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WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

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WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Yukiyoshi Nakkajima

***Abstract.* —** We construct a theory of weights on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$ by using the log crystalline cohomology of a split proper hypercovering of the scheme. We also calculate the slope filtration on the rigid cohomology by using the cohomology of the log de Rham-Witt complex of the hypercovering.

***Résumé.* —** Nous construisons une théorie des poids sur la cohomologie rigide d'un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$ en utilisant la cohomologie log-cristalline d'un hyperrecouvrement propre scindé du schéma. Nous calculons aussi la filtration par les pentes sur la cohomologie rigide en utilisant la cohomologie du complexe de de Rham-Witt logarithmique de l'hyperrecouvrement.

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1. Introduction

J.-P. Serre has conjectured the existence of the virtual Betti number of a separated scheme of finite type over a field (see [**33**]). Furthermore, A. Grothendieck has conjectured the existence of, so to speak, the virtual slope number and the virtual Hodge number of the scheme (see [*loc. cit.*]). Let us recall the numbers briefly as follows.

Let κ be an algebraically closed field of characteristic $p \geq 0$. Let $\mathbf{S}(\kappa)$ be the set of isomorphism classes of separated schemes of finite type over κ. For a separated scheme U of finite type over κ, we denote by $[U]$ the element of $\mathbf{S}(\kappa)$ defined by U. For a function $f\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$, we denote $f([U])$ by $f(U)$ for simplicity of notation. Let r be a nonnegative integer. Then Serre has conjectured that there exists a unique function

$$h^r_{\mathrm{c}}\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

satisfying the following two properties (see [**33**]):

▹ For a closed subscheme Z of U,

$$h^r_{\mathrm{c}}(U) = h^r_{\mathrm{c}}(Z) + h^r_{\mathrm{c}}(U \setminus Z). \tag{I-a}$$

▹ If U is a proper smooth scheme over κ, then

$$h^r_{\mathrm{c}}(U) = (-1)^r \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell) \tag{I-b}$$

for a prime number $\ell \neq p$.

Here the letter 'c' in the notation h^r_{c} stands for the compact support. Heuristically $h^r_{\mathrm{c}}(U)$ is given by the formula (see [*loc. cit.*]):

$$h^r_{\mathrm{c}}(U) = \begin{cases} \sum_{m=0}^{\infty}(-1)^m \dim_{\mathbb{Q}_\ell} \mathrm{gr}^P_r H^m_{\text{ét},c}(U, \mathbb{Q}_\ell) & (p > 0), \\ \sum_{m=0}^{\infty}(-1)^m \dim_{\mathbb{Q}} \mathrm{gr}^P_r H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}), \end{cases} \tag{1.0.1}$$

where P is the 'weight filtration' on $H^m_{\text{ét},c}(U, \mathbb{Q}_\ell)$ (resp. $H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q})$). (For the cohomology without compact support, see (1.0.3) below for the definition of the weight filtration P when κ is $\overline{\mathbb{F}}_p$ (resp. $\mathbb{C}$).) Set

$$H^m_{\mathrm{c}}(U) = \begin{cases} H^m_{\text{ét},c}(U, \mathbb{Q}_\ell) & (p > 0), \\ H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}). \end{cases}$$

Heuristically (I-a) is obtained from the strict exactness with respect to P of the exact sequence

$$\cdots \longrightarrow H^m_{\mathrm{c}}(U \setminus Z) \longrightarrow H^m_{\mathrm{c}}(U) \longrightarrow H^m_{\mathrm{c}}(Z) \longrightarrow \cdots .$$

Serre has called $h^r_{\mathrm{c}}(U)$ the virtual Betti number of $[U]$. He has also remarked that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^r_{c}. Serre's conjecture was the starting point of Grothendieck's theory of weights (see [**33**]).

In [**33**] Grothendieck has conjectured, for two nonnegative integers i and j, there exists a function

$$h^{ij}_{\mathrm{c}} \colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

such that

$$\text{(II-a)} \qquad h^r_{\mathrm{c}}(U) = \sum_{i+j=r} h^{ij}_{\mathrm{c}}(U).$$

In the case $p > 0$, let $\mathcal{W}$ be the Witt ring of κ, K_0 the fraction field of $\mathcal{W}$ and $\mathcal{W}\Omega^i_U$ the de Rham-Witt sheaf of U of degree i (see [**47**]). We call $h^{ij}_{\mathrm{c}}(U)$ the *virtual slope number* and the *virtual Hodge number* of $[U]$ if $p > 0$ and $p = 0$, respectively, because we require that h^{ij}_{c} satisfies the equation:

$$\text{(II-b)} \qquad h^{ij}_{\mathrm{c}}(U) = \begin{cases} (-1)^{i+j} \dim_{K_0}(H^j(U, \mathcal{W}\Omega^i_U) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ (-1)^{i+j} \dim_{\kappa} H^j(U, \Omega^i_{U/\kappa}) & (p = 0) \end{cases}$$

if U is a proper smooth scheme over κ. We also require that h^{ij}_{c} satisfies the equation

$$\text{(II-c)} \qquad h^{ij}_{\mathrm{c}}(U) = h^{ij}_{\mathrm{c}}(Z) + h^{ij}_{\mathrm{c}}(U \setminus Z).$$

Furthermore, we conjecture that h^{ij}_{c} is uniquely determined by the equations (II-b) and (II-c). It is easy to check that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^{ij}_{c}. The existence of h^{ij}_{c} $(i, j \in \mathbb{N})$ implies the existence of h^r_{c} $(r \in \mathbb{N})$ because (I-a) is clear by (II-a) and (II-c) and because (I-b) holds as follows: if U is a proper smooth scheme over κ, then we have the equations

$$\text{(1.0.2)} \qquad \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell) = \begin{cases} \dim_{K_0}(H^r_{\mathrm{crys}}(U/\mathcal{W}) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \dim_{\kappa} H^r_{\mathrm{dR}}(U/\kappa) & (p = 0), \end{cases}$$
$$= \begin{cases} \sum_{i+j=r} \dim_{K_0}(H^j(U, \mathcal{W}\Omega^i_U) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \sum_{i+j=r} \dim_{\kappa} H^j(U, \Omega^i_{U/\kappa}) & (p = 0). \end{cases}$$

The first (resp. the second) equation in (1.0.2) in the case $p > 0$ has been obtained in [**54**], [**16**] and [**69**] (resp. [**47**]). The first equation in (1.0.2) for the case $\kappa = \mathbb{C}$ is obtained by the following way:

$$\begin{aligned}\dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell) &= \dim_{\mathbb{Q}} H^r(U_{\text{an}}, \mathbb{Q}) = \dim_{\mathbb{C}} H^r(U_{\text{an}}, \mathbb{C}) \\ &= \dim_{\mathbb{C}} H^r_{\text{dR}}(U_{\text{an}}/\mathbb{C}) = \dim_{\mathbb{C}} H^r_{\text{dR}}(U/\mathbb{C})\end{aligned}$$

(see [**36**], the Poincaré lemma, GAGA [**41**]). The first equation in (1.0.2) in the case $p = 0$ is reduced to the case $\kappa = \mathbb{C}$. The second equation in (1.0.2) in the case $p = 0$ has been obtained in [**22**].

In this book, we solve the following variants of Serre's and Grothendieck's conjectures about the existence of the functions h^r_{c} and h^{ij}_{c}.

Let $\mathbf{CS}(\kappa)$ be the set of isomorphism classes of pairs of separated schemes of finite type over κ and their closed subschemes. For a closed subscheme Z of a separated scheme U of finite type over κ, we denote by $[(Z, U)]$ the element of $\mathbf{CS}(\kappa)$ defined by U and Z. Then we prove that there exists a function

$$h^r_?(?) \colon \mathbf{CS}(\kappa) \ni [(Z, U)] \longmapsto h^r_Z(U) \in \mathbb{Z}$$

(set $h^r(U) := h^r_U(U)$) satisfying the following four properties:

▹ For a closed subscheme Z' of Z,

$$h^r_Z(U) = h^r_{Z'}(U) + h^r_{Z \setminus Z'}(U \setminus Z'). \tag{III-a}$$

▹ For an open subscheme U' of U which contains Z as a closed subscheme,

$$h^r_Z(U') = h^r_Z(U). \tag{III-b}$$

▹ If U and Z are smooth over κ and if Z is of pure codimension c ($c \in \mathbb{N}$) in U, then

$$h^r_Z(U) = h^{r-2c}(Z). \tag{III-c}$$

▹ If U is a proper smooth scheme over κ, then

$$h^r(U) = (-1)^r \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell). \tag{III-d}$$

Heuristically (III-a), (III-b) and (III-c) stem from the exact sequence

$$\cdots \longrightarrow H^m_{Z'}(U, \mathbb{Q}_\ell) \longrightarrow H^m_Z(U, \mathbb{Q}_\ell) \longrightarrow H^m_{Z \setminus Z'}(U \setminus Z', \mathbb{Q}_\ell) \longrightarrow \cdots,$$

the isomorphism

$$H^m_Z(U, \mathbb{Q}_\ell) \xrightarrow{\sim} H^m_Z(U', \mathbb{Q}_\ell) \quad (m \in \mathbb{N})$$

and the Gysin isomorphism

$$G_{Z/U} \colon H^{m-2c}(Z, \mathbb{Q}_\ell)(-c) \xrightarrow{\sim} H^m_Z(U, \mathbb{Q}_\ell) \quad (m \in \mathbb{N}),$$

respectively.

Moreover, we prove that, for two nonnegative integers i and j, there exists a function

$$h^{ij}_{?}(?)\colon \mathbf{CS}(\kappa) \ni \big[(Z,U)\big] \longmapsto h^{ij}_{Z}(U) \in \mathbb{Z}$$

(set $h^{ij}(U) := h^{ij}_{U}(U)$) satisfying the following five properties:

▷ For a closed subscheme Z' of Z,

$$h^{ij}_{Z}(U) = h^{ij}_{Z'}(U) + h^{ij}_{Z\setminus Z'}(U \setminus Z'). \tag{IV-a}$$

▷ For an open subscheme U' of U which contains Z as a closed subscheme,

$$h^{ij}_{Z}(U') = h^{ij}_{Z}(U). \tag{IV-b}$$

▷ If U and Z are smooth over κ and if Z is of pure codimension c ($c \in \mathbb{N}$) in U, then

$$h^{ij}_{Z}(U) = h^{i-c,j-c}(Z). \tag{IV-c}$$

▷ If U is a proper smooth scheme over κ, then

$$h^{ij}(U) = \begin{cases} (-1)^{i+j}\dim_{K_0}(H^j(U,\mathcal{W}\Omega^i_U) \otimes_{\mathcal{W}} K_0) & (p>0),\\ (-1)^{i+j}\dim_{\kappa} H^j(U,\Omega^i_{U/\kappa}) & (p=0), \end{cases} \tag{IV-d}$$

$$h^{r}_{Z}(U) = \sum_{i+j=r} h^{ij}_{Z}(U). \tag{IV-e}$$

Let $\mathbf{SCS}(\kappa)$ be the set of isomorphism classes of pairs of separated smooth schemes of finite type over κ and their closed subschemes over κ. In the text, we also show that the (conjectural) embedded resolution of singularities for any variety with any closed subscheme (in the positive characteristic case) implies the uniqueness of the restriction $h^{ij}_{?}(?)|_{\mathbf{SCS}(\kappa)}$ of $h^{ij}_{?}(?)$.

By the philosophy of weights above, Grothendieck has born the conjectural theory of motives (see [**33**]). Motivated by the philosophy of weights, by the conjectural theory of motives and by Grothendieck's isomorphism between the Betti cohomology with coefficients in $\mathbb{C}$ of the analytification of a smooth scheme over $\mathbb{C}$ and its de Rham cohomology (see [**34**]), P. Deligne has given a translation between concepts on ℓ-adic cohomologies and those on the mixed Hodge structure in [**23**], and he has constructed the theory of Hodge-Deligne in [**24**] and [**25**]. Especially he has given the following translation for a separated scheme U (and V) of finite type over a field κ:

(1.0.3)

ℓ-adic objects/$\mathbb{F}_q$ (for simplicity), $(\ell, q) = 1$	objects/$\mathbb{C}$
$H^h_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \quad (h \in \mathbb{Z})$	$H^h(U_{\text{an}}, \mathbb{Q}) \quad (h \in \mathbb{Z})$
F: geometric Frobenius $\curvearrowright H^h_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$; there exists a unique finite increasing filtration $\{P_k\}_{k\in\mathbb{Z}}$ characterized by the following: $P_k H^h_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is the principal subspace of $H^h_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ where the eigenvalues α's of F satisfy the following: $\lvert\sigma(\alpha)\rvert \leq q^{k/2}$ $(\forall\sigma\colon \overline{\mathbb{Q}} \overset{\subset}{\longrightarrow} \mathbb{C})$ (cf. [**50**], [**31**], (12.20) below)	(MHS/$\mathbb{Q}$) on $H^h(U_{\text{an}}, \mathbb{Q})$:= (mixed Hodge structures/$\mathbb{Q}$) on $H^h(U_{\text{an}}, \mathbb{Q})$:= (weight filtrations/$\mathbb{Q}$)+(Hodge ones) on $H^h(U_{\text{an}}, \mathbb{Q})$.
A morphism $H^h_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \longrightarrow H^h_{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell[F]$-modules is strictly compatible with P_k's.	A morphism in the category (MHS's/$\mathbb{Q}$) is strictly compatible with the weight filtration and the Hodge filtration

Let Z be a closed subscheme of U. In the case $\kappa = \mathbb{C}$, we can endow $H^h_{Z_{\text{an}}}(U_{\text{an}}, \mathbb{Q})$ with the mixed Hodge structure as a corollary of the theory of Hodge-Deligne, and we can solve the variants of Serre's and Grothendieck's conjectures in the case $\kappa = \mathbb{C}$.

Let us consider the case where κ is a perfect field of characteristic $p > 0$. Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with residue field κ and let K be the fraction field of $\mathcal{V}$. Let $\mathcal{W}$ be the Witt ring of κ and let K_0 be the fraction field of $\mathcal{W}$. P. Berthelot has defined the rigid cohomology $H^h_{\text{rig},Z}(U/K)$ $(h \in \mathbb{N})$ with closed support (see [**6**], [**9**], [**19**], [**62**]), which corresponds to $H^h_{Z_{\text{an}}}(U_{\text{an}}, \mathbb{Q})$ in the case $\kappa = \mathbb{C}$. He has also defined the rigid cohomology $H^h_{\text{rig,c}}(U/K)$ with compact support (see [**6**]). The main results in this book are to endow $H^h_{\text{rig},Z}(U/K)$ with a weight filtration and to calculate the slope filtration on $H^h_{\text{rig},Z}(U/K_0)$. As a corollary of the main results, we solve the variants of Serre's and Grothendieck's conjectures about the existence of the functions $h^r_?(?)$ and $h^{ij}_?(?)$ in the case $\text{ch}(\kappa) > 0$. If U is a closed subscheme of a separated smooth scheme over κ, then we also endow $H^h_{\text{rig,c}}(U/K)$ with a weight filtration and calculate the slope filtration on $H^h_{\text{rig,c}}(U/K_0)$. As a corollary, we solve original Serre's and Grothendieck's conjectures about the existence of the functions h^r_{c} and h^{ij}_{c} in the case $\text{ch}(\kappa) > 0$

for separated schemes of finite type over κ which are closed subschemes of separated smooth schemes over κ.

Before explaining our results in this book in more details, we give the following picture:

$$\begin{array}{ccc} [\mathbf{24}]\ (\text{see } [\mathbf{21}]) & \longrightarrow & [\mathbf{72}] \\ \downarrow & & \downarrow \\ [\mathbf{25}]\ (\text{see } [\mathbf{21}]) & \longrightarrow & \text{this book.} \end{array}$$

Here we mean the movement from the case $\mathrm{ch}(\kappa) = 0$ to the case $\mathrm{ch}(\kappa) > 0$ by the horizontal arrows, and we mean that the sources of the vertical arrows are bases for the targets.

First we recall Deligne's result in [**24**] quickly.

Let X be a proper smooth scheme over $\mathbb{C}$ with an SNCD (= simple normal crossing divisor) D. Set $U := X \setminus D$. Let $j\colon U \hookrightarrow X$ be the natural open immersion. Set $D^{(0)} := X$ and, for a positive integer k, let $D^{(k)}$ be the disjoint union of all k-fold intersections of the different irreducible components of D. Let P be the weight filtration on the sheaf $\Omega^i_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}})$ ($i \in \mathbb{N}$) of logarithmic differential forms on X_{an} along D_{an} which is obtained by counting the numbers of the logarithmic poles along D_{an} of local sections of $\Omega^i_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}})$. Let Fil_H be the stupid filtration on $\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}})$:

$$\mathrm{Fil}^i_H\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}) := \Omega^{\bullet\geq i}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}) \quad (i \in \mathbb{Z}).$$

Let τ be the canonical filtration on a complex. Set

$$(K^\bullet_{\mathbb{Q}}, \tau) := \big(Rj_{\mathrm{an}*}(\mathbb{Q}_{U_{\mathrm{an}}}), \tau\big)$$

in the derived category $\mathrm{D}^+\mathrm{F}(\mathbb{Q}_{X_{\mathrm{an}}})$ of bounded below filtered complexes of $\mathbb{Q}_{X_{\mathrm{an}}}$-modules; set

$$(K^\bullet_{\mathbb{C}}, P, F) := \big(\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), P, \mathrm{Fil}_H\big)$$

in the derived category $\mathrm{D}^+\mathrm{F}_2(\mathbb{C}_{X_{\mathrm{an}}})$ of bounded below biregular bifiltered complexes of $\mathbb{C}_{X_{\mathrm{an}}}$-modules (see [**25**]). The main result in [**24**] is: the pair

$$\big((K^\bullet_{\mathbb{Q}}, \tau), (K^\bullet_{\mathbb{C}}, P, F)\big)$$

is a cohomological mixed $\mathbb{Q}$-Hodge complex on X_{an}, that is,

$$(K^\bullet_{\mathbb{Q}}, \tau) \otimes_{\mathbb{Q}} \mathbb{C} \simeq (K^\bullet_{\mathbb{C}}, P) \text{ in } \mathrm{D}^+\mathrm{F}(\mathbb{C}_{X_{\mathrm{an}}})$$

and

$$\big(H^h(X_{\mathrm{an}}, \mathrm{gr}^\tau_k K^\bullet_{\mathbb{Q}}), (H^h(X_{\mathrm{an}}, \mathrm{gr}^P_k K^\bullet_{\mathbb{C}}), F)\big) \quad (h, k \in \mathbb{Z}) \tag{1.0.4}$$

is a $\mathbb{Q}$-Hodge structure of pure weight $h+k$. Here F in (1.0.4) is the induced filtration on $H^h(X_{\mathrm{an}}, \mathrm{gr}_k^P K_{\mathbb{C}}^{\bullet})$ by Fil_H. Let $a^{(k)}\colon D^{(k)} \to X$ ($k \in \mathbb{N}$) be the natural morphism of schemes over $\mathbb{C}$. Let $\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})$ be the orientation sheaf of $D_{\mathrm{an}}^{(k)}/\mathbb{C}$: $\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})(-k) := \epsilon_{\mathbb{Z}}^k$ in [**24**]; $\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})$ is isomorphic to $\mathbb{Z}_{D_{\mathrm{an}}^{(k)}}$ non-canonically. By using the exponential sequence on U_{an} and the cup product, we have the purity isomorphism

$$(1.0.5) \qquad R^k j_{\mathrm{an}*}(\mathbb{Z}_{U_{\mathrm{an}}}) \xleftarrow{\sim} a_{\mathrm{an}*}^{(k)}\bigl(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})\bigr)(-k) \quad (k \in \mathbb{N})$$

(cf. [**53**, (1.5.1)]). By using the isomorphism

$$(1.0.6) \qquad R^k j_{\mathrm{an}*}(\mathbb{Z}_{U_{\mathrm{an}}}) \underset{\sim}{\xrightarrow{(1.0.5)}} a_{\mathrm{an}*}^{(k)}\bigl(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})\bigr)(-k) \underset{\sim}{\xrightarrow{(-1)^k}} a_{\mathrm{an}*}^{(k)}\bigl(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})\bigr)(-k)$$

instead of (1.0.5) (the isomorphism (1.0.6) is equal to the isomorphism in [**24**, (3.1.9)]), we have the weight spectral sequence

$$(1.0.7) \quad E_1^{-k,h+k} = H^{h-k}\bigl((D_{\mathrm{an}})^{(k)}, \varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C})\bigr)(-k) \otimes_{\mathbb{Z}} \mathbb{Q} \Longrightarrow H^h(U_{\mathrm{an}}, \mathbb{Q}).$$

In fact, Deligne has proved the existence of the weight filtration on the higher direct image of the constant sheaf $\mathbb{Q}$ for a family of open smooth varieties with good compactifications in characteristic 0 (see [**21**]).

Next we recall Deligne's result in [**25**] quickly.

Let U be a separated scheme of finite type over $\mathbb{C}$. Let $U_{\bullet} = U_{\bullet \in \mathbb{N}}$ be a proper hypercovering of U, that is, $U_{\bullet}$ is a separated simplicial scheme of finite type over $\mathbb{C}$ with an augmentation $U_{\bullet} \to U$ over $\mathbb{C}$ and the natural morphism $U_{n+1} \to \mathrm{cosk}_n^U(U_{\bullet \leq n})_{n+1}$ ($n \geq -1$) is proper and surjective (see [**25**]). Then, by proper cohomological descent, the natural morphism

$$R\Gamma(U_{\mathrm{an}}, \mathbb{Q}) \longrightarrow R\Gamma\bigl((U_{\bullet})_{\mathrm{an}}, \mathbb{Q}\bigr)$$

is an isomorphism in the derived category $\mathrm{D}^+(\mathbb{Q})$ of bounded below complexes of $\mathbb{Q}$-vector spaces (see [*loc. cit.*]). Assume that $U_{\bullet}$ is the complement of a simplicial SNCD $D_{\bullet}$ on a proper smooth simplicial scheme $X_{\bullet}$ over $\mathbb{C}$. Let $j_{\bullet}\colon U_{\bullet} \hookrightarrow X_{\bullet}$ be the natural open immersion. Let P (resp. $\mathrm{Fil_H}$) be the weight filtration (resp. Hodge filtration) on $\Omega^{\bullet}_{(X_{\bullet})_{\mathrm{an}}/\mathbb{C}}(\log(D_{\bullet})_{\mathrm{an}})$. Set

$$(K_{\mathbb{Q}}^{\bullet\bullet}, \tau) := (Rj_{\bullet\mathrm{an}*}(\mathbb{Q}), \tau) \in \mathrm{D}^+\mathrm{F}\bigl(\mathbb{Q}_{(X_{\bullet})_{\mathrm{an}}}\bigr),$$
$$(K_{\mathbb{C}}^{\bullet\bullet}, P, F) := \bigl(\Omega^{\bullet}_{(X_{\bullet})_{\mathrm{an}}/\mathbb{C}}(\log(D_{\bullet})_{\mathrm{an}}), P, \mathrm{Fil_H}\bigr) \in \mathrm{D}^+\mathrm{F}_2(\mathbb{Q}_{(X_{\bullet})_{\mathrm{an}}}).$$

(The left (resp. right) degree is the cosimplicial degree (resp. complex degree).) Then, by the result in [**24**], the pair $((K_{\mathbb{Q}}^{\bullet\bullet}, \tau), (K_{\mathbb{C}}^{\bullet\bullet}, P, F))$ is a cohomological mixed $\mathbb{Q}$-Hodge complex on $(X_\bullet)_{\mathrm{an}}$, that is,

$$(K_{\mathbb{Q}}^{\bullet\bullet}, \tau) \otimes_{\mathbb{Q}} \mathbb{C} \simeq (K_{\mathbb{C}}^{\bullet\bullet}, P) \text{ in } \mathrm{D}^+\mathrm{F}(\mathbb{C}_{(X_\bullet)_{\mathrm{an}}})$$

and, on each $(X_t)_{\mathrm{an}}$, $((K_{\mathbb{Q}}^{t\bullet}, \tau), (K_{\mathbb{C}}^{t\bullet}, P, F))$ is a cohomological mixed $\mathbb{Q}$-Hodge complex on $(X_t)_{\mathrm{an}}$. Let L be the filtrations on

$$R\Gamma\big((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{Q}}^{\bullet\bullet})\big) = \mathbf{s}\big(R\Gamma^\bullet((X_\bullet)_{\mathrm{an}}, K_{\mathbb{Q}}^{\bullet\bullet})\big)$$

and $R\Gamma((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{C}}^{\bullet\bullet}))$ defined in [**24**] (we recall the definition of L in §2 below), where $\mathbf{s}$ means the single complex. Let $\delta(\tau, L)$ (resp. $\delta(P, L)$) be the diagonal filtration of the induced filtration by τ (resp. P) and L on the complex $R\Gamma((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{Q}}^{\bullet\bullet}))$ (resp. $R\Gamma((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{C}}^{\bullet\bullet}))$) (we recall the definition of the diagonal filtration in §2 below). Then Deligne's results are the following:

$$\Big(\big(R\Gamma((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{Q}}^{\bullet\bullet})), \delta(\tau, L)\big), \big(R\Gamma((X_\bullet)_{\mathrm{an}}, \mathbf{s}(K_{\mathbb{C}}^{\bullet\bullet})), \delta(P, L), F\big)\Big)$$

is a cohomological mixed $\mathbb{Q}$-Hodge complex, and there exists the following weight spectral sequence

$$E_1^{-k,h+k} = \bigoplus_{t\geq 0} H^{h-2t-k}\big((D_t^{(t+k)})_{\mathrm{an}}, \varpi^{(t+k)}((D_t)_{\mathrm{an}}/\mathbb{C})\big)(-t-k) \otimes_{\mathbb{Z}} \mathbb{Q} \Longrightarrow H^h\big((U_\bullet)_{\mathrm{an}}, \mathbb{Q}\big) = H^h(U_{\mathrm{an}}, \mathbb{Q}). \tag{1.0.8}$$

Now we quickly recall some results in [**72**] corresponding to [**24**] and [**21**] as follows.

Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a PD-scheme on which p is locally nilpotent. Assume that $\mathcal{I}$ is quasi-coherent. Set

$$S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I}).$$

Let (X, D) be a smooth scheme over S_0 with a relative SNCD over S_0 (see [**72**]). Set $U := X \setminus D$. Let $f \colon X \to S_0$ be the structural morphism; by abuse of notation, f also denotes the composite structural morphism $X \to S_0 \hookrightarrow S$. In [**72**] we have defined a fine and saturated log structure $M(D)$ with a natural inclusion morphism $M(D) \hookrightarrow (\mathcal{O}_X, *)$ associated to $(X, D)/S_0$ (cf. the DF-log structure in [**51**]) which will be recalled in §3 below. Here, following [**52**], we mean by a log structure on a scheme Y a pair (M, α), where M is a sheaf of monoids on the Zariski site Y_{zar} and α is a morphism $M \to (\mathcal{O}_Y, *)$ which induces an isomorphism $\alpha^{-1}(\mathcal{O}_Y^*) \overset{\sim}{\to} \mathcal{O}_Y^*$. The composite morphism

$$\mathcal{O}_Y^* \overset{\sim}{\longrightarrow} \alpha^{-1}(\mathcal{O}_Y^*) \longrightarrow M$$

induces a morphism $\epsilon_{(Y,M)} \colon (Y, M) \to (Y, \mathcal{O}_Y^*)$ of log schemes. We denote $(X, M(D))$ simply by (X, D) by abuse of notation when there arises no confusion. Let $((\widetilde{X, D})/S)^{\log}_{\mathrm{crys}}$ (resp. $(\widetilde{X/S})_{\mathrm{crys}}$) be the log (resp. classical) crystalline topos of $(X,D)/S(=(X,M(D))/S)$ (see [**51**]) (resp. X/S (see [**5**])). Let

$$\epsilon_{(X,D)/S}\colon \big(\widetilde{(X,D)/S}\big)^{\log}_{\rm crys} \longrightarrow (\widetilde{X/S})_{\rm crys}$$

be the natural morphism of topoi induced by the morphism

$$\epsilon_{(X,M(D))}\colon \big(X,M(D)\big) \longrightarrow (X,\mathcal{O}_X^*).$$

Let $u_{(X,D)/S}\colon (\widetilde{(X,D)/S})^{\log}_{\rm crys} \to \widetilde{X}_{\rm zar}$ (resp. $u_{X/S}\colon (\widetilde{X/S})_{\rm crys} \to \widetilde{X}_{\rm zar}$) be the natural projection. Let $(\widetilde{X/S})_{\rm Rcrys}$ be the restricted crystalline topos defined in [**5**] and let $Q_{X/S}\colon (\widetilde{X/S})_{\rm Rcrys} \to (\widetilde{X/S})_{\rm crys}$ be the morphism of topoi such that $Q^*_{X/S}(E)$ is the natural restriction functor of E ($E \in (\widetilde{X/S})_{\rm crys}$) (see [**5**]). Let $\mathcal{O}_{(X,D)/S}$ (resp. $\mathcal{O}_{X/S}$) be the structure sheaf in $(\widetilde{(X,D)/S})^{\log}_{\rm crys}$ (resp. $(\widetilde{X/S})_{\rm crys}$). As in [**5**], set

$$\bar{u}_{X/S} := u_{X/S}\circ Q_{X/S}\colon (\widetilde{X/S})_{\rm Rcrys} \longrightarrow \widetilde{X}_{\rm zar}.$$

The morphisms $\epsilon_{(X,D)/S}$, $u_{(X,D)/S}$, $u_{X/S}$, $Q_{X/S}$ and $\bar{u}_{X/S}$ induce the following morphisms of ringed topoi

$$\begin{array}{ccccc}
((\widetilde{(X,D)/S})^{\log}_{\rm crys}, \mathcal{O}_{(X,D)/S}) & \xrightarrow{\epsilon_{(X,D)/S}} & ((\widetilde{X/S})_{\rm crys}, \mathcal{O}_{X/S}) & \xleftarrow{Q_{X/S}} & ((\widetilde{X/S})_{\rm Rcrys}, Q^*_{X/S}(\mathcal{O}_{X/S})) \\
{\scriptstyle u_{(X,D)/S}}\downarrow & & {\scriptstyle u_{X/S}}\downarrow & & {\scriptstyle \bar{u}_{X/S}}\downarrow \\
(\widetilde{X}_{\rm zar}, f^{-1}(\mathcal{O}_S)) & = & (\widetilde{X}_{\rm zar}, f^{-1}(\mathcal{O}_S)) & = & (\widetilde{X}_{\rm zar}, f^{-1}(\mathcal{O}_S)).
\end{array}$$

Let ${\rm D}^+{\rm F}(\mathcal{O}_{X/S})$, ${\rm D}^+{\rm F}(Q^*_{X/S}(\mathcal{O}_{X/S}))$ and ${\rm D}^+{\rm F}(f^{-1}(\mathcal{O}_S))$ be the derived categories of bounded below filtered complexes of $\mathcal{O}_{X/S}$-modules, $Q^*_{X/S}(\mathcal{O}_{X/S})$-modules and $f^{-1}(\mathcal{O}_S)$-modules, respectively. In [**72**] we have defined two filtered complexes defined by the formula

$$\big(E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P\big) := \big(R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}), \tau\big) \in {\rm D}^+{\rm F}(\mathcal{O}_{X/S}),$$
$$\big(E_{\rm zar}(\mathcal{O}_{(X,D)/S}), P\big) := Ru_{X/S*}\big((E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P)\big) \in {\rm D}^+{\rm F}\big(f^{-1}(\mathcal{O}_S)\big).$$

In [**72**] we have called $(E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P)$ and $(E_{\rm zar}(\mathcal{O}_{(X,D)/S}), P)$ the preweight-filtered vanishing cycle crystalline complex of $(X,D)/S$ and the preweight-filtered vanishing cycle zariskian complex of $(X,D)/S$, respectively. We have called $Q^*_{X/S}(E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P)$ the (pre)weight-filtered vanishing cycle restricted crystalline complex of $(X,D)/S$.

Let $\Delta := \{D_\lambda\}_{\lambda\in\Lambda}$ be a decomposition of D by smooth components of D, where Λ is a set: each D_λ is smooth over S_0 and $D = \sum_{\lambda\in\Lambda} D_\lambda$ in the monoid of effective Cartier divisors on X/S_0. Set $D^{(0)} := X$ and

$$D^{(k)} := \coprod_{\{\lambda_1,\ldots,\lambda_k \mid \lambda_i\neq\lambda_j \ (i\neq j)\}} D_{\lambda_1}\cap\cdots\cap D_{\lambda_k}$$

for a positive integer k; $D^{(k)}$ is independent of the choice of the decomposition of D by smooth components of D (see [**72**]). Let $a^{(k)}\colon D^{(k)}\to X$ be the natural morphism of schemes over S_0. In [**72**], by using another filtered complex $(C_{\rm crys}(\mathcal{O}_{(X,D)/S}),P)\in {\rm D}^+{\rm F}(Q^*_{X/S}(\mathcal{O}_{X/S}))$ which is isomorphic to $Q^*_{X/S}(E_{\rm crys}(\mathcal{O}_{(X,D)/S}),P)$ (we recall $(C_{\rm crys}(\mathcal{O}_{(X,D)/S}),P)$ in (3.3), 1) below), we have proved the following p-adic purity (cf. (1.0.6)):

$$\begin{aligned}(1.0.9)\qquad & Q^*_{X/S}{\rm gr}^P_k E_{\rm crys}(\mathcal{O}_{(X,D)/S})\\ &= Q^*_{X/S}R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})\{-k\}\\ &\overset{\sim}{\longrightarrow} Q^*_{X/S}a^{(k)}_{\rm crys*}\big(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\rm crys}(D/S)\big)\{-k\},\end{aligned}$$

where $\varpi^{(k)}_{\rm crys}(D/S)$ is the crystalline orientation sheaf of $D^{(k)}$ defined in [**72**], which is isomorphic to $\mathbb{Z}_{D^{(k)}}$ on $(D^{(k)}/S)_{\rm crys}$ non-canonically and $\{-k\}$ is the shift of a complex defined in the Convention 1) below. By applying the derived functor $R\bar{u}_{X/S*}$ to (1.0.9) and by considering the action of the relative Frobenius when S_0 is of characteristic p, we have the isomorphism

$$\begin{aligned}(1.0.10)\qquad & {\rm gr}^P_k E_{\rm zar}(\mathcal{O}_{(X,D)/S})\\ &\overset{\sim}{\longrightarrow} a^{(k)}_{\rm zar*}\big(Ru_{D^{(k)}/S*}(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\rm crys}(D/S))(-k)\big)\{-k\},\end{aligned}$$

where $(-k)$ means the Tate twist. Set

$$\begin{aligned}f_{(X,D)/S} := f\circ u_{(X,D)/S}\colon &\big((\widetilde{(X,D)}/S)^{\log}_{\rm crys},\mathcal{O}_{(X,D)/S}\big)\\ &\longrightarrow (\widetilde{X}_{\rm zar},f^{-1}(\mathcal{O}_S))\longrightarrow(\widetilde{S}_{\rm zar},\mathcal{O}_S).\end{aligned}$$

Because $E_{\rm zar}(\mathcal{O}_{(X,D)/S}) = Ru_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$, the filtered complex $(E_{\rm zar}(\mathcal{O}_{(X,D)/S}),P)$ produces an increasing filtration on the log crystalline cohomology $R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ $(h\in\mathbb{N})$. By (1.0.10) we obtain the spectral sequence

$$\begin{aligned}(1.0.11)\qquad E_1^{-k,h+k} = R^{h-k}f_{D^{(k)}/S*}\big(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\rm crys}(D/S)\big)(-k)&\\ \Longrightarrow R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}),&\end{aligned}$$

which we call the *preweight spectral sequence* of $(X,D)/S$ (cf. (1.0.7)).

Let κ be a perfect field of characteristic $p>0$ and let $\mathcal{W}_n$ be the Witt ring of κ of length $n>0$. In the case where $S_0={\rm Spec}(\kappa)$ and $S={\rm Spec}(\mathcal{W}_n)$, Mokrane has constructed the preweight-filtered log de Rham-Witt complex $(\mathcal{W}_n\Omega^\bullet_X(\log D), P) := (\mathcal{W}_n\Omega^\bullet_X(\log D), \{P_k\mathcal{W}_n\Omega^\bullet_X(\log D)\}_{k\in\mathbb{Z}})$ and has proved

$$\mathrm{gr}_k^P\mathcal{W}_n\Omega^\bullet_X(\log D) = \mathcal{W}_n\Omega^{\bullet-k}_{D^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D/\kappa)(-k) \quad (k \in \mathbb{Z})$$

in [**65**] (see also [**69**]), where $\varpi^{(k)}_{\mathrm{zar}}(D/\kappa)$ is the zariskian orientation sheaf of $D^{(k)}$, which is isomorphic to $\mathbb{Z}_{D^{(k)}}$ on $D^{(k)}$ non-canonically. Consequently he has constructed the spectral sequence in [*loc. cit.*]:

$$E_1^{-k,h+k} = H^{h-k}\big((\widetilde{D^{(k)}/\mathcal{W}_n})_{\mathrm{crys}}, \mathcal{O}_{D^{(k)}/\mathcal{W}_n} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D/\mathcal{W}_n)\big)(-k) \Longrightarrow H^h_{\text{log-crys}}((X,D)/\mathcal{W}_n). \tag{1.0.12}$$

In [**72**] we have proved that there exists a canonical filtered isomorphism

$$\big(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P\big) \overset{\sim}{\longrightarrow} \big(\mathcal{W}_n\Omega^\bullet_X(\log D), P\big) \tag{1.0.13}$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$. As a corollary, it turns out that the spectral sequence (1.0.11) is a generalization of the spectral sequence (1.0.12).

Let (X,D) be as above or an analogous log scheme over $\mathbb{C}$. Then we have the translation of Table 1 (see next page).

Here $(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}$ (resp. $((\widetilde{X_{\mathrm{an}}, D_{\mathrm{an}}}))^{\log}_{\text{ét}}$) is the real blow up (resp. the log étale topos) of $(X_{\mathrm{an}}, D_{\mathrm{an}})$ defined in [**53**] (resp. [**48**]), $\widetilde{X_{\mathrm{an}}}$ is the topos defined by the local isomorphisms to X_{an}, ϵ_{top} is the natural morphism of topological spaces which is denoted by τ in [**53**], and ϵ_{an} is the natural morphism forgetting the log structure.

In [**18**] Chiarellotto and Le Stum have constructed the weight filtration on the rigid cohomology of an open smooth variety over κ which is the complement of an SNCD on a proper smooth scheme over κ which is embedded into a formally smooth scheme over $\mathcal{V}$. In [**18**] they have used local rigid cohomologies in order to obtain their weight filtration. The method in [**72**] is different from theirs.

The purposes of this book are the following:

1) To construct the theory of the weight filtration on the log crystalline cohomology of a family of simplicial open smooth varieties in characteristic $p > 0$.

2) To construct the theory of the slope filtration on the log crystalline cohomology with the weight filtration of a split simplicial open smooth variety in characteristic $p > 0$.

3) To define and study the weight filtration on the rigid cohomology of a separated scheme of finite type over a perfect field κ of characteristic p.

(1.0.14)

/$\mathbb{C}$	crystal
$U_{\rm an}$, $(X_{\rm an},D_{\rm an})^{\log}$ (see [**53**]) $(\widetilde{(X_{\rm an},D_{\rm an})})^{\log}_{\text{ét}}$ (see [**48**])	$(\widetilde{(X,D)/S})^{\log}_{\rm crys}$
$X_{\rm an}$, $\widetilde{X_{\rm an}}$	$\widetilde{(X/S)}_{\rm crys}$
$j_{\rm an}\colon U_{\rm an}\hookrightarrow X_{\rm an}$, $\epsilon_{\rm top}\colon (X_{\rm an},D_{\rm an})^{\log}\to X_{\rm an}$, $\epsilon_{\rm an}\colon (\widetilde{(X_{\rm an},D_{\rm an})})^{\log}_{\text{ét}}\to \widetilde{X_{\rm an}}$	$\epsilon_{(X,D)/S}\colon (\widetilde{(X,D)/S})^{\log}_{\rm crys}\to \widetilde{(X/S)}_{\rm crys}$
$Rj_{{\rm an}*}(\mathbb{Z})=R\epsilon_{{\rm top}*}(\mathbb{Z})$ (see [**53**]), $R\epsilon_{{\rm top}*}(\mathbb{Z}/n)=R\epsilon_{{\rm an}*}(\mathbb{Z}/n)$ $(n\in\mathbb{Z})$ (see [**68**])	$Q^*_{X/S}R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$
$X_{\rm an}\to X$	$u_{X/S}\colon \widetilde{(X/S)}_{\rm crys}\to \widetilde{X}_{\rm zar}$
$\mathbb{Z}_{(X_{\rm an},D_{\rm an})^{\log}}$ $(\mathbb{Z}/n)_{(X_{\rm an},D_{\rm an})^{\log}}$, $(\mathbb{Z}/n)_{(\widetilde{(X_{\rm an},D_{\rm an})})^{\log}_{\text{et}}}$ $(n\in\mathbb{Z})$	$\mathcal{O}_{(X,D)/S}$
$\mathbb{Z}_{X_{\rm an}}$ $(\mathbb{Z}/n)_{X_{\rm an}}$ $(n\in\mathbb{Z})$	$\mathcal{O}_{X/S}$
$(R\epsilon_{{\rm an}*}(\mathbb{Z}/n),\tau)$ $(n\in\mathbb{Z})$	$(E_{\rm crys}(\mathcal{O}_{(X,D)/S}),P)$
$(\Omega^{\bullet}_{X/\mathbb{C}}(\log D),P)$	$(E_{\rm zar}(\mathcal{O}_{(X,D)/S}),P)$

Table 1

4) To calculate the slope filtration on the rigid cohomology above.

1) is a straight generalization of [**72**]. Let N be a nonnegative integer. The key point for 2) is a comparison theorem between the split N-truncated cosimplicial preweight-filtered vanishing cycle zariskian complex and the split N-truncated cosimplicial preweight-filtered log de Rham-Witt complex (see (1.0.16) below). The existence of the weight filtration in 3) follows from 1), de Jong's alteration theorem (see [**50**]), Tsuzuki's proper cohomological descent in rigid cohomology (see [**86**]) and a generalization of Shiho's comparison theorem (see [**82**]) between the rigid cohomology of the scheme in 3) and the log crystalline cohomology of a certain proper hypercovering of the scheme (see (1.0.17) below). 4) is an immediate consequence of 2) and the generalization of Shiho's comparison theorem.

We outline Part I of this book, which treats 1) and 2) above.

Let $f\colon X_\bullet \to S_0$ be a smooth simplicial scheme and let $D_\bullet$ be a simplicial relative SNCD on $X_\bullet/S_0$; by abuse of notation, f also denotes the composite structural morphism $X_\bullet \to S_0 \hookrightarrow S$. Let

$$\epsilon_{(X_\bullet,D_\bullet)/S}\colon \big((\widetilde{X_\bullet, D_\bullet})/S\big)^{\log}_{\rm crys} \longrightarrow (\widetilde{X_\bullet/S})_{\rm crys}$$

be the natural morphism of topoi forgetting the log structure. Let $u_{X_\bullet/S}\colon (\widetilde{X_\bullet/S})_{\rm crys} \to \widetilde{X}_{\bullet\rm zar}$ be the natural projection. Then we have the two filtered complexes

$$\big(E_{\rm crys}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}), P\big) := \big(R\epsilon_{(X_\bullet,D_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}), \tau\big) \in {\rm D}^+{\rm F}(\mathcal{O}_{X_\bullet/S}),$$
$$\big(E_{\rm zar}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}), P\big) := Ru_{X_\bullet/S*}\big((E_{\rm crys}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}), P)\big) \in {\rm D}^+{\rm F}(f^{-1}(\mathcal{O}_S)).$$

By (1.0.10) for each $(E_{\rm zar}(\mathcal{O}_{(X_t,D_t)/S}), P)$ $(t \in \mathbb{N})$ and by using the diagonal filtration in [**25**] (see also §2 below), we obtain the spectral sequence:

$$E_1^{-k,h+k} = \bigoplus_{t\geq 0} R^{h-2t-k} f_{D_t^{(t+k)}/S*}\big(\mathcal{O}_{D_t^{(t+k)}/S} \otimes_{\mathbb{Z}} \varpi^{(t+k)}_{\rm crys}(D_t/S)\big)(-t-k)$$
$$\Longrightarrow R^h f_{(X_\bullet,D_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}), \tag{1.0.15}$$

which we call the *preweight spectral sequence* of $(X_\bullet, D_\bullet)/S$. The spectral sequence (1.0.15) is a generalization of (1.0.11).

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field κ of characteristic $p > 0$ and let K be the fraction field of $\mathcal{V}$. For a p-adic formal $\mathcal{V}$-scheme S in the sense of [**74**] and $S_0 := \underline{\rm Spec}_S(\mathcal{O}_S/p)$, we obtain (1.0.9), (1.0.10), (1.0.11) and (1.0.15). In this case, we call (1.0.15) the *p-adic weight spectral sequence* of $(X_\bullet, D_\bullet)/S$ when $X_\bullet$ is proper over S. This spectral sequence is the p-adic analogue of the weight spectral sequence (1.0.8).

Using techniques developed in [**72**], we prove the fundamental properties:

1) the convergence (in the sense of [**74**]) of the weight filtration on

$$R^h f_{(X_\bullet,D_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet)/S})_K := R^h f_{(X_\bullet,D_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}) \otimes_{\mathcal{V}} K,$$

2) the E_2-degeneration of the p-adic weight spectral sequence of $(X_\bullet, D_\bullet)/S$ modulo torsion and

3) the strict compatibility of the induced morphism of a morphism of proper smooth simplicial schemes with simplicial relative SNCD's with respect to the weight filtration on $R^h f_{(X_\bullet,D_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet)/S})_K$.

These are straight generalizations of results in [**72**].

We conclude the explanation for Part I by stating one more another result.

Let N be a nonnegative integer. In the case where S_0 is the spectrum of a perfect field κ of characteristic $p > 0$ and $S = \mathrm{Spec}(\mathcal{W}_n)$ $(n > 0)$, we can generalize (1.0.13) as follows: if $(X_{\bullet \leq N}, D_{\bullet \leq N})$ is split, then there exists a canonical filtered isomorphism

$$(1.0.16) \qquad \big(E_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N})/S}), P\big) \xrightarrow{\sim} \big(\mathcal{W}_n\Omega^{\bullet}_{X_{\bullet \leq N}}(\log D_{\bullet \leq N}), P\big)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}_{\bullet \leq N}(\mathcal{W}_n))$, where $f_{\bullet \leq N}\colon X_{\bullet \leq N} \to \mathrm{Spec}(\mathcal{W}_n)$ is the structural morphism. Though to prove that there exists the isomorphism

$$E_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N})/S}) \xrightarrow{\sim} \mathcal{W}_n\Omega^{\bullet}_{X_{\bullet \leq N}}(\log D_{\bullet \leq N})$$

is not difficult (see [**72**, (3.5)]) if one uses Tsuzuki's functor Γ in [**19**] and [**86**] (we recall Γ in (6.4) below), the proof of (1.0.16) is involved. The point of the proof of (1.0.16) is to construct the morphism (1.0.16). To construct it, we need the explicit descriptions of $(E_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N})/S}), P)$ and $(\mathcal{W}_n\Omega^{\bullet}_{X_{\bullet \leq N}}(\log D_{\bullet \leq N}), P)$, which are generalizations of the explicit descriptions of $(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ and $(\mathcal{W}_n\Omega^{\bullet}_X(\log D), P)$. (Even in the N-truncated constant simplicial case, the descriptions in this book are generalizations of the descriptions in [**72**]: in [*loc. cit.*] we have defined and used the admissible immersion for the explicit descriptions of $(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ and $(\mathcal{W}_n\Omega^{\bullet}_X(\log D), P)$; in this book we do not use it for the explicit descriptions.)

Next, we outline Part II of this book.

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field κ of characteristic $p > 0$ and let K be the fraction field of $\mathcal{V}$. Let $\mathcal{W}$ be the Witt ring of κ and K_0 the fraction field of $\mathcal{W}$. Let U be a separated scheme of finite type over κ. Let $j\colon U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Then, using a general formalism in [**35**, $\mathrm{V}^{\mathrm{bis}}$], we can construct a split proper hypercovering $(U_\bullet, X_\bullet)$ of $(U, \overline{U})$ in the sense of [**86**], that is, $(U_\bullet, X_\bullet)$ is split, $U_\bullet$ is a proper hypercovering of U, $X_\bullet$ is a proper simplicial scheme over $\overline{U}$ and $U_\bullet = X_\bullet \times_{\overline{U}} U$. Moreover, using de Jong's alteration theorem (see [**50**]), we can require that $X_\bullet$ is a proper smooth simplicial scheme over κ and that $U_\bullet$ is the complement of a simplicial SNCD $D_\bullet$ on $X_\bullet$ over κ. We call such a split proper hypercovering $(U_\bullet, X_\bullet)$ of $(U, \overline{U})$ a *gs* (= *good and split*) *proper hypercovering* of $(U, \overline{U})$. We prove

$$(1.0.17) \qquad R\Gamma_{\mathrm{rig}}(U/K) = R\Gamma\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_K$$

by using

1) Berthelot's base change theorem in rigid cohomology (the reduction to the case $\mathcal{V} = \mathcal{W}$) (see [**9**]),

2) a technique of the construction of a proper hypercovering in [**86**],

3) Tsuzuki's proper cohomological descent in rigid cohomology for a proper hypercovering of a triple (see [**86**]) and

4) Shiho's comparison theorem (see [**82**]):

$$H^h_{\mathrm{rig}}(U_t/K_0) = H^h\big((X_t, D_t)/\mathcal{W}\big)_{K_0} \quad (h, t \in \mathbb{N}).$$

Especially, the right hand side of (1.0.17) depends only on U and K; this solves a problem raised in [**50**, Introduction] for the split case in a stronger form: not only the log crystalline cohomology on the right hand side of (1.0.17) but also the complex itself on the right hand side depends only on U and K. (In [**1**], in a different method from ours, Andreatta and Barbieri-Viale have proved that $H^1((X_\bullet, D_\bullet)/\mathcal{W})$ depends only on U/κ for the case where $p \geq 3$ and where the augmentation morphism $X_0 \setminus D_0 \to U$ is generically etale even for the non-split case.) We also come to know that the log crystalline cohomology theory is a good cohomology theory for nonproper and nonsmooth schemes by considering the larger category of simplicial log schemes which appear as (split) proper hypercoverings of schemes. On the other hand, (1.0.17) tells us that the rigid cohomology of the trivial coefficient is interpreted by the cohomology of an abelian sheaf in a (non-canonical) topos (see [**63**] for another interpretation of the rigid cohomology by the cohomology of an abelian sheaf in a topos).

