}$$

Proof. – From (6.15) and (6.16), we get (6.17).

For $u \in C^\infty(X;\, \Lambda^{0,q+1}T^*(X))$, we have

$$\begin{aligned}0 &= (\Box_b^{(q+1)}\pi_b^{(q+1)}u \mid \pi_b^{(q+1)}u)\\ &= (\overline{\partial}_b^{(q+1)}\pi_b^{(q+1)}u \mid \overline{\partial}_b^{(q+1)}\pi_b^{(q+1)}u) + (\overline{\partial}_b^{(q),*}\pi_b^{(q+1)}u \mid \overline{\partial}_b^{(q),*}\pi_b^{(q+1)}u).\end{aligned}$$

Thus,

$$\overline{\partial}_b^{(q+1)}\pi_b^{(q+1)} = 0,\quad \overline{\partial}_b^{(q),*}\pi_b^{(q+1)} = 0. \tag{6.22}$$

Hence, by taking the formal adjoints

$$\pi_b^{(q+1)}\overline{\partial}_b^{(q+1),*} = 0,\quad \pi_b^{(q+1)}\overline{\partial}_b^{(q)} = 0. \tag{6.23}$$

Similarly,

$$\overline{\partial}_b^{(q-1)}\pi_b^{(q-1)} = 0,\quad \pi_b^{(q-1)}\overline{\partial}_b^{(q-1),*} = 0. \tag{6.24}$$

Note that

$$\overline{\partial}_b^{(q),*}\Box_b^{(q+1)} = \Box_b^{(q)}\overline{\partial}_b^{(q),*},\quad \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)} = \Box_b^{(q)}\overline{\partial}_b^{(q-1)}. \tag{6.25}$$

Now,

$$\begin{aligned}&\Box_b^{(q)}(\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*})\\ &\quad= \overline{\partial}_b^{(q),*}\Box_b^{(q+1)}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &\quad= \overline{\partial}_b^{(q),*}(I - \pi_b^{(q+1)})\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(I - \pi_b^{(q-1)})\overline{\partial}_b^{(q-1),*}\\ &\quad= \Box_b^{(q)}.\end{aligned}$$

Here we used (6.22), (6.24) and (6.25). Hence, $\Box_b^{(q)}\pi = 0$. We have that $\pi\Box_b^{(q)} = (\Box_b^{(q)}\pi)^* = 0$, where $(\Box_b^{(q)}\pi)^*$ is the formal adjoint of $\Box_b^{(q)}\pi$. We get (6.18).

Now,

$$\begin{aligned}\Box_b^{(q)}M &= \overline{\partial}_b^{(q),*}\Box_b^{(q+1)}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}\Box_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*}\\ &= \overline{\partial}_b^{(q),*}(I-\pi_b^{(q+1)})N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(I-\pi_b^{(q-1)})N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &= \overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &= I-\pi.\end{aligned}$$

Here we used (6.22), (6.24) and (6.25). Thus, $\Box_b^{(q)}M + \pi = I$. We have $\pi + M\Box_b^{(q)} = (\Box_b^{(q)}M+\pi)^* = I$, where $(\Box_b^{(q)}M+\pi)^*$ is the formal adjoint of $\Box_b^{(q)}M+\pi$. We get (6.19).

Now,

$$\begin{aligned}M(I-\pi) &= M(\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*})\\ &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)}\overline{\partial}_b^{(q)}\\ &\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}.\end{aligned}$$

From (6.22), (6.23) and (6.25), we have

$$\begin{aligned}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)} &= (I-\pi_b^{(q+1)})\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)}\\ &= N_b^{(q+1)}\Box_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}N_b^{(q+1)}\\ &= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}\Box_b^{(q+1)}N_b^{(q+1)}\\ &= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}(I-\pi_b^{(q+1)})\\ &= N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}.\end{aligned}$$

Similarly, we have $\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}N_b^{(q-1)} = N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}$. Hence,

$$\begin{aligned}M(I-\pi) &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2N_b^{(q+1)}\overline{\partial}_b^{(q)}\overline{\partial}_b^{(q),*}\overline{\partial}_b^{(q)}\\ &\quad + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2N_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\overline{\partial}_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2N_b^{(q+1)}\Box_b^{(q+1)}\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2N_b^{(q-1)}\Box_b^{(q-1)}\overline{\partial}_b^{(q-1),*}\\ &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2(I-\pi_b^{(q+1)})\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2(I-\pi_b^{(q-1)})\overline{\partial}_b^{(q-1),*}\\ &= \overline{\partial}_b^{(q),*}(N_b^{(q+1)})^2\overline{\partial}_b^{(q)} + \overline{\partial}_b^{(q-1)}(N_b^{(q-1)})^2\overline{\partial}_b^{(q-1),*}\\ &= M.\end{aligned}$$

Here we used (6.23) and (6.24). Thus, $M\pi = 0$. We have $\pi M = (M\pi)^* = 0$, where $(M\pi)^*$ is the formal adjoint of $M\pi$. We get (6.20).

Finally, $\pi = (\Box_b^{(q)} M + \pi)\pi = \pi^2$. We get (6.21). The lemma follows. $\Box$

Lemma 6.13. – *If we restrict π to $L^2(X;\, \Lambda^{0,q}T^*(X))$, then $\pi = \pi^{(q)}$. That is, π is the orthogonal projection onto $\operatorname{Ker}\Box_b^{(q)}$. Thus, π is well-defined as a continuous operator: $\pi : L^2(X;\, \Lambda^{0,q}T^*(X)) \to L^2(X;\, \Lambda^{0,q}T^*(X))$.*

Proof. – From (6.18), we get $\operatorname{Ran}\pi \subset \operatorname{Ker}\Box_b^{(q)}$ in the space of distributions. From (6.19), we get $\pi u = u$, when $u \in \operatorname{Ker}\Box_b^{(q)}$, so $\operatorname{Ran}\pi = \operatorname{Ker}\Box_b^{(q)}$ and $\pi^2 = \pi = \pi^*\pi = \pi^*$. For $\varphi, \phi \in C^\infty(X;\, \Lambda^{0,q}T^*(X))$, we get $((1-\pi)\varphi \mid \pi\phi) = 0$ so $\operatorname{Ran}(I-\pi) \perp \operatorname{Ran}\pi$ and $\varphi = (I-\pi)\varphi + \pi\varphi$ is the orthogonal decomposition. It follows that π restricted to $L^2(X;\, \Lambda^{0,q}T^*(X))$ is the orthogonal projection onto $\operatorname{Ker}\Box_b^{(q)}$. $\Box$

Lemma 6.14. – *If we consider $\Box_b^{(q)}$ as an unbounded operator*

$$\Box_b^{(q)} : L^2(X;\, \Lambda^{0,q}T^*(X)) \supset \operatorname{Dom}\Box_b^{(q)} \to L^2(X;\, \Lambda^{0,q}T^*(X)),$$

then $\Box_b^{(q)}$ has closed range and $M : L^2(X;\, \Lambda^{0,q}T^*(X)) \to \operatorname{Dom}\Box_b^{(q)}$ is the partial inverse.

Proof. – From (6.19) and Lemma 6.13, we see that $M : L^2(X;\, \Lambda^{0,q}T^*(X)) \to \operatorname{Dom}\Box_b^{(q)}$ and $\operatorname{Ran}\Box_b^{(q)} \supset \operatorname{Ran}(I-\pi)$. If $\Box_b^{(q)}u = v$, $u, v \in L^2(X;\, \Lambda^{0,q}T^*(X))$, then $(I-\pi)v = (I-\pi)\Box_b^{(q)}u = v$ since $\pi\Box_b^{(q)} = \Box_b^{(q)}\pi = 0$. Hence $\operatorname{Ran}\Box_b^{(q)} \subset \operatorname{Ran}(I-\pi)$ so $\Box_b^{(q)}$ has closed range.

From (6.20), we know that $M\pi = \pi M = 0$. Thus, M is the partial inverse. $\Box$

From Lemma 6.13 and Lemma 6.14 we get the following classical result

Proposition 6.15. – *We assume that $Y(q)$ fails but that $Y(q-1)$ and $Y(q+1)$ hold. Then $\Box_b^{(q)}$ has closed range. Let M and π be as in (6.15) and (6.16) respectively. Then M is the partial inverse of $\Box_b^{(q)}$ and $\pi = \pi^{(q)}$.*

CHAPTER 7

THE SZEGÕ PROJECTION

In this chapter, we assume that $Y(q)$ fails. From Proposition 6.5, we know that, for every local coordinate patch X_j, there exist

$$G_j \in L^{-1}_{\frac{1}{2},\frac{1}{2}}\big(X_j;\, \Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j)\big), \quad S_j \in L^{0}_{\frac{1}{2},\frac{1}{2}}\big(X_j;\, \Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j)\big)$$

such that

$$\begin{cases} S_j + \Box_b^{(q)} G_j \equiv I \\ \Box_b^{(q)} S_j \equiv 0 \end{cases} \tag{7.1}$$

in the space $\mathscr{D}'\big(X_j \times X_j;\, \mathscr{L}(\Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j))\big)$. Furthermore, the distribution kernel K_{S_j} of S_j is of the form

$$K_{S_j}(x,y) = \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} a(\infty,x,\eta)d\eta, \tag{7.2}$$

where $\psi(\infty,x,\eta)$, $a(\infty,x,\eta)$ are as in Proposition 3.3 and (6.4). From now on, we assume that S_j and G_j are properly supported operators.

We assume that $X = \bigcup_{j=1}^{k} X_j$. Let χ_j be a C^∞ partition of unity subordinate to $\{X_j\}$. From (7.1), we have

$$\begin{cases} S_j\chi_j + \Box_b^{(q)} G_j\chi_j \equiv \chi_j \\ \Box_b^{(q)} S_j\chi_j \equiv 0 \end{cases} \tag{7.3}$$

in the space $\mathscr{D}'\big(X_j \times X_j;\, \mathscr{L}(\Lambda^{0,q}T^*(X_j), \Lambda^{0,q}T^*(X_j))\big)$. Thus,

$$\begin{cases} S + \Box_b^{(q)} G \equiv I \\ \Box_b^{(q)} S \equiv 0 \end{cases} \tag{7.4}$$

in $\mathscr{D}'\big(X \times X;\, \mathscr{L}(\Lambda^{0,q}T^*(X), \Lambda^{0,q}T^*(X))\big)$, where

$$S, G : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)),$$

$$\begin{cases} Su = \sum_{j=1}^{k} S_j(\chi_j u), & u \in \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)), \\ Gu = \sum_{j=1}^{k} G_j(\chi_j u), & u \in \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)). \end{cases} \tag{7.5}$$

From (7.4), we can repeat the method of Beals and Greiner (see pages 173–176 of [**1**]) with minor change to show that S is the Szegő projection (up to some smoothing operators) if $\Box_b^{(q)}$ has closed range.

Let S^*, $G^* : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X))$ be the formal adjoints of S and G respectively. As in Lemma 6.6, we see that S^* and G^* are well-defined as continuous operators

$$\begin{cases} S^* : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^s(X;\, \Lambda^{0,q}T^*(X)), \\ G^* : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+1}(X;\, \Lambda^{0,q}T^*(X)), \end{cases} \tag{7.6}$$

for all $s \in \mathbb{R}$. We have the following

Lemma 7.1. – *Let S be as in* (7.4) *and* (7.5). *We have $S \equiv S^*S$. It follows that $S \equiv S^*$ and $S^2 \equiv S$.*

Proof. – From (7.4), it follows that $S^* + G^*\Box_b^{(q)} \equiv I$. We have $S \equiv (S^* + G^*\Box_b^{(q)}) \circ S \equiv S^*S + G^*\Box_b^{(q)}S \equiv S^*S$. The lemma follows. □

Let

$$H = (I - S) \circ G. \tag{7.7}$$

H is well-defined as a continuous operator

$$H : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+1}(X;\, \Lambda^{0,q}T^*(X))$$

for all $s \in \mathbb{R}$. The formal adjoint H^* is well-defined as a continuous operator: $H^* : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+1}(X;\, \Lambda^{0,q}T^*(X))$, for all $s \in \mathbb{R}$.

Lemma 7.2. – *Let S and H be as in* (7.4), (7.5) *and* (7.7). *Then*

$$SH \equiv 0, \tag{7.8}$$

$$S + \Box_b^{(q)}H \equiv I. \tag{7.9}$$

Proof. – We have $SH \equiv S(I - S)G \equiv (S - S^2)G \equiv 0$ since $S^2 \equiv S$, where G is as in (7.4). From (7.4), it follows that $S + \Box_b^{(q)}H = S + \Box_b^{(q)}(I - S)G \equiv I - \Box_b^{(q)}SG \equiv I$. The lemma follows. □

Lemma 7.3. – *Let H be as in* (7.7). *Then $H \equiv H^*$.*

Proof. – Taking the adjoint in (7.9), we get $S^* + H^*\Box_b^{(q)} \equiv I$. Hence

$$H \equiv (S^* + H^*\Box_b^{(q)})H \equiv S^*H + H^*\Box_b^{(q)}H.$$

From Lemma 7.1, Lemma 7.2, we have $S^*H \equiv SH \equiv 0$. Hence $H \equiv H^*\Box_b^{(q)}H \equiv H^*$. ☐

Summing up, we get the following

PROPOSITION 7.4. – *We assume that $Y(q)$ fails. Let*

$$\begin{cases} S : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \\ H : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \end{cases}$$

be as in (7.5) *and* (7.7). *Then, S and H are well-defined as continuous operators*

$$S : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^s(X;\, \Lambda^{0,q}T^*(X)), \tag{7.10}$$

$$H : H^s(X;\, \Lambda^{0,q}T^*(X)) \to H^{s+1}(X;\, \Lambda^{0,q}T^*(X)), \tag{7.11}$$

for all $s \in \mathbb{R}$. Moreover, we have

$$H\Box_b^{(q)} + S \equiv S + \Box_b^{(q)}H \equiv I, \tag{7.12}$$

$$\Box_b^{(q)}S \equiv S\Box_b^{(q)} \equiv 0, \tag{7.13}$$

$$S \equiv S^* \equiv S^2, \tag{7.14}$$

$$SH \equiv HS \equiv 0, \tag{7.15}$$

$$H \equiv H^*. \tag{7.16}$$

REMARK 7.5. – If S', $H' : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) \to \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X))$ satisfy (7.10)–(7.16), then $S' \equiv (H\Box_b^{(q)} + S)S' \equiv SS' \equiv S(\Box_b^{(q)}H' + S') \equiv S$ and

$$H' \equiv (H\Box_b^{(q)} + S)H' \equiv (H\Box_b^{(q)} + S')H' \equiv H\Box_b^{(q)}H' \equiv H(\Box_b^{(q)}H' + S') \equiv H.$$

Thus, (7.10)–(7.16) determine S and H uniquely up to smoothing operators.

Now we can prove the following

PROPOSITION 7.6. – *We assume that $Y(q)$ fails. Suppose $\Box_b^{(q)}$ has closed range. Let N be the partial inverse of $\Box_b^{(q)}$. Then $N = H + F$ and $\pi^{(q)} = S + K$, where H, S are as in Proposition 7.4, F, K are smoothing operators.*

Proof. – We may replace S by $I - \Box_b^{(q)}H$ and we have $\Box_b^{(q)}H + S = I = H^*\Box_b^{(q)} + S^*$. Now,

$$\pi^{(q)} = \pi^{(q)}(\Box_b^{(q)}H + S) = \pi^{(q)}S, \tag{7.17}$$

hence

$$(\pi^{(q)})^* = S^*(\pi^{(q)})^* = \pi^{(q)} = S^*\pi^{(q)}. \tag{7.18}$$

Similarly,

$$S = (N\Box_b^{(q)} + \pi^{(q)})S = \pi^{(q)}S + NF_1, \tag{7.19}$$

where F_1 is a smoothing operator. From (7.17) and (7.19), we have

$$S - \pi^{(q)} = S - \pi^{(q)}S = NF_1. \tag{7.20}$$

Hence $(S^* - \pi^{(q)})(S - \pi^{(q)}) = F_1^* N^2 F_1$. On the other hand,

$$\begin{aligned}(S^* - \pi^{(q)})(S - \pi^{(q)}) &= S^*S - S^*\pi^{(q)} - \pi^{(q)}S + (\pi^{(q)})^2 \\ &= S^*S - \pi^{(q)} \\ &= S - \pi^{(q)} + F_2,\end{aligned}$$

where F_2 is a smoothing operator. Here we used (7.17) and (7.18). Now,

$$\begin{aligned}F_1^* N^2 F_1 : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) &\to C^\infty(X;\, \Lambda^{0,q}T^*(X)) \\ &\to L^2(X;\, \Lambda^{0,q}T^*(X)) \to C^\infty(X;\, \Lambda^{0,q}T^*(X)).\end{aligned}$$

Hence $F_1^* N^2 F_1$ is smoothing. Thus $S - \pi^{(q)}$ is smoothing.

We have,

$$\begin{aligned}N - H &= N(\Box_b^{(q)} H + S) - H \\ &= (I - \pi^{(q)})H + NS - H \\ &= NS - \pi^{(q)}H \\ &= N(S - \pi^{(q)}) + F_3 \\ &= NF_4 + F_3\end{aligned}$$

where F_4 and F_3 are smoothing operators. Now,

$$\begin{aligned}N - H^* &= N^* - H^* \\ &= F_4^* N + F_3^* \\ &= F_4^*(NF_4 + F_3 + H) + F_3^*.\end{aligned}$$

Note that

$$\begin{aligned}F_4^* N F_4 : \mathscr{D}'(X;\, \Lambda^{0,q}T^*(X)) &\to C^\infty(X;\, \Lambda^{0,q}T^*(X)) \\ &\to L^2(X;\, \Lambda^{0,q}T^*(X)) \to C^\infty(X;\, \Lambda^{0,q}T^*(X)).\end{aligned}$$

and $F_4^* H : H^s(X\ ; \Lambda^{0,q}T^*(X)) \to H^{s+m}(X\ ; \Lambda^{0,q}T^*(X))$ for all $s \in \mathbb{R}$ and $m \geq 0$. Hence $N - H^*$ is smoothing and so is $(N - H^*)^* = N - H$. $\Box$

From Proposition 7.4 and Proposition 7.6, we obtain the following

THEOREM 7.7. – *We recall that we work with the assumption that $Y(q)$ fails. Let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of the Levi form L. Suppose $\Box_b^{(q)}$ has closed range. Then for every local coordinate patch $U \subset X$, the distribution kernel of $\pi^{(q)}$ on $U \times U$ is of the form*

$$K_{\pi^{(q)}}(x,y) \equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta)d\eta, \tag{7.21}$$

$a(\infty,x,\eta) \in S^0_{1,0}\big(T^*(U)\;;\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))\big)$,

$$a(\infty,x,\eta) \sim \sum_0^\infty a_j(\infty,x,\eta)$$

in Hörmander symbol space $S^0_{1,0}\big(T^*(U);\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))\big)$, where $a_j(\infty,x,\eta) \in C^\infty\big(T^*(U);\mathscr{L}(\Lambda^{0,q}T^*(U),\Lambda^{0,q}T^*(U))\big)$, $j=0,1,\ldots$, $a_j(\infty,x,\lambda\eta) = \lambda^{-j}a_j(\infty,x,\eta)$, $\lambda \geq 1$, $|\eta| \geq 1$, $j=0,1,\ldots$. Here $\psi(\infty,x,\eta)$ is as in Proposition 3.3 and (3.8). *We recall that $\psi(\infty,x,\eta) \in C^\infty(T^*(U))$, $\psi(\infty,x,\lambda\eta) = \lambda\psi(\infty,x,\eta)$, $\lambda > 0$,*

$$\operatorname{Im}\psi(\infty,x,\eta) \asymp |\eta| \left(\operatorname{dist}\left((x,\frac{\eta}{|\eta|}),\Sigma\right)\right)^2$$

and

$$\psi(\infty,x,\eta) = -\overline{\psi}(\infty,x,-\eta). \tag{7.22}$$

Moreover, for all $j=0,1,\ldots$,

$$\begin{cases} a_j(\infty,x,\eta)=0 \text{ in a conic neighborhood of } \Sigma^+, \text{ if } q=n_-,\ n_-\neq n_+, \\ a_j(\infty,x,\eta)=0 \text{ in a conic neighborhood of } \Sigma^-, \text{ if } q=n_+,\ n_-\neq n_+. \end{cases} \tag{7.23}$$

In the rest of this chapter, we will study the singularities of the distribution kernel of the Szegő projection. We need

DEFINITION 7.8. – Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. Let U be an open set in M, $U \bigcap \Lambda \neq \varnothing$. We let $C^\infty_\Lambda(U)$ denote the set of equivalence classes of $f \in C^\infty(U)$ under the equivalence relation

$$f \equiv g \text{ in the space } C^\infty_\Lambda(U)$$

if for every $z_0 \in \Lambda \bigcap U$, there exists a neighborhood $W \subset U$ of z_0 such that $f = g + h$ on W, where $h \in C^\infty(W)$ and h vanishes to infinite order on $\Lambda \bigcap W$.

In view of Proposition 3.3, we see that $\psi(\infty,x,\eta)$ has a uniquely determined Taylor expansion at each point of Σ. Thus, we can define $\psi(\infty,x,\eta)$ as an element in $C^\infty_\Sigma(T^*(X))$. We also write $\psi(\infty,x,\eta)$ for the equivalence class of $\psi(\infty,x,\eta)$ in the space $C^\infty_\Sigma(T^*(X))$.

Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. If $x_0 \in \Lambda$, we let $A(\Lambda, n, x_0)$ be the set

$$\begin{aligned} A(\Lambda, n, x_0) = \Big\{ &(U, f_1, \ldots, f_n);\ U \text{ is an open neighborhood of } x_0, \\ &f_1, \ldots, f_n \in C^\infty_\Lambda(U), f_j|_{\Lambda \bigcap U} = 0,\ j = 1, \ldots, n, df_1, \ldots, df_n \\ &\text{are linearly independent over } \mathbb{C} \text{ at each point of } U \Big\}. \end{aligned} \tag{7.24}$$

Definition 7.9. – If $x_0 \in \Lambda$, we let $A_{x_0}(\Lambda, n, x_0)$ denote the set of equivalence classes of $A(\Lambda, n, x_0)$ under the equivalence relation

$$\Gamma_1 = (U, f_1, \ldots, f_n) \sim \Gamma_2 = (V, g_1, \ldots, g_n), \quad \Gamma_1, \Gamma_2 \in A(\Lambda, n, x_0),$$

if there exists an open set $W \subset U \bigcap V$ of x_0 such that $g_j \equiv \sum_{k=1}^n a_{j,k} f_k$ in the space $C^\infty_\Lambda(W)$, $j = 1, \ldots, n$, where $a_{j,k} \in C^\infty_\Lambda(W)$, j, $k = 1, \ldots, n$, and $(a_{j,k})_{j,k=1}^n$ is invertible.

If $(U, f_1, \ldots, f_n) \in A(\Lambda, n, x_0)$, we write $(U, f_1, \ldots, f_n)_{x_0}$ for the equivalence class of $(U, f_1, \ldots, f_n)$ in the set $A_{x_0}(\Lambda, n, x_0)$, which is called the germ of $(U, f_1, \ldots, f_n)$ at x_0.

Definition 7.10. – Let M be a real paracompact C^∞ manifold and let Λ be a C^∞ closed submanifold of M. A formal manifold Ω of codimensin k at Λ associated to M is given by:

> For each point of $x \in \Lambda$, we assign a germ $\Gamma_x \in A_x(\Lambda, k, x)$ in such a way that for every point $x_0 \in \Lambda$ has an open neighborhood U such that there exist $f_1, \ldots, f_k \in C^\infty_\Lambda(U)$, $f_j|_{\Lambda \bigcap U} = 0$, $j = 1, \ldots, k$, $df_1, \ldots, df_k$ are linearly independent over $\mathbb{C}$ at each point of U, having the following property: whatever $x \in U$, the germ $(U, f_1, \ldots, f_k)_x$ is equal to Γ_x.

Formally, we write $\Omega = \{\Gamma_x;\ x \in \Lambda\}$. If the codimension of Ω is 1, we call Ω a formal hypersurface at Λ.

Let $\Omega = \{\Gamma_x;\ x \in \Lambda\}$ and $\Omega_1 = \{\widetilde{\Gamma}_x;\ x \in \Lambda\}$ be two formal manifolds at Λ. If $\Gamma_x = \widetilde{\Gamma}_x$, for all $x \in \Lambda$, we write $\Omega = \Omega_1$ at Λ.

Definition 7.11. – Let $\Omega = \{\Gamma_x;\ x \in \Lambda\}$ be a formal manifold of codimensin k at Λ associated to M, where Λ and M are as above. The tangent space of Ω at $x_0 \in \Lambda$ is given by:

$$\text{the tangent space of } \Omega \text{ at } x_0 = \{u \in \mathbb{C}T_{x_0}(M);\ \langle df_j(x_0), u \rangle = 0, j = 1, \ldots, k\},$$

where $\mathbb{C}T_{x_0}(M)$ is the complexified tangent space of M at x_0, $(U, f_1, \ldots, f_k)$ is a representative of Γ_{x_0}. We write $T_{x_0}(\Omega)$ to denote the tangent space of Ω at x_0.

Let (x, y) be some coordinates of $X \times X$. From now on, we use the notations ξ and η for the dual variables of x and y respectively.

REMARK 7.12. – For each point $(x_0,\eta_0,x_0,\eta_0)\in\operatorname{diag}(\Sigma\times\Sigma)$, we assign a germ

$$\Gamma_{(x_0,\eta_0,x_0,\eta_0)}=\big(T^*(X)\times T^*(X),\xi-\psi'_x(\infty,x,\eta),y-\psi'_\eta(\infty,x,\eta)\big)_{(x_0,\eta_0,x_0,\eta_0)}. \tag{7.25}$$

Let C_∞ be the formal manifold at $\operatorname{diag}(\Sigma\times\Sigma)$:

$$C_\infty=\left\{\Gamma_{(x,\eta,x,\eta)};\ (x,\eta,x,\eta)\in\operatorname{diag}(\Sigma\times\Sigma)\right\}.$$

C_∞ is strictly positive in the sense that

$$\frac{1}{i}\sigma(v,\overline{v})>0,\quad\forall v\in T_\rho(C_\infty)\setminus\mathbb{C}T_\rho(\operatorname{diag}(\Sigma\times\Sigma)),$$

where $\rho\in\operatorname{diag}(\Sigma\times\Sigma)$. Here σ is the canonical two form on $\mathbb{C}T^*_\rho(X)\times\mathbb{C}T^*_\rho(X)$ (see (2.11)).

The following is well-known (see Chapter 1 of **[29]** for the proof).