(1.0.17) gives us some deep results; for example, one can directly derive the finiteness theorem of Berthelot–Große-Klönne (see [**9**], [**32**], [**86**]; see also [**55**] for the generalization to the case of the overconvergent F-isocrystal) without using any result of the finiteness of rigid cohomology since we have the spectral sequence

$$E_1^{ij} = H^j\big((X_i, D_i)/\mathcal{W}\big)_K \Longrightarrow H^{i+j}_{\mathrm{rig}}(U/K).$$

We can also derive a generalization of Berthelot's Künneth formula without support (see [**8**]) from the Künneth formula of log crystalline cohomology (see [**51**]). More importantly, by results of Part I and (1.0.17), we can define a weight filtration on $H^h_{\mathrm{rig}}(U/K)$ which must be independent of the choice of the gs proper hypercovering $(U_\bullet, X_\bullet)$ of $(U, \overline{U})$. We prove this independence by using Grothendieck's idea for the reduction of geometric problems to arithmetic problems (see [**40**], [**33**]) in the following way:

1) For two gs proper hypercoverings $(U^1_\bullet, X^1_\bullet)$ and $(U^2_\bullet, X^2_\bullet)$ of $(U, \overline{U})$, there exists another gs proper hypercovering $(U^3_\bullet, X^3_\bullet)$ of $(U, \overline{U})$ fitting into the commutative diagram

$$\begin{array}{ccc} (U^3_\bullet, X^3_\bullet) & \longrightarrow & (U^1_\bullet, X^1_\bullet) \\ \big\downarrow & & \big\downarrow \\ (U^2_\bullet, X^2_\bullet) & \longrightarrow & (U, \overline{U}) \end{array}$$

(a general formalism in [**35**, $\mathrm{V}^{\mathrm{bis}}$]).

2) The existence of a model of the split N-truncated simplicial family $(X_{\bullet \leq N}, D_{\bullet \leq N})$ $(N \in \mathbb{N})$ over the spectrum of a smooth ring of finite type over a finite field (standard log deformation theory).

3) The reduction to the case of finite fields (the specialization argument of Deligne-Illusie, see [**46**], [**69**], [**72**]).

4) The purity of the eigenvalues of the Frobenius on the classical crystalline cohomology of a proper smooth scheme over a finite field (see [**54**], [**16**], [**69**]).

As a conclusion, we capture remarkable subspaces of $H^h_{\mathrm{rig}}(U/K)$ and, from a different point of view, we can say that we capture subspaces of $H^h((X_\bullet, D_\bullet)/\mathcal{W})_{K_0}$ which depend only on U and K_0.

Summing up, our method is a hard p-adic version of [**25**] except some techniques. Indeed, we give a comparison theorem between different cohomology theories in (1.0.17), while Deligne has worked in the same cohomology theory.

Once one obtains the isomorphisms

$$(1.0.18) \quad H^h_{\mathrm{rig}}(U/K) \xrightarrow{\sim} H^h\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_K \xleftarrow{\sim} H^h\big(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)\big)_K$$

obtained from (1.0.16) and (1.0.17), one may think that one can use the log de Rham-Witt complex $\mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)$ of $(X_\bullet, D_\bullet)/\kappa$ (cf. [**65**], [**69**], [**72**]) for the construction of the weight filtration on $H^h_{\mathrm{rig}}(U/K)$. However, to prove that the weight filtration is independent of the choice of $(X_\bullet, D_\bullet)$, the method using the log de Rham-Witt complex is not enough; we need $(E_{\mathrm{zar}}(\mathcal{O}_{(X_\bullet, D_\bullet)/S}), P)$ and some techniques developed in [**72**].

We prove some fundamental properties of the weight filtration on $H^h_{\mathrm{rig}}(U/K)$, especially, the determination of the possible range of the weights of $H^h_{\mathrm{rig}}(U/K)$ and the strict compatibility of the induced morphism of rigid cohomologies by a morphism of schemes.

Using (1.0.18) and using the slope decomposition

$$H^h\big(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)\big)_{K_0} = \bigoplus_{i+j=h} H^j\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0}$$

(this is an easy generalization of the slope decomposition in [**47**]), we can immediately calculate the slope filtration on $H^h_{\mathrm{rig}}(U/K_0)$ by

$$H^{h-i}\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0} \quad (i \in \mathbb{N});$$

conversely we can show that $H^j(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))_{K_0}$ depends only on U/κ.

If one ignores the weight filtration, one can generalize (1.0.17) to certain coefficients and to certain gs proper hypercoverings as follows.

Let E be an overconvergent F-isocrystal on $(U, \overline{U})/K$. Shiho's conjecture in [**82**] says that E extends to an F-isocrystal with logarithmic singularities if one makes a suitable alteration and pulls back E to the alteration. Assuming that Shiho's conjecture holds and using a standard argument in [**25**], we see that there exist a gs proper hypercovering $\pi\colon (U_\bullet, X_\bullet) \to (U, \overline{U})$ and an F-isocrystal $E^\bullet_{\mathrm{conv}}$ in $\mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}((X_\bullet, D_\bullet)/\mathcal{V})$ such that $\pi^*(E) = j^\dagger(E^\bullet_{\mathrm{conv}})$, where

$$j^\dagger\colon \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}\big((X_\bullet, D_\bullet)/\mathcal{V}\big) \longrightarrow \mathrm{Isoc}^\dagger\big((U_\bullet, X_\bullet)/K\big)$$

is the simplicial version of the functor defined in [**82**, Proposition 2.4.1]. (In the constant simplicial case $(U_n, X_n) = (U_0, X_0)$ $(\forall n \in \mathbb{N})$ with trivial degeneracy morphisms and trivial face morphisms, this is nothing but Shiho's conjecture.) Let

$$\Xi\colon \mathrm{Isoc}_{\mathrm{conv}}\big((X_\bullet, D_\bullet)/\mathcal{W}\big) \longrightarrow \mathrm{Isoc}_{\mathrm{crys}}\big((X_\bullet, D_\bullet)/\mathcal{W}\big)$$

be the simplicial version of the functor defined in [**81**, Theorem 5.3.1] and denoted by Φ in [*loc. cit.*]. Assume that $\mathcal{V} = \mathcal{W}$. Then, by the same proof as that of (1.0.17), we can prove that

$$R\Gamma_{\mathrm{rig}}(U/K, E) = R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi(E^\bullet_{\mathrm{conv}})\big). \tag{1.0.19}$$

Recently Kedlaya has proved Shiho's conjecture (see [**56**], [**57**], [**58**], [**59**, (2.4.4)]).

Finally, we outline Part III.

In Part III, by using (1.0.17), we define the weight filtration on the cohomology of the mapping fiber and the mapping cone of the induced morphism of the complexes of rigid cohomologies by a morphism of separated schemes of finite type over κ. We also calculate the slope filtration on the cohomology of the mapping fiber and the mapping cone.

Let Z be a closed subscheme of U/κ. As an example of the theory of the mapping fiber, we interpret the rigid cohomology $H^h_{\mathrm{rig},Z}(U/K)$ by the cohomology of the mapping fiber of a morphism of the complexes of cosimplicial log crystalline cohomologies, and we endow $H^h_{\mathrm{rig},Z}(U/K)$ with the weight filtration and calculate the slope filtration on it. As a corollary of the existence of the weight filtration on $H^h_{\mathrm{rig},Z}(U/K)$ and as a consequence of a geometric interpretation of the slope filtration on $H^h_{\mathrm{rig},Z}(U/K)$, we prove the variants of Serre's and Grothendieck's conjectures about the existence of the desired functions.

We determine the possible range of the weights (resp. slopes) of $H^h_{\mathrm{rig},Z}(U/K)$ (resp. $H^h_{\mathrm{rig},Z}(U/K_0)$); the determination of the range of the weights (resp. slopes) is a generalization of the determination in [**15**] (resp. [**17**]).

It is known that, by using Tsuzuki's proper descent in rigid cohomology, Kedlaya's Künneth formula for overconvergent F-isocrystals on smooth schemes over κ (see [**55**]) can be generalized to the case of overconvergent F-isocrystals on separated schemes of finite type over κ (see [*loc. cit.*]) (Kedlaya's Künneth formula is a generalization of Berthelot's Künneth formula, see [**8**]). In the case of the trivial coefficient, we prove that the generalized Künneth formula is compatible with the weight filtration.

In the last section, by using Berthelot's Poincaré duality (see [**8**]), we first define the weight filtration on $H^h_{\mathrm{rig,c}}(U/K)$ for a separated smooth scheme U of finite type over κ. Moreover, using Shiho's recent result for a relative version of Shiho's comparison theorem (see [**80**]), we endow $H^h_{\mathrm{rig,c}}(Z/K)$ with a well-defined weight filtration for a separated scheme Z of finite type over κ which can be embedded into a smooth scheme over κ as a closed subscheme. In the case where Z is a smooth closed subscheme of a separated smooth scheme U over κ, we prove that the Gysin isomorphism is an isomorphism of weight-filtered vector spaces. We also prove that the Künneth isomorphism for rigid cohomology with compact support is compatible with the weight filtration. As a corollary of the existence of the weight filtration on the rigid cohomology with compact support and as a consequence of a geometric interpretation of the slope filtration on it, we prove Serre's and Grothendieck's conjectures about the existence of the functions h^{ij}_{c} and h^r_{c} $(i,j,r\in\mathbb{N})$ for separated schemes of finite type over κ which can be embedded into smooth schemes over κ.

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numerous (English and French) mistakes, a big gap in the proof of the comparison theorem (1.0.16) and two mistakes in the proofs of (10.1) and (10.5) below.

Notations

1) For a log scheme Y, $\overset{\circ}{Y}$ (resp. M_Y) denotes the underlying scheme (resp. the log structure) of Y. For a morphism $g\colon Y \longrightarrow Y'$ of log schemes, $\overset{\circ}{g}\colon \overset{\circ}{Y} \to \overset{\circ}{Y}{}'$ denotes the underlying morphism of g.

2) (S)NCD = (simple) normal crossing divisor.

3) Let $(\mathcal{T}, \mathcal{A})$ be a ringed topos.

(a) $\mathrm{C}(\mathcal{A})$ (resp. $\mathrm{C}^{\pm}(\mathcal{A})$, $\mathrm{C}^{\mathrm{b}}(\mathcal{A})$): the category of (resp. bounded below, bounded above, bounded) complexes of $\mathcal{A}$-modules.

(b) $\mathrm{K}(\mathcal{A})$ (resp. $\mathrm{K}^{\pm}(\mathcal{A})$, $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$): the category of (resp. bounded below, bounded above, bounded) complexes of $\mathcal{A}$-modules modulo homotopy.

(c) $\mathrm{D}(\mathcal{A})$ (resp. $\mathrm{D}^{\pm}(\mathcal{A})$, $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$): the derived category of (resp. bounded below, bounded above, bounded) complexes of $\mathcal{A}$-modules. For an object $E^\bullet$ of $\mathrm{C}(\mathcal{A})$ (resp. $\mathrm{C}^{\pm}(\mathcal{A})$, $\mathrm{C}^{\mathrm{b}}(\mathcal{A})$), we denote simply by $E^\bullet$ the corresponding object to $E^\bullet$ in $\mathrm{D}(\mathcal{A})$ (resp. $\mathrm{D}^{\pm}(\mathcal{A})$, $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$).

(d) The additional notation F to the categories above means "filtered ". Here the filtration is an increasing filtration indexed by $\mathbb{Z}$ or a decreasing filtration indexed by $\mathbb{Z}$ which is determined in context. For example, $\mathrm{K}^{+}\mathrm{F}(\mathcal{A})$ is the category of bounded below filtered complexes modulo filtered homotopy. As in [**4**] (and [**72**]), the filtration in this book is not necessarily exhaustive nor separated.

(e) $\mathrm{DF}_2(\mathcal{A})$ (resp. $\mathrm{D}^{\pm}\mathrm{F}_2(\mathcal{A})$, $\mathrm{D}^{\mathrm{b}}\mathrm{F}_2(\mathcal{A})$): the derived category of (resp. bounded below, bounded above, bounded) biregular bifiltered complexes of $\mathcal{A}$-modules (see [**24**, (1.3.1), (1.3.6)], [**25**, (7.1.1)]).

Conventions. — We make the following conventions about signs (cf. [**11**], [**20**]). Let $\mathcal{A}$ be an exact additive category.

1) For a complex $(E^\bullet, d^\bullet)$ of objects of $\mathcal{A}$ and for an integer n, $(E^{\bullet+n}, d^{\bullet+n})$ or $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the complex:

$$\cdots \longrightarrow \underset{q-1}{E^{q-1+n}} \xrightarrow{d^{q-1+n}} \underset{q}{E^{q+n}} \xrightarrow{d^{q+n}} \underset{q+1}{E^{q+1+n}} \xrightarrow{d^{q+1+n}} \cdots .$$

Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

For a morphism $f\colon (E^\bullet, d_E^\bullet) \to (F^\bullet, d_F^\bullet)$ of complexes of objects of $\mathcal{A}$, $f\{n\}$ denotes the natural morphism $(E^\bullet\{n\}, d_E^\bullet\{n\}) \to (F^\bullet\{n\}, d_F^\bullet\{n\})$ induced by f. For a morphism $f\colon (E^\bullet, d_E^\bullet) \to (F^\bullet, d_F^\bullet)$ in the derived category $\mathrm{D}^\star(\mathcal{A})$ ($\star = \mathrm{b}, +, -, \text{nothing}$) of the complexes of objects of $\mathcal{A}$, there exists a naturally induced morphism in $\mathrm{D}^\star(\mathcal{A})$:

$$f\{n\}\colon \big(E^\bullet\{n\}, d_E^\bullet\{n\}\big) \longrightarrow \big(F^\bullet\{n\}, d_F^\bullet\{n\}\big).$$

2) For a complex $(E^\bullet, d^\bullet)$ of objects of $\mathcal{A}$ and for an integer n, $(E^\bullet[n], d^\bullet[n])$ denotes the complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphism $d^\bullet[n] = (-1)^n d^{\bullet+n}$.

For a morphism $f\colon (E^\bullet, d_E^\bullet) \to (F^\bullet, d_F^\bullet)$ of complexes of objects of $\mathcal{A}$, $f[n]$ denotes the natural morphism $(E^\bullet[n], d_E^\bullet[n]) \to (F^\bullet[n], d_F^\bullet[n])$ induced by f without change of signs. This operation is well-defined in the derived category as in 1).

3) (see [**11**, 0.3.2], [**20**, (1.3.2)]) For a short exact sequence

$$0 \to (E^\bullet, d_E^\bullet) \xrightarrow{f} (F^\bullet, d_F^\bullet) \xrightarrow{g} (G^\bullet, d_G^\bullet) \to 0$$

of bounded below complexes of objects of $\mathcal{A}$, let

$$\mathrm{MC}(f) := \big(E^\bullet[1], d_E^\bullet[1]\big) \oplus (F^\bullet, d_F^\bullet)$$

be the mapping cone of f. We fix an isomorphism

$$\text{“}(E^\bullet[1], d_E^\bullet[1]) \oplus (F^\bullet, d_F^\bullet) \ni (x, y) \longmapsto g(y) \in (G^\bullet, d_G^\bullet)\text{”}$$

in the derived category $\mathrm{D}^+(\mathcal{A})$.

Let $\mathrm{MF}(g) := (F^\bullet, d_F^\bullet) \oplus (G^\bullet[-1], d_G^\bullet[-1])$ be the mapping fiber of g. We fix an isomorphism

$$\text{“}\big(E^\bullet, d_E^\bullet\big) \ni x \longmapsto \big(f(x), 0\big) \in (F^\bullet, d_F^\bullet) \oplus (G^\bullet[-1], d_G^\bullet[-1])\text{”}$$

in the derived category $\mathrm{D}^+(\mathcal{A})$.

4) (see [**11**, 0.3.2], [**20**, (1.3.3)]) Under the situation 3), the boundary morphism $(G^\bullet, d_G^\bullet) \to (E^\bullet[1], d_E^\bullet[1])$ in $\mathrm{D}^+(\mathcal{A})$ is the following composite morphism

$$(G^\bullet, d_G^\bullet) \overset{\sim}{\leftarrow} \big(E^\bullet[1], d_E^\bullet[1]\big) \oplus (F^\bullet, d_F^\bullet) \xrightarrow{\text{proj}} \big(E^\bullet[1], d_E^\bullet[1]\big) \xrightarrow{(-1)\times} \big(E^\bullet[1], d_E^\bullet[1]\big).$$

5) Assume that $\mathcal{A}$ is an abelian category with enough injectives. Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Then, under the situation 3), the boundary morphism $\partial\colon R^q\mathcal{F}((G^\bullet, d_G^\bullet)) \to R^{q+1}\mathcal{F}((E^\bullet, d_E^\bullet))$ of cohomologies is, by definition, the induced morphism by the morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in 4). By taking injective resolutions $(I^\bullet, d_I^\bullet)$, $(J^\bullet, d_J^\bullet)$ and $(K^\bullet, d_K^\bullet)$ of $(E^\bullet, d_E^\bullet)$, $(F^\bullet, d_F^\bullet)$ and $(G^\bullet, d_G^\bullet)$, respectively, which fit into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I^\bullet, d_I^\bullet) & \longrightarrow & (J^\bullet, d_J^\bullet) & \longrightarrow & (K^\bullet, d_K^\bullet) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (E^\bullet, d_E^\bullet) & \longrightarrow & (F^\bullet, d_F^\bullet) & \longrightarrow & (G^\bullet, d_G^\bullet) & \longrightarrow & 0 \end{array} \tag{1.0.20}$$

of complexes of objects in $\mathcal{A}$, it is easy to check that the boundary morphism ∂ above is equal to the usual boundary morphism obtained by the upper short exact sequence of (1.0.20). (For a short exact sequence in 3), the existence of the commutative diagram (1.0.20) has been proved in, *e.g.*, [**72**, (1.1.7)] as a very special case.)

6) For a complex $(E^\bullet, d^\bullet)$ of objects of $\mathcal{A}$, the identity id: $E^q \to E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $\mathcal{H}^q((E^\bullet, -d^\bullet)) \xrightarrow{\sim} \mathcal{H}^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies.

7) We often denote a complex $(E^\bullet, d^\bullet)$ simply by $(E^\bullet, d)$ or $E^\bullet$ as usual when there is no risk of confusion.

8) Let $r \geq 2$ be a positive integer. As usual, an r-uple complex of objects of $\mathcal{A}$ is, by definition, a pair $(E^{\bullet\cdots\bullet}, \{d_i\}_{i=1}^r)$ such that $E^{m_1\cdots m_r}$ $(m_i \in \mathbb{Z})$ is an object of $\mathcal{A}$ with morphisms

$$d_i\colon E^{\bullet\cdots\bullet, m_i, \bullet\cdots\bullet} \longrightarrow E^{\bullet\cdots\bullet, m_i+1, \bullet\cdots\bullet}$$

satisfying the relations $d_i^2 = 0$ and $d_i d_j + d_j d_i = 0$ $(i \neq j)$.

PART I. WEIGHT FILTRATION ON THE LOG CRYSTALLINE COHOMOLOGY OF A SIMPLICIAL FAMILY OF OPEN SMOOTH VARIETIES IN CHARACTERISTIC $p > 0$

2. Preliminaries on filtered derived categories

Though some facts in this section hold in an abelian category with enough injectives, we are content with the framework of ringed topoi.

Let $(\mathcal{T}, \mathcal{A})$ be a ringed topos. For a positive integer r, let

$$(\mathcal{T}_{t_1\cdots t_r}, \mathcal{A}^{t_1\cdots t_r})_{t_1,\ldots,t_r\in\mathbb{N}}$$

be the constant r-simplicial ringed topos defined by $(\mathcal{T}, \mathcal{A})$:

$$\mathcal{T}_{t_1\cdots t_r} = \mathcal{T}, \quad \mathcal{A}^{t_1\cdots t_r} = \mathcal{A}.$$

Let M be an object of $\mathrm{C}(\mathcal{A}^{\bullet\cdots\bullet})$ (r-points). For simplicity of notation, for nonnegative integers $t_1, \ldots, t_j$ ($1 \leq j \leq r$), set

$$\underline{t}_j := t_1 + \cdots + t_j, \quad \text{and} \quad \underline{t}_0 := 0.$$

We also set $\underline{t} := (t_1, \ldots, t_r)$ and $\underline{\bullet} := \bullet\cdots\bullet$ (r-points). The object M defines an $(r+1)$-uple complex $M^{\underline{\bullet}\bullet} = (M^{t_1\cdots t_r\bullet})_{t_1,\ldots,t_r\in\mathbb{N}}$ of $\mathcal{A}$-modules whose boundary morphisms will be fixed in (2.0.1) below. Let $d_M\colon M^{\underline{t}s} \to M^{\underline{t},s+1}$ be the boundary morphism arising from the boundary morphism of the complex M and let $\delta^i_j\colon M^{t_1\cdots t_j\cdots t_r s} \to M^{t_1\cdots t_{j-1},t_j+1,t_{j+1}\cdots t_r s}$ ($1 \leq j \leq r, 0 \leq i \leq t_j + 1$) be a standard coface morphism. Consider the single complex $\mathbf{s}(M)$ with the following boundary morphism (cf. [**25**, (5.1.9.1), (5.1.9.2)]):

$$\mathbf{s}(M)^n = \bigoplus_{\underline{t}_r+s=n} M^{\underline{t}s} = \bigoplus_{t_1+\cdots+t_r+s=n} M^{t_1\cdots t_r s}, \tag{2.0.1}$$

$$d(x^{\underline{t}s}) = \sum_{i=0}^{t_1+1}(-1)^i\delta_1^i(x^{\underline{t}s}) + (-1)^{t_1}\sum_{i=0}^{t_2+1}(-1)^i\delta_2^i(x^{\underline{t}s}) + \cdots + (-1)^{\underline{t}_{r-1}}\sum_{i=0}^{t_r+1}(-1)^i\delta_r^i(x^{\underline{t}s}) + (-1)^{\underline{t}_r}d_M(x^{\underline{t}s}) \quad (x^{\underline{t}s}\in M^{\underline{t}s}).$$

If $r=1$, our convention on the place of the cosimplicial degrees is the same as that in [**73**, (2.3)] and different from that in [**25**, (5.1.9) (IV)] and [**19**, (3.9)] (for any $r\in\mathbb{Z}_{>0}$); our convention on the signs of the boundary morphisms of $\mathbf{s}(M)$ is better than that in [**25**, (5.1.9.2)] and [**19**, (3.9)]: see (10.15) 1) below for the reason. We also note that the diagram in [**25**, p. 35] is not a part of a double complex; it is mistaken since it is commutative. Furthermore, the "Gysin" in the diagram in [*loc. cit.*] is not clear; in (5.3.2) and (5.3.3) below, we shall give an explicit expression of the p-adic analogue of the "Gysin" as in [**66**, (4.9)] (see also [**69**, (10.3)]).

For a morphism $f\colon M\to N$ in $\mathrm{C}(\mathcal{A}^{\underline{\bullet}}) = \mathrm{C}(\mathcal{A}^{\bullet\cdots\bullet})$, we define $\mathbf{s}(f)\colon \mathbf{s}(M)\to \mathbf{s}(N)$ as the naturally induced morphism by f (without change of signs); $\mathbf{s}$ gives a functor

$$\mathbf{s}\colon \mathrm{C}(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{C}(\mathcal{A}). \tag{2.0.2}$$

Lemma 2.1 (cf. [**35**, $\mathrm{V}^{\mathrm{bis}}$ (2.3.2.2)]). — *The functor* (2.0.2) *induces the following functors*:

$$\mathbf{s}\colon \mathrm{K}^\star(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{K}^\star(\mathcal{A}) \quad (\star = + \text{ or nothing}), \tag{2.1.1}$$

$$\mathbf{s}\colon \mathrm{D}^\star(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{D}^\star(\mathcal{A}). \tag{2.1.2}$$

Proof. — We have only to pay attention to signs.

Let $f\colon (M,d_M)\to(N,d_N)$ be a morphism in $\mathrm{C}(\mathcal{A}^{\underline{\bullet}})$ which is homotopic to zero. Let $H\colon M\to N\{-1\}$ be a homotopy from f to the zero morphism: $Hd_M + d_NH = f$. Let $f^{\underline{\bullet}\bullet}$ (resp. $H^{\underline{\bullet}\bullet}$) be the induced morphism of $(r+1)$-uple complexes by f (resp. H). Then the morphisms

$$(-1)^{\underline{t}_r}H^{t_1\cdots t_rs}\colon M^{t_1\cdots t_rs}\longrightarrow N^{t_1\cdots t_r,s-1} \quad (\underline{t}\in\mathbb{N}^r, s\in\mathbb{Z}) \tag{2.1.3}$$

define a homotopy from $f^{\underline{\bullet}\bullet}$ to the zero. We leave the rest of the proof to the reader. □

We can obtain $\mathbf{s}$ inductively in the following way (cf. [**25**, (8.1.22)]).

Fix $t_1,\dots,t_{r-1}\in\mathbb{N}$. Let

$$\mathbf{s}_{t_1\cdots t_{r-1}}\colon \mathrm{C}(\mathcal{A}^{t_1\cdots t_{r-1}\bullet}) \longrightarrow \mathrm{C}(\mathcal{A}^{t_1\cdots t_{r-1}})$$

be the functor defined in (2.0.1) for a simplicial ringed topos $(\mathcal{T}_{t_1\cdots t_{r-1}\bullet}, \mathcal{A}^{t_1\cdots t_{r-1}\bullet})$. The family $\{\mathbf{s}_{t_1\cdots t_{r-1}}\}_{t_1,\ldots,t_{r-1}\in\mathbb{N}}$ induces the functor

$$\mathbf{s}_r\colon \mathrm{C}(\mathcal{A}^{\bullet_1\cdots\bullet_r}) \longrightarrow \mathrm{C}(\mathcal{A}^{\bullet_1\cdots\bullet_{r-1}}). \tag{2.1.4}$$

Then, as in the same way above, we have the following functor

$$\mathbf{s}_i\colon \mathrm{C}(\mathcal{A}^{\bullet_1\cdots\bullet_i}) \longrightarrow \mathrm{C}(\mathcal{A}^{\bullet_1\cdots\bullet_{i-1}}) \quad (1 \le i \le r-1). \tag{2.1.5}$$

Then we have the following equality

$$\mathbf{s} = \mathbf{s}_1 \circ \cdots \circ \mathbf{s}_r\colon \mathrm{C}(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{C}(\mathcal{A}). \tag{2.1.6}$$

We also have the equality (2.1.6) for $\mathrm{K}^{\star}(\mathcal{A}^{\underline{\bullet}})$ and $\mathrm{D}^{\star}(\mathcal{A}^{\underline{\bullet}})$.

Lemma 2.2. — *Let h be a nonnegative integer. Let N be a nonnegative integer satisfying the inequality*

(2.2.1)
$$N > \max\left\{i + 2^{-1}(h-i+1)(h-i+2) \mid 0 \le i \le h\right\} = 2^{-1}(h+1)(h+2).$$

Let $M = M^{\bullet\bullet}$ be an object of $\mathrm{C}^{+}(\mathcal{A}^{\bullet})$. Assume that $M^{ij} = 0$ for $j < 0$ $(i \in \mathbb{N})$. Let $\tau^{(1)}_{\bullet\le N}(M) = \tau^{(1)}_{\bullet\le N}(M^{\bullet\bullet})$ be a sub double complex of $M^{\bullet\bullet}$ defined by

$$\tau^{(1)}_{\bullet\le N}(M)^{ij} := M^{ij} \quad \textit{for } i < N,$$
$$\mathrm{Ker}\Big(\sum_{k=0}^{N+1}(-1)^k\delta_1^k\colon M^{Nj} \longrightarrow M^{N+1,j}\Big) \quad \textit{for } i = N,$$
$$\tau^{(1)}_{\bullet\le N}(M)^{ij} := 0 \quad \textit{for } i > N.$$

Then the natural morphism $\mathbf{s}(\tau^{(1)}_{\bullet\le N}(M)) \to \mathbf{s}(M)$ induces an isomorphism

$$\mathcal{H}^h\big(\mathbf{s}(\tau^{(1)}_{\bullet\le N}(M))\big) \overset{\sim}{\longrightarrow} \mathcal{H}^h\big(\mathbf{s}(M)\big). \tag{2.2.2}$$

Proof. — Consider the convergent spectral sequence

$$E_1^{ij} = \mathcal{H}^j(M^{i\bullet}) \Longrightarrow \mathcal{H}^{i+j}\big(\mathbf{s}(M)\big). \tag{2.2.3}$$

Let $0 \le i \le h$ be an integer. Let $r_{\max}$ be the largest integer such that $r_{\max}-1 \le h-i$, that is, $r_{\max} = h-i+1$. Then, if $d_r^{i,h-i}\colon E_r^{i,h-i} \longrightarrow E_r^{i+r,h-i-(r-1)}$ $(r \in \mathbb{Z}_{\ge 1})$ is nonzero, then $1 \le r \le r_{\max}$ by the assumption. The term $E_r^{i+r,h-i-(r-1)}$ may depend on the term $E_{r-1}^{i+r+(r-1),h-i-\{(r-1)+(r-2)\}}$ (of course, if $E_{r-1}^{i+r+(r-1),h-i-\{(r-1)+(r-2)\}} = 0$, then $E_r^{i+r,h-i-(r-1)}$ may depend on the term $E_{r-2}^{i+r+(r-2),h-i-\{(r-1)+(r-3)\}}$), and the term $E_{r-1}^{i+r+(r-1),h-i-\{(r-1)+(r-2)\}}$

may depend on the term $E_{r-2}^{i+\sum_{j=r-2}^{r} j, h-i-\sum_{j=r-3}^{r-1} j}$. Hence, for a fixed $h \in \mathbb{N}$, $\mathcal{H}^h(\mathbf{s}(M))$ depends only on $M^{\bullet \leq \{i+2^{-1}r(r+1)\}, \bullet}$. Since

$$N > i + 2^{-1}(h-i+1)(h-i+2) \geq i + 2^{-1}r(r+1),$$

we have (2.2). □

Let L_j be the stupid filtration on $\mathbf{s}(M)$ with respect to the j-th index $(1 \leq j \leq r)$:

$$L_j^{t_j}\big(\mathbf{s}(M)\big) = \bigoplus_{t'_j \geq t_j} M^{\bullet \cdots \bullet t'_j \bullet \cdots \bullet} \quad (t_j \in \mathbb{N}). \tag{2.2.4}$$

Let us also define the following stupid filtration $\underline{L}$ on $\mathbf{s}(M)$:

$$\underline{L}^t\big(\mathbf{s}(M)\big) = \bigoplus_{\underline{t}_r \geq t} M^{t_1 \cdots t_r \bullet} \quad (t \in \mathbb{N}). \tag{2.2.5}$$

If M is quasi-isomorphic to an object of $\mathrm{C}^+(\mathcal{A}^{\underline{\bullet}})$, then we have the convergent spectral sequence

$$E_1^{t, h-t} = \bigoplus_{\underline{t}_r = t} \mathcal{H}^{h-t}(M^{t_1 \cdots t_r \bullet}) \Longrightarrow \mathcal{H}^h\big(\mathbf{s}(M)\big). \tag{2.2.6}$$

Next we consider the filtered version of the above.

Let $(M, P) := (M, \{P_k M\}_{k \in \mathbb{Z}})$ be a complex of increasingly filtered $\mathcal{A}^{\underline{\bullet}}$-modules. Then (M, P) defines an $(r+1)$-uple filtered complex

$$\Big(\bigoplus_{\underline{t} \geq 0, s} M^{\underline{t} s}, \Big\{\bigoplus_{\underline{t} \geq 0, s} P_k M^{\underline{t} s}\Big\}_{k \in \mathbb{Z}}\Big)$$

of $\mathcal{A}$-modules by (2.0.1). Here $\underline{t} \geq 0$ means that $t_j \geq 0$ $(1 \leq \forall j \leq r)$. Then we have the following functor

$$\mathbf{s}\colon \mathrm{D}^+\mathrm{F}(\mathcal{A}^{\underline{\bullet}}) \ni \big[(M, P)\big] \longmapsto \big[(\mathbf{s}(M), \{\mathbf{s}(P_k M)\}_{k \in \mathbb{Z}})\big] \in \mathrm{D}^+\mathrm{F}(\mathcal{A}). \tag{2.2.7}$$

Let $\delta(\underline{L}, P) := \delta(L_1, \ldots, L_r, P)$ be the diagonal filtration of $L_1, \ldots, L_r$ and P on $\mathbf{s}(M)$ (cf. [**25**, (7.1.6.1), (8.1.22)]):

$$\begin{aligned} \delta\big(\underline{L}, P)_k(\mathbf{s}(M)\big) &= \bigoplus_{\underline{t} \geq 0, s} P_{\underline{t}_r + k} M^{\underline{t} s} \\ &= \sum_{\underline{t} \geq 0} L_1^{t_1}(\mathbf{s}(M)) \cap L_2^{t_2}(\mathbf{s}(M)) \cap \cdots \cap L_r^{t_r}(\mathbf{s}(M)) \cap \mathbf{s}(P_{\underline{t}_r + k} M). \end{aligned} \tag{2.2.8}$$

(In the case $r = 1$, the first formula $\delta(W, L)_n(\mathbf{s}(K)) = \bigoplus_{p,q} W_{n+p}(K^{p,q})$ in [**25**, (7.1.6.1)] have to be replaced by a formula $\delta(W, L)_n(\mathbf{s}(K)) = \bigoplus_{p,q} W_{n+p}(K^{q,p})$

(cf. [**25**, (5.1.9) (IV)]).) Then we have

$$\operatorname{gr}_k^{\delta(\underline{L},P)}\big(\mathbf{s}(M)\big) = \bigoplus_{\underline{t}\geq 0} \operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}[-\underline{t}_r]. \tag{2.2.9}$$

Assume that the filtration $\delta(\underline{L}, P)$ is exhaustive and complete, that $\operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}$ is quasi-isomorphic to an object of $\mathrm{C}^+(\mathcal{A}^{\underline{\bullet}})$ and that the spectral sequence arising from the filtration $\delta(\underline{L}, P)$ is regular and bounded below. Here we say that $\delta(\underline{L}, P)$ is complete if $\mathbf{s}(M) = \varprojlim_k \mathbf{s}(M)/(\delta(\underline{L}, P)_{-k}(\mathbf{s}(M)))$ (cf. [**89**, (5.4.4)]). Then we have the following convergent spectral sequence by the Convention 6) (cf. [**89**, (5.5.10)], [**25**, (8.1.15)]):

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \Longrightarrow \mathcal{H}^h\big(\mathbf{s}(M)\big). \tag{2.2.10}$$

We can obtain the filtered complex $(\mathbf{s}(M), \delta(\underline{L}, P))$ inductively by the following formula (cf. [**25**, (8.1.22)]):

$$\begin{aligned}\big(\mathbf{s}, \delta(\underline{L}, P)\big) = \big(&\mathbf{s}_1, \delta(L_1, \delta(L_2, \dots, \delta(L_r, P))\cdots)\big) \circ \cdots \\ &\circ \big(\mathbf{s}_{r-1}, \delta(L_{r-1}, \delta(L_r, P))\big) \circ \big(\mathbf{s}_r, \delta(L_r, P)\big).\end{aligned} \tag{2.2.11}$$

Let

$$\partial\colon \mathcal{H}^{h-\underline{t}_r}(\operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \to \mathcal{H}^{h-\underline{t}_r+1}(\operatorname{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet})$$

be the boundary morphism of the exact sequence

$$0 \to \operatorname{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet} \longrightarrow (P_{\underline{t}_r+k}/P_{\underline{t}_r+k-2}) M^{\underline{t}\bullet} \longrightarrow \operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet} \to 0.$$

Then the boundary morphism $d_1\colon E_1^{-k,h+k} \to E_1^{-k+1,h+k}$ of the E_1-terms of (2.2.10) is the sum of the following morphisms:

$$\begin{aligned}(-1)^{\underline{t}_{j-1}} \sum_{i=0}^{t_j+1} (-1)^i \delta^i_j \colon &\mathcal{H}^{h-\underline{t}_r}(\operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \\ &\longrightarrow \mathcal{H}^{h-\underline{t}_r}(\operatorname{gr}^P_{\underline{t}_r+k} M^{(t_1\dots t_{j-1}, t_j+1, t_{j+1}\dots t_r\bullet)})\end{aligned} \tag{2.2.12}$$

for $1 \leq j \leq r$ and

$$(-1)^{\underline{t}_r} \partial\colon \mathcal{H}^{h-\underline{t}_r}(\operatorname{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \longrightarrow \mathcal{H}^{h-\underline{t}_r+1}(\operatorname{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet}). \tag{2.2.13}$$

Let $(M, F) := (M, \{F^i M\}_{i\in\mathbb{Z}})$ be a complex of decreasingly filtered $\mathcal{A}^{\underline{\bullet}}$-modules. Then the filtration F induces the following filtration on $\mathbf{s}(M)$:

$$F^i\big(\mathbf{s}(M)\big) := \bigoplus_{\underline{t}\geq 0, s} F^i M^{\underline{t}s}. \tag{2.2.14}$$

Let σ be the stupid filtration on M. Then we have $\mathrm{gr}^i_\sigma M = M^i\{-i\}$. The complex $M^i\{-i\}$ defines an r-uple complex $M^{\underline{\bullet} i}\{-i\}$ of $\mathcal{A}$-modules; the complex $M^{\underline{\bullet} i}\{-i\}$ defines a single complex $\mathbf{s}(M^i\{-i\})$ with boundary morphism is equal to d with $d_M = 0$ in (2.0.1). Assume that M is bounded below. Then we have the following convergent spectral sequence

$$E_1^{i,h-i} = \mathcal{H}^{h-i}(\mathbf{s}(M^i)) \Longrightarrow \mathcal{H}^h(\mathbf{s}(M)). \tag{2.2.15}$$

Next we work in filtered derived categories of multi-simplicial ringed topoi.

Let $\mathrm{D}^+\mathrm{F}(\mathcal{A})$ (resp. $\mathrm{D}^+\mathrm{F}(\mathcal{A}^{\underline{\bullet}})$) be the derived category of bounded below filtered complexes of $\mathcal{A}$-modules (resp. $\mathcal{A}^{\underline{\bullet}}$-modules). Let $\mathrm{D}^+\mathrm{F}_2(\mathcal{A})$ (resp. $\mathrm{D}^+\mathrm{F}_2(\mathcal{A}^{\underline{\bullet}})$) be the derived category of bounded below biregular bifiltered complexes of $\mathcal{A}$-modules (resp. $\mathcal{A}^{\underline{\bullet}}$-modules) (see [**24**, (1.3.1), (1.3.6)], [**25**, (7.1.1)]). Then, by using (2.0.1), as in [**25**, (7.1.6.3), (7.1.7.1)], we have the following functors

$$(\mathbf{s},\delta)\colon \mathrm{D}^+\mathrm{F}(\mathcal{A}^{\underline{\bullet}}) \ni [(M,P)] \longmapsto [(\mathbf{s}(M), \delta(\underline{L}, P))] \in \mathrm{D}^+\mathrm{F}(\mathcal{A}), \tag{2.2.16}$$

$$\begin{aligned}(\mathbf{s},\delta)\colon \mathrm{D}^+\mathrm{F}_2(\mathcal{A}^{\underline{\bullet}}) \ni [(M,P,F)]& \\ \longmapsto [(\mathbf{s}(M), \delta(\underline{L}, P), F)] &\in \mathrm{D}^+\mathrm{F}_2(\mathcal{A}).\end{aligned} \tag{2.2.17}$$

Let $[(M,P)]$ be an object of $\mathrm{D}^+\mathrm{F}(\mathcal{A}^{\underline{\bullet}})$. If the filtration $\delta(\underline{L},P)$ on $\mathbf{s}(M)$ satisfies the assumptions before (2.2.10), then we have the convergent spectral sequence (2.2.10) for (M,P); the resulting spectral sequence is independent of the choice of the representative of $[(M,P)]$.

For a family $\{n_j\}_{j=1}^u$ $(1 \leq u \leq r)$ of nonnegative integers and for $i =$ nothing or 2, we have a natural restriction functor

$$\begin{aligned}&e^{-1}_{\bullet\cdots\bullet n_1\bullet\cdots\bullet n_2\bullet\cdots\cdots\bullet n_{u-1}\bullet\cdots\bullet n_u\bullet\cdots\bullet}\colon \\ &\qquad \mathrm{D}^+\mathrm{F}_i(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{D}^+\mathrm{F}_i(\mathcal{A}^{\bullet\cdots\bullet n_1\bullet\cdots\bullet n_2\bullet\cdots\cdots\bullet n_{u-1}\bullet\cdots\bullet n_u\bullet\cdots\bullet}).\end{aligned} \tag{2.2.18}$$

Let $f\colon (\mathcal{T}_{\underline{\bullet}}, \mathcal{A}^{\underline{\bullet}}) \longrightarrow (\mathcal{T}', \mathcal{A}')$ be an augmented (not necessarily constant) r-simplicial ringed topos. Then f induces the following morphism:

$$Rf_*\colon \mathrm{D}^+\mathrm{F}_i(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{D}^+\mathrm{F}_i(\mathcal{A}') \quad (i = \text{nothing or } 2). \tag{2.2.19}$$

For a nonnegative integer t_j $(1 \leq j \leq r)$, the morphism f induces the following augmentation of the $(r-1)$-simplicial ringed topos:

$$f_{t_j}\colon (\mathcal{T}_{\bullet\cdots\bullet t_j\bullet\cdots\bullet}, \mathcal{A}^{\bullet\cdots\bullet t_j\bullet\cdots\bullet}) \longrightarrow (\mathcal{T}', \mathcal{A}'). \tag{2.2.20}$$

Let $(\mathcal{T}'_{\underline{\bullet}}, \mathcal{A}'^{\underline{\bullet}})$ be the constant r-simplicial ringed topos defined by $(\mathcal{T}', \mathcal{A}')$. The morphism f also induces the following morphism

$$f_{\underline{\bullet}}\colon (\mathcal{T}_{\underline{\bullet}}, \mathcal{A}^{\underline{\bullet}}) \longrightarrow (\mathcal{T}'_{\underline{\bullet}}, \mathcal{A}'^{\underline{\bullet}}) \tag{2.2.21}$$

of r-simplicial ringed topoi. The following composite functor

$$\mathbf{s}Rf_{\underline{\bullet}*}\colon \mathrm{D}^+\mathrm{F}_i(\mathcal{A}^{\underline{\bullet}}) \longrightarrow \mathrm{D}^+\mathrm{F}_i(\mathcal{A}') \quad (i = \text{nothing or } 2) \tag{2.2.22}$$

(cf. [**25**, (5.2.6.1), (7.1.4)]) is canonically isomorphic to Rf_*. Hence we have the following convergent spectral sequence

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} R^{h-\underline{t}_r} f_{\underline{t}*}(\mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \Longrightarrow R^h f_*(M) \tag{2.2.23}$$

for an object $[(M,P)] \in \mathrm{D}^+\mathrm{F}(\mathcal{A}^{\underline{\bullet}})$ if the filtration $\delta(\underline{L},P)$ is exhaustive and complete and if (2.2.23) is regular and bounded below.

Let M be an object of $\mathrm{D}^+(\mathcal{A}^{\underline{\bullet}})$. Then, for a nonnegative integer t_j $(1 \leq j \leq r)$, we have the pull-back $M^{t_j} \in \mathrm{D}^+(\mathcal{A}^{\bullet\cdots\bullet t_j\bullet\cdots\bullet})$ by (2.2.18). Then we have the following spectral sequence

$$E_1^{t_j,s} = R^s f_{t_j*}(M^{t_j}) \Longrightarrow R^{s+t_j} f_*(M). \tag{2.2.24}$$

The boundary morphism between the E_1-terms of (2.2.24) is

$$(-1)^{\underline{t}_{j-1}} \sum_{i=0}^{t_j+1} (-1)^i \delta_j^i.$$

REMARKS 2.3. — 1) Let N_i $(1 \leq i \leq r)$ be a nonnegative integer or ∞. Set $\underline{N} := (N_1, \ldots, N_r)$. Let $(\mathcal{T}, \mathcal{A})$ be a ringed topos. Let

$$(\mathcal{T}_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}^{\underline{\bullet}\leq\underline{N}}) := (\mathcal{T}_{\bullet\leq N_1,\cdots,\bullet\leq N_r}, \mathcal{A}^{\bullet\leq N_1,\cdots,\bullet\leq N_r})$$

be the constant $(N_1, \ldots, N_r)$-truncated r-simplicial ringed topos defined by $(\mathcal{T}, \mathcal{A})$. Then the analogues of all results in this section hold for $(\mathcal{T}_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}^{\underline{\bullet}\leq\underline{N}})$.

2) Let $(\mathcal{T}_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}^{\underline{\bullet}\leq\underline{N}})$ be an $\underline{N}$-truncated r-simplicial ringed topos. Let $f\colon (\mathcal{T}_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}^{\underline{\bullet}\leq\underline{N}}) \to (\mathcal{T}', \mathcal{A}')$ be a morphism of ringed topoi. Then the analogues of all results in this section hold for f.

3. Review on (pre)weight-filtered vanishing cycle (restricted) crystalline and zariskian complexes

In this section we review some results in [**72**] briefly.

Let p be a prime number. Let S be a scheme on which p is locally nilpotent. Let $(S, \mathcal{I}, \gamma)$ be a PD-scheme with quasi-coherent PD-ideal sheaf $\mathcal{I}$ and with PD-structure γ on $\mathcal{I}$. Set $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I})$. Let $f\colon X \to S_0$ be a smooth scheme with a relative SNCD D on X over S_0 (see [**72**, (2.1.7)]). By abuse of notation, we also denote by f the composite morphism $X \longrightarrow S_0 \hookrightarrow S$. Let $\mathrm{Div}(X/S_0)_{\geq 0}$ be the monoid of effective Cartier divisors on X over S_0.

Let $\Delta_D := \{D_\lambda\}_{\lambda\in\Lambda}$ be a decomposition of D by smooth components of D: each D_λ is smooth over S_0 and $D = \sum_\lambda D_\lambda$ in $\mathrm{Div}(X/S_0)_{\geq 0}$. Set $D^{(0)} := X$ and, for a positive integer k, set

$$D^{(k)} := \coprod_{\{\lambda_1,\ldots,\lambda_k \mid \lambda_i\neq\lambda_j\ (i\neq j)\}} D_{\lambda_1}\cap\cdots\cap D_{\lambda_k}.$$

In [**72**, (2.2.15)] we have proved that $D^{(k)}$ is independent of the choice of Δ_D. Let Z be a relative SNCD on X over S_0 which intersects D transversally. Let

$$\Delta_Z := \{Z_\mu\}_\mu$$

be a decomposition of Z by smooth components of Z. Then

$$\Delta_{D,Z} := \{D_\lambda, Z_\mu\}_{\lambda,\mu}$$

is a decomposition of $D\cup Z$ by smooth components of $D\cup Z$. The pair $(X, D\cup Z)$ (resp. (X,Z)) defines an fs (= fine and saturated) log structure $M(D\cup Z)$ (resp. $M(Z)$) which has been defined in [**72**, (2.1)] (cf. [**51**, pp. 222–223]). We recall this log structure as follows.

Let $\mathrm{Div}_D(X/S_0)_{\geq 0}$ be a submonoid of $\mathrm{Div}(X/S_0)_{\geq 0}$ consisting of elements E's of $\mathrm{Div}(X/S_0)_{\geq 0}$ such that there exists an open covering $X = \bigcup_{i\in I} V_i$ (depending on E) of X such that $E|_{V_i}$ is contained in the submonoid of $\mathrm{Div}(V_i/S_0)_{\geq 0}$ generated by $D_\lambda|_{V_i}$ ($\lambda\in\Lambda$). By [**72**, Proposition A.0.1] we see that the definition of $\mathrm{Div}_D(X/S_0)_{\geq 0}$ is independent of the choice of Δ_D.

The pair (X,D) gives a natural fs (= fine and saturated) log structure in $\widetilde{X}_{\mathrm{zar}}$ as follows (cf. [**51**, pp. 222–223]).

Let $M(D)'$ be a presheaf of monoids in $\widetilde{X}_{\mathrm{zar}}$ defined as follows: for an open subscheme V of X,

$$\Gamma(V, M(D)') := \big\{(E,a)\in \mathrm{Div}_{D|_V}(V/S_0)_{\geq 0}\times\Gamma(V,\mathcal{O}_X) \mid a \text{ is a generator of } \Gamma(V,\mathcal{O}_X(-E))\big\}$$

with a monoid structure defined by $(E,a)\cdot(E',a') := (E+E', aa')$. The natural morphism $M(D)'\to\mathcal{O}_X$ defined by $(E,a)\mapsto a$ induces a morphism $M(D)'\to(\mathcal{O}_X,*)$ of presheaves of monoids in $\widetilde{X}_{\mathrm{zar}}$. The log structure $M(D)$ is, by definition, the associated log structure to the sheafification of $M(D)'$. Because $\mathrm{Div}_D(X/S_0)_{\geq 0}$ is independent of the choice of Δ_D, $M(D)$ is independent of the choice of Δ_D. We say that the log scheme $(X, M(D))$ is the log scheme obtained from the pair (X,D) and that $M(D)$ is the log structure associated to D.

Henceforth, by abuse of notation, we denote the log scheme $(X, M(D\cup Z))$ and $(X, M(Z))$ simply by $(X, D\cup Z)$ and (X, Z), respectively, unless we need to make the log structures explicit.

We also recall the preweight filtration $P^D_{\bullet} = \{P^D_k\}_{k\in\mathbb{Z}}$ on the sheaf $\Omega^i_{X/S_0}(\log(D\cup Z))$ of log differential forms ($i\in\mathbb{N}$) on $X_{\rm zar}$ with respect to D (see [**72**, (2.2.15.2)]):

(3.0.1) $$P^D_k\Omega^i_{X/S_0}\big(\log(D\cup Z)\big) =$$
$$\begin{cases} 0 & (k<0),\\ \mathrm{Im}(\Omega^k_{X/S_0}(\log(D\cup Z))\otimes_{\mathcal{O}_X}\Omega^{i-k}_{X/S_0}(\log Z)\to\Omega^i_{X/S_0}(\log(D\cup Z))) & (0\le k\le i),\\ \Omega^i_{X/S_0}(\log(D\cup Z)) & (k>i).\end{cases}$$

Let $((X,\widetilde{D\cup Z})/S)^{\log}_{\rm crys}$ (resp. $((\widetilde{X,Z})/S)^{\log}_{\rm crys}$) be the log crystalline topos of $(X, D\cup Z)$ (resp. (X,Z)) over $(S,\mathcal{I},\gamma)$ (see [**51**, §5]). Let

$$\epsilon_{(X,D\cup Z,Z)/S}\colon \big((X,\widetilde{D\cup Z})/S\big)^{\log}_{\rm crys}\longrightarrow\big((\widetilde{X,Z})/S\big)^{\log}_{\rm crys}$$

be the natural morphism of topoi which is induced by a natural morphism $(X, D\cup Z)\longrightarrow(X,Z)$ of log schemes. Let

$$u_{(X,D\cup Z)/S}\colon \big((X,\widetilde{D\cup Z})/S\big)^{\log}_{\rm crys}\longrightarrow \widetilde{X}_{\rm zar},$$
$$u_{(X,Z)/S}\colon \big((\widetilde{X,Z})/S\big)^{\log}_{\rm crys}\longrightarrow \widetilde{X}_{\rm zar}$$

be the natural projections. Then we have the following commutative diagram of topoi:

$$\begin{array}{ccc} \big((X,\widetilde{D\cup Z})/S\big)^{\log}_{\rm crys} & \xrightarrow{\epsilon_{(X,D\cup Z,Z)/S}} & \big((\widetilde{X,Z})/S\big)^{\log}_{\rm crys}\\ {\scriptstyle u_{(X,D\cup Z)/S}}\big\downarrow & & \big\downarrow{\scriptstyle u_{(X,Z)/S}}\\ \widetilde{X}_{\rm zar} & = & \widetilde{X}_{\rm zar}.\end{array}$$

Let $\mathcal{O}_{(X,D\cup Z)/S}$ (resp. $\mathcal{O}_{(X,Z)/S}$) be the structure sheaf in the log crystalline topos $((X,\widetilde{D\cup Z})/S)^{\log}_{\rm crys}$ (resp. $((\widetilde{X,Z})/S)^{\log}_{\rm crys}$).

Let $\varpi^{(k)\log}_{\rm crys}(D/S;Z)$ ($k\in\mathbb{N}$) be the log crystalline orientation sheaf of $D^{(k)}/(S,\mathcal{I},\gamma)$ in $((D^{(k)},\widetilde{Z|_{D^{(k)}}})/S)^{\log}_{\rm crys}$ defined in [**72**, (2.2.18)] and let $\varpi^{(k)}_{\rm zar}(D/S_0)$ be the zariskian orientation sheaf of $D^{(k)}/S_0$ in $\widetilde{D}^{(k)}_{\rm zar}$ defined in [*loc. cit.*] (cf. [**24**, (3.1.4)]).

The abelian sheaves $\varpi^{(k)\log}_{\rm crys}(D/S;Z)$ and $\varpi^{(k)}_{\rm zar}(D/S_0)$ are isomorphic to the constant sheaf $\mathbb{Z}$.

If S_0 is of characteristic $p>0$, then we have defined the Frobenius action on the orientation sheaves in [**72**, (2.9)]. Let us recall it.

Let $F\colon (X,D\cup Z)\to (X',D'\cup Z')$ be the relative Frobenius morphism over S_0. The morphism F induces relative Frobenius morphisms

$$F_{(X,Z)}\colon (X,Z)\longrightarrow (X',Z') \quad \text{and} \quad F^{(k)}\colon (D^{(k)},Z|_{D^{(k)}})\longrightarrow (D'^{(k)},Z'|_{D'^{(k)}}).$$

Let $a^{(k)}\colon (D^{(k)},Z|_{D^{(k)}})\to (X,Z)$ and $a^{(k)\prime}\colon (D'^{(k)},Z'|_{D'^{(k)}})\to (X',Z')$ be the natural morphisms. Then we define the following two Frobenius morphisms

$$\Phi^{(k)}\colon a^{(k)\prime\log}_{\mathrm{crys}*}\varpi^{(k)\log}_{\mathrm{crys}}(D'/S;Z')\longrightarrow F^{\log}_{(X,Z)\mathrm{crys}*}a^{(k)\log}_{\mathrm{crys}*}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z), \tag{3.0.2}$$

$$\Phi^{(k)}\colon a^{(k)\prime}_{*}\varpi^{(k)}_{\mathrm{zar}}(D'/S_0)\longrightarrow \overset{\circ}{F}_{*}a^{(k)}_{*}\varpi^{(k)}_{\mathrm{zar}}(D/S_0) \tag{3.0.3}$$

by the identities under the natural identifications

$$\varpi^{(k)\log}_{\mathrm{crys}}(D'/S;Z')\xrightarrow{\ \sim\ } F^{(k)\log}_{\mathrm{crys}*}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z),$$
$$\varpi^{(k)}_{\mathrm{zar}}(D'/S_0)\xrightarrow{\ \sim\ } \overset{\circ}{F}{}^{(k)}_{*}\varpi^{(k)}_{\mathrm{zar}}(D/S_0).$$

Let $(\mathcal{X},\mathcal{D}\cup\mathcal{Z})$ be a smooth scheme over S with transversal relative SNCD's over S. Let $\iota\colon (X,D\cup Z)\hookrightarrow (\mathcal{X},\mathcal{D}\cup\mathcal{Z})$ be an exact immersion over S. In [**72**, (2.1.10)] we have called $\iota\colon (X,D\cup Z)\hookrightarrow (\mathcal{X},\mathcal{D}\cup\mathcal{Z})$ an *admissible immersion over* S *with respect to* $\Delta_{D,Z}$ if there exists a decomposition $\Delta_{\mathcal{D},\mathcal{Z}}:=\{\mathcal{D}_\lambda,\mathcal{Z}_\mu\}_{\lambda,\mu}$ of $\mathcal{D}\cup\mathcal{Z}$ by smooth components ($\mathcal{D}=\bigcup_\lambda\mathcal{D}_\lambda$, $\mathcal{Z}=\bigcup_\mu\mathcal{Z}_\mu$) such that ι induces isomorphisms $D_\lambda\overset{\sim}{\to}\mathcal{D}_\lambda\times_{\mathcal{X}}X$ and $Z_\mu\overset{\sim}{\to}\mathcal{Z}_\mu\times_{\mathcal{X}}X$ for all λ's and μ's. Locally on X_{zar}, there exists an admissible immersion of $(X,D\cup Z)$ over S with respect to $\Delta_{D,Z}$. In fact, locally on X_{zar}, there exists a lift of $(X,D\cup Z)$ over S (see [**72**, (2.3.14)]).

In [**72**] we have defined two increasingly filtered complexes

$$\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big)\in \mathrm{D}^+\mathrm{F}(\mathcal{O}_{(X,Z)/S}),$$
$$\big(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big)\in \mathrm{D}^+\mathrm{F}\big(f^{-1}(\mathcal{O}_S)\big)$$

by

$$\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big):=\big(R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),\tau\big),$$
$$\big(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big):=Ru_{(X,Z)/S*}\big((E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\big).$$

Here τ is the canonical filtration on a complex. In [*loc. cit.*] we have called the filtered complex $(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ (resp. $(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$) the preweight-filtered vanishing cycle crystalline complex (resp. the preweight-filtered vanishing cycle zariskian complex) of $(X,D\cup Z)/S$ with respect to D.

Let $(T,\mathcal{J},\delta)$ be a fine log PD-scheme with quasi-coherent PD-ideal sheaf $\mathcal{J}$ and with PD-structure δ on $\mathcal{J}$. Assume that p is locally nilpotent on $\overset{\circ}{T}$. Let $f\colon Y\to T$ be a morphism of fine log schemes such that δ extends to $\overset{\circ}{Y}$. Assume that $\overset{\circ}{f}$ is locally of finite presentation. Let $(Y/T)^{\log}_{\mathrm{Rcrys}}$ be the restricted log crystalline site of $Y/(T,\mathcal{J},\delta)$ (see [**85**, (6.2)] = a log version of [**5**, IV, Déf. 1.7.1] (cf. [*loc. cit.*, IV, Prop. 1.5.5])): $(Y/T)^{\log}_{\mathrm{Rcrys}}$ is a full subcategory of the log crystalline site $(Y/T)^{\log}_{\mathrm{crys}}$ whose objects are isomorphic to triples

$$\big(V,\mathfrak{D}_V(\mathcal{V}),[\ ]\big)\text{'s},$$

where V is a log open subscheme of Y, $\iota\colon V\hookrightarrow\mathcal{V}$ is an immersion into a log smooth scheme over T and $\mathfrak{D}_V(\mathcal{V})$ is the log PD-envelope of ι over $(T,\mathcal{J},\delta)$ (see [**51**, (5.4)]); the topology of $(Y/T)^{\log}_{\mathrm{Rcrys}}$ is the induced topology by that of $(Y/T)^{\log}_{\mathrm{crys}}$. Let $(\widetilde{Y/T})^{\log}_{\mathrm{Rcrys}}$ be the topos associated to $(Y/T)^{\log}_{\mathrm{Rcrys}}$. Let

$$Q_{Y/T}\colon(\widetilde{Y/T})^{\log}_{\mathrm{Rcrys}}\longrightarrow(\widetilde{Y/T})^{\log}_{\mathrm{crys}}$$

be the natural morphism of topoi which is the log version of the morphism in [**5**, IV, (2.1.1)]: $Q_{Y/T}$ is a morphism of topoi such that $Q^*_{Y/T}(E)$ $(E\in(\widetilde{Y/T})^{\log}_{\mathrm{crys}})$ is the natural restriction of E. Let

$$u_{Y/T}\colon(\widetilde{Y/T})^{\log}_{\mathrm{crys}}\longrightarrow\widetilde{Y}_{\mathrm{zar}}$$

be the canonical projection and set $\bar{u}_{Y/T}:=u_{Y/T}\circ Q_{Y/T}$.

We call $Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ the *preweight-filtered vanishing cycle restricted crystalline complex* of $(X,D\cup Z)/S$ with respect to D. In [**72**] we have proved the following two theorems:

Theorem 3.1 (p-adic purity). — *Let k be a nonnegative integer. Then the following hold*:

1) (see [**72**, (2.7.1)]; *see also* (3.5.13) *and* 3.6) *below.*) *There exists in* $\mathrm{D}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$ *an isomorphism*

$$(3.1.1)\quad Q^*_{(X,Z)/S}\mathrm{gr}^{P^D}_k E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})\xrightarrow{\sim} Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}.$$

2) (see [**72**, (2.6.1.2), (2.7.5), (2.9.3)]) *There exists in* $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$ *the following functorial isomorphism*

$$(3.1.2)\qquad \mathrm{gr}^{P^D}_k E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S})\xrightarrow{\sim} a^{(k)}_{\mathrm{zar}*}(Ru_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S,Z)),(-k))\{-k\}$$

where $(-k)$ *means the Tate twist, which is considered when* S_0 *is of characteristic* p.

THEOREM 3.2. — *Let* X_0 *be the disjoint union of an open covering of* X. *Let* $\pi_0\colon X_0 \to X$ *be the natural augmentation morphism. Set* $D_0 := \pi_0^{-1}(D)$ *and* $Z_0 := \pi_0^{-1}(Z)$. *Let* Δ_0 *be the decomposition of* $D_0 \cup Z_0$ *by smooth components of* $D_0 \cup Z_0$ *which is obtained from* $\Delta_{D,Z}$. *Set*

$$(X_\bullet, D_\bullet)_{\bullet\in\mathbb{N}} := \mathrm{cosk}_0^{(X,D)}(X_0, D_0) \quad \textit{and} \quad (X_\bullet, Z_\bullet)_{\bullet\in\mathbb{N}} := \mathrm{cosk}_0^{(X,Z)}(X_0, Z_0).$$

Let $\pi_\bullet\colon X_\bullet \to X$ *be the augmentation morphism. Set*

$$\Delta_\bullet := \left\{\pi_\bullet^{-1}(D_\lambda), \pi_\bullet^{-1}(Z_\mu)\right\}_{\lambda,\mu}.$$

Then the following hold:

1) (See [**72**, (2.5.7)].) *If each member of the open covering of* X *is an affine scheme, then there exists a simplicial admissible immersion*

$$(X_\bullet, D_\bullet \cup Z_\bullet)_{\bullet\in\mathbb{N}} \longhookrightarrow (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)_{\bullet\in\mathbb{N}} \tag{3.2.1}$$

into a smooth simplicial scheme with transversal simplicial relative SNCD's $\mathcal{D}_\bullet$ *and* $\mathcal{Z}_\bullet$ *over* S *with respect to* $\Delta_\bullet$ *such that the immersion* (3.2.1) *induces the following simplicial admissible immersions*

$$(X_\bullet, D_\bullet)_{\bullet\in\mathbb{N}} \longhookrightarrow (\mathcal{X}_\bullet, \mathcal{D}_\bullet)_{\bullet\in\mathbb{N}} \quad \textit{and} \quad (X_\bullet, Z_\bullet)_{\bullet\in\mathbb{N}} \longhookrightarrow (\mathcal{X}_\bullet, \mathcal{Z}_\bullet)_{\bullet\in\mathbb{N}}$$

over S.

2) (See [**72**, (2.2.17) (1)].) *Assume that there exists the simplicial admissible immersion* (3.2.1). *Let* $\mathfrak{D}_\bullet$ *be the simplicial PD-envelope of the simplicial immersion* $X_\bullet \hookrightarrow \mathcal{X}_\bullet$ *over* $(S, \mathcal{I}, \gamma)$. *Then the natural morphism*

$$\begin{aligned} &\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} P_k^{\mathcal{D}_\bullet}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \\ &\qquad \longrightarrow \mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} \Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \quad (k \in \mathbb{Z}) \end{aligned} \tag{3.2.2}$$

is injective.

3) (See [**72**, (2.2.17) (2)].) *Let the notations and the assumption be as in* 1). *Let* $L_{(X_\bullet,Z_\bullet)/S}$ *be the log linearization functor for* $\mathcal{O}_{\mathcal{X}_\bullet}$*-modules. Set*

$$\begin{aligned} &P_k^{D_\bullet} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) \\ &\qquad := L_{(X_\bullet,Z_\bullet)/S}(P_k^{\mathcal{D}_\bullet}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) \quad (k \in \mathbb{Z}). \end{aligned}$$

Then the natural morphism

$$\begin{aligned} &Q^*_{(X_\bullet,Z_\bullet)/S} P_k^{D_\bullet} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}\big(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))\big) \\ &\qquad \longrightarrow Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}\big(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))\big) \end{aligned} \tag{3.2.3}$$

is injective.

4) (See [**72**, (2.4.6), (2.7.3)]; *see also* (3.6), 4) *below.) Let the notations and the assumption be as in* 1) *and* 3). *Denote by*

$$\big(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))), Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}\big)$$

the filtered complex

$$\big(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))), \{Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}_k L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)))\}_{k\in\mathbb{Z}}\big).$$

Let $(((\widetilde{X_\bullet,Z_\bullet})/S)^{\log}_{\rm Rcrys}, Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S}))$ *be the ringed topos constructed in* [**72**, (1.6)] *and let*

$$\pi_{\rm Rcrys}\colon\big(((\widetilde{X_\bullet,Z_\bullet})/S)^{\log}_{\rm Rcrys}, Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S})\big)\longrightarrow\big(((\widetilde{X,Z})/S)^{\log}_{\rm Rcrys}, Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})\big)$$

be the natural augmentation morphism constructed in [*loc. cit.*]. *Then there exists a canonical isomorphism*

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)\overset{\sim}{\longrightarrow} R\pi_{{\rm Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))), Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}). \tag{3.2.4}$$

5) (See [**72**, (2.5.8), (2.7.5)].) *Let the notations and the assumption be as in* 2). *Denote by*

$$\big(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)), P^{\mathcal{D}_\bullet}\big)$$

the filtered complex

$$\big(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)), \{\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}P^{\mathcal{D}_\bullet}_k\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\}_{k\in\mathbb{Z}}\big).$$

Let $f_\bullet\colon X_\bullet\to S$ *be the structural morphism and* $\pi_{\rm zar}\colon(\widetilde{X}_{\bullet\rm zar}, f^{-1}_\bullet(\mathcal{O}_S))\to(\widetilde{X}_{\rm zar}, f^{-1}(\mathcal{O}_S))$ *the natural augmentation morphism. Then there exists a canonical isomorphism*

$$\big(E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big)\overset{\sim}{\longrightarrow} R\pi_{{\rm zar}*}\big(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)), P^{\mathcal{D}_\bullet}\big). \tag{3.2.5}$$

Remarks 3.3. — 1) In [**72**] we have denoted the right hand side on (3.2.4) (resp. (3.2.5)) by $(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ (resp. $(C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$). When $Z=\varnothing$, we have denoted it

$$(C_{\rm Rcrys}(\mathcal{O}_{(X,D)/S}), P)\ (\text{resp.}\ (C_{\rm zar}(\mathcal{O}_{(X,D)/S}), P)).$$

2) Let $(S',\mathcal{I}',\gamma')$ be a PD-scheme with a quasi-coherent PD-ideal sheaf and a PD-structure. Let $u\colon (S',\mathcal{I}',\gamma')\longrightarrow (S,\mathcal{I},\gamma)$ be a PD-morphism. Set $S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}')$. The morphism u induces a morphism $S'_0\to S_0$. Consider the following commutative diagram

$$\begin{array}{ccc}
(X',D'\cup Z') & \xrightarrow{g} & (X,D\cup Z)\\
\big\downarrow & & \big\downarrow\\
(X',Z') & \xrightarrow{g_Z} & (X,Z)\\
\big\downarrow & & \big\downarrow\\
S'_0 & \longrightarrow & S_0,
\end{array}$$

where g and g_Z are morphisms of log schemes over the morphism $S'_0\to S_0$. As a corollary of (3.2), we see that $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ is functorial. That is, we have a natural morphism

$$g^*\colon \bigl(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\bigr)\longrightarrow Rg_*\bigl(C^{\log,Z'}_{\mathrm{zar}}(\mathcal{O}_{(X',D'\cup Z')/S'}),P^{D'}\bigr). \tag{3.3.1}$$

3) For a (not necessarily affine) P-adic formal scheme S in the sense of [**12**, 7.17, Def.], we have an abelian sheaf $\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)$ and two complexes

$$\bigl(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\bigr)\quad\text{and}\quad\bigl(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\bigr)$$

and we have the obvious analogues of (3.1) and (3.2). We also have

$$\bigl(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\bigr)\quad\text{and}\quad\bigl(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\bigr).$$

The following theorem is the crystalline Poincaré lemma of a vanishing cycle sheaf (see [**72**, (2.3.10)]):

Theorem 3.4 (Poincaré lemma of a vanishing cycle sheaf)

Let T be a fine log scheme on which a prime number is locally nilpotent. Let $(T,\mathcal{J},\delta)$ be a log PD-scheme such that $\mathcal{J}$ is a quasi-coherent PD-ideal sheaf of $\mathcal{O}_T$. Let (Y,M) be a fine log scheme over T. Let $N\subset M$ be also a fine log structure on Y_{zar}. Let $\epsilon_{(Y,M,N)/T}\colon (Y,M)\to (Y,N)$ be a natural morphism of log schemes over T. Let

$$\begin{array}{ccc}
(Y,M) & \xrightarrow{\iota_{\mathcal{M}}} & (\mathcal{Y},\mathcal{M})\\
{\scriptstyle \epsilon_{(Y,M,N)/T}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{(\mathcal{Y},\mathcal{M},\mathcal{N})/T}}\\
(Y,N) & \xrightarrow{\iota_{\mathcal{N}}} & (\mathcal{Y},\mathcal{N})
\end{array} \tag{3.4.1}$$

be a commutative diagram whose horizontal morphisms are closed immersions into log smooth schemes over T such that $\mathcal{N}\subset\mathcal{M}$. Let $\mathfrak{D}_{\mathcal{M}}$ and $\mathfrak{D}_{\mathcal{N}}$ be the *log PD-envelopes of* $\iota_{\mathcal{M}}$ *and* $\iota_{\mathcal{N}}$ *over* $(T,\mathcal{J},\delta)$*, respectively, with the natural following commutative diagram*:

$$\begin{array}{ccc} \mathfrak{D}_{\mathcal{M}} & \xrightarrow{g_{\mathcal{M}}} & (\mathcal{Y},\mathcal{M}) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{(\mathcal{Y},\mathcal{M},\mathcal{N})/T}} \\ \mathfrak{D}_{\mathcal{N}} & \xrightarrow{g_{\mathcal{N}}} & (\mathcal{Y},\mathcal{N}). \end{array} \tag{3.4.2}$$

Assume that the underlying morphism $\overset{\circ}{h}$ *of schemes is the identity. Let* E *be a crystal of* $\mathcal{O}_{(Y,N)/S}$*-modules and let* $(\mathcal{E},\nabla)$ *be the* $\mathcal{O}_{\mathfrak{D}_{\mathcal{M}}}$*-module with integrable connection corresponding to* $\epsilon^*_{(Y,M,N)/S}(E)$*. Let* $L^{\rm PD}_{(Y,N)/T}$ *be the linearization functor for* $\mathcal{O}_{\mathfrak{D}_{\mathcal{N}}}$*-modules. Then there exists in* $\mathrm{D}^+(\mathcal{O}_{(Y,N)/T})$ *a canonical isomorphism*

$$R\epsilon_{(Y,M,N)/T*}\big(\epsilon^*_{(Y,M,N)/S}(E)\big) \overset{\sim}{\longrightarrow} L^{\rm PD}_{(Y,N)/T}\big(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/T}(\log(\mathcal{M}/M_T))\big). \tag{3.4.3}$$

The following has been proved in the proof of [**72**, (2.7.3)] implicitly by using $(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$.

Proposition 3.5. — *Let the notations be as in* (3.2)*. Set*

$$\big(E^{\log,Z_\bullet}_{\rm crys}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big) := \big(R\epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),\tau\big),$$

$$\big(E^{\log,Z_\bullet}_{\rm zar}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big) := Ru_{(X_\bullet,Z_\bullet)/S*}\big((E^{\log,Z_\bullet}_{\rm crys}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet})\big).$$

Then the following hold:

1) *There exists the following natural filtered isomorphism*

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} R\pi_{{\rm Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}\big(E^{\log,Z_\bullet}_{\rm crys}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big). \tag{3.5.1}$$

2) *There exists the following natural filtered isomorphism*

$$\big(E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} R\pi_{{\rm zar}*}\big(E^{\log,Z_\bullet}_{\rm zar}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big). \tag{3.5.2}$$

Proof. — For the completeness of this book, we give a proof of (3.5); the following argument is necessary for the proofs of (6.16) and (6.17) below. Let

$$\pi^{\log}_{\rm crys}\colon \big(((X_\bullet,\widetilde{D_\bullet\cup Z_\bullet})/S)^{\log}_{\rm crys},\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}\big) \longrightarrow \big(((X,\widetilde{D\cup Z})/S)^{\log}_{\rm crys},\mathcal{O}_{(X,D\cup Z)/S}\big)$$

$$\pi_{\rm crys}\colon \big(((\widetilde{X_\bullet,Z_\bullet})/S)^{\log}_{\rm crys},\mathcal{O}_{(X_\bullet,Z_\bullet)/S}\big) \longrightarrow \big(((\widetilde{X,Z})/S)^{\log}_{\rm crys},\mathcal{O}_{(X,Z)/S}\big)$$

be natural morphisms of ringed topoi. Since we have a natural morphism

$$\mathcal{O}_{(X,D\cup Z)/S} \longrightarrow \pi^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),$$

we have the following natural morphism

$$\begin{aligned}(3.5.3)\quad \big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) &= \big(R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),\tau\big)\\ &\longrightarrow \big(R\epsilon_{(X,D\cup Z,Z)/S*}R\pi^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),\tau\big).\end{aligned}$$

Let $f\colon (\mathcal{T},\mathcal{A}) \to (\mathcal{T}',\mathcal{A}')$ be a morphism of ringed topoi. Let $E^\bullet$ be an object of $\mathrm{D}^+(\mathcal{A})$. In [**72**, (2.7.2)] we have proved that there exists a canonical morphism

$$(3.5.4)\qquad \big(Rf_*(E^\bullet),\tau\big) \longrightarrow Rf_*\big((E^\bullet,\tau)\big)$$

in $\mathrm{D}^+\mathrm{F}(\mathcal{A}')$. Hence we have the following morphism

$$\begin{aligned}(3.5.5)\quad &\big(R\epsilon_{(X,D\cup Z,Z)/S*}R\pi^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),\tau\big)\\ &= \big(R\pi_{\mathrm{crys}*}R\epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),\tau\big)\\ &\longrightarrow R\pi_{\mathrm{crys}*}\big(R\epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),\tau\big)\\ &= R\pi_{\mathrm{crys}*}\big(E^{\log,Z_\bullet}_{\mathrm{crys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big).\end{aligned}$$

By (3.5.3) and (3.5.5), we have the composite morphism

$$(3.5.6)\quad \big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \longrightarrow R\pi_{\mathrm{crys}*}\big(E^{\log,Z_\bullet}_{\mathrm{crys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}\big).$$

By [**72**, (1.6.4.1)], we have

$$(3.5.7)\qquad Q^*_{(X,Z)/S}R\pi_{\mathrm{crys}*} \xrightarrow{=} R\pi_{\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}.$$

Hence we have the following morphism by (3.5.6):

$$\begin{aligned}(3.5.8)\quad &Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\\ &\longrightarrow R\pi_{\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}(E^{\log,Z_\bullet}_{\mathrm{crys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}).\end{aligned}$$

By the log Poincaré lemma (see [**72**, (2.2.7)]), we have the following quasi-isomorphism

$$\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S} \xrightarrow{\sim} L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big).$$

Let $I^\bullet$ be an injective resolution of $\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}$. Then we have a quasi-isomorphism

$$L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big) \xrightarrow{\sim} I^\bullet.$$

By the simplicial version of [**72**, (2.3.7)], there exists a natural quasi-isomorphism

$$L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}\big(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)\big) \\ \overset{\sim}{\longrightarrow} \epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}\big(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)\big).$$

Hence we have the following composite morphism

$$L_{(X_\bullet,Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big) \\ \overset{\sim}{\longrightarrow} \epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big) \\ \longrightarrow \epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(I^\bullet).$$

By (3.4) this composite morphism is a quasi-isomorphism. Hence

$$R\epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}) \\ = L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}\big(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big). \tag{3.5.9}$$

Consequently the morphism (3.5.8) is equal to

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) \\ \longrightarrow R\pi_{{\rm Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}\big(L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau\big). \tag{3.5.10}$$

We also have the natural morphism

$$Q^*_{(X_\bullet,Z_\bullet)/S}\big(L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau\big) \\ \longrightarrow \big(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big),Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}\big). \tag{3.5.11}$$

By (3.5.10) and (3.5.11), we have the composite morphism

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) \longrightarrow \\ R\pi_{{\rm Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}). \tag{3.5.12}$$

Let $a^{(k)}_\bullet\colon (D^{(k)}_\bullet, Z|_{D^{(k)}_\bullet}) \longrightarrow (X_\bullet, Z_\bullet)$ be the natural morphism of simplicial log schemes. By [**72**, (1.3.4.1)], [*loc. cit.*, (2.2.21.2)], the log Poincaré lemma, (3.5.7) and the cohomological descent, we have the following (however see (3.6),

2) below for the second equality):

(3.5.13)

$$
\begin{aligned}
&\mathrm{gr}_k R\pi_{\mathrm{Rcrys}*}\big(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}\big)\\
&=R\pi_{\mathrm{Rcrys}*}\mathrm{gr}_k^{Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}\big(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))\big)\\
&=R\pi_{\mathrm{Rcrys}*}\big(Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}L_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}(\Omega^\bullet_{\mathcal{D}^{(k)}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}^{(k)}_\bullet}))\\
&\qquad\qquad\qquad\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S,Z_\bullet))(-k)\{-k\}\big)\\
&=R\pi_{\mathrm{Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}\big(\mathcal{O}_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S,Z_\bullet)\big)(-k))\{-k\}\\
&=Q^*_{(X,Z)/S}\big(a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S,Z))(-k)\big)\{-k\}.
\end{aligned}
$$

Now, by (3.5.13) and the argument in [**72**, (2.7.1), (2.7.3)], we see that the morphism (3.5.12) is a filtered isomorphism (To avoid the circular reasoning, we do not use (3.1.1) here. See (3.6) , 3) below.). By (3.5.9), (3.5.13) and the argument in [**72**, (2.7.1), (2.7.3)], we see that, for each $t\in\mathbb{N}$, the morphism

$$
\begin{aligned}
&Q^*_{(X_t,Z_t)/S}(L_{(X_t,Z_t)/S}(\Omega^\bullet_{\mathcal{X}_t/S}(\log(\mathcal{D}_t\cup\mathcal{Z}_t))),\tau)\\
&\qquad\longrightarrow\big(Q^*_{(X_t,Z_t)/S}L_{(X_t,Z_t)/S}(\Omega^\bullet_{\mathcal{X}_t/S}(\log(\mathcal{D}_t\cup\mathcal{Z}_t))),Q^*_{(X_t,Z_t)/S}P^{D_t}\big)
\end{aligned}
$$

is a filtered quasi-isomorphism. Hence the morphism (3.5.11) is also a filtered quasi-isomorphism. As a result, the morphism (3.5.10) is also a filtered isomorphism. We can complete the proof of 1).

2) By applying $R\bar{u}_{(X,Z)/S*}$ to (3.5.1) and using relations

$$
\bar{u}_{(X,Z)/S}\circ\pi_{\mathrm{Rcrys}}=\pi_{\mathrm{zar}}\circ\bar{u}_{(X_\bullet,Z_\bullet)/S},
$$
$$
Ru_{(X_\bullet,Z_\bullet)/S*}=R\bar{u}_{(X_\bullet,Z_\bullet)/S*}Q^*_{(X_\bullet,Z_\bullet)/S},
$$

we have the filtered isomorphism (3.5.2). □

REMARKS 3.6. — 1) The second equality in (3.5.13) is missing in [**72**]. We had to use the (analogous) equality for the calculation of

$$
\mathrm{gr}_k^{P^D}C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})\quad(k\in\mathbb{N})
$$

in the third equality in [**72**, (2.6.1.4)]; strictly speaking, the proof of [**72**, (2.6.1)] is not perfect. To obtain the (analogous) equality, we had to prove

(a) the existence of the diagram $\mathcal{D}^{(k)}_\bullet$ of schemes in [*loc. cit.*] and

(b) the functoriality of the Poincaré residue isomorphism.

To prove (a), we had to prove (4.3) below and then to use [**72**, (2.4.2)] (one may use (4.14) below instead of [**72**, (2.4.2)]). To prove (b), we had to prove (4.8) below.

2) By (4.3), (4.14) and (4.8) below and by the argument in (3.5), we obtain the second equality in (3.5.13).

3) We obtain (3.1.1) by using (3.5.13) and the argument in [**72**, (2.7.1)].

4) We obtain (3.2.4) by using (3.5.13) and the arguments in [**72**, (2.7.1), (2.7.3)].

The following is a relation between $(E_{\rm zar}^{\log, Z}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ and a filtered log de Rham-Witt complex:

THEOREM 3.7 (see [**72**, (2.12)]). — *Let the notations and the assumption be as in* (3.2) 1), 2) and 3). *Assume that S_0 is the spectrum of a perfect field κ of characteristic p and that S is the spectrum of the Witt ring $\mathcal{W}_n$ of κ of finite length $n > 0$. Then the following hold*:

1) (*A generalization of the filtered complex* $(\mathcal{W}_n\Omega_X^{\bullet}(\log D), P)$ *in* [**65**].) *There exists a filtered complex* $(\mathcal{W}_n\Omega_X^{\bullet}(\log(D\cup Z)), P^D)$ *of* $f^{-1}(\mathcal{W}_n)$*-modules such that*

$$(3.7.1)\qquad \big(\mathcal{W}_n\Omega_X^q(\log(D\cup Z)), P^D\big) \overset{\sim}{\longrightarrow} R\pi_{{\rm zar}*}\big(\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))), \{\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} P_k^{\mathcal{D}_\bullet}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)))\}_{k\in\mathbb{Z}}\big)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$ *for each* $q\in\mathbb{N}$.

2) *There exists a canonical isomorphism*

$$(3.7.2)\qquad \big(E_{\rm zar}^{\log,Z}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big) \overset{\sim}{\longrightarrow} \big(\mathcal{W}_n\Omega_X^\bullet(\log(D\cup Z)), P^D\big)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$. *This isomorphism is compatible with the transition morphisms.*

Finally, we state the base change theorem of the preweight-filtered vanishing cycle crystalline complex and the Künneth formula of it.

THEOREM 3.8 (see [**72**, (2.10.6)]). — *Let* $u\colon (S',\mathcal{I}',\gamma')\to(S,\mathcal{I},\gamma)$ *be a morphism of PD-schemes. Assume that* $\mathcal{I}$ *and* $\mathcal{I}'$ *are quasi-coherent ideal sheaves of* $\mathcal{O}_S$ *and* $\mathcal{O}_{S'}$, *respectively. Set*

$$S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I}) \quad \text{and} \quad S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}').$$

Let Y (resp. Y') be a quasi-compact smooth scheme over S_0 (resp. S'_0) (with trivial log structure). Let $f\colon (X, D \cup Z) \to Y$ be a morphism of log schemes such that $\overset{\circ}{f}\colon X \to Y$ is smooth, quasi-compact and quasi-separated and such that D and Z are transversal relative SNCD's on X over Y. Let

$$\begin{array}{ccc} (X', D' \cup Z') & \xrightarrow{g} & (X, D \cup Z) \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y \\ \big\downarrow & & \big\downarrow \\ (S', \mathcal{I}', \gamma') & \xrightarrow{u} & (S, \mathcal{I}, \gamma) \end{array}$$

be a commutative diagram of (log) schemes such that the upper rectangle is cartesian. Let $f_{(X,Z)}\colon (X, Z) \longrightarrow Y$ and $f'_{(X',Z')}\colon (X', Z') \to Y'$ be the induced morphisms. Then the base change morphism

$$(3.8.1) \qquad Lh^*_{\mathrm{crys}} Rf^{\log}_{(X,Z)\mathrm{crys}*}\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big) \longrightarrow Rf'^{\log}_{(X',Z')\mathrm{crys}*}\big(E^{\log,Z'}_{\mathrm{crys}}(\mathcal{O}_{(X',D'\cup Z')/S'}), P^{D'}\big)$$

is an isomorphism.

Theorem 3.9 (see [**72**, (2.13.3)]). — *Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p > 0$. Let K be the fraction field of $\mathcal{V}$. Assume that S is a p-adic formal $\mathcal{V}$-scheme in the sense of* [**74**, §1]. *Set $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. Assume that the morphism $\overset{\circ}{X} \to S_0$ is proper. Let h and k be two integers. Then the image*

$$(3.9.1) \qquad P^D_k R^h f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})_K := \mathrm{Im}\big(R^h f_{(X,Z)/S*}(P^D_k E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))_K \to R^h f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})_K\big)$$

prolongs to a convergent F-isocrystal on $S/\mathcal{V}$.

We denote the resulting convergent F-isocrystal in (3.9) by

$$P^D_k R^h f_*(\mathcal{O}_{(X,D\cup Z)/K}).$$

Especially we obtain $R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$.

THEOREM 3.10 (see [**72**, (2.10.14)]). — *Let* Y *and* $f_i\colon (X_i, D_i \cup Z_i) \to Y$ $(i = 1, 2)$ *be as in* (3.8). *Set* $f_3 := f_1 \times_Y f_2$,

$$X_3 := X_1 \times_Y X_2,$$
$$D_3 := (D_1 \times_Y X_2) \cup (X_1 \times_Y D_2),$$
$$Z_3 := (Z_1 \times_Y X_2) \cup (X_1 \times_Y Z_2).$$

Denote $Rf^{\log}_{i(X_i,Z_i)\mathrm{crys}*}(E^{\log,Z_i}_{\mathrm{crys}}(\mathcal{O}_{(X_i,D_i\cup Z_i)/S}), P^{D_i})$ $(i = 1, 2, 3)$ *by*

$$Rf^{\log}_{(X_i,Z_i)\mathrm{crys}*}(E^{\log,Z_i}_{\mathrm{crys}}(\mathcal{O}_{(X_i,D_i\cup Z_i)/S}), P^{D_i}).$$

Then there exists a canonical isomorphism

$$(3.10.1)\quad Rf^{\log}_{(X_1,Z_1)\mathrm{crys}*}\big(E^{\log,Z_1}_{\mathrm{crys}}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}), P^{D_1}\big) \otimes^{L}_{\mathcal{O}_{Y/S}} Rf^{\log}_{(X_2,Z_2)\mathrm{crys}*}\big(E^{\log,Z_2}_{\mathrm{crys}}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}), P^{D_2}\big) \overset{\sim}{\longrightarrow} Rf^{\log}_{(X_3,Z_3)\mathrm{crys}*}\big(E^{\log,Z_3}_{\mathrm{crys}}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}), P^{D_3}\big).$$

The isomorphism (3.10.1) *is compatible with the base change isomorphism* (3.8.1).

THEOREM 3.11 (see [**72**, (2.13.7)]). — *Let* S *and* S_0 *be as in* (3.9). *Let* $(X_i, D_i \cup Z_i)/S_0$ $(i = 1, 2, 3)$ *be as in* (3.10), *where* $Y := S_0$. *Assume that the morphism* $X_i \longrightarrow S_0$ $(i = 1, 2)$ *is proper. Then there exists the following canonical isomorphism*

$$(3.11.1)\quad \bigoplus_{i+j=h} R^i f_*(\mathcal{O}_{(X_1,D_1\cup Z_1)/K}) \otimes_{\mathcal{O}_{S/K}} R^j f_*(\mathcal{O}_{(X_2,D_2\cup Z_2)/K}) \overset{\sim}{\longrightarrow} R^h f_*(\mathcal{O}_{(X_3,D_3\cup Z_3)/K})$$

of convergent F*-isocrystals on* $S/\mathcal{V}$. *The isomorphism above is compatible with the filtrations* P^{D_i} $(i = 1, 2, 3)$.

4. Generalized descriptions of preweight-filtered vanishing cycle (restricted) crystalline and zariskian complexes

Let the notations be as in the beginning of §3. In this section we generalize (3.2). In (4.9) and (4.10) below, we give explicit descriptions of

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big) \quad \text{and} \quad \big(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big)$$

for a certain case. These descriptions give us the descriptions of

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big) \quad \text{and} \quad \big(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big)$$

for the general case without using admissible immersions ((4.15.1), (4.15.2)). The last descriptions are generalizations of (3.2.4) and (3.2.5), respectively. When S is the spectrum of the Witt ring $\mathcal{W}_n$ of a perfect field of characteristic $p>0$ of length $n>0$, we also give for a certain case an explicit description of

$$\big(\mathcal{W}_n\Omega^{\bullet}_X(\log(D\cup Z)), P^D\big).$$

This description also gives an explicit description of $(\mathcal{W}_n\Omega^{\bullet}_X(\log(D\cup Z)), P^D)$ for the general case without using admissible immersions ((4.21.1)). The last description is a generalization of (3.7.1). We need the explicit descriptions of $(E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ and $(\mathcal{W}_n\Omega^{\bullet}_X(\log(D\cup Z)), P^D)$ for the main results (6.17) in §6 and (7.6) in §7.

Let $(S,\mathcal{I},\gamma)$, S_0 and $(X,D\cup Z)/S_0$ be as in the beginning of §3. Let

$$\begin{array}{ccc} (X,D\cup Z) & \overset{\subset}{\longrightarrow} & \mathcal{P} \\ {\scriptstyle \epsilon_{(X,D\cup Z,Z)/S}}\big\downarrow & & \big\downarrow \\ (X,Z) & \overset{\subset}{\longrightarrow} & \mathcal{Q} \end{array} \tag{4.0.1}$$

be a commutative diagram of log schemes over S such that the horizontal morphisms are exact immersions into log smooth schemes over S. Assume that $\overset{\circ}{\mathcal{P}}=\overset{\circ}{\mathcal{Q}}$ and that the underlying morphism $\overset{\circ}{\mathcal{P}}\to\overset{\circ}{\mathcal{Q}}$ of the morphism $\mathcal{P}\to\mathcal{Q}$ is the identity ${\rm id}_{\overset{\circ}{\mathcal{P}}}$. Let $\mathfrak{D}$ be the log PD-envelope of the exact immersion $(X,Z)\hookrightarrow\mathcal{Q}$ over $(S,\mathcal{I},\gamma)$. Let $M_{\mathcal{P}}$ and $M_{\mathcal{Q}}$ be the log structures of $\mathcal{P}$ and $\mathcal{Q}$, respectively.

First we generalize [**72**, (2.2.17)] (= (3.2) 2), 3) for the constant simplicial case) to give the descriptions of

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big) \quad\text{and}\quad \big(E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D\big)$$

for the case (4.0.1) satisfying the equality (4.3.2) below and the two conditions immediately after (4.3.2).

For a fine log smooth scheme Y over a fine log scheme T, $\Lambda^{\bullet}_{Y/T}$ denotes the log de Rham complex of Y/T by following Friedman (see [**28**]). (In [**51**] $\Lambda^{\bullet}_{Y/T}$ has been denoted by $\omega^{\bullet}_{Y/T}$.)

For a nonnegative integer i and an integer k, set

$$P^{\mathcal{P}/\mathcal{Q}}_k\Lambda^i_{\mathcal{P}/S}=\begin{cases}0 & (k<0),\\ {\rm Im}(\Lambda^k_{\mathcal{P}/S}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^{i-k}_{\mathcal{Q}/S}\to\Lambda^i_{\mathcal{P}/S}) & (0\leq k\leq i),\\ \Lambda^i_{\mathcal{P}/S} & (k>i).\end{cases} \tag{4.0.2}$$

Then we have a filtration $P^{\mathcal{P}/\mathcal{Q}} := \{P_k^{\mathcal{P}/\mathcal{Q}}\}_{k\in\mathbb{Z}}$ on $\Lambda^i_{\mathcal{P}/S}$. Let $L_{(X,Z)/S}$ be the log linearization functor for $\mathcal{O}_{\mathcal{P}}$-modules. Set

$$P_k^{\mathcal{P}/\mathcal{Q}} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}) := L_{(X,Z)/S}(P_k^{\mathcal{P}/\mathcal{Q}}\Lambda^\bullet_{\mathcal{P}/S}) \quad (k \in \mathbb{Z}).$$

For a commutative monoid R with unit element, denote by $\mathrm{Spec}^{\log}(\mathbb{Z}[R])$ the log scheme whose underlying scheme is $\mathrm{Spec}(\mathbb{Z}[R])$ and whose log structure is associated to the natural morphism $R \longrightarrow \mathbb{Z}[R]$ of monoids.

In [**72**] we have proved the following:

Proposition 4.1 (see [**72**, (2.1.5)]). — *Let $T_0 \hookrightarrow T$ be a closed immersion of fine log schemes. Let Z_0 (resp. Y) be a log smooth scheme over T_0 (resp. T), which can be considered as the log scheme over T. Let $\iota\colon Z_0 \hookrightarrow Y$ be an exact closed immersion over T. Let $\mathbb{A}^n_T$ ($n \in \mathbb{N}$) be a log scheme whose underlying scheme is the scheme $\mathbb{A}^n_{\overset{\circ}{T}}$ and whose log structure is the pull-back of that of T by the natural projection $\mathbb{A}^n_{\overset{\circ}{T}} \to \overset{\circ}{T}$. Let z be a point of $\overset{\circ}{Z}_0$. Assume that there exists a chart*

$$(Q \to M_T, R \to M_{Z_0}, Q \xrightarrow{\rho} R)$$

of $Z_0 \to T$ on a neighborhood of z such that ρ is injective, such that $\mathrm{Coker}(\rho^{\mathrm{gp}})$ is torsion free and such that the natural homomorphism $\mathcal{O}_{Z_0,z} \otimes_{\mathbb{Z}} (R^{\mathrm{gp}}/Q^{\mathrm{gp}}) \longrightarrow \Lambda^1_{Z_0/T_0,z}$ is an isomorphism. Then, locally around z, there exist a nonnegative integer c and the following cartesian diagrams:

(4.1.1)
$$\begin{array}{ccc} Z_0 & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ T_0 \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[Q])} \mathrm{Spec}^{\log}(\mathbb{Z}[R]) & \overset{\subset}{\longrightarrow} & T \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[Q])} \mathrm{Spec}^{\log}(\mathbb{Z}[R]) \end{array}$$

$$\begin{array}{cc} \longrightarrow & Y \\ & \downarrow \\ \overset{\subset}{\longrightarrow} & T \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[Q])} \mathrm{Spec}^{\log}(\mathbb{Z}[R]) \times_T \mathbb{A}^c_T, \end{array}$$

where the vertical morphisms are strict and log étale and the lower second horizontal morphism is the base change of the zero section $T \hookrightarrow \mathbb{A}^c_T$ and $Y' := Y \times_{\mathbb{A}^c_T} T$.

Though the following lemma immediately follows from [**72**, (2.2.17)], it is a key lemma for (4.9) and (4.10) below.

LEMMA 4.2. — 1) *The following natural morphism is injective*:

$$\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P_k^{\mathcal{P}/\mathcal{Q}} \Lambda^\bullet_{\mathcal{P}/S} \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/S}. \tag{4.2.1}$$

2) *The following natural morphism is injective*:

$$Q^*_{(X,Z)/S} P_k^{\mathcal{P}/\mathcal{Q}} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}) \longrightarrow Q^*_{(X,Z)/S} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}). \tag{4.2.2}$$

Proof. — 1): The question is local. Let x be a point of $\overset{\circ}{\mathfrak{D}}$. Because the points of $\overset{\circ}{\mathfrak{D}}$ are the points of X, x is also a point of X. Identify the image of x in $\overset{\circ}{\mathcal{P}}$ with x. Then $M_{\mathcal{P},x}/\mathcal{O}^*_{\mathcal{P},x} = M(D \cup Z)_x/\mathcal{O}^*_{X,x} \simeq \mathbb{N}^s$ for some $s \in \mathbb{N}$. We have a chart $\mathbb{N}^{s'} \to M_{\mathcal{P}}$ around x for some $s' \geq s$ such that the induced morphism $\mathcal{O}_{X,x} \otimes_{\mathbb{Z}} \mathbb{Z}^{s'} \to \Omega^1_{X/S_0}(\log(D \cup Z))$ by the composite morphism $\mathbb{N}^{s'} \to M_{\mathcal{P}} \to M(D \cup Z)$ is an isomorphism. In the notation (4.1), we can take $Q = \{0\}$ and $R = \mathbb{N}^{s'}$ for the exact closed immersion $(X, D \cup Z) \hookrightarrow \mathcal{P}$ (see also the proof of [**72**, (2.1.4)]). In particular, we may assume that $\mathcal{X} := \overset{\circ}{\mathcal{P}}$ is a smooth scheme over S. Set $e_k := (0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0) \in \mathbb{N}^{s'}$ $(1 \leq k \leq s')$. Consider the composite morphism

$$g\colon \mathbb{N}^{s'} \longrightarrow M(D \cup Z) \longrightarrow M(D \cup Z)_x/\mathcal{O}^*_{X,x} \simeq \mathbb{N}^s.$$

By a simple calculation, we see that there exist $k_1, \ldots, k_s$ such that $g(e_{k_\ell}) = (0, \ldots, 0, \overset{\ell}{1}, 0, \ldots, 0)$ $(1 \leq \ell \leq s)$. The images of $e_{k_1}, \ldots, e_{k_s}$ in $\mathcal{O}_{\mathcal{X}}$ by the composite morphism $\mathbb{N}^{s'} \to \mathcal{M}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{X}}$ give us transversal relative SNCD's $\mathcal{D}$ and $\mathcal{Z}$ on $\mathcal{X}/S$ such that $\mathcal{P} = (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ and $\mathcal{Q} = (\mathcal{X}, \mathcal{Z})$ locally around x. In this local case,

$$\Lambda^\bullet_{\mathcal{P}/S} = \Omega^\bullet_{\mathcal{X}/S}\big(\log(\mathcal{D} \cup \mathcal{Z})\big) \quad \text{and} \quad P_k^{\mathcal{P}/\mathcal{Q}} \Lambda^\bullet_{\mathcal{P}/S} = P_k^{\mathcal{D}} \Omega^\bullet_{\mathcal{X}/S}\big(\log(\mathcal{D} \cup \mathcal{Z})\big).$$

In this case, 1) is nothing but [**72**, (2.2.17), (1)].

2): this follows immediately from 1) as in [**72**, (2.2.17), (2)]. □

We denote the filtered complexes

$$\big(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/S}, \{\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P_k^{\mathcal{P}/\mathcal{Q}} \Lambda^\bullet_{\mathcal{P}/S}\}_{k \in \mathbb{Z}}\big)$$

$$(Q^*_{(X,Z)/S} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}), \{Q^*_{(X,Z)/S} P_k^{\mathcal{P}/\mathcal{Q}} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S})\}_{k \in \mathbb{Z}})$$

by $(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/S}, P^{\mathcal{P}/\mathcal{Q}})$ and $(Q^*_{(X,Z)/S} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}), Q^*_{(X,Z)/S} P^{\mathcal{P}/\mathcal{Q}})$, respectively.

When $Z = \varnothing$, we denote

$$(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/S}, P^{\mathcal{P}/\mathcal{Q}}) \quad \text{and} \quad \big(Q^*_{(X,Z)/S} L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}), Q^*_{(X,Z)/S} P^{\mathcal{P}/\mathcal{Q}}\big)$$

by $(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^{\bullet}_{\mathcal{P}/S}, P)$ and $(Q^*_{X/S} L_{X/S}(\Lambda^{\bullet}_{\mathcal{P}/S}), Q^*_{X/S} P)$, respectively.

Next we give a generalization of the classical Poincaré residue isomorphism. To give it, we have to generalize $D^{(k)}$ and $\varpi^{(k)}_{\mathrm{zar}}(D/S_0)$ in §3. For a finitely generated monoid Q and a set $\{q_1, \ldots, q_n\}$ ($n \in \mathbb{Z}_{\geq 1}$) of generators of Q, we say that $\{q_1, \ldots, q_n\}$ is *minimal* if n is minimal. Let $Y = (\overset{\circ}{Y}, M_Y)$ be an fs log scheme. Let y be a point of $\overset{\circ}{Y}$. Let $m_{1,y}, \ldots, m_{r,y}$ be local sections of M_Y around y whose images in $M_{Y,y}/\mathcal{O}^*_{Y,y}$ form a minimal set of generators of $M_{Y,y}/\mathcal{O}^*_{Y,y}$. Let $D(M_Y)_i$ ($1 \leq i \leq r$) be the local closed subscheme of $\overset{\circ}{Y}$ defined by the ideal sheaf generated by the image of $m_{i,y}$ in $\mathcal{O}_Y$. For a positive integer k, let $D^{(k)}(M_Y)$ be the disjoint union of the k-fold intersections of different $D(M_Y)_i$'s. Assume that

$$M_{Y,y}/\mathcal{O}^*_{Y,y} \simeq \mathbb{N}^r \tag{4.2.3}$$

for any point y of $\overset{\circ}{Y}$ and for some $r \in \mathbb{N}$ depending on y. Note that $\mathrm{Aut}(\mathbb{N}^r) \simeq \mathfrak{S}_r$. Indeed, set $e_i := (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \in \mathbb{N}^r$. For the monoid $\mathbb{N}^r$ ($r \in \mathbb{Z}_{\geq 1}$), observe that $\{e_i\}_{i=1}^r$ is the unique minimal set of generators of $\mathbb{N}^r$. Indeed, we can prove this by induction on r as follows. Let $\{f_i\}_{i=1}^s$ ($s \in \mathbb{Z}_{\geq 1}$) be a minimal set of generators of $\mathbb{N}^r$. Then $s = r$. We may assume that the r-th component of f_r is nonzero. Let $\overline{f}_i$ be the image of f_i by the projection $\mathbb{N}^r \ni (a_1, \ldots, a_r) \mapsto (a_1, \ldots, a_{r-1}) \in \mathbb{N}^{r-1}$. Then, by the inductive assumption, $\{\overline{f}_i\}_{i=1}^{r-1} = \{\overline{e}_i\}_{i=1}^{r-1}$. By renumbering i ($1 \leq i \leq r-1$), we may assume that $\overline{f}_i = \overline{e}_i$ ($1 \leq i \leq r-1$). If the r-th component of some f_i ($1 \leq i \leq r-1$) is nonzero, then e_i does not belong to the submonoid generated by $f_1, \ldots, f_r$. Hence $f_i = e_i$ ($1 \leq i \leq r-1$). By the analogous argument, we immediately see that $f_r = e_r$. Hence $\mathrm{Aut}(\mathbb{N}^r) \simeq \mathfrak{S}_r$. Consequently $\mathrm{Aut}(M_{Y,y}/\mathcal{O}^*_{Y,y}) \simeq \mathfrak{S}_r$. Hence the scheme $D^{(k)}(M_Y)$ is independent of the choice of $m_{1,y}, \ldots, m_{r,y}$ and it is globalized. We denote this globalized scheme by the same symbol $D^{(k)}(M_Y)$. Set $D^{(0)}(M_Y) := \overset{\circ}{Y}$. Let $c^{(k)} \colon D^{(k)}(M_Y) \to \overset{\circ}{Y}$ be the natural morphism.

As in [**24**, (3.1.4)] and [**72**, (2.2.18)], we have an orientation sheaf $\varpi^{(k)}_{\mathrm{zar}}(D(M_Y))$ ($k \in \mathbb{N}$) in $\widetilde{\overset{\circ}{Y}}{}^{(k)}_{\mathrm{zar}}$ associated to the set $D(M_Y)_i$'s. We have the equality

$$c^{(k)}_* \varpi^{(k)}_{\mathrm{zar}}(D(M_Y)) = \bigwedge^k (M^{\mathrm{gp}}_Y/\mathcal{O}^*_Y) \tag{4.2.4}$$

as sheaves of abelian groups on $\overset{\circ}{Y}_{\mathrm{zar}}$.