PROPOSITION 7.13. – *There exists a formal manifold $J_+=\left\{J_{(x,\eta)};\ (x,\eta)\in\Sigma\right\}$ at Σ associated to $T^*(X)$ such that* $\operatorname{codim}J_+=n-1$ *and* $\forall(x_0,\eta_0)\in\Sigma$, *if* $(U,f_1,\ldots,f_{n-1})$ *is a representative of* $J_{(x_0,\eta_0)}$, *then*

$$\{f_j,f_k\}\equiv 0 \text{ in the space } C^\infty_\Sigma(U),\quad j,k=1,\ldots,n-1, \tag{7.26}$$

$$p_0\equiv\sum_{j=1}^{n-1}g_jf_j \text{ in the space } C^\infty_\Sigma(U), \tag{7.27}$$

where $g_j\in C^\infty_\Sigma(U)$, $j=1,\ldots,n-1$, *and*

$$\frac{1}{i}\sigma(H_{f_j},H_{\overline{f}_j})>0 \text{ at } (x_0,\eta_0)\in\Sigma,\quad j=1,\ldots,n-1. \tag{7.28}$$

We also write f_j to denote an almost analytic extension of f_j. Then,

$$f_j(x,\psi'_x(\infty,x,\eta)) \text{ vanishes to infinite order on } \Sigma,\quad j=1,\ldots,n-1. \tag{7.29}$$

Moreover, we have

$$\begin{aligned}T_\rho(C_\infty)=\Big\{\Big(v+\sum_{j=1}^{n-1}t_jH_{f_j}(x_0,\eta_0),v+\sum_{j=1}^{n-1}s_jH_{\overline{f}_j}(x_0,\eta_0)\Big);\\ v\in T_{(x_0,\eta_0)}(\Sigma),\ t_j,s_j\in\mathbb{C},j=1,\ldots,n-1\Big\},\end{aligned} \tag{7.30}$$

where $\rho=(x_0,\eta_0,x_0,\eta_0)\in\operatorname{diag}(\Sigma\times\Sigma)$ *and* C_∞ *is as in Remark 7.12.*

We return to our problem. We need the following

LEMMA 7.14. – *We have*

$$\psi''_{\eta\eta}(\infty, p, \omega_0(p))\omega_0(p) = 0 \tag{7.31}$$

and

$$\operatorname{Rank}\Big(\psi''_{\eta\eta}(\infty, p, \omega_0(p))\Big) = 2n - 2, \tag{7.32}$$

for all $p \in X$.

Proof. – Since $\psi'_\eta(\infty, x, \eta)$ is positively homogeneous of degree 0, it follows that $\psi''_{\eta\eta}(\infty, p, \omega_0(p))\omega_0(p) = 0$. We conclude that $\operatorname{Rank}\big(\psi''_{\eta\eta}(\infty, p, \omega_0(p))\big) \leq 2n-2$. From $\operatorname{Im}\psi(\infty, x, \eta) \asymp |\eta| \operatorname{dist}\big((x, \frac{\eta}{|\eta|});\ \Sigma\big)^2$, we have $\operatorname{Im}\psi''_{\eta\eta}(\infty, p, \omega_0(p))V \neq 0$, if $V \notin \{\lambda\omega_0(p);\ \lambda \in \mathbb{C}\}$. Thus, for all $V \notin \{\lambda\omega_0(p);\ \lambda \in \mathbb{C}\}$, we have

$$\begin{aligned}\big\langle \psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\big\rangle &= \big\langle \operatorname{Re}\psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\big\rangle + i\,\big\langle \operatorname{Im}\psi''_{\eta\eta}(\infty, p, \omega_0(p))V, \overline{V}\big\rangle \\ &\neq 0.\end{aligned}$$

We get (7.32). □

Until further notice, we assume that $q = n_+$. For $p \in X$, we take local coordinates $x = (x_1, x_2, \ldots, x_{2n-1})$ defined on some neighborhood Ω of $p \in X$ such that

$$\omega_0(p) = dx_{2n-1},\ \ x(p) = 0 \tag{7.33}$$

and $\Lambda^{0,1}T_p(X) \oplus \Lambda^{1,0}T_p(X) = \Big\{\sum_{j=1}^{2n-2} a_j \frac{\partial}{\partial x_j};\ a_j \in \mathbb{C},\ \ j = 1, \ldots, 2n-2\Big\}$. We take Ω so that if $x_0 \in \Omega$ then $\eta_{0,2n-1} > 0$ where $\omega_0(x_0) = (\eta_{0,1}, \ldots, \eta_{0,2n-1})$.

Until further notice, we work in Ω and we work with the local coordinates x. Choose $\chi(x, \eta) \in C^\infty(T^*(X))$ so that $\chi(x, \eta) = 1$ in a conic neighborhood of $(p, \omega_0(p))$, $\chi(x, \eta) = 0$ outside $T^*(\Omega)$, $\chi(x, \eta) = 0$ in a conic neighborhood of Σ^- and $\chi(x, \lambda\eta) = \chi(x, \eta)$ when $\lambda > 0$. We introduce the cut-off functions $\chi(x, \eta)$ and $(1 - \chi(x, \eta))$ in the integral (7.21): $K_{\pi^{(q)}}(x, y) \equiv K_{\pi_+^{(q)}}(x, y) + K_{\pi_-^{(q)}}(x, y)$,

$$\begin{aligned} K_{\pi_+^{(q)}}(x, y) &\equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} \chi(x, \eta) a(\infty, x, \eta) d\eta, \\ K_{\pi_-^{(q)}}(x, y) &\equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,\eta) - \langle y,\eta\rangle)} \big(1 - \chi(x, \eta)\big) a(\infty, x, \eta) d\eta. \end{aligned} \tag{7.34}$$

Now, we study $K_{\pi_+^{(q)}}$. We write t to denote η_{2n-1}. Put $\eta' = (\eta_1, \ldots, \eta_{2n-2})$. We have

$$\begin{aligned} K_{\pi_+^{(q)}} &\equiv \frac{1}{(2\pi)^{2n-1}} \int e^{i(\psi(\infty,x,(\eta',t)) - \langle y,(\eta',t)\rangle)} \chi\big(x, (\eta', t)\big) a\big(\infty, x, (\eta', t)\big) d\eta' dt \\ &= \frac{1}{(2\pi)^{2n-1}} \int_0^\infty \Big(\int e^{it(\psi(\infty,x,(w,1)) - \langle y,(w,1)\rangle)} t^{2n-2} \chi\big(x, (tw, t)\big) \\ &\quad \times a\big(\infty, x, (tw, t)\big) dw \Big) dt, \end{aligned} \tag{7.35}$$

where $\eta' = tw$, $w \in \mathbb{R}^{2n-2}$. The stationary phase method of Melin and Sjöstrand (see page 148 of[**28**]) then permits us to carry out the w integration in (7.35), to get

$$K_{\pi_+^{(q)}}(x,y) \equiv \int_0^\infty e^{it\phi_+(x,y)} s_+(x,y,t)dt \tag{7.36}$$

with

$$s_+(x,y,t) \sim \sum_{j=0}^\infty s_+^j(x,y) t^{n-1-j} \tag{7.37}$$

in $S^{n-1}_{1,0}\big(\Omega\times\Omega\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))\big)$, where

$$s_+^j(x,y) \in C^\infty\big(\Omega\times\Omega;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X),\Lambda^{0,q}T^*_x(X))\big),\quad j=0,1,\ldots,$$

and $\phi_+(x,y) \in C^\infty(\Omega\times\Omega)$ is the corresponding critical value. For $x\in\Omega$, let $\sigma(x) \in \mathbb{R}^{2n-2}$ be the vector:

$$(x,(\sigma(x),1)) \in \Sigma^+. \tag{7.38}$$

Since $d_w(\psi(\infty,x,(w,1)) - \langle y,(w,1)\rangle) = 0$ at $x=y$, $w=\sigma(x)$, it follows that when $x=y$, the corresponding critical point is $w=\sigma(x)$ and consequently

$$\phi_+(x,x) = 0, \tag{7.39}$$

$$(\phi_+)'_x(x,x) = \psi'_x(\infty,x,(\sigma(x),1)) = (\sigma(x),1),\quad (\phi_+)'_y(x,x) = -(\sigma(x),1). \tag{7.40}$$

The following is essentially well-known (see page 147 of [**28**] or Proposition B.14 of paper I in [**20**]).

Proposition 7.15. – *In some open neighborhood Q of p in Ω, we have*

(7.41)
$$\operatorname{Im}\phi_+(x,y) \geq c \inf_{w\in W}\Big(\operatorname{Im}\psi(\infty,x,(w,1)) + \left|d_w(\psi(\infty,x,(w,1)) - \langle y,(w,1)\rangle)\right|^2\Big),$$
$$(x,y)\in Q\times Q,$$

where c is a positive constant and W is some open set of the origin in $\mathbb{R}^{2n-2}$.

We have the following

Proposition 7.16. – *In some open neighborhood Q of p in Ω, there is a constant $c>0$ such that*

$$\operatorname{Im}\phi_+(x,y) \geq c\left|x'-y'\right|^2,\quad (x,y)\in Q\times Q, \tag{7.42}$$

where $x' = (x_1,\ldots,x_{2n-2})$, $y' = (y_1,\ldots,y_{2n-2})$ and $\left|x'-y'\right|^2 = (x_1-y_1)^2+\cdots+(x_{2n-2}-y_{2n-2})^2$.

Proof. – From $\psi(\infty, x, (w,1)) - \langle y, (w,1)\rangle = \langle x-y, (w,1)\rangle + O(|w-\sigma(x)|^2)$ we can check that

$$d_w\big(\psi(\infty, x, (w,1)) - \langle y, (w,1)\rangle\big) = \langle x'-y', dw\rangle + O(|w-\sigma(x)|),$$

where $\sigma(x)$ is as in (7.38) and $x' = (x_1, \ldots, x_{2n-2})$, $y' = (y_1, \ldots, y_{2n-2})$. Thus, there are constants c_1, $c_2 > 0$ such that

$$\left|d_w\big(\psi(\infty, x, (w,1)) - \langle y, (w,1)\rangle\big)\right|^2 \geq c_1 \left|x'-y'\right|^2 - c_2 \left|w-\sigma(x)\right|^2$$

for (x, w) in some compact set of $\Omega \times \dot{\mathbb{R}}^{2n-2}$. If $\frac{c_1}{2}\left|(x'-y')\right|^2 \geq c_2 \left|w-\sigma(x)\right|^2$, then

$$\left|d_\omega\big(\psi(\infty, x, \omega) - \langle y, \omega\rangle\big)\right|^2 \geq \frac{c_1}{2}\left|(x'-y')\right|^2. \tag{7.43}$$

Now, we assume that $\left|(x'-y')\right|^2 \leq \frac{2c_2}{c_1}\left|w-\sigma(x)\right|^2$. We have

$$\operatorname{Im}\psi(\infty, x, (w,1)) \geq c_3\left|w-\sigma(x)\right|^2 \geq \frac{c_1c_3}{2c_2}\left|(x'-y')\right|^2, \tag{7.44}$$

for (x, w) in some compact set of $\Omega \times \dot{\mathbb{R}}^{2n-2}$, where c_3 is a positive constant. From (7.43), (7.44) and Proposition 7.15, we have $\operatorname{Im}\phi_+(x,y) \geq c\left|(x'-y')\right|^2$ for x, y in some neighborhood of p, where c is a positive constant. We get the proposition. □

REMARK 7.17. – For each point $(x_0, x_0) \in \operatorname{diag}(\Omega \times \Omega)$, we assign a germ

$$H_{+,(x_0,x_0)} = (\Omega \times \Omega, \phi_+(x,y))_{(x_0,x_0)}.$$

Let H_+ be the formal hypersurface at $\operatorname{diag}(\Omega \times \Omega)$:

$$H_+ = \left\{H_{+,(x,x)};\ (x,x) \in \operatorname{diag}(\Omega \times \Omega)\right\}. \tag{7.45}$$

The formal conic conormal bundle $\Lambda_{\phi_+ t}$ of H_+ is given by: For each point

$$(x_0, \eta_0, x_0, \eta_0) \in \operatorname{diag}\big((\Sigma^+ \textstyle\bigcap T^*(\Omega)) \times (\Sigma^+ \textstyle\bigcap T^*(\Omega))\big),$$

we assign a germ

$$\begin{aligned}\Lambda_{(x_0,\eta_0,x_0,\eta_0)} = \Big(&T^*(U) \times T^*(U), \xi_j - (\phi_+)'_{x_j} t, \quad j = 1, \ldots, 2n-1,\\ &\eta_k - (\phi_+)'_{y_k} t, \quad k = 1, \ldots, 2n-2, \quad \phi_+(x,y)\Big)_{(x_0,\eta_0,x_0,\eta_0)},\end{aligned}$$

where $t = \frac{\eta_{2n-1}}{(\phi_+)'_{y_{2n-1}}}$ and $U \subset \Omega$ is an open set of x_0 such that $(\phi_+)'_{y_{2n-1}} \neq 0$ on $U \times U$. Then,

$$\Lambda_{\phi_+ t} = \left\{\Lambda_{(x,\eta,x,\eta)};\ (x,\eta,x,\eta) \in \operatorname{diag}\big((\Sigma^+ \textstyle\bigcap T^*(\Omega)) \times (\Sigma^+ \textstyle\bigcap T^*(\Omega))\big)\right\}. \tag{7.46}$$

$\Lambda_{\phi_+ t}$ is a formal manifold at $\operatorname{diag}\big((\Sigma^+ \bigcap T^*(\Omega)) \times (\Sigma^+ \bigcap T^*(\Omega))\big)$. In fact, $\Lambda_{\phi_+ t}$ is the positive Lagrangean manifold associated to $\phi_+ t$ in the sense of Melin and Sjöstrand (see Chapter 3 of [**28**]).

Let

$$\big(W, f_1(x,\xi,y,\eta), \ldots, f_{4n-2}(x,\xi,y,\eta)\big)$$

be a representative of $\Gamma_{(x_0,\eta_0,x_0,\eta_0)}$, where $\Gamma_{(x_0,\eta_0,x_0,\eta_0)}$ is as in (7.25). Put

$$\Gamma'_{(x_0,\eta_0,x_0,\eta_0)} = \big(W, f_1(x,\xi,y,-\eta),\ldots,f_{4n-2}(x,\xi,y,-\eta)\big)_{(x_0,\eta_0,x_0,\eta_0)}.$$

Let C'_∞ be the formal manifold at $\operatorname{diag}(\Sigma^+\times\Sigma^+)$:

$$C'_\infty = \Big\{\Gamma'_{(x,\eta,x,\eta)};\ (x,\eta,x,\eta)\in\operatorname{diag}(\Sigma^+\times\Sigma^+)\Big\}. \tag{7.47}$$

We notice that $\psi(\infty,x,\eta)-\langle y,\eta\rangle$ and $\phi_+(x,y)t$ are equivalent at each point of

$$\operatorname{diag}\big((\Sigma^+\textstyle\bigcap T^*(\Omega))\times(\Sigma^+\bigcap T^*(\Omega))\big)$$

in the sense of Melin-Sjöstrand (see **[28]**). From the global theory of Fourier integral operators (see **[28]**), we get

$$\Lambda_{\phi_+t} = C'_\infty \text{ at } \operatorname{diag}\big((\Sigma^+\textstyle\bigcap T^*(\Omega))\times(\Sigma^+\bigcap T^*(\Omega))\big) \tag{7.48}$$

(we refer the reader to Proposition B.7 and Proposition B.21 of paper I in **[20]** for the details). Formally,

$$C_\infty = \big\{(x,\xi,y,\eta);\ (x,\xi,y,-\eta)\in\Lambda_{\phi_+t}\big\}.$$

Put $\hat{\phi}_+(x,y) = -\overline{\phi}_+(y,x)$. We claim that

$$\Lambda_{\phi_+t} = \Lambda_{\hat{\phi}_+t} \text{ at } \operatorname{diag}\big((\Sigma^+\textstyle\bigcap T^*(\Omega))\times(\Sigma^+\bigcap T^*(\Omega))\big), \tag{7.49}$$

where $\Lambda_{\hat{\phi}_+t}$ is defined as in (7.46). From Proposition 3.5, it follows that $\phi_+(x,y)t$ and $-\overline{\phi}_+(y,x)t$ are equivalent at each point of $\operatorname{diag}\big((\Sigma^+\bigcap T^*(\Omega))\times(\Sigma^+\bigcap T^*(\Omega))\big)$ in the sense of Melin-Sjöstrand. Again from the global theory of Fourier integral operators we get (7.49).

From (7.49), we get the following

Proposition 7.18. – *There is a function $f\in C^\infty(\Omega\times\Omega)$, $f(x,x)\neq 0$, such that*

$$\phi_+(x,y)+f(x,y)\overline{\phi}_+(y,x) \tag{7.50}$$

vanishes to infinite order on $x=y$.

From (7.50), we can replace $\phi_+(x,y)$ by $\frac{1}{2}\big(\phi_+(x,y)-\overline{\phi}_+(y,x)\big)$. Thus, we have

$$\phi_+(x,y) = -\overline{\phi}_+(y,x). \tag{7.51}$$

From (7.40), we see that $\big(x,d_x\phi_+(x,x)\big)\in\Sigma^+$, $d_y\phi_+(x,x) = -d_x\phi_+(x,x)$. We can replace $\phi_+(x,y)$ by $\frac{2\phi_+(x,y)}{\|d_x\phi_+(x,x)\|+\|d_x\phi_+(y,y)\|}$. Thus,

$$d_x\phi_+(x,x) = \omega_0(x),\quad d_y\phi_+(x,x) = -\omega_0(x). \tag{7.52}$$

Similarly,

$$K_{\pi_-^{(q)}}(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}s_-(x,y,t)dt,$$

where $K_{\pi_-^{(q)}}(x,y)$ is as in (7.34). From (7.22), it follows that when $q = n_- = n_+$, we can take $\phi_-(x,y)$ so that $\phi_+(x,y) = -\overline{\phi}_-(x,y)$.

Our method above only works locally. From above, we know that there exist open sets X_j, $j = 1, 2, \ldots, k$, $X = \bigcup_{j=1}^k X_j$, such that

$$K_{\pi_+^{(q)}}(x,y) \equiv \int_0^\infty e^{i\phi_{+,j}(x,y)t} s_{+,j}(x,y,t)dt$$

on $X_j \times X_j$, where $\phi_{+,j}$ satisfies (7.39), (7.41), (7.42), (7.48), (7.51), (7.52) and $s_{+,j}(x,y,t)$, $j = 0, 1, \ldots$, are as in (7.37). From the global theory of Fourier integral operators, we have

$$\Lambda_{\phi_{+,j}t} = C'_\infty = \Lambda_{\phi_{+,k}t} \text{ at diag}\left((\Sigma^+ \bigcap T^*(X_j \bigcap X_k)) \times (\Sigma^+ \bigcap T^*(X_j \bigcap X_k))\right),$$

for all j, k, where $\Lambda_{\phi_{+,j}t}$, $\Lambda_{\phi_{+,k}t}$ are defined as in (7.46) and C'_∞ is as in (7.47). Thus, there is a function $f_{j,k} \in C^\infty\left((X_j \bigcap X_k) \times (X_j \bigcap X_k)\right)$, such that

$$\phi_{+,j}(x,y) - f_{j,k}(x,y)\phi_{+,k}(x,y) \tag{7.53}$$

vanishes to infinite order on $x = y$, for all j, k. Let $\chi_j(x,y)$ be a C^∞ partition of unity subordinate to $\{X_j \times X_j\}$ with $\chi_j(x,y) = \chi_j(y,x)$ and set

$$\phi_+(x,y) = \sum \chi_j(x,y)\phi_{+,j}(x,y).$$

From (7.53) and the global theory of Fourier integral operators, it follows that $\phi_{+,j}(x,y)t$ and $\phi_+(x,y)t$ are equivalent at each point of

$$\text{diag}\left((\Sigma^+ \bigcap T^*(X_j)) \times (\Sigma^+ \bigcap T^*(X_j))\right)$$

in the sense of Melin-Sjöstrand, for all j. Again, from the global theory of Fourier integral operators, we get the main result of this work

THEOREM 7.19. – *We assume that the Levi form has signature (n_-, n_+), $n_- + n_+ = n - 1$. Let $q = n_-$ or n_+. Suppose $\Box_b^{(q)}$ has closed range. Then,*

$$\begin{aligned} K_{\pi^{(q)}} &= K_{\pi_+^{(q)}} && \text{if } n_+ = q \neq n_-, \\ K_{\pi^{(q)}} &= K_{\pi_-^{(q)}} && \text{if } n_- = q \neq n_+, \\ K_{\pi^{(q)}} &= K_{\pi_+^{(q)}} + K_{\pi_-^{(q)}} && \text{if } n_+ = q = n_-, \end{aligned}$$

where $K_{\pi_+^{(q)}}(x,y)$ satisfies

$$K_{\pi_+^{(q)}}(x,y) \equiv \int_0^\infty e^{i\phi_+(x,y)t} s_+(x,y,t)dt$$

with

$$s_+(x,y,t) \in S^{n-1}_{1,0}\big(X \times X \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big),$$

(7.54)
$$s_+(x,y,t) \sim \sum_{j=0}^{\infty} s^j_+(x,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(X \times X \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big)$, *where*

$$s^j_+(x,y) \in C^\infty\big(X \times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big),\ j = 0,1,\ldots,$$

$$\phi_+(x,y) \in C^\infty(X \times X), \quad \operatorname{Im}\phi_+(x,y) \geq 0,$$

$$\phi_+(x,x) = 0, \quad \phi_+(x,y) \neq 0 \quad \textit{if} \quad x \neq y,$$

$$d_x\phi_+ \neq 0, \quad d_y\phi_+ \neq 0 \quad \textit{where} \quad \operatorname{Im}\phi_+ = 0,$$

$$\phi_+(x,y) = -\overline{\phi}_+(y,x),$$

(7.55)
$$d_x\phi_+(x,y)|_{x=y} = \omega_0(x), \quad d_y\phi_+(x,y)|_{x=y} = -\omega_0(x).$$

Moreover, $\phi_+(x,y)$ *satisfies* (7.48). *Similarly,*

$$K_{\pi^{(q)}_-}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t}s_-(x,y,t)dt$$

with

$$s_-(x,y,t) \sim \sum_{j=0}^{\infty} s^j_-(x,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(X \times X \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big)$, *where*

$$s^j_-(x,y) \in C^\infty\big(X \times X;\ \mathscr{L}(\Lambda^{0,q}T^*_y(X), \Lambda^{0,q}T^*_x(X))\big),\ j = 0,1,\ldots,$$

and when $q = n_- = n_+$, $\phi_-(x,y) = -\overline{\phi}_+(x,y)$.

CHAPTER 8

THE LEADING TERM OF THE SZEGŐ PROJECTION

To compute the leading term of the Szegő projection, we have to know the tangential Hessian of ϕ_+ at each point of $\operatorname{diag}(X \times X)$ (see (8.1)), where ϕ_+ is as in Theorem 7.19. We work with local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on an open set $\Omega \subset X$. The tangential Hessian of $\phi_+(x, y)$ at $(p, p) \in \operatorname{diag}(X \times X)$ is the bilinear map:

$$
\begin{aligned}
&T_{(p,p)}H_+ \times T_{(p,p)}H_+ \to \mathbb{C}, \\
&\qquad (u, v) \to \big\langle (\phi''_+)(p, p)u, v \big\rangle, \quad u, v \in T_{(p,p)}H_+,
\end{aligned}
\tag{8.1}
$$

where H_+ is as in (7.45) and $(\phi_+)'' = \begin{bmatrix} (\phi_+)''_{xx} & (\phi_+)''_{xy} \\ (\phi_+)''_{yx} & (\phi_+)''_{yy} \end{bmatrix}$.

From (7.55), we can check that $T_{(p,p)}H_+$ is spanned by

$$
(u, v), \quad \big(Y(p), Y(p)\big), \quad u, v \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X). \tag{8.2}
$$

Now, we compute the the tangential Hessian of ϕ_+ at $(p, p) \in \operatorname{diag}(X \times X)$. We need to understand the tangent space of the formal manifold

$$
J_+ = \big\{ J_{(x,\eta)};\ (x, \eta) \in \Sigma \big\}
$$

at $\rho = (p, \lambda\omega_0(p)) \in \Sigma^+$, $\lambda > 0$, where J_+ is as in Proposition 7.13.

Let λ_j, $j = 1, \ldots, n-1$, be the eigenvalues of the Levi form L_p. We recall that $2i\,|\lambda_j|\,|\sigma_{iY}(\rho)|$, $j = 1, \ldots, n-1$ and $-2i\,|\lambda_j|\,|\sigma_{iY}(\rho)|$, $j = 1, \ldots, n-1$, are the non-vanishing eigenvalues of the fundamental matrix F_ρ (see (2.14)). Let $\Lambda^+_\rho \subset \mathbb{C}T_\rho(T^*(X))$ be the span of the eigenspaces of F_ρ corresponding to $2i\,|\lambda_j|\,|\sigma_{iY}(\rho)|$, $j = 1, \ldots, n-1$. It is well known (see **[36]**, **[29]** and Boutet de Monvel-Guillemin **[33]**) that

$$
T_\rho(J_+) = \mathbb{C}T_\rho(\Sigma) \oplus \Lambda^+_\rho, \quad \Lambda^+_\rho = T_\rho(J_+)^\perp,
$$

where $T_\rho(J_+)^\perp$ is the orthogonal to $T_\rho(J_+)$ in $\mathbb{C}T_\rho(T^*(X))$ with respect to the canonical two form σ. We need the following

LEMMA 8.1. – *Let $\rho = (p, \lambda\omega_0(p)) \in \Sigma^+$, $\lambda > 0$. Let $\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$ be an orthonormal frame of $\Lambda^{1,0}T_x(X)$ varying smoothly with x in a neighborhood of p, for which the Levi form is diagonalized at p. Let $q_j(x,\xi)$, $j = 1, \ldots, n-1$, be the principal symbols of $Z_j(x)$, $j = 1, \ldots, n-1$. Then, Λ_ρ^+ is spanned by*

$$\begin{cases} H_{q_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) > 0, \\ H_{\overline{q}_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) < 0. \end{cases} \tag{8.3}$$

We recall that (see (2.9)*)* $\frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) = -2\lambda L_p(\overline{Z}_j, Z_j)$.

Proof. – In view of (2.13), we see that $H_{q_j}(\rho)$ and $H_{\overline{q}_j}(\rho)$ are the eigenvectors of the fundamental matrix F_ρ corresponding to $\left\{q_j, \overline{q}_j\right\}(\rho)$ and $\left\{\overline{q}_j, q_j\right\}(\rho)$, for all j. Since Λ_ρ^+ is the span of the eigenspaces of the fundamental matrix F_ρ corresponding to $2i\lambda\left|\lambda_j\right|$, $j = 1, \ldots, n-1$, where λ_j, $j = 1, \ldots, n-1$, are the eigenvalues of the Levi form L_p. Thus, Λ_ρ^+ is spanned by

$$\begin{cases} H_{q_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) > 0, \\ H_{\overline{q}_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) < 0. \end{cases}$$

□

We assume that $(U, f_1, \ldots, f_{n-1})$ is a representative of J_ρ. We also write f_j to denote an almost analytic extension of f_j, for all j. It is well known that (see **[29]** and (7.29)) there exist $h_j(x,y) \in C^\infty(X \times X)$, $j = 1, \ldots, n-1$, such that

$$f_j(x, (\phi_+)'_x) - h_j(x,y)\phi_+(x,y) \tag{8.4}$$

vanishes to infinite order on $x = y$, $j = 1, \ldots, n-1$. From Lemma 8.1, we may assume that

$$\begin{cases} H_{f_j}(\rho) = H_{q_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) > 0, \\ H_{f_j}(\rho) = H_{\overline{q}_j}(\rho), & \text{if } \frac{1}{i}\left\{q_j, \overline{q}_j\right\}(\rho) < 0. \end{cases} \tag{8.5}$$

Here q_j, $j = 1, \ldots, n-1$, are as in Lemma 8.1.

We take local coordinates

$$x = (x_1, \ldots, x_{2n-1}), \quad z_j = x_{2j-1} + ix_{2j}, \quad j = 1, \ldots, n-1,$$

defined on some open neighborhood of p such that $\omega_0(p) = \sqrt{2}dx_{2n-1}$, $x(p) = 0$, $(\frac{\partial}{\partial x_j}(p) \mid \frac{\partial}{\partial x_k}(p)) = 2\delta_{j,k}$, $j, k = 1, \ldots, 2n-1$ and

$$\overline{Z}_j = \frac{\partial}{\partial z_j} - \frac{1}{\sqrt{2}} i\lambda_j \overline{z}_j \frac{\partial}{\partial x_{2n-1}} - \frac{1}{\sqrt{2}} c_j x_{2n-1} \frac{\partial}{\partial x_{2n-1}} + O(|x|^2), \quad j = 1, \ldots, n-1,$$

where $\overline{Z}_j$, $j = 1, \ldots, n-1$, are as in Lemma 8.1, $c_j \in \mathbb{C}$, $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}})$, $j = 1, \ldots, n-1$ and λ_j, $j = 1, \ldots, n-1$, are the eigenvalues of L_p (this is always possible. see pages 157–160 of **[1]**).

We may assume that $\lambda_j > 0$, $j = 1, \ldots, q$, $\lambda_j < 0$, $j = q+1, \ldots, n-1$. Let $\xi = (\xi_1, \ldots, \xi_{2n-1})$ denote the dual variables of x. From (8.5), we can check that

(8.6)
$$
\begin{aligned}
f_j(x,\xi) &= -\frac{i}{2}(\xi_{2j-1} - i\xi_{2j}) - \frac{1}{\sqrt{2}}\lambda_j \overline{z}_j \xi_{2n-1} + i\frac{1}{\sqrt{2}} c_j x_{2n-1}\xi_{2n-1} + O(\left|(x,\xi')\right|^2),\\
&\quad j = 1, \ldots, q, \quad \xi' = (\xi_1, \ldots, \xi_{2n-2}),\\
f_j(x,\xi) &= \frac{i}{2}(\xi_{2j-1} + i\xi_{2j}) - \frac{1}{\sqrt{2}}\lambda_j z_j \xi_{2n-1} - i\frac{1}{\sqrt{2}} \overline{c}_j x_{2n-1}\xi_{2n-1} + O(\left|(x,\xi')\right|^2),\\
&\quad j = q+1, \ldots, n-1, \quad \xi' = (\xi_1, \ldots, \xi_{2n-2}).
\end{aligned}
$$

We write $y = (y_1, \ldots, y_{2n-1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n-1$, $\frac{\partial}{\partial w_j} = \frac{1}{2}(\frac{\partial}{\partial y_{2j-1}} - i\frac{\partial}{\partial y_{2j}})$, $\frac{\partial}{\partial \overline{w}_j} = \frac{1}{2}(\frac{\partial}{\partial y_{2j-1}} + i\frac{\partial}{\partial y_{2j}})$, $j = 1, \ldots, n-1$ and $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}})$, $j = 1, \ldots, n-1$. From (8.4) and (8.6), we have

(8.7)
$$
\begin{aligned}
-i\frac{\partial \phi_+}{\partial z_j} - \frac{1}{\sqrt{2}}(\lambda_j \overline{z}_j - ic_j x_{2n-1})\frac{\partial \phi_+}{\partial x_{2n-1}} &= h_j\phi_+ + O(\left|(x,y)\right|^2),\ j = 1, \ldots, q,\\
i\frac{\partial \phi_+}{\partial \overline{z}_j} - \frac{1}{\sqrt{2}}(\lambda_j z_j + i\overline{c}_j x_{2n-1})\frac{\partial \phi_+}{\partial x_{2n-1}} &= h_j\phi_+ + O(\left|(x,y)\right|^2),\ j = q+1, \ldots, n-1.
\end{aligned}
$$

From (8.7), it is straight forward to see that

$$
\begin{aligned}
\frac{\partial^2 \phi_+}{\partial z_j \partial z_k}(0,0) &= \frac{\partial^2 \phi_+}{\partial z_j \partial \overline{z}_k}(0,0) - i\lambda_j\delta_{j,k} = \frac{\partial^2 \phi_+}{\partial z_j \partial w_k}(0,0)\\
&= \frac{\partial^2 \phi_+}{\partial z_j \partial \overline{w}_k}(0,0) = \frac{\partial^2 \phi_+}{\partial z_j \partial x_{2n-1}}(0,0) + \frac{\partial^2 \phi_+}{\partial z_j \partial y_{2n-1}}(0,0) - c_j\\
&= 0, \quad 1 \le j \le q, \quad 1 \le k \le n-1,
\end{aligned}
\tag{8.8}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial \overline{z}_k}(0,0) &= \frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial z_k}(0,0) + i\lambda_j\delta_{j,k} = \frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial w_k}(0,0)\\
&= \frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial \overline{w}_k}(0,0) = \frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial x_{2n-1}}(0,0) + \frac{\partial^2 \phi_+}{\partial \overline{z}_j \partial y_{2n-1}}(0,0) - \overline{c}_j\\
&= 0, \quad q+1 \le j \le n-1, \quad 1 \le k \le n-1.
\end{aligned}
\tag{8.9}
$$

Since $d_x\phi_+|_{x=y} = \omega_0(x)$, we have $\overline{f}_j(x, (\phi_+)'_x(x,x)) = 0$, $j = 1, \ldots, n-1$. Thus,

(8.10)
$$
\begin{aligned}
i\frac{\partial \phi_+}{\partial \overline{z}_j}(x,x) - \frac{1}{\sqrt{2}}(\lambda_j z_j + i\overline{c}_j x_{2n-1})\frac{\partial \phi_+}{\partial x_{2n-1}}(x,x) &= O(|x|^2),\ j = 1, \ldots, q,\\
-i\frac{\partial \phi_+}{\partial z_j}(x,x) - \frac{1}{\sqrt{2}}(\lambda_j \overline{z}_j - ic_j x_{2n-1})\frac{\partial \phi_+}{\partial x_{2n-1}}(x,x) &= O(|x|^2),\ j = q+1, \ldots, n-1.
\end{aligned}
$$

From (8.10), it is straight forward to see that

$$
\begin{aligned}
\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial\overline{z}_k}(0,0)+\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial\overline{w}_k}(0,0) &= \frac{\partial^2\phi_+}{\partial\overline{z}_j\partial z_k}(0,0)+\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial w_k}(0,0)+i\lambda_j\delta_{j,k}\\
&= \frac{\partial^2\phi_+}{\partial\overline{z}_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial y_{2n-1}}(0,0)-\overline{c}_j\\
&= 0,\quad 1\leq j\leq q,\quad 1\leq k\leq n-1,
\end{aligned}
\tag{8.11}
$$

and

$$
\begin{aligned}
\frac{\partial^2\phi_+}{\partial z_j\partial z_k}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial w_k}(0,0) &= \frac{\partial^2\phi_+}{\partial z_j\partial\overline{z}_k}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial\overline{w}_k}(0,0)-i\lambda_j\delta_{j,k}\\
&= \frac{\partial^2\phi_+}{\partial z_j\partial x_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial z_j\partial y_{2n-1}}(0,0)-c_j\\
&= 0,\quad q+1\leq j\leq n-1,\quad 1\leq k\leq n-1.
\end{aligned}
\tag{8.12}
$$

Since $\phi_+(x,y)=-\overline{\phi}_+(y,x)$, from (8.9), we have

$$
\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial w_k}(0,0)=-\frac{\partial^2\overline{\phi}_+}{\partial\overline{w}_j\partial z_k}(0,0)=-\overline{\frac{\partial^2\phi_+}{\partial w_j\partial\overline{z}_k}(0,0)}=0,\quad q+1\leq k\leq n-1.
$$

Combining this with (8.11), we get

$$
\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial z_k}(0,0)=0,\quad 1\leq j\leq q,\quad q+1\leq k\leq n-1.
\tag{8.13}
$$

Similarly,

$$
\begin{aligned}
&\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial\overline{z}_k}(0,0)=0,\quad 1\leq j\leq q,\quad 1\leq k\leq q,\\
&\frac{\partial^2\phi_+}{\partial z_j\partial z_k}(0,0)=0,\quad q+1\leq j\leq n-1,\quad q+1\leq k\leq n-1.
\end{aligned}
\tag{8.14}
$$

From (8.8) and (8.11), we have

$$
\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial w_k}(0,0)=-i\lambda_j\delta_{j,k}-\frac{\partial^2\phi_+}{\partial\overline{z}_j\partial z_k}(0,0)=-2i\lambda_j\delta_{j,k},\quad 1\leq j,k\leq q.
\tag{8.15}
$$

Similarly,

$$
\frac{\partial^2\phi_+}{\partial z_j\partial\overline{w}_k}(0,0)=2i\lambda_j\delta_{j,k},\quad q+1\leq j,k\leq n-1.
\tag{8.16}
$$

Since $\phi_+(x,x)=0$, we have

$$
\frac{\partial^2\phi_+}{\partial x_{2n-1}\partial x_{2n-1}}(0,0)+2\frac{\partial^2\phi_+}{\partial x_{2n-1}\partial y_{2n-1}}(0,0)+\frac{\partial^2\phi_+}{\partial y_{2n-1}\partial y_{2n-1}}(0,0)=0.
\tag{8.17}
$$

Combining (8.8), (8.9) and (8.11)–(8.17), we completely determine the tangential Hessian of $\phi_+(x,y)$ at (p,p).

THEOREM 8.2. – *With the notations used before, in some small neighborhood of $(p,p)\in X\times X$, we have*

(8.18)

$$\begin{aligned}\phi_+(x,y) &= \sqrt{2}(x_{2n-1}-y_{2n-1})+i\sum_{j=1}^{n-1}\left|\lambda_j\right|\left|z_j-w_j\right|^2\\ &+\sum_{j=1}^{n-1}\Big(i\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+c_j(z_jx_{2n-1}-w_jy_{2n-1})+\overline{c}_j(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\Big)\\ &+\sqrt{2}(x_{2n-1}-y_{2n-1})f(x,y)+O(\left|(x,y)\right|^3),\ \ f\in C^\infty, f(0,0)=0, f(x,y)=\overline{f}(y,x).\end{aligned}$$

We have the classical formulas

(8.19)

$$\int_0^\infty e^{-tx}t^m dt=\begin{cases} m!x^{-m-1}, & \text{if } m\in\mathbb{Z},\quad m\geq 0,\\ \frac{(-1)^m}{(-m-1)!}x^{-m-1}(\log x+c-\sum_1^{-m-1}\frac{1}{j}), & \text{if } m\in\mathbb{Z},\quad m<0.\end{cases}$$

Here $x\neq 0$, $\operatorname{Re}x\geq 0$ and c is the Euler constant, i.e. $c=\lim_{m\to\infty}(\sum_1^m\frac{1}{j}-\log m)$. Note that

(8.20)

$$\int_0^\infty e^{i\phi_+(x,y)t}\sum_{j=0}^\infty s_+^j(x,y)t^{n-1-j}dt=\lim_{\varepsilon\to 0+}\int_0^\infty e^{-(-i(\phi_+(x,y)+i\varepsilon)t)}\sum_{j=0}^\infty s_+^j(x,y)t^{n-1-j}dt.$$

We have the following corollary of Theorem 7.19

COROLLARY 8.3. – *There exist*

$$F_+,G_+,F_-,G_-\in C^\infty\big(X\times X;\ \mathscr{L}(\Lambda^{0,q}T_y^*(X),\Lambda^{0,q}T_x^*(X))\big)$$

such that

$$\begin{aligned}K_{\pi_+^{(q)}}&=F_+(-i(\phi_+(x,y)+i0))^{-n}+G_+\log(-i(\phi_+(x,y)+i0)),\\ K_{\pi_-^{(q)}}&=F_-(-i(\phi_-(x,y)+i0))^{-n}+G_-\log(-i(\phi_-(x,y)+i0)).\end{aligned}$$

Moreover, we have

(8.21)

$$\begin{aligned}F_+&=\sum_0^{n-1}(n-1-k)!s_+^k(x,y)(-i\phi_+(x,y))^k+f_+(x,y)(\phi_+(x,y))^n,\\ F_-&=\sum_0^{n-1}(n-1-k)!s_-^k(x,y)(-i\phi_-(x,y))^k+f_-(x,y)(\phi_-(x,y))^n,\\ G_+&\equiv\sum_0^\infty\frac{(-1)^{k+1}}{k!}s_+^{n+k}(x,y)(-i\phi_+(x,y))^k,\\ G_-&\equiv\sum_0^\infty\frac{(-1)^{k+1}}{k!}s_-^{n+k}(x,y)(-i\phi_-(x,y))^k,\end{aligned}$$

where s_+^k, $k = 0, 1, \ldots$, *are as in* (7.54) *and*

$$f_+(x,y), f_-(x,y) \in C^\infty\big(X \times X; \mathscr{L}(\Lambda^{0,q}T_y^*(X), \Lambda^{0,q}T_x^*(X))\big).$$

In the rest of this chapter, we assume that $q = n_+$. We will compute the leading term of $K_{\pi_+^{(q)}}$. For a given point $p \in X$, let $x = (x_1, x_2, \ldots, x_{2n-1})$ be the local coordinates as in Theorem 8.2. We recall that $w_0(p) = \sqrt{2}dx_{2n-1}$, $x(p) = 0$, $\Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X) = \Big\{\sum_j^{2n-2} a_j \frac{\partial}{\partial x_j};\ a_j \in \mathbb{C}\Big\}$, $(\frac{\partial}{\partial x_j}(p) \mid \frac{\partial}{\partial x_k}(p)) = 2\delta_{j,k}$, $j, k = 1, \ldots, 2n-1$. We identify Ω with some open set in $\mathbb{R}^{2n-1}$. We represent the Hermitian inner product $(\ |\ )$ on $\mathbb{C}T(X)$ by $(u \mid v) = \langle Hu, \overline{v}\rangle$, where $u, v \in \mathbb{C}T(X)$, H is a positive definite Hermitian matrix and $\langle , \rangle$ is given by $\langle s, t\rangle = \sum_{j=1}^{2n-1} s_j t_j$, $s = (s_1, \ldots, s_{2n-1}) \in \mathbb{C}^{2n-1}$, $t = (t_1, \ldots, t_{2n-1}) \in \mathbb{C}^{2n-1}$. Here we identify $\mathbb{C}T(X)$ with $\mathbb{C}^{2n-1}$. Let $h(x)$ denote the determinant of H. The induced volume form on X is given by $\sqrt{h(x)}dx$. We have $h(p) = 2^{2n-1}$. Now,

$$(K_{\pi_+^{(q)}} \circ K_{\pi_+^{(q)}})(x,y) \equiv \int_0^\infty \int_0^\infty \Big(\int e^{it\phi_+(x,w)+is\phi_+(w,y)} s_+(x,w,t) s_+(w,y,s)\sqrt{h(w)}dw\Big) dtds.$$

Let $s = t\sigma$, we get

$$(K_{\pi_+^{(q)}} \circ K_{\pi_+^{(q)}})(x,y) \equiv \int_0^\infty \int_0^\infty \Big(\int e^{it\phi(x,y,w,\sigma)} s_+(x,w,t) s_+(w,y,t\sigma) t\sqrt{h(w)}dw\Big) d\sigma dt,$$

where $\phi(x,y,w,\sigma) = \phi_+(x,w) + \sigma\phi_+(w,y)$. It is easy to see that

$$\operatorname{Im}\phi(x,y,w,\sigma) \geq 0, \quad d_w\phi(x,y,w,\sigma)|_{x=y=w} = (\sigma-1)\omega_0(x).$$

Thus, $x = y = w$, $\sigma = 1$, x is real, are real critical points.

Now, we will compute the Hessian of ϕ at $x = y = w = p$, p is real, $\sigma = 1$. We write $H_\phi(p)$ to denote the Hessian of ϕ at $x = y = w = p$, p is real, $\sigma = 1$. $H_\phi(p)$ has the following form: $H_\phi(p) = \begin{bmatrix} 0 & {}^t(\phi_+)'_x \\ (\phi_+)'_x & (\phi_+)''_{xx} + (\phi_+)''_{yy}\end{bmatrix}$. Since $(\phi_+)'_x(p) = \omega_0(p) = \sqrt{2}dx_{2n-1}$, we have $H_\phi(p) = \begin{bmatrix} 0, & 0, \ldots, 0, & \sqrt{2} \\ \vdots & A, & * \\ \sqrt{2} & * & * \end{bmatrix}$, where A is the linear map

$$A : \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X) \to \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X),$$
$$\langle Au, v\rangle = \big\langle ((\phi_+)''_{xx} + (\phi_+)''_{yy})u, v\big\rangle, \quad \forall\ u, v \in \Lambda^{1,0}T_p(X) \oplus \Lambda^{0,1}T_p(X).$$

From eq8.18, it follows that A has the eigenvalues:

$$4i\,|\lambda_1(p)|\,, 4i\,|\lambda_1(p)|\,, \ldots, 4i\,|\lambda_{n-1}(p)|\,, 4i\,|\lambda_{n-1}(p)|\,, \tag{8.22}$$

where $\lambda_j(p)$, $j = 1, \cdots, (n-1)$, are the eigenvalues of the Levi form L_p. We have,

$$\det\Big(\frac{H_\phi(p)}{i}\Big) = 2^{4n-3}\,|\lambda_1(p)|^2 \cdots |\lambda_{n-1}(p)|^2\,. \tag{8.23}$$

From the stationary phase formula of Melin-Sjöstrand (see page 148 of [**28**]), we get

$$(K_{\pi_+^{(q)}} \circ K_{\pi_+^{(q)}})(x, y) \equiv \int_0^\infty e^{it\phi_1(x,y)} a(x, y, t) dt,$$

where

$$a(x, y, t) \sim \sum_{j=0}^\infty a_j(x, y) t^{n-1-j}$$

in the symbol space $S^{n-1}_{1,0}\big(\Omega \times \Omega \times [0, \infty[; \mathscr{L}(\Lambda^{0,q} T^*_y(X), \Lambda^{0,q} T^*_x(X))\big)$, $a_j(x, y) \in C^\infty\big(\Omega \times \Omega; \mathscr{L}(\Lambda^{0,q} T^*(X), \Lambda^{0,q} T^*(X))\big)$, $j = 0, 1, \ldots$, $\phi_1(x, y)$ is the corresponding critical value. Moreover, we have

$$\begin{aligned} a_0(p, p) &= \left(\det \frac{H_\phi(p)}{2\pi i}\right)^{-\frac{1}{2}} s^0_+(p, p) \circ s^0_+(p, p) \sqrt{h(p)} \\ &= 2 \left|\lambda_1(p)\right|^{-1} \cdots \left|\lambda_{n-1}(p)\right|^{-1} \pi^n s^0_+(p, p) \circ s^0_+(p, p), \end{aligned} \tag{8.24}$$

where s^0_+ is as in (7.54). We notice that

$$\phi_1(x, x) = 0, \quad (\phi_1)'_x(x, x) = (\phi_+)'_x(x, x), \quad (\phi_1)'_y(x, x) = (\phi_+)'_y(x, x). \tag{8.25}$$

From ((8.19) and (8.20), it follows that

(8.26)
$$\begin{aligned} (K_{\pi_+^{(q)}} \circ K_{\pi_+^{(q)}})(x, y) &\equiv F_1(-i(\phi_1(x, y) + i0))^{-n} + G_1 \log(-i(\phi_1(x, y) + i0)) \\ &\equiv F_+(-i(\phi_+(x, y) + i0))^{-n} + G_+ \log(-i(\phi_+(x, y) + i0)), \end{aligned}$$

where

$$F_1 = \sum_0^{n-1} (n - 1 - k)! a_j(-i\phi_1)^k + f_1 \phi_1^n,$$

$f_1, G_1 \in C^\infty\big(\Omega \times \Omega; \mathscr{L}(\Lambda^{0,q} T^*_y(X), \Lambda^{0,q} T^*_x(X))\big)$, F_+ and G_+ are as in Corollary 8.3. From (8.25) and (8.26), we see that $s^0_+(p, p) = a_0(p, p)$. From this and (8.24), we get

$$2 \left|\lambda_1(p)\right|^{-1} \cdots \left|\lambda_{n-1}(p)\right|^{-1} \pi^n s^0_+(p, p) \circ s^0_+(p, p) = s^0_+(p, p). \tag{8.27}$$

Let $\mathscr{N}_x(p^s_0 + \frac{1}{2} \widetilde{\operatorname{tr}} F) = \big\{u \in \Lambda^{0,q} T^*_x(X);\ (p^s_0 + \frac{1}{2} \widetilde{\operatorname{tr}} F)(x, \omega_0(x)) u = 0\big\}$, where p^s_0 is the subprincipal symbol of $\Box^{(q)}_b$ and F is the fundamental matrix of $\Box^{(q)}_b$. From the asymptotic expansion of $\Box^{(q)}_b(e^{i\phi_+} s_+)$, we see that $s^0_+(p, p) u \in \mathscr{N}_p(p^s_0 + \frac{1}{2} \widetilde{\operatorname{tr}} F)$ for all $u \in \Lambda^{0,q} T^*_p(X)$ (see Chapter 4). Let

$$I_1 = (\frac{1}{2} \left|\lambda_1\right| \cdots \left|\lambda_{n-1}\right|)^{-1} \pi^n s^0_+(p, p).$$

From (8.27), we see that

$$I_1^2 = I_1. \tag{8.28}$$

Since $K_{(\pi_+^{(q)})^*} \equiv K_{\pi_+^{(q)}}$ and $\phi_+(x,y) = -\overline{\phi}_+(y,x)$, we have $(s_+^0)^*(p,p) = s_+^0(p,p)$ and hence

$$I_1^* = I_1, \tag{8.29}$$

where $(\pi_+^{(q)})^*$ is the adjoint of $\pi_+^{(q)}$, $(s_+^0)^*(p,p)$, I_1^* are the adjoints of $s_+^0(p,p)$ and I_1 in the space $\mathscr{L}\big(\Lambda^{0,q}T_p^*(X), \Lambda^{0,q}T_p^*(X)\big)$ with respect to $(\ |\ )$ respectively. Note that $\dim \mathscr{N}_p(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\, F) = 1$ (see Chapter 2). Combining this with (8.28), (8.29) and $s_+^0(p,p) \neq 0$, it follows that $I_1 : \Lambda^{0,q}T_p^*(X) \to \Lambda^{0,q}T_p^*(X)$ is the orthogonal projection onto $\mathscr{N}_p(p_0^s + \frac{1}{2}\widetilde{\operatorname{tr}}\, F)$.

For a given point $p \in X$, let $\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$ be an orthonormal frame of $\Lambda^{1,0}T_x(X)$, for which the Levi form is diagonalized at p. Let $e_j(x)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1}T_x^*(X)$, which is dual to $Z_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that $\lambda_j(p) > 0$ if $1 \leq j \leq n_+$. Then $I_1 = \prod_{j=1}^{j=n_+} e_j(p)^\wedge e_j^{\wedge,*}(p)$ at p (see Chapter 2). Summing up, we have proved

Proposition 8.4. – *For a given point $p \in X$, let $\overline{Z}_1(x), \ldots, \overline{Z}_{n-1}(x)$ be an orthonormal frame of $\Lambda^{1,0}T_x(X)$, for which the Levi form is diagonalized at p. Let $e_j(x)$, $j = 1, \ldots, n-1$ denote the basis of $\Lambda^{0,1}T_x^*(X)$, which is dual to $Z_j(x)$, $j = 1, \ldots, n-1$. Let $\lambda_j(x)$, $j = 1, \ldots, n-1$ be the eigenvalues of the Levi form L_x. We assume that $q = n_+$ and that $\lambda_j(p) > 0$ if $1 \leq j \leq n_+$. Then*

$$F_+(p,p) = (n-1)!\frac{1}{2}\,|\lambda_1(p)| \cdots |\lambda_{n-1}(p)|\,\pi^{-n} \prod_{j=1}^{j=n_+} e_j(p)^\wedge e_j(p)^{\wedge,*}.$$

PART II

ON THE SINGULARITIES OF THE BERGMAN PROJECTION FOR $(0, q)$ FORMS

CHAPTER 1

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper, we assume that all manifolds are paracompact (for the precise definition, see page 156 of Kelley [**21**]). Let M be a relatively compact open subset with C^∞ boundary Γ of a complex manifold M' of dimension n with a smooth Hermitian metric $(\ |\ )$ on its holomorphic tangent bundle (see (1.1)). The Hermitian metric induces a Hermitian metric on the bundle of $(0, q)$ forms of M' (see the discussion after (1.1)) and a positive density (dM') (see (1.4)). Let $\Box$ be the $\overline{\partial}$-Neumann Laplacian on M (see Folland-Kohn [**10**] or (1.7)) and let $\Box^{(q)}$ denote the restriction to $(0, q)$ forms. For $p \in \Gamma$, let L_p be the Levi form of Γ at p (see (1.10)). Given q, $0 \leq q \leq n-1$, the Levi form is said to satisfy condition $Z(q)$ at $p \in \Gamma$ if it has at least $n - q$ positive or at least $q + 1$ negative eigenvalues. When condition $Z(q)$ holds at each point of Γ, Kohn's L^2 estimates give the hypoellipicity with loss of one dervative for the solutions of $\Box^{(q)}u = f$ (see [**10**] or Theorem 2.6). The Bergman projection is the orthogonal projection onto the kernel of $\Box^{(q)}$ in the L^2 space. When condition $Z(q)$ fails at some point of Γ, one is interested in the Bergman projection on the level of $(0, q)$ forms. When $q = 0$ and the Levi form is positive definite, the existence of the complete asymptotic expansion of the singularities of the Bergman projection was obtained by Fefferman [**9**] on the diagonal and subsequently by Boutet de Monvel-Sjöstrand (see [**34**]) in complete generality. If $q = n - 1$ and the Levi form is negative definite, Hörmander [**19**] obtained the corresponding asymptotics for the Bergman projection in the distribution sense. We have been influenced by these works.

We now start to formulate the main results. First, we introduce some standard notations. Let Ω be a C^∞ manifold equipped with a smooth density of integration. We let $T(\Omega)$ and $T^*(\Omega)$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T(\Omega)$ and $\mathbb{C}T^*(\Omega)$ respectively. We write $\langle\ ,\rangle$ to denote the pointwise duality between $T(\Omega)$ and $T^*(\Omega)$. We extend $\langle\ ,\rangle$ bilinearly to

$\mathbb{C}T(\Omega) \times \mathbb{C}T^*(\Omega)$. Let E be a C^∞ vector bundle over Ω. The fiber of E at $x \in \Omega$ will be denoted by E_x. Let $Y \subset\subset \Omega$ be an open set. The spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y; E)$ and $\mathscr{D}'(Y; E)$ respectively. Let $\mathscr{E}'(Y; E)$ be the subspace of $\mathscr{D}'(Y; E)$ of sections with compact support in Y. For $s \in \mathbb{R}$, we let $H^s(Y; E)$ denote the Sobolev space of order s of sections of E over Y. Put

$$H^s_{\rm loc}(Y; E) = \{u \in \mathscr{D}'(Y; E);\, \varphi u \in H^s(Y; E),\ \forall \varphi \in C^\infty_0(Y)\}$$

and $H^s_{\rm comp}(Y; E) = H^s_{\rm loc}(Y; E) \bigcap \mathscr{E}'(Y; E)$.