Proposition 4.3. — *Let $g\colon Y \to Y'$ be a morphism of fs log schemes satisfying the condition* (4.2.3). *Assume that, for each point $y \in \overset{\circ}{Y}$ and for each member m of the minimal generators of $M_{Y,y}/\mathcal{O}^*_{Y,y}$, there exists a unique member m' of the minimal generators of $M_{Y',\overset{\circ}{g}(y)}/\mathcal{O}^*_{Y',\overset{\circ}{g}(y)}$ such that $g^*(m') \in m^{\mathbb{Z}_{>0}}$. Then there exists a canonical morphism $g^{(k)}\colon D^{(k)}(M_Y) \longrightarrow D^{(k)}(M_{Y'})$ fitting into the following commutative diagram of schemes*:

(4.3.1)
$$\begin{array}{ccc} D^{(k)}(M_Y) & \xrightarrow{g^{(k)}} & D^{(k)}(M_{Y'}) \\ \big\downarrow & & \big\downarrow \\ \overset{\circ}{Y} & \xrightarrow{\overset{\circ}{g}} & \overset{\circ}{Y'}. \end{array}$$

Proof. — (The proof is a variant of the argument in [**77**, p. 614].) First we consider the local case. Let $\{m_{i,y}\}_{i=1}^{r}$ and $\{D(M_Y)_i\}_{i=1}^{r}$ be as above. Set $y' := \overset{\circ}{g}(y)$. Let $\{m_{i',y'}\}_{i'=1}^{r'}$ and $\{D(M_{Y'})_{i'}\}_{i'=1}^{r'}$ be the similar objects for Y' around $\overset{\circ}{g}(y)$. We may assume that $g^*(m'_{i,y'}) \in m_{i,y}^{\mathbb{Z}_{>0}}$ $(1 \leq i \leq r)$. This means that g induces a natural local morphism $D(M_Y)_i \to D(M_{Y'})_i$ $(1 \leq i \leq r)$. Let $D^{(k)}(M_Y; g)$ be the disjoint union of all k-fold intersections of $D(M_{Y'})_1, \dots, D(M_{Y'})_r$. Then we have a natural morphism $D^{(k)}(M_Y) \to D^{(k)}(M_Y; g)$. By composing this with the natural inclusion morphism $D^{(k)}(M_Y; g) \to D^{(k)}(M_{Y'})$, we have a morphism $D^{(k)}(M_Y) \to D^{(k)}(M_{Y'})$. This is a desired local morphism. This local morphism is compatible with localization. Hence we have a desired global morphism $g^{(k)}\colon D^{(k)}(M_Y) \to D^{(k)}(M_{Y'})$. □

Let (X, D) be a smooth scheme with an SNCD over a scheme S_0. If $Y = (X, D)$, then $D^{(k)}(M_Y) = D^{(k)}$ $(k \in \mathbb{N})$ and $\varpi^{(k)}_{\mathrm{zar}}(D(M_Y)) = \varpi^{(k)}_{\mathrm{zar}}(D/S_0)$.

Let M_1 and M_2 be two fine log structures on $\overset{\circ}{Y}$ such that $M_Y = M_1 \oplus_{\mathcal{O}^*_Y} M_2$. Assume that the condition (4.2.3) is satisfied for M_1. Let $m_{1,y}, \dots, m_{r,y}$ be local sections of M_1 around y whose images in $M_{1,y}/\mathcal{O}^*_{Y,y}$ form a minimal set of generators of $M_{1,y}/\mathcal{O}^*_{Y,y}$. Set $Y_i := (\overset{\circ}{Y}, M_i)$ $(i = 1, 2)$. Endow $D^{(k)}(M_1)$ with the pull-back of the log structure of M_2 and denote the resulting log scheme by $(D^{(k)}(M_1), M_2)$ by abuse of notation. Let

$$b^{(k)}\colon (D^{(k)}(M_1), M_2) \longrightarrow (\overset{\circ}{Y}, M_2)$$

be the natural morphism. By abuse of notation, we denote by the same symbol $b^{(k)}$ the underlying morphism $D^{(k)}(M_1) \to \overset{\circ}{Y}$.

Now we come back to the situation (4.0.1). We assume that there exists an fs sub log structure $\mathcal{M}$ of $M_{\mathcal{P}}$ such that

$$M_{\mathcal{P}} = \mathcal{M} \oplus_{\mathcal{O}_{\mathcal{P}}^*} M_{\mathcal{Q}}, \tag{4.3.2}$$

such that the pull-back of $\mathcal{M}$ to X is equal to $M(D)$ and such that the condition (4.2.3) is satisfied for $(\overset{\circ}{\mathcal{P}}, \mathcal{M})$.

Let $b^{(k)} \colon (D^{(k)}(\mathcal{M}), M_{\mathcal{Q}}) \to \mathcal{Q}$ be the natural morphism. By abuse of notation, we also denote the underlying morphism $D^{(k)}(\mathcal{M}) \to \overset{\circ}{\mathcal{Q}}$ by $b^{(k)}$.

Proposition 4.4. — 1) *Let $\mathcal{M}'$ be another fs sub log structure of $M_{\mathcal{P}}$ satisfying the equality* (4.3.2)*, such that the pull-back of $\mathcal{M}'$ to X is equal to $M(D)$ and such that the condition* (4.2.3) *is satisfied for $(\overset{\circ}{\mathcal{P}}, \mathcal{M}')$. Let $b'^{(k)} \colon (D^{(k)}(\mathcal{M}'), M_{\mathcal{Q}}) \to \mathcal{Q}$ be the natural morphism. Then there exists a canonical isomorphism*

$$\begin{aligned} &\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b_*^{(k)} \big(\Lambda^{\bullet}_{(D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big) \\ &\qquad \xrightarrow{\sim} \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b'^{(k)}_* \big(\Lambda^{\bullet}_{(D^{(k)}(\mathcal{M}'), M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}'))\big). \end{aligned}$$

2) *Identify the points of $\overset{\circ}{\mathfrak{D}}$ with those of X. Identify also the images of the points of X in $\overset{\circ}{\mathcal{P}}$ with the points of X. Let x be a point of $\overset{\circ}{\mathfrak{D}}$. For any different $i_1, \dots, i_k$ $(1 \leq i_1, \dots, i_k \leq r)$, denote by $(D(\mathcal{M})_{i_1 \cdots i_k}, M_{\mathcal{Q}})$ the local log closed subscheme of $\mathcal{Q}$ whose underlying scheme is $D(\mathcal{M})_{i_1} \cap \cdots \cap D(\mathcal{M})_{i_k}$ and whose log structure is the pull-back of $\mathcal{M}_{\mathcal{Q}}$. Denote the natural local exact closed immersion $(D(\mathcal{M})_{i_1 \cdots i_k}, M_{\mathcal{Q}}) \hookrightarrow \mathcal{Q}$ by $b_{i_1 \cdots i_k}$. We denote by $b_{i_1 \cdots i_k}$ the underlying morphism $D(\mathcal{M})_{i_1 \cdots i_k} \hookrightarrow \overset{\circ}{\mathcal{Q}}$. Then, for a nonnegative integer k, the morphism*

$$\begin{aligned} &\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P_k^{\mathcal{P}/\mathcal{Q}} \Lambda^{\bullet}_{\mathcal{P}/S} \\ &\qquad \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b_*^{(k)} \big(\Lambda^{\bullet - k}_{(D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big), \\ &f \otimes \omega d\log m_{i_1, x} \cdots d\log m_{i_k, x} \qquad (f \in \mathcal{O}_{\mathfrak{D}}, \omega \in P_0^{\mathcal{P}/\mathcal{Q}} \Lambda^{\bullet}_{\mathcal{P}/S}) \\ &\qquad \longmapsto f \otimes b^*_{i_1 \cdots i_k}(\omega)(\text{orientation } (i_1 \cdots i_k)\big) \end{aligned} \tag{4.4.1}$$

(cf. [**25**, (3.1.5)]) *induces the following "Poincaré residue isomorphism"*

$$\begin{aligned} &\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathrm{gr}_k^{P^{\mathcal{P}/\mathcal{Q}}} \Lambda^{\bullet}_{\mathcal{P}/S} \\ &\qquad \xrightarrow{\sim} \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b_*^{(k)} \big(\Lambda^{\bullet - k}_{(D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big). \end{aligned} \tag{4.4.2}$$

Proof. — 1): Since the points of $\overset{\circ}{\mathfrak{D}}$ are equal to the points of X, we have only to prove that there exists a log open subscheme $\mathcal{U}$ of $\mathcal{P}$ containing X such that $\mathcal{M}|_{\mathcal{U}} = \mathcal{M}'|_{\mathcal{U}}$. Let x be any point of X. Because the pull-backs of $\mathcal{M}$ and $\mathcal{M}'$ to X are equal to $M(D)$, $\mathcal{M}_x/\mathcal{O}^*_{\mathcal{P},x} = M(D)_x/\mathcal{O}^*_{X,x} = \mathcal{M}'_x/\mathcal{O}^*_{\mathcal{P},x}$. Hence $\mathcal{M}_x = \mathcal{M}'_x$. This means that there exists a log open subscheme $\mathcal{U}(x)$ containing x such that $\mathcal{M}|_{\mathcal{U}(x)} = \mathcal{M}'|_{\mathcal{U}(x)}$. Set $\mathcal{U} := \bigcup_{x\in X}\mathcal{U}(x)$, which is a desired log open subscheme of $\mathcal{P}$.

2): As in the classical case, we can easily check that the morphism (4.4.1) is well-defined. Let x be a point of $\overset{\circ}{\mathfrak{D}}$. Because the question is local, we may assume that there exist a smooth scheme $\mathcal{X}$ over S and transversal relative SNCD's $\mathcal{D}$ and $\mathcal{Z}$ on $\mathcal{X}/S$ such that $\mathcal{P} = (\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ and $\mathcal{Q} = (\mathcal{X}, \mathcal{Z})$ by (4.1) as in the proof of (4.2). Then the isomorphism (4.4.2) is the Poincaré residue isomorphism in [**72**, (2.2.21.1)]. ☐

By (4.2) and (4.4), we have the following lemma:

Lemma 4.5 (see cf. [**72**, (2.2.21) (1), (2)]). — 1) *For a nonnegative integer k, the following sequence is exact*:

(4.5.1)
$$\begin{aligned}0 \to \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_{k-1}\Lambda^{\bullet}_{\mathcal{P}/S} &\\ \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_{k}\Lambda^{\bullet}_{\mathcal{P}/S} &\\ \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b^{(k)}_{*}\big(\Lambda^{\bullet-k}_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big) &\to 0.\end{aligned}$$

2) *For a nonnegative integer k, the following sequence is exact*:

(4.5.2)
$$\begin{aligned}0 \to Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}_{k-1}L_{(X,Z)/S}(\Lambda^{\bullet}_{\mathcal{P}/S}) &\\ \longrightarrow Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}_{k}L_{(X,Z)/S}(\Lambda^{\bullet}_{\mathcal{P}/S}) &\\ \longrightarrow Q^*_{(X,Z)/S}L_{(X,Z)/S}(b^{(k)}_{*}\big(\Lambda^{\bullet-k}_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big)) &\to 0.\end{aligned}$$

Lemma 4.6 (cf. [**72**, (2.2.16)]). — *For a nonnegative integer k, the following hold*:

1) *There exists a natural exact immersion $(D^{(k)}, Z|_{D^{(k)}}) \hookrightarrow (D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})$ over S.*

2) *The log scheme $\mathfrak{D} \times_{\mathcal{Q}} (D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})$ with the PD-structure induced by the PD-structure of $\mathfrak{D}$ is the log PD-envelope of the exact immersion in* 1) *over $(S, \mathcal{I}, \gamma)$.*

Proof. — 1): Because the pull-back of $\mathcal{M}$ (resp. $M_{\mathcal{Q}}$) to X is equal to $M(D)$ (resp. $M(Z)$), 1) immediately follows from the definitions of $(D^{(k)}(\mathcal{M}), M_{\mathcal{Q}})$ and $(D^{(k)}, Z|_{D^{(k)}})$.

2): By the universality of the log PD-envelope, the question is local. Let x be a point of X. By (4.1) and the proof of (4.2), 1), we may assume that there exists a smooth scheme $\mathcal{X}$ over S with transversal relative SNCD's $\mathcal{D}$ and $\mathcal{Z}$ on $\mathcal{X}/S$ such that $\mathcal{P} = (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ and $\mathcal{Q} = (\mathcal{X}, \mathcal{Z})$ around x. In this case, 2) is equivalent to [**72**, (2.2.16), (2)], which has already been proved in [*loc. cit.*]. □

Corollary 4.7 (cf. [**72**, (2.2.21.2)]). — *In* $\mathrm{D}^+(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$ *there exists the following isomorphism*:

$$\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}} Q^*_{(X,Z)/S} L_{(X,Z)/S}(\Lambda^{\bullet}_{\mathcal{P}/S}) \xrightarrow{\sim} Q^*_{(X,Z)/S} a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}, Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}. \tag{4.7.1}$$

Proof. — By [**72**, (2.2.12)] and (4.6), 2),

$$\begin{aligned} L_{(X,Z)/S}\big(b^{(k)}_*(\Lambda^{\bullet}_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M})))\big) \\ = a^{(k)\log}_{\mathrm{crys}*} L_{(D^{(k)}, Z|_{D^{(k)}})/S}\big(\Lambda^{\bullet}_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M}))\big). \end{aligned} \tag{4.7.2}$$

Because the pull-back of $\mathcal{M}$ to X is equal to $M(D)$, the right hand side is equal to $a^{(k)\log}_{\mathrm{crys}*}(L_{(D^{(k)}, Z|_{D^{(k)}})/S}(\Lambda^{\bullet}_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S}) \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))$ by [**72**, (2.2.20)]. By the log Poincaré lemma (see [**72**, (2.2.7)]), the last complex is isomorphic to

$$a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}, Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))$$

in $\mathrm{D}^+(\mathcal{O}_{(X,Z)/S})$. Hence (4.7) follows from (4.5), 2). □

Proposition 4.8. — *Let* $u\colon (S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ *be a morphism of PD-schemes such that* $\mathcal{I}$ *and* $\mathcal{I}'$ *are quasi-coherent ideal sheaves of* $\mathcal{O}_S$ *and* $\mathcal{O}_{S'}$, *respectively. Let* S_0 *and* S'_0 *be as in* (3.8). *Let*

$$\begin{array}{ccc} (X', D' \cup Z') & \xrightarrow{\subset} & \mathcal{P}' \\ {\scriptstyle \epsilon_{(X', D' \cup Z', Z')/S'}}\big\downarrow & & \big\downarrow \\ (X', Z') & \xrightarrow{\subset} & \mathcal{Q}' \end{array} \tag{4.8.1}$$

be a similar diagram to (4.0.1) *over* S'. *Assume that there exists an fs sub log structure* $\mathcal{M}'$ *of* $M_{\mathcal{P}'}$ *such that*

$$M_{\mathcal{P}'} = \mathcal{M}' \oplus_{\mathcal{O}^*_{\mathcal{P}'}} M_{\mathcal{Q}'}, \tag{4.8.2}$$

such that the pull-back of $\mathcal{M}'$ *to* X' *is equal to* $M(D')$ *and such that the condition* (4.2.3) *is satisfied for* $(\overset{\circ}{\mathcal{P}}{}', \mathcal{M}')$. *Let* $a'^{(k)}\colon (D'^{(k)}, Z|_{D'^{(k)}}) \to (X', Z')$ $(k \in \mathbb{N})$ *be the natural morphism of log schemes over* S'_0. *Assume that there exists a morphism from the commutative diagram* (4.0.1) *to the commutative diagram* (4.8.1). *Let* $g\colon (\overset{\circ}{\mathcal{P}}, \mathcal{M}) \longrightarrow (\overset{\circ}{\mathcal{P}}{}', \mathcal{M}')$ *and* $g_{(X,Z)}\colon (X, Z) \to (X', Z')$ *be the induced morphisms. Assume that, for each point* $x \in \overset{\circ}{\mathcal{P}}$ *and for each member* m *of the minimal generators of* $\mathcal{M}_x/\mathcal{O}^*_{\mathcal{P},x}$, *there exists a unique member* m' *of the minimal generators of* $\mathcal{M}'_{\overset{\circ}{g}(x)}/\mathcal{O}^*_{\mathcal{P}',\overset{\circ}{g}(x)}$ *such that* $g^*(m') = m$ *and such that the image of the other minimal generators of* $\mathcal{M}'_{\overset{\circ}{g}(x)}/\mathcal{O}^*_{\mathcal{P}',\overset{\circ}{g}(x)}$ *by* g^* *are the trivial element of* $\mathcal{M}_x/\mathcal{O}^*_{\mathcal{P},x}$. *Let* $b'^{(k)}\colon D^{(k)}(\mathcal{M}') \to \overset{\circ}{\mathcal{P}}{}'$ *be the similar morphism to* $b^{(k)}$. *Then the following hold*:

1) *The following diagram is commutative*:

(4.8.3)

$$\begin{array}{ccc} g_*(\mathrm{Res}^{\mathcal{P}/\mathcal{Q}}) : g_*(\mathrm{gr}_k^{P^{\mathcal{P}/\mathcal{Q}}}\Lambda^\bullet_{\mathcal{P}/S}) & \xrightarrow{\sim} & g_*b^{(k)}_*(\Lambda^\bullet_{D^{(k)}(\mathcal{M})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D^{(k)}(\mathcal{M})))\{-k\}) \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathrm{Res}^{\mathcal{P}'/\mathcal{Q}'} : \mathrm{gr}_k^{P^{\mathcal{P}'/\mathcal{Q}'}}\Lambda^\bullet_{\mathcal{P}'/S'} & \xrightarrow{\sim} & b'^{(k)}_*(\Lambda^\bullet_{D^{(k)}(\mathcal{M}')/S'} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D^{(k)}(\mathcal{M}')))\{-k\}). \end{array}$$

2) *The following diagram is commutative*:

$$\begin{array}{cc} g^{\log}_{(X,Z)\mathrm{crys}*}(\mathrm{Res}) : g^{\log}_{(X,Z)\mathrm{crys}*}L_{(X,Z)/S}(\mathrm{gr}_k^{P^{\mathcal{P}/\mathcal{Q}}}\Lambda^\bullet_{\mathcal{P}/S}) & \xrightarrow{\sim} \\ \uparrow & \\ \mathrm{Res} : L_{(X',Z')/S'}(\mathrm{gr}_k^{P^{\mathcal{P}'/\mathcal{Q}'}}\Lambda^\bullet_{\mathcal{P}'/S'}) & \xrightarrow{\sim} \end{array} \tag{4.8.4}$$

$$\begin{array}{c} g^{\log}_{(X,Z)\mathrm{crys}*}a^{(k)\log}_{\mathrm{crys}*}(L_{(D^{(k)},Z|_{D^{(k)}})/S}(\Lambda^\bullet_{D^{(k)}(\mathcal{M})/S}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D/S;Z))\{-k\} \\ \uparrow \\ a'^{(k)\log}_{\mathrm{crys}*}(L_{(D'^{(k)},Z'|_{D'^{(k)}})/S}(\Lambda^\bullet_{D^{(k)}(\mathcal{M}')/S'}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D'/S';Z'))\{-k\} \end{array}$$

Proof. — 1): We immediately obtain 1) from the condition $g^*(m') = m$ and by the definition of the morphism $g^{(k)}\colon D^{(k)}(\mathcal{M}) \to D^{(k)}(\mathcal{M}')$ in (4.3).

2): This follows from 1) and (4.7.2). □

The following theorem and corollary are main results in this section.

THEOREM 4.9. — *There exists the following isomorphism*

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} \big(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}\big). \tag{4.9.1}$$

Proof. — By (3.4.3) we have a canonical isomorphism

$$\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} \big(L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),\tau\big).$$

Hence we have an isomorphism

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} Q^*_{(X,Z)/S}\big(L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),\tau\big).$$

Because we have a natural morphism

$$Q^*_{(X,Z)/S}\big(L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),\tau\big) \longrightarrow \big(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}\big),$$

we have the following composite morphism

$$Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \longrightarrow \big(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}),Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}\big). \tag{4.9.2}$$

By taking the graded parts of both sides of (4.9.2), we have the following commutative diagram by the p-adic purity (3.1.1) (see (3.6), 3)) and (4.7.1) for a nonnegative integer k:

$$\begin{array}{ccc}
\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D} Q^*_{(X,Z)/S}E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}) & \longrightarrow & \mathrm{gr}_k^{Q^*_{(X,Z)/S}P^{\mathcal{P}/\mathcal{Q}}} Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Lambda^\bullet_{\mathcal{P}/S}) \\
\| & & \| \\
Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\rm crys}(D/S;Z)) & = & Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\rm crys}(D/S;Z)).
\end{array}$$

We can complete the proof of (4.9). □

COROLLARY 4.10. — *There exists the following isomorphism*

$$(E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) \overset{\sim}{\longrightarrow} (\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^\bullet_{\mathcal{P}/S},P^{\mathcal{P}/\mathcal{Q}}). \tag{4.10.1}$$

Proof. — The Corollary immediately follows from (4.9). □

Lemma 4.11. — *Let Y be a fine log (formal) scheme. Assume that $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)\simeq\mathbb{N}^r$ for some $r\in\mathbb{N}$ and that, for any point y of $\overset{\circ}{Y}$, $M_{Y,y}/\mathcal{O}_{Y,y}^*\simeq\mathbb{N}^{r_y}$ for some $r_y\leq r$. Assume that the natural morphism $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)\to M_{Y,y}/\mathcal{O}_{Y,y}^*$ is surjective. Let $\iota\colon Y\hookrightarrow\mathcal{R}$ be a closed immersion of fine log (formal) schemes. Assume that $\mathcal{R}$ has a global chart $R\to M_{\mathcal{R}}$. Let R' be the inverse image of $\Gamma(Y,M_Y)$ by the composite morphism $R^{\mathrm{gp}}\longrightarrow\Gamma(\mathcal{R},M_{\mathcal{R}})^{\mathrm{gp}}\to\Gamma(Y,M_Y)^{\mathrm{gp}}$. Set*

$$\mathcal{R}':=\mathcal{R}\times_{\mathrm{Spec}^{\log}(\mathbb{Z}[R])}\mathrm{Spec}^{\log}(\mathbb{Z}[R']).$$

If the natural morphism $R'\to M_{\mathcal{R}'}$ induces a surjection

$$R'\longrightarrow\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*),$$

then $\mathcal{R}'$ is fine and the induced closed immersion $\iota'\colon Y\hookrightarrow\mathcal{R}'$ is exact. In particular, if the chart $R\to M_{\mathcal{R}}$ induces a surjection $R\to\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)$, then the conclusions above hold.

Proof. — (Cf. the proof of [**51**, (4.10)].) By using the natural morphism $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)\to M_{Y,y}/\mathcal{O}_{Y,y}^*$ and the isomorphisms

$$\mathbb{N}^r\overset{\sim}{\longrightarrow}\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)\quad\text{and}\quad M_{Y,y}/\mathcal{O}_{Y,y}^*\overset{\sim}{\longrightarrow}\mathbb{N}^{r_y},$$

we have a surjection $\lambda\colon\mathbb{N}^r\to\mathbb{N}^{r_y}$. Set $e_k:=(0,\dots,0,\overset{k}{1},0,\dots,0)\in\mathbb{N}^r$ $(1\leq k\leq r)$. By a simple calculation, we see that there exist $k_1,\dots,k_{r_y}$ such that $\lambda(e_{k_\ell})=(0,\dots,0,\overset{\ell}{1},0,\dots,0)$ $(1\leq\ell\leq r_y)$. Let $\{e_{k_{r_y+1}},\dots,e_{k_r}\}$ be the complement of $\{e_{k_1},\dots,e_{k_{r_y}}\}$ in $\{e_k\}_{k=1}^r$. Let $\lambda^{\mathrm{gp}}\colon\mathbb{Z}^r\longrightarrow\mathbb{Z}^{r_y}$ be the induced morphism by λ. We denote the usual addition $+$ (resp. $\pm$) in $\mathbb{N}^r$ (resp. $\mathbb{Z}^r$) by the multiplicative notation. Then, in $\mathbb{Z}^r$, there exists an element u_{k_m} $(r_y<m\leq r)$ of $\mathrm{Ker}(\lambda^{\mathrm{gp}})$ such that $e_{k_m}=u_{k_m}e_{k_1}^{\alpha_1}\cdots e_{k_{r_y}}^{\alpha_{r_y}}$ for some $\alpha_1,\dots,\alpha_{r_y}\in\mathbb{N}$. Let P be a free monoid generated by $e_{k_1},\dots,e_{k_{r_y}}$ and

$$u_{k_{r_y+1}},\dots,u_{k_r}\colon P=\prod_{\ell=1}^{r_y}e_{k_\ell}^{\mathbb{N}}\times\prod_{m=r_y+1}^{r}u_{k_m}^{\mathbb{N}}.$$

By mapping $\mathbb{N}^r\ni e_{k_\ell}\mapsto e_{k_\ell}\in P$ and $\mathbb{N}^r\ni e_{k_m}\mapsto(e_1^{\alpha_1}\cdots e_{k_{r_y}}^{\alpha_{r_y}},u_{k_m})\in P$, we have the commutative diagram

$$\begin{array}{ccc}\mathbb{N}^r & \overset{\sim}{\longrightarrow} & P\simeq\mathbb{N}^{r_y}\times\mathbb{N}^{r-r_y}\\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\text{1st.proj}}\\ \mathbb{N}^{r_y} & = & \mathbb{N}^{r_y}.\end{array}$$

Hence we have the following commutative diagram

$$\begin{array}{ccc} \Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*) & \xrightarrow{\sim} & \mathbb{N}^r \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{1st.proj}} \\ M_{Y,y}/\mathcal{O}_{Y,y}^* & \xrightarrow{\sim} & \mathbb{N}^{r_y}. \end{array} \tag{4.11.1}$$

Set $K := \mathrm{Ker}(R^{\mathrm{gp}} \to \Gamma(Y,M_Y)^{\mathrm{gp}}/\Gamma(Y,\mathcal{O}_Y^*))$. Since K is finitely generated as an abelian group, K is finitely generated as a monoid. Let $f_1,\dots,f_r$ be elements of R' whose images in $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)$ are generators of $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)$. Then R' is generated by $\{f_1,\dots,f_r\}$ and K. In particular, R' is finitely generated. In fact, R' is fine. Let $g\colon R' \to \Gamma(Y,M_Y)$ be the induced morphism by the chart $R' \to M_{\mathcal{R}'}$. Let $s\colon \Gamma(Y,M_Y) \to M_{Y,y}$ and $t\colon R' \to (\iota'^* M_{\mathcal{R}'})_y$ be the natural morphisms. We have to prove that, if $\iota'^*(a) \in \iota'^*(b)M_{Y,y}$ for any elements a and b of $(\iota'^* M_{\mathcal{R}'})_y$, then $a \in b(\iota'^* M_{\mathcal{R}'})_y$. We may assume that $a = t(a_0)$ and $b = t(b_0)$ for elements a_0 and b_0 of R'. Let $\frac{m_0}{n_0}$ $(m_0, n_0 \in \Gamma(Y,M_Y))$ be an element of $\Gamma(Y,M_Y)^{\mathrm{gp}}$ whose image in $M_{Y,y}^{\mathrm{gp}}/\mathcal{O}_{Y,y}^*$ belongs to $M_{Y,y}/\mathcal{O}_{Y,y}^*$. Let $e'_1,\dots,e'_r$ be elements of $\Gamma(Y,M_Y)$ whose images in $\Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*) \xrightarrow{\sim} \mathbb{N}^r$ (given by the upper horizontal isomorphism in (4.11.1)) are $(1,0,\dots,0),\dots,(0,\dots,0,1)$, respectively. Then the power of e'_i $(1 \le i \le r_y)$ in m_0 is greater than or equal to that in n_0. Hence we can write $\frac{m_0}{n_0} = \frac{m'_0}{n'_0}$ $(m'_0, n'_0 \in \Gamma(Y,M_Y))$ such that $s(n'_0) \in \mathcal{O}_{Y,y}^*$. Now we can find elements c_0 and d_0 of $\Gamma(Y,M_Y)$ such that $g(a_0)c_0 = g(b_0)d_0$ and such that $s(c_0) \in \mathcal{O}_{Y,y}^*$. Because the induced morphism $R' \to \Gamma(Y,M_Y)/\Gamma(Y,\mathcal{O}_Y^*)$ by g is surjective, $c_0 = g(e_0)u$ for elements $e_0 \in R'$ and $u \in \Gamma(Y,\mathcal{O}_Y^*)$. Hence $g(a_0e_0/b_0) = d_0u^{-1}$. By the definition of R', $a_0e_0 \in b_0R'$. Because $sg(e_0) \in \mathcal{O}_{Y,y}^*$, $t(e_0) \in \mathcal{O}_{Y,y}^*$. Thus we have $a \in b(\iota'^* M_{\mathcal{R}'})_y$. □

Remark 4.12. — In [**51**, (4.10)] Kato has obtained an exactification of a closed immersion of fine log schemes with global charts; for the definition of $\mathcal{R}'$, we have not used a (global) chart of Y. This different point plays an important role in (4.15), (6.10), (6.11), (6.12) and (6.17) below.

Lemma 4.13. — *Let $S_0 \hookrightarrow S$ be a nil-immersion of schemes. Let Y be a smooth scheme over S_0 and let E be a relative SNCD on Y/S_0. Let $\iota\colon (Y,E) \hookrightarrow \mathcal{Y}$ be a closed immersion into a log scheme obtained from a smooth scheme with a relative SNCD over S. Assume that $\Gamma(Y,M(E))/\Gamma(Y,\mathcal{O}_Y^*) \simeq \mathbb{N}^r$ for some $r \in \mathbb{N}$ and that $\overset{\circ}{\mathcal{Y}}$ has a system of global coordinates $y_1,\dots,y_d$ such that $M_{\mathcal{Y}} \simeq \mathcal{O}_{\mathcal{Y}}^* y_1^{\mathbb{N}} \cdots y_s^{\mathbb{N}}$ for some $0 \le s \le d$ which induces a composite* *surjection*

$$\mathbb{N}^s \longrightarrow \Gamma(\mathcal{Y}, M_{\mathcal{Y}}) \longrightarrow \Gamma(Y, M(E))/\Gamma(Y, \mathcal{O}_Y^*) \simeq \mathbb{N}^r.$$

Let P be the inverse image of $\Gamma(Y, M(E))/\Gamma(Y, \mathcal{O}_Y^*)$ *by the morphism*

$$\mathbb{Z}^s \longrightarrow \Gamma(Y, M(E))^{\mathrm{gp}}/\Gamma(Y, \mathcal{O}_Y^*).$$

Set $\mathcal{Y}^{\mathrm{ex}} := \mathcal{Y} \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[\mathbb{N}^s])} \mathrm{Spec}^{\log}(\mathbb{Z}[P])$. *Then the following hold:*

1) *The natural morphism* $(Y, E) \longrightarrow \mathcal{Y}^{\mathrm{ex}}$ *is an exact closed immersion over* S.

2) $\overset{\circ}{\mathcal{Y}}{}^{\mathrm{ex}}$ *is smooth over* S *and the log structure of* $\mathcal{Y}^{\mathrm{ex}}$ *is associated to a relative SNCD on* $\overset{\circ}{\mathcal{Y}}{}^{\mathrm{ex}}/S$.

3) *Assume that* $E = E_1 \cup E_2$ *for transversal relative SNCD's* E_1 *and* E_2 *on* Y/S_0. *Let* $\iota_j\colon (Y, E_j) \hookrightarrow \mathcal{Y}_j$ $(j = 1, 2)$ *be a closed immersion into a log scheme obtained from a smooth scheme with a relative SNCD over* S. *Assume that* $\overset{\circ}{\mathcal{Y}}_1 = \overset{\circ}{\mathcal{Y}}_2$ *and that* $\mathcal{Y} = (\overset{\circ}{\mathcal{Y}}_1, M_{\mathcal{Y}_1} \oplus_{\mathcal{O}^*_{\mathcal{Y}_1}} M_{\mathcal{Y}_2})$. *Endow* $\overset{\circ}{\mathcal{Y}}{}^{\mathrm{ex}}$ *with the pull-back of the log structure of* $\mathcal{Y}_j$. *Then the resulting log scheme* $\mathcal{Y}_j^{\mathrm{ex}}$ *is obtained from a smooth scheme with a relative SNCD over* S *and the natural morphism* $(Y, E_j) \longrightarrow \mathcal{Y}_j^{\mathrm{ex}}$ *is an exact closed immersion.*

Proof. — 1): Let y be a point of Y. Since the composite morphism

$$\Gamma(\mathcal{Y}, M_{\mathcal{Y}})/\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^*) \longrightarrow M_{\mathcal{Y},y}/\mathcal{O}^*_{\mathcal{Y},y} \longrightarrow M(E)_y/\mathcal{O}^*_{Y,y}$$

is surjective, the morphism $\Gamma(Y, M(E))/\Gamma(Y, \mathcal{O}_Y^*) \to M(E)_y/\mathcal{O}^*_{Y,y}$ is surjective. There exists an isomorphism $M(E)_y/\mathcal{O}^*_{Y,y} \overset{\sim}{\longrightarrow} \mathbb{N}^{r_y}$ for some $r_y \leq r$. Hence 1) follows from (4.11).

2): Let $h\colon \mathbb{N}^s \longrightarrow \mathbb{N}^r$ be the surjection in the statement (4.13). Set $e_k := (0, \dots, 0, \overset{k}{1}, 0, \dots, 0) \in \mathbb{N}^s$ $(1 \leq k \leq s)$. By a simple calculation, we see that there exist $k_1, \dots, k_r$ such that $h(e_{k_\ell}) = (0, \dots, 0, \overset{\ell}{1}, 0, \dots, 0)$ $(1 \leq \ell \leq r)$. Let $e_{k_{r+1}}, \dots, e_{k_s}$ be elements of $\mathbb{N}^s$ such that

$$\{e_{k_{r+1}}, \dots, e_{k_s}\} = \{e_k\}_{k=1}^s \setminus \{e_{k_1}, \dots, e_{k_r}\}.$$

Let $h^{\mathrm{gp}}\colon \mathbb{Z}^s \to \mathbb{Z}^r$ be the induced morphism by h. Then

$$P = \langle e_{k_1}, \dots, e_{k_r} \rangle \mathrm{Ker}(h^{\mathrm{gp}}).$$

Set $\overset{\circ}{\mathcal{Y}}_0 = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[y_1, \ldots, y_d])$. Then $\overset{\circ}{\mathcal{Y}}{}^{\mathrm{ex}}$ is étale over

$$(4.13.1)\qquad \overset{\circ}{\mathcal{Y}}_0 \otimes_{\mathbb{Z}[\mathbb{N}^s]} \mathbb{Z}[\langle e_{k_1}, \ldots, e_{k_r}\rangle \mathrm{Ker}(h^{\mathrm{gp}})] \overset{\sim}{\longrightarrow} \underline{\mathrm{Spec}}_S(\mathcal{O}_S[y_{k_1}, \ldots, y_{k_r}, y_{s+1}, \ldots, y_d] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Ker}(h^{\mathrm{gp}})]).$$

Hence $\overset{\circ}{\mathcal{Y}}{}^{\mathrm{ex}}$ is smooth over S. The log structure of $\mathcal{Y}^{\mathrm{ex}}$ is associated to the relative SNCD defined by the equation $y_{k_1} \cdot \cdots \cdot y_{k_r} = 0$.

3): This follows immediately from the description (4.13.1). □

By using (4.13), we would like to give the descriptions of

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \quad \text{and} \quad (E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$$

without using admissible immersions as follows; we generalize (4.9).

Until (4.16) we do not assume the existence of the commutative diagram (4.0.1).

Let $X = \bigcup_i X_i$ be an affine open covering of X such that

$$\Gamma(X_i, M(D\cup Z))/\Gamma(X_i, \mathcal{O}^*_{X_i})$$

is isomorphic to a finite direct sum of $\mathbb{N}$. Set

$$D_i := D|_{X_i} \quad \text{and} \quad Z_i := Z|_{X_i}.$$

Let $\mathcal{P}'_i$ be a lift of $(X_i, D_i \cup Z_i)$ over S such that $\mathcal{P}'_i$ has a global chart $P'_i \to M_{\mathcal{P}'_i}$ satisfying the assumptions in (4.13) for the closed immersion $(X_i, D_i \cup Z_i) \overset{\subset}{\longrightarrow} \mathcal{P}'_i$. Set

$$X_{i_0\cdots i_r} := X_{i_0} \cap \cdots \cap X_{i_r}, \qquad D_{i_0\cdots i_r} := D|_{X_{i_0\cdots i_r}},$$
$$Z_{i_0\cdots i_r} := Z|_{X_{i_0\cdots i_r}}, \qquad \mathcal{P}'_{i_0\cdots i_r} := \mathcal{P}'_{i_0} \times_S \cdots \times_S \mathcal{P}'_{i_r}.$$

We obtain a global chart $\bigoplus_{k=0}^r P'_{i_k} \to M_{\mathcal{P}'_{i_0\cdots i_r}}$ of $\mathcal{P}'_{i_0\cdots i_r}$. Take a log open subscheme $\mathcal{P}''_{i_0\cdots i_r}$ of $\mathcal{P}'_{i_0\cdots i_r}$ such that there exists a natural closed immersion $(X_{i_0\cdots i_r}, D_{i_0\cdots i_r} \cup Z_{i_0\cdots i_r}) \hookrightarrow \mathcal{P}''_{i_0\cdots i_r}$. The global chart above gives us a global chart of $\mathcal{P}''_{i_0\cdots i_r}$ satisfying the assumptions in (4.13) for the closed immersion $(X_{i_0\cdots i_r}, D_{i_0\cdots i_r} \cup Z_{i_0\cdots i_r}) \hookrightarrow \mathcal{P}''_{i_0\cdots i_r}$. Then we have the induced morphism of monoids

$$\bigoplus_{k=0}^r (P'_{i_k})^{\mathrm{gp}} \longrightarrow \Gamma\big(X_{i_0\cdots i_r}, M(D\cup Z)\big)^{\mathrm{gp}}.$$

Let $P_{i_0\cdots i_r}$ be the inverse image of $\Gamma(X_{i_0\cdots i_r}, M(D\cup Z))$ in $\bigoplus_{k=0}^r (P'_{i_k})^{\mathrm{gp}}$. Set

$$\mathcal{P}_{i_0\cdots i_r} := \mathcal{P}'_{i_0\cdots i_r} \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[\bigoplus_{k=0}^r P'_{i_k}])} \mathrm{Spec}^{\log}\big(\mathbb{Z}[P_{i_0\cdots i_r}]\big).$$

Then we have an exact immersion

$$(X_{i_0\cdots i_r}, (D \cup Z)|_{X_{i_0\cdots i_r}}) \longrightarrow \mathcal{P}_{i_0\cdots i_r} \tag{4.13.2}$$

by (4.13), 1). Though the immersion above is not necessarily closed, we see that $\mathcal{P}_{i_0\cdots i_r}$ is a log scheme obtained from a smooth scheme with a relative SNCD over S by the proof of (4.13), 2) (see the expression (4.13.1)). The projection $\mathcal{P}'_{i_0\cdots i_r} \to \mathcal{P}'_{i_0\cdots \widehat{i_j}\cdots i_r}$ induces the following natural morphism $\mathcal{P}_{i_0\cdots i_r} \to \mathcal{P}_{i_0\cdots \widehat{i_j}\cdots i_r}$; the immersion $\mathcal{P}'_{i_0\cdots \widehat{i_j}\cdots i_r} \to \mathcal{P}'_{i_0\cdots i_{j-1}i_k i_{j+1}\cdots i_r}$ $(k = j \pm 1)$ induces the following natural morphism $\mathcal{P}_{i_0\cdots \widehat{i_j}\cdots i_r} \to \mathcal{P}_{i_0\cdots i_{j-1}i_k i_{j+1}\cdots i_r}$ (here $\widehat{i_j}$ means to omit i_j). By using the exact closed immersions (4.13.2) for various $i_0, \ldots i_r$, we have a simplicial exact immersion $(X_\bullet, D_\bullet \cup Z_\bullet) \hookrightarrow \mathcal{P}_\bullet$ into a log smooth simplicial log scheme over S. By (4.13), 3) we have a simplicial exact immersion $(X_\bullet, Z_\bullet) \hookrightarrow \mathcal{Q}_\bullet$ fitting into the following commutative diagram

$$\begin{array}{ccc} (X_\bullet, D_\bullet \cup Z_\bullet) & \xrightarrow{\subset} & \mathcal{P}_\bullet \\ \downarrow & & \downarrow \\ (X_\bullet, Z_\bullet) & \xrightarrow{\subset} & \mathcal{Q}_\bullet \end{array}$$

such that $\overset{\circ}{\mathcal{P}}_\bullet = \overset{\circ}{\mathcal{Q}}_\bullet$ and such that the underlying morphism $\overset{\circ}{\mathcal{P}}_\bullet \longrightarrow \overset{\circ}{\mathcal{Q}}_\bullet$ is $\mathrm{id}_{\overset{\circ}{\mathcal{P}}_\bullet}$. Let $\mathfrak{D}_\bullet$ be the log PD-envelope of the immersion $(X_\bullet, Z_\bullet) \hookrightarrow \mathcal{Q}_\bullet$ over $(S, \mathcal{I}, \gamma)$. The condition (4.3.2) and the assumptions after (4.3.2) are satisfied for $M_{\mathcal{P}_n}$ and $M_{\mathcal{Q}_n}$ for each $n \in \mathbb{N}$. That is, there exists an fs sub log structure $\mathcal{M}_n$ of $M_{\mathcal{P}_n}$ such that

$$M_{\mathcal{P}_n} = \mathcal{M}_n \oplus_{\mathcal{O}^*_{\mathcal{P}_n}} M_{\mathcal{Q}_n}, \tag{4.13.3}$$

such that the pull-back of $\mathcal{M}_n$ to X_n is equal to $M(D_n)$ and such that the condition (4.2.3) is satisfied for $(\overset{\circ}{\mathcal{P}}_n, \mathcal{M}_n)$. In fact we may assume that we have a simplicial log scheme $(\overset{\circ}{\mathcal{P}}_\bullet, \mathcal{M}_\bullet)$. We have the following filtered complexes by (4.2):

$$(\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{P}_\bullet}} \Lambda^\bullet_{\mathcal{P}_\bullet/S}, P^{\mathcal{D}_\bullet}) := (\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{P}_\bullet}} \Lambda^\bullet_{\mathcal{P}_\bullet/S}, \{\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{P}_\bullet}} P_k^{\mathcal{P}_\bullet/\mathcal{Q}_\bullet} \Lambda^\bullet_{\mathcal{P}_\bullet/S}\}_{k\in\mathbb{Z}}),$$

$$(Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Lambda^\bullet_{\mathcal{P}_\bullet/S}), Q^*_{(X_\bullet,Z_\bullet)/S} P^{\mathcal{P}_\bullet/\mathcal{Q}_\bullet}) :=$$
$$(Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Lambda^\bullet_{\mathcal{P}_\bullet/S}), \{Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(P_k^{\mathcal{P}_\bullet/\mathcal{Q}_\bullet} \Lambda^\bullet_{\mathcal{P}_\bullet/S})\}_{k\in\mathbb{Z}}).$$

Let Δ be the standard simplicial category: an object of Δ is denoted by

$$[n] := \{0, \ldots, n\} \quad (n \in \mathbb{N});$$

a morphism in Δ is a non-decreasing function $[n] \to [m]$ $(n, m \in \mathbb{N})$.

LEMMA 4.14. — *Let the notations be as above. Let k be a nonnegative integer. Then the following hold*:

1) *For the morphism* $g\colon (\overset{\circ}{\mathcal{P}}_n, \mathcal{M}_n) \to (\overset{\circ}{\mathcal{P}}_{n'}, \mathcal{M}_{n'})$ *corresponding to a morphism* $[n'] \to [n]$ *in* Δ, *the assumptions in* (4.8) *are satisfied for*

$$(\overset{\circ}{\mathcal{P}}, \mathcal{M}) = (\overset{\circ}{\mathcal{P}}_n, \mathcal{M}_n) \quad \text{and} \quad (\overset{\circ}{\mathcal{P}}', \mathcal{M}') = (\overset{\circ}{\mathcal{P}}_{n'}, \mathcal{M}_{n'}).$$

2) *The family* $\{D^{(k)}(\mathcal{M}_n)\}_{n\in\mathbb{N}}$ *gives a simplicial scheme* $D^{(k)}(\mathcal{M}_\bullet)$ *with natural morphism* $b_\bullet^{(k)}\colon D^{(k)}(\mathcal{M}_\bullet) \to \overset{\circ}{\mathcal{P}}_\bullet$ *of simplicial schemes.*

Proof. — 1): Let ℓ be a nonnegative integer and let m_ℓ be a member of the local minimal generators of $\mathcal{M}_\ell/\mathcal{O}^*_{\mathcal{P}_\ell}$. Then, by the description (4.13.1), the image $\overline{m}_\ell$ of m_ℓ in $M(D_\ell)/\mathcal{O}^*_{X_\ell}$ is also a member of the local minimal generators of $M(D_\ell)/\mathcal{O}^*_{X_\ell}$. Because the standard degeneracy morphism $s_i\colon (X_\ell, D_\ell) \to (X_{\ell+1}, D_{\ell+1})$ $(\ell \in \mathbb{N}, 0 \leq i \leq \ell)$ and the standard face morphism $p_i\colon (X_\ell, D_\ell) \to (X_{\ell-1}, D_{\ell-1})$ $(\ell > 0, 0 \leq i \leq \ell)$ are obtained from open immersions, we can easily check that there exists a unique member $\overline{m}_{\ell\pm1}$ of the local minimal generators of $M(D_{\ell\pm1})/\mathcal{O}^*_{X_{\ell\pm1}}$ such that $s_i^*(\overline{m}_{\ell+1}) = \overline{m}_\ell$ and $p_i^*(\overline{m}_{\ell-1}) = \overline{m}_\ell$. We can also easily check that the other minimal generators of $M(D_{\ell\pm1})/\mathcal{O}^*_{X_{\ell\pm1}}$ are mapped to the trivial element of $M(D_\ell)/\mathcal{O}^*_{X_\ell}$ by s_i^* and p_i^*, respectively. Let m_n be a member of the local minimal generators of $\mathcal{M}_n/\mathcal{O}^*_{\mathcal{P}_n}$. Let $\overline{g}\colon (X_n, D_n \cup Z_n) \longrightarrow (X_{n'}, D_{n'} \cup Z_{n'})$ be the morphism corresponding to a morphism $[n'] \to [n]$. Since the morphism $[n'] \longrightarrow [n]$ is a composite morphism of standard degeneracy morphisms and standard face morphisms, we see that there exists a unique member $\overline{m}_{n'}$ of the local minimal generators of $M_{n'}/\mathcal{O}^*_{X_{n'}}$ such that $\overline{g}^*(\overline{m}_{n'}) = \overline{m}_n$. Let $\iota_\bullet\colon (X_\bullet, D_\bullet \cup Z_\bullet) \overset{\subset}{\longrightarrow} \mathcal{P}_\bullet$ be the immersion constructed before this lemma. Then we have the following commutative diagram:

$$\begin{array}{ccc} g_*(\mathcal{M}_n/\mathcal{O}^*_{\mathcal{P}_n}) & \longrightarrow & g_*\iota_{n*}(M(D_n)/\mathcal{O}^*_{X_n}) \\ \uparrow & & \uparrow \\ \mathcal{M}_{n'}/\mathcal{O}^*_{\mathcal{P}_{n'}} & \longrightarrow & \iota_{n'*}(M(D_{n'})/\mathcal{O}^*_{X_{n'}}). \end{array} \tag{4.14.1}$$

By the description (4.13.1) again, the horizontal morphisms in (4.14.1) are isomorphisms. Hence there exists a unique member $m_{n'}$ of the local minimal generators of $\mathcal{M}_{n'}/\mathcal{O}^*_{\mathcal{P}_{n'}}$ such that $g^*(m_{n'}) = m_n$. Thus we have checked the assumptions in (4.8).

2): This follows immediately from 1) and (4.3). □

Corollary 4.15. — *Let* π_{Rcrys} *and* π_{zar} *be the morphisms of ringed topoi in* (3.2), 4) *and* 5)*, respectively. Then the following hold*:

1) *There exist the following canonical isomorphism*:

$$(4.15.1)\quad Q^*_{(X,Z)/S}\big(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} R\pi_{\mathrm{Rcrys}*}\big(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Lambda^\bullet_{\mathcal{P}_\bullet/S}),Q^*_{(X_\bullet,Z_\bullet)/S}P^{\mathcal{P}_\bullet/\mathcal{Q}_\bullet}\big).$$

2) *There exist the following canonical isomorphism*:

$$(4.15.2)\quad \big(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big) \overset{\sim}{\longrightarrow} R\pi_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{P}_\bullet}}\Lambda^\bullet_{\mathcal{P}_\bullet/S},P^{\mathcal{P}_\bullet/\mathcal{Q}_\bullet}).$$

Proof. — 1): By (4.14) 1), 2) and (4.8) 2), we have the isomorphism

$$(4.15.3)\quad \mathrm{gr}_k^{Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}} Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Lambda^\bullet_{\mathcal{P}_\bullet/S}) = Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}(L_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}(\Lambda^\bullet_{D^{(k)}(\mathcal{M}_\bullet)/S}) \otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet))(-k)\{-k\}$$

As in (3.5.13), by using (4.7), we see that the derived direct image of the left hand side of (4.15.3) by $R\pi_{\mathrm{Rcrys}*}$ is equal to

$$Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S,Z))(-k)\{-k\}.$$

Using (3.1.1) (see (3.6), 3)), we obtain 1).

2): This follows immediately from 1). □

Remark 4.16. — In (4.15) we have given the descriptions of

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\quad\text{and}\quad(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$$

without using admissible immersions. Thus we can simplify the definitions of the preweight-filtered restricted crystalline complex

$$\big(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big)\quad\big(\simeq Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\big)$$

and the preweight-filtered zariskian complex

$$\big(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D\big)\quad\big(\simeq(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\big)$$

in [**72**, (2.4)] by considering the right hand sides of (4.15.1) and (4.15.2) as the definitions of $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ and $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$, respectively.

We come back to the situation (4.0.1). But, now we assume that $S_0=\mathrm{Spec}(\kappa)$ and $S=\mathrm{Spec}(\mathcal{W}_n)$, where κ is a perfect field of characteristic $p>0$ and $\mathcal{W}_n$ is the Witt ring of κ of length $n>0$. We give an explicit description of $(\mathcal{W}_n\Omega^\bullet_X(\log(D\cup Z)), P^D)$, which is a generalization of the description in [**72**, (2.12.4)].