Let F be a C^∞ vector bundle over M'. Let $C^\infty(\overline{M}; F)$, $\mathscr{D}'(\overline{M}; F)$, $H^s(\overline{M}; F)$ denote the spaces of restrictions to M of elements in the spaces $C^\infty(M'; F)$, $\mathscr{D}'(M'; F)$ and $H^s(M'; F)$ respectively. Let $C^\infty_0(M; F)$ be the subspace of $C^\infty(\overline{M}; F)$ of sections with compact support in M.

Let $\Lambda^{1,0}T(M')$ and $\Lambda^{0,1}T(M')$ be the holomorphic tangent bundle of M' and the anti-holomorphic tangent boundle of M' respectively. In local coordinates $z = (z_1, \ldots, z_n)$, we represent the Hermitian metric on $\Lambda^{1,0}T(M')$ by

$$(1.1)\qquad (u \mid v) = g(u, \overline{v}),\quad u, v \in \Lambda^{1,0}T(M'),\quad g = \sum_{j,k=1}^n g_{j,k}(z) dz_j \otimes d\overline{z}_k,$$

where $g_{j,k}(z) = \overline{g}_{k,j}(z) \in C^\infty,\ j, k = 1, \ldots, n$, and $(g_{j,k}(z))^n_{j,k=1}$ is positive definite at each point. We extend the Hermitian metric $(\,|\,)$ to $\mathbb{C}T(M')$ in a natural way by requiring $\Lambda^{1,0}T(M')$ to be orthogonal to $\Lambda^{0,1}T(M')$ and satisfy $\overline{(u \mid v)} = (\overline{u} \mid \overline{v})$, $u, v \in \Lambda^{0,1}T(M')$.

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*(M')$ that we shall also denote by $(\,|\,)$. For $q \in \mathbb{N}$, let $\Lambda^{0,q}T^*(M')$ be the bundle of $(0, q)$ forms of M'. The Hermitian metric $(\,|\,)$ on $\mathbb{C}T^*(M')$ induces a Hermitian metric on $\Lambda^{0,q}T^*(M')$ also denoted by $(\,|\,)$.

Let $r \in C^\infty(M')$ be a defining function of Γ such that r is real, $r = 0$ on Γ, $r < 0$ on M and $dr \neq 0$ near Γ. From now on, we take a defining function r so that $\|dr\| = 1$ on Γ.

The Hermitian metric $(\,|\,)$ on $\mathbb{C}T(M')$ induces a Hermitian metric $(\,|\,)$ on $\mathbb{C}T(\Gamma)$. For $z \in \Gamma$, we identify $\mathbb{C}T^*_z(\Gamma)$ with the space

$$(1.2)\qquad \{u \in \mathbb{C}T^*_z(M');\, (u \mid dr) = 0\}.$$

For $q \in \mathbb{N}$, let $\Lambda^{0,q}T^*(\Gamma)$ be the bundle of $(0, q)$ forms of Γ. We recall that

$$(1.3)\qquad \Lambda^{0,q}T^*_z(\Gamma) = \left\{u \in \Lambda^{0,q}T^*_z(M');\, (u \mid \overline{\partial} r(z) \wedge g) = 0,\quad \forall g \in \Lambda^{0,q-1}T^*_z(M')\right\},\quad z \in \Gamma.$$

We associate to the Hermitian metric $\sum_{j,k=1}^{n} g_{j,k}(z)dz_j \otimes d\overline{z}_k$ a real $(1,1)$ form (see page 144 of Kodaira [**22**]) $\omega = i\sum_{j,k=1}^{n} g_{j,k}dz_j \wedge d\overline{z}_k$. Let

$$dM' = \frac{\omega^n}{n!} \tag{1.4}$$

be the volume element and let $(\ |\ )_M$ be the inner product on $C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$ defined by

$$(f \mid h)_M = \int_M (f \mid h)(dM') = \int_M (f \mid h)\frac{\omega^n}{n!},\quad f, h \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')). \tag{1.5}$$

Similarly, we take $(d\Gamma)$ as the induced volume form on Γ and let $(\ |\ )_\Gamma$ be the inner product on $C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$ defined by

$$(f \mid g)_\Gamma = \int_\Gamma (f \mid g)d\Gamma,\quad f, g \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')). \tag{1.6}$$

Let $\overline{\partial} : C^\infty(M';\, \Lambda^{0,q}T^*(M')) \to C^\infty(M';\, \Lambda^{0,q+1}T^*(M'))$ be the part of the exterior differential operator which maps forms of type $(0,q)$ to forms of type $(0,q+1)$ and we denote by $\overline{\partial_f}^* : C^\infty(M';\, \Lambda^{0,q+1}T^*(M')) \to C^\infty(M';\, \Lambda^{0,q}T^*(M'))$ the formal adjoint of $\overline{\partial}$. That is

$$(\overline{\partial} f \mid h)_{M'} = (f \mid \overline{\partial_f}^* h)_{M'},$$

$f \in C_0^\infty(M';\, \Lambda^{0,q}T^*(M'))$, $h \in C^\infty(M';\, \Lambda^{0,q+1}T^*(M'))$, where $(\ |\ )_{M'}$ is defined by $(g \mid k)_{M'} = \int_M (g \mid k)(dM')$, $g, k \in C_0^\infty(M';\, \Lambda^{0,q}T^*(M'))$. We shall also use the notation $\overline{\partial}$ for the closure in L^2 of the $\overline{\partial}$ operator, initially defined on $C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$ and $\overline{\partial}^*$ for the Hilbert space adjoint of $\overline{\partial}$.

The $\overline{\partial}$-Neumann Laplacian on $(0,q)$ forms is then the self-adjoint operator in the space $L^2(M;\, \Lambda^{0,q}T^*(M'))$ (see chapter I of [**10**])

$$\Box^{(q)} = \overline{\partial}\, \overline{\partial}^* + \overline{\partial}^* \overline{\partial}. \tag{1.7}$$

We notice that

$$\begin{aligned} \operatorname{Dom} \Box^{(q)} = \Big\{ & u \in L^2(M;\, \Lambda^{0,q}T^*(M'));\, u \in \operatorname{Dom} \overline{\partial}^* \bigcap \operatorname{Dom} \overline{\partial}, \\ & \overline{\partial}^* u \in \operatorname{Dom} \overline{\partial},\ \overline{\partial} u \in \operatorname{Dom} \overline{\partial}^* \Big\} \end{aligned} \tag{1.8}$$

and $C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) \bigcap \operatorname{Dom} \Box^{(q)}$ is dense in $\operatorname{Dom} \Box^{(q)}$ for the norm

$$u \in \operatorname{Dom} \Box^{(q)} \to \|u\| + \left\|\overline{\partial} u\right\| + \left\|\overline{\partial}^* u\right\|$$

(see also page 14 of [**10**]).

Let $\Box_f^{(q)} = \overline{\partial}\, \overline{\partial_f}^* + \overline{\partial_f}^* \overline{\partial} : C^\infty(M';\, \Lambda^{0,q}T^*(M')) \to C^\infty(M';\, \Lambda^{0,q}T^*(M'))$ denote the complex Laplace-Beltrami operator on $(0,q)$ forms and denote by $\sigma_{\Box_f^{(q)}}$ the principal symbol of $\Box_f^{(q)}$.

Let $\frac{\partial}{\partial r}$ be the dual vector of dr. That is $(u \mid \frac{\partial}{\partial r}) = \langle u, dr\rangle$, for all $u \in \mathbb{C}T(M')$. Put

$$\omega_0 = J^t(dr), \tag{1.9}$$

where J^t is the complex structure map for the cotangent bundle.

Let $\Lambda^{1,0}T(\Gamma)$ and $\Lambda^{0,1}T(\Gamma)$ be the holomorphic tangent bundle of Γ and the anti-holomorphic tangent bundle of Γ respectively. The Levi form $L_p(Z, \overline{W})$, $p \in \Gamma$, Z, $W \in \Lambda^{1,0}T_p(\Gamma)$, is the Hermitian quadratic form on $\Lambda^{1,0}T_p(\Gamma)$ defined as follows:

(1.10) For any Z, $W \in \Lambda^{1,0}T_p(\Gamma)$, pick $\widetilde{Z}$, $\widetilde{W} \in C^\infty(\Gamma; \Lambda^{1,0}T(\Gamma))$ that satisfy $\widetilde{Z}(p) = Z$, $\widetilde{W}(p) = W$. Then $L_p(Z, \overline{W}) = \dfrac{1}{2i}\left\langle [\widetilde{Z}\ , \overline{\widetilde{W}}](p)\ , \omega_0(p)\right\rangle$.

The eigenvalues of the Levi form at $p \in \Gamma$ are the eigenvalues of the Hermitian form L_p with respect to the inner product $(\mid)$ on $\Lambda^{1,0}T_p(\Gamma)$. If the Levi form is non-degenerate at $p \in \Gamma$, let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of L_p. Then $Z(q)$ holds at p if and only if $q \neq n_-$.

We recall the Hörmander symbol spaces

Definition 1.1. – Let $m \in \mathbb{R}$. Let U be an open set in $M' \times M'$.

$$S^m_{1,0}\big(U\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_y(M'), \Lambda^{0,q}T^*_x(M'))\big)$$

is the space of all $a(x,y,t) \in C^\infty\big(U\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_y(M'), \Lambda^{0,q}T^*_x(M'))\big)$ such that for all compact sets $K \subset U$ and all $\alpha \in \mathbb{N}^{2n}$, $\beta \in \mathbb{N}^{2n}$, $\gamma \in \mathbb{N}$, there is a constant $c > 0$ such that $\left|\partial^\alpha_x \partial^\beta_y \partial^\gamma_t a(x,y,t)\right| \le c(1+|t|)^{m-|\gamma|}$, $(x,y,t) \in K\times]0,\infty[$. $S^m_{1,0}$ is called the space of symbols of order m type $(1,0)$. We write $S^{-\infty}_{1,0} = \bigcap S^m_{1,0}$.

Let $S^m_{1,0}\Big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big)$ denote the space of restrictions to $U\bigcap(M\times M)\times]0,\infty[$ of elements in

$$S^m_{1,0}\big(U\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\big).$$

Let

$$a_j \in S^{m_j}_{1,0}\Big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big),\ j = 0,1,2,\ldots,$$

with $m_j \searrow -\infty$, $j \to \infty$. Then there exists

$$a \in S^{m_0}_{1,0}\Big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big)$$

such that

$$a - \sum_{0\le j<k} a_j \in S^{m_k}_{1,0}\Big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big),$$

for every $k \in \mathbb{N}$ (see Proposition 1.8 of Grigis-Sjöstrand [**12**]).

If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \text{ in } S^{m_0}_{1,0}\Big(U \bigcap (\overline{M} \times \overline{M}) \times [0, \infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big).$$

Let

$$\Pi^{(q)} : L^2(M;\, \Lambda^{0,q}T^*(M')) \to \operatorname{Ker} \Box^{(q)}$$

be the Bergman projection, i.e. the orthogonal projection onto the kernel of $\Box^{(q)}$. Let $K_{\Pi^{(q)}}(z, w) \in \mathscr{D}'\big(M \times M;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\big)$ be the distribution kernel of $\Pi^{(q)}$.

Let X and Y be C^∞ vector bundles over M'. Let

$$C, D : C^\infty_0(M;\, X) \to \mathscr{D}'(M;\, Y)$$

with distribution kernels $K_C(z, w), K_D(z, w) \in \mathscr{D}'(M \times M;\, \mathscr{L}(X_w, Y_z))$. We write $C \equiv D \mod C^\infty\big(U \bigcap (\overline{M} \times \overline{M})\big)$ if $K_C(z, w) = K_D(z, w) + F(z, w)$, where $F(z, w)|_U \in C^\infty\big(U \bigcap (\overline{M} \times \overline{M});\, \mathscr{L}(X_w, Y_z)\big)$ and U is an open set in $M' \times M'$.

Given q, $0 \leq q \leq n - 1$. Put

$$\Gamma_q = \{z \in \Gamma;\, Z(q) \text{ fails at } z\}\,. \tag{1.11}$$

If the Levi form is non-degenerate at each point of Γ, then Γ_q is a union of connected components of Γ.

The main result of this work is the following

THEOREM 1.2. – *Let M be a relatively compact open subset with C^∞ boundary Γ of a complex analytic manifold M' of dimension n. We assume that the Levi form is non-degenerate at each point of Γ. Let q, $0 \leq q \leq n - 1$. Suppose that $Z(q)$ fails at some point of Γ and that $Z(q - 1)$ and $Z(q + 1)$ hold at each point of Γ. Then*

$$K_{\Pi^{(q)}}(z, w) \in C^\infty\Big(\overline{M} \times \overline{M} \setminus \operatorname{diag}(\Gamma_q \times \Gamma_q);\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big).$$

Moreover, in a neighborhood U of $\operatorname{diag}(\Gamma_q \times \Gamma_q)$, $K_{\Pi^{(q)}}(z, w)$ *satisfies*

$$K_{\Pi^{(q)}}(z, w) \equiv \int_0^\infty e^{i\phi(z,w)t} b(z, w, t)dt \mod C^\infty\big(U \bigcap (\overline{M} \times \overline{M})\big) \tag{1.12}$$

(for the precise meaning of the oscillatory integral $\int_0^\infty e^{i\phi(z,w)t} b(z, w, t)dt$, see Remark 1.3 below) with

$$b(z, w, t) \in S^n_{1,0}\Big(U \bigcap (\overline{M} \times \overline{M}) \times]0, \infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big),$$

$$b(z, w, t) \sim \sum_{j=0}^{\infty} b_j(z, w) t^{n-j}$$

in the space $S^n_{1,0}\Big(U \bigcap (\overline{M} \times \overline{M}) \times]0, \infty[;\, \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\Big)$,

$$b_0(z, z) \neq 0,\ \ z \in \Gamma_q,$$

where $b_j(z,w)\in C^\infty\Big(U\bigcap(\overline{M}\times\overline{M});\,\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\Big)$*,* $j=0,1,\ldots,$ *and*

$$\phi(z,w)\in C^\infty(U\bigcap(\overline{M}\times\overline{M})),\quad \operatorname{Im}\phi\geq 0,\tag{1.13}$$

$$\phi(z,z)=0,\quad z\in\Gamma_q,\quad \phi(z,w)\neq 0\ \ \textit{if}\ \ (z,w)\notin\operatorname{diag}(\Gamma_q\times\Gamma_q),\tag{1.14}$$

$$\operatorname{Im}\phi(z,w)>0\ \ \textit{if}\ \ (z,w)\notin\Gamma\times\Gamma,\tag{1.15}$$

$$\phi(z,w)=-\overline{\phi}(w,z).\tag{1.16}$$

For $p\in\Gamma_q$*, we have*

$$\begin{aligned}&\sigma_{\Box^{(q)}_f}(z,d_z\phi(z,w))\ \ \textit{vanishes to infinite order at}\ \ z=p,\\&(z,w)\ \textit{is in some neighborhood of}\ (p,p)\ \textit{in}\ M'.\end{aligned}\tag{1.17}$$

For $z=w$*,* $z\in\Gamma_q$*, we have* $d_z\phi=-\omega_0-idr$*,* $d_w\phi=\omega_0-idr$*.*

Moreover, we have $\phi(z,w)=\phi_-(z,w)$ *if* $z,w\in\Gamma_q$*, where* $\phi_-(z,w)\in C^\infty(\Gamma_q\times\Gamma_q)$ *is the phase appearing in the description of the Szegő projection in Part I (see also Theorem 6.15 below). More properties of the phase* $\phi(z,w)$ *will be given in Theorem 1.4.*

Remark 1.3. – Let ϕ and $b(z,w,t)$ be as in Theorem 1.2. Let $y=(y_1,\ldots,y_{2n-1})$ be local coordinates on Γ and extend $y_1,\ldots,y_{2n-1}$ to real smooth functions in some neighborhood of Γ. We work with local coordinates

$$w=(y_1,\ldots,y_{2n-1},r)$$

defined on some neighborhood U of $p\in\Gamma_q$. Let $u\in C^\infty_0(U;\,\Lambda^{0,q}T^*(M'))$. Choose a cut-off function $\chi(t)\in C^\infty(\mathbb{R})$ so that $\chi(t)=1$ when $|t|<1$ and $\chi(t)=0$ when $|t|>2$. Set

$$(B_\varepsilon u)(z)=\iint_0^\infty e^{i\phi(z,w)t}b(z,w,t)\chi(\varepsilon t)u(w)dtdw.$$

Since $d_y\phi\neq 0$ where $\operatorname{Im}\phi=0$ (see (6.21)), we can integrate by parts in y and t and obtain $\lim_{\varepsilon\to 0}(B_\varepsilon u)(z)\in C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))$. This means that $B=\lim_{\varepsilon\to 0}B_\varepsilon:C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))$ is continuous. We write $B(z,w)$ to denote the distribution kernel of B. Formally, $B(z,w)=\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt$.

From (1.17) and Remark 1.5 of Part I it follows that

Theorem 1.4. – *Under the assumptions of Theorem 1.2, let* $p\in\Gamma_q$*. We choose local complex analytic coordinates* $z=(z_1,\ldots,z_n)$*,* $z_j=x_{2j-1}+ix_{2j}$*,* $j=1,\ldots,n$*, vanishing at* p *such that the metric on* $\Lambda^{1,0}T(M')$ *is* $\sum_{j=1}^n dz_j\otimes d\overline{z}_j$ *at* p *and* $r(z)=\sqrt{2}\operatorname{Im}z_n+\sum_{j=1}^{n-1}\lambda_j\,|z_j|^2+O(|z|^3)$*, where* λ_j*,* $j=1,\ldots,n-1$*, are the eigenvalues of*

L_p (this is always possible. see Lemma 3.2 of **[19]***). We also write $w=(w_1,\dots,w_n)$, $w_j=y_{2j-1}+iy_{2j}$, $j=1,\dots,n$. Then, we can take $\phi(z,w)$ so that*

$$
\begin{aligned}
\phi(z,w) &= -\sqrt{2}x_{2n-1}+\sqrt{2}y_{2n-1}-ir(z)\Big(1+\sum_{j=1}^{2n-1}a_jx_j+\frac{1}{2}a_{2n}x_{2n}\Big)\\
&\quad -ir(w)\Big(1+\sum_{j=1}^{2n-1}\overline{a}_jy_j+\frac{1}{2}\overline{a}_{2n}y_{2n}\Big)+i\sum_{j=1}^{n-1}|\lambda_j|\,|z_j-w_j|^2\\
&\quad +\sum_{j=1}^{n-1}i\lambda_j(\overline{z}_jw_j-z_j\overline{w}_j)+O\big(|(z,w)|^3\big)
\end{aligned}
\tag{1.18}
$$

in some neighborhood of (p,p) in $M'\times M'$, where $a_j=\frac{1}{2}\frac{\partial\sigma_{\Box_f^{(q)}}}{\partial x_j}(p,-\omega_0(p)-idr(p))$, $j=1,\dots,2n$.

We have the following corollary of Theorem 1.2

COROLLARY 1.5. – *Under the assumptions of Theorem 1.2 and let U be a small neighborhood of $\operatorname{diag}(\Gamma_q\times\Gamma_q)$. Then there exist smooth functions $F,G\in C^\infty\Big(U\bigcap(\overline{M}\times\overline{M}));\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\Big)$ such that*

$$K_{\Pi^{(q)}}=F(-i(\phi(z,w)+i0))^{-n-1}+G\log(-i(\phi(z,w)+i0)).$$

Moreover, we have

$$
\begin{aligned}
F &= \sum_{j=0}^{n}(n-j)!b_j(z,w)(-i\phi(z,w))^j+f(z,w)(\phi(z,w))^{n+1},\\
G &\equiv \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}b_{n+j+1}(z,w)(-i\phi(z,w))^j \mod C^\infty\big(U\textstyle\bigcap(\overline{M}\times\overline{M})\big)
\end{aligned}
\tag{1.19}
$$

where $f(z,w)\in C^\infty\Big(U\bigcap(\overline{M}\times\overline{M});\mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\Big)$.

If $w\in\Lambda^{0,1}T^*_z(M')$, let $w^{\wedge,*}:\Lambda^{0,q+1}T^*_z(M')\to\Lambda^{0,q}T^*_z(M')$ be the adjoint of left exterior multiplication $w^\wedge:\Lambda^{0,q}T^*_z(M')\to\Lambda^{0,q+1}T^*_z(M')$. That is,

$$(w^\wedge u\mid v)=(u\mid w^{\wedge,*}v),\tag{1.20}$$

for all $u\in\Lambda^{0,q}T^*_z(M')$, $v\in\Lambda^{0,q+1}T^*_z(M')$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

PROPOSITION 1.6. – *Under the assumptions of Theorem 1.2, let $p\in\Gamma_q$, $q=n_-$. Let $U_1(z),\dots,U_{n-1}(z)$ be an orthonormal frame of $\Lambda^{1,0}T_z(\Gamma)$, $z\in\Gamma$, for which the Levi form is diagonalized at p. Let $e_j(z)$, $j=1,\dots,n-1$ denote the basis of $\Lambda^{0,1}T^*_z(\Gamma)$, $z\in\Gamma$, which is dual to $\overline{U}_j(z)$, $j=1,\dots,n-1$. Let $\lambda_j(z)$, $j=1,\dots,n-1$ be the* *eigenvalues of the Levi form L_z, $z \in \Gamma$. We assume that $\lambda_j(p) < 0$ if $1 \leq j \leq n_-$. Then*

(1.21)

$$F(p,p) = n!\,|\lambda_1(p)|\cdots|\lambda_{n-1}(p)|\,\pi^{-n}2\Big(\prod_{j=1}^{j=n_-} e_j(p)^\wedge e_j^{\wedge,*}(p)\Big) \circ (\overline{\partial}r(p))^{\wedge,*}(\overline{\partial}r(p))^\wedge,$$

where F is as in Corollary 1.5.

In the rest of this chapter, we outline the proof of Theorem 1.2. We assume that the Levi form is non-degenerate at each point of Γ. We pause and recall a general fact of distribution theory (see Hörmander [**18**]). Let E, F be C^∞ vector bundles over C^∞ manifolds G and H respectively. We take smooth densities of integration on G and H respectively. If $A : C_0^\infty(G;\, E) \to \mathscr{D}'(H;\, F)$ is continuous, we write $K_A(x,y)$ or $A(x,y)$ to denote the distribution kernel of A. Then the following two statements are equivalent

(a) A is continuous: $\mathscr{E}'(G;\, E) \to C^\infty(H;\, F)$,
(b) $K_A \in C^\infty(H \times G;\, \mathscr{L}(E_y, F_x))$.

If A satisfies (a) or (b), we say that A is smoothing. Let $B : C_0^\infty(G;\, E) \to \mathscr{D}'(H;\, F)$. We write $A \equiv B$ if $A - B$ is a smoothing operator.

Let γ denote the operator of restriction to the boundary Γ. Let us consider the map

$$F^{(q)} : H^2(\overline{M};\, \Lambda^{0,q}T^*(M')) \to H^0(\overline{M};\, \Lambda^{0,q}T^*(M')) \oplus H^{\frac{3}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')),\quad u \to (\Box_f^{(q)}u, \gamma u). \tag{1.22}$$

It is well-known that $\dim \operatorname{Ker} F^{(q)} < \infty$ and $\operatorname{Ker} F^{(q)} \subset C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$. Let

$$K^{(q)} : H^2(\overline{M};\, \Lambda^{0,q}T^*(M')) \to \operatorname{Ker} F^{(q)} \tag{1.23}$$

be the orthogonal projection with respect to $(\ |\ )_M$. Then,

$$K^{(q)} \in C^\infty\big(\overline{M} \times \overline{M};\, \mathscr{L}(\Lambda^{0,q}T^*(M'), \Lambda^{0,q}T^*(M'))\big). \tag{1.24}$$

Put

$$\widetilde{\Box}_f^{(q)} = \Box_f^{(q)} + K^{(q)} \tag{1.25}$$

and consider the map

$$\widetilde{F}^{(q)} : H^2(\overline{M};\, \Lambda^{0,q}T^*(M')) \to H^0(\overline{M};\, \Lambda^{0,q}T^*(M')) \oplus H^{\frac{3}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')),\quad u \to (\widetilde{\Box}_f^{(q)}u, \gamma u). \tag{1.26}$$

We can check that $\widetilde{F}^{(q)}$ is injective (see Chapter 3). Let

$$\widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) \tag{1.27}$$

be the Poisson operator for $\widetilde{\Box}^{(q)}_f$ which is well-defined since (1.26) is injective. It is well-known that $\widetilde{P}$ extends continuously

$$\widetilde{P} : H^s(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{s+\frac{1}{2}}(\overline{M};\, \Lambda^{0,q}T^*(M')),\quad \forall\, s \in \mathbb{R}$$

(see page 29 of Boutet de Monvel [**31**]). Let

$$\widetilde{P}^* : \mathscr{E}'(\overline{M};\, \Lambda^{0,q}T^*(M')) \to \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(M'))$$

be the operator defined by $(\widetilde{P}^*u \mid v)_\Gamma = (u \mid \widetilde{P}v)_M, u \in \mathscr{E}'(\overline{M};\, \Lambda^{0,q}T^*(M')), v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$. It is well-known (see page 30 of [**31**]) that $\widetilde{P}^*$ is continuous: $\widetilde{P}^* : L^2(M;\, \Lambda^{0,q}T^*(M')) \to H^{\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$ and

$$\widetilde{P}^* : C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')).$$

We use the inner product $[\ |\ ]$ on $H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$ defined as follows:

$$[u \mid v] = (\widetilde{P}u \mid \widetilde{P}v)_M,$$

where $u,\, v \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$. We consider $(\overline{\partial}r)^{\wedge,*}$ as an operator

$$(\overline{\partial}r)^{\wedge,*} : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q-1}T^*(M')).$$

Note that $(\overline{\partial}r)^{\wedge,*}$ is the pointwise adjoint of $\overline{\partial}r$ with respect to $(\ |\ )$. Let

$$T : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to \operatorname{Ker}(\overline{\partial}r)^{\wedge,*} \tag{1.28}$$

be the orthogonal projection onto $\operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$ with respect to $[\ |\ ]$. That is, if $u \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$, then $(\overline{\partial}r)^{\wedge,*}Tu = 0$ and $[(I-T)u \mid g] = 0$, for all $g \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$. In Chapter 3, we will show that T is a classical pseudodifferential operator of order 0 with principal symbol $2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^\wedge$. If $u \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$, then $u \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$ if and only if $u \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$. Put

$$\overline{\partial_\beta} = T\gamma\overline{\partial}\widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\, \Lambda^{0,q+1}T^*(\Gamma)). \tag{1.29}$$

$\overline{\partial_\beta}$ is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q+1)$ forms. It is easy to see that $\overline{\partial_\beta} = \overline{\partial_b}+$lower order terms, where $\overline{\partial_b}$ is the tangential Cauchy-Riemann operator (see [**10**] or Chapter 5). In Chapter 5, we will show that $(\overline{\partial_\beta})^2 = 0$. Let

$$\overline{\partial_\beta}^\dagger : C^\infty(\Gamma;\, \Lambda^{0,q+1}T^*(\Gamma)) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$$

be the formal adjoint of $\overline{\partial_\beta}$ with respect to $[\ |\ ]$. $\overline{\partial_\beta}^\dagger$ is a classical pseudodifferential operator of order one from boundary $(0,q+1)$ forms to boundary $(0,q)$ forms. In Chapter 5, we will show that $\overline{\partial_\beta}^\dagger = \gamma\overline{\partial_f}^*\widetilde{P}$.

Put

$$\Box^{(q)}_\beta = \overline{\partial_\beta}\ \overline{\partial_\beta}^\dagger + \overline{\partial_\beta}^\dagger\overline{\partial_\beta} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)).$$

For simplicity, we assume that $\Gamma=\Gamma_q$, $\Gamma_q\neq\Gamma_{n-1-q}$ (Γ_q is given by (1.11)). We can repeat the method of Part I (see Chapter 6) to construct

$$A\in L^{-1}_{\frac{1}{2},\frac{1}{2}}\big(\Gamma;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)\big),\quad B\in L^{0}_{\frac{1}{2},\frac{1}{2}}\big(\Gamma;\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma)\big)$$

such that $A\Box^{(q)}_\beta+B\equiv B+\Box^{(q)}_\beta A\equiv I$, $\overline{\partial_\beta}B\equiv 0$, $\overline{\partial_\beta}^\dagger B\equiv 0$, and $B\equiv B^\dagger\equiv B^2$, where $L^m_{\frac{1}{2},\frac{1}{2}}$ is the space of pseudodifferential operators of order m type $(\frac{1}{2},\frac{1}{2})$ (see Definition 6.11) and $B^\dagger$ is the formal adjoint of B with respect to $[\ |\ ]$. Moreover, $K_B(x,y)$ satisfies $K_B(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}b(x,y,t)dt$, where $\phi_-(x,y)$ and $b(x,y,t)$ are as in Theorem 6.15. In Chapter 7, we will show that

$$\Pi^{(q)}\equiv\widetilde{P}BT(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\quad\mathrm{mod}\ C^\infty(\overline{M}\times\overline{M})$$

and $\widetilde{P}BT(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)dt\ \mathrm{mod}\ C^\infty(\overline{M}\times\overline{M})$, where $\phi(z,w)$ and $b(z,w,t)$ are as in Theorem 1.2.

CHAPTER 2

THE $\overline{\partial}$-NEUMANN PROBLEM, A REVIEW

In this chapter, we will give a brief discussion of the $\overline{\partial}$-Neumann problem in a setting appropriate for our purpose. General references for this chapter are the books by Hörmander [**17**], [**10**] and Chen-Shaw [**6**].

As in Chapter 1, let M be a relatively compact open subset with smooth boundary Γ of a complex manifold M' of dimension n with a smooth Hermitian metric on its holomorphic tangent bundle. We will use the same notations as before. We have the following (see page 13 of [**10**] for the proof).

Lemma 2.1. – *For all $f \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$, $g \in C^\infty(\overline{M};\, \Lambda^{0,q+1}T^*(M'))$,*

$$(\overline{\partial} f \mid g)_M = (f \mid \overline{\partial_f}^* g)_M + (\gamma f \mid \gamma(\overline{\partial} r)^{\wedge,*} g)_\Gamma, \tag{2.1}$$

where $(\overline{\partial} r)^{\wedge,*}$ is defined by (1.20). *We recall that $\overline{\partial_f}^*$ is the formal adjoint of $\overline{\partial}$ and γ is the operator of restriction to the boundary Γ.*

From Lemma 2.1, it follows that

$$\operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) = \left\{u \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'));\ \gamma(\overline{\partial} r)^{\wedge,*} u = 0\right\} \tag{2.2}$$

and

$$\overline{\partial}^* = \overline{\partial_f}^* \text{ on } \operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')). \tag{2.3}$$

The $\overline{\partial}$-Neumann Laplacian on $(0,q)$ forms is the operator in $L^2(M;\, \Lambda^{0,q}T^*(M'))$

$$\Box^{(q)} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

We notice that $\Box^{(q)}$ is self-adjoint (see chapter I of [**10**]). We have

$$\operatorname{Dom}\Box^{(q)} = \Big\{u \in L^2(M;\, \Lambda^{0,q}T^*(M'));\ u \in \operatorname{Dom}\overline{\partial}^* \bigcap \operatorname{Dom}\overline{\partial},$$
$$\overline{\partial}^* u \in \operatorname{Dom}\overline{\partial},\ \overline{\partial} u \in \operatorname{Dom}\overline{\partial}^*\Big\}.$$

Put $D^{(q)} = \mathrm{Dom}\,\Box^{(q)} \bigcap C^\infty(\overline{M}\ ; \Lambda^{0,q}T^*(M'))$. From (2.2), we have

$$D^{(q)} = \left\{u \in C^\infty(\overline{M};\ \Lambda^{0,q+1}T^*(M'));\ \gamma(\overline{\partial}r)^{\wedge,*}u = 0,\ \gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}u = 0\right\}. \tag{2.4}$$

In view of (1.3), we see that $u \in D^{(q)}$ if and only if $\gamma u \in C^\infty(\Gamma;\ \Lambda^{0,q}T^*(\Gamma))$ and $\gamma\overline{\partial}u \in C^\infty(\Gamma;\ \Lambda^{0,q+1}T^*(\Gamma))$. We have the following

Lemma 2.2. – *Let $q \geq 1$. For every $u \in \mathrm{Dom}\,\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q+1}T^*(M'))$, we have*

$$\overline{\partial}^*u \in \mathrm{Dom}\,\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')).$$

Proof. – Let

$$u \in \mathrm{Dom}\,\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q+1}T^*(M')).$$

For $g \in C^\infty(\overline{M};\ \Lambda^{0,q-1}T^*(M'))$, we have

$$\begin{aligned} 0 = (\overline{\partial_f}^*\overline{\partial}^*u \mid g)_M &= (\overline{\partial}^*u \mid \overline{\partial}g)_M - (\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}^*u \mid \gamma g)_\Gamma \\ &= (u \mid \overline{\partial}\overline{\partial}g)_M - (\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}^*u \mid \gamma g)_\Gamma \\ &= -(\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}^*u \mid \gamma g)_\Gamma. \end{aligned}$$

Here we used (2.1). Thus, $\gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}^*u = 0$. The lemma follows. □

Definition 2.3. – The boundary conditions

$$\gamma(\overline{\partial}r)^{\wedge,*}u = 0,\quad \gamma(\overline{\partial}r)^{\wedge,*}\overline{\partial}u = 0,\quad u \in C^\infty(\overline{M}\ , \Lambda^{0,q}T^*(M'))$$

are called $\overline{\partial}$-Neumann boundary conditions.

Definition 2.4. – The $\overline{\partial}$-Neumann problem in M is the problem of finding, given a form $f \in C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$, another form $u \in D^{(q)}$ verifying $\Box^{(q)}u = f$.

Definition 2.5. – Given q, $0 \leq q \leq n-1$. The Levi form is said to satisfy condition $Z(q)$ at $p \in \Gamma$ if it has at least $n-q$ positive or at least $q+1$ negative eigenvalues. If the Levi form is non-degenerate at $p \in \Gamma$, let (n_-, n_+), $n_- + n_+ = n-1$, be the signature of L_p. Then $Z(q)$ holds at p if and only if $q \neq n_-$.

The following classical results are due to Kohn. For the proofs, see **[10]**.

Theorem 2.6. – *We assume that $Z(q)$ holds at each point of Γ. Then* $\mathrm{Ker}\,\Box^{(q)}$ *is a finite dimensional subspace of $C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$, $\Box^{(q)}$ has closed range and $\Pi^{(q)}$ is a smoothing operator. That is, the distribution kernel*

$$K_{\Pi^{(q)}}(z,w) \in C^\infty\big(\overline{M}\times\overline{M};\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'), \Lambda^{0,q}T^*_z(M'))\big).$$

Moreover, there exists an continuous operator

$$N^{(q)} : L^2(M;\ \Lambda^{0,q}T^*(M')) \to \mathrm{Dom}\,\Box^{(q)}$$

such that $N^{(q)}\Box^{(q)} + \Pi^{(q)} = I$ on $\mathrm{Dom}\,\Box^{(q)}$,

$$\Box^{(q)}N^{(q)} + \Pi^{(q)} = I$$

on $L^2(M;\Lambda^{0,q}T^*(M'))$. *Furthermore,*

$$N^{(q)}\Big(C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\Big)\subset C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$$

and for each $s\in\mathbb{R}$ *and all* $f\in C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))$, *there is a constant* $c>0$, *such that* $\left\|N^{(q)}f\right\|_{s+1}\leq c\left\|f\right\|_s$ *where* $\|\ \|_s$ *denotes any of the equivalent norms defining* $H^s(\overline{M};\Lambda^{0,q}T^*(M'))$.

THEOREM 2.7. – *Suppose that* $Z(q)$ *fails at some point of* Γ *and that* $Z(q-1)$ *and* $Z(q+1)$ *hold at each point of* Γ. *Then,*

$$\Pi^{(q)}u=(I-\overline{\partial}N^{(q-1)}\overline{\partial}^*-\overline{\partial}^*N^{(q+1)}\overline{\partial})u,\quad u\in\operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M')),\tag{2.5}$$

where $N^{(q+1)}$ *and* $N^{(q-1)}$ *are as in Theorem 2.6. In particular,*

$$\Pi^{(q)}:\operatorname{Dom}\overline{\partial}^*\bigcap C^\infty(\overline{M};\Lambda^{0,q}T^*(M'))\to D^{(q)}.$$

CHAPTER 3

THE OPERATOR T

First, we claim that $\widetilde{F}^{(q)}$ is injective, where $\widetilde{F}^{(q)}$ is given by (1.26). If $u \in \operatorname{Ker} \widetilde{F}^{(q)}$, then $u \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$, $\widetilde{\Box}^{(q)}_f u = 0$ and $\gamma u = 0$. We can check that

$$\begin{aligned}(\widetilde{\Box}^{(q)}_f u \mid u)_M &= (\Box^{(q)}_f u \mid u)_M + (K^{(q)} u \mid u)_M \\ &= (\overline{\partial} u \mid \overline{\partial} u)_M + (\overline{\partial}^*_f u \mid \overline{\partial}^*_f u)_M + (K^{(q)} u \mid K^{(q)} u)_M = 0.\end{aligned}$$

Thus, $u \in \operatorname{Ker} F^{(q)} \bigcap \operatorname{Ker} K^{(q)}$ ($F^{(q)}$ is given by (1.22)). We get $u = 0$. Hence, $\widetilde{F}^{(q)}$ is injective. The Poisson operator

$$\widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$$

of $\widetilde{\Box}^{(q)}_f$ is well-defined. That is, if $u \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$, then

$$\widetilde{P}u \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')),\quad \widetilde{\Box}^{(q)}_f \widetilde{P}u = 0,\quad \gamma\widetilde{P}u = u.$$

Moreover, if $v \in C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M'))$ and $\widetilde{\Box}^{(q)}_f v = 0$, then $v = \widetilde{P}\gamma v$. Furthermore, it is straight forward to see that

$$\overline{\partial}\widetilde{P}u = \widetilde{P}\gamma\overline{\partial}\widetilde{P}u,\quad \overline{\partial}^*_f\widetilde{P}u = \widetilde{P}\gamma\overline{\partial}^*_f\widetilde{P}u,\quad u \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')). \tag{3.1}$$

We recall that (see Chapter 1) $\widetilde{P}$ extends continuously

$$\widetilde{P} : H^s(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{s+\frac{1}{2}}(\overline{M};\, \Lambda^{0,q}T^*(M')),\quad \forall\ s \in \mathbb{R}.$$

As in Chapter 1, let $\widetilde{P}^* : \mathscr{E}'(\overline{M};\, \Lambda^{0,q}T^*(M')) \to \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(M'))$ be the operator defined by

$$(\widetilde{P}^* u \mid v)_\Gamma = (u \mid \widetilde{P}v)_M,$$

$u \in \mathscr{E}'(\overline{M};\, \Lambda^{0,q}T^*(M'))$, $v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$. We recall that (see Chapter 1) $\widetilde{P}^*$ is continuous:

$$\widetilde{P}^* : L^2(M;\, \Lambda^{0,q}T^*(M')) \to H^{\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$$

and $\widetilde{P}^* : C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$.

Let L be a classical pseudodifferential operator on a C^∞ manifold. From now on, we let σ_L denote the principal symbol of L. The operator

$$\widetilde{P}^*\widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$$

is a classical elliptic pseudodifferential operator of order -1 and invertible since $\widetilde{P}$ is injective (see Boutet de Monvel [**30**]). Let $\triangle_\Gamma$ be the real Laplacian on Γ and let $\sqrt{-\triangle_\Gamma}$ be the square root of $-\triangle_\Gamma$. It is well-known (see [**30**]) that

$$\sigma_{\widetilde{P}^*\widetilde{P}} = \sigma_{(2\sqrt{-\triangle_\Gamma})^{-1}}. \tag{3.2}$$

Let $(\widetilde{P}^*\widetilde{P})^{-1} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$ be the inverse of $\widetilde{P}^*\widetilde{P}$. $(\widetilde{P}^*\widetilde{P})^{-1}$ is a classical elliptic pseudodifferential operator of order 1 with scalar principal symbol. We have

$$\sigma_{(\widetilde{P}^*\widetilde{P})^{-1}} = \sigma_{2\sqrt{-\triangle_\Gamma}}. \tag{3.3}$$

Definition 3.1. – The Neumann operator $\mathscr{N}^{(q)}$ is the operator on $C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$ defined as follows:

$$\mathscr{N}^{(q)}f = \gamma\frac{\partial}{\partial r}\widetilde{P}f, \quad f \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')).$$

The following is well-known (see page 95 of Greiner-Stein [**11**])

Lemma 3.2. – $\mathscr{N}^{(q)} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M'))$ *is a classical elliptic pseudodifferential operator of order* 1 *with scalar principal symbol and we have*

$$\sigma_{\mathscr{N}^{(q)}} = \sigma_{\sqrt{-\triangle_\Gamma}}. \tag{3.4}$$

We use the inner product $[\ |\ ]$ on $H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$ defined as follows:

$$[u \mid v] = (\widetilde{P}u \mid \widetilde{P}v)_M = (\widetilde{P}^*\widetilde{P}u \mid v)_\Gamma, \tag{3.5}$$

where $u,\, v \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$. We consider $(\overline{\partial}r)^{\wedge,*}$ as an operator

$$(\overline{\partial}r)^{\wedge,*} : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q-1}T^*(M')).$$

Let

$$T : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to \operatorname{Ker}(\overline{\partial}r)^{\wedge,*} = H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \tag{3.6}$$

be the orthogonal projection onto $\operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$ with respect to $[\ |\ ]$. That is, if $u \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$, then $(\overline{\partial}r)^{\wedge,*}Tu = 0$ and $[(I-T)u \mid g] = 0,\ \ \forall\, g \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$.

Lemma 3.3. – *T is a classical pseudodifferential operator of order* 0 *with principal symbol* $2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}$. *Moreover,*

$$I - T = (\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^{\wedge}R, \tag{3.7}$$

where $R : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\, \Lambda^{0,q-1}T^*(M'))$ *is a classical pseudodifferential operator of order* -1.

Proof. – Let $E = 2(\overline{\partial}r)^{\wedge,*}\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}+2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}$,

$$E : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')),$$

where $\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}$ is the formal adjoint of $(\overline{\partial}r)^{\wedge,*}$ with respect to $[\,|\,]$. That is,

$$[(\overline{\partial}r)^{\wedge,*}u \mid v] = [u \mid \big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}v],$$

$u \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$, $v \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q-1}T^*(M'))$. We can check that

$$\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger} = (\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^{\wedge}(\widetilde{P}^*\widetilde{P}). \tag{3.8}$$

Thus, the principal symbol of E is $2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge} + 2(\overline{\partial}r)^{\wedge}(\overline{\partial}r)^{\wedge,*}$. Since

$$\|dr\| = 1 = (\|\bar{\partial}r\|^2 + \|\partial r\|^2)^{\frac{1}{2}}$$

on Γ, we have

$$\|\bar{\partial}r\|^2 = \|\partial r\|^2 = \frac{1}{2} \ \text{ on } \ \Gamma. \tag{3.9}$$

From this, we can check that

$$2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge} + 2(\overline{\partial}r)^{\wedge}(\overline{\partial}r)^{\wedge,*} = I : H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')) \to H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')),$$

where I is the identity map. E is a classical elliptic pseudodifferential operator with principal symbol I. Then $\dim \operatorname{Ker} E < \infty$. Let G be the orthogonal projection onto $\operatorname{Ker} E$ and N be the partial inverse. Then G is a smoothing operator and N is a classical elliptic pseudodifferential operator of order 0 with principal symbol I (up to some smoothing operator). We have

$$EN + G = 2\Big((\overline{\partial}r)^{\wedge,*}\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}+2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}\Big)N + G = I \tag{3.10}$$

on $H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'))$. Put $\widetilde{T} = 2(\overline{\partial}r)^{\wedge,*}\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}N + G$. Note that

$$\operatorname{Ker} E = \Big\{u \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M'));\ (\overline{\partial}r)^{\wedge,*}u = 0,\ \big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}u = 0\Big\}.$$

From this and $(\overline{\partial}r)^{\wedge,*} \circ (\overline{\partial}r)^{\wedge,*} = 0$, we see that

$$\widetilde{T}g \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}, \quad g \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')).$$

From (3.10), we have $I - \widetilde{T} = 2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}N$ and

$$\begin{aligned}
[(I-\widetilde{T})g \mid u] &= [2\big((\overline{\partial}r)^{\wedge,*}\big)^{\dagger}(\overline{\partial}r)^{\wedge,*}Ng \mid u] \\
&= [2(\overline{\partial}r)^{\wedge,*}Ng \mid (\overline{\partial}r)^{\wedge,*}u] \\
&= 0, \quad u \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}, g \in H^{-\frac{1}{2}}(\Gamma;\, \Lambda^{0,q}T^*(M')).
\end{aligned}$$

Thus, $g = \widetilde{T}g + (I-\widetilde{T})g$ is the orthogonal decomposition with respect to $[\,|\,]$. Hence, $\widetilde{T} = T$. The lemma follows. □

Now, we assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Put

$$\widetilde{\Pi}^{(q)} = \Pi^{(q)}(I - K^{(q)}), \tag{3.11}$$

where $K^{(q)}$ is as in (1.23). It is straight forward to see that

$$\widetilde{\Pi}^{(q)} = \Pi^{(q)} - K^{(q)} = (I - K^{(q)})\Pi^{(q)} = \widetilde{P}\gamma\widetilde{\Pi}^{(q)} \tag{3.12}$$

and $(\widetilde{\Pi}^{(q)})^2 = \widetilde{\Pi}^{(q)}$, $(\widetilde{\Pi}^{(q)})^* = \widetilde{\Pi}^{(q)}$, where $(\widetilde{\Pi}^{(q)})^*$ is the formal adjoint of $\widetilde{\Pi}^{(q)}$ with respect to $(\ |\ )_M$.

Proposition 3.4. – *We assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then,*

$$\begin{aligned}\widetilde{\Pi}^{(q)}u &= \widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u\\ &= (I - K^{(q)})(I - \overline{\partial}N^{(q-1)}\overline{\partial}^* - \overline{\partial}^*N^{(q+1)}\overline{\partial})\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u,\\ & u \in C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')),\end{aligned} \tag{3.13}$$

where $N^{(q+1)}$, $N^{(q-1)}$ are as in Theorem 2.6, T is as in (3.6). In particular,

$$\widetilde{\Pi}^{(q)} : C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')) \to D^{(q)}.$$

Proof. – Let $v \in \operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$. From Theorem 2.7 and (3.12), we see that $\widetilde{\Pi}^{(q)}v \in D^{(q)}$ and $\widetilde{\Pi}^{(q)}v = \widetilde{P}\gamma\widetilde{\Pi}^{(q)}v$. Note that

$$(f - \widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*f \mid \widetilde{P}\gamma g)_M = 0, \quad f, g \in C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')).$$

We have

$$\begin{aligned}(\widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{\Pi}^{(q)}v)_M &= (\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{\Pi}^{(q)}v)_M\\ &= (\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{P}\gamma\widetilde{\Pi}^{(q)}v)_M\\ &= (\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{P}\gamma\widetilde{\Pi}^{(q)}v)_M\\ &= (u \mid \widetilde{\Pi}^{(q)}v)_M.\end{aligned} \tag{3.14}$$

Thus,

$$(u - \widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{\Pi}^{(q)}v)_M = 0.$$

Since $\operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$ is dense in $L^2(M;\ \Lambda^{0,q}T^*(M'))$, we get

$$(u - \widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid \widetilde{\Pi}^{(q)}v)_M = 0,$$

for all $v \in L^2(M;\ \Lambda^{0,q}T^*(M'))$. Thus, $\widetilde{\Pi}^{(q)}u = \widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u$.

Since

$$\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \in \operatorname{Dom}\overline{\partial}^* \bigcap C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')),$$

from (2.5) and (3.12), we get the last identity in (3.13). The proposition follows. ☐

CHAPTER 4

THE PRINCIPAL SYMBOLS OF $\widetilde{\gamma\overline{\partial}P}$ AND $\widetilde{\gamma\overline{\partial_f}^{*}P}$

Let J be the complex structure map for the tangent bundle $T(M')$. Put

$$Y = J(\frac{\partial}{\partial r}). \tag{4.1}$$

We notice that $J(iY + \frac{\partial}{\partial r}) = J\Big(iJ(\frac{\partial}{\partial r}) + \frac{\partial}{\partial r}\Big) = -i(iY + \frac{\partial}{\partial r})$. Thus, $iY + \frac{\partial}{\partial r} \in \Lambda^{0,1}T(M')$. Near Γ, put

$$T_z^{*,0,1} = \big\{u \in \Lambda^{0,1}T_z^*(M');\ (u \mid \overline{\partial}r(z)) = 0\big\} \tag{4.2}$$

and

$$T_z^{0,1} = \Big\{u \in \Lambda^{0,1}T_z(M');\ (u \mid (iY + \frac{\partial}{\partial r})(z)) = 0\Big\}. \tag{4.3}$$

We have the orthogonal decompositions with respect to $(\mid)$

$$\Lambda^{0,1}T_z^*(M') = T_z^{*,0,1} \oplus \big\{\lambda(\overline{\partial}r)(z);\ \lambda \in \mathbb{C}\big\}, \tag{4.4}$$

$$\Lambda^{0,1}T_z(M') = T_z^{0,1} \oplus \Big\{\lambda(iY + \frac{\partial}{\partial r})(z);\ \lambda \in \mathbb{C}\Big\}. \tag{4.5}$$

Note that $T_z^{*,0,1} = \Lambda^{0,1}T_z^*(\Gamma)$, $T_z^{0,1} = \Lambda^{0,1}T_z(\Gamma)$, $z \in \Gamma$.

First, we compute the principal symbols of $\overline{\partial}$ and $\overline{\partial_f}^{*}$. For each point $z_0 \in \Gamma$, we can choose an orthonormal frame $t_1(z), \ldots, t_{n-1}(z)$ for $T_z^{*,0,1}$ varying smoothly with z in a neighborhood of z_0. Then

$$t_1(z), \ldots, t_{n-1}(z), t_n(z) := \frac{\overline{\partial}r(z)}{\|\overline{\partial}r(z)\|}$$

is an orthonormal frame for $\Lambda^{0,1}T_z^*(M')$. Let

$$T_1(z), \ldots, T_{n-1}(z), T_n(z)$$

denote the basis of $\Lambda^{0,1}T_z(M')$ which is dual to $t_1(z),\dots,t_n(z)$. We have $T_n = \frac{iY+\frac{\partial}{\partial r}}{\|iY+\frac{\partial}{\partial r}\|}$. Note that

$$T_1(z),\dots,T_{n-1}(z) \text{ is an orthonormal frame for } \Lambda^{0,1}T_z(\Gamma),\ z\in\Gamma, \tag{4.6}$$

and

$$t_1(z),\dots,t_{n-1}(z) \text{ is an orthonormal frame for } \Lambda^{0,1}T^*_z(\Gamma),\ z\in\Gamma. \tag{4.7}$$

We have $\overline{\partial} f = \Big(\sum_{j=1}^n t_j^\wedge T_j\Big)f,\ f\in C^\infty(M')$. If

$$f(z)t_{j_1}(z)\wedge\cdots\wedge t_{j_q}(z)\in C^\infty(M';\ \Lambda^{0,q}T^*(M'))$$

is a typical term in a general $(0,q)$ form, we have

$$\overline{\partial} f = \sum_{j=1}^n (T_jf)t_j^\wedge t_{j_1}\wedge\cdots\wedge t_{j_q} + \sum_{k=1}^q(-1)^{k-1}f(z)t_{j_1}\wedge\cdots\wedge(\overline{\partial}t_{j_k})\wedge\cdots\wedge t_{j_q}.$$

So for the given orthonormal frame we have

$$\begin{aligned}\overline{\partial} &= \sum_{j=1}^n t_j^\wedge\circ T_j + \text{lower order terms}\\ &= \sum_{j=1}^{n-1} t_j^\wedge\circ T_j + \frac{(\overline{\partial}r)^\wedge}{\|\overline{\partial}r\|}\circ\frac{iY+\frac{\partial}{\partial r}}{\|iY+\frac{\partial}{\partial r}\|} + \text{lower order terms}\end{aligned} \tag{4.8}$$

and correspondingly

$$\overline{\partial_f}^* = \sum_{j=1}^{n-1} t_j^{\wedge,*}\circ T_j^* + \frac{(\overline{\partial}r)^{\wedge,*}}{\|\overline{\partial}r\|}\circ\frac{iY-\frac{\partial}{\partial r}}{\|iY+\frac{\partial}{\partial r}\|} + \text{lower order terms}. \tag{4.9}$$

We consider

$$\gamma\overline{\partial}\widetilde{P} : C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M'))\to C^\infty(\Gamma;\ \Lambda^{0,q+1}T^*(M'))$$

and

$$\gamma\overline{\partial_f}^*\widetilde{P} : C^\infty(\Gamma;\ \Lambda^{0,q+1}T^*(M'))\to C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M')).$$

$\gamma\overline{\partial}\widetilde{P}$ and $\gamma\overline{\partial_f}^*\widetilde{P}$ are classical pseudodifferential operators of order 1. From (3.9), we know that $\|\overline{\partial}r\| = \frac{1}{\sqrt{2}}$ on Γ. We can check that $\|iY+\frac{\partial}{\partial r}\| = \sqrt{2}$ on Γ. Combining this with (4.8), (4.9) and (3.4), we get

$$\gamma\overline{\partial}\widetilde{P} = \sum_{j=1}^{n-1} t_j^\wedge\circ T_j + (\overline{\partial}r)^\wedge\circ(iY+\sqrt{-\triangle_\Gamma}) + \text{lower order terms} \tag{4.10}$$

and

$$\gamma\overline{\partial_f}^*\widetilde{P} = \sum_{j=1}^{n-1} t_j^{\wedge,*}\circ T_j^* + (\overline{\partial}r)^{\wedge,*}\circ(iY-\sqrt{-\triangle_\Gamma}) + \text{lower order terms}. \tag{4.11}$$

From Lemma 2.2, it follows that

$$\gamma\overline{\partial}_f^{\,*}\widetilde{P} : C^\infty(\Gamma;\ \Lambda^{0,q+1}T^*(\Gamma)) \to C^\infty(\Gamma;\ \Lambda^{0,q}T^*(\Gamma)). \tag{4.12}$$

Put

$$\begin{aligned} \Sigma^+ &= \{(x, \lambda\omega_0(x)) \in T^*(\Gamma) \setminus 0;\ \lambda > 0\}, \\ \Sigma^- &= \{(x, \lambda\omega_0(x)) \in T^*(\Gamma) \setminus 0;\ \lambda < 0\}. \end{aligned} \tag{4.13}$$

We recall that $\omega_0 = J^t(dr)$. In Chapter 7, we need the following

PROPOSITION 4.1. – *The map* $\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} : C^\infty(\Gamma;\ \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\ \Lambda^{0,q}T^*(\Gamma))$ *is a classical pseudodifferential operator of order one from boundary* $(0,q)$ *forms to boundary* $(0,q)$ *forms and we have*

$$\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} = \frac{1}{2}(iY + \sqrt{-\triangle_\Gamma}) + \textit{lower order terms}. \tag{4.14}$$

In particular, it is elliptic outside Σ^-.