Lemma 4.17. — *Assume that the closed immersion $D(\mathcal{M})_{i_1\cdots i_k} \hookrightarrow \overset{\circ}{\mathcal{P}}$ has a local retraction etale locally. Then the long exact sequence associated to* (4.5.1) *is decomposed into the exact sequence* $(q\in\mathbb{Z})$

$$(4.17.1)\quad \begin{aligned} 0 \to \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_{k-1}\Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}) &\longrightarrow \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_{k}\Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}) \\ &\longrightarrow \mathcal{H}^{q-k}(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} b^{(k)}_*(\Lambda^\bullet_{(D^{(k)}(\mathcal{M}),M_{\mathcal{Q}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D(\mathcal{M})))) \to 0. \end{aligned}$$

Proof. — The proof is the same as that of [**72**, (2.12.2)] by using (4.4) and (4.2) (cf. [**66**, 1.2]). □

We denote $(\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}), P^{\mathcal{P}/\mathcal{Q}})$ the filtered sheaf

$$\big(\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}), \{\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_k\Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n})\}_{k\in\mathbb{Z}}\big).$$

Now assume that X is affine. Then there exists a lift $\iota'\colon (X, D\cup Z) \hookrightarrow (\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ of $(X, D\cup Z)$ over $\mathcal{W}_n$ which induces lifts $(X,D) \hookrightarrow (\mathcal{X},\mathcal{D})$ and $(X,Z) \hookrightarrow (\mathcal{X},\mathcal{Z})$ of (X,D) and (X,Z) over $\mathcal{W}_n$, respectively. Denote the immersion $(X, D\cup Z) \hookrightarrow \mathcal{P}$ by ι. Because $\mathcal{P}$ and $\mathcal{Q}$ are log smooth over $\mathcal{W}_n$ and because ι' is a nilpotent immersion, there exists a morphism $\mathfrak{f}\colon (\mathcal{X}, \mathcal{D}\cup\mathcal{Z}) \to \mathcal{P}$ such that $\mathfrak{f}$ induces two morphisms $(\mathcal{X},\mathcal{D}) \to (\overset{\circ}{\mathcal{P}}, \mathcal{M})$ and $(\mathcal{X},\mathcal{Z}) \to \mathcal{Q}$ and such that $\mathfrak{f}\circ\iota' = \iota$.

By [**51**, (6.4)] we have the following commutative diagram:

$$(4.17.2)\quad \begin{array}{ccc} R^q u_{(X,D\cup Z)/\mathcal{W}_n *}(\mathcal{O}_{(X,D\cup Z)/\mathcal{W}_n}) & \xrightarrow{\sim} & \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}) \\ \| & & \big\downarrow {\scriptstyle \mathcal{H}^q(\mathfrak{f}^*)} \\ R^q u_{(X,D\cup Z)/\mathcal{W}_n *}(\mathcal{O}_{(X,D\cup Z)/\mathcal{W}_n}) & \xrightarrow{\sim} & \mathcal{H}^q(\Omega^\bullet_{\mathcal{X}/\mathcal{W}_n}(\log(\mathcal{D}\cup\mathcal{Z}))). \end{array}$$

Lemma 4.18. — *Let k be a nonnegative integer. Then $\mathcal{H}^q(\mathfrak{f}^*)$ induces an isomorphism*

$$\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} P^{\mathcal{P}/\mathcal{Q}}_k\Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}) \xrightarrow{\sim} \mathcal{H}^q\big(P^{\mathcal{D}}_k\Omega^\bullet_{\mathcal{X}/\mathcal{W}_n}(\log(\mathcal{D}\cup\mathcal{Z}))\big).$$

Proof. — The lemma immediately follows from (4.15.2). □

The following is a generalization of [**72**, (2.12.4)]:

Corollary 4.19. — *Let i be a nonnegative integer. Then*

$$(4.19.1)\qquad \big(\mathcal{W}_n\Omega^i_X(\log(D\cup Z)), P^D\big) = \big(\mathcal{H}^i(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^\bullet_{\mathcal{P}/\mathcal{W}_n}), P^{\mathcal{P}/\mathcal{Q}}\big).$$

Now assume that $S_0 := \mathrm{Spec}(\kappa)$ and $S = \mathrm{Spf}(\mathcal{W})$, where $\mathcal{W}$ is the Witt ring of κ. In this p-adic case, we also have analogous objects and facts already stated in this section. Let $\mathcal{P}$ and $\mathcal{Q}$ be fine log p-adic formal $\mathcal{W}$-schemes in (4.0.1) satisfying (4.3.2) and the assumptions after (4.3.2). For a positive integer n, set

$$\mathcal{P}_n := \mathcal{P} \otimes_{\mathcal{W}} \mathcal{W}_n, \quad \mathcal{Q}_n := \mathcal{Q} \otimes_{\mathcal{W}} \mathcal{W}_n$$

and let $\mathfrak{D}_n$ be the log PD-envelope of the immersion $(X, Z) \hookrightarrow \mathcal{Q}_n$ over $(\mathrm{Spec}(\mathcal{W}_n), p\mathcal{W}_n, [\ ])$. Then we have the exact sequence (see [**45**, p. 243]):

$$0 \to \mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{\bullet}_{\mathcal{P}_n/\mathcal{W}_n} \xrightarrow{p^n} \mathcal{O}_{\mathfrak{D}_{2n}} \otimes_{\mathcal{O}_{\mathcal{P}_{2n}}} \Lambda^{\bullet}_{\mathcal{P}_{2n}/\mathcal{W}_{2n}} \longrightarrow \mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{\bullet}_{\mathcal{P}_n/\mathcal{W}_n} \to 0.$$

As in [**49**, III (1.5)], we have the boundary morphism

$$d\colon \mathcal{H}^i(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{\bullet}_{\mathcal{P}_n/\mathcal{W}_n}) \longrightarrow \mathcal{H}^{i+1}(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{\bullet}_{\mathcal{P}_n/\mathcal{W}_n})$$

of the long exact sequence obtained from the exact sequence above. This boundary morphism preservers the preweight filtration $P^{\mathcal{P}_n/\mathcal{Q}_n}$ on $\mathcal{H}^j(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{\bullet}_{\mathcal{P}_n/\mathcal{W}_n})$ $(j = i, i+1)$. Hence we have a filtered complex

$$\big(\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{*}_{\mathcal{P}_n/\mathcal{W}_n}), P^{\mathcal{P}_n/\mathcal{Q}_n}\big).$$

As in [**49**, III, (1.5)], we also have a projection

$$\pi\colon \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_{n+1}} \otimes_{\mathcal{O}_{\mathcal{P}_{n+1}}} \Lambda^{*}_{\mathcal{P}_{n+1}/W_{n+1}}) \longrightarrow \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{*}_{\mathcal{P}_n/\mathcal{W}_n})$$

(see [**45**, (4.2)]). We see that this projection preservers the preweight filtration $P^{\mathcal{P}_n/\mathcal{Q}_n}$ as in [**69**, (8.7), (1)] by using (4.17.1). By using the definitions of d and π, we have the following as a corollary of (4.19):

Corollary 4.20. — *Let $f\colon X \to \mathrm{Spec}(\mathcal{W}_n)$ be the structural morphism. Then there exists an isomorphism*

$$\text{(4.20.1)} \quad \big(\mathcal{W}_n\Omega^{\bullet}_X(\log(D \cup Z)), P^D\big) \xrightarrow{\sim} \big(\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{P}_n}} \Lambda^{*}_{\mathcal{P}_n/\mathcal{W}_n}), P^{\mathcal{P}_n/\mathcal{Q}_n}\big).$$

of filtered complexes of $f^{-1}(\mathcal{W}_n)$-modules, which is compatible with the projections.

Remark 4.21. — Let the notations be before (4.14). Assume that $S_0 := \mathrm{Spec}(\kappa)$ and $S = \mathrm{Spf}(\mathcal{W})$. Set

$$\mathcal{P}_{\bullet n} := \mathcal{P}_{\bullet} \otimes_{\mathcal{W}} \mathcal{W}_n \quad \text{and} \quad \mathcal{Q}_{\bullet n} := \mathcal{Q}_{\bullet} \otimes_{\mathcal{W}} \mathcal{W}_n.$$

Let $\mathfrak{D}_{\bullet n}$ be the simplicial log PD-envelope of the simplicial immersion

$$(X_{\bullet}, D_{\bullet} \cup Z_{\bullet}) \longrightarrow \mathcal{P}_{\bullet n}$$

over $(\mathrm{Spec}(\mathcal{W}_n), p\mathcal{W}_n, [\;])$. By the filtered cohomological descent (see [**72**, (1.5.1)]), we have

$$\begin{aligned}(4.21.1)\qquad &\big(\mathcal{W}_n\Omega^\bullet_X(\log(D\cup Z)), P^D\big)\\ &\qquad = R\pi_{\mathrm{zar}*}\big(\mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{D}_{\bullet n}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet n}}}\Lambda^*_{\mathcal{P}_{\bullet n}/\mathcal{W}_n}), P^{\mathcal{P}_{\bullet n}/\mathcal{Q}_{\bullet n}}\big).\end{aligned}$$

This equality is compatible with the projections as in [**72**, (2.12.11)].

LEMMA 4.22. — 1) *Let L be a fine log structure on* $\mathrm{Spec}(\kappa)$. *Let $\mathcal{W}_n(L)$ be the canonical lift of L over* $(\mathrm{Spec}(\mathcal{W}_n)$*:*

$$\mathrm{Spec}(\mathcal{W}_n)\colon \mathcal{W}_n(L) = L\oplus \mathrm{Ker}(\mathcal{W}_n^*\to\kappa^*)$$

(see [**45**, (3.1)]). *Let Y (resp. $\mathcal{Y}$) be a log smooth log scheme of Cartier type over* $(\mathrm{Spec}(\kappa), L)$ *(resp. a log smooth log scheme over* $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$). *Let $Y\hookrightarrow\mathcal{Y}$ be an immersion over* $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$ *and let $\mathfrak{E}$ be the log PD-envelope of this immersion over* $((\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L)), p\mathcal{W}_n, [\;])$. *Let*

$$g\colon Y\longrightarrow \big(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L)\big)$$

be the structural morphism. Let $\mathcal{W}_n(Y)$ be the canonical lift of Y over $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$ *(see [loc. cit.]). Assume that $\overset{\circ}{Y}$ is affine. Then the quasi-isomorphism*

$$(4.22.1)\qquad \mathcal{O}_{\mathfrak{E}}\otimes_{\mathcal{O}_\mathcal{Y}}\Lambda^\bullet_{\mathcal{Y}/\mathcal{W}_n}\longrightarrow \mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{E}}\otimes_{\mathcal{O}_\mathcal{Y}}\Lambda^*_{\mathcal{Y}/\mathcal{W}_n}) = \mathcal{W}_n\Lambda^\bullet_Y$$

constructed in the proof of [**45**, (4.19)] *in* $\mathrm{C}^+(\overset{\circ}{g}{}^{-1}(\mathcal{W}_n))$ *is functorial in the following sense: for a log smooth affine log scheme Y_i of Cartier type over* $(\mathrm{Spec}(\kappa), L)$ $(i=1,2)$ *and for a morphism $\mathcal{W}_n(Y_i)\longrightarrow\mathcal{Y}_i$ to a log smooth log scheme $\mathcal{Y}_i$ over* $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$ *such that the composite morphism $Y_i\hookrightarrow\mathcal{W}_n(Y_i)\longrightarrow\mathcal{Y}_i$ is an immersion and for the following commutative diagram*

$$\begin{array}{ccc}\mathcal{W}_n(Y_1) & \longrightarrow & \mathcal{Y}_1\\ \big\downarrow & & \big\downarrow h\\ \mathcal{W}_n(Y_2) & \longrightarrow & \mathcal{Y}_2\end{array}$$

of log schemes over $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$ *and for the log PD-envelope $\mathfrak{E}_i$ $(i=1,2)$ of the immersion $Y_i\hookrightarrow\mathcal{Y}_i$ over* $((\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L)), p\mathcal{W}_n, [\;])$*, the following diagram is commutative:*

$$(4.22.2)\qquad \begin{array}{ccc}\mathcal{O}_{\mathfrak{E}_1}\otimes_{\mathcal{O}_{\mathcal{Y}_1}}\Lambda^\bullet_{\mathcal{Y}_1/\mathcal{W}_n} & \longrightarrow & \mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{E}_1}\otimes_{\mathcal{O}_{\mathcal{Y}_1}}\Lambda^*_{\mathcal{Y}_1/\mathcal{W}_n})\\ {\scriptstyle h^*}\big\uparrow & & \big\uparrow{\scriptstyle h^*}\\ h^*(\mathcal{O}_{\mathfrak{E}_2}\otimes_{\mathcal{O}_{\mathcal{Y}_2}}\Lambda^\bullet_{\mathcal{Y}_2/\mathcal{W}_n}) & \longrightarrow & h^*\mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{E}_2}\otimes_{\mathcal{O}_{\mathcal{Y}_2}}\Lambda^*_{\mathcal{Y}_2/\mathcal{W}_n}).\end{array}$$

2) *Let the notations be as in* (4.0.1). *Assume that* $S = \mathrm{Spec}(\mathcal{W}_n)$ *and* $S_0 = \mathrm{Spec}(\kappa)$. *Assume also that* X *is affine. Then the morphism* (4.22.1) *for the immersion* $(X, D \cup Z) \hookrightarrow \mathcal{P}$ *induces the following morphism*

$$(4.22.3) \qquad (\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^{\bullet}_{\mathcal{P}/\mathcal{W}_n}, P^{\mathcal{P}/\mathcal{Q}}) \longrightarrow \big(\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{P}}} \Lambda^{*}_{\mathcal{P}/\mathcal{W}_n}), P^{\mathcal{P}/\mathcal{Q}}\big) = \big(\mathcal{W}_n\Omega^{\bullet}_X(\log(D \cup Z)), P^D\big)$$

in $\mathrm{C}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$, *which is a filtered quasi-isomorphism if* $M_{\mathcal{P}}$ *and* $M_{\mathcal{Q}}$ *satisfy* (4.3.2) *and the assumptions after* (4.3.2). *The filtered morphism* (4.22.3) *is functorial for the following commutative diagram*

$$\begin{array}{ccc} (\mathcal{W}_n(X), \mathcal{W}_n(D \cup Z)) & \longrightarrow & \mathcal{P} \\ {\scriptstyle \epsilon_{(\mathcal{W}_n(X), \mathcal{W}_n(D\cup Z), \mathcal{W}_n(Z))/\mathcal{W}_n}} \big\downarrow & & \big\downarrow \\ (\mathcal{W}_n(X), \mathcal{W}_n(Z)) & \longrightarrow & \mathcal{Q} \end{array}$$

which induces the commutative diagram (4.0.1) *for the case* $S = \mathrm{Spec}(\mathcal{W}_n)$. *Here we denote by* $(\mathcal{W}_n(X), \mathcal{W}_n(D \cup Z))$ (*resp.* $(\mathcal{W}_n(X), \mathcal{W}_n(Z))$) *the canonical lift* $\mathcal{W}_n((X, M(D \cup Z)))$ (*resp.* $\mathcal{W}_n((X, M(Z)))$) (*cf.* [**72**, (2.12.7)]). *In the p-adic case after* (4.19), *the morphism* (4.22.3) *is compatible with the projections.*

Proof. — 1): Because $\mathcal{Y}$ is log smooth over $(\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$, we have the following commutative diagram

$$\begin{array}{ccc} Y & \overset{\subset}{\longrightarrow} & \mathcal{W}_n(Y) \\ \big\| & & \big\downarrow \\ Y & \overset{\subset}{\longrightarrow} & \mathcal{Y}. \end{array}$$

By the proof of [**45**, (4.19)], we have a natural morphism $\mathcal{W}_n(Y) \to \mathfrak{E}$. Then the morphism (4.22.1) for each $i \in \mathbb{N}$ is the following composite morphism

$$(4.22.4) \quad \mathcal{O}_{\mathfrak{E}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^i_{\mathcal{Y}/\mathcal{W}_n} \longrightarrow \Lambda^i_{\mathcal{W}_n(Y)/(\mathcal{W}_n, \mathcal{W}_n(L)), [\ ]} \longrightarrow \mathcal{H}^i(\mathcal{O}_{\mathfrak{E}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^*_{\mathcal{Y}/\mathcal{W}_n}).$$

Here $\bigoplus_{i \geq 0} \Lambda^i_{\mathcal{W}_n(Y)/(\mathcal{W}_n, \mathcal{W}_n(L)), [\ ]}$ is a sheaf of differential graded algebras over $\mathcal{W}_n$ which is a quotient of $\bigoplus_{i \geq 0} \Lambda^i_{\mathcal{W}_n(Y)/(\mathcal{W}_n, \mathcal{W}_n(L))}$ divided by a $\mathcal{W}_n$-submodule generated by the local sections of the form $da^{[j]} - a^{[j-1]}da$ $(a \in \mathrm{Ker}(\mathcal{W}_n(\mathcal{O}_Y) \to \mathcal{O}_Y),\ j \geq 1)$ (cf. [**47**, 0. (3.1.2)]). The first morphism in (4.22.4) in $\mathrm{C}^+(\overset{\circ}{g}{}^{-1}(\mathcal{W}_n))$ depends on the morphism $\mathcal{W}_n(Y) \to \mathcal{Y}$. The second morphism has been constructed in [**45**, (4.9)]. Here we note only that a local section $d\log m$ $(m \in M_Y)$ is mapped to the cohomology class of $d\log \widetilde{m}$ by the second morphism in (4.22.4), where $\widetilde{m} \in M_{\mathcal{Y}}$ is any lift of m.

In fact, the morphism (4.22.4) induces the morphism of complexes

$$(4.22.5)\quad \mathcal{O}_{\mathfrak{E}}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Lambda^{\bullet}_{\mathcal{Y}/\mathcal{W}_n}\longrightarrow \Lambda^{\bullet}_{\mathcal{W}_n(Y)/(\mathcal{W}_n,\mathcal{W}_n(L)),[\ ]}\longrightarrow \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{E}}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Lambda^{*}_{\mathcal{Y}/\mathcal{W}_n}).$$

Indeed, by [**51**, (6.4)] and the definition of $\mathcal{W}_n\Lambda^{\bullet}_Y$ (see [**45**, (4.1)]), we have $\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{E}}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Lambda^{*}_{\mathcal{Y}/\mathcal{W}_n})=\mathcal{W}_n\Lambda^{\bullet}_Y$, and by [**69**, (7.18.2)], we have the commutative diagram:

$$\begin{array}{ccc}\Lambda^{i}_{\mathcal{W}_n(Y)/(\mathcal{W}_n,\mathcal{W}_n(L)),[\ ]} & \longrightarrow & \mathcal{W}_n\Lambda^{i}_Y\\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d}\\ \Lambda^{i+1}_{\mathcal{W}_n(Y)/(\mathcal{W}_n,\mathcal{W}_n(L)),[\ ]} & \longrightarrow & \mathcal{W}_n\Lambda^{i+1}_Y.\end{array}$$

Hence the morphism (4.22.4) is a morphism of complexes.

The functoriality of the first morphism in (4.22.5) follows from the universality of the log PD-envelope. The functoriality of the second morphism in (4.22.5) is clear by the construction in [loc. cit.].

2): Because X is affine, we have a morphism $(\mathcal{W}_n(X),\mathcal{W}_n(D\cup Z))\to\mathcal{P}$ such that the composite morphism $(X,D\cup Z)\hookrightarrow(\mathcal{W}_n(X),\mathcal{W}_n(D\cup Z))\to\mathcal{P}$ is the given immersion. The morphism $(\mathcal{W}_n(X),\mathcal{W}_n(D\cup Z))\to\mathcal{P}$ gives us the commutative diagram

$$(4.22.6)\qquad \begin{array}{ccc}(\mathcal{W}_n(X),\mathcal{W}_n(D\cup Z)) & \longrightarrow & \mathcal{P}\\ \big\downarrow & & \big\downarrow\\ (\mathcal{W}_n(X),\mathcal{W}_n(Z)) & \longrightarrow & \mathcal{Q}.\end{array}$$

Let $\mathfrak{D}(\mathcal{P})$ be the log PD-envelope of the exact immersion $(X,D\cup Z)\hookrightarrow\mathcal{P}$ over $(\mathrm{Spec}(\mathcal{W}_n),p\mathcal{W}_n,[\ ])$. Note that $\overset{\circ}{\mathfrak{D}(\mathcal{P})}=\overset{\circ}{\mathfrak{D}}$. By the functoriality of the morphism (4.22.5), we have the commutative diagram

$$(4.22.7)\qquad \begin{array}{ccc}\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^{\bullet}_{\mathcal{P}/\mathcal{W}_n} & \longrightarrow & \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^{*}_{\mathcal{P}/\mathcal{W}_n})\\ \big\uparrow & & \big\uparrow\\ \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{Q}}}\Lambda^{\bullet}_{\mathcal{Q}/\mathcal{W}_n} & \longrightarrow & \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{Q}}}\Lambda^{*}_{\mathcal{Q}/\mathcal{W}_n}).\end{array}$$

By this commutative diagram and by the note about the second morphism in (4.22.4), we have the morphism

$$(4.22.8)\qquad \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}P^{\mathcal{P}/\mathcal{Q}}_k\Lambda^{\bullet}_{\mathcal{P}/\mathcal{W}_n}\longrightarrow\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}P^{\mathcal{P}/\mathcal{Q}}_k\Lambda^{*}_{\mathcal{P}/\mathcal{W}_n})\quad(k\in\mathbb{Z}).$$

In this way, we have the filtered morphism (4.22.3).

By [**72**, (2.12.4.2)] we have the following quasi-isomorphism

$$\mathrm{Res}^{D}\colon \mathrm{gr}^{P^D}_k\mathcal{W}_n\Omega^{\bullet}_X(\log(D\cup Z))\overset{\sim}{\longrightarrow}\mathcal{W}_n\Omega^{\bullet}_{D^{(k)}}(\log Z|_{D^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(D/\kappa).$$

By (4.5.1), (4.6), 2), (4.17.1), (4.20.1) and [**45**, (4.19)] = [**69**, (7.19)], we see that the filtered morphism (4.22.3) is a filtered quasi-isomorphism.

The functoriality of (4.22.3) follows from the proof of 1).

The compatibility of the morphism (4.22.3) with the projections follows from [**69**, (7.18)]. □

Remark 4.23. — In [**72**, (2.12.6)] we have assumed the existence of an endomorphism of $\mathcal{P}$ lifting the Frobenius endomorphism of $\mathcal{P}\otimes_{\mathcal{W}_n}\kappa$ and have used the endomorphism to prove that the morphism

$$\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^{\bullet}_{\mathcal{P}/\mathcal{W}_n}\longrightarrow \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{P}}}\Lambda^{*}_{\mathcal{P}/\mathcal{W}_n})$$

preserves the preweight filtration with respect to $\mathcal{P}$ in a slightly more special case. In this paper we have not used the lift. Consequently it has turned out that we can dispense with the log version of a lemma of Dwork-Dieudonné-Cartier (see [**69**, (7.10)]) in the proof of the preservation of the preweight filtration with respect to $\mathcal{P}$.

5. p-adic weight spectral sequence

Let $(S,\mathcal{I},\gamma)$ and S_0 be as in §3. Let r be a positive integer. Let $S_{0\underline{\bullet}}$ (resp. $S_{\underline{\bullet}}$) be the constant r-simplicial scheme defined by S_0 (resp. S).

Let $f\colon (X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})\to S_0$ be a smooth r-simplicial scheme with transversal r-simplicial relative SNCD's $D_{\underline{\bullet}}$ and $Z_{\underline{\bullet}}$ over S_0. Let $f_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})}\colon (X_{\underline{\bullet}},Z_{\underline{\bullet}})\to S_0$ be the induced morphism by f. Let $f_{\underline{\bullet}}\colon (X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})\to S_{0\underline{\bullet}}$ be the natural morphism induced by f. By abuse of notation, we also denote by f (resp. $f_{\underline{\bullet}}$) the composite morphism

$$(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})\longrightarrow S_0\lhook\joinrel\longrightarrow S\quad(\text{resp. } (X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})\longrightarrow S_{0\underline{\bullet}}\lhook\joinrel\longrightarrow S_{\underline{\bullet}}).$$

Furthermore, for simplicity of notation, we often denote simply by f the morphism $f_{\underline{t}}\colon (X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})\to S_0$ ($\underline{t}\in\mathbb{N}^r$) and also by f the composite morphism $f_{\underline{t}}\colon (X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})\to S_0\hookrightarrow S$. We apply the same rule for $f_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})}$. Let $((X_{\underline{\bullet}},\widetilde{D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}})/S)^{\log}_{\mathrm{crys}}$ and $((\widetilde{X_{\underline{\bullet}},Z_{\underline{\bullet}}})/S)^{\log}_{\mathrm{crys}}$ be the log crystalline topoi of $(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})$ and $(X_{\underline{\bullet}},Z_{\underline{\bullet}})$ over $(S,\mathcal{I},\gamma)$, respectively. We have four morphisms of topoi:

$$f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S_{\underline{\bullet}}}\colon \big((X_{\underline{\bullet}},\widetilde{D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}})/S\big)^{\log}_{\mathrm{crys}}\longrightarrow \widetilde{S}_{\underline{\bullet}\mathrm{zar}},\tag{5.0.1}$$

$$f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}\colon \big((X_{\underline{\bullet}},\widetilde{D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}})/S\big)^{\log}_{\mathrm{crys}}\longrightarrow \widetilde{S}_{\mathrm{zar}},\tag{5.0.2}$$

$$f_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S_{\underline{\bullet}}}\colon \big((\widetilde{X_{\underline{\bullet}},Z_{\underline{\bullet}}})/S\big)^{\log}_{\mathrm{crys}}\longrightarrow \widetilde{S}_{\underline{\bullet}\mathrm{zar}},\tag{5.0.3}$$

$$f_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S}\colon \big((\widetilde{X_{\underline{\bullet}},Z_{\underline{\bullet}}})/S\big)^{\log}_{\mathrm{crys}} \longrightarrow \widetilde{S}_{\mathrm{zar}}. \tag{5.0.4}$$

Let $\mathcal{O}_{(X_{\underline{\bullet}},\widetilde{D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}})/S}$ and $\mathcal{O}_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S}$ be the structure sheaves in $((X_{\underline{\bullet}},\widetilde{D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}})/S)^{\log}_{\mathrm{crys}}$ and $((\widetilde{X_{\underline{\bullet}},Z_{\underline{\bullet}}})/S)^{\log}_{\mathrm{crys}}$, respectively.

Let $I^{\underline{\bullet}\bullet}$ be a flasque resolution of $\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}$, which exists by [SGA 4-2, $\mathrm{V}^{\mathrm{bis}}$ (1.3.10)] (see also [**72**, (1.5.0.2)]). Then, set

$$\begin{aligned}\big(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}}\big)&\\ := \big(\epsilon_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}&(I^{\underline{\bullet}\bullet}),\{\tau_k\epsilon_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}(I^{\underline{\bullet}\bullet})\}_{k\in\mathbb{Z}}\big)\end{aligned}$$

in $\mathrm{D}^+\mathrm{F}(\mathcal{O}_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S})$ and let $(J^{\underline{\bullet}\bullet},\{J^{\underline{\bullet}\bullet}_k\}_{k\in\mathbb{Z}})$ be a filtered flasque resolution of the filtered complex on the right hand side above. Set

$$\big(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}}\big) := u_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}\big((J^{\underline{\bullet}\bullet},\{J^{\underline{\bullet}\bullet}_k\}_{k\in\mathbb{Z}})\big)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}_{\underline{\bullet}}(\mathcal{O}_S))$.

Definition 5.1. — We call

$$\big(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}}\big)\quad\big(\text{resp. }(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}})\big)$$

the *r-cosimplicial preweight-filtered vanishing cycle crystalline complex* of $(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S$ *with respect to* $D_{\underline{\bullet}}$ (resp. *r-cosimplicial preweight-filtered vanishing cycle zariskian complex* of $(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S$ *with respect to* $D_{\underline{\bullet}}$).

Let $(S',\mathcal{I}',\gamma')$ be a PD-scheme with quasi-coherent PD-ideal sheaf $\mathcal{I}'$ and with PD-structure γ' on $\mathcal{I}'$. Set

$$S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}').$$

Let $f'\colon (X'_{\underline{\bullet}},D'_{\underline{\bullet}}\cup Z'_{\underline{\bullet}})\to S'_0$ be a smooth r-simplicial scheme with transversal r-simplicial relative SNCD's $D'_{\underline{\bullet}}$ and $Z'_{\underline{\bullet}}$ over S'_0. Obviously

$$\big(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}}\big)\quad\text{and}\quad\big(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}),P^{D_{\underline{\bullet}}}\big)$$

are functorial with respect to the following commutative diagram

$$\begin{array}{ccc}(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}) & \longrightarrow & (X'_{\underline{\bullet}},D'_{\underline{\bullet}}\cup Z'_{\underline{\bullet}})\\ {\scriptstyle\epsilon_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}},Z_{\underline{\bullet}})/S}}\big\downarrow & & \big\downarrow{\scriptstyle\epsilon_{(X'_{\underline{\bullet}},D'_{\underline{\bullet}}\cup Z'_{\underline{\bullet}},Z'_{\underline{\bullet}})/S}}\\ (X_{\underline{\bullet}},Z_{\underline{\bullet}}) & \longrightarrow & (X'_{\underline{\bullet}},Z'_{\underline{\bullet}})\\ \big\downarrow & & \big\downarrow\\ S_0 & \longrightarrow & S'_0.\end{array}$$

Denote by $(\mathcal{K}, P)$ (a representative of) the filtered complex

$$Rf_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S_{\underline{\bullet}}*}\big(E^{\log,Z_{\underline{\bullet}}}_{\rm crys}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}), P^{D_{\underline{\bullet}}}\big).$$

Since

$$\mathrm{gr}^t_{\underline{L}}\mathbf{s}(\mathcal{K}) = \bigoplus_{\underline{t}_r=t} Rf_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S*}\big(E^{\log,Z_{\underline{t}}}_{\rm crys}(\mathcal{O}_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S})\big)[-t],$$

$$Ru_{(X_{\underline{t}},Z_{\underline{t}})/S*}\big(E^{\log,Z_{\underline{t}}}_{\rm crys}(\mathcal{O}_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S})\big) = Ru_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S*}\big(\mathcal{O}_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S}\big),$$

we have the following spectral sequence by (2.2.6):

$$(5.1.1)\qquad E_1^{ts} = \bigoplus_{\underline{t}_r=t} R^s f_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S*}(\mathcal{O}_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S}) \Longrightarrow R^{s+t}f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}).$$

Now let us construct a spectral sequence of $R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})$ $(h \in \mathbb{N})$. Let $\delta(\underline{L}, P)$ be the diagonal filtration of $L_1, \ldots, L_r$ and P on $\mathbf{s}(\mathcal{K})$ (see (2.2.8)). It is easy to see that the filtration $\delta(\underline{L}, P)$ is exhaustive and complete. By virtue of (2.2.9) and (3.1.2), we have

$$\begin{aligned}\mathcal{H}^h(\mathrm{gr}_k^{\delta(\underline{L},P)}\mathbf{s}(\mathcal{K})) &= \bigoplus_{\underline{t}\geq 0}\mathcal{H}^h\big(\mathrm{gr}^P_{\underline{t}_r+k}\mathcal{K}^{\underline{t}\bullet}[-\underline{t}_r]\big)\\ (5.1.2)\qquad &= \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k} f_{(D^{(\underline{t}_r+k)}_{\underline{t}},Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S*}\big(\mathcal{O}_{(D^{(\underline{t}_r+k)}_{\underline{t}},Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S}\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\rm crys}(D_{\underline{t}}/S;Z_{\underline{t}})\big)\big(-(\underline{t}_r+k)\big).\end{aligned}$$

Let $\{E_r^{**}((X_{\underline{\bullet}}, D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S)\}_{r\geq 1}$ be the E_r-terms of the spectral sequence arising from the filtration $\delta(\underline{L}, P)$. Since $E_1^{-k,h+k}((X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S) = 0$ for $|k| > h$, the spectral sequence is bounded below and regular. By (5.1.2) we have the following convergent spectral sequence

$$(5.1.3)\quad E_1^{-k,h+k}\big((X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S\big) = \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k} f_{(D^{(\underline{t}_r+k)}_{\underline{t}},Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S*} \big(\mathcal{O}_{(D^{(\underline{t}_r+k)}_{\underline{t}},Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S}\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\rm crys}(D_{\underline{t}}/S;Z_{\underline{t}})\big)\big(-(\underline{t}_r+k)\big) \Longrightarrow R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}).$$

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p > 0$. Let K be the fraction field of $\mathcal{V}$. Let S be a p-adic formal $\mathcal{V}$-scheme in the sense of [**74**, §1]. Set $S_0 = \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. In this case we also have the spectral sequence (5.1.3).

CONVENTION 5.2. — We usually denote by $P^{D_{\bullet}} := \{P^{D_{\bullet}}_k\}_{k\in\mathbb{Z}}$ (instead of $\delta(\underline{L}, P^{D_{\bullet}}) := \{\delta(\underline{L}, P^{D_{\bullet}})_k\}_{k\in\mathbb{Z}}$) the induced filtration on

$$R^h f_{(X_{\bullet},D_{\bullet}\cup Z_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet}\cup Z_{\bullet})/S})$$

by the spectral sequence (5.1.3) by abuse of notation. If there is a risk of confusion, we denote it by $\delta(\underline{L}, P^{D_{\bullet}})$.

DEFINITION 5.3. — We call the spectral sequence (5.1.3) the *preweight spectral sequence of* $(X_{\bullet}, D_{\bullet} \cup Z_{\bullet})/(S,\mathcal{I},\gamma)$ *with respect to* $D_{\bullet}$. If $Z_{\bullet} = \phi$, then we call (5.1.3) the *preweight spectral sequence of* $(X_{\bullet}, D_{\bullet})/(S,\mathcal{I},\gamma)$. In this case, we denote by $P = \{P_k\}_{k\in\mathbb{Z}}$ the induced filtration on

$$R^h f_{(X_{\bullet},D_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet})/S})$$

if there is no risk of confusion. We call P the *preweight filtration* on $R^h f_{(X_{\bullet},D_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet})/S})$. If S is a p-adic formal $\mathcal{V}$-scheme in the sense of [**74**, §1], if $S_0 = \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$ and if $\overset{\circ}{f} : X_{\bullet} \to S_0$ is proper, then we call (5.1.3) the *p-adic weight spectral sequence* of $R^h f_{(X_{\bullet},D_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet})/S})$ and call the induced filtration P the *weight filtration* on $R^h f_{(X_{\bullet},D_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet})/S})$.

Next, we describe the boundary morphism between the E_1-terms of (5.1.3). Fix $\underline{t} = (t_1, \dots, t_r) \in \mathbb{N}^r$ and a decomposition $\Delta_{\underline{t}} := \{D_{\lambda_{\underline{t}}}\}_{\lambda_{\underline{t}}}$ of $D_{\underline{t}}$ by smooth components of $D_{\underline{t}}$. Let m be a positive integer. Set

$$\begin{aligned}
\underline{\lambda}_{\underline{t}} &:= \{\lambda_{\underline{t}0}, \dots, \lambda_{\underline{t},m-1}\},\\
\underline{\lambda}_{\underline{t}j} &:= \{\lambda_{\underline{t}0}, \dots, \widehat{\lambda}_{\underline{t}j}, \dots, \lambda_{\underline{t},m-1}\} \quad (0 \le j \le m-1),\\
D_{\underline{\lambda}_{\underline{t}}} &:= D_{\lambda_{\underline{t}0}} \cap \cdots \cap D_{\lambda_{\underline{t},m-1}},\\
D_{\underline{\lambda}_{\underline{t}j}} &:= D_{\lambda_{\underline{t}0}} \cap \cdots \cap \widehat{D}_{\lambda_{\underline{t}j}} \cap \cdots \cap D_{\lambda_{\underline{t},m-1}}.
\end{aligned}$$

Here $\widehat{\ }$ means the elimination. Then $D_{\underline{\lambda}_{\underline{t}}}$ is a smooth divisor on $D_{\underline{\lambda}_{\underline{t}j}}/S_0$. For a nonnegative integer k and an integer m, let

$$\begin{aligned}
(5.3.1)\quad (-1)^j G^{\underline{\lambda}_{\underline{t}j}}_{\underline{\lambda}_{\underline{t}}} \colon R^k f_{(D_{\underline{\lambda}_{\underline{t}}}, Z_{\underline{t}}|_{D_{\underline{\lambda}_{\underline{t}}}})/S*}(\mathcal{O}_{(D_{\underline{\lambda}_{\underline{t}}}, Z_{\underline{t}}|_{D_{\underline{\lambda}_{\underline{t}}}})/S} \\
\otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_{\underline{t}}\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}}))(-m) \\
\longrightarrow R^{k+2} f_{(D_{\underline{\lambda}_{\underline{t}j}}, Z_{\underline{t}}|_{D_{\underline{\lambda}_{\underline{t}j}}})/S*}(\mathcal{O}_{(D_{\underline{\lambda}_{\underline{t}j}}, Z_{\underline{t}}|_{D_{\underline{\lambda}_{\underline{t}j}}})/S} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_{\underline{t}j}\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}}))(-(m-1))
\end{aligned}$$

be the Gysin morphism defined in [**72**, (2.8.4.5)]. Here

$$\varpi^{\log}_{\underline{\lambda}_{\underline{t}}\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}}) \quad \text{and} \quad \varpi^{\log}_{\underline{\lambda}_{\underline{t}j}\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}})$$

are the crystalline orientations sheaves of $D_{\underline{\lambda}_{\underline{t}}}$ and $D_{\underline{\lambda}_{\underline{t}j}}$ in $((D_{\underline{\lambda}_{\underline{t}}}, \widetilde{Z|_{D_{\underline{\lambda}_{\underline{t}}}}})/S)^{\log}_{\text{crys}}$ and $((D_{\underline{\lambda}_{\underline{t}j}}, \widetilde{Z|_{D_{\underline{\lambda}_{\underline{t}j}}}})/S)^{\log}_{\text{crys}}$, respectively, defined in [**72**, (2.2.18), (2.8)]. Set

$$G_{\underline{t}} := \sum_{\{\lambda_{\underline{t}0},\ldots,\lambda_{\underline{t},m-1} \mid \lambda_{\underline{t}i}\neq\lambda_{\underline{t}j}\ (i\neq j)\}} \sum_{j=0}^{m-1} (-1)^j G^{\lambda_{\underline{t}j}}_{\underline{\lambda}_{\underline{t}}}. \tag{5.3.2}$$

Set $e_j := (0,\ldots,0,\overset{j}{1},0,\ldots,0)$ $(1 \leq j \leq r)$. We can identify the morphisms (2.2.12) and (2.2.13) for the case $(M,P) = (\mathcal{K},P)$ with the following diagram (we use [**72**, (2.8.5)] for the morphism (2.2.13)):

$$\begin{array}{c}
R^{h-2\underline{t}_r-k} f_{(D^{(\underline{t}_r+k)}_{(\underline{t}+e_j)}, Z|_{D^{(\underline{t}_r+k)}_{(\underline{t}+e_j)}})/S*} (\mathcal{O}_{(D^{(\underline{t}_r+k)}_{(\underline{t}+e_j)}, Z|_{D^{(\underline{t}_r+k)}_{(\underline{t}+e_j)}})/S} \\
\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\text{crys}}(D_{(\underline{t}+e_j)}/S; Z_{(\underline{t}+e_j)}))(-(\underline{t}_r+k)) \\
\Big\uparrow {\scriptstyle (-1)^{\underline{t}_j-1}\sum_{i=0}^{\underline{t}_j+1}(-1)^i\delta^i_j \ (1\leq j\leq r)} \\
R^{h-2\underline{t}_r-k} f_{(D^{(\underline{t}_r+k)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S*} (\mathcal{O}_{(D^{(\underline{t}_r+k)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S} \\
\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\text{crys}}(D_{\underline{t}}/S; Z_{\underline{t}}))(-(\underline{t}_r+k)) \\
\Big\downarrow {\scriptstyle -(-1)^{\underline{t}_r}G_{\underline{t}}} \\
R^{h-2\underline{t}_r-k+2} f_{(D^{(\underline{t}_r+k-1)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k-1)}_{\underline{t}}})/S*} (\mathcal{O}_{(D^{(\underline{t}_r+k-1)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k-1)}_{\underline{t}}})/S} \\
\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k-1)\log}_{\text{crys}}(D_{\underline{t}}/S; Z_{\underline{t}}))(-(\underline{t}_r+k-1)).
\end{array} \tag{5.3.3}$$

Proposition 5.4. — *Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p > 0$. Let K be the fraction field of $\mathcal{V}$. Let S be a p-adic formal $\mathcal{V}$-scheme in the sense of* [**74**, §1]. *Set $S_0 = \underline{\text{Spec}}_S(\mathcal{O}_S/p)$. Assume that $\overset{\circ}{f}\colon X_{\bullet} \to S_0$ is proper. Then the log isocrystalline cohomology $R^h f_{(X_{\bullet},D_{\bullet}\cup Z_{\bullet})/S*}(\mathcal{O}_{(X_{\bullet},D_{\bullet}\cup Z_{\bullet})/S})_K$ $(h \in \mathbb{N})$ prolongs to a convergent F-isocrystal $R^h f_*(\mathcal{O}_{(X_{\bullet},D_{\bullet}\cup Z_{\bullet})/K})$ on $S/\mathcal{V}$.*

Proof. — The log iso-crystalline cohomology $R^s f_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S*}(\mathcal{O}_{(X_{\underline{t}},D_{\underline{t}}\cup Z_{\underline{t}})/S})_K$ prolongs to a convergent F-isocrystal by [**75**, Theorem 4]; in particular, for a morphism $g\colon T' \longrightarrow T$ of p-adic enlargements of $S/\mathcal{V}$, the natural morphism

$$g^* R^s f_{(X_{\underline{t}}, D_{\underline{t}} \cup Z_{\underline{t}})_{T_1}/T*}(\mathcal{O}_{(X_{\underline{t}}, D_{\underline{t}} \cup Z_{\underline{t}})_{T_1}/T})_K \longrightarrow R^s f_{(X_{\underline{t}}, D_{\underline{t}} \cup Z_{\underline{t}})_{T'_1}/T'*}(\mathcal{O}_{(X_{\underline{t}}, D_{\underline{t}} \cup Z_{\underline{t}})_{T'_1}/T'})_K$$

is an isomorphism, where $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/p)$ and $T'_1 := \underline{\mathrm{Spec}}_{T'}(\mathcal{O}_{T'}/p)$. Because (5.1.1) is functorial, the natural morphism

$$g^* R^{s+t} f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})_{T_1}/T*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})_{T_1}/T})_K \longrightarrow R^{s+t} f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})_{T'_1}/T'*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})_{T'_1}/T'})_K \tag{5.4.1}$$

is an isomorphism. Furthermore, by (5.1.1) and [**45**, (2.24)], the relative Frobenius on $R^{s+t} f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S})_K$ is bijective. Hence the log isocrystalline cohomology $R^h f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S})_K$ ($h \in \mathbb{N}$) prolongs to a convergent F-isocrystal $R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/K})$ on $S/\mathcal{V}$ by [**74**, (2.18)] as in [**74**, (3.7)]. □

PROPOSITION 5.5. — *Let the notations be as in* (5.4). *For $h, k \in \mathbb{Z}$, the E_2-term $E_2^{-k,h+k}((X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/S)_K$ of* (5.1.3) *tensorized with K prolongs to a convergent F-isocrystal on $S/\mathcal{V}$.*

Proof. — By [**72**, (2.13.10)], the Gysin morphism in (5.3.3) extends to a morphism of convergent F-isocrystals on $S/\mathcal{V}$. (5.5) immediately follows from the diagram (5.3.3), [**75**, Theorem 4] and [**74**, (2.10)]. □

THEOREM 5.6 (E_2-degeneration). — *Assume that $Z_{\underline{\bullet}} = \phi$ and that $\overset{\circ}{f}\colon X_{\underline{\bullet}} \to S_0$ is proper. Then the following hold*:

1) *Let κ be a perfect field of characteristic $p > 0$. If $S_0 = \mathrm{Spec}(\kappa)$ and if $S = \mathrm{Spf}(\mathcal{W})$, then* (5.1.3) *degenerates at E_2 modulo torsion.*

2) *Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p > 0$. If S is a p-adic formal $\mathcal{V}$-scheme and if $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$, then* (5.1.3) *degenerates at E_2 modulo torsion.*

Proof. — 1): The proof is the same as that of [**72**, (2.15.4)] by using (8.1) below.

2): By the same proof as that of [**72**, (2.17.2)], 2) follows from (5.4), (5.5), 1) and the log deformation invariance (8.3) below. □

COROLLARY 5.7. — 1) *Let k and h be two integers. Then*

$$P_k R^h f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}})/S})_K$$

prolongs to a convergent F-isocrystal $P_k R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}})/K})$ on $S/\mathcal{V}$.

2) *There exists the following spectral sequence of convergent F-isocrystals on $S/\mathcal{V}$:*

(5.7.1)

$$\begin{aligned} &E_1^{-k,h+k}\big((X_{\underline{\bullet}}, D_{\underline{\bullet}})/K\big) \\ &\quad = \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k} f_*(\mathcal{O}_{D_{\underline{t}}^{(\underline{t}_r+k)}/K} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}\big(D_{\underline{t}}/K)\big)\big(-(\underline{t}_r+k)\big) \\ &\qquad \Longrightarrow R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}})/K}). \end{aligned}$$

Here $R^h f_*(\mathcal{O}_{D_{\underline{t}}^{(\underline{t}_r+k)}/K} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}(D_{\underline{t}}/K))$ $(h \in \mathbb{Z})$ *is a convergent F-isocrystal on $S/\mathcal{V}$ characterized by the formula*

$$\begin{aligned} &R^h f_*\big(\mathcal{O}_{D_{\underline{t}}^{(\underline{t}_r+k)}/K} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}(D_{\underline{t}}/K)\big)_T \\ &\quad = R^h f_{(D_{\underline{t}}^{(\underline{t}_r+k)})_{T_1}/T*}\big(\mathcal{O}_{(D_{\underline{t}}^{(\underline{t}_r+k)})_{T_1}/T} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(\underline{t}_r+k)}((D_{\underline{t}})_{T_1}/T)\big) \end{aligned}$$

for a p-adic enlargement T of $S/\mathcal{V}$, where $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/p)$. *The spectral sequence* (5.7.1) *degenerates at* E_2.

Proof. — 1) and 2) immediately follow from (5.5) and (5.6), 2). □

Definition 5.8. — We call the spectral sequence (5.7.1) the *weight spectral sequence* of $R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}})/K})$.

As to the convergence in (5.7), 1), we can say more:

Theorem 5.9. — 1) *Let* $u\colon (S', \mathcal{I}', \gamma') \to (S, \mathcal{I}, \gamma)$ *and* $h\colon Y' \to Y$ *be as in* (3.8). *Let* $N_1, \dots, N_r$ *be nonnegative integers. Set* $\underline{N} := (N_1, \dots, N_r)$. *Let*

$$f\colon (X_{\underline{\bullet}\leq\underline{N}}, D_{\underline{\bullet}\leq\underline{N}} \cup Z_{\underline{\bullet}\leq\underline{N}}) \longrightarrow Y$$

be a morphism from a smooth $\underline{N}$-truncated r-simplicial scheme with transversal $\underline{N}$-truncated r-simplicial relative SNCD's $D_{\underline{\bullet}\leq\underline{N}}$ *and* $Z_{\underline{\bullet}\leq\underline{N}}$ *such that the morphism* $\overset{\circ}{f}\colon X_{\underline{\bullet}\leq\underline{N}} \to Y$ *is smooth, quasi-compact and quasi-separated. Let* $(X'_{\underline{\bullet}\leq\underline{N}}, D'_{\underline{\bullet}\leq\underline{N}} \cup Z'_{\underline{\bullet}\leq\underline{N}})$ *be the base change of* $(X_{\underline{\bullet}\leq\underline{N}}, D_{\underline{\bullet}\leq\underline{N}} \cup Z_{\underline{\bullet}\leq\underline{N}})$ *with respect to the morphism h and let* $f'\colon (X'_{\underline{\bullet}\leq\underline{N}}, D'_{\underline{\bullet}\leq\underline{N}} \cup Z'_{\underline{\bullet}\leq\underline{N}}) \to Y'$ *be the structural morphism. Then the base change morphism*

(5.9.1)

$$\begin{aligned} &Lh^*_{\mathrm{crys}}[Rf^{\log}_{(X_{\underline{\bullet}\leq\underline{N}}, Z_{\underline{\bullet}\leq\underline{N}})\mathrm{crys}*}(E_{\mathrm{crys}}^{\log, Z_{\underline{\bullet}\leq\underline{N}}}(\mathcal{O}_{(X_{\underline{\bullet}\leq\underline{N}}, D_{\underline{\bullet}\leq\underline{N}} \cup Z_{\underline{\bullet}\leq\underline{N}})/S})), \delta(\underline{L}, P^{D_{\underline{\bullet}\leq\underline{N}}})] \\ &\longrightarrow [Rf'^{\log}_{(X'_{\underline{\bullet}\leq\underline{N}}, Z'_{\underline{\bullet}\leq\underline{N}})\mathrm{crys}*}(E_{\mathrm{crys}}^{\log, Z'_{\underline{\bullet}\leq\underline{N}}}(\mathcal{O}_{(X'_{\underline{\bullet}\leq\underline{N}}, D'_{\underline{\bullet}\leq\underline{N}} \cup Z'_{\underline{\bullet}\leq\underline{N}})/S'})), \delta(\underline{L}, P^{D'_{\underline{\bullet}\leq\underline{N}}})] \end{aligned}$$

is an isomorphism in the filtered derived category $\mathrm{DF}(\mathcal{O}_{Y'/S'})$.