Proof. – Note that

$$\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} = \gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} + \gamma\overline{\partial}\widetilde{P}(\overline{\partial} r)^{\wedge,*} \tag{4.15}$$

on the space $C^\infty(\Gamma;\ \Lambda^{0,q}T^*(\Gamma))$. From (4.10), we have

$$\gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} = \sum_{j=1}^{n-1}\Big((\overline{\partial} r)^{\wedge,*}t_j^\wedge\Big)\circ T_j + \Big((\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^\wedge\Big)\circ(iY+\sqrt{-\triangle_\Gamma}) + \text{lower order terms}$$

and

$$\gamma\overline{\partial}\widetilde{P}(\overline{\partial} r)^{\wedge,*} = \sum_{j=1}^{n-1}\Big(t_j^\wedge(\overline{\partial} r)^{\wedge,*}\Big)\circ T_j + \Big((\overline{\partial} r)^\wedge(\overline{\partial} r)^{\wedge,*}\Big)\circ(iY+\sqrt{-\triangle_\Gamma}) + \text{lower order terms}.$$

Thus,

$$\begin{aligned} \gamma(\overline{\partial} r)^{\wedge,*}\overline{\partial}\widetilde{P} + \gamma\overline{\partial}\widetilde{P}(\overline{\partial} r)^{\wedge,*} &= \sum_{j=1}^{n-1}\Big(t_j^\wedge(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}t_j^\wedge\Big)\circ T_j \\ &\quad + \Big((\overline{\partial} r)^\wedge(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^\wedge\Big)\circ(iY + \sqrt{-\triangle_\Gamma}) \\ &\quad + \text{lower order terms}. \end{aligned} \tag{4.16}$$

Note that

$$t_j^\wedge(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}t_j^\wedge = 0,\quad j = 1, \ldots, n-1, \tag{4.17}$$

and

$$(\overline{\partial} r)^\wedge(\overline{\partial} r)^{\wedge,*} + (\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^\wedge = \frac{1}{2}. \tag{4.18}$$

Combining this with (4.16) and (4.15), we get (4.14).

Note that $\sigma_{iY+\sqrt{-\triangle_\Gamma}}(x,\xi) = -\langle Y,\xi\rangle + \|\xi\| = \|\xi\| + (\omega_0 \mid \xi) \geq 0$ with equality precisely when $\xi = -\lambda\omega_0$, $\lambda > 0$. The proposition follows. □

For $z \in \Gamma$, put

$$I^{0,q}T_z^*(M') = \left\{u \in \Lambda^{0,q}T_z^*(M');\ u = (\overline{\partial}r)^\wedge g,\ g \in \Lambda^{0,q-1}T_z^*(M')\right\}. \tag{4.19}$$

$I^{0,q}T_z^*(M)$ is orthogonal to $\Lambda^{0,q}T_z^*(\Gamma)$. In Chapter 6, we need the following

Proposition 4.2. – *The operator*

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} : C^\infty(\Gamma;\ I^{0,q}T^*(M')) \to C^\infty(\Gamma;\ I^{0,q}T^*(M'))$$

is a classical pseudodifferential operator of order one,

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} = (iY - \sqrt{-\triangle_\Gamma})\sqrt{-\triangle_\Gamma} + \textit{lower order terms.} \tag{4.20}$$

It is elliptic outside Σ^+.

Proof. – Note that

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} = \gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} + \gamma\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^\wedge \tag{4.21}$$

on the space $C^\infty(\Gamma;\ I^{0,q}T^*(\Gamma))$. From (4.11) and (3.3), we have

$$\begin{aligned}\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} &= \sum_{j=1}^{n-1}\Big((\overline{\partial}r)^\wedge t_j^{\wedge,*}\Big)\circ\Big(T_j^*\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\Big((\overline{\partial}r)^\wedge(\overline{\partial}r)^{\wedge,*}\Big)\circ\Big((iY - \sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\text{lower order terms}\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\gamma\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^\wedge &= \sum_{j=1}^{n-1}\Big(t_j^{\wedge,*}(\overline{\partial}r)^\wedge\Big)\circ\Big(T_j^*\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\Big((\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^\wedge\Big)\circ\Big((iY - \sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\text{lower order terms.}\end{aligned} \tag{4.23}$$

Thus,

$$\begin{aligned}&\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} + \gamma\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^\wedge\\ &\quad= \sum_{j=1}^{n-1}\Big(t_j^{\wedge,*}(\overline{\partial}r)^\wedge + (\overline{\partial}r)^\wedge t_j^{\wedge,*}\Big)\circ\Big(T_j^*\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\Big((\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^\wedge + (\overline{\partial}r)^\wedge(\overline{\partial}r)^{\wedge,*}\Big)\circ\Big((iY - \sqrt{-\triangle_\Gamma})\circ 2\sqrt{-\triangle_\Gamma}\Big)\\ &\quad+\text{lower order terms.}\end{aligned} \tag{4.24}$$

Combining this with (4.21), (4.17) and (4.18), we get (4.20). The proposition follows. □

CHAPTER 5

THE OPERATOR $\Box_\beta^{(q)}$

Put

$$\overline{\partial_\beta} = T\gamma\overline{\partial}\widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\, \Lambda^{0,q+1}T^*(\Gamma)). \tag{5.1}$$

We recall that (see (3.6)) the orthogonal projection T onto $\operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$ with respect to $[\ |\ ]$ is a classical pseudodifferential operator of order 0 with principal symbol $2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}$ (see Lemma 3.3). $\overline{\partial_\beta}$ is a classical pseudodifferential operator of order one from boundary $(0,q)$ forms to boundary $(0,q+1)$ forms.

Lemma 5.1. – *We have* $(\overline{\partial_\beta})^2 = 0$.

Proof. – Let $u,\, v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$. We claim that

$$[T\gamma\overline{\partial}\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid v] = 0. \tag{5.2}$$

We have

$$\begin{aligned}
[T\gamma\overline{\partial}\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid v] &= [\gamma\overline{\partial}\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid v] && (\text{since } v \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*})\\
&= (\widetilde{P}\gamma\overline{\partial}\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid \widetilde{P}v)_M\\
&= (\overline{\partial}\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid \widetilde{P}v)_M && (\text{here we used (3.1)})\\
&= (\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}u \mid \overline{\partial_f}^*\widetilde{P}v)_M && (\text{since } \widetilde{P}v \in \operatorname{Dom}\overline{\partial}^*)\\
&= [(I-T)\gamma\overline{\partial}\widetilde{P}u \mid \gamma\overline{\partial_f}^*\widetilde{P}v] && (\text{here we used (3.1)}).
\end{aligned}$$

From Lemma 2.2, we have $\gamma\overline{\partial_f}^*\widetilde{P}v \in \operatorname{Ker}(\overline{\partial}r)^{\wedge,*}$. Thus,

$$[(I-T)\gamma\overline{\partial}\widetilde{P}u \mid \gamma\overline{\partial_f}^*\widetilde{P}v] = 0.$$

We get (5.2), and hence $T\gamma\overline{\partial}\widetilde{P}\gamma\overline{\partial}\widetilde{P}u = T\gamma\overline{\partial}\widetilde{P}T\gamma\overline{\partial}\widetilde{P}u$, $u \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$.

Now, $(\overline{\partial_\beta})^2 = T\gamma\overline{\partial}\widetilde{P}T\gamma\overline{\partial}\widetilde{P} = T\gamma\overline{\partial}\widetilde{P}\gamma\overline{\partial}\widetilde{P} = T\gamma\overline{\partial}^2\widetilde{P} = 0$. The lemma follows. □

We pause and recall the tangential Cauchy-Riemann operator. For $z \in \Gamma$, let $\pi_z^{0,q} : \Lambda^{0,q}T_z^*(M') \to \Lambda^{0,q}T_z^*(\Gamma)$ be the orthogonal projection map (with respect to $(\ |\ )$). We can check that $\pi_z^{0,q} = 2(\overline{\partial}r(z))^{\wedge,*}(\overline{\partial}r(z))^{\wedge}$. For an open set $U \subset \Gamma$, the tangential Cauchy-Riemann operator: $\overline{\partial_b} : C^\infty(U; \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(U; \Lambda^{0,q+1}T^*(\Gamma))$ is now defined as follows: for any $\phi \in C^\infty(U; \Lambda^{0,q}T^*(\Gamma))$, let $\widetilde{U}$ be an open set in M' with $\widetilde{U} \cap \Gamma = U$ and pick $\phi_1 \in C^\infty(\widetilde{U}; \Lambda^{0,q}T^*(M'))$ that satisfies $\pi_z^{0,q}(\phi_1(z)) = \phi(z)$, for all $z \in U$. Then $\overline{\partial_b}\phi$ is defined to be a smooth section in $C^\infty(U; \Lambda^{0,q+1}T^*(\Gamma))$: $z \to \pi_z^{0,q}(\gamma\overline{\partial}\phi_1(z))$. It is not difficult to check that the definition of $\overline{\partial_b}$ is independent of the choice of ϕ_1. Since $\overline{\partial}^2 = 0$, we have $\overline{\partial_b}^2 = 0$. Let $\overline{\partial_b}^*$ be the formal adjoint of $\overline{\partial_b}$ with respect to $(\ |\ )_\Gamma$, that is $(\overline{\partial}_b f \mid h)_\Gamma = (f \mid \overline{\partial_b}^* h)_\Gamma$, $f \in C_0^\infty(U\ ; \Lambda^{0,q}T^*(\Gamma))$, $h \in C^\infty(U\ ; \Lambda^{0,q+1}T^*(\Gamma))$. $\overline{\partial_b}^*$ is a differential operator of order one from boundary $(0, q+1)$ forms to boundary $(0, q)$ forms and $(\overline{\partial_b}^*)^2 = 0$.

From the definition of $\overline{\partial_b}$, we know that $\overline{\partial_b} = 2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}\gamma\overline{\partial}\widetilde{P}$. Since the principal symbol of T is $2(\overline{\partial}r)^{\wedge,*}(\overline{\partial}r)^{\wedge}$, it follows that

$$\overline{\partial_\beta} = \overline{\partial_b} + \text{lower order terms.} \tag{5.3}$$

Let

$$\overline{\partial_\beta}^\dagger : C^\infty(\Gamma; \Lambda^{0,q+1}T^*(\Gamma)) \to C^\infty(\Gamma; \Lambda^{0,q}T^*(\Gamma)), \tag{5.4}$$

be the formal adjoint of $\overline{\partial_\beta}$ with respect to $[\ |\ ]$, that is $[\overline{\partial_\beta} f \mid h] = [f \mid \overline{\partial_\beta}^\dagger h]$, $f \in C^\infty(\Gamma; \Lambda^{0,q}T^*(\Gamma))$, $h \in C^\infty(\Gamma; \Lambda^{0,q+1}T^*(\Gamma))$. $\overline{\partial_\beta}^\dagger$ is a classical pseudodifferential operator of order one from boundary $(0, q+1)$ forms to boundary $(0, q)$ forms.

Lemma 5.2. – *We have* $\overline{\partial_\beta}^\dagger = \gamma\overline{\partial_f}^*\widetilde{P}$.

Proof. – Let $u \in C^\infty(\Gamma; \Lambda^{0,q}T^*(\Gamma))$, $v \in C^\infty(\Gamma; \Lambda^{0,q+1}T^*(\Gamma))$. We have

$$\begin{aligned}[\overline{\partial_\beta} u \mid v] &= [T\gamma\overline{\partial}\widetilde{P}u \mid v] = [\gamma\overline{\partial}\widetilde{P}u \mid v] \\ &= (\widetilde{P}\gamma\overline{\partial}\widetilde{P}u \mid \widetilde{P}v)_M = (\overline{\partial}\widetilde{P}u \mid \widetilde{P}v)_M \\ &= (\widetilde{P}u \mid \overline{\partial_f}^*\widetilde{P}v)_M = [u \mid \gamma\overline{\partial_f}^*\widetilde{P}v],\end{aligned}$$

and the lemma follows. □

Remark 5.3. – We can check that on boundary $(0, q)$ forms, we have

$$\overline{\partial_\beta}^\dagger = \gamma\overline{\partial_f}^*\widetilde{P} = \overline{\partial_b}^* + \text{lower order terms.} \tag{5.5}$$

Set

$$\Box_\beta^{(q)} = \overline{\partial_\beta}^\dagger\overline{\partial_\beta} + \overline{\partial_\beta}\ \overline{\partial_\beta}^\dagger : \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(\Gamma)) \to \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(\Gamma)). \tag{5.6}$$

$\Box_\beta^{(q)}$ is a classical pseudodifferential operator of order two from boundary $(0, q)$ forms to boundary $(0, q)$ forms. We recall that the Kohn Laplacian on Γ is given by $\Box_b^{(q)} = \overline{\partial_b}\ \overline{\partial_b}^* + \overline{\partial_b}^*\overline{\partial_b} : \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(\Gamma)) \to \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(\Gamma))$. From (5.3) and (5.5), we see that $\sigma_{\Box_b^{(q)}} = \sigma_{\Box_\beta^{(q)}}$ and the characteristic manifold of $\Box_\beta^{(q)}$ is $\Sigma = \Sigma^+ \bigcup \Sigma^-$, where Σ^+, Σ^- are given by (4.13) (see Chapter 2 of Part I). Moreover, $\sigma_{\Box_\beta^{(q)}}$ vanishes to second order on Σ and we have

$$\Box_\beta^{(q)} = \Box_b^{(q)} + L_1, \tag{5.7}$$

where L_1 is a classical pseudodifferential operator of order one with

$$\sigma_{L_1} = 0 \text{ at each point of } \Sigma. \tag{5.8}$$

The following is well-known (see the proof of Lemma 2.4 of Part I)

LEMMA 5.4. – *Σ is a symplectic submanifold of $T^*(\Gamma)$ if and only if the Levi form is non-degenerate at each point of Γ (for the precise definition of symplectic manifold, see chapter* XVIII *of Hörmander* [**15**]*).*

Let p_β^s denote the subprincipal symbol of $\Box_\beta^{(q)}$ (invariantly defined on Σ) and let $F_\beta(\rho)$ denote the fundamental matrix of $\sigma_{\Box_\beta^{(q)}}$ at $\rho \in \Sigma$. We write $\widetilde{\operatorname{tr}}\, F_\beta(\rho)$ to denote $\sum |\lambda_j|$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of $F_\beta(\rho)$. From (5.7) and (5.8), we see that $p_\beta^s + \frac{1}{2}\widetilde{\operatorname{tr}}\, F_\beta = p_b^s + \frac{1}{2}\widetilde{\operatorname{tr}}\, F_b$ on Σ, where p_b^s is the subprincipal symbol of $\Box_b^{(q)}$ and F_b is the fundamental matrix of $\sigma_{\Box_b^{(q)}}$ (for the precise meanings of subprincipal symbol and fundamental matrix, see Chapter 2 of Part I). We have the following

LEMMA 5.5. – *Let $\rho = (p, \xi) \in \Sigma$. Then*

$$\frac{1}{2}\widetilde{\operatorname{tr}}\, F_\beta + p_\beta^s = \sum_{j=1}^{n-1} |\lambda_j|\, |\sigma_{iY}| + \Big(\sum_{j=1}^{n-1} L_p(\overline{T}_j, T_j) - \sum_{j,k=1}^{n-1} 2t_j^{\wedge} t_k^{\wedge,*} L_p(\overline{T}_k, T_j)\Big)\sigma_{iY} \text{ at } \rho, \tag{5.9}$$

where λ_j, $j = 1, \ldots, n-1$, are the eigenvalues of L_p and T_j, t_j, $j = 1, \ldots, n-1$, are as in (4.6) *and* (4.7)*.*

Proof. – See Chapter 2 of Part I. □

It is not difficult to see that on Σ the action of $\frac{1}{2}\widetilde{\operatorname{tr}}\, F_\beta + p_\beta^s$ on boundary $(0, q)$ forms has the eigenvalues

$$\begin{gathered}\sum_{j=1}^{n-1} |\lambda_j|\, |\sigma_{iY}| + \sum_{j \notin J} \lambda_j \sigma_{iY} - \sum_{j \in J} \lambda_j \sigma_{iY},\ |J| = q,\\ J = (j_1, j_2, \ldots, j_q),\ 1 \le j_1 < j_2 < \cdots < j_q \le n-1\end{gathered} \tag{5.10}$$

(see Chapter 2 of Part I). We assume that the Levi form is non-degenerate at $p \in \Gamma$. Let (n_-, n_+), $n_- + n_+ = n - 1$, be the signature of L_p. Since $\langle Y, \omega_0 \rangle = -1$, we have $\sigma_{iY} > 0$ on Σ^+, $\sigma_{iY} < 0$ on Σ^-.

Let

$$\inf\,(p_\beta^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)(\rho)=\inf\left\{\lambda;\ \lambda:\ \text{eigenvalue of } (p_\beta^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)(\rho)\right\},\ \ \rho\in\Sigma.$$

From (5.10), we see that at $(p,\omega_0(p))\in\Sigma^+$,

$$\inf\,(p_\beta^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)\begin{cases}=0, & q=n_+,\\ >0, & q\neq n_+.\end{cases}\tag{5.11}$$

At $(p,-\omega_0(p))\in\Sigma^-$,

$$\inf\,(p_\beta^s+\frac{1}{2}\widetilde{\operatorname{tr}}\,F_\beta)\begin{cases}=0, & q=n_-,\\ >0, & q\neq n_-.\end{cases}\tag{5.12}$$

Definition 5.6. – Given q, $0\leq q\leq n-1$, the Levi form is said to satisfy condition $Y(q)$ at $p\in\Gamma$ if for any $|J|=q$, $J=(j_1,j_2,\ldots,j_q)$, $1\leq j_1<j_2<\cdots<j_q\leq n-1$, we have

$$\left|\sum_{j\notin J}\lambda_j-\sum_{j\in J}\lambda_j\right|<\sum_{j=1}^{n-1}|\lambda_j|\,,$$

where λ_j, $j=1,\ldots,(n-1)$, are the eigenvalues of L_p. If the Levi form is non-degenerate at p, then the condition is equivalent to $q\neq n_+,\ n_-$, where (n_-,n_+), $n_-+n_+=n-1$, is the signature of L_p.

From now on, we assume that

Assumption 5.7. – *The Levi form is non-degenerate at each point of* Γ.

By classical works of Boutet de Monvel [**32**] and Sjöstrand [**36**], we get the following

Proposition 5.8. – $\Box_\beta^{(q)}$ *is hypoelliptic with loss of one derivative if and only if* $Y(q)$ *holds at each point of* Γ.

CHAPTER 6

THE HEAT EQUATION FOR $\Box_\beta^{(q)}$

In this chapter, we will apply some results of Menikoff-Sjöstrand [**29**] to construct approximate orthogonal projection for $\Box_\beta^{(q)}$. Our presentation is essentially taken from Part I. The reader who is familiar with Part I may go directly to Theorem 6.15.

Until further notice, we work with coordinates $x = (x_1, x_2, \ldots, x_{2n-1})$ defined on a connected open set $\Omega \subset \Gamma$. We identify $T^*(\Omega)$ with $\Omega \times \mathbb{R}^{2n-1}$. Thus, the Levi form has constant signature on Ω. For any C^∞ function f, we also write f to denote an almost analytic extension (for the precise meaning of almost analytic functions, we refer the reader to Definition 1.1 of Melin-Sjöstrand [**28**]). We let the full symbol of $\Box_\beta^{(q)}$ be:

$$\text{full symbol of } \Box_\beta^{(q)} \sim \sum_{j=0}^{\infty} q_j(x, \xi),$$

where $q_j(x, \xi)$ is positively homogeneous of order $2 - j$.

First, we consider the characteristic equation for $\partial_t + \Box_\beta^{(q)}$. We look for solutions $\psi(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega) \setminus 0)$ of the problem

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - iq_0(x, \psi'_x) = O(|\operatorname{Im} \psi|^N), \ \forall N \geq 0, \\ \psi|_{t=0} = \langle x \ , \eta \rangle \end{cases} \tag{6.1}$$

with $\operatorname{Im} \psi(t, x, \eta) \geq 0$.

Let U be an open set in $\mathbb{R}^n$ and let $f, g \in C^\infty(U)$. We write $f \asymp g$ if for every compact set $K \subset U$ there is a constant $c_K > 0$ such that $f \leq c_K g$, $g \leq c_K f$ on K. We have the following

Proposition 6.1. – *There exists $\psi(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*(\Omega) \setminus 0)$ such that $\operatorname{Im} \psi \geq 0$ with equality precisely on $(\{0\} \times T^*(\Omega) \setminus 0) \bigcup (\mathbb{R}_+ \times \Sigma)$ and such that (6.1) holds* *where the error term is uniform on every set of the form* $[0,T]\times K$ *with* $T>0$ *and* $K\subset T^*(\Omega)\setminus 0$ *compact. Furthermore,*

$$\psi(t,x,\eta)=\langle x,\eta\rangle \ \text{ on } \Sigma,\ d_{x,\eta}(\psi-\langle x,\eta\rangle)=0 \ \text{ on } \Sigma,$$
$$\psi(t,x,\lambda\eta)=\lambda\psi(\lambda t,x,\eta),\quad \lambda>0,$$

and

$$\operatorname{Im}\psi(t,x,\eta)\asymp\Big(|\eta|\,\frac{t\,|\eta|}{1+t\,|\eta|}\Big)\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2,\quad t\geq 0,\ |\eta|\geq 1. \tag{6.2}$$

PROPOSITION 6.2. – *There exists a function* $\psi(\infty,x,\eta)\in C^\infty(T^*(\Omega)\setminus 0)$ *with a uniquely determined Taylor expansion at each point of* Σ *such that*

> *For every compact set* $K\subset T^*(\Omega)\setminus 0$ *there is a* $c_K>0$ *such that* $\operatorname{Im}\psi(\infty,x,\eta)\geq c_K\,|\eta|\,(\operatorname{dist}((x,\frac{\eta}{|\eta|}),\Sigma))^2$, $d_{x,\eta}(\psi(\infty,x,\eta)-\langle x,\eta\rangle)=0$ *on* Σ.

If $\lambda\in C(T^*(\Omega)\setminus 0)$, $\lambda>0$ *and* $\lambda|_\Sigma<\min|\lambda_j|$, *where* $\pm i\,|\lambda_j|$ *are the non-vanishing eigenvalues of the fundamental matrix of* $\Box_\beta^{(q)}$, *then the solution* $\psi(t,x,\eta)$ *of* (6.1) *can be chosen so that for every compact set* $K\subset T^*(\Omega)\setminus 0$ *and all indices* α, β, γ, *there is a constant* $c_{\alpha,\beta,\gamma,K}$ *such that*

$$\big|\partial_x^\alpha\partial_\eta^\beta\partial_t^\gamma(\psi(t,x,\eta)-\psi(\infty,x,\eta))\big|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(x,\eta)t}\ \text{ on } \overline{\mathbb{R}}_+\times K. \tag{6.3}$$

For the proofs of Proposition 6.1, Proposition 6.2, we refer the reader to **[29]**.

DEFINITION 6.3. – We say that $a\in C^\infty(\overline{\mathbb{R}}_+\times T^*(\Omega))$ is quasi-homogeneous of degree j if $a(t,x,\lambda\eta)=\lambda^j a(\lambda t,x,\eta)$ for all $\lambda>0$.

We consider the problem

$$\begin{cases}(\partial_t+\Box_\beta^{(q)})u(t,x)=0 & \text{in } \mathbb{R}_+\times\Omega,\\ u(0,x)=v(x).\end{cases} \tag{6.4}$$

We shall start by making only a formal construction. We look for an approximate solution of (6.4) of the form $u(t,x)=A(t)v(x)$,

$$A(t)v(x)=\frac{1}{(2\pi)^{2n-1}}\iint e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)v(y)dyd\eta \tag{6.5}$$

where formally

$$a(t,x,\eta)\sim\sum_{j=0}^\infty a_j(t,x,\eta),$$

$a_j(t,x,\eta)\in C^\infty\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$, $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$.

We apply $\partial_t+\Box_\beta^{(q)}$ formally under the integral in (6.5) and then introduce the asymptotic expansion of $\Box_\beta^{(q)}(ae^{i\psi})$. Setting $(\partial_t+\Box_\beta^{(q)})(ae^{i\psi})\sim 0$ and regrouping the terms according to the degree of quasi-homogeneity. We obtain the transport equations

$$(6.6)\qquad \begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(|\operatorname{Im}\psi|^N),\ \forall N, \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(|\operatorname{Im}\psi|^N),\ \forall N. \end{cases}$$

Here

$$T(t,x,\eta,\partial_t,\partial_x) = \partial_t - i\sum_{j=1}^{2n-1}\frac{\partial q_0}{\partial \xi_j}(x,\psi'_x)\frac{\partial}{\partial x_j} + q(t,x,\eta)$$

where

$$q(t,x,\eta) = q_1(x,\psi'_x) + \frac{1}{2i}\sum_{j,k=1}^{2n-1}\frac{\partial^2 q_0(x,\psi'_x)}{\partial\xi_j\partial\xi_k}\frac{\partial^2\psi(t,x,\eta)}{\partial x_j\partial x_k}$$

and $l_j(t,x,\eta)$ is a linear differential operator acting on $a_0, a_1, \ldots, a_{j-1}$. We can repeat the method of Part I (see Proposition 4.6 of Part I) to get the following

Proposition 6.4. – *Let (n_-,n_+), $n_-+n_+=n-1$, be the signature of the Levi form on Ω. We can find solutions*

$$a_j(t,x,\eta)\in C^\infty\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),\quad j=0,1,\ldots,$$

of the system (6.6) with $a_0(0,x,\eta)=I$, $a_j(0,x,\eta)=0$ when $j>0$, where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$, such that a_j has unique Taylor expansions on Σ. Moreover, we can find

$$a_j(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),\quad j=0,1,\ldots,$$

where $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$, $\varepsilon_0>0$ such that for all indices α, β, γ, j, every compact set $K\subset\Omega$, there exists $c>0$, such that

$$(6.7)\qquad \big|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta\big(a_j(t,x,\eta)-a_j(\infty,x,\eta)\big)\big|\le ce^{-\varepsilon_0 t|\eta|}(1+|\eta|)^{-j-|\beta|+\gamma}$$

on $\overline{\mathbb{R}}_+\times\Big((K\times\mathbb{R}^{2n-1})\bigcap\Sigma\Big)$, $|\eta|\ge 1$.

Furthermore, for all $j=0,1,\ldots,$

$$(6.8)\qquad \begin{cases} \textit{all derivatives of } a_j(\infty,x,\eta) \textit{ vanish at } \Sigma^+ \textit{ if } q\neq n_+, \\ \textit{all derivatives of } a_j(\infty,x,\eta) \textit{ vanish at } \Sigma^- \textit{ if } q\neq n_-, \end{cases}$$

and

$$(6.9)\qquad \begin{cases} a_0(\infty,x,\eta)\neq 0 \textit{ at each point of } \Sigma^+ \textit{ if } q=n_+, \\ a_0(\infty,x,\eta)\neq 0 \textit{ at each point of } \Sigma^- \textit{ if } q=n_-. \end{cases}$$

Definition 6.5. – Let $r(x,\eta)$ be a non-negative real continuous function on $T^*(\Omega)$. We assume that $r(x,\eta)$ is positively homogeneous of degree 1, that is, $r(x,\lambda\eta) = \lambda r(x,\eta)$, for $\lambda \geq 1$, $|\eta| \geq 1$. For $0 \leq q \leq n-1$ and $k \in \mathbb{R}$, we say that

$$a \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$$

if $a \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$ and for all indices α, β, γ, every compact set $K \subset \Omega$ and every $\varepsilon > 0$, there exists a constant $c > 0$ such that

$$\left|\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a(t,x,\eta)\right| \leq c e^{t(-r(x,\eta)+\varepsilon|\eta|)}(1+|\eta|)^{k+\gamma-|\beta|},\ x \in K,\ |\eta| \geq 1.$$

Remark 6.6. – It is easy to see that we have the following properties:

(a) If $a \in \hat{S}^k_{r_1}$, $b \in \hat{S}^l_{r_2}$ then $ab \in \hat{S}^{k+l}_{r_1+r_2}$, $a+b \in \hat{S}^{\max(k,l)}_{\min(r_1,r_2)}$.

(b) If $a \in \hat{S}^k_r$ then $\partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta a \in \hat{S}^{k-|\beta|+\gamma}_r$.

(c) If $a_j \in \hat{S}^{k_j}_r$, $j = 0,1,2,\ldots$ and $k_j \searrow -\infty$ as $j \to \infty$, then there exists $a \in \hat{S}^{k_0}_r$ such that $a - \sum_0^{v-1} a_j \in \hat{S}^{k_v}_r$, for all $v = 1,2,\ldots$. Moreover, if $\hat{S}^{-\infty}_r$ denotes $\bigcap_{k\in\mathbb{R}} \hat{S}^k_r$ then a is unique modulo $\hat{S}^{-\infty}_r$.

If a and a_j have the properties of (c), we write

$$a \sim \sum_0^\infty a_j \text{ in the symbol space } \hat{S}^{k_0}_r.$$

From Proposition 6.4 and the standard Borel construction, we get the following

Proposition 6.7. – *Let (n_-,n_+), $n_-+n_+ = n-1$, be the signature of the Levi form on Ω. We can find solutions*

$$a_j(t,x,\eta) \in C^\infty\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big),\quad j = 0,1,\ldots,$$

of the system (6.6) with $a_0(0,x,\eta) = I$, $a_j(0,x,\eta) = 0$ when $j > 0$, where $a_j(t,x,\eta)$ is a quasi-homogeneous function of degree $-j$, such that for some $r > 0$ as in Definition 6.5,

$$a_j(t,x,\eta) - a_j(\infty,x,\eta) \in \hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big),\quad j = 0,1,\ldots,$$

where

$$a_j(\infty,x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big),\quad j = 0,1,\ldots,$$

and $a_j(\infty,x,\eta)$ is a positively homogeneous function of degree $-j$.

Furthermore, for all $j = 0,1,\ldots$,

$$\begin{cases} a_j(\infty,x,\eta) = 0 \text{ in a conic neighborhood of } \Sigma^+, \text{ if } q \neq n_+, \\ a_j(\infty,x,\eta) = 0 \text{ in a conic neighborhood of } \Sigma^-, \text{ if } q \neq n_-. \end{cases}$$

Remark 6.8. – Let $b(t,x,\eta)\in\hat S^k_r\big(\overline{\mathbb R}_+\times T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$ with $r>0$. We assume that $b(t,x,\eta)=0$ when $|\eta|\le 1$. Let $\chi\in C_0^\infty(\mathbb R^{2n-1})$ be equal to 1 near the origin. Put $B_\varepsilon(x,y)=\int\big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\big)\chi(\varepsilon\eta)d\eta$. For $u\in C_0^\infty(\Omega;\,\Lambda^{0,q}T^*(\Gamma))$, we can show that

$$\lim_{\varepsilon\to 0}\Big(\int B_\varepsilon(x,y)u(y)dy\Big)\in C^\infty(\Omega;\,\Lambda^{0,q}T^*(\Gamma))$$

and

$$\begin{aligned}B: C_0^\infty(\Omega;\,\Lambda^{0,q}T^*(\Gamma))&\to C^\infty(\Omega;\,\Lambda^{0,q}T^*(\Gamma))\\ u&\to \lim_{\varepsilon\to 0}\Big(\int B_\varepsilon(x,y)u(y)dy\Big)\end{aligned}$$

is continuous. Formally,

$$B(x,y)=\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)dt\Big)d\eta.$$

Moreover, B has a unique continuous extension:

$$B:\mathscr E'(\Omega;\,\Lambda^{0,q}T^*(\Gamma))\to\mathscr D'(\Omega;\,\Lambda^{0,q}T^*(\Gamma))$$

and $B(x,y)\in C^\infty\big(\Omega\times\Omega\smallsetminus\operatorname{diag}(\Omega\times\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$. For the details, we refer the reader to Proposition 5.6 of Part I.

Remark 6.9. – Let $a(t,x,\eta)\in\hat S^k_0\big(\overline{\mathbb R}_+\times T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$. We assume $a(t,x,\eta)=0$, if $|\eta|\le 1$ and

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat S^k_r\big(\overline{\mathbb R}_+\times T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$$

with $r>0$, where $a(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$. Then we can also define

$$A(x,y)=\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)dt\Big)d\eta \tag{6.10}$$

as an oscillatory integral by the following formula:

$$A(x,y)=\int\Big(\int_0^\infty e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}(-t)\big(i\psi_t'(t,x,\eta)a(t,x,\eta)+a_t'(t,x,\eta)\big)dt\Big)d\eta.$$

We notice that $(-t)\big(i\psi_t'(t,x,\eta)a(t,x,\eta)+a_t'(t,x,\eta)\big)\in\hat S^{k+1}_r$, $r>0$.

We recall the following

Definition 6.10. – Let $k\in\mathbb R$. $S^k_{\frac12,\frac12}\big(T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$ is the space of all $a\in C^\infty\big(T^*(\Omega);\,\mathscr L(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$ such that for every compact sets $K\subset\Omega$ and all $\alpha\in\mathbb N^{2n-1}$, $\beta\in\mathbb N^{2n-1}$, there is a constant $c_{\alpha,\beta,K}>0$ such that

$$\Big|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\Big|\le c_{\alpha,\beta,K}(1+|\xi|)^{k-\frac{|\beta|}{2}+\frac{|\alpha|}{2}},$$

$(x,\xi)\in T^*(\Omega)$, $x\in K$. $S^k_{\frac12,\frac12}$ is called the space of symbols of order k type $(\frac12,\frac12)$.

Definition 6.11. – Let $k \in \mathbb{R}$. A pseudodifferential operator of order k type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$ is a continuous linear map $A : C_0^\infty(\Omega; \Lambda^{0,q}T^*(\Gamma)) \to \mathscr{D}'(\Omega; \Lambda^{0,q}T^*(\Gamma))$ such that the distribution kernel of A is

$$K_A = A(x,y) = \frac{1}{(2\pi)^{2n-1}} \int e^{i\langle x-y,\xi\rangle} a(x,\xi) d\xi$$

with $a \in S^k_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$. We call $a(x,\xi)$ the symbol of A. We shall write $L^k_{\frac{1}{2},\frac{1}{2}}(\Omega; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$ to denote the space of pseudodifferential operators of order k type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$.

We recall the following classical proposition of Calderon-Vaillancourt (for a proof, see [**15**]).

Proposition 6.12. – *If $A \in L^k_{\frac{1}{2},\frac{1}{2}}(\Omega; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$. Then, for every $s \in \mathbb{R}$, A is continuous: $A : H^s_{\rm comp}(\Omega; \Lambda^{0,q}T^*(\Gamma)) \to H^{s-k}_{\rm loc}(\Omega; \Lambda^{0,q}T^*(\Gamma))$.*

We have the following

Proposition 6.13. – *Let $a(t,x,\eta) \in \hat{S}^k_0\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$. We assume $a(t,x,\eta) = 0$, if $|\eta| \leq 1$ and*

$$a(t,x,\eta) - a(\infty,x,\eta) \in \hat{S}^k_r\big(\overline{\mathbb{R}}_+ \times T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$$

with $r > 0$, where $a(\infty,x,\eta) \in C^\infty\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$. Let

$$A(x,y) = \frac{1}{(2\pi)^{2n-1}} \int \Big(\int_0^\infty \big(e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta) \big) dt \Big) d\eta$$

be as in (6.10). Then $A \in L^{k-1}_{\frac{1}{2},\frac{1}{2}}(\Omega; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$ with symbol

$$q(x,\eta) = \int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle x,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle x,\eta\rangle)} a(\infty,x,\eta) \Big) dt$$

in $S^{k-1}_{\frac{1}{2},\frac{1}{2}}\big(T^*(\Omega); \mathscr{L}(\Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))\big)$.

Proof. – See Lemma 5.14 and Lemma 5.16 of Part I. □

From now on, we write

$$\frac{1}{(2\pi)^{2n-1}} \int \Big(\int_0^\infty \big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)} a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)} a(\infty,x,\eta) \big) dt \Big) d\eta$$

to denote the kernel of pseudodifferential operator of order $k-1$ type $(\frac{1}{2}, \frac{1}{2})$ from sections of $\Lambda^{0,q}T^*(\Gamma)$ to sections of $\Lambda^{0,q}T^*(\Gamma)$. Here $a(t,x,\eta)$, $a(\infty,x,\eta)$ are as in Proposition 6.13.

The following is essentially well-known (see page 72 of [**29**]).

PROPOSITION 6.14. – *Let Q be a properly supported pseudodifferential operator on Ω of order $k>0$ with classical symbol $q(x,\xi)\in C^\infty(T^*(\Omega))$. Let*

$$b(t,x,\eta)\in \hat S^m_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big).$$

We assume that $b(t,x,\eta)=0$ when $|\eta|\leq 1$ and that

$$b(t,x,\eta)-b(\infty,x,\eta)\in \hat S^m_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$$

with $r>0$, where $b(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$ is a classical symbol of order m. Then,

$$Q\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)\big)=e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)+d(t,x,\eta), \tag{6.11}$$

where $c(t,x,\eta)\in \hat S^{k+m}_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$,

$$c(t,x,\eta)\sim\sum_\alpha\frac{1}{\alpha!}q^{(\alpha)}(x,\psi'_x(t,x,\eta))(R_\alpha(\psi,D_x)b)$$

in the symbol space $\hat S^{k+m}_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$,

$$c(t,x,\eta)-c(\infty,x,\eta)\in \hat S^{k+m}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),$$

where $r>0$,

$$d(t,x,\eta)\in \hat S^{-\infty}_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),$$

$d(t,x,\eta)-d(\infty,x,\eta)\in \hat S^{-\infty}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$, *$r>0$. Here*

$$c(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$$

is a classical symbol of order $k+m$,

$$d(\infty,x,\eta)\in S^{-\infty}_{1,0}\big(T^*(\Omega);\,\mathscr{L}(\Lambda^{0,q}T^*(M'),\Lambda^{0,q}T^*(M'))\big)$$

(for the precise meaning of $S^{-\infty}_{1,0}$, see Definition 1.1) and

$$R_\alpha(\psi,D_x)b=D^\alpha_y\Big\{e^{i\phi_2(t,x,y,\eta)}b(t,y,\eta)\Big\}\Big|_{y=x},$$

$\phi_2(t,x,y,\eta)=(x-y)\psi'_x(t,x,\eta)-\big(\psi(t,x,\eta)-\psi(t,y,\eta)\big)$. Moreover, put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}b(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}b(\infty,x,\eta)\big)dt\Big)d\eta,$$

$$C(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}c(t,x,\eta)-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}c(\infty,x,\eta)\big)dt\Big)d\eta.$$

We have $Q\circ B\equiv C$.

As in Chapter 1, we put $\Gamma_q=\{z\in\Gamma;\ Z(q)\text{ fails at }z\}$ and set

$$\Sigma^-(q)=\big\{(x,\xi)\in\Sigma^-;\ Z(q)\text{ fails at }x\big\},$$
$$\Sigma^+(q)=\big\{(x,\xi)\in\Sigma^+;\ Z(q)\text{ fails at }x\big\}.$$

From Proposition 6.7 and Proposition 6.14, we can repeat the method of Part I to get the following

THEOREM 6.15. – *We recall that we work with the Assumption 5.7. Given q, $0 \le q \le n-1$. Suppose that $Z(q)$ fails at some point of Γ. Then there exist*

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma)), \quad B_-, B_+ \in L^0_{\frac{1}{2},\frac{1}{2}}(\Gamma; \Lambda^{0,q}T^*(\Gamma), \Lambda^{0,q}T^*(\Gamma))$$

such that

$$\begin{aligned} &\mathrm{WF}'(K_{B_-}) = \mathrm{diag}\left(\Sigma^-(q) \times \Sigma^-(q)\right), \\ &\mathrm{WF}'(K_{B_+}) = \mathrm{diag}\left(\Sigma^+(n-1-q) \times \Sigma^+(n-1-q)\right) \end{aligned} \tag{6.12}$$

and

$$A\Box_\beta^{(q)} + B_- + B_+ \equiv B_- + B_+ + \Box_\beta^{(q)} A \equiv I, \tag{6.13}$$

$$\overline{\partial_\beta} B_- \equiv 0,\ \overline{\partial_\beta}^\dagger B_- \equiv 0, \tag{6.14}$$

$$\overline{\partial_\beta} B_+ \equiv 0,\ \overline{\partial_\beta}^\dagger B_+ \equiv 0, \tag{6.15}$$

$$B_- \equiv B_-^\dagger \equiv B_-^2, \tag{6.16}$$

$$B_+ \equiv B_+^\dagger \equiv B_+^2, \tag{6.17}$$

where $B_-^\dagger$ and $B_+^\dagger$ are the formal adjoints of B_- and B_+ with respect to $[\,|\,]$ respectively and

$$\mathrm{WF}'(K_{B_-}) = \left\{(x,\xi,y,\eta) \in T^*(\Gamma) \times T^*(\Gamma);\ (x,\xi,y,-\eta) \in \mathrm{WF}(K_{B_-})\right\}.$$

Here $\mathrm{WF}(K_{B_-})$ *is the wave front set of K_{B_-} in the sense of Hörmander.*

Moreover near $\mathrm{diag}(\Gamma_q \times \Gamma_q)$, *$K_{B_-}(x,y)$ satisfies*

$$K_{B_-}(x,y) \equiv \int_0^\infty e^{i\phi_-(x,y)t} b(x,y,t)dt$$

with

(6.18)

$$\begin{aligned} &b(x,y,t) \in S^{n-1}_{1,0}\left(\Gamma \times \Gamma \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma), \Lambda^{0,q}T^*_x(\Gamma))\right), \\ &b(x,y,t) \sim \sum_{j=0}^\infty b_j(x,y)t^{n-1-j} \text{ in } S^{n-1}_{1,0}\left(\Gamma \times \Gamma \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma), \Lambda^{0,q}T^*_x(\Gamma))\right), \\ &b_0(x,x) \neq 0 \text{ if } x \in \Gamma_q, \end{aligned}$$

(a formula for $b_0(x,x)$ will be given in Proposition 6.17) where $S^m_{1,0}$, $m \in \mathbb{R}$, is the Hörmander symbol space (see Definition 1.1),

$$b_j(x,y) \in C^\infty\left(\Gamma \times \Gamma;\ \mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma), \Lambda^{0,q}T^*_x(\Gamma))\right), \quad j = 0, 1, \dots,$$

and

$$\phi_-(x,y)\in C^\infty(\Gamma\times\Gamma),\quad \operatorname{Im}\phi_-(x,y)\geq 0, \tag{6.19}$$

$$\phi_-(x,x)=0,\quad \phi_-(x,y)\neq 0 \quad \textit{if}\quad x\neq y, \tag{6.20}$$

$$d_x\phi_-\neq 0,\quad d_y\phi_-\neq 0\quad \textit{where}\quad \operatorname{Im}\phi_-=0, \tag{6.21}$$

$$d_x\phi_-(x,y)|_{x=y}=-\omega_0(x),\quad d_y\phi_-(x,y)|_{x=y}=\omega_0(x), \tag{6.22}$$

$$\phi_-(x,y)=-\overline{\phi}_-(y,x). \tag{6.23}$$

Similarly, near diag $(\Gamma_{n-1-q}\times\Gamma_{n-1-q})$,

$$K_{B_+}(x,y)\equiv\int_0^\infty e^{i\phi_+(x,y)t}c(x,y,t)dt$$

with $c(x,y,t)\in S^{n-1}_{1,0}\big(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))\big)$,

$$c(x,y,t)\sim\sum_{j=0}^\infty c_j(x,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(\Gamma\times\Gamma\times]0,\infty[;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))\big)$, *where*

$$c_j(x,y)\in C^\infty\big(\Gamma\times\Gamma;\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_x(\Gamma))\big),\quad j=0,1,\ldots,$$

and $-\overline{\phi}_+(x,y)$ *satisfies* (6.19)–(6.23).

We only give the outline of the proof of Theorem 6.15. For all the details, we refer the reader to Chapter 6 and Chapter 7 of Part I. Let

$$a_j(t,x,\eta)\in\hat{S}^{-j}_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),\quad j=0,1,\ldots,$$

and $a_j(\infty,x,\eta)\in C^\infty\big(T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$, $j=0,1,\ldots$, be as in Proposition 6.7. We recall that for some $r>0$

$$a_j(t,x,\eta)-a_j(\infty,x,\eta)\in\hat{S}^{-j}_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big),\quad j=0,1,\ldots.$$

Let

$$a(\infty,x,\eta)\sim\sum_{j=0}^\infty a_j(\infty,x,\eta)$$

in $S^0_{1,0}(T^*(\Omega);\mathscr{L}\big(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$. Let

$$a(t,x,\eta)\sim\sum_{j=0}^\infty a_j(t,x,\eta)$$

in $\hat{S}^0_0\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)$. We take $a(t,x,\eta)$ so that for every compact set $K\subset\Omega$ and all indices α, β, γ, k, there exists $c>0$, c is independent of

t, such that

$$\left|\partial_t^\gamma\partial_x^\alpha\partial_\eta^\beta\big(a(t,x,\eta)-\sum_{j=0}^k a_j(t,x,\eta)\big)\right|\leq c(1+|\eta|)^{-k-1+\gamma-|\beta|}, \tag{6.24}$$

where $t\in\overline{\mathbb{R}}_+$, $x\in K$, $|\eta|\geq 1$, and

$$a(t,x,\eta)-a(\infty,x,\eta)\in\hat{S}^0_r\big(\overline{\mathbb{R}}_+\times T^*(\Omega);\mathscr{L}(\Lambda^{0,q}T^*(\Gamma),\Lambda^{0,q}T^*(\Gamma))\big)\text{ with } r>0.$$

Choose $\chi\in C_0^\infty(\mathbb{R}^{2n-1})$ so that $\chi(\eta)=1$ when $|\eta|<1$ and $\chi(\eta)=0$ when $|\eta|>2$. Set

$$\begin{aligned}A(x,y)=\frac{1}{(2\pi)^{2n-1}}\int\Big(\int_0^\infty\big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta)\\-e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\big)(1-\chi(\eta))dt\Big)d\eta.\end{aligned} \tag{6.25}$$

Put

$$B(x,y)=\frac{1}{(2\pi)^{2n-1}}\int e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)d\eta. \tag{6.26}$$

Since $a_j(t,x,\eta)$, $j=0,1,\ldots$, solve the transport equations (6.6), we can check that $B+\Box^{(q)}_\beta A\equiv I$, $\Box^{(q)}_\beta B\equiv 0$. From the global theory of Fourier integral operators (see [**28**]), we get $K_B\equiv K_{B_-}+K_{B_+}$, wher K_{B_-} and K_{B_+} are as in Theorem 6.15. By using a partition of unity we get the global result.

Remark 6.16. – For more properties of the phase $\phi_-(x,y)$, see Theorem 1.4 and Remark 1.5 of Part I.

We can repeat the computation of the leading term of the Szegő projection (see Chapter 8 of Part I), to get the following

Proposition 6.17. – *Let $p\in\Gamma_q$, $q=n_-$. Let $U_1(x),\ldots,U_{n-1}(x)$ be an orthonormal frame of $\Lambda^{1,0}T_x(\Gamma)$, for which the Levi form is diagonalized at p. Let $e_j(x)$, $j=1,\ldots,n-1$, denote the basis of $\Lambda^{0,1}T^*_x(\Gamma)$, which is dual to $\overline{U}_j(x)$, $j=1,\ldots,n-1$. Let $\lambda_j(x)$, $j=1,\ldots,n-1$, be the eigenvalues of the Levi form L_x. We assume that $\lambda_j(p)<0$ if $1\leq j\leq n_-$. Then*

$$b_0(p,p)=\frac{1}{2}\left|\lambda_1(p)\right|\cdots\left|\lambda_{n-1}(p)\right|\pi^{-n}\prod_{j=1}^{j=n_-}e_j(p)^\wedge e_j(p)^{\wedge,*},$$

where b_0 is as in (6.18).

In Chapter 7, we need the following

PROPOSITION 6.18. – *Suppose that $Z(q)$ fails at some point of Γ. Let B_- be as in Theorem 6.15. Then,*

$$\gamma\overline{\partial}\widetilde{P}B_- \equiv 0. \tag{6.27}$$

Proof. – In view of Theorem 6.15, we know that

$$T\gamma\overline{\partial}\widetilde{P}B_- = \overline{\partial_\beta}B_- \equiv 0, \quad \gamma\overline{\partial_f}^*\widetilde{P}B_- = \overline{\partial_\beta}^\dagger B_- \equiv 0.$$

Combining this with $\gamma(\overline{\partial}\overline{\partial_f}^* + \overline{\partial_f}^*\overline{\partial})\widetilde{P} \equiv 0$, we have

$$\gamma\overline{\partial_f}^*\widetilde{P}\gamma\overline{\partial}\widetilde{P}B_- \equiv -\gamma\overline{\partial}\widetilde{P}\gamma\overline{\partial_f}^*\widetilde{P}B_- \equiv 0$$

and

$$\gamma\overline{\partial_f}^*\widetilde{P}(I-T)\gamma\overline{\partial}\widetilde{P}B_- = \gamma\overline{\partial_f}^*\widetilde{P}\gamma\overline{\partial}\widetilde{P}B_- - \gamma\overline{\partial_f}^*\widetilde{P}T\gamma\overline{\partial}\widetilde{P}B_- \equiv 0. \tag{6.28}$$

Combining this with (3.7), we get $\gamma\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^\wedge R\gamma\overline{\partial}\widetilde{P}B_- \equiv 0$. Thus,

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}(\overline{\partial}r)^\wedge R\gamma\overline{\partial}\widetilde{P}B_- \equiv 0.$$

In view of Proposition 4.2, we know that

$$\gamma(\overline{\partial}r)^\wedge\overline{\partial_f}^*\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1} : C^\infty(\Gamma;\, I^{0,q}T^*(M')) \to C^\infty(\Gamma;\, I^{0,q}T^*(M'))$$

is elliptic near Σ^-, where $I^{0,q}T_z^*(M')$ is as in (4.19). Since

$$\mathrm{WF}'(K_{B_-}) \subset \mathrm{diag}\,(\Sigma^- \times \Sigma^-),$$

we get $(\overline{\partial}r)^\wedge R\gamma\overline{\partial}\widetilde{P}B_- \equiv 0$. Thus, by (3.7), $(I-T)\gamma\overline{\partial}\widetilde{P}B_- \equiv 0$. The proposition follows. □

CHAPTER 7

THE BERGMAN PROJECTION

Given q, $0 \leq q \leq n-1$. In this chapter, we assume that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. In view of Proposition 3.4, we know that $\widetilde{\Pi}^{(q)} : C^\infty(\overline{M};\, \Lambda^{0,q}T^*(M')) \to D^{(q)}$. Put

$$K = \gamma \widetilde{\Pi}^{(q)} \widetilde{P} : C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)). \tag{7.1}$$

Let $K^\dagger$ be the formal adjoint of K with respect to $[\ |\ ]$. That is,

$$K^\dagger : \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$$
$$[K^\dagger u \mid v] = [u \mid Kv],\ u \in \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)),\ v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)).$$

Lemma 7.1. – *We have $K^\dagger v = Kv$, $v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$.*

Proof. – For u, $v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$, we have

$$\begin{aligned} [Ku \mid v] &= [\gamma \widetilde{\Pi}^{(q)} \widetilde{P} u \mid v] \\ &= (\widetilde{\Pi}^{(q)} \widetilde{P} u \mid \widetilde{P} v)_M \\ &= (\widetilde{P} u \mid \widetilde{\Pi}^{(q)} \widetilde{P} v)_M \\ &= [u \mid Kv]. \end{aligned}$$

Thus, $K^\dagger v = Kv$. The lemma follows. □

We can extend K to $\mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma)) \to \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$ by the following formula: $[Ku \mid v] = [u \mid K^\dagger v]$, $u \in \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$, $v \in C^\infty(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$.

Lemma 7.2. – *Let $u \in \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$. We have* $\mathrm{WF}\,(Ku) \subset \Sigma^-$.

Proof. – Let $u \in \mathscr{D}'(\Gamma;\, \Lambda^{0,q}T^*(\Gamma))$. We have $(\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} \widetilde{P})(Ku) = 0$. In view of Proposition 4.1, we know that $\gamma(\overline{\partial} r)^{\wedge,*} \overline{\partial} \widetilde{P}$ is elliptic outside Σ^-. The lemma follows. □

Lemma 7.3. – *Let B_- be as in Theorem 6.15. We have $B_- K \equiv K B_- \equiv K$.*

Proof. – Let A, B_- and B_+ be as in Theorem 6.15. In view of Theorem 6.15, we have

$$B_- + B_+ + A\Box_\beta^{(q)} \equiv I.$$

We may replace B_+ by $I - A\Box_\beta^{(q)} - B_-$ and get $B_- + B_+ + A\Box_\beta^{(q)} = I$. It is easy to see that $\Box_\beta^{(q)} K = 0$. Thus,

$$K = (B_- + B_+ + A\Box_\beta^{(q)})K = (B_- + B_+)K. \tag{7.2}$$

Let $u \in \mathscr{D}'(\Gamma; \Lambda^{0,q}T^*(\Gamma))$. From Lemma 7.2, we know that $\mathrm{WF}\,(Ku) \subset \Sigma^-$. Note that $\mathrm{WF}'(K_{B_+}) \subset \mathrm{diag}\,(\Sigma^+ \times \Sigma^+)$. Thus, $B_+Ku \in C^\infty$, so B_+K is smoothing and $(B_- + B_+)K \equiv B_-K$. From this and (7.2), we get $K \equiv B_-K$ and $K = K^\dagger \equiv K^\dagger B_-^\dagger \equiv KB_-$. The lemma follows. □

We pause and introduce some notations. Let X and Y be C^∞ vector bundles over M' and Γ respectively. Let $C, D : C^\infty(\Gamma; Y) \to \mathscr{D}'(M; X)$ with distribution kernels $K_C(z,y)$, $K_D(z,y) \in \mathscr{D}'(M \times \Gamma; \mathscr{L}(Y_y, X_z))$. We write $C \equiv D \mod C^\infty(\overline{M} \times \Gamma)$ if $K_C(z,y) = K_D(z,y) + F(z,y)$, where $F(z,y) \in C^\infty(\overline{M} \times \Gamma; \mathscr{L}(Y_y, X_z))$.