2) *Let* u *and* h *be as in* (3.8). *Let* $f\colon (X_{\underline{\bullet}}, D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}) \longrightarrow Y$ *be a morphism from a smooth* r*-simplicial scheme with transversal* r*-simplicial relative SNCD's* $D_{\underline{\bullet}}$ *and* $Z_{\underline{\bullet}}$ *such that the morphism* $\overset{\circ}{f}\colon X_{\underline{\bullet}} \longrightarrow Y$ *is smooth, quasi-compact and quasi-separated. Assume that*

$$\big(Rf^{\log}_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})\mathrm{crys}*}(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{D_{\underline{\bullet}}})\big)$$

is isomorphic to a filtered bounded above complex of $\mathcal{O}_{Y/S}$*-modules. Then the obvious analogue of* 1) *holds.*

3) *Let* $\mathcal{V}$, K, S, S_0 *and* $f\colon (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \to S_0$ *be as in* (5.4). *Let* k *and* h *be two integers. Then* $P^{D_{\underline{\bullet}}}_k R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})_K$ *prolongs to a convergent* F*-isocrystal* $P^{D_{\underline{\bullet}}}_k R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/K})$ *on* $S/\mathcal{V}$ *(recall the convention* (5.2)*).*

Proof. — 1): Let $h_{\underline{\bullet}\leq\underline{N}}\colon Y'_{\underline{\bullet}\leq\underline{N}} \to Y_{\underline{\bullet}\leq\underline{N}}$ be the morphism of constant $\underline{N}$-truncated r-simplicial schemes defined by h. Let

$$\big[Rf^{\log}_{(X_{\underline{\bullet}\leq\underline{N}},Z_{\underline{\bullet}\leq\underline{N}})_{\underline{\bullet}\leq\underline{N}}\mathrm{crys}*}(E^{\log,Z_{\underline{\bullet}\leq\underline{N}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}\leq\underline{N}},D_{\underline{\bullet}\leq\underline{N}}\cup Z_{\underline{\bullet}\leq\underline{N}})/S}), P^{D_{\underline{\bullet}\leq\underline{N}}})\big]$$

be the filtered cohomological complex in $\mathrm{D}^+\mathrm{F}(\mathcal{O}_{Y_{\underline{\bullet}\leq\underline{N}}/S})$ of the (pre)weight-filtered crystalline complex $(E^{\log,Z_{\underline{\bullet}\leq\underline{N}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}\leq\underline{N}},D_{\underline{\bullet}\leq\underline{N}}\cup Z_{\underline{\bullet}\leq\underline{N}})/S}), P^{D_{\underline{\bullet}\leq\underline{N}}})$. In fact, this filtered complex belongs to $\mathrm{D}^{\mathrm{b}}\mathrm{F}(\mathcal{O}_{Y_{\underline{\bullet}\leq\underline{N}}/S})$ by [**72**, (2.10.2) (2)]. Let

$$\big[Rf'^{\log}_{(X'_{\underline{\bullet}\leq\underline{N}},Z'_{\underline{\bullet}\leq\underline{N}})_{\underline{\bullet}\leq\underline{N}}\mathrm{crys}*}(E^{\log,Z'_{\underline{\bullet}\leq\underline{N}}}_{\mathrm{crys}}(\mathcal{O}_{(X'_{\underline{\bullet}\leq\underline{N}},D'_{\underline{\bullet}\leq\underline{N}}\cup Z'_{\underline{\bullet}\leq\underline{N}})/S}), P^{D'_{\underline{\bullet}\leq\underline{N}}})\big]$$

be the analogous filtered complex in $\mathrm{D}^+\mathrm{F}(\mathcal{O}_{Y'_{\underline{\bullet}\leq\underline{N}}/S})$. Then, by the base change theorem for the constant simplicial case (see [**72**, (2.10.3)]), the base change morphism

(5.9.2)

$$Lh^*_{\underline{\bullet}\leq\underline{N}\mathrm{crys}}[Rf^{\log}_{(X_{\underline{\bullet}\leq\underline{N}},Z_{\underline{\bullet}\leq\underline{N}})_{\underline{\bullet}\leq\underline{N}}\mathrm{crys}*}(E^{\log,Z_{\underline{\bullet}\leq\underline{N}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}\leq\underline{N}},D_{\underline{\bullet}\leq\underline{N}}\cup Z_{\underline{\bullet}\leq\underline{N}})/S}), P^{D_{\underline{\bullet}\leq\underline{N}}})]$$
$$\longrightarrow [Rf'^{\log}_{(X'_{\underline{\bullet}\leq\underline{N}},Z'_{\underline{\bullet}\leq\underline{N}})_{\underline{\bullet}\leq\underline{N}}\mathrm{crys}*}(E^{\log,Z'_{\underline{\bullet}\leq\underline{N}}}_{\mathrm{crys}}(\mathcal{O}_{(X'_{\underline{\bullet}\leq\underline{N}},D'_{\underline{\bullet}\leq\underline{N}}\cup Z'_{\underline{\bullet}\leq\underline{N}})/S}), P^{D'_{\underline{\bullet}\leq\underline{N}}})]$$

is an isomorphism. Here note that, for a morphism $g\colon (\mathcal{T}_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}_{\underline{\bullet}\leq\underline{N}}) \to (\mathcal{T}'_{\underline{\bullet}\leq\underline{N}}, \mathcal{A}'_{\underline{\bullet}\leq\underline{N}})$ of $\underline{N}$-truncated r-simplicial ringed topoi, the derived functor

$$Lg^*_{\underline{\bullet}\leq\underline{N}}\colon \mathrm{D}^-\mathrm{F}(\mathcal{A}'_{\underline{\bullet}\leq\underline{N}}) \longrightarrow \mathrm{D}^-\mathrm{F}(\mathcal{A}_{\underline{\bullet}\leq\underline{N}}) \tag{5.9.3}$$

is defined by using the functor $L^{\bullet}$ in [**12**, 7.7–7.8] (cf. [**72**, (1.1.20)]). By applying the functor $(\mathbf{s}, \delta)$ ((2.2.16)) to (5.9.2) and using an easily-verified equation $(\mathbf{s}Lh^*_{\underline{\bullet}\leq\underline{N}\mathrm{crys}}, \delta) = Lh^*_{\mathrm{crys}}(\mathbf{s}, \delta)$, we obtain the isomorphism (5.9.1).

2): The proof is the same as that of 1).

3): Follows from 1) by the argument in [**72**, (2.13.3)]. □

Let N be a nonnegative integer. Let $(X^i_{\bullet\leq N}, D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})/Y$ $(i=1,2,3)$ be an N-truncated simplicial version of $(X^i, D^i\cup Z^i)/Y$ in (3.10). Let $f^i\colon (X^i_{\bullet\leq N}, D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})\to Y$ $(i=1,2,3)$ be the structural morphism. Denote

$$\big[Rf^{i\,\log}_{(X^i_{\bullet\leq N},Z^i_{\bullet\leq N})\mathrm{crys}*}(E^{\log,Z^i_{\bullet\leq N}}_{\mathrm{crys}}(\mathcal{O}_{(X^i_{\bullet\leq N},D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})/S})),\delta(L,P^{D^i_{\bullet\leq N}})\big]$$

by

$$\big[Rf^{\log}_{(X^i_{\bullet\leq N},Z^i_{\bullet\leq N})\mathrm{crys}*}(E^{\log,Z^i_{\bullet\leq N}}_{\mathrm{crys}}(\mathcal{O}_{(X^i_{\bullet\leq N},D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})/S})),\delta(L,P^{D^i_{\bullet\leq N}})\big].$$

For an r-simplicial version $(X^i_{\underline{\bullet}}, D^i_{\underline{\bullet}}\cup Z^i_{\underline{\bullet}})/Y$ of $(X^i, D^i\cup Z^i)/Y$ $(i=1,2,3)$ in (3.10), we use the analogous notation. Then the following holds:

Theorem 5.10. — 1) *For $i=1,2,3$, set*

$$(F^\bullet_i,\delta_i):=\big[Rf^{\log}_{(X^i_{\bullet\leq N},Z^i_{\bullet\leq N})\mathrm{crys}*}(E^{\log,Z^i_{\bullet\leq N}}_{\mathrm{crys}}(\mathcal{O}_{(X^i_{\bullet\leq N},D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})/S})),\delta(L,P^{D^i_{\bullet\leq N}})\big].$$

Then there exists the following canonical isomorphism:

$$\mathcal{H}^h\big[(\delta_1\otimes^L_{\mathcal{O}_S}\delta_2)_k\{(F^\bullet_1,\delta_1)\otimes^L_{\mathcal{O}_S}(F^\bullet_2,\delta_2)\}\big]\xrightarrow{\ \sim\ }\mathcal{H}^h\big[(\delta_3)_k(F^\bullet_3,\delta_3)\big] \tag{5.10.1}$$

for a nonnegative integer h such that $2^{-1}(h+1)(h+2)<N$.

2) *Assume that*

$$\big[Rf^{\log}_{(X^i_{\underline{\bullet}},Z^i_{\underline{\bullet}})\mathrm{crys}*}(E^{\log,Z^i_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X^i_{\underline{\bullet}},D^i_{\underline{\bullet}}\cup Z^i_{\underline{\bullet}})/S})),\delta(L,P^{D^i_{\underline{\bullet}}})\big]\quad(i=1,2)$$

is isomorphic to a filtered bounded above complex of $\mathcal{O}_{Y/S}$-modules. Then there exists the following canonical isomorphism

$$\begin{aligned}\big[Rf^{\log}_{(X^1_{\underline{\bullet}},Z^1_{\underline{\bullet}})\mathrm{crys}*}&(E^{\log,Z^1_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X^1_{\underline{\bullet}},D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}})/S})),\delta(L,P^{D^1_{\underline{\bullet}}})\big]\otimes^L_{\mathcal{O}_{Y/S}}\\ &\big[Rf^{\log}_{(X^2_{\underline{\bullet}},Z^2_{\underline{\bullet}})\mathrm{crys}*}(E^{\log,Z^2_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X^2_{\underline{\bullet}},D^2_{\underline{\bullet}}\cup Z^2_{\underline{\bullet}})/S})),\delta(L,P^{D^2_{\underline{\bullet}}})\big]\\ &\xrightarrow{\ \sim\ }\big[Rf^{\log}_{(X^3_{\underline{\bullet}},Z^3_{\underline{\bullet}})\mathrm{crys}*}(E^{\log,Z^3_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X^3_{\underline{\bullet}},D^3_{\underline{\bullet}}\cup Z^3_{\underline{\bullet}})/S})),\delta(L,P^{D^3_{\underline{\bullet}}})\big].\end{aligned} \tag{5.10.2}$$

The isomorphism (5.10.2) *is compatible with the base change isomorphism in* (5.9), 2).

3) *Let $\mathcal{V}$, K, S and S_0 be as in* (5.4). *Assume that $X^i_\bullet\to S_0$ $(i=1,2)$ is proper. Then the following canonical morphism*

$$\begin{aligned}\bigoplus_{i+j=h}R^if_*(\mathcal{O}_{(X^1_\bullet,D^1_\bullet\cup Z^1_\bullet)/K})\otimes_{\mathcal{O}_{S/K}}R^jf_*(\mathcal{O}_{(X^2_\bullet,D^2_\bullet\cup Z^2_\bullet)/K})\\ \longrightarrow R^hf_*(\mathcal{O}_{(X^3_\bullet,D^3_\bullet\cup Z^3_\bullet)/K})\end{aligned} \tag{5.10.3}$$

is an isomorphism of convergent F-isocrystals on $S/\mathcal{V}$. The isomorphism above is compatible with the weight filtration $P^{D^i_\bullet}$ $(i=1,2,3)$.

Proof. — 1): Let $(\mathcal{R}_i^{\bullet\leq N},\delta_i)$ $(i=1,2,3)$ be a representative of

$$\big[Rf^{\log}_{(X^i_{\bullet\leq N},Z^i_{\bullet\leq N})\mathrm{crys}*}(E^{\log,Z^i_{\bullet\leq N}}_{\mathrm{crys}}(\mathcal{O}_{(X^i_{\bullet\leq N},D^i_{\bullet\leq N}\cup Z^i_{\bullet\leq N})/S})),\delta(L,P^{D^i_{\bullet\leq N}})\big].$$

Let $(\mathcal{L}_i^{\bullet\leq N},\delta_i)$ $(i=1,2)$ be a filtered flat resolution of $(\mathcal{R}_i^{\bullet\leq N},\delta_i)$ (see [**72**, (1.1.17) (2)]). Let $\mathrm{CF}(\mathcal{O}_S)$ be the category of filtered complexes of $\mathcal{O}_S$-modules. Let $\mathrm{CF}(\mathcal{O}_{S_\bullet})$ be the category of cosimplicial filtered complexes of $\mathcal{O}_{S_\bullet}$-modules and $\mathrm{CF}(\mathcal{O}_{S_{\bullet\leq N}})$ the category of N-truncated cosimplicial filtered complexes of $\mathcal{O}_{S_{\bullet\leq N}}$-modules. Consider the opposite categories $\mathrm{CF}(\mathcal{O}_S)^o$, $\mathrm{CF}(\mathcal{O}_{S_\bullet})^o$ and $\mathrm{CF}(\mathcal{O}_{S_{\bullet\leq N}})^o$ of $\mathrm{CF}(\mathcal{O}_S)$, $\mathrm{CF}(\mathcal{O}_{S_\bullet})$ and $\mathrm{CF}(\mathcal{O}_{S_{\bullet\leq N}})$, respectively. Because the finite projective limit exists in $\mathrm{CF}(\mathcal{O}_S)^o$, we have the coskeleton

$$\mathrm{cosk}_N\colon \mathrm{CF}(\mathcal{O}_{S_{\bullet\leq N}})^o \longrightarrow \mathrm{CF}(\mathcal{O}_{S_\bullet})^o.$$

Let

$$\mathrm{cosk}^o_N\colon \mathrm{CF}(\mathcal{O}_{S_{\bullet\leq N}}) \longrightarrow \mathrm{CF}(\mathcal{O}_{S_\bullet})$$

be the resulting functor. Set $(\mathcal{M}_i^{\bullet\bullet},\delta_i) := \mathrm{cosk}^o_N((\mathcal{L}_i^{\bullet\leq N},\delta_i))$. Then, by Eilenberg-Zilber's theorem [**89**, Theorem (8.5.1)] (cf. [*loc. cit.*, Theorem (8.3.8)]), we have a canonical isomorphism

$$\big(\mathbf{s}(\mathcal{M}_1^{\bullet_1\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{\bullet_2\bullet}),\delta_1\otimes\delta_2\big) \xrightarrow{\sim} \big(\mathbf{s}(\mathcal{M}_1^{\bullet_1\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{\bullet_1\bullet}),\delta_1\otimes\delta_2\big)$$

in $\mathrm{D}^+\mathrm{F}(\mathcal{O}_S)$. For an integer k, consider the spectral sequence

$$E_1^{ij} := \mathcal{H}^j\big((\delta_1\otimes\delta_2)_k(\mathbf{s}(\mathcal{M}_1^{i\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{i\bullet}))\big) \Longrightarrow \mathcal{H}^{i+j}\big((\delta_1\otimes\delta_2)_k(\mathbf{s}(\mathcal{M}_1^{\bullet_1\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{\bullet_1\bullet}))\big).$$

Because h and N satisfy the inequality in (2.2.1), the natural morphism

$$\begin{aligned}(5.10.4)\qquad &\mathcal{H}^h\big((\delta_1\otimes\delta_2)_k(\mathbf{s}(\mathcal{M}_1^{\bullet_1\leq N,\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{\bullet_1\leq N,\bullet}))\big)\\ &\qquad\longrightarrow \mathcal{H}^h\big((\delta_1\otimes\delta_2)_k(\mathbf{s}(\mathcal{M}_1^{\bullet_1\bullet}\otimes_{\mathcal{O}_S}\mathcal{M}_2^{\bullet_1\bullet}))\big)\end{aligned}$$

is an isomorphism by (2.2). The left hand side of (5.10.4) is equal to $\mathcal{H}^h((\delta_1\otimes\delta_2)_k(\mathbf{s}(\mathcal{L}_1^{\bullet_1\leq N,\bullet}\otimes_{\mathcal{O}_S}\mathcal{L}_2^{\bullet_1\leq N,\bullet})))$, and this is equal to $\mathcal{H}^h((\delta_3)_k\mathcal{R}_3^{\bullet\leq N})$ by the filtered Künneth formula for the constant case ((3.10)).

2): Follows immediately from the inductive formula (2.2.11), Eilenberg-Zilber's theorem and (3.10).

3): Follows from 1) and (5.9), 3). □

6. Explicit description of the truncated cosimplicial preweight-filtered vanishing cycle zariskian complex

In this section we give an explicit description of the truncated cosimplicial preweight-filtered vanishing cycle zariskian complex. To give the explicit description, we have to construct a nice embedding system. To construct this, we use Tsuzuki's functor in [**19**] and [**86**]. The concepts of the decomposition datum of a fine log scheme and the quasi-global chart of it, which will be defined in (6.7) below, also play a fundamental role in the construction. The Čech-Alexander complex in [**5**] is also indispensable for the proof of the explicit description.

Let Δ and $[n]$ ($n \in \mathbb{N}$) be as before (4.14). Let Y be a fine log (formal) scheme over a fine log (formal) scheme T. Let $Y_\bullet$ be a fine simplicial log (formal) scheme over Y. Let $s_i^{m-1}\colon Y_{m-1} \longrightarrow Y_m$ ($m \in \mathbb{Z}_{>0}, 0 \leq i \leq m-1$) be the degeneracy morphism corresponding to the standard degeneracy map $\partial_m^i\colon [m] \to [m-1]$: $\partial_m^i(j) = j$ $(0 \leq j \leq i)$, $\partial_m^i(j) = j-1$ $(i < j \leq m)$. Following [**35**, $\mathrm{V}^{\mathrm{bis}}$ §5] and [**25**, (6.2.1.1)], set $N(Y_0) := Y_0$ and let $N(Y_m)$ ($m \in \mathbb{Z}_{>0}$) be the intersection of the complements of $s_i^{m-1}(Y_{m-1})$ ($0 \leq i \leq m-1$). The simplicial log (formal) scheme $Y_\bullet$ is said to be split (see [*loc. cit.*]) if $Y_m = \coprod_{0\leq \ell \leq m} \coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)$, where the subscripts $[m] \twoheadrightarrow [\ell]$'s run through the surjective non-decreasing morphisms $[m] \to [\ell]$'s. In this section we assume that $Y_\bullet$ is split.

The following is one of key lemmas for (11.6), 3) below which is one of main results in this book.

LEMMA 6.1. — *There exists a fine split simplicial log* (*formal*) *scheme* $Y'_\bullet$ *with a morphism* $Y'_\bullet \longrightarrow Y_\bullet$ *of simplicial log* (*formal*) *schemes over* Y *satisfying the following conditions*:

(6.1.1) Y'_m ($m \in \mathbb{N}$) *is the disjoint union of log affine open* (*formal*) *subschemes which cover* Y_m *and whose images in* Y *are contained in log affine open* (*formal*) *subschemes of* Y.

(6.1.2) *If* $\overset{\circ}{Y}_m$ ($m \in \mathbb{N}$) *is quasi-compact, then the number of the log affine open* (*formal*) *subschemes in* (6.1.1) *can be assumed to be finite.*

Set $Y_{mn} := \mathrm{cosk}_0^{Y_m}(Y'_m)_n$ ($m, n \in \mathbb{N}$). *Then there exists a natural morphism* $Y_{\bullet\bullet} \to Y_\bullet$ *over* Y. *For each* $n \in \mathbb{N}$, $Y_{\bullet n}$ *is split.*

Proof. — Let $g\colon T' \longrightarrow T$ be a morphism of ringed spaces. Let $\{T_i\}_i$ be an open covering of T. Then we have a natural morphism $\coprod_i g^{-1}(T_i) \longrightarrow \coprod_i T_i$.

Let $\{T'_{j(i)}\}_{j(i)}$ be an open covering of $g^{-1}(T_i)$ $(\forall i)$. Then we have the following natural composite morphism

$$\coprod_i \coprod_{j(i)} T'_{j(i)} \longrightarrow \coprod_i g^{-1}(T_i) \longrightarrow \coprod_i T_i.$$

Let Y'_0 be the disjoint union of log affine open (formal) subschemes which cover Y_0 and whose images in Y are contained in log affine open (formal) subschemes of Y. Let m be a positive integer. Assume that we are given $Y'_{\bullet\leq m-1}$. We construct Y'_m. Let Δ^+ be a subcategory of Δ whose objects are those of Δ and whose morphisms are injective nondecreasing functions. For a nonnegative integer r, let $\Delta_{[r]}$ (resp. $\Delta^+_{[r]}$) be the category of objects of Δ (resp. Δ^+) augmented to $[r]$. For a nonnegative integer N, let $\Delta_{N[r]}$ (resp. $\Delta^+_{N[r]}$) be a full subcategory of $\Delta_{[r]}$ (resp. $\Delta^+_{[r]}$) whose objects are $[q]\to[r]$'s such that $q\leq N$. The inverse limit of a finite inverse system exists in the category of fine log (formal) schemes (over a fine log (formal) scheme) (see [**51**, (1.6), (2.8)]). Recall the following explicit description of the coskeleton (e.g., [**35**, $\mathrm{V}^{\mathrm{bis}}$, (3.0.1.2)]):

$$\mathrm{cosk}^Y_N(Y_{\bullet\leq N})_r = \varprojlim_{\Delta_{N[r]}} Y_q = \varprojlim_{\Delta^+_{N[r]}} Y_q \quad (N,r\in\mathbb{N}), \tag{6.1.3}$$

where the projective limits are taken in the category of log (formal) schemes over Y. The log scheme $\mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_m$ is the disjoint union of the members of a log affine open covering of $\mathrm{cosk}^Y_{m-1}(Y_{\bullet\leq m-1})_m$. Consider the natural composite morphism $N(Y_m)\hookrightarrow Y_m\to \mathrm{cosk}^Y_{m-1}(Y_{\bullet\leq m-1})_m$ and a log affine open covering of $N(Y_m)$ which refines the inverse image of the open covering of $\mathrm{cosk}^Y_{m-1}(Y_{\bullet\leq m-1})_m$. Then the image of each log affine open (formal) subscheme of $N(Y_m)$ in Y is contained in a log affine open (formal) subscheme of Y. Let $N(Y_m)'$ be the disjoint union of the members of this open covering. Then we have the following commutative diagram:

$$\begin{array}{ccc} N(Y_m)' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_m \\ \downarrow & & \downarrow \\ N(Y_m)(\subset Y_m) & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y_{\bullet\leq m-1})_m. \end{array} \tag{6.1.4}$$

Set $Y'_m = \coprod_{0\leq\ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)'$, where the subscripts $[m]\twoheadrightarrow[\ell]$'s run through the surjective non-decreasing morphisms $[m]\longrightarrow[\ell]$'s. Since $Y_\bullet$ is split, Y'_m is a desired log (formal) scheme over Y, and we have a natural morphism $Y'_\bullet\longrightarrow Y_\bullet$ of simplicial log (formal) schemes over Y by (6.1.4) and [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.3)].

The finiteness claimed in (6.1.2) is easy to check.

The splitness of $Y_{\bullet n}$ is clear by the construction. Indeed, express $Y_m = \coprod_{0\leq\ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)$ as in [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.1)]. Then, by the construction, according to the decomposition of Y_m, we have the decomposition $Y'_m = \coprod_{0\leq\ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)'$. Since $N(Y_\ell)'$ and $N(Y_{\ell'})'$ $((\gamma\colon [m]\to[\ell]) \neq (\gamma'\colon [m]\to[\ell']))$ have no intersection in Y_m,

$$\mathrm{cosk}_0^{Y_m}(Y'_m)_n \simeq \coprod_{0\leq\ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]} \mathrm{cosk}_0^{N(Y_\ell)}(N(Y_\ell)')_n \quad (n\in\mathbb{N}).$$

Thus $Y_{\bullet n}$ is split for each $n \in \mathbb{N}$. □

Definition 6.2. — Let Y be as in (6.1).

1) We call the simplicial log (formal) scheme $Y'_\bullet$ satisfying (6.1.1) and (6.1.2) the *disjoint union of the members of an affine simplicial open covering* of $Y_\bullet/Y$.

2) We call the bisimplicial scheme $Y_{\bullet\bullet}$ in (6.1) the *Čech diagram* of $Y'_\bullet$ over $Y_\bullet/Y$.

3) In 1) and 2), if $Y = T$, then we say “over T” instead of “over Y”.

Proposition 6.3. — 1) *Let $Z_\bullet \longrightarrow Y_\bullet$ be a morphism of fine split simplicial log* (*formal*) *schemes over a morphism $Z \longrightarrow Y$ of fine log* (*formal*) *schemes over T. Then there exist the disjoint unions of the members of affine simplicial open coverings $Y'_\bullet$ and $Z'_\bullet$ of $Y_\bullet/Y$ and $Z_\bullet/Z$, respectively, which fit into the following commutative diagram*:

$$\begin{array}{ccc} Z'_\bullet & \longrightarrow & Y'_\bullet \\ \big\downarrow & & \big\downarrow \\ Z_\bullet & \longrightarrow & Y_\bullet. \end{array} \tag{6.3.1}$$

2) *Let $Y_\bullet/Y/T$ be as in* (6.1). *Let $Y'_\bullet$ and $Y''_\bullet$ be two disjoint unions of the members of affine simplicial open coverings of $Y_\bullet/Y$. Then there exists a disjoint union of the members of an affine simplicial open covering $Y'''_\bullet$ of $Y_\bullet/Y$ fitting into the following commutative diagram*:

$$\begin{array}{ccc} Y'''_\bullet & \longrightarrow & Y''_\bullet \\ \big\downarrow & & \big\downarrow \\ Y'_\bullet & \longrightarrow & Y_\bullet. \end{array} \tag{6.3.2}$$

Proof. — 1): Take the disjoint union of the members of an affine simplicial open covering $Y'_\bullet$ of $Y_\bullet/Y$. Let $Z''_\bullet$ be the fiber product of $Z_\bullet$ and $Y'_\bullet$ over $Y_\bullet$.

Then

$$(6.3.3)\quad Z''_m = \coprod_{0\leq \ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]}(N(Z_\ell)\times_{Y_m} Y'_m) = \coprod_{0\leq \ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]}(N(Z_\ell)\times_{Y_\ell} Y'_\ell).$$

(Here note that the morphism $Y_\ell \to Y_m$ corresponding to a morphism $[m] \twoheadrightarrow [\ell]$ is an immersion.) Using the argument in the proof of (6.1), we obtain 1).

2): Take Y_0''' as an affine refinement of the disjoint unions Y_0' and Y_0'' of the open coverings of Y_0. For a positive integer m, assume that we are given $Y'''_{\bullet\leq m-1}$. Let $N(Y_m)'''$ be the scheme constructed in the proof of (6.1) for $Y'''_{\bullet\leq m-1}$ fitting into the commutative diagram

$$\begin{array}{ccc}
N(Y_m)' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_m \\
\uparrow & & \uparrow \\
N(Y_m)''' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y'''_{\bullet\leq m-1})_m \\
\downarrow & & \downarrow \\
N(Y_m)'' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y''_{\bullet\leq m-1})_m .
\end{array}$$

Set $Y'''_m = \coprod_{0\leq\ell\leq m}\coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)'''$. In this way, we obtain $Y'''_\bullet$. ☐

Next we need the obvious log version $\Gamma^T_N(Y)^{\leq \ell}$ $(N, \ell \in \mathbb{N})$ of the simplicial scheme in [**19**, §11].

Let $\mathcal{C}$ be a category which has finite inverse limits. Let ℓ be a nonnegative integer. Following [**19**, §11], define a set $\mathrm{Hom}^{\leq \ell}_{\Delta}([n],[m])$ $(n, m \in \mathbb{N})$: the set $\mathrm{Hom}^{\leq \ell}_{\Delta}([n],[m])$, by definition, consists of the morphisms $\gamma\colon [n] \to [m]$ in Δ such that the cardinality of the set $\gamma([n])$ is less than or equal to ℓ.

Let N be a nonnegative integer. Let us recall the definition of the simplicial object $\Gamma := \Gamma^{\mathcal{C}}_N(X)^{\leq \ell}$ in $\mathcal{C}$ for an object $X \in \mathcal{C}$ (see [**87**, (7.3.1)]): for an object $[m]$, set

$$\Gamma_m := \prod_{\gamma\in \mathrm{Hom}^{\leq \ell}_{\Delta}([N],[m])} X_\gamma$$

with $X_\gamma = X$; for a morphism $\alpha\colon [m'] \to [m]$ in Δ, $\alpha_\Gamma\colon \Gamma_m \to \Gamma_{m'}$ is defined to be the following: "$(c_\gamma) \mapsto (d_\beta)$" with "$d_\beta = c_{\alpha\beta}$". In fact, we have a functor

$$(6.3.4)\qquad \Gamma^{\mathcal{C}}_N(?)^{\leq \ell}\colon \mathcal{C} \longrightarrow \mathcal{C}^{\Delta} := \{\text{simplicial objects of } \mathcal{C}\}.$$

Set $\Gamma^{\mathcal{C}}_N(X) = \Gamma^{\mathcal{C}}_N(X)^{\leq N}$. Then $\Gamma^{\mathcal{C}}_N(X)^{\leq \ell} = \Gamma^{\mathcal{C}}_N(X)$ for $\ell \geq N$. Obviously the functor $\Gamma^{\mathcal{C}}_N(?)^{\leq \ell}$ commutes with finite inverse limits.

Definition 6.4. — We call the functor $\Gamma^{\mathcal{C}}_N(?)$ *Tsuzuki's functor*; for an object X of $\mathcal{C}$, we call $\Gamma^{\mathcal{C}}_N(X)$ *Tsuzuki's simplicial object* of X.

As in the proof of [**19**, (11.2.5)], we have the following by the adjointness of the coskeleton and the skeleton:

LEMMA 6.5 (see [**87**, (7.3.2)]). — $\mathrm{cosk}_\ell^{\mathcal{C}}(\Gamma_N^{\mathcal{C}}(X)_{\bullet\leq\ell}) = \Gamma_N^{\mathcal{C}}(X)^{\leq\ell}$.

Let N be a nonnegative integer. Let $X_{\bullet\leq N}$ be an N-truncated simplicial object of $\mathcal{C}$ and let $f\colon X_N \to Y$ be a morphism in $\mathcal{C}$. Then we have a morphism

$$X_{\bullet\leq N} \longrightarrow \Gamma_N^{\mathcal{C}}(Y)_{\bullet\leq N} \tag{6.5.1}$$

defined by the following commutative diagram:

$$\begin{array}{ccc} X_m & \longrightarrow & \prod_{\gamma\in\mathrm{Hom}_\Delta([N],[m])} Y_\gamma \\ {\scriptstyle X(\gamma)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{proj}} \\ X_N & \xrightarrow{\ f\ } & Y_\gamma = Y. \end{array} \tag{6.5.2}$$

Let T be a fine log (formal) scheme. Let $\mathcal{C}_T$ be the category of fine log (formal) schemes over T. Set $\Gamma_N^T(?) := \Gamma_N^{\mathcal{C}_T}(?)$ and $\mathrm{cosk}_N^T(?) = \mathrm{cosk}_N^{\mathcal{C}_T}(?)$. We now limit ourselves to the case $\mathcal{C} = \mathcal{C}_T$. As remarked in the proof of (6.1), the finite inverse limit exists in $\mathcal{C}_T$ (see [**51**, (1.6), (2.8)]). Hence we can apply the constructions above to $\mathcal{C}_T$.

The following are immediate generalizations of [**19**, (11.2.4), (11.2.6)]:

LEMMA 6.6. — *Let N be a nonnegative integer. Then the following hold*:

1) *Let $X_{\bullet\leq N}$ be a fine N-truncated simplicial log* (*formal*) *scheme over T. If $X_N \to Y$ is a closed immersion of fine log* (*formal*) *schemes over T, then the morphism $X_{\bullet\leq N} \to \Gamma_N^T(Y)_{\bullet\leq N}$ in* (6.5.1) *is an immersion of N-truncated fine log* (*formal*) *schemes over T. Moreover, if $\overset{\circ}{X}_{\bullet\leq N}$ and $\overset{\circ}{Y}$ are separated over $\overset{\circ}{T}$, then the morphism $X_{\bullet\leq N} \to \Gamma_N^T(Y)_{\bullet\leq N}$ is a closed immersion.*

2) *Let $X \to T$ be a morphism of fine log* (*formal*) *schemes. Assume that the morphism*

$$\underbrace{X \times_T \cdots \times_T X}_{n\,\text{times}} \xrightarrow{\mathrm{proj}} \underbrace{X \times_T \cdots \times_T X}_{m\,\text{times}}$$

for any $n \geq m$ satisfies a condition (P). *Then the natural morphism*

$$\mathrm{cosk}_\ell^T\bigl(\Gamma_N^T(X)_{\bullet\leq\ell}\bigr)_m \longrightarrow \mathrm{cosk}_{\ell'}^T\bigl(\Gamma_N^T(X)_{\bullet\leq\ell'}\bigr)_m$$

satisfies (P) *for $\ell' \leq \ell$ and for any $m \in \mathbb{N}$.*

Proof. — Because the proof is almost the same as that of [*loc. cit.*], we mention only why we require the separateness of $\overset{\circ}{X}_{\bullet\leq N}/\overset{\circ}{T}$ in the latter statement of 1), which is not assumed in [*loc. cit.*]. Let $\overset{\circ}{s}_i \colon \overset{\circ}{X}_m \to \overset{\circ}{X}_{m+1}$ $(0 \leq m \leq N-1,$ $0 \leq i \leq m)$ be the standard degeneracy morphism; let $\overset{\circ}{\delta}_i \colon \overset{\circ}{X}_{m+1} \to \overset{\circ}{X}_m$ $(0 \leq m \leq N-1,\ 0 \leq i \leq m+1)$ be the standard face morphism. Then, by [**37**, (5.5.1) (v)], $\overset{\circ}{\delta}_i$ is separated. Since $\overset{\circ}{s}_i$ is a section of $\overset{\circ}{\delta}_i$, $\overset{\circ}{s}_i$ is a closed immersion by [**37**, (5.4.6)]. In fact, s_i is a closed immersion. This is why we assume the separateness of $\overset{\circ}{X}_{\bullet \leq N}/T$. □

Next we define the decomposition datum of a fine log scheme and the quasi-global chart of it.

Definition 6.7. — 1) Let Z be a fine log scheme over a fine log scheme Y. We call a direct sum $Z = \coprod_{\lambda \in \Lambda} Z_\lambda$ by log open and closed subschemes of Z the *decomposition* of Z. We call the set $\mathfrak{D} := \{Z_\lambda\}_{\lambda \in \Lambda}$ the *decomposition datum* of Z. We call the decomposition datum $\{Z\}$ of Z the *trivial decomposition datum* of Z. Let $\mathfrak{D}' := \{Z'_\mu\}_{\mu \in M}$ be a decomposition datum of another fine log scheme Z' over Y. We define a morphism $f \colon (Z', \mathfrak{D}') \to (Z, \mathfrak{D})$ of fine log schemes over Y with decomposition data as a morphism $f \colon Z' \to Z$ of log schemes over Y such that there exists a function $\ell \colon M \longrightarrow \Lambda$ such that $f|_{Z'_\mu}$ factors through a morphism $f_\mu \colon Z'_\mu \to Z_{\ell(\mu)}$ for any $\mu \in M$. Let (LSchD/Y) be the category of fine log schemes over Y with decomposition data (see (6.8), 1) below).

2) Let the notations be as in 1). We say that Z has a *quasi-global chart* with respect to $\mathfrak{D}$ if Z_λ has a global chart $P_\lambda \longrightarrow M_{Z_\lambda}$ for any $\lambda \in \Lambda$. In this case, we also say that $\mathfrak{D}$ has a *quasi-global chart.* Set

$$\mathfrak{C} := \{P_\lambda\}_{\lambda \in \Lambda} := \{P_\lambda \longrightarrow M_{Z_\lambda}\}_{\lambda \in \Lambda}.$$

We define the *quasi-global chart* of the morphism $f \colon (Z', \mathfrak{D}') \to (Z, \mathfrak{D})$ as a family of charts

$$\mathfrak{F} := \big\{(P'_\mu \longrightarrow M_{Z'_\mu}, P_{\ell(\mu)} \longrightarrow M_{Z_{\ell(\mu)}}, P_{\ell(\mu)} \longrightarrow P'_\mu)\big\}_{\mu \in M}$$

of f_μ. Set $\mathfrak{C}' := \{P'_\mu\}_{\mu \in M}$. More precisely, we call the triple $(\mathfrak{C}', \mathfrak{C}, \mathfrak{F})$ the quasi-global chart of f. In this case, we say that we have a morphism $f \colon (Z', \mathfrak{D}', \mathfrak{C}') \longrightarrow (Z, \mathfrak{D}, \mathfrak{C})$ of fine log schemes with decomposition data and quasi-global charts with respect to the decomposition data. Let (LSchDC/Y) be the category of fine log schemes over Y with decomposition data and quasi-global charts with respect to the decomposition data (see (6.8), 4) below).

3) Let the notations be as in 1). Assume that Z has a quasi-global chart $P_\lambda \to M_{Z_\lambda}$ $(\forall \lambda \in \Lambda)$ with respect to $\mathfrak{D}$. Let $P^{\mathrm{ex}}_{\ell(\mu)}$ be the inverse image

of $\Gamma(Z'_\mu, M_{Z'_\mu})$ by the composite morphism $P^{\mathrm{gp}}_{\ell(\mu)} \to \Gamma(Z_{\ell(\mu)}, M_{Z_{\ell(\mu)}})^{\mathrm{gp}} \to \Gamma(Z'_\mu, M_{Z'_\mu})^{\mathrm{gp}}$. By abuse of notation, we denote simply by Z^{qe} the following fine log scheme

$$(6.7.1) \qquad \coprod_{\mu\in M} \big(Z_{\ell(\mu)} \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[P_{\ell(\mu)}])} \mathrm{Spec}^{\log}(\mathbb{Z}[P^{\mathrm{ex}}_{\ell(\mu)}])\big) \sqcup \coprod_{\lambda\neq\ell(\mu)} \big(Z_\lambda \times_{\mathrm{Spec}^{\log}(\mathbb{Z}[P_\lambda])} \mathrm{Spec}^{\log}(\mathbb{Z}[P^{\mathrm{gp}}_\lambda])\big)$$

and we call the log scheme (6.7.1) the *quasi-exactification* of f with respect to the decomposition data $\mathfrak{D}$ and $\mathfrak{D}'$ and the global charts $P_\lambda \to M_{Z_\lambda}$ ($\lambda \in \Lambda$). We denote by $\mathfrak{D}^{\mathrm{qe}}$ the decomposition datum of Z^{qe} which is naturally obtained from the direct sum (6.7.1). (The decomposition datum $\mathfrak{D}^{\mathrm{qe}}$ has a natural quasi-global chart.)

4) Let r be a positive integer. Let N_i ($1 \leq i \leq r$) be a nonnegative integer or ∞. Set $\underline{N} := (N_1, \ldots, N_r)$. Let $\Delta^r(\underline{N})$ be a full subcategory of Δ^r whose objects are $[\underline{n}] := ([n_1], \ldots, [n_r])$'s such that $n_i \leq N_i$ ($1 \leq i \leq r$). Let $Z_{\underline{\bullet}\leq\underline{N}}$ be a fine $\underline{N}$-truncated r-simplicial log scheme over a fine log scheme Y. Consider a functor

$$(Z_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}_{\underline{\bullet}\leq\underline{N}}) \colon \big(\Delta^r(\underline{N})\big)^o \ni [\underline{n}] \longmapsto (Z_{\underline{n}}, \mathfrak{D}_{\underline{n}}) \in (\mathrm{LSchD}/Y).$$

We call the family $\mathfrak{D}_{\underline{\bullet}\leq\underline{N}}$ an $\underline{N}$-truncated r-*simplicial decomposition datum* of $Z_{\underline{\bullet}\leq\underline{N}}$ over Y. We define a morphism

$$(Z'_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}'_{\underline{\bullet}\leq\underline{N}}) \to (Z_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}_{\underline{\bullet}\leq\underline{N}})$$

of fine $\underline{N}$-truncated r-simplicial log schemes over Y with $\underline{N}$-truncated r-simplicial decomposition data in an obvious way.

Consider also a functor

$$(Z_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}_{\underline{\bullet}\leq\underline{N}}, \mathfrak{C}_{\underline{\bullet}\leq\underline{N}}) \colon \big(\Delta^r(\underline{N})\big)^o \ni [\underline{n}] \longmapsto (Z_{\underline{n}}, \mathfrak{D}_{\underline{n}}, \mathfrak{C}_{\underline{n}}) \in (\mathrm{LSchDC}/Y).$$

We call the family $\mathfrak{C}_{\underline{\bullet}\leq\underline{N}}$ an $\underline{N}$-truncated r-*simplicial quasi-global chart* with respect to $\mathfrak{D}_{\underline{\bullet}\leq\underline{N}}$. We define a morphism

$$(Z'_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}'_{\underline{\bullet}\leq\underline{N}}, \mathfrak{C}'_{\underline{\bullet}\leq\underline{N}}) \longrightarrow (Z_{\underline{\bullet}\leq\underline{N}}, \mathfrak{D}_{\underline{\bullet}\leq\underline{N}}, \mathfrak{C}_{\underline{\bullet}\leq\underline{N}})$$

of fine $\underline{N}$-truncated r-simplicial log schemes over Y with $\underline{N}$-truncated r-simplicial decomposition data and $\underline{N}$-truncated r-simplicial quasi-global charts in an obvious way.

We easily obtain the following (we omit the proof; however the proof of (8) in (6.8) let us know that the definition P^{gp}_λ (not P_λ) in the component for $\lambda \neq l(\mu)$ in (6.7.1) is appropriate.):

Proposition 6.8. — *Let Y be a fine log scheme. Then the following hold*:

1) *For two morphisms $(Z_1, \mathfrak{D}_1) \to (Z_2, \mathfrak{D}_2)$ and $(Z_2, \mathfrak{D}_2) \to (Z_3, \mathfrak{D}_3)$ of fine log schemes over Y with decomposition data, there exists the natural composite morphism $(Z_1, \mathfrak{D}_1) \longrightarrow (Z_3, \mathfrak{D}_3)$.*

2) *For a finite projective family $\{(Z_i, \mathfrak{D}_i)\}_{i\in I}$ of fine log schemes over Y with decomposition data, the inverse limit $\varprojlim_{i\in I}(Z_i, \mathfrak{D}_i)$ exists in the category of fine log schemes over Y with decomposition data.*

3) *Let $f\colon I \to J$ be a map of finite ordered sets. Let $\{(Z_i, \mathfrak{D}_i)\}_{i\in I}$ be as in 2) and let $\{(Z'_j, \mathfrak{D}'_j)\}_{j\in J}$ be another finite projective family of fine log schemes over Y with decomposition data. Assume that we are given the following commutative diagrams*

$$\begin{array}{ccc} (Z'_{f(i)}, \mathfrak{D}'_{f(i)}) & \longleftarrow & (Z'_{f(i')}, \mathfrak{D}'_{f(i')}) \\ \big\downarrow & & \big\downarrow \\ (Z_i, \mathfrak{D}_i) & \longleftarrow & (Z_{i'}, \mathfrak{D}_{i'}) \end{array}$$

for all pairs $i \leq i'$ which are compatible for all triples $i \leq i' \leq i''$. Then there exists a natural morphism $\varprojlim_{j\in J}(Z'_j, \mathfrak{D}'_j) \to \varprojlim_{i\in I}(Z_i, \mathfrak{D}_i)$.

4) *Let $(Z_1, \mathfrak{D}_1, \mathfrak{C}_1) \to (Z_2, \mathfrak{D}_2, \mathfrak{C}_2)$ and $(Z_2, \mathfrak{D}_2, \mathfrak{C}_2) \to (Z_3, \mathfrak{D}_3, \mathfrak{C}_3)$ be two morphisms of fine log schemes over Y with decomposition data and quasi-global charts with respect to the decomposition data. Then there exists the natural composite morphism $(Z_1, \mathfrak{D}_1, \mathfrak{C}_1) \to (Z_3, \mathfrak{D}_3, \mathfrak{C}_3)$ with decomposition data and quasi-global charts with respect to the decomposition data.*

5) *For a finite projective family $\{(Z_i, \mathfrak{D}_i, \mathfrak{C}_i)\}_{i\in I}$ of fine log schemes over Y with decomposition data and quasi-global charts with respect to the decomposition data, $\varprojlim_{i\in I}(Z_i, \mathfrak{D}_i, \mathfrak{C}_i)$ exists.*

6) *Let $f\colon I \to J$ be a map of finite ordered sets. Let Y and $\{(Z_i, \mathfrak{D}_i, \mathfrak{C}_i)\}_{i\in I}$ be as in 5) and let $\{(Z'_j, \mathfrak{D}'_j, \mathfrak{C}'_j)\}_{j\in J}$ be another finite projective family of fine log schemes over Y with decomposition data and quasi-global charts with respect to the decomposition data. Assume that we are given the following commutative diagrams*

$$\begin{array}{ccc} (Z'_{f(i)}, \mathfrak{D}'_{f(i)}, \mathfrak{C}'_{f(i)}) & \longleftarrow & (Z'_{f(i')}, \mathfrak{D}'_{f(i')}, \mathfrak{C}'_{f(i')}) \\ \big\downarrow & & \big\downarrow \\ (Z_i, \mathfrak{D}_i, \mathfrak{C}_i) & \longleftarrow & (Z_{i'}, \mathfrak{D}_{i'}, \mathfrak{C}'_{i'}) \end{array}$$

for all pairs $i \leq i'$ which are compatible for all triples $i \leq i' \leq i''$. Then there exists a natural morphism $\varprojlim_{j\in J}(Z'_j, \mathfrak{D}'_j, \mathfrak{C}'_j) \to \varprojlim_{i\in I}(Z_i, \mathfrak{D}_i, \mathfrak{C}_i)$.

7) *Let the notations be as in* (6.7), 3). *Then there exists the following natural commutative diagram*

$$\begin{array}{ccc} (Z',\mathfrak{D}') & \longrightarrow & (Z^{\mathrm{qe}},\mathfrak{D}^{\mathrm{qe}}) \\ \| & & \downarrow \\ (Z',\mathfrak{D}') & \longrightarrow & (Z,\mathfrak{D}). \end{array}$$

8) *Let*

$$\begin{array}{ccc} (Z',\mathfrak{D}') & \longrightarrow & (Z,\mathfrak{D}) \\ \downarrow & & \downarrow \\ (W',\mathfrak{E}') & \longrightarrow & (W,\mathfrak{E}) \end{array}$$

be a commutative diagram of fine log schemes over Y *with decomposition data such that the morphism* $(Z,\mathfrak{D}) \to (W,\mathfrak{E})$ *has a quasi-global chart. Let* $(Z^{\mathrm{qe}},\mathfrak{D}^{\mathrm{qe}})$ (*resp.* $(W^{\mathrm{qe}},\mathfrak{E}^{\mathrm{qe}})$) *be the quasi-exactification of the upper* (*resp. lower*) *horizontal morphism of the commutative diagram above. Then there exists a natural morphism* $(Z^{\mathrm{qe}},\mathfrak{D}^{\mathrm{qe}}) \to (W^{\mathrm{qe}},\mathfrak{E}^{\mathrm{qe}})$ *which has a quasi-global chart.*

LEMMA 6.9. — *Let the notations be as in* (6.1). *Then the simplicial log scheme* $Y'_\bullet$ *has a natural simplicial decomposition datum which will be constructed in the proof below. The components of the decomposition datum are log affine* (*formal*) *schemes.*

Proof. — Let m be a positive integer. By the construction of Y'_0, Y'_0 has a natural decomposition datum. Assume that we are given an $(m-1)$-truncated simplicial log scheme $Y'_{\bullet\leq m-1}$ with an $(m-1)$-truncated simplicial decomposition datum. Endow Y with the trivial decomposition datum. By (6.1.3) and (6.8), 2), the log scheme $\mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_a$ ($a \in \mathbb{N}$) has a natural decomposition datum. Consider the commutative diagram (6.1.4). By the proof of (6.1), we can endow $N(Y_m)'$ with a decomposition datum such that the morphism $N(Y_m)' \longrightarrow \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_m$ is an underlying morphism of fine log schemes with decomposition data.

Let $s^{m-1}_i \colon Y'_{m-1} \longrightarrow Y'_m$ ($m \in \mathbb{Z}_{>0}, 0 \leq i \leq m-1$) be the degeneracy morphism corresponding to the standard degeneracy map $\partial^i_m \colon [m] \longrightarrow [m-1]$ before the proof of (6.1). Then, by the proof of [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.3)],

$$s^{m-1}_i \colon Y'_{m-1} = \coprod_{0\leq \ell\leq m-1} \coprod_{[m-1]\twoheadrightarrow[\ell]} N(Y_\ell)' \longrightarrow Y'_m = \coprod_{0\leq \ell\leq m} \coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)'$$

is induced by the identity $\mathrm{id}\colon N(Y_\ell)' \to N(Y_\ell)'$ corresponding to the components with indexes $[m-1] \twoheadrightarrow [\ell]$ and $[m] \xrightarrow{\partial^i_m} [m-1] \twoheadrightarrow [\ell]$. Hence s_i^{m-1} is an underlying morphism of fine log schemes with decomposition data.

Let $\delta'^m_i\colon Y'_m \longrightarrow Y'_{m-1}$ $(0 \le i \le m)$ be the standard face morphism corresponding to the morphism $\sigma^m_i\colon [m-1] \to [m]$ defined by the formulas $\sigma^m_i(j) = j$ $(0 \le j < i)$ and $\sigma^m_i(j) = j+1$ $(i \le j \le m)$. We prove that δ'^m_i is an underlying morphism of fine log schemes with decomposition data. Let $\delta''^m_i\colon \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_m \to \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_{m-1} = Y'_{m-1}$ be the standard face morphism. Then, by the proof of [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.3)], the face morphism

$$\delta'^m_i\colon Y'_m = \coprod_{0\le\ell\le m}\ \coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)' \longrightarrow Y'_{m-1} = \coprod_{0\le\ell\le m-1}\ \coprod_{[m-1]\twoheadrightarrow[\ell]} N(Y_\ell)'$$

is induced by the following composite morphisms for a surjective non-decreasing morphism $[m] \twoheadrightarrow [\ell]$:

$$N(Y_\ell)' \hookrightarrow Y'_\ell = \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_\ell \longrightarrow \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_m \xrightarrow{\delta''^m_i} Y'_{m-1}$$

$(0 \le \ell < m)$ and

$$N(Y_m)' \longrightarrow \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_m \xrightarrow{\delta''^m_i} Y'_{m-1}.$$

Here the morphism $\mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_\ell \to \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_m$ is induced by the surjective non-decreasing morphism $[m] \twoheadrightarrow [\ell]$. By (6.1.3) and (6.8), 3), we see that the morphism $\mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_b \longrightarrow \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\le m-1})_a$ corresponding to a morphism $[a] \to [b]$ in Δ is an underlying morphism of fine log schemes with decomposition data. Hence, by (6.8), 1), $\delta'^m_i\colon Y'_m \to Y'_{m-1}$ is an underlying morphism of fine log schemes with decomposition data. Because any morphism $Y'_\ell \to Y'_m$ $(0 \le \ell, m \le N)$ is a composite morphism of degeneracy morphisms and face morphisms (see [**89**, (8.1.2)]), we can complete the proof by (6.8), 1). □

PROPOSITION 6.10. — *Let the notations be as in* (6.1). *Let $T \hookrightarrow \mathcal{T}$ be a nil-immersion of fine log schemes. Let N be a nonnegative integer. Assume that Y_N is log smooth over T. Then there exists the disjoint union $Y'_{\bullet\le N}$ of the members of an affine open covering of $Y_{\bullet\le N}$ and an N-truncated simplicial immersion $Y'_{\bullet\le N} \hookrightarrow \Gamma'_{\bullet\le N}$ into a log smooth N-truncated simplicial log scheme over $\mathcal{T}$ which is an underlying morphism of fine N-truncated simplicial log schemes over $\mathcal{T}$ with decomposition data such that the decomposition datum of $\Gamma'_{\bullet\le N}$ has an N-truncated simplicial quasi-global chart.*

Proof. — Endow T and $\mathcal{T}$ with the trivial decomposition data. Because Y_N is log smooth over T, Y'_N is also log smooth over T. There exists a closed immersion $Y'_N \hookrightarrow \mathcal{Y}'_N$ into a log smooth scheme over $\mathcal{T}$ (in fact, Y'_N has a lift $\mathcal{Y}'_N$ (see [**72**, (2.3.14)])) and we may assume that this closed immersion is an underlying morphism of fine log schemes with decomposition data over $\mathcal{T}$. Moreover, we can assume that the morphism $\mathcal{Y}'_N$ has a quasi-global chart with respect to the decomposition data of $\mathcal{Y}'_N$. Set $\Gamma'_\bullet := \Gamma^{\mathcal{T}}_N(\mathcal{Y}'_N)$. By (6.8), 4), 5), 6), $\Gamma'_\bullet$ has a natural simplicial quasi-global chart with respect to a natural simplicial decomposition data of $\Gamma'_\bullet$. The diagonal immersion $Y'_m \hookrightarrow \Pi_{\gamma\in \mathrm{Hom}_\Delta([N],[m])} Y'_m$ is an underlying morphism of fine log schemes with decomposition data. The immersion (6.5.1) is obtained from the following composite morphism

$$Y'_m \hookrightarrow \Pi_{\gamma\in \mathrm{Hom}_\Delta([N],[m])} Y'_m \longrightarrow \Pi_{\gamma\in \mathrm{Hom}_\Delta([N],[m])} Y'_N \hookrightarrow \Pi_{\gamma\in \mathrm{Hom}_\Delta([N],[m])} \mathcal{Y}'_N.$$

By (6.9), (6.8), 1) and 3), we have a natural immersion $Y'_m \hookrightarrow \Gamma'_m$ $(0 \leq m \leq N)$ which is an underlying morphism of fine log schemes over $\mathcal{T}$ with decomposition data. In fact, by the following commutative diagram

$$\begin{array}{ccccc} Y'_\ell & \longrightarrow & Y'_N & \xrightarrow{\subset} & \mathcal{Y}'_N \\ \big\downarrow & & \big\| & & \big\| \\ Y'_m & \longrightarrow & Y'_N & \xrightarrow{\subset} & \mathcal{Y}'_N \end{array}$$

for morphisms $[m] \to [\ell]$ and $[N] \to [m]$ in Δ, we have a desired N-truncated simplicial immersion $Y'_{\bullet\leq N} \hookrightarrow \Gamma'_{\bullet\leq N}$ with N-truncated simplicial decomposition data. ☐

PROPOSITION 6.11. — *Let the notations and the assumptions be as in* (6.10). *Let* $Y_{\bullet\bullet}$ *be the affine Čech diagram of* $Y'_\bullet$ *over* $Y_\bullet/Y$. *Assume that each log open and closed subscheme of* Y_{mn} $(0 \leq m \leq N, n \in \mathbb{N})$ *obtained from the decomposition data of* Y_{mn} *satisfies the condition in* (4.11). *If each component of the decomposition datum of* Y'_m $(0 \leq m \leq N)$ *is small, then there exists an exact immersion* $Y_{\bullet\leq N,\bullet} \hookrightarrow G_{\bullet\leq N,\bullet}$ *into an* (N,∞)*-truncated log smooth scheme over* $\mathcal{T}$.