Lemma 7.4. – *We have*

$$\widetilde{\Pi}^{(q)}\widetilde{P}B_- \equiv \widetilde{\Pi}^{(q)}\widetilde{P} \mod C^\infty(\overline{M} \times \Gamma). \tag{7.3}$$

Proof. – From Lemma 7.3, we have $K = \gamma\widetilde{\Pi}^{(q)}\widetilde{P} \equiv KB_- = \gamma\widetilde{\Pi}^{(q)}\widetilde{P}B_-$. Thus, $\widetilde{\Pi}^{(q)}\widetilde{P} = \widetilde{P}\gamma\widetilde{\Pi}^{(q)}\widetilde{P} \equiv \widetilde{P}\gamma\widetilde{\Pi}^{(q)}\widetilde{P}B_- \mod C^\infty(\overline{M} \times \Gamma)$. We get (7.3). □

Put

$$Q = \widetilde{P}B_-T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* : C^\infty(\overline{M}; \Lambda^{0,q}T^*(M')) \to C^\infty(\overline{M}; \Lambda^{0,q}T^*(M')), \tag{7.4}$$

where T is as in (3.6).

Proposition 7.5. – *We have* $Q \equiv \widetilde{\Pi}^{(q)} \mod C^\infty(\overline{M} \times \overline{M})$. *From* (3.12), *it follows that* $Q \equiv \Pi^{(q)} \mod C^\infty(\overline{M} \times \overline{M})$.

Proof. – We have

$$\widetilde{\Pi}^{(q)}Q = \widetilde{\Pi}^{(q)}\widetilde{P}B_-T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* \equiv \widetilde{\Pi}^{(q)}\widetilde{P}T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* \mod C^\infty(\overline{M} \times \overline{M}). \tag{7.5}$$

Here we used (7.3). From (7.5) and the first part of (3.13), we get

$$\widetilde{\Pi}^{(q)}Q \equiv \widetilde{\Pi}^{(q)} \mod C^\infty(\overline{M} \times \overline{M}). \tag{7.6}$$

From Theorem 2.7, we have $\widetilde{\Pi}^{(q)}Q = (I - K^{(q)})(I - \overline{\partial}^* N^{(q+1)}\overline{\partial} - \overline{\partial}N^{(q-1)}\overline{\partial}^*)Q$, where $N^{(q+1)}$ and $N^{(q-1)}$ are as in Theorem 2.6. From (6.27), (6.14) and Lemma 5.2, we see that $\overline{\partial}Q \equiv 0$, $\overline{\partial}^*Q \equiv 0 \mod C^\infty(\overline{M} \times \overline{M})$. Thus, $\widetilde{\Pi}^{(q)}Q = (I - K^{(q)})(I - \overline{\partial}^* N^{(q+1)}\overline{\partial} - \overline{\partial}N^{(q-1)}\overline{\partial}^*)Q \equiv Q \mod C^\infty(\overline{M} \times \overline{M})$. From this and (7.6), the proposition follows. □

Let $x = (x_1, \ldots, x_{2n-1})$ be a system of local coordinates on Γ and extend the functions $x_1, \ldots, x_{2n-1}$ to real smooth functions in some neighborhood of Γ. We write $(\xi_1, \ldots, \xi_{2n-1}, \theta)$ to denote the dual variables of (x, r). We write $z = (x_1, \ldots, x_{2n-1}, r)$, $x = (x_1, \ldots, x_{2n-1}, 0)$, $\xi = (\xi_1, \ldots, \xi_{2n-1})$, $\zeta = (\xi, \theta)$. Until further notice, we work with the local coordinates $z = (x, r)$ defined on some neighborhood of $p \in \Gamma$.

We represent the Riemannian metric on $T(M')$ by

$$h = \sum_{j,k=1}^{2n} h_{j,k}(z) dx_j \otimes dx_k, \quad dx_{2n} = dr,$$

where $h_{j,k}(z) = h_{k,j}(z)$, j, $k = 1, \ldots, n$, and $(h_{j,k}(z))_{1 \leq j,k \leq 2n}$ is positive definite at each point of M'. Put $(h_{j,k}(z))^{-1}_{1 \leq j,k \leq 2n} = \left(h^{j,k}(z)\right)_{1 \leq j,k \leq 2n}$. It is well-known (see page 99 of Morrow-Kodaira [**35**]) that

$$\Box^{(q)}_f = -\frac{1}{2}\Big(h^{2n,2n}(z)\frac{\partial^2}{\partial r^2} + 2\sum_{j=1}^{2n-1} h^{2n,j}(z)\frac{\partial^2}{\partial r \partial x_j} + T(r)\Big) + \text{lower order terms}, \tag{7.7}$$

where

$$T(r) = \sum_{j,k=1}^{2n-1} h^{j,k}(z)\frac{\partial^2}{\partial x_j \partial x_k}. \tag{7.8}$$

Note that $T(0) = \triangle_\Gamma +$ lower order terms and

$$h^{2n,2n}(x) = 1, \quad h^{2n,j}(x) = 0, \quad j = 1, \ldots, 2n-1.$$

We let the full symbol of $\widetilde{\Box}^{(q)}_f$ be:

$$\text{full symbol of } \widetilde{\Box}^{(q)}_f = \sum_{j=0}^{2} q_j(z, \zeta)$$

where $q_j(z, \zeta)$ is a homogeneous polynomial of order $2 - j$ in ζ (we recall that $\widetilde{\Box}^{(q)}_f = \Box^{(q)}_f + K^{(q)}$). We have the following

Proposition 7.6. – *Let $\phi_- \in C^\infty(\Gamma \times \Gamma)$ be as in Theorem 6.15. Then, in some neighborhood U of* $\operatorname{diag}(\Gamma_q \times \Gamma_q)$ *in $M' \times M'$, there exists a smooth function $\widetilde{\phi}(z, y) \in C^\infty\big((\overline{M} \times \Gamma) \bigcap U\big)$ such that*

$$\begin{aligned} &\widetilde{\phi}(x, y) = \phi_-(x, y), \quad \operatorname{Im} \widetilde{\phi} \geq 0, \\ &d_z\widetilde{\phi} \neq 0, \ d_y\widetilde{\phi} \neq 0 \ \textit{where} \operatorname{Im} \widetilde{\phi} = 0, \\ &\operatorname{Im} \widetilde{\phi} > 0 \ \textit{if}\ r \neq 0, \end{aligned} \tag{7.9}$$

and $q_0(z,\widetilde{\phi}'_z)$ vanishes to infinite order on $r=0$. We write $\frac{\partial}{\partial r(z)}$ to denote $\frac{\partial}{\partial r}$ acting in the z variables. We have

$$\frac{\partial}{\partial r(z)}\widetilde{\phi}(z,y)|_{r=0} = -i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)} \tag{7.10}$$

in some neighborhood of $x=y$, where $\mathrm{Re}\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}>0$.

Proof. – From (7.7) and (7.8), we have

$$\begin{aligned} q_0(z,\zeta) &= \frac{1}{2}h^{2n,2n}(z)\theta^2 + \sum_{j=1}^{2n-1} h^{2n,j}(z)\theta\xi_j + g(z,\xi), \\ g(x,\xi) &= -\frac{1}{2}\sigma_{\triangle_\Gamma}, \end{aligned} \tag{7.11}$$

where $g(z,\xi)$ is the principal symbol of $-\frac{1}{2}T(r)$.

We consider the Taylor expansion of $q_0(z,\zeta)$ with respect to r,

$$q_0(z,\zeta) = \frac{1}{2}\theta^2 - \frac{1}{2}\sigma_{\triangle_\Gamma} + \sum_{j=1}^{\infty} g_j(x,\xi)r^j + \sum_{j=1}^{\infty} s_j(x,\zeta)\theta r^j. \tag{7.12}$$

We introduce the Taylor expansion of $\widetilde{\phi}(z,y)$ with respect to r,

$$\widetilde{\phi}(z,y) = \phi_-(x,y) + \sum_1^\infty \phi_j(x,y)r^j.$$

Let $\phi_1(x,y) = -i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}$. Since $(\phi_-)'_x|_{x=y} = -\omega_0(x)$ is real, we choose the branch of $\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}$ so that $\mathrm{Re}\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}>0$ in some neighborhood of $x=y$, $r=0$. Put $\widetilde{\phi}_1(z,y) = \phi_-(x,y)+r\phi_1(x,y)$. We have $q_0(z,(\widetilde{\phi}_1)'_z) = O(r)$. Similarly, we can find $\phi_2(x,y)$ so that $q_0(z,(\widetilde{\phi}_2)'_z) = O(r^2)$, where $\widetilde{\phi}_2(z,y) = \phi_-(x,y) + r\phi_1(x,y) + r^2\phi_2(x,y)$. Continuing in this way we get the phase $\widetilde{\phi}(z,y)$ such that $\widetilde{\phi}(x,y) = \phi_-(x,y)$ and $q_0(z,\widetilde{\phi}'_z)$ vanishes to infinite order on $r=0$. The proposition follows. □

Remark 7.7. – Let $\widetilde{\phi}(z,y)$ be as in Proposition 7.6 and let

$$d(z,y,t)\in S^m_{1,0}\big(\overline{M}\times\Gamma\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))\big)$$

with support in some neighborhood of diag $(\Gamma_q\times\Gamma_q)$ (for the meaning of the space $S^m_{1,0}\big(\overline{M}\times\Gamma\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))\big)$, see Definition 1.1). Choose a cut-off function $\chi(t)\in C^\infty(\mathbb{R})$ so that $\chi(t)=1$ when $|t|<1$ and $\chi(t)=0$ when $|t|>2$. For all $u\in C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M'))$, set

$$(D_\varepsilon u)(z) = \iint_0^\infty e^{i\widetilde{\phi}(z,y)t}d(z,y,t)\chi(\varepsilon t)u(y)dtdy.$$

Since $\operatorname{Im}\widetilde{\phi}\geq 0$ and $d_y\widetilde{\phi}\neq 0$ where $\operatorname{Im}\widetilde{\phi}=0$, we can integrate by parts in y, t and obtain $\lim_{\varepsilon\to 0}(D_\varepsilon u)(z)\in C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))$. This means that $D=\lim_{\varepsilon\to 0}D_\varepsilon:$ $C^\infty(\Gamma;\,\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))$ is continuous. Formally,

$$D(z,y)=\int_0^\infty e^{i\widetilde{\phi}(z,y)t}d(z,y,t)dt.$$

PROPOSITION 7.8. – *Let*

$$B_-(x,y)=\int_0^\infty e^{i\phi_-(x,y)t}b(x,y,t)dt$$

be as in Theorem 6.15. We have

$$\widetilde{P}B_-(z,y)\equiv\int_0^\infty e^{i\widetilde{\phi}(z,y)t}\widetilde{b}(z,y,t)dt\quad\operatorname{mod}C^\infty(\overline{M}\times\Gamma)$$

with $\widetilde{b}(z,y,t)\in S^{n-1}_{1,0}\big(\overline{M}\times\Gamma\times]0,\infty[;\,\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_z(M'))\big)$,

$$\widetilde{b}(z,y,t)\sim\sum_{j=0}^\infty\widetilde{b}_j(z,y)t^{n-1-j}$$

in $S^{n-1}_{1,0}\big(\overline{M}\times\Gamma\times]0,\infty[;\,\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_z(M'))\big)$, *where*

$$\widetilde{b}_j(z,y)\in C^\infty\big(\overline{M}\times\Gamma;\,\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_z(M'))\big),\quad j=0,1,\ldots.$$

Proof. – Put $b(x,y,t)\sim\sum_{j=0}^\infty b_j(x,y)t^{n-1-j}$ and formally set

$$\widetilde{b}(z,y,t)\sim\sum_{j=0}^\infty\widetilde{b}_j(z,y)t^{n-1-j}.$$

We notice that $B_-(x,y)\in C^\infty\big(\Gamma\times\Gamma\setminus\operatorname{diag}(\Gamma_q\times\Gamma_q);\,\mathscr{L}(\Lambda^{0,q}T^*_y(\Gamma),\Lambda^{0,q}T^*_z(\Gamma))\big)$. For simplicity, we may assume that $b(x,y,t)=0$ outside some small neighborhood of $\operatorname{diag}(\Gamma_q\times\Gamma_q)\times\overline{\mathbb{R}}_+$. Put $\widetilde{\Box}^{(q)}_f(\widetilde{b}(z,y,t)e^{i\widetilde{\phi}t})=\widetilde{c}(z,y,t)e^{i\widetilde{\phi}t}$. From (6.20) and (7.9), we know that near $\operatorname{diag}(\Gamma_q\times\Gamma_q)$, $\widetilde{\phi}(z,y)=0$ if and only if $x=y$, $r=0$. From this observation, we see that if $\widetilde{c}(z,y,t)$ vanishes to infinite order on $\operatorname{diag}(\Gamma_q\times\Gamma_q)\times\overline{\mathbb{R}}_+$, we can integrate by parts and obtain

$$\lim_{\varepsilon\to 0}\int_0^\infty e^{i\widetilde{\phi}t}\widetilde{c}(z,y,t)\chi(\varepsilon t)dt\equiv 0\quad\operatorname{mod}C^\infty(\overline{M}\times\Gamma),$$

where $\chi(t)$ is as in Remark 7.7. Thus, we only need to consider the Taylor expansion of $\widetilde{b}(z,y,t)$ on $x=y$, $r=0$. We introduce the asymptotic expansion of $\widetilde{\Box}^{(q)}_f(\widetilde{b}e^{i\widetilde{\phi}t})$. Setting $\widetilde{\Box}^{(q)}_f(\widetilde{b}e^{i\widetilde{\phi}t})\sim 0$ and regrouping the terms according to the degree of homogeneity. We obtain the transport equations

$$\begin{cases}T(z,y,\partial_z)\widetilde{b}_0(z,y)=0,\\ T(z,y,\partial_z)\widetilde{b}_j(z,y)+l_j(z,y,\widetilde{b}_0(z,y),\ldots,\widetilde{b}_{j-1}(z,y))=0,\quad j=1,2,\ldots.\end{cases}\tag{7.13}$$

Here

$$T(z,y,\partial_z)=-i\sum_{j=1}^{2n-1}\frac{\partial q_0}{\partial\xi_j}(z,\widetilde{\phi}'_z)\frac{\partial}{\partial x_j}-i\frac{\partial q_0}{\partial\theta}(z,\widetilde{\phi}'_z)\frac{\partial}{\partial r}+R(z,y),$$

where

$$R(z,y)=q_1(z,\widetilde{\phi}'_z)+\frac{1}{2i}\sum_{j,k=1}^{2n}\frac{\partial^2 q_0(z,\widetilde{\phi}'_z)}{\partial\xi_j\partial\xi_k}\frac{\partial^2\widetilde{\phi}}{\partial x_j\partial x_k},\quad x_{2n}=r,\quad \xi_{2n}=\theta,$$

and l_j is a linear differential operator acting on $\widetilde{b}_0(z,y),\ldots,\widetilde{b}_{j-1}(z,y)$.

We introduce the Taylor expansion of $\widetilde{b}_0(z,y)$ with respect to r,

$$\widetilde{b}_0(z,y)=b_0(x,y)+\sum_1^\infty b_0^j(x,y)r^j.$$

Since $\frac{\partial q_0}{\partial\theta}|_{r=0}=\theta$ and $\widetilde{\phi}'_r|_{r=0}=-i\sqrt{-\sigma_{\triangle_\Gamma}(x,(\phi_-)'_x)}$, we have $\frac{\partial q_0}{\partial\theta}(z,\widetilde{\phi}'_z)|_{r=0}\neq 0$ in some neighborhood of $x=y$. Thus, we can find $b_0^1(x,y)r$ such that

$$T(z,y,\partial_z)\big(b_0(x,y)+b_0^1(x,y)r\big)=O(|r|)$$

in some neighborhood of $r=0$, $x=y$. We can repeat the procedure above to find $b_0^2(x,y)$ such that $T(z,y,\partial_z)(b_0(x,y)+\sum_{k=1}^2 b_0^k(x,y)r^k)=O(|r|^2)$ in some neighborhood of $r=0$, $x=y$. Continuing in this way we solve the first transport equation to infinite order at $r=0$, $x=y$.

For the second transport equation, we can repeat the method above to solve the second transport equation to infinite order at $r=0$, $x=y$. Continuing in this way we solve (7.13) to infinite order at $r=0$, $x=y$.

Put $\widetilde{B}(z,y)=\int_0^\infty e^{i\widetilde{\phi}(z,y)t}\widetilde{b}(z,y,t)dt$. From the construction above, we see that

$$\widetilde{\Box}_f^{(q)}\widetilde{B}\equiv 0\mod C^\infty(\overline{M}\times\Gamma),\quad \gamma\widetilde{B}\equiv B_-. \tag{7.14}$$

It is well-known (see chapter XX of [**15**]) that there exists

$$G:C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))\to C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M'))$$

such that

$$G\widetilde{\Box}_f^{(q)}+\widetilde{P}\gamma=I\text{ on }C^\infty(\overline{M};\,\Lambda^{0,q}T^*(M')). \tag{7.15}$$

From this and (7.14), we have $\widetilde{B}=(G\widetilde{\Box}_f^{(q)}+\widetilde{P}\gamma)\widetilde{B}\equiv\widetilde{P}B_-\mod C^\infty(\overline{M}\times\Gamma)$. The proposition follows. □

From Proposition 7.8, we have

$$C(z,y):=\widetilde{P}B_-T(\widetilde{P}^*\widetilde{P})^{-1}(z,y)\equiv\int_0^\infty e^{i\widetilde{\phi}(z,y)t}c(z,y,t)dt\mod C^\infty(\overline{M}\times\Gamma)$$

with $c(z,y,t) \in S^n_{1,0}\big(\overline{M} \times \Gamma \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))\big)$,

$$c(z,y,t) \sim \sum_{j=0}^{\infty} c_j(z,y)t^{n-j}$$

in the space $S^n_{1,0}\big(\overline{M} \times \Gamma \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_y(M'),\Lambda^{0,q}T^*_z(M'))\big)$. Let

$$C^* : C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')) \to \mathscr{D}'(\Gamma;\ \Lambda^{0,q}T^*(M'))$$

be the operator defined by $(C^*u \mid v)_\Gamma = (u \mid Cv)_M$, $u \in C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$, $v \in C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M'))$. The distribution kernel of C^* is

$$C^*(y,z) \equiv \int_0^\infty e^{-i\overline{\widetilde{\phi}}(z,y)t} c^*(y,z,t)dt \mod C^\infty(\Gamma \times \overline{M}) \tag{7.16}$$

where

$$c^*(y,z,t) \in S^n_{1,0}(\Gamma \times \overline{M} \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_y(M'))),$$
$$(c^*(y,z,t)\mu \mid \nu) = (\mu \mid c(z,y,t)\nu),\quad \mu \in \Lambda^{0,q}T^*_z(M'),\ \nu \in \Lambda^{0,q}T^*_y(M'),$$
$$c^*(y,z,t) \sim \sum_{j=0}^{\infty} c^*_j(y,z)t^{n-j}$$

in $S^n_{1,0}\big(\Gamma \times \overline{M} \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_y(M'))\big)$. The integral (7.16) is defined as follows: Let $u \in C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M'))$. Set

$$(C^*_\varepsilon u)(y) = \iint_0^\infty e^{-i\overline{\widetilde{\phi}(z,y)}t} c^*(y,z,t)\chi(\varepsilon t)u(z)dtdz,$$

where χ is as in Remark 7.7. Since $d_x\widetilde{\phi} \neq 0$ where $\operatorname{Im}\widetilde{\phi} = 0$, we can integrate by parts in x and t and obtain $\lim_{\varepsilon\to 0}(C^*_\varepsilon u)(y) \in C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M'))$. This means that $C^* = \lim_{\varepsilon\to 0} C^*_\varepsilon : C^\infty(\overline{M};\ \Lambda^{0,q}T^*(M')) \to C^\infty(\Gamma;\ \Lambda^{0,q}T^*(M'))$ is continuous.

We also write $w = (y_1,\ldots,y_{2n-1},r)$. We can repeat the proof of Proposition 7.6 to find $\phi(z,w) \in C^\infty(\overline{M} \times \overline{M})$ such that $\phi(z,y) = \widetilde{\phi}(z,y)$, $\operatorname{Im}\phi \geq 0$, $\operatorname{Im}\phi > 0$ if $(z,w) \notin \Gamma \times \Gamma$ and $q_0(w, -\overline{\phi}'_w)$ vanishes to infinite order on $r = 0$. Since $\phi_-(x,y) = -\overline{\phi}_-(y,x)$, we can take $\phi(z,w)$ so that $\phi(z,w) = -\overline{\phi}(w,z)$. As in the proof of Proposition 7.8, we can find

$$a^*(w,z,t) \in S^n_{1,0}\big(\overline{M} \times \overline{M} \times [0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_w(M'))\big),$$
$$a^*(w,z,t) \sim \sum_{j=0}^{\infty} a^*_j(w,z)t^{n-j}$$

in $S^n_{1,0}\big(\overline{M} \times \overline{M} \times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_z(M'),\Lambda^{0,q}T^*_w(M'))\big)$, such that

$$a^*(y,z,t) = c^*(y,z,t)$$

and $\widetilde{\Box}^{(q)}_f(a^*(w,z,t)e^{-i\overline{\phi}(z,w)t})$ vanishes to infinite order on $\operatorname{diag}(\Gamma_q\times\Gamma_q)\times\overline{\mathbb{R}}_+$. From (7.15), we have $\widetilde{P}C^*(w,z)\equiv\int_0^\infty e^{-i\overline{\phi}(z,w)t}a^*(w,z,t)dt \mod C^\infty(\overline{M}\times\overline{M})$. Thus,

$$C\widetilde{P}^*(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}a(z,w,t)dt \mod C^\infty(\overline{M}\times\overline{M}),$$
$$a(z,w,t)\in S^n_{1,0}\big(\overline{M}\times\overline{M}\times[0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big),$$
$$a(z,w,t)\sim\sum_{j=0}^\infty a_j(z,w)t^{n-j}$$

in the space $S^n_{1,0}\big(\overline{M}\times\overline{M}\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big)$. Note that $C\widetilde{P}^*=\widetilde{P}B_-T(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*$. From this and Proposition 7.5, we get the main result of this work

THEOREM 7.9. – *Given q, $0\leq q\leq n-1$. Suppose that $Z(q)$ fails at some point of Γ and that $Z(q-1)$ and $Z(q+1)$ hold at each point of Γ. Then*

$$K_{\Pi^{(q)}}(z,w)\in C^\infty\big(\overline{M}\times\overline{M}\setminus\operatorname{diag}(\Gamma_q\times\Gamma_q);\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big).$$

Moreover, in a neighborhood U of $\operatorname{diag}(\Gamma_q\times\Gamma_q)$, $K_{\Pi^{(q)}}(z,w)$ satisfies

$$K_{\Pi^{(q)}}(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}a(z,w,t)dt \mod C^\infty\big(U\textstyle\bigcap(\overline{M}\times\overline{M})\big) \tag{7.17}$$

with $a(z,w,t)\in S^n_{1,0}\big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big)$,

$$a(z,w,t)\sim\sum_{j=0}^\infty a_j(z,w)t^{n-j}$$

in the space $S^n_{1,0}\big(U\bigcap(\overline{M}\times\overline{M})\times]0,\infty[;\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big)$,

$$a_0(z,z)\neq 0,\ \ z\in\Gamma_q,$$

where $a_j(z,w)\in C^\infty\big(U\bigcap(\overline{M}\times\overline{M});\ \mathscr{L}(\Lambda^{0,q}T^*_w(M'),\Lambda^{0,q}T^*_z(M'))\big)$, $j=0,1,\dots$, and

$$\phi(z,w)\in C^\infty(U\textstyle\bigcap(\overline{M}\times\overline{M})),\quad \operatorname{Im}\phi\geq 0, \tag{7.18}$$
$$\phi(z,z)=0,\quad z\in\Gamma_q,\quad \phi(z,w)\neq 0\ \ \textit{if}\ \ (z,w)\notin\operatorname{diag}(\Gamma_q\times\Gamma_q), \tag{7.19}$$
$$\operatorname{Im}\phi(z,w)>0\ \ \textit{if}\ \ (z,w)\notin\Gamma\times\Gamma, \tag{7.20}$$
$$\phi(z,w)=-\overline{\phi}(w,z). \tag{7.21}$$

For $p\in\Gamma_q$, we have $\sigma_{\Box^{(q)}_f}(z,d_z\phi(z,w))$ vanishes to infinite order at $z=p$, where (z,w) is in some neighborhood of (p,p) in M'.

For $z=w$, $z\in\Gamma_q$, we have $d_z\phi=-\omega_0-idr$, $d_w\phi=\omega_0-idr$.

As before, we put $B_-(x,y)\equiv\int_0^\infty e^{i\phi_-(x,y)t}b(x,y,t)dt$,

$$b(x,y,t)\sim\sum_{j=0}^\infty b_j(x,y)t^{n-1-j}$$

and

$$K_{\Pi^{(q)}}(z,w) \equiv \int_0^\infty e^{i\phi(z,w)t} a(z,w,t)dt,$$

$a(z,w,t) \sim \sum_{j=0}^\infty a_j(z,w)t^{n-j}$. Since

$$\Pi^{(q)} \equiv \widetilde{P} B_- T(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^*,$$

$(\widetilde{P}^*\widetilde{P})^{-1} = 2\sqrt{-\triangle_\Gamma}$ + lower order terms and

$$T = 2(\overline{\partial} r)^{\wedge,*}(\overline{\partial} r)^\wedge + \text{lower order terms},$$

we have

$$a_0(x,x) = 2\sigma_{\sqrt{-\triangle_\Gamma}}(x,(\phi_-)'_y(x,x))b_0(x,x)2(\overline{\partial} r(x))^{\wedge,*}(\overline{\partial} r(x))^\wedge, \quad x \in \Gamma.$$

Since $(\phi_-)'_y(x,x) = \omega_0(x)$ and $\|\omega_0\| = 1$ on Γ, it follows that

$$a_0(x,x) = 4b_0(x,x)(\overline{\partial} r(x))^{\wedge,*}(\overline{\partial} r(x))^\wedge. \tag{7.22}$$

From this and Proposition 6.17, we get the following

Proposition 7.10. – *Under the assumptions of Theorem 7.9, let $p \in \Gamma_q$, $q = n_-$. Let $U_1(z),\ldots,U_{n-1}(z)$ be an orthonormal frame of $\Lambda^{1,0}T_z(\Gamma)$, $z \in \Gamma$, for which the Levi form is diagonalized at p. Let $e_j(z)$, $j = 1,\ldots,n-1$ denote the basis of $\Lambda^{0,1}T^*_z(\Gamma)$, $z \in \Gamma$, which is dual to $\overline{U}_j(z)$, $j = 1,\ldots,n-1$. Let $\lambda_j(z)$, $j = 1,\ldots,n-1$ be the eigenvalues of the Levi form L_z, $z \in \Gamma$. We assume that $\lambda_j(p) < 0$ if $1 \leq j \leq n_-$. Then*

(7.23)

$$a_0(p,p) = |\lambda_1(p)|\cdots|\lambda_{n-1}(p)|\,\pi^{-n}2\Big(\prod_{j=1}^{j=n_-} e_j(p)^\wedge e_j^{\wedge,*}(p)\Big)\circ(\overline{\partial} r(p))^{\wedge,*}(\overline{\partial} r(p))^\wedge.$$

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