Proof. — Let the notations be as in the proof of (6.10). Set $G'_{mn} := \mathrm{cosk}^{\mathcal{T}}_0(\Gamma'_m)_n$ $(0 \leq m \leq N, n \in \mathbb{N})$. Let $G_{mn} := \mathrm{cosk}^{\mathcal{T}}_0(\Gamma'_m)^{\mathrm{qe}}_n$ be the quasi-exactification of the immersion $Y_{mn} \hookrightarrow G'_{mn}$ with respect to the decomposition data and the natural quasi-global chart of G'_{mn}. We take Y'_m such that each component of the decomposition datum of Y_{mn} $(\forall n \in \mathbb{N})$ and each global chart of the quasi-global chart of G'_{mn} $(\forall n \in \mathbb{N})$ satisfies the condition in

(4.11). Then (6.11) immediately follows from (6.10), (6.8), 3), 5), 7), 8) and (4.11). □

COROLLARY 6.12. — *Let $S_0 \hookrightarrow S$ be a nil-immersion of schemes. Let N be a nonnegative integer. Let $(X_{\bullet \leq N}, D_{\bullet \leq N} \cup Z_{\bullet \leq N})$ be a smooth split N-truncated simplicial scheme with transversal split N-truncated simplicial relative SNCD's over S_0. Then there exist the following data*:

1) *the disjoint union of the members of an affine simplicial open covering $(X'_{\bullet \leq N}, D'_{\bullet \leq N} \cup Z'_{\bullet \leq N})$ of $(X_{\bullet \leq N}, D_{\bullet \leq N} \cup Z_{\bullet \leq N})$ over S_0,*

2) *(Set $(X_{mn}, D_{mn} \cup Z_{mn}) := \mathrm{cosk}_0^{(X_m, D_m \cup Z_m)}(X'_m, D'_m \cup Z'_m)_n$.) an (N,∞)-truncated bisimplicial exact immersion $(X_{\bullet \leq N, \bullet}, D_{\bullet \leq N, \bullet} \cup Z_{\bullet \leq N, \bullet}) \hookrightarrow \mathcal{P}_{\bullet \leq N, \bullet}$ into a log smooth (N,∞)-truncated bisimplicial log scheme over S. In fact, $\mathcal{P}_{\bullet \leq N, \bullet}$ is an (N,∞)-truncated bisimplicial log scheme which is obtained from a smooth (N,∞)-truncated bisimplicial scheme with a relative (N,∞)-truncated bisimplicial SNCD over S.*

Proof. — The Corollary follows from the construction of $G_{\bullet \leq N, \bullet}$ in (6.11) and (4.13), 1) and 2). □

PROBLEM 6.13. — Let the notations be as in (6.12). Let Δ_m $(0 \leq m \leq N)$ be a decomposition of smooth components of D_m in the sense of §3. The scheme D_{mn} has the decomposition Δ_{mn} induced by the decomposition Δ_m. The following problem is fundamental: does there exist an (N,∞)-truncated bisimplicial admissible immersion $(X_{\bullet \leq N, \bullet}, D_{\bullet \leq N, \bullet}) \hookrightarrow (\mathcal{X}_{\bullet \leq N, \bullet}, \mathcal{D}_{\bullet \leq N, \bullet})$ with respect to $\Delta_{\bullet \leq N, \bullet}$ over S. (I have not yet known the answer for this problem.)

Let the notations be as in the proof of (6.12). Let $(X'_N, D'_N \cup Z'_N) \hookrightarrow (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ be a closed immersion into a smooth scheme over S with transversal relative SNCD's over S which induces closed immersions $(X'_N, D'_N) \hookrightarrow (\mathcal{X}'_N, \mathcal{D}'_N)$ and $(X'_N, Z'_N) \hookrightarrow (\mathcal{X}'_N, \mathcal{Z}'_N)$. We may assume that the closed immersion $(X'_N, D'_N \cup Z'_N) \hookrightarrow (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ is an underlying morphism of log schemes with decomposition data and that $(\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ has a quasi-global chart with respect to the decomposition datum of $(\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$. Set

$$\mathcal{P}'_{\bullet \leq N, \bullet} := \mathrm{cosk}_0^S(\Gamma_N^S((\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N))_{\bullet \leq N}),$$
$$\mathcal{Q}'_{\bullet \leq N, \bullet} := \mathrm{cosk}_0^S(\Gamma_N^S((\mathcal{X}'_N, \mathcal{Z}'_N))_{\bullet \leq N}).$$
$$\epsilon_{\bullet \leq N, \bullet} := \epsilon_{(X_{\bullet \leq N, \bullet}, D_{\bullet \leq N, \bullet} \cup Z_{\bullet \leq N, \bullet}, Z_{\bullet \leq N, \bullet})/S_0}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet}\cup Z_{\bullet\leq N,\bullet}) & \overset{\subset}{\longrightarrow} & \mathcal{P}'_{\bullet\leq N,\bullet}\\ {\scriptstyle\epsilon_{\bullet\leq N,\bullet}}\big\downarrow & & \big\downarrow\\ (X_{\bullet\leq N,\bullet}, Z_{\bullet\leq N,\bullet}) & \overset{\subset}{\longrightarrow} & \mathcal{Q}'_{\bullet\leq N,\bullet}.\end{array}$$

Note that $\overset{\circ}{\mathcal{P}}{}'_{\bullet\leq N,\bullet} = \overset{\circ}{\mathcal{Q}}{}'_{\bullet\leq N,\bullet}$. Let $\mathcal{P}_{mn}$ $(0\leq m\leq N, n\in\mathbb{N})$ be the quasi-exactification of the immersion $(X_{mn}, D_{mn}\cup Z_{mn})\hookrightarrow\mathcal{P}'_{mn}$ with respect to the natural decomposition data of $(X_{mn}, D_{mn}\cup Z_{mn})$ and $\mathcal{P}'_{mn}$ and with respect to the natural quasi-global chart of $\mathcal{P}'_{mn}$. Endow $\overset{\circ}{\mathcal{P}}_{mn}$ with the pull-back of the log structure of $\mathcal{Q}'_{mn}$ by the natural morphism $\overset{\circ}{\mathcal{P}}_{mn}\longrightarrow\overset{\circ}{\mathcal{P}}{}'_{mn}=\overset{\circ}{\mathcal{Q}}{}'_{mn}$ and let $\mathcal{Q}_{\bullet\leq N,\bullet}$ be the resulting log smooth (N,∞)-truncated bisimplicial log scheme. Then, by (4.13), 3), we have the following commutative diagram

$$\begin{array}{ccc}(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet}\cup Z_{\bullet\leq N,\bullet}) & \overset{\subset}{\longrightarrow} & \mathcal{P}_{\bullet\leq N,\bullet}\\ {\scriptstyle\epsilon_{\bullet\leq N,\bullet}}\big\downarrow & & \big\downarrow\\ (X_{\bullet\leq N,\bullet}, Z_{\bullet\leq N,\bullet}) & \overset{\subset}{\longrightarrow} & \mathcal{Q}_{\bullet\leq N,\bullet},\end{array} \tag{6.13.1}$$

where the horizontal morphisms are exact immersions into log smooth (N,∞)-truncated bisimplicial log schemes over S which are obtained from smooth (N,∞)-truncated bisimplicial schemes over S and (N,∞)-truncated bisimplicial relative SNCD's over S. The log structures $M_{\mathcal{P}_{mn}}$ and $M_{\mathcal{Q}_{mn}}$ $(0\leq m\leq N, n\in\mathbb{N})$ satisfy the condition in (4.3.2):

$$M_{\mathcal{P}_{mn}} = \mathcal{M}_{mn}\oplus_{\mathcal{O}^*_{\mathcal{P}_{mn}}} M_{\mathcal{Q}_{mn}},$$

where $\mathcal{M}_{mn}$ is an fs sub log structure of $M_{\mathcal{P}_{mn}}$, and $(\overset{\circ}{\mathcal{P}}_{mn}, \mathcal{M}_{mn})$ (and $M(D_{mn})$) satisfies the assumptions after (4.3.2). Then we have the log de Rham complexes $\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}$ and $\Lambda^{\bullet}_{\mathcal{Q}_{\bullet\leq N,\bullet}/S}$ of $\mathcal{P}_{\bullet\leq N,\bullet}/S$ and $\mathcal{Q}_{\bullet\leq N,\bullet}/S$, respectively.

By (4.2), 1) we have a filtered complex

$$(\mathcal{O}_{\mathfrak{D}_{\bullet\leq N,\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet\leq N,\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}, P^{\mathcal{P}_{\bullet\leq N,\bullet}/\mathcal{Q}_{\bullet\leq N,\bullet}}).$$

Let Y be a fine log scheme over a fine log PD-scheme $(T,\mathcal{J},\delta)$ and let $Y\hookrightarrow\mathcal{Y}$ be an immersion into a log smooth fine log scheme over T. Let $\mathfrak{D}_Y(\mathcal{Y}^{\nu})$ $(\nu\in\mathbb{Z}_{\geq 1})$ be the log PD-envelope of the natural immersion $Y\hookrightarrow\mathcal{Y}^{\nu}$ over $(T,\mathcal{J},\delta)$. Let $g\colon Y\longrightarrow T$ be the structural morphism. For a family

$(E, \{E_k\}_{k\in\mathbb{Z}})$ of sheaves of $\mathcal{O}_{Y/T}$-modules with commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_k & \longrightarrow & E_{k+1} & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow & & \\ \cdots & \longrightarrow & E & =\!= & E & \longrightarrow & \cdots \end{array}$$

and for $\circ = k \in \mathbb{Z}$ or nothing, we have the obvious log version $\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E_\circ)$ of the Čech-Alexander complex defined in [**5**, V 1.2.3]:

$$\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E_\circ) := \big((E_\circ)_{\mathfrak{D}_Y(\mathcal{Y})} \xrightarrow{\partial^0} (E_\circ)_{\mathfrak{D}_Y(\mathcal{Y}^2)} \xrightarrow{\partial^1} (E_\circ)_{\mathfrak{D}_Y(\mathcal{Y}^3)} \longrightarrow \cdots\big).$$

(We set $\check{\mathrm{C}}\mathrm{A}^{\nu-1}_{\mathcal{Y}}(E_\circ) = (E_\circ)_{\mathfrak{D}_Y(\mathcal{Y}^\nu)}$.) We have the following commutative diagram of complexes of $g^{-1}(\mathcal{O}_T)$-modules:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E_k) & \longrightarrow & \check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E_{k+1}) & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow & & \\ \cdots & \longrightarrow & \check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E) & =\!= & \check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E) & \longrightarrow & \cdots . \end{array}$$

By abuse of notation, we denote this commutative diagram by

$$\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}\big((E, \{E_k\}_{k\in\mathbb{Z}})\big) := \big(\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E), \{\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E_k)\}_{k\in\mathbb{Z}}\big),$$

which we call the *commutative diagram of the Čech-Alexander complexes* of $(E, \{E_k\}_{k\in\mathbb{Z}})$. Let $(E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}})$ be a family of complexes of $\mathcal{O}_{Y/T}$-modules with commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E^\bullet_k & \longrightarrow & E^\bullet_{k+1} & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow & & \\ \cdots & \longrightarrow & E^\bullet & =\!= & E^\bullet & \longrightarrow & \cdots . \end{array}$$

Let $\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E^\bullet_\circ)$ be the single complex of the following double complex

$$\check{\mathrm{C}}\mathrm{A}^0_{\mathcal{Y}}(E^\bullet_\circ) \longrightarrow \check{\mathrm{C}}\mathrm{A}^1_{\mathcal{Y}}(E^\bullet_\circ)[1] \longrightarrow \check{\mathrm{C}}\mathrm{A}^2_{\mathcal{Y}}(E^\bullet_\circ)[2] \longrightarrow \cdots .$$

(We add $\mathbf{s}$ to the notation of the Čech-Alexander complex in [**5**, V 1.2.3].) We have the diagram $(\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E^\bullet), \{\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}(E^\bullet_k)\}_{k\in\mathbb{Z}})$ of the single complexes, which we denote by $\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}((E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}}))$ and call the *diagram of the* Č*ech-Alexander complexes* of $(E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}})$. If $(E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}})$ is a filtered complex of $\mathcal{O}_{Y/T}$-modules, then $\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}((E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}}))$ is a filtered complex of $g^{-1}(\mathcal{O}_T)$-modules. In this case, we call $\mathbf{s}\check{\mathrm{C}}\mathrm{A}^\bullet_{\mathcal{Y}}((E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}}))$ the *Čech-Alexander complex* of $(E^\bullet, \{E^\bullet_k\}_{k\in\mathbb{Z}})$. Similarly we can define the Čech-Alexander complex of a filtered complex of $Q^*_{Y/T}(\mathcal{O}_{Y/T})$-modules.

LEMMA 6.14. — *Set*

$$\mathcal{Y}^{\nu} := \underbrace{\mathcal{Y} \times_T \cdots \times_T \mathcal{Y}}_{\nu\,\text{times}}.$$

Let $q_i\colon \mathfrak{D}_Y(\mathcal{Y}^{\nu}) \to \mathfrak{D}_Y(\mathcal{Y})$ $(1 \leq i \leq \nu)$ *be the induced morphism by the* i*-th projection* $p_i\colon \mathcal{Y}^{\nu} \to \mathcal{Y}$. *Then* $\overset{\circ}{q}_i\colon \overset{\circ}{\mathfrak{D}}_Y(\mathcal{Y}^{\nu}) \to \overset{\circ}{\mathfrak{D}}_Y(\mathcal{Y})$ *is flat.*

Proof. — The proof is the same as that of [**51**, (6.5)]. □

The following has been implicitly used in [**72**, (1.6.3)]:

LEMMA 6.15. — *Let the notations be before* (3.1). *Let* $(E^{\bullet}, \{E^{\bullet}_k\}_{k\in\mathbb{Z}})$ *be a filtered complex of* $\mathcal{O}_{Y/T}$*-modules* (*resp.* $Q^*_{Y/T}(\mathcal{O}_{Y/T})$*-modules*). *Set* $v_{Y/T} := u_{Y/T}$ (*resp.* $\bar{u}_{Y/T}$). *Then there exists a canonical filtered isomorphism*

$$Rv_{Y/T*}(E^{\bullet}, \{E^{\bullet}_k\}_{k\in\mathbb{Z}}) \overset{\sim}{\longrightarrow} \mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{Y}}\big((E^{\bullet}, \{E^{\bullet}_k\}_{k\in\mathbb{Z}})\big) \tag{6.15.1}$$

in $\mathrm{D}^+\mathrm{F}(g^{-1}(\mathcal{O}_T))$.

Proof. — As in [**72**, (1.6.3)], we have only to prove (6.15.1) for the case of the trivial filtrations. In this case, we can give the proof as the obvious log version of the proof of [**5**, V Théorème 1.2.5] (resp. [**5**, V Proposition 1.3.1]). □

PROPOSITION 6.16. — *Let* $f_{\bullet\leq N,\bullet}\colon X_{\bullet\leq N,\bullet} \longrightarrow S$ *and* $f_{\bullet\leq N}\colon X_{\bullet\leq N} \longrightarrow S$ *be the structural morphisms. Let*

$$\eta_{\mathrm{zar}}\colon \big((\widetilde{X}_{\bullet\leq N,\bullet})_{\mathrm{zar}}, f^{-1}_{\bullet\leq N,\bullet}(\mathcal{O}_S)\big) \longrightarrow \big((\widetilde{X}_{\bullet\leq N})_{\mathrm{zar}}, f^{-1}_{\bullet\leq N}(\mathcal{O}_S)\big)$$

be the natural morphism of ringed topoi. Then there exists the following natural filtered isomorphism

$$\begin{aligned}(6.16.1)\qquad &\big(E^{\log, Z_{\bullet\leq N}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet\leq N}, D_{\bullet\leq N}\cup Z_{\bullet\leq N})/S}), P^{D_{\bullet\leq N}}\big)\\ &\qquad\overset{\sim}{\longrightarrow} R\eta_{\mathrm{zar}*}\big(E^{\log, Z_{\bullet\leq N,\bullet}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet}\cup Z_{\bullet\leq N,\bullet})/S}), P^{D_{\bullet\leq N,\bullet}}\big)\end{aligned}$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}_{\bullet\leq N}(\mathcal{O}_S))$.

Proof. — Let

$$\begin{aligned}\lambda^{\log}_{\mathrm{crys}}\colon &\big(((X_{\bullet\leq N,\bullet}, \widetilde{D_{\bullet\leq N,\bullet}} \cup Z_{\bullet\leq N,\bullet})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet}\cup Z_{\bullet\leq N,\bullet})/S}\big)\\ &\longrightarrow \big(((X_{\bullet\leq N}, \widetilde{D_{\bullet\leq N}} \cup Z_{\bullet\leq N})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\leq N}, D_{\bullet\leq N}\cup Z_{\bullet\leq N})/S}\big),\\ \eta^{\log}_{\mathrm{crys}}\colon &\big(((X_{\bullet\leq N,\bullet}, \widetilde{Z_{\bullet\leq N,\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\leq N,\bullet}, Z_{\bullet\leq N,\bullet})/S}\big)\\ &\longrightarrow \big(((X_{\bullet\leq N}, \widetilde{Z_{\bullet\leq N}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\leq N}, Z_{\bullet\leq N})/S}\big)\end{aligned}$$

be the natural morphisms of ringed topoi. Since we have a natural morphism

$$\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N} \cup Z_{\bullet \leq N})/S} \longrightarrow \lambda^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(X_{\bullet \leq N, \bullet}, D_{\bullet \leq N, \bullet} \cup Z_{\bullet \leq N, \bullet})/S}),$$

we have the following morphism

$$\begin{aligned} (6.16.2) \quad & (E^{\log, Z_{\bullet \leq N}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N} \cup Z_{\bullet \leq N})/S}), P^{D_{\bullet \leq N}}) \\ & \longrightarrow R\eta^{\log}_{\mathrm{crys}*}(E^{\log, Z_{\bullet \leq N, \bullet}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\bullet \leq N, \bullet}, D_{\bullet \leq N, \bullet} \cup Z_{\bullet \leq N, \bullet})/S}), P^{D_{\bullet \leq N, \bullet}}) \end{aligned}$$

as in the proof of (3.5) ((3.5.3), (3.5.4), (3.5.5), (3.5.6)). By applying $Ru_{(X_{\bullet \leq N}, Z_{\bullet \leq N})/S*}$ to (6.16.2) and using a relation

$$u_{(X_{\bullet \leq N}, Z_{\bullet \leq N})/S} \circ \eta^{\log}_{\mathrm{crys}} = \eta_{\mathrm{zar}} \circ u_{(X_{\bullet \leq N, \bullet}, Z_{\bullet \leq N, \bullet})/S},$$

we have the morphism (6.16.1).

For each $0 \leq t \leq N$, let

$$\eta_{t\mathrm{zar}} \colon \big((\widetilde{X}_{t\bullet})_{\mathrm{zar}}, f^{-1}_{t\bullet}(\mathcal{O}_S)\big) \longrightarrow \big((\widetilde{X}_t)_{\mathrm{zar}}, f^{-1}_t(\mathcal{O}_S)\big)$$

be the natural morphism of ringed topoi. For a filtered complex

$$\big(E^{\bullet \leq N, \bullet\bullet}, \{E^{\bullet \leq N, \bullet\bullet}_k\}_{k \in \mathbb{Z}}\big)$$

of $f^{-1}_{\bullet \leq N, \bullet}(\mathcal{O}_S)$-modules, there exists a filtered flasque resolution

$$\big(I^{\bullet \leq N, \bullet\bullet}, \{I^{\bullet \leq N, \bullet\bullet}_k\}_{k \in \mathbb{Z}}\big)$$

of $(E^{\bullet \leq N, \bullet\bullet}, \{E^{\bullet \leq N, \bullet\bullet}_k\}_{k \in \mathbb{Z}})$ such that $(I^{t\bullet\bullet}, \{I^{t\bullet\bullet}_k\}_{k \in \mathbb{Z}})$ is a filtered flasque resolution of $(E^{t\bullet\bullet}, \{E^{t\bullet\bullet}_k\}_{k \in \mathbb{Z}})$ (cf. [**72**, (1.5)]). Hence, to prove that the morphism (6.16.1) is a filtered isomorphism, it suffices to prove that the natural morphism

$$\begin{aligned} (6.16.3) \quad & (E^{\log, Z_t}_{\mathrm{zar}}(\mathcal{O}_{(X_t, D_t \cup Z_t)/S}), P^{D_t}) \\ & \longrightarrow R\eta_{t\mathrm{zar}*}(E^{\log, Z_{t\bullet}}_{\mathrm{zar}}(\mathcal{O}_{(X_{t\bullet}, D_{t\bullet} \cup Z_{t\bullet})/S}), P^{D_{t\bullet}}) \end{aligned}$$

is a filtered isomorphism, which is nothing but (3.5), 2). As a consequence, we see that the morphism (6.16.1) is a filtered isomorphism. □

The following is the main result in this section (this is necessary for the proof of the comparison theorem (7.6) below):

THEOREM 6.17 (Explicit description of the N-truncated cosimplicial preweight-filtered vanishing cycle zariskian complex)

Let the notations be as in (6.16). *Then in* $\mathrm{D}^+\mathrm{F}(f^{-1}_{\bullet \leq N}(\mathcal{O}_S))$, *one has*

$$\begin{aligned} (6.17.1) \quad & (E^{\log, Z_{\bullet \leq N}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet \leq N}, D_{\bullet \leq N} \cup Z_{\bullet \leq N})/S}), P^{D_{\bullet \leq N}}) \\ & = R\eta_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_{\bullet \leq N, \bullet}} \otimes_{\mathcal{O}_{\mathcal{P}_{\bullet \leq N, \bullet}}} \Lambda^{\bullet}_{\mathcal{P}_{\bullet \leq N, \bullet}/S}, P^{\mathcal{P}_{\bullet \leq N, \bullet}/\mathcal{Q}_{\bullet \leq N, \bullet}}). \end{aligned}$$

Proof. — By (6.16.1) we obtain

$$(6.17.2)\quad \big(E_{\rm zar}^{\log, Z_{\bullet\le N}}(\mathcal{O}_{(X_{\bullet\le N},D_{\bullet\le N}\cup Z_{\bullet\le N})/S}), P^{D_{\bullet\le N}}\big)$$
$$\overset{\sim}{\longrightarrow} R\eta_{{\rm zar}*}(E_{\rm zar}^{\log, Z_{\bullet\le N,\bullet}}\big(\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}\big), P^{D_{\bullet\le N,\bullet}})$$
$$= R\eta_{{\rm zar}*}Ru_{(X_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}$$
$$\big(E_{\rm crys}^{\log, Z_{\bullet\le N,\bullet}}(\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}), P^{D_{\bullet\le N,\bullet}}\big)$$
$$= R\eta_{{\rm zar}*}Ru_{(X_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}$$
$$\big(R\epsilon_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}(\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}),\tau\big).$$

By the log Poincaré lemma (see [**72**, (2.2.7)]), we have the following quasi-isomorphism

$$\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}\overset{\sim}{\longrightarrow} L_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\le N,\bullet}/S}).$$

Let $I^{\bullet\le N,\bullet\bullet}$ be an injective resolution of $\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}$. Then we have a quasi-isomorphism

$$L_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\le N,\bullet}/S})\overset{\sim}{\longrightarrow} I^{\bullet\le N,\bullet\bullet}.$$

As in the proof of (3.5), we have

$$R\epsilon_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}(\mathcal{O}_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})/S})$$
$$=\epsilon_{(X_{\bullet\le N,\bullet},D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}(I^{\bullet\le N,\bullet\bullet})$$
$$= L_{(X_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\le N,\bullet}/S}).$$

By (6.17.2), we have

$$(6.17.3)\quad \big(E_{\rm zar}^{\log, Z_{\bullet\le N}}(\mathcal{O}_{(X_{\bullet\le N},D_{\bullet\le N}\cup Z_{\bullet\le N})/S}), P^{D_{\bullet\le N}}\big)$$
$$= R\eta_{{\rm zar}*}Ru_{(X_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S*}(L_{(X_{\bullet\le N,\bullet},Z_{\bullet\le N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\le N,\bullet}/S}),\tau).$$

For a nonnegative integer $n \leq N$, we have the following diagram of filtered complexes of $f^{-1}_{n\bullet}(\mathcal{O}_S)$-modules:

(6.17.4;n)

$$\begin{array}{c}
u_{(X_{n\bullet},Z_{n\bullet})/S*}(L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau) \\
\| \\
\bar{u}_{(X_{n\bullet},Z_{n\bullet})/S*}Q^{*}_{(X_{n\bullet},Z_{n\bullet})/S}(L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau) \\
\cap\downarrow \\
\bar{u}_{(X_{n\bullet},Z_{n\bullet})/S*}(Q^{*}_{(X_{n\bullet},Z_{n\bullet})/S}L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),Q^{*}_{(X_{n\bullet},Z_{n\bullet})/S}P^{D_{n\bullet}}) \\
\| \\
(u_{(X_{n\bullet},Z_{n\bullet})/S*}L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\{u_{(X_{n\bullet},Z_{n\bullet})/S*}L_{(X_{n\bullet},Z_{n\bullet})/S}(P^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}_{k}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S})\}_{k\in\mathbb{Z}}) \\
\| \\
(\mathcal{O}_{\mathfrak{D}_{n\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{n\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S},P^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}).
\end{array}$$

Hence we have the following diagram of filtered complexes of $f^{-1}_{n\bullet}(\mathcal{O}_S)$-modules:

(6.17.5;n)

$$\begin{array}{c}
\mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{n\bullet}}((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau)) \\
\downarrow \\
\mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{n\bullet}}((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\{L_{(X_{n\bullet},Z_{n\bullet})/S}(P^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}_{k}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S})\}_{k\in\mathbb{Z}})) \\
\uparrow \\
(\mathcal{O}_{\mathfrak{D}_{n\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{n\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S},P^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}).
\end{array}$$

Note that

$$\mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{n\bullet}}\big((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\{L_{(X_{n\bullet},Z_{n\bullet})/S}(P^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}_{k}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S})\}_{k\in\mathbb{Z}})\big)$$

is indeed a filtered complex by (4.2), 1) and the flatness of $\mathring{\mathfrak{D}}_{(X_{n\bullet},Z_{n\bullet})}(\mathcal{Q}^{\nu}_{n\bullet})$ over $\mathring{\mathfrak{D}}_{(X_{n\bullet},Z_{n\bullet})}(\mathcal{Q}_{n\bullet})$ ((6.14)).

For a morphism $\alpha\colon [n]\to[m]$ $(0\leq n,m\leq N)$ in Δ, let $X(\alpha)\colon (\widetilde{X}_{m\bullet})_{\mathrm{zar}}\to(\widetilde{X}_{n\bullet})_{\mathrm{zar}}$ be the corresponding morphism. Then we have the morphism (6.17.5;n) $\to X(\alpha)_*((6.17.5;m))$ of diagrams of filtered complexes of $f^{-1}_{n\bullet}(\mathcal{O}_S)$-modules. Hence we have the following diagram of filtered complexes of

$f^{-1}_{\bullet\leq N,\bullet}(\mathcal{O}_S)$-modules:

$$(6.17.6)\qquad \begin{array}{c} \mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}}((L_{(X_{\bullet\leq N,\bullet},Z_{\bullet\leq N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}),\tau)) \\ \downarrow \\ \mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}}((L_{(X_{\bullet\leq N,\bullet},Z_{\bullet\leq N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}), \\ \{L_{(X_{\bullet\leq N,\bullet},Z_{\bullet\leq N,\bullet})/S}(P_k^{\mathcal{P}_{\bullet\leq N,\bullet}/\mathcal{Q}_{\bullet\leq N,\bullet}}\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S})\}_{k\in\mathbb{Z}})) \\ \uparrow \\ (\mathcal{O}_{\mathfrak{D}_{\bullet\leq N,\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet\leq N,\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}, P^{\mathcal{P}_{\bullet\leq N,\bullet}/\mathcal{Q}_{\bullet\leq N,\bullet}}). \end{array}$$

We claim that the two morphisms in (6.17.6) are filtered quasi-isomorphisms; we have only to prove that the two morphisms in (6.17.5;n) are filtered quasi-isomorphisms. Indeed, by (6.15) for the log crystalline case, we have

$$(6.17.7)\qquad \begin{aligned} &\mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{n\bullet}}\big((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau)\big) \\ &\qquad = Ru_{(X_{n\bullet},Z_{n\bullet})/S*}\big((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau)\big) \end{aligned}$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}_{n\bullet}(\mathcal{O}_S))$. By (6.15) for the restricted log crystalline case, we have

$$(6.17.8)\qquad \begin{aligned} &\mathbf{s}\check{\mathrm{C}}\mathrm{A}^{\bullet}_{\mathcal{P}_{n\bullet}}((L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}), \\ &\qquad \{L_{(X_{n\bullet},Z_{n\bullet})/S}(P_k^{\mathcal{P}_{n\bullet}/\mathcal{Q}_{n\bullet}}\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S})\}_{k\in\mathbb{Z}})) \\ &= R\bar{u}_{(X_{n\bullet},Z_{n\bullet})/S*}(Q^*_{(X_{n\bullet},Z_{n\bullet})/S}L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}), \\ &\qquad Q^*_{(X_{n\bullet},Z_{n\bullet})/S}(P^{D_{n\bullet}})) \end{aligned}$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}_{n\bullet}(\mathcal{O}_S))$. By (3.4.3) and (4.9.1),

$$(6.17.9)\qquad \begin{aligned} &Ru_{(X_{n\bullet},Z_{n\bullet})/S*}(L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}),\tau) \\ &= R\bar{u}_{(X_{n\bullet},Z_{n\bullet})/S*}(Q^*_{(X_{n\bullet},Z_{n\bullet})/S}L_{(X_{n\bullet},Z_{n\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{n\bullet}/S}), \\ &\qquad Q^*_{(X_{n\bullet},Z_{n\bullet})/S}(P^{D_{n\bullet}})). \end{aligned}$$

Hence the first morphism in (6.17.6) is a filtered quasi-isomorphism. Furthermore, by (4.10.1) and (6.17.8), the second morphism in (6.17.5;n) is a filtered quasi-isomorphism. Hence the second morphism in (6.17.6) is a filtered quasi-isomorphism. Now, by (6.17.3) and (6.17.7), we have

$$(6.17.10)\qquad \begin{aligned} &(E^{\log,Z_{\bullet\leq N}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet\leq N},D_{\bullet\leq N}\cup Z_{\bullet\leq N})/S}), P^{D_{\bullet\leq N}}) \\ &= R\eta_{\mathrm{zar}*}Ru_{(X_{\bullet\leq N,\bullet},Z_{\bullet\leq N,\bullet})/S*}(L_{(X_{\bullet\leq N,\bullet},Z_{\bullet\leq N,\bullet})/S}(\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}),\tau) \\ &= R\eta_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_{\bullet\leq N,\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet\leq N,\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{\bullet\leq N,\bullet}/S}, P^{\mathcal{P}_{\bullet\leq N,\bullet}/\mathcal{Q}_{\bullet\leq N,\bullet}}). \qquad \square \end{aligned}$$

7. Cosimplicial filtered log de Rham-Witt complex

In this section we give a comparison theorem between the truncated cosimplicial (pre)weight-filtered log crystalline cohomology and the cohomology of the truncated cosimplicial (pre)weight-filtered log de Rham-Witt complex.

Before giving the comparison theorem, we give results on the cohomology of the multi-cosimplicial (pre)weight-filtered log de Rham-Witt complex.

Let κ be a perfect field of characteristic $p > 0$. Let $\mathcal{W}$ be the Witt ring of κ and K_0 the fraction field of $\mathcal{W}$. Let n be a positive integer and let $\mathcal{W}_n$ be the Witt ring of κ of length $n > 0$. Let r be a positive integer. Let $(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})$ be a smooth r-simplicial scheme with transversal r-simplicial SNCD's $D_{\underline{\bullet}}$ and $Z_{\underline{\bullet}}$ over κ. By the functoriality of the log de Rham-Witt complex (see [**69**, (9.1)]), we have a complex $\mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))$ of $\mathcal{W}_n$-modules in $\widetilde{X}_{\underline{\bullet}\mathrm{zar}}$, an r-cosimplicial complex $R\Gamma^{\underline{\bullet}}(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})))$ of $\mathcal{W}_n$-modules and an isomorphism

$$R\Gamma\big(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big) \xrightarrow{\ \sim\ } \mathbf{s}R\Gamma^{\underline{\bullet}}\big(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big)$$

(cf. [**25**, (5.2)]). The Frobenius acts on $\mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))$ by [**69**, (9.1), (2)], and we have the transition morphism (cf. [**44**, p. 301])

$$\pi\colon \mathcal{W}_{n+1}\Omega^{\bullet}_{X_{\underline{\bullet}}}\big(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})\big) \longrightarrow \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}\big(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})\big).$$

The morphism π is a surjection. Indeed, let $s\colon \prod_{\underline{t}\in\mathbb{N}^r} \widetilde{X}_{\underline{t}\mathrm{zar}} \to \widetilde{X}_{\underline{\bullet}\mathrm{zar}}$ be the natural morphism of topoi. Then, as in the proof of [**5**, V, Proposition 3.4.4], we have only to prove that $s^*(\operatorname{Coker}\pi) = 0$. Because s^* is exact, this vanishing follows from the surjectivity of

$$\pi\colon \mathcal{W}_{n+1}\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}})) \longrightarrow \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}}))$$

for all $\underline{t} \in \mathbb{N}^r$. Set

$$\mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})) := \varprojlim_{\pi}{}_n \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})).$$

Henceforth, we assume that $X_{\underline{\bullet}}$ is proper over κ.

Lemma 7.1. — *The canonical morphism*

$$(7.1.1)\qquad H^h(X_{\underline{\bullet}}, \mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}\big(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big) \longrightarrow \varprojlim_{n} H^h\big(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big)$$

is an isomorphism.

Proof. — Consider the spectral sequence

$$(7.1.2)\qquad E_1^{t,h-t} = \bigoplus_{\underline{t}_r=t} H^{h-t}\big(X_{\underline{t}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}}))\big) \Longrightarrow H^h\big(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).$$

Because $H^{h-t}(X_{\underline{t}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}})))$ and $H^h(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})))$ are $\mathcal{W}_n$-modules of finite length, we have the following spectral sequence

$$(7.1.3)\qquad E_1^{t,h-t} = \bigoplus_{\underline{t}_r=t} \varprojlim_n H^{h-t}\big(X_{\underline{t}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{t}}}(\log D_{\underline{t}})\big) \Longrightarrow \varprojlim_n H^h\big(X_{\underline{\bullet}}, \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).$$

On the other hand, we have the spectral sequence

$$(7.1.4)\qquad E_1^{t,h-t} = \bigoplus_{\underline{t}_r=t} H^{h-t}\big(X_{\underline{t}}, \mathcal{W}\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}}))\big) \Longrightarrow H^h\big(X_{\underline{\bullet}}, \mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).$$

Because the E_1-terms of (7.1.3) and (7.1.4) are isomorphic as in [**47**, II (2.1)], the canonical isomorphism in (7.1.1) is an isomorphism. □

By considering the stupid filtration on the complex $\mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))$, we have the following spectral sequence

$$(7.1.5)\qquad E_1^{ij} = H^j\big(X_{\underline{\bullet}}, \mathcal{W}\Omega^{i}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big) \Longrightarrow H^{i+j}\big(X_{\underline{\bullet}}, \mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).$$

Definition 7.2. — We call (7.1.5) the *slope spectral sequence* of $(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/\mathcal{W}$.

By (2.2.6) we can calculate the E_1-term of (7.1.5) by the spectral sequence

$$(7.2.1)\qquad E_1^{t,j-t} = \bigoplus_{\underline{t}_r=t} H^{j-t}(X_{\underline{t}}, \mathcal{W}\Omega^{i}_{X_{\underline{t}}}\big(\log(D_{\underline{t}} \cup Z_{\underline{t}}))\big) \Longrightarrow H^j\big(X_{\underline{\bullet}}, \mathcal{W}\Omega^{i}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).$$

As in [**47**], we have :

Proposition 7.3. — *The slope spectral sequence* (7.1.5) *degenerates at* E_1 *modulo torsion, and there exists the following slope decomposition*:

$$(7.3.1)\qquad H^h(X_{\underline{\bullet}}, \mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})))_{K_0} = \bigoplus_{i=0}^{h} H^{h-i}(X_{\underline{\bullet}}, \mathcal{W}\Omega^{i}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})))_{K_0}.$$

Proof. — The Proposition follows from (7.2.1) and [**47**, II (3.2)]. □

Next we construct a spectral sequence of $H^{h-i}(X_{\underline{\bullet}}, \mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})))$ as follows. Let $P^{D_{\underline{\bullet}}}$ be the weight filtration on $\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))$ with respect to $D_{\underline{\bullet}}$ (see [**72**, (2.12.4)], [**69**, (8.7), (9.3) (2)]) and let $\delta(\underline{L}, P^{D_{\underline{\bullet}}})$ be the diagonal filtration of $L_1, \ldots, L_r$ and $P^{D_{\underline{\bullet}}}$ ((2.2.8)). Then we have the following by [**72**, (1.3.4.1)]:

$$\begin{aligned}(7.3.2)\qquad \mathrm{gr}_k^{\delta(\underline{L},P^{D_{\underline{\bullet}}})}\mathbf{s}R\Gamma^{\bullet}(X_{\underline{\bullet}}, \mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))) &= \bigoplus_{\underline{t}\geq 0} \mathrm{gr}^{P^{D_{\underline{t}}}}_{\underline{t}_r+k} R\Gamma\big(X_{\underline{t}}, \mathcal{W}\Omega^i_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}}))\big)[-\underline{t}_r] \\ &= \bigoplus_{\underline{t}\geq 0} R\Gamma\big(X_{\underline{t}}, \mathrm{gr}^{P^{D_{\underline{t}}}}_{\underline{t}_r+k}\mathcal{W}\Omega^i_{X_{\underline{t}}}(\log(D_{\underline{t}} \cup Z_{\underline{t}}))\big)[-\underline{t}_r].\end{aligned}$$

By [**72**, (2.12.4.2)] (cf. [**66**, 1.4.5], [**69**, (9.6.2)]), we have the formula

$$\begin{aligned}(7.3.3)\qquad \mathrm{gr}_k^{P^{D_{\underline{t}}}}\mathcal{W}\Omega^{\bullet}_{X_{\underline{t}}}\big(\log(D_{\underline{t}} \cup Z_{\underline{t}})\big) = \mathcal{W}\Omega^{\bullet-k}_{D^{(k)}_{\underline{t}}}(\log Z_{\underline{t}}|_{D^{(k)}_{\underline{t}}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa)(-k).\end{aligned}$$

Hence the last formula in (7.3.2) is equal to

$$\bigoplus_{\underline{t}\geq 0} R\Gamma(D^{(\underline{t}_r+k)}_{\underline{t}}, \mathcal{W}\Omega^{i-\underline{t}_r-k}_{D^{(\underline{t}_r+k)}_{\underline{t}}}(\log Z_{\underline{t}}|_{D^{(\underline{t}_r+k)}_{\underline{t}}}) \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa))(-(\underline{t}_r+k))[-\underline{t}_r].$$

Therefore we have by the Convention (6) the spectral sequence

$$\begin{aligned}(7.3.4)\qquad E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} H^{h-i-\underline{t}_r}\big(D^{(\underline{t}_r+k)}_{\underline{t}}, \mathcal{W}\Omega^{i-\underline{t}_r-k}_{D^{(\underline{t}_r+k)}_{\underline{t}}}(\log Z_{\underline{t}}|_{D^{(\underline{t}_r+k)}_{\underline{t}}}) &\otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa)\big)\big(-(\underline{t}_r+k)\big) \\ &\Longrightarrow H^{h-i}\big(X_{\underline{\bullet}}, \mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))\big).\end{aligned}$$

Definition 7.4. — We call the spectral sequence (7.3.4) the *weight spectral sequence* of $\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}))$ with respect to $D_{\underline{\bullet}}$. In the case where $Z_{\underline{\bullet}} = \phi$, we call (7.3.4) the *weight spectral sequence* of $\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log D_{\underline{\bullet}})$.

As in (5.3.3), we can give an explicit description of the boundary morphisms between the E_1-terms of (7.3.4) as follows. Let the notations be after (5.3). We set $S := \mathrm{Spf}(\mathcal{W})$ and $S_0 := \mathrm{Spec}(\kappa)$ here. Let

$$\begin{aligned}(7.4.1)\qquad (-1)^j G^{\underline{\lambda}_{\underline{t}j}}_{\underline{\lambda}_{\underline{t}}} \colon H^k(D_{\underline{\lambda}_{\underline{t}}}, \mathcal{W}\Omega^i_{D_{\underline{\lambda}_{\underline{t}}}}(\log Z|_{D_{\underline{\lambda}_{\underline{t}}}}) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_{\underline{t}},\mathrm{zar}}(D_{\underline{t}}/\kappa))(-m) &\longrightarrow H^{k+1}(D_{\underline{\lambda}_{\underline{t}j}}, \mathcal{W}\Omega^{i+1}_{D_{\underline{\lambda}_{\underline{t}j}}}(\log Z|_{D_{\underline{\lambda}_{\underline{t}j}}}) \\ &\otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_{\underline{t}j},\mathrm{zar}}(D_{\underline{t}}/\kappa))(-(m-1)) \quad (m \in \mathbb{Z})\end{aligned}$$

be the Gysin morphism with signs in log Hodge-Witt cohomologies associated to the closed immersion $(D_{\underline{\lambda}_{\underline{t}}}, Z|_{D_{\underline{\lambda}_{\underline{t}}}}) \hookrightarrow (D_{\underline{\lambda}_{\underline{t}j}}, Z|_{D_{\underline{\lambda}_{\underline{t}j}}})$ (see [**72**, (2.12.13)], cf. [**69**, (4.4.5; +)]). Here $\varpi_{\underline{\lambda}\mathrm{zar}}(D_{\underline{t}}/\kappa)$ and $\varpi_{\underline{\lambda}_{\underline{t}j}\mathrm{zar}}(D_{\underline{t}}/\kappa)$ are zariskian orientation sheaves in the zariski topoi $\widetilde{D}_{\underline{\lambda}_{\underline{t}}\mathrm{zar}}$ and $\widetilde{D}_{\underline{\lambda}_{\underline{t}j}\mathrm{zar}}$, respectively (see [**72**, (2.2), (2.8)], cf. [**24**, (3.1.4)]).

Set

$$G_{\underline{t}} := \sum_{\{\lambda_{\underline{t}0},\ldots,\lambda_{\underline{t},m-1} \mid \lambda_{\underline{t}i}\neq\lambda_{\underline{t}j}\ (i\neq j)\}} \sum_{j=0}^{m-1}(-1)^j G^{\underline{\lambda}_{\underline{t}j}}_{\underline{\lambda}_{\underline{t}}} : \tag{7.4.2}$$

$$H^k(D^{(m)}_{\underline{t}}, \mathcal{W}\Omega^i_{D^{(m)}_{\underline{t}}}(\log Z|_{D^{(m)}_{\underline{t}}}) \otimes_{\mathbb{Z}} \varpi^{(m)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa))(-m) \longrightarrow$$
$$H^{k+1}(D^{(m-1)}_{\underline{t}}, \mathcal{W}\Omega^{i+1}_{D^{(m-1)}_{\underline{t}}}(\log Z|_{D^{(m-1)}_{\underline{t}}}) \otimes_{\mathbb{Z}} \varpi^{(m-1)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa))(-(m-1)).$$

Then we can describe the boundary morphisms between the E_1-terms of (7.3.4) as the following diagram:

$$\begin{array}{c} H^{h-i-\underline{t}_r}(D^{(\underline{t}_r+k)}_{\underline{t}+e_j}, \mathcal{W}\Omega^{i-\underline{t}_r-k}_{D^{(\underline{t}_r+k)}_{\underline{t}+e_j}}(\log Z|_{D^{(\underline{t}_r+k)}_{\underline{t}+e_j}}) \\ \qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}_{\mathrm{zar}}(D_{\underline{t}+e_j}/\kappa)))(-(\underline{t}_r+k)) \\ {\scriptstyle (-1)^{t_{j-1}}\sum_{i=0}^{t_j+1}(-1)^i\delta^i_j} \Big\uparrow {\scriptstyle (1\leq j\leq r)} \\ H^{h-i-\underline{t}_r}(D^{(\underline{t}_r+k)}_{\underline{t}}, \mathcal{W}\Omega^{i-\underline{t}_r-k}_{D^{(\underline{t}_r+k)}_{\underline{t}}}(\log Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}}) \\ \qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa))(-(\underline{t}_r+k)) \\ {\scriptstyle -(-1)^{\underline{t}_r}G_{\underline{t}}} \Big\downarrow \\ H^{h-i-\underline{t}_r+1}(D^{(\underline{t}_r+k-1)}_{\underline{t}}, \mathcal{W}\Omega^{i-\underline{t}_r-k+1}_{D^{(\underline{t}_r+k-1)}_{\underline{t}}}(\log Z|_{D^{(\underline{t}_r+k-1)}_{\underline{t}}}) \\ \qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k-1)}_{\mathrm{zar}}(D_{\underline{t}}/\kappa))(-(\underline{t}_r+k-1)). \end{array} \tag{7.4.3}$$

The next lemma is necessary for the proof of (7.6):

Lemma 7.5. — *Let Y (resp. $\mathcal{P}$) be a fine log scheme over* $\mathrm{Spec}(\kappa)$ *(resp.* $\mathrm{Spec}(\mathcal{W}_n)$*). Let $\mathcal{W}_n(Y) \to \mathcal{P}$ be a morphism of log schemes over $\mathcal{W}_n$ which is an underlying morphism of log schemes with decomposition data. Assume that $\mathcal{P}$ has a quasi-global chart with respect to the decomposition datum of $\mathcal{P}$. Let $Y \to \mathcal{P}$ be the composite morphism $Y \hookrightarrow \mathcal{W}_n(Y) \longrightarrow \mathcal{P}$. Let $\mathcal{P}^{\mathrm{qe}}$ be the* *quasi-exactification of this composite morphism with respect to the decomposition data and the quasi-global chart of* $\mathcal{P}$. *Then there exists a morphism* $\mathcal{W}_n(Y) \to \mathcal{P}^{\rm qe}$ *such that the composite morphism* $Y \hookrightarrow \mathcal{W}_n(Y) \to \mathcal{P}^{\rm qe}$ *is the natural morphism* $Y \to \mathcal{P}^{\rm qe}$. *The morphism* $\mathcal{W}_n(Y) \to \mathcal{P}^{\rm qe}$ *is functorial for the following commutative diagram*

$$\begin{array}{ccc} \mathcal{W}_n(Y) & \longrightarrow & \mathcal{P} \\ \big\downarrow & & \big\downarrow \\ \mathcal{W}_n(Z) & \longrightarrow & \mathcal{Q} \end{array}$$

of fine log schemes over $\mathcal{W}_n$ *with decomposition data and with quasi-global charts of* $\mathcal{P}$ *and* $\mathcal{Q}$ *with respect to the decomposition data of* $\mathcal{P}$ *and* $\mathcal{Q}$.

Proof. — Because $\mathcal{W}_n(Z\coprod Z') = \mathcal{W}_n(Z)\coprod\mathcal{W}_n(Z')$ for the disjoint union $Z\coprod Z'$ of log schemes over κ, we may assume that the given decomposition data of $\mathcal{W}_n(Y)$ and $\mathcal{P}$ are trivial and that $\mathcal{P}$ has a global chart $P \to M_{\mathcal{P}}$. The morphism $\mathcal{W}_n(Y) \to \mathcal{P}$ induces a morphism $P \to \Gamma(\mathcal{W}_n(Y), M_{\mathcal{W}_n(Y)}) = \Gamma(Y, M_Y) \oplus \mathrm{Ker}(\Gamma(\mathcal{W}_n(Y), \mathcal{O}^*_{\mathcal{W}_n(Y)}) \to \Gamma(Y, \mathcal{O}^*_Y))$. Since we have a natural morphism $P^{\rm qe} \longrightarrow \Gamma(Y, M_Y)$, the restriction of the morphism $P^{\rm gp} \to \Gamma(\mathcal{W}_n(Y), M_{\mathcal{W}_n(Y)})^{\rm gp}$ to $P^{\rm qe}$ factors through a morphism $P^{\rm qe} \to \Gamma(\mathcal{W}_n(Y), M_{\mathcal{W}_n(Y)})$. Hence we have a desired morphism $\mathcal{W}_n(Y) \longrightarrow \mathcal{P}^{\rm qe}$. The claimed functoriality is clear by the argument above. □

Henceforth, assume that $r = 1$. The following is a main result in this section.

THEOREM 7.6. — *Let* N *be a nonnegative integer. Let* $f\colon X_{\bullet\leq N} \to \mathrm{Spec}(\mathcal{W}_n)$ *be the structural morphism. Assume that* $(X_{\bullet\leq N}, D_{\bullet\leq N} \cup Z_{\bullet\leq N})$ *is split. Then there exists a filtered isomorphism*

$$(7.6.1)\qquad \big(E^{\log, Z_{\bullet\leq N}}_{\rm zar}(\mathcal{O}_{(X_{\bullet\leq N}, D_{\bullet\leq N}\cup Z_{\bullet\leq N})/\mathcal{W}_n}), P^{D_{\bullet\leq N}}\big) \overset{\sim}{\longrightarrow} \big(\mathcal{W}_n\Omega^\bullet_{X_{\bullet\leq N}}(\log(D_{\bullet\leq N}\cup Z_{\bullet\leq N})), P^{D_{\bullet\leq N}}\big)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$. *The isomorphism* (7.6.1) *is functorial and compatible with the projections.*

Proof. — Let the notations be as in (6.10), (6.12) and (6.17) and after (6.13). First we construct the morphism (7.6.1). Denote

$$\mathcal{W}_n\big((X'_N, M(D'_N \cup Z'_N))\big)$$

by $(\mathcal{W}_n(X'_N), \mathcal{W}_n(D'_N \cup Z'_N))$. Since X'_N is affine, there exists a morphism $(\mathcal{W}_n(X'_N), \mathcal{W}_n(D'_N \cup Z'_N)) \to (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ over $\mathcal{W}_n$ such that the composite morphism

$$(X'_N, D'_N \cup Z'_N) \hookrightarrow \big(\mathcal{W}_n(X'_N), \mathcal{W}_n(D'_N \cup Z'_N)\big) \longrightarrow (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$$

is the given closed immersion. Let m be a nonnegative integer less than or equal to N. For a morphism $\gamma\colon [N] \to [m]$ in Δ, let $X(\gamma)$ be the corresponding morphism $(X'_m, D'_m \cup Z'_m) \to (X'_N, D'_N \cup Z'_N)$. Then we have the following commutative diagram

(7.6.2)

$$\begin{array}{ccccc}
(X'_m, D'_m \cup Z'_m) & \xrightarrow{\subset} & (\mathcal{W}_n(X'_m), \mathcal{W}_n(D'_m \cup Z'_m)) & & \\
{\scriptstyle X(\gamma)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{W}_n(X(\gamma))} & & \\
(X'_N, D'_N \cup Z'_N) & \xrightarrow{\subset} & (\mathcal{W}_n(X'_N), \mathcal{W}_n(D'_N \cup Z'_N)) & \longrightarrow & (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N).
\end{array}$$

By this commutative diagram, we have a natural morphism

$$\big(\mathcal{W}_n(X'_{\bullet\leq N}), \mathcal{W}_n(D'_{\bullet\leq N} \cup Z'_{\bullet\leq N})\big) \longrightarrow \Gamma_N^{\mathcal{W}_n}\big((\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)\big)_{\bullet\leq N} \tag{7.6.3}$$

of N-truncated simplicial log schemes. Let

$$\big(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})\big)$$

be the Čech diagram of $(\mathcal{W}_n(X'_{\bullet\leq N}), \mathcal{W}_n(D'_{\bullet\leq N} \cup Z'_{\bullet\leq N}))$ over

$$\big(\mathcal{W}_n(X_{\bullet\leq N}), \mathcal{W}_n(D_{\bullet\leq N} \cup Z_{\bullet\leq N})\big).$$

It is easy to check that $(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet}))$ is the canonical lift of $(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})$. Set

$$\mathcal{P}'_{\bullet\leq N,\bullet} := \operatorname{cosk}_0^{\mathcal{W}_n}\big(\Gamma_N^{\mathcal{W}_n}((\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N))_{\bullet\leq N}\big)$$

as after (6.13). By (7.6.3) we have the morphism

$$\big(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})\big) \longrightarrow \mathcal{P}'_{\bullet\leq N,\bullet} \tag{7.6.4}$$

of (N,∞)-truncated bisimplicial log schemes. By (7.5) we have the following two horizontal morphisms fitting into the following commutative diagram

$$\begin{array}{ccc}
(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})) & \longrightarrow & \mathcal{P}_{\bullet\leq N,\bullet} \\
\big\downarrow & & \big\downarrow \\
(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(Z_{\bullet\leq N,\bullet})) & \longrightarrow & \mathcal{Q}_{\bullet\leq N,\bullet}
\end{array} \tag{7.6.5}$$

such that the composite morphisms

$$(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet}) \hookrightarrow \big(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})\big) \to \mathcal{P}_{\bullet\leq N,\bullet}$$

and

$$(X_{\bullet\leq N,\bullet}, Z_{\bullet\leq N,\bullet}) \hookrightarrow \big(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(Z_{\bullet\leq N,\bullet})\big) \to \mathcal{Q}_{\bullet\leq N,\bullet}$$

are the exact immersions in (6.13.1) (in the case $S = \mathrm{Spec}(\mathcal{W}_n)$). Hence, by (4.22), 2), we have the following morphism of filtered complexes:

$$\begin{aligned}&\big(\mathcal{O}_{\mathfrak{D}_{\bullet\le N,\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet\le N,\bullet}}}\Lambda^{\bullet}_{\mathcal{P}_{\bullet\le N,\bullet}/\mathcal{W}_n}, P^{\mathcal{P}_{\bullet\le N,\bullet}/\mathcal{Q}_{\bullet\le N,\bullet}}\big)\\ &\qquad\longrightarrow \big(\mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_{\bullet\le N,\bullet}}\otimes_{\mathcal{O}_{\mathcal{P}_{\bullet\le N,\bullet}}}\Lambda^{*}_{\mathcal{P}_{\bullet\le N,\bullet}/\mathcal{W}_n}), P^{\mathcal{P}_{\bullet\le N,\bullet}/\mathcal{Q}_{\bullet\le N,\bullet}}\big)\\ &\qquad\quad= \big(\mathcal{W}_n\Omega^{\bullet}_{X_{\bullet\le N,\bullet}}(\log(D_{\bullet\le N,\bullet}\cup Z_{\bullet\le N,\bullet})), P^{D_{\bullet\le N,\bullet}}\big).\end{aligned}$$

Applying $R\eta_{\mathrm{zar}*}$ to this morphism and using (6.17.1), we obtain a morphism

$$\begin{aligned}&\big(E^{\log, Z_{\bullet\le N}}_{\mathrm{zar}}(\mathcal{O}_{(X_{\bullet\le N}, D_{\bullet\le N}\cup Z_{\bullet\le N})/\mathcal{W}_n}), P^{D_{\bullet\le N}}\big)\\ &\qquad\longrightarrow \big(\mathcal{W}_n\Omega^{\bullet}_{X_{\bullet\le N}}(\log(D_{\bullet\le N}\cup Z_{\bullet\le N})), P^{D_{\bullet\le N}}\big)\end{aligned}\tag{7.6.6}$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{W}_n))$. This morphism is a filtered isomorphism by (3.7.2).

Next we prove that the isomorphism (7.6.6) is independent of the choice of the quasi-global chart of $(\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ and that it is is independent of the choice of the disjoint union $(X'_N, D'_N \cup Z'_N)$ of an affine open covering of $(X_N, D_N \cup Z_N)$ at the same time. Assume that we are given the disjoint union $(X''_N, D''_N \cup Z''_N)$ of another affine open covering of $(X_N, D_N \cup Z_N)$. Let $(X''_N, D''_N \cup Z''_N) \hookrightarrow (\mathcal{X}''_N, \mathcal{D}''_N \cup \mathcal{Z}''_N)$ be a closed immersion into a smooth scheme over S with transversal relative SNCD's over S which is an underlying morphism of log schemes with decomposition data such that $(\mathcal{X}''_N, \mathcal{D}''_N \cup \mathcal{Z}''_N)$ has a quasi-global chart with respect to the decomposition datum of $(\mathcal{X}''_N, \mathcal{D}''_N\cup \mathcal{Z}''_N)$. Then, by considering the refinement $(X'''_N, D'''_N \cup Z'''_N)$ of $(X'_N, D'_N \cup Z'_N)$ and $(X''_N, D''_N \cup Z''_N)$ and the product $(\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N) \times_S (\mathcal{X}''_N, \mathcal{D}''_N \cup \mathcal{Z}''_N)$ and by using (4.13) 1), 2), there exists the following commutative diagram

$$\begin{array}{ccc}(X'_N, D'_N \cup Z'_N) & \xrightarrow{\subset} & (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)\\ \uparrow & & \uparrow\\ (X'''_N, D'''_N \cup Z'''_N) & \xrightarrow{\subset} & (\mathcal{X}'''_N, \mathcal{D}'''_N \cup \mathcal{Z}'''_N)\\ \downarrow & & \downarrow\\ (X''_N, D''_N \cup Z''_N) & \xrightarrow{\subset} & (\mathcal{X}''_N, \mathcal{D}''_N \cup \mathcal{Z}''_N).\end{array}$$

Here the target of the middle immersion is a smooth scheme with relative transversal SNCD's over $\mathcal{W}_n$ and $(\mathcal{X}'''_N, \mathcal{D}'''_N \cup \mathcal{Z}'''_N)$ has a quasi-global chart with respect to the decomposition datum of $(\mathcal{X}'''_N, \mathcal{D}'''_N \cup \mathcal{Z}'''_N)$ and that the two right vertical morphisms are underlying morphisms of morphisms of log schemes with decomposition data and quasi-global charts with respect to the decomposition data. Now it is a routine work to check the desired independence.

Lastly we prove the functoriality. Let

$$(X_{\bullet\leq N}, D_{\bullet\leq N} \cup Z_{\bullet\leq N}) \longrightarrow (Y_{\bullet\leq N}, E_{\bullet\leq N} \cup W_{\bullet\leq N})$$

be a morphism of split N-truncated simplicial log schemes over κ which are obtained from smooth split N-truncated simplicial schemes and transversal split N-truncated simplicial SNCD's. Let us consider lifts $(\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N)$ and $(\mathcal{Y}'_N, \mathcal{E}'_N \cup \mathcal{W}'_N)$ of $(X'_N, D'_N \cup Z'_N)$ and $(Y'_N, E'_N \cup W'_N)$, respectively, which fits into the following commutative diagram over $\mathcal{W}_n$:

$$\begin{array}{ccc} (X'_N, D'_N \cup Z'_N) & \xrightarrow{\subset} & (\mathcal{X}'_N, \mathcal{D}'_N \cup \mathcal{Z}'_N) \\ \big\downarrow & & \big\downarrow \\ (Y'_N, E'_N \cup W'_N) & \xrightarrow{\subset} & (\mathcal{Y}'_N, \mathcal{E}'_N \cup \mathcal{W}'_N). \end{array}$$

Here we may assume that the two horizontal morphisms above are underlying morphisms of log schemes with decomposition data and the right vertical morphism is an underlying morphism of log schemes with the decomposition data and with quasi-global charts with respect to the decomposition data. Set

$$\mathcal{R}'_{\bullet\leq N,\bullet} := \mathrm{cosk}_0^{\mathcal{W}_n}(\Gamma_N^{\mathcal{W}_n}((\mathcal{Y}'_N, \mathcal{E}'_N \cup \mathcal{W}'_N))_{\bullet\leq N})$$

and let $\mathcal{R}_{\bullet\leq N,\bullet}$ be the quasi-exactification of the immersion

$$(Y_{\bullet\leq N,\bullet}, E_{\bullet\leq N,\bullet} \cup W_{\bullet\leq N,\bullet}) \longrightarrow \mathcal{R}'_{\bullet\leq N,\bullet}. \tag{7.6.8}$$

Then, by (6.8) (6) and (8), we have the morphisms

$$\begin{aligned} \big(\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})\big) &\longrightarrow \mathcal{P}_{\bullet\leq N,\bullet}, \\ \big(\mathcal{W}_n(Y_{\bullet\leq N,\bullet}), \mathcal{W}_n\big(E_{\bullet\leq N,\bullet} \cup W_{\bullet\leq N,\bullet})\big) &\longrightarrow \mathcal{R}_{\bullet\leq N,\bullet} \end{aligned}$$

fitting into the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{W}_n(X_{\bullet\leq N,\bullet}), \mathcal{W}_n(D_{\bullet\leq N,\bullet} \cup Z_{\bullet\leq N,\bullet})) & \longrightarrow & \mathcal{P}_{\bullet\leq N,\bullet} \\ \big\downarrow & & \big\downarrow \\ (\mathcal{W}_n(Y_{\bullet\leq N,\bullet}), \mathcal{W}_n(E_{\bullet\leq N,\bullet} \cup W_{\bullet\leq N,\bullet})) & \longrightarrow & \mathcal{R}_{\bullet\leq N,\bullet}. \end{array}$$

Now the desired functoriality is clear by (4.22), 2).

By a well-known argument (e.g., [**69**, (7.1), (7.19)]), the isomorphism (7.6.1) is compatible with the projections. □

Corollary 7.7. — *Assume that* $(X_\bullet, D_\bullet \cup Z_\bullet)$ *is split. Then there exists the following canonical isomorphism*

$$H^h\big((X_\bullet, D_\bullet \cup Z_\bullet)/\mathcal{W}\big) \overset{\sim}{\longrightarrow} H^h\big(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log(D_\bullet \cup Z_\bullet))\big) \quad (h \in \mathbb{N}), \tag{7.7.1}$$

which is compatible with the weight filtrations with respect to $D_\bullet$.

Remarks 7.8. — 1) [**45**, (4.19)], [**69**, (7.19)] and (3.7.2) imply (7.7.1) for the constant simplicial case. However, as pointed out by the referee, they do not imply (7.7.1) in the non-constant split simplicial case.

2) I do not know whether an analogous theorem to (7.7) holds in the non-split (multi-)simplicial case.

Corollary 7.9 (Slope decomposition). — *Assume that* $(X_\bullet, D_\bullet \cup Z_\bullet)$ *is split. Then there exists the following canonical isomorphism*:

$$\begin{aligned} H^j\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log(D_\bullet \cup Z_\bullet))\big)_{K_0} \\ \overset{\sim}{\longrightarrow} \big(H^{i+j}((X_\bullet, D_\bullet \cup Z_\bullet)/\mathcal{W})_{K_0}\big)_{[i,i+1)}. \end{aligned} \tag{7.9.1}$$

Consequently, there exists the following canonical decomposition:

$$H^h\big((X_\bullet, D_\bullet \cup Z_\bullet)/\mathcal{W}\big)_{K_0} = \bigoplus_{i=0}^{h} H^{h-i}\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log(D_\bullet \cup Z_\bullet))\big)_{K_0}. \tag{7.9.2}$$

Proof. — The proof is the same as that of [**47**, II (3.5)]: (7.9) follows from (7.7.1) and (7.3.1). □

Theorem 7.10. — *Assume that* $Z_\bullet = \phi$. *Then the spectral sequence* (7.3.4) *degenerates at* E_2 *modulo torsion.*

Proof. — By (7.9.2), (5.6), 1), (7.3.4) and the two diagrams (5.3.3) and (7.4.3), the proof is the same as that of [**69**, (5.9), (4.7)]. □

Remark 7.11. — To the reader, we leave the results in characteristic 0 which are analogous to those in this section

We conclude this section by stating the following nontrivial generalization of [**45**, (4.19)] and [**69**, (7.19)] whose proof has essentially been given in [**72**, (3.5)] by using (6.1) and Tsuzuki's functor Γ. (Because the detailed proof is much easier than that of (7.6), we omit the proof.)

Theorem 7.12. — *Let* L *be a fine log structure on* $\mathrm{Spec}(\kappa)$. *Let* N *be a nonnegative integer. Let* $\mathcal{W}_n(L)$ *be the canonical lift of* L *over* $\mathrm{Spec}(\mathcal{W}_n)$ (see

[**45**, (3.1)]). *Set* $S_n := (\mathrm{Spec}(\mathcal{W}_n), \mathcal{W}_n(L))$. *Let* $Y_{\bullet\leq N}$ *be a log smooth split* N*-truncated simplicial log scheme of Cartier type over* S_1. *Let*

$$f_{\bullet\leq N}\colon Y_{\bullet\leq N} \longrightarrow S_1 \hookrightarrow S_n$$

be the structural morphism. Then there exists a canonical isomorphism

$$Ru_{Y_{\bullet\leq N}/S_n *}(\mathcal{O}_{Y_{\bullet\leq N}/S_n}) \xrightarrow{\sim} \mathcal{W}_n\Lambda^{\bullet}_{Y_{\bullet\leq N}} \tag{7.12.1}$$

in $\mathrm{D}^+(f^{-1}_{\bullet\leq N}(\mathcal{W}_n))$. *The isomorphism* (7.12.1) *is compatible with projections.*

Remark 7.13. — I do not know whether (7.12) holds in the non-split N-truncated (multi-)simplicial case.

8. Complements

In this section we state the r-simplicial versions of some results in [**72**]; we sketch or omit the proofs of the propositions and the theorems in this section because they are the same as those in [**72**]. We have already used (8.1) and (8.3) below in the proofs of (5.6), 1) and 2), respectively.

Let p be a prime number. Let T be a noetherian formal scheme with an ideal sheaf of definition $a\mathcal{O}_T$, where a is a global section of $\Gamma(T, \mathcal{O}_T)$. Assume that there exists a positive integer n such that $p\mathcal{O}_T = a^n\mathcal{O}_T$. Assume that $\mathcal{O}_T$ is a-torsion-free and that the ideal sheaf $a\mathcal{O}_T$ has a PD-structure γ. We call $T = (T, a\mathcal{O}_T, \gamma)$ above an *a-adic formal PD-scheme*. We define the notion of a morphism $T' \longrightarrow T$ of a-adic formal PD-schemes in an obvious way.

Proposition 8.1. — *Let the notations be as above. Assume that, for each affine open subscheme* $\mathrm{Spf}(R)$ *of* T, aR *is a prime ideal of* R *and that the localization ring* R_a *at* aR *is a discrete valuation ring. Let* $g\colon T' \longrightarrow T$ *be a morphism of a-adic formal PD-schemes. Let* $(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})$ *be a proper smooth r-simplicial scheme with transversal r-simplicial relative SNCD's* $D_{\underline{\bullet}}$ *and* $Z_{\underline{\bullet}}$ *over* $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/a)$. *Set*

$$T'_1 := \underline{\mathrm{Spec}}_{T'}(\mathcal{O}_{T'}/a) \quad \textit{and} \quad (X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}}) := (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \times_{T_1} T'_1.$$

If T is small, then the canonical morphism

$$\begin{aligned} &g^* P^{D_{\underline{\bullet}}}_k R^h f_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T}) \\ &\qquad \longrightarrow P^{D'_{\underline{\bullet}}}_k R^h f_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T'*}(\mathcal{O}_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T'}) \quad (h \in \mathbb{Z}) \end{aligned}$$

is an isomorphism.

Proof. — By using the weight spectral sequence (5.1.3) with respect to $D_{\underline{\bullet}}$ for the p-adic formal schemes T and T', the proof in [**69**, (3.2)] works (cf. [**72**, (2.14.2)]). $\square$

The following is an r-simplicial log filtered version of [**13**, (2.1)]:

Proposition 8.2. — *Let S be a scheme of characteristic $p > 0$ and let $S_0 \hookrightarrow S$ be a nilpotent immersion. Let $S \hookrightarrow T$ be a PD-closed immersion into a formal scheme with p-adic topology such that $\mathcal{O}_T$ is p-torsion-free. Let $f\colon (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \to S$ and $f'\colon (X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}}) \to S$ be smooth r-simplicial schemes with transversal r-simplicial relative SNCD's over S. Assume that $X_{\underline{\bullet}}$, $X'_{\underline{\bullet}}$, S and T are noetherian. Set*

$$(X^0_{\underline{\bullet}}, D^0_{\underline{\bullet}} \cup Z^0_{\underline{\bullet}}) := (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \times_S S_0,$$
$$(X'^0_{\underline{\bullet}}, D'^0_{\underline{\bullet}} \cup Z'^0_{\underline{\bullet}}) := (X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}}) \times_S S_0.$$

Let $g\colon (X'^0_{\underline{\bullet}}, D'^0_{\underline{\bullet}} \cup Z'^0_{\underline{\bullet}}) \to (X^0_{\underline{\bullet}}, D^0_{\underline{\bullet}} \cup Z^0_{\underline{\bullet}})$ be a morphism of log schemes over S_0. Then there exists a canonical filtered morphism

$$\begin{aligned} g^* : (Rf_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}, P^{D_{\underline{\bullet}}}) \\ \longrightarrow (Rf'_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T*}(\mathcal{O}_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}, P^{D'_{\underline{\bullet}}}). \end{aligned}$$

The following is an r-simplicial log filtered version of [**13**, (2.2)]:

Corollary 8.3. — *If $(X^0_{\underline{\bullet}}, D^0_{\underline{\bullet}} \cup Z^0_{\underline{\bullet}}) = (X'^0_{\underline{\bullet}}, D'^0_{\underline{\bullet}} \cup Z'^0_{\underline{\bullet}})$, then*

$$\begin{aligned} &\big(Rf_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T*}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})/T}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}, P^{D_{\underline{\bullet}}}\big) \\ &\quad = \big(Rf_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T*}(\mathcal{O}_{(X'_{\underline{\bullet}}, D'_{\underline{\bullet}} \cup Z'_{\underline{\bullet}})/T}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}, P^{D'_{\underline{\bullet}}}\big). \end{aligned}$$

The following is an r-simplicial log filtered version of [**13**, (2.4), (2.5)] and [**74**, (3.8)]:

Theorem 8.4 (Filtered log Berthelot-Ogus isomorphism)

*Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic p. Set $K := \operatorname{Frac}\mathcal{V}$. Let S be a p-adic formal $\mathcal{V}$-scheme in the sense of [**74**]. Let $f\colon (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \to S$ be a proper formally smooth r-simplicial scheme with transversal r-simplicial relative SNCD's $D_{\underline{\bullet}}$ and $Z_{\underline{\bullet}}$ over S. Let T be an enlargement of $S/\mathcal{V}$ with morphism*

$$z\colon T_0 := (\underline{\operatorname{Spec}}_T(\mathcal{O}_T/p))_{\mathrm{red}} \longrightarrow S$$

over $\operatorname{Spf}(\mathcal{V})$. *Set $T_1 := \underline{\operatorname{Spec}}_T(\mathcal{O}_T/p)$. Let*

$$f_0\colon (X^0_{\underline{\bullet}}, D^0_{\underline{\bullet}} \cup Z^0_{\underline{\bullet}}) := (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}}) \times_{S,z} T_0 \longrightarrow T_0$$

be the base change of f. If there exists a log smooth lift $f_1\colon (X^1_{\underline{\bullet}}, D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}}) \to T_1$ of f_0, then there exists the following canonical filtered isomorphism

$$\sigma_T\colon (R^h f_{(X^1_{\underline{\bullet}},D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}})/T*}(\mathcal{O}_{(X^1_{\underline{\bullet}},D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}})/T})_K, P^{D^1_{\underline{\bullet}}}) \xrightarrow{\sim} (R^h f_*(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/K})_T, P^{D_{\underline{\bullet}}}).$$

Proof. — The proof is the same as that of [**74**, (3.8)]. See also [**72**, (2.16.3)]. □

THEOREM 8.5 (Strict compatibility). — *Let $\mathcal{V}$, K, S and S_1 be as in* (8.4). *Let $f\colon (X_{\underline{\bullet}}, D_{\underline{\bullet}}) \to S_1$ and $f'\colon (X'_{\underline{\bullet}}, D'_{\underline{\bullet}}) \to S_1$ be proper smooth r-simplicial schemes with r-simplicial relative SNCD's over S_1. Let h be an integer. Let $g\colon (X'_{\underline{\bullet}}, D'_{\underline{\bullet}}) \to (X_{\underline{\bullet}}, D_{\underline{\bullet}})$ be a morphism of r-simplicial log schemes over S_1. Then the induced morphism*

$$g^*\colon R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}})/S})_K \longrightarrow R^h f_{(X'_{\underline{\bullet}},D'_{\underline{\bullet}})/S*}(\mathcal{O}_{(X'_{\underline{\bullet}},D'_{\underline{\bullet}})/S})_K$$

is strictly compatible with the weight filtration.

Proof. — The proof is the same as that of [**72**, (2.18.2) (1)]. □

REMARK 8.6. — All the results in this section hold in the multi-truncated multi-simplicial case.

PART II. WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A SEPARATED SCHEME OF FINITE TYPE OVER A PERFECT FIELD OF CHARACTERISTIC $p > 0$

9. Proper hypercoverings

Let U be a separated scheme of finite type over a (not necessarily perfect) field κ of characteristic $p > 0$. In this section we construct a proper hypercovering of U which computes the rigid cohomology of U and which will be useful for the construction of the weight filtration on the rigid cohomology and the calculation of the slope filtration on the rigid cohomology.

By Nagata's theorem (see [**67**]), there exists an open immersion $U \hookrightarrow \overline{U}$ into a proper scheme over κ. In this section we start with the following, which is a corollary of de Jong's alteration theorem (see [**50**, (4.1)]):

Proposition 9.1 (see [**50**, Introduction]). — *There exists the following cartesian diagram*

(9.1.1)
$$\begin{array}{ccc} U_0 & \overset{\subset}{\longrightarrow} & X_0 \\ \big\downarrow & & \big\downarrow \\ U & \overset{\subset}{\longrightarrow} & \overline{U} \end{array}$$

such that the left vertical morphism is proper and surjective and such that U_0 is the complement of an SNCD D_0 in a projective regular scheme X_0 over κ.

Proof. — Take the reduced pair $(U_{\rm red}, \overline{U}_{\rm red})$ of $(U, \overline{U})$. Let $\overline{U}_{{\rm red},i}$ be an irreducible component of $\overline{U}_{\rm red}$ such that $\overline{U}_{{\rm red},i} \setminus U_{\rm red} \subsetneqq \overline{U}_{{\rm red},i}$. Set $C_i := \overline{U}_{{\rm red},i} \setminus U_{\rm red}$. Then, applying [**50**, (4.1)] to the closed subscheme C_i of $\overline{U}_{{\rm red},i}$, there exists an alteration $X_{0i} \longrightarrow \overline{U}_{{\rm red},i}$ such that X_{0i} is a projective regular variety over κ and such that $D_{0i} := X_{0i} \times_{\overline{U}_{{\rm red},i}} C_i$ is an SNCD on X_{0i}.

Set $U_{0i} := X_{0i} \setminus D_{0i}$. Then the pair $(U_0, X_0) := (\coprod_i U_{0i}, \coprod_i X_{0i})$ is a desired pair. By the construction of U_0, the morphism $U_0 \longrightarrow U$ is proper and surjective. □

Proposition 9.2. — *There exists a proper hypercovering $(U_\bullet, \overline{U}_\bullet)_{\bullet \in \mathbb{N}}$ of $(U, \overline{U})$ in the sense of* [**86**, (2.1.1) (2)], *that is, the natural morphism $U_{n+1} \to \mathrm{cosk}_n^U(U_{\bullet \leq n})_{n+1}$ $(n \geq -1)$ is proper and surjective, the morphism $\overline{U}_\bullet \to \overline{U}$ is proper and $U_\bullet = \overline{U}_\bullet \times_{\overline{U}} U$ (strictness over $(U, \overline{U})$). Moreover, one can take the pair $(U_\bullet, \overline{U}_\bullet)_{\bullet \in \mathbb{N}}$ satisfying the following conditions*:

(9.2.1) $X_\bullet := \overline{U}_\bullet$ *is regular,*

(9.2.2) $D_\bullet := X_\bullet \setminus U_\bullet$ *is a simplicial SNCD on* $X_\bullet$,

(9.2.3) $U_\bullet$ *and* $\overline{U}_\bullet$ *are split* ([**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.1)], [**25**, (6.2.2)]),

(9.2.4) $\overline{U}_\bullet$ *is projective over* κ.

Proof. — Let the notations be as in (9.1). Then

$$(X_0 \times_{\overline{U}} X_0) \times_{\overline{U}} U = U_0 \times_U U_0.$$

By (9.1) there exists a proper surjective morphism $N_1^X \to X_0 \times_{\overline{U}} X_0$ over κ from a projective regular scheme over κ fitting into the following cartesian diagram:

$$\begin{array}{ccc} N_1^U & \xrightarrow{\subset} & N_1^X \\ \big\downarrow & & \big\downarrow \\ U_0 \times_U U_0 & \xrightarrow{\subset} & X_0 \times_{\overline{U}} X_0. \end{array} \tag{9.2.5}$$

Here N_1^U is the complement of an SNCD on N_1^X. As in [**25**, (6.2.5)], we have a pair $({}_1U_\bullet, {}_1X_\bullet)_{\bullet \leq 1}$ of 1-truncated simplicial schemes over $(U, \overline{U})$: for $V = U$ or X, ${}_1V_0 := V_0$ and ${}_1V_1 := V_0 \coprod N_1^V$ with natural morphisms.

For a nonnegative integer n, assume that we are given a pair $({}_nU_\bullet, {}_nX_\bullet)_{\bullet \leq n}$ of split n-truncated simplicial schemes over κ such that ${}_nX_{\bullet \leq n}$ is projective over κ and regular, and such that ${}_nU_{\bullet \leq n}$ is the complement of a split n-truncated simplicial SNCD on ${}_nX_{\bullet \leq n}$. By the expression of the coskeleton in (6.1.3), the natural morphism

$$\mathrm{cosk}_n^U({}_nU_{\bullet \leq n})_{n+1} \longrightarrow \mathrm{cosk}_n^{\overline{U}}({}_nX_{\bullet \leq n})_{n+1}$$

is an open immersion and $\mathrm{cosk}_n^{\overline{U}}({}_nX_{\bullet \leq n})_{n+1} \times_{\overline{U}} U = \mathrm{cosk}_n^U({}_nU_{\bullet \leq n})_{n+1}$. Hence, by the same technique as that in [**24**, (6.2)] and by (9.1), we have an open immersion $U_\bullet \hookrightarrow X_\bullet$ such that $U_\bullet$ is the complement of a split simplicial SNCD

$D_\bullet$ in a projective regular split simplicial scheme $X_\bullet$ over κ, such that $U_{n+1} \to \operatorname{cosk}_n^U(U_{\bullet\leq n})_{n+1}$ is proper and surjective, and such that $U_\bullet = X_\bullet \times_{\overline{U}} U$. $\square$

Definition 9.3. — 1) We say that a proper hypercovering of $(U, \overline{U})$ satisfying (9.2.1) and (9.2.2) is *good*. We say that a proper hypercovering of $(U, \overline{U})$ satisfying (9.2.1) $\sim$ (9.2.3) is *gs* (= *good and split*).

2) We say that a proper hypercovering $U_\bullet$ of U is *good* if $U_\bullet$ is the complement of a simplicial SNCD $D_\bullet$ on a proper regular simplicial scheme $X_\bullet$ over κ.

Proposition 9.4. — *The following hold*:

1) *Any two proper hypercoverings of* $(U, \overline{U})$ *satisfying* (9.2.1) $\sim$ (9.2.i) $(i = 1, \ldots, 4)$ *are covered by a proper hypercovering of* $(U, \overline{U})$ *satisfying* (9.2.1) $\sim$ (9.2.i).

2) *Let* $U' \hookrightarrow \overline{U}'$ *be an open immersion from a separated scheme of finite type over* κ *into a proper scheme over* κ. *For a morphism* $(U', \overline{U}') \longrightarrow (U, \overline{U})$ *of pairs over* κ *and for a proper hypercovering* $(U_\bullet, \overline{U}_\bullet)$ *of* $(U, \overline{U})$ *satisfying* (9.2.1) $\sim$ (9.2.i) $(i = 1, \ldots, 4)$, *there exist a proper hypercovering* $(U'_\bullet, \overline{U}'_\bullet)$ *of* $(U', \overline{U}')$ *satisfying* (9.2.1) $\sim$ (9.2.i) *and a morphism* $(U'_\bullet, \overline{U}'_\bullet) \longrightarrow (U_\bullet, \overline{U}_\bullet)$ *of pairs fitting into the following commutative diagram*:

$$\begin{array}{ccc} (U'_\bullet, \overline{U}'_\bullet) & \longrightarrow & (U_\bullet, \overline{U}_\bullet) \\ \big\downarrow & & \big\downarrow \\ (U', \overline{U}') & \longrightarrow & (U, \overline{U}). \end{array} \tag{9.4.1}$$

3) *Let* $U^j \hookrightarrow \overline{U}^j$ $(j = 1, 2)$ *be an open immersion from a separated scheme of finite type over* κ *into a proper scheme over* κ. *Set*

$$(U^{12}, \overline{U}^{12}) := (U^1 \times_\kappa U^2, \overline{U}^1 \times_\kappa \overline{U}^2).$$

If $(U^j_\bullet, \overline{U}^j_\bullet)$ *is a proper hypercovering of* $(U^j, \overline{U}^j)$ *satisfying* (9.2.1) $\sim$ (9.2.i) $(i = 1, \ldots, 4)$, *then there exists a proper hypercovering* $(U^{12}_\bullet, \overline{U}^{12}_\bullet)$ *of* $(U^{12}, \overline{U}^{12})$ *satisfying* (9.2.1) $\sim$ (9.2.i) *and fitting into the following commutative diagram for* $j = 1$ *and* 2 :

$$\begin{array}{ccc} (U^{12}_\bullet, \overline{U}^{12}_\bullet) & \longrightarrow & (U^j_\bullet, \overline{U}^j_\bullet) \\ \big\downarrow & & \big\downarrow \\ (U^{12}, \overline{U}^{12}) & \xrightarrow[\text{proj.}]{} & (U^j, \overline{U}^j). \end{array} \tag{9.4.2}$$

4) *Let* $U^j \hookrightarrow \overline{U}^j$ $(j=1,2)$ *be an open immersion from a separated scheme of finite type over* κ *into a proper scheme over* κ. *Set*

$$(U^{12}, \overline{U}^{12}) := (U^1 \times_\kappa U^2, \overline{U}^1 \times_\kappa \overline{U}^2).$$

If $(U^j_\bullet, \overline{U}^j_\bullet)$ *is a proper hypercovering of* $(U^j, \overline{U}^j)$ *satisfying* (9.2.i) $(i \neq 3)$, *then* $(U^1_\bullet \times_\kappa U^2_\bullet, \overline{U}^1_\bullet \times_\kappa \overline{U}^2_\bullet)$ *is a proper hypercovering of* $(U^{12}, \overline{U}^{12})$ *satisfying* (9.2.i).

Proof. — 1), 2): We use a general formalism in [**35**, $\mathrm{V}^{\mathrm{bis}}$ §5]. Here we give a proof for proper hypercoverings satisfying (9.2.1) $\sim$ (9.2.4). (By using (9.9), 2) below, we have only to prove (9.4) in this case.)

Let $\mathcal{S}$ be the category of open immersions $\iota\colon U \hookrightarrow \overline{U}$ from separated schemes of finite type over κ into proper schemes over κ; we define a morphism $\iota' \to \iota$ in $\mathcal{S}$ in an obvious way. For an object ι above of $\mathcal{S}$, we often denote it by $(U, \overline{U})$. Let $\pi\colon \mathcal{E} \to \mathcal{S}$ be a category over $\mathcal{S}$ defined by the following: for an object $\iota \in \mathcal{S}$, the fiber category $\mathcal{E}_\iota$ consists of objects $(Y, \overline{Y})$'s, where $\overline{Y}$ is proper over $\overline{U}$ and the morphism $(Y, \overline{Y}) \to (U, \overline{U})$ is strict: $Y = \overline{Y} \times_{\overline{U}} U$. It is easy to check that $\mathcal{E}_\iota$ has finite projective limits and finite disjoint sums. For a morphism $t\colon \iota' \longrightarrow \iota$ in $\mathcal{S}$ and for an object $(Y, \overline{Y})$ (resp. $(Y', \overline{Y}')$) of $\mathcal{E}_\iota$ (resp. $\mathcal{E}_{\iota'}$), we define a morphism $(Y', \overline{Y}') \to (Y, \overline{Y})$ over the morphism t by the following commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \cap\big\downarrow & & \big\downarrow\cap \\ \overline{Y}' & \longrightarrow & \overline{Y} \end{array} \tag{9.4.3}$$

over the following commutative diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ {\scriptstyle\iota'}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ \overline{U}' & \longrightarrow & \overline{U}. \end{array} \tag{9.4.4}$$

We say that an object $(Y, \overline{Y}) \in \mathcal{E}_\iota$ satisfies the property Q if $\overline{Y}$ is projective over κ and regular, and if Y is the complement of an SNCD on $\overline{Y}$. We say that a morphism $(Y', \overline{Y}') \to (Y, \overline{Y})$ in $\mathcal{E}$ satisfies the property P if $(Y', \overline{Y}') \in Q$ and if $Y' \to Y$ is proper and surjective. Obviously the properties Q and P are stable under isomorphisms in $\mathcal{E}$. As in the proof of [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.3.4)], we check that (π, Q, P) satisfies the conditions of a), b), c), d), e) and f) in [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.4)] as follows.

a): For an object $\iota \in \mathcal{S}$, we have an object of $\mathcal{E}_\iota$ satisfying the property Q by (9.1).

b): Let $\iota\colon U \hookrightarrow \overline{U}$ be an object of $\mathcal{S}$ and $(Y^j, \overline{Y}^j)$ $(j = 1, 2)$ an object of $\mathcal{E}_\iota$. Then $(Y^1 \times_U Y^2, \overline{Y}^1 \times_{\overline{U}} \overline{Y}^2)$ is an object of $\mathcal{E}_\iota$. Hence, by (9.1), there exists the following cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \subset\ } & \overline{X} \\ \big\downarrow & & \big\downarrow \\ Y^1 \times_U Y^2 & \xrightarrow{\ \subset\ } & \overline{Y}^1 \times_{\overline{U}} \overline{Y}^2, \end{array}$$

where the composite morphism $(X, \overline{X}) \to (Y^1 \times_U Y^2, \overline{Y}^1 \times_{\overline{U}} \overline{Y}^2) \to (U, \overline{U})$ is an element of $\mathcal{E}_\iota$ satisfying the condition Q.

c): The proof for checking this condition is analogous to b). We have only to note the following: for a morphism $\iota' \to \iota$ in $\mathcal{S}$ and for an object $(Y, \overline{Y})$ of $\mathcal{E}_\iota$, the pair $(Y \times_U U', \overline{Y} \times_{\overline{U}} \overline{U}')$ is an object of $\mathcal{E}_{\iota'}$ and to use (9.1).

d): Follows immediately from (9.1).

e): The proof for checking this condition is analogous to b); we have only to take suitable fiber products three times and to use (9.1).

f): Since the proof for checking this condition is analogous to b), we leave the detail to the reader.

Hence 1) and 2) follow from [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.7), (5.1.3)].

3): This immediately follows from 1) and 2).

4): The proof is easy. □

Remarks 9.5. — 1) In [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.3)], for a separated scheme S of finite type over a field of characteristic 0 and for an object $(\overline{X}, X, i)$ of E_S in [*loc. cit.*], X is assumed to be dense in $\overline{X}$. We have not assumed this condition for an object of $\mathcal{E}_\iota$ in the proof of (9.4).

2) In [**35**, p. 158, N.B.], there is a mistyped equation (we omit to point out other mistypes in [**35**, p. 158]): one has to replace the formula $\overline{h}^{-1}(i(X)) = i(X')$ in [**35**, p. 158, N.B.] by $\overline{h}^{-1}(i(X)) = i'(X')$. Even if one corrects this mistype, there is obviously a counter-example for the corrected formula: take $\overline{X}' = \overline{X}$, $\overline{h} = \mathrm{id}$, $\overline{X}$: a variety over the base field, $X' := X \setminus \{P\}$ (P: a closed point of X), i', i and h in [*loc. cit.*]: the natural open immersions. If $X \neq \{P\}$, the corrected formula does not hold.

3) As is well-known (e.g., [**9**, p. 340]), for a morphism $U' \longrightarrow U$ of separated schemes of finite type over κ, we have the following commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{\subset} & \overline{U}' \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\subset} & \overline{U}, \end{array} \tag{9.5.1}$$

where the two horizontal morphisms are open immersions into proper schemes over κ; in fact, we can take the two horizontal morphisms as compactifications.

4) More generally, for a commutative diagram

$$\begin{array}{ccc} U_3 & \longrightarrow & U_4 \\ \big\downarrow & & \big\downarrow \\ U_1 & \longrightarrow & U_2 \end{array}$$

of separated schemes of finite type over κ, by using the argument in [**9**, p. 340] repeatedly, we can easily prove that there exists a commutative diagram

$$\begin{array}{ccc} \overline{U}_3 & \longrightarrow & \overline{U}_4 \\ \big\downarrow & & \big\downarrow \\ \overline{U}_1 & \longrightarrow & \overline{U}_2 \end{array} \tag{9.5.2}$$

of proper schemes over κ with four open immersions $U_i \hookrightarrow \overline{U}_i$ $(i = 1, 2, 3, 4)$ which makes the four new diagrams commutative.

The following is an easy corollary of (6.1).

Lemma 9.6. — *Let $U_\bullet \hookrightarrow Y_\bullet$ be an open immersion of simplicial schemes over an open immersion $U \hookrightarrow Y$ of schemes over a scheme S. Assume that $(U_\bullet, Y_\bullet)$ is split. Then there exists a pair $(U'_\bullet, Y'_\bullet)$ of split simplicial schemes over (U, Y) with a natural morphism $(U'_\bullet, Y'_\bullet) \longrightarrow (U_\bullet, Y_\bullet)$ of the pairs of the simplicial schemes satisfying the following conditions*:

(9.6.1) U'_m $(m \in \mathbb{N})$ *is the disjoint union of open subschemes of U_m which cover U_m and which are open subschemes of affine open subschemes of Y_m.*

(9.6.2) Y'_m $(m \in \mathbb{N})$ *is the disjoint union of affine open subschemes which cover Y_m and whose images in Y are contained in affine open subschemes of Y.*

(9.6.3) $U'_m = U_m \times_{Y_m} Y'_m$.

(9.6.4) *If* $(U_\bullet, Y_\bullet)$ *is strict over* (U, Y)*, that is,* $U_\bullet = Y_\bullet \times_Y U$*, then* $(U'_\bullet, Y'_\bullet)$ *is strict over* (U, Y)*.*

(9.6.5) *If* Y_m $(m \in \mathbb{N})$ *is quasi-compact, then the number of the open subschemes in* (9.6.1) *and* (9.6.2) *can be assumed to be finite.*

Set $(U_{mn}, Y_{mn}) := (\operatorname{cosk}_0^{U_m}(U'_m)_n, \operatorname{cosk}_0^{Y_m}(Y'_m)_n)$ $(m, n \in \mathbb{N})$*. Then there exists a natural morphism* $(U_{\bullet\bullet}, Y_{\bullet\bullet}) \to (U_\bullet, Y_\bullet)$ *over* (U, Y)*. For each* $n \in \mathbb{N}$*,* $(U_{\bullet n}, Y_{\bullet n})$ *is split.*

Proof. — By (6.1) we obtain a split simplicial scheme $Y'_\bullet$ satisfying (9.6.2) and (9.6.5). Set $U'_\bullet := Y'_\bullet \times_{Y_\bullet} U_\bullet$. If $U_\bullet = Y_\bullet \times_Y U$, then $U'_\bullet = Y'_\bullet \times_Y U$. By (6.1) we obtain the splitness of $(U_{\bullet n}, Y_{\bullet n})$. □

Definition 9.7. — 1) We call the simplicial scheme $(U'_\bullet, Y'_\bullet)$ satisfying (9.6.1) $\sim$ (9.6.5) the *disjoint union of the members of an affine simplicial open covering* of $(U_\bullet, Y_\bullet)/(U, Y)$.

2) We call the bisimplicial scheme $(U_{\bullet\bullet}, Y_{\bullet\bullet})$ in (9.6) the *Čech diagram* of $(U'_\bullet, Y'_\bullet)$ over $(U_\bullet, Y_\bullet)/(U, Y)$.

3) In 1) and 2), if $U = Y = S$, then we say "over S" instead of "over (U, Y)".

Proposition 9.8. — 1) *Let* $(V_\bullet, Z_\bullet) \to (U_\bullet, Y_\bullet)$ *be a morphism of pairs of split simplicial schemes over a morphism* $(V, Z) \to (U, Y)$ *of open immersions of schemes over a scheme* S*. Assume that* $(U_\bullet, Y_\bullet)$ *is strict over* (U, Y)*. Then there exist the disjoint union of the members of affine simplicial open coverings* $(U'_\bullet, Y'_\bullet)$ *and* $(V'_\bullet, Z'_\bullet)$ *of* $(U_\bullet, Y_\bullet)/(U, Y)$ *and* $(V_\bullet, Z_\bullet)/(V, Z)$*, respectively, which fit into the following commutative diagram*:

(9.8.1)
$$\begin{array}{ccc} (V'_\bullet, Z'_\bullet) & \longrightarrow & (U'_\bullet, Y'_\bullet) \\ \downarrow & & \downarrow \\ (V_\bullet, Z_\bullet) & \longrightarrow & (U_\bullet, Y_\bullet). \end{array}$$

2) *Let* $U \hookrightarrow Y$ *be an open immersion of schemes over a scheme* S*. Let* $(U_\bullet, Y_\bullet)$ *be a pair of split simplicial schemes over* (U, Y)*. Let* $(U'_\bullet, Y'_\bullet)$ *and* $(U''_\bullet, Y''_\bullet)$ *be two disjoint unions of the members of affine simplicial open coverings of* $(U_\bullet, Y_\bullet)/(U, Y)$*. Then there exists a disjoint union of the members of an affine simplicial open covering* $(U'''_\bullet, Y'''_\bullet)$ *of* $(U_\bullet, Y_\bullet)/(U, Y)$ *fitting into*

the following commutative diagram:

$$\begin{array}{ccc} (U'''_\bullet, Y'''_\bullet) & \longrightarrow & (U''_\bullet, Y''_\bullet) \\ \downarrow & & \downarrow \\ (U'_\bullet, Y'_\bullet) & \longrightarrow & (U_\bullet, Y_\bullet). \end{array} \tag{9.8.2}$$

Proof. — 1): Let $(U'_\bullet, Y'_\bullet)$ be the disjoint union of the members of an affine simplicial open covering of $(U_\bullet, Y_\bullet)/(U, Y)$. Let $(V''_\bullet, Z''_\bullet)$ be the fiber product of $(V_\bullet, Z_\bullet)$ and $(U'_\bullet, Y'_\bullet)$ over $(U_\bullet, Y_\bullet)$. If $(V_\bullet, Z_\bullet)$ is strict over (V, Z), then so is $(V''_\bullet, Z''_\bullet)$. Using the argument in the proof of (6.1), (6.3) and (9.6), we obtain 1).

2): Take Y'''_0 as an affine refinement of the disjoint unions Y'_0 and Y''_0 of the open coverings of Y_0. For a positive integer m, assume that we are given $Y'''_{\bullet\leq m-1}$. Let $N(Y_m)'''$ be the scheme constructed in the proof of (6.1) for $Y'''_{\bullet\leq m-1}$ fitting into the following commutative diagram

$$\begin{array}{ccc} N(Y_m)' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y'_{\bullet\leq m-1})_m \\ \uparrow & & \uparrow \\ N(Y_m)''' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y'''_{\bullet\leq m-1})_m \\ \downarrow & & \downarrow \\ N(Y_m)'' & \longrightarrow & \mathrm{cosk}^Y_{m-1}(Y''_{\bullet\leq m-1})_m. \end{array}$$

Set $Y'''_m = \coprod_{0\leq \ell\leq m} \coprod_{[m]\twoheadrightarrow[\ell]} N(Y_\ell)'''$. In this way, we obtain $Y'''_\bullet$. Set $U'''_\bullet := Y'''_\bullet \times_Y U$. □

Lemma 9.9. — 1) *Let (U, Y) be as in* (9.6). *Let $(U_\bullet, Y_\bullet)$ be a pair of simplicial schemes over (U, Y). Then there exists a split pair $(V_\bullet, Z_\bullet)$ of simplicial schemes over (U, Y) with a morphism $(V_\bullet, Z_\bullet) \to (U_\bullet, Y_\bullet)$ of pairs of simplicial schemes over (U, Y).*

2) *Let the notations be as in* (9.6). *Assume that $S = \mathrm{Spec}(\kappa)$ and $Y = \overline{U}$ in the beginning of this section. If $(U_\bullet, Y_\bullet)$ is a proper hypercovering of $(U, \overline{U})$, then one can take $(V_\bullet, Z_\bullet)$ as a gs proper hypercovering of $(U, \overline{U})$.*

Proof. — 1): Follows immediately from [**35**, V$^{\mathrm{bis}}$ (5.1.3)].

2): Follows from the proof of 1) and (9.1) (cf. the proof of (6.1)). □

In §15 below, we shall need the following variant of (9.4).

As in the proof of (9.4), we denote by $(U, \overline{U})$ an open immersion $\iota\colon U \hookrightarrow \overline{U}$ from a separated scheme of finite type over κ into a proper scheme over κ.

Let $\mathcal{S}$ and $\mathcal{E}$ be as in the proof of (9.4). Let $\mathcal{T}$ be the category of the morphisms in $\mathcal{S}$; we define the morphisms in $\mathcal{T}$ in an obvious way. Let $\pi\colon \mathcal{H} \to \mathcal{T}$ be a category over $\mathcal{T}$ defined by the following: for an object $\alpha\colon (U', \overline{U}') \to (U, \overline{U})$ of $\mathcal{T}$, the fiber category $\mathcal{H}_\alpha$ consists of $\beta\colon (Y', \overline{Y}') \to (Y, \overline{Y})$ over α such that $(Y', \overline{Y}') \in \mathcal{E}_{\iota'}$ and $(Y, \overline{Y}) \in \mathcal{E}_\iota$, where ι' is the open immersion $U' \hookrightarrow \overline{U}'$. It is easy to check that $\mathcal{H}_\alpha$ has finite projective limits and finite disjoint sums. We define the morphisms in $\mathcal{H}$ in an obvious way.

We say that an object $\beta\colon (Y', \overline{Y}') \to (Y, \overline{Y})$ of $\mathcal{H}$ satisfies the property HQ if $(Y', \overline{Y}')$ and $(Y, \overline{Y})$ satisfy Q in the proof of (9.4). We call a morphism

$$\big[(Y'_1, \overline{Y}'_1) \to (Y'_2, \overline{Y}'_2)\big] \longrightarrow \big[(Y_1, \overline{Y}_1) \to (Y_2, \overline{Y}_2)\big]$$

in $\mathcal{H}$ over

$$\big[(U'_1, \overline{U}'_1) \to (U'_2, \overline{U}'_2)\big] \longrightarrow \big[(U_1, \overline{U}_1) \to (U_2, \overline{U}_2)\big]$$

satisfies the property HP if $[(Y'_1, \overline{Y}'_1) \to (Y'_2, \overline{Y}'_2)] \in HQ$ and if the morphisms $Y'_i \to U'_i$ and $Y_i \to U_i$ $(i = 1, 2)$ are proper and surjective. Then the properties HQ and HP are stable under isomorphisms in $\mathcal{H}$, and, as in the proof of (9.4), we see that (E, HQ, HP) satisfies the conditions a), b), c), d), e) and f) in [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.4)]; we leave the checking to the reader.

Definition 9.10. — Let $\alpha\colon (U', \overline{U}') \to (U, \overline{U})$ be an object of $\mathcal{T}$. Let $\alpha_\bullet\colon (U'_\bullet, \overline{U}'_\bullet) \to (U_\bullet, \overline{U}_\bullet)$ be a morphism of simplicial open immersions from separated simplicial schemes of finite type over κ into proper simplicial schemes over κ which lies over α. Then we call $\alpha_\bullet$ a *proper hypercovering* of α if $(U'_\bullet, \overline{U}'_\bullet)$ and $(U_\bullet, \overline{U}_\bullet)$ are proper hypercoverings of $(U', \overline{U}')$ and $(U, \overline{U})$, respectively.

By (9.2) and (9.4), 2), for an object $\alpha\colon (U', \overline{U}') \to (U, \overline{U})$ of $\mathcal{T}$, there exist a proper hypercovering of $\alpha_\bullet\colon (U'_\bullet, \overline{U}'_\bullet) \to (U_\bullet, \overline{U}_\bullet)$ such that $(U'_\bullet, \overline{U}'_\bullet)$ and $(U_\bullet, \overline{U}_\bullet)$ satisfy the conditions (9.2.1) $\sim$ (9.2.4)

By the argument before (9.10) and by [**35**, $\mathrm{V}^{\mathrm{bis}}$ (5.1.7), (5.1.3)], we have the following:

Proposition 9.11. — 1) *Let $\alpha\colon (U', \overline{U}') \to (U, \overline{U})$ be an object of $\mathcal{T}$. Any two proper hypercoverings of α whose sources and targets satisfy* (9.2.1) $\sim$ (9.2.i) $(i = 1, \ldots, 4)$ *are covered by a proper hypercovering of α satisfying the same conditions.*

2) *For a morphism $[(U^{1'}, \overline{U}^{1'}) \to (U^1, \overline{U}^1)] \to [(U^{2'}, \overline{U}^{2'}) \to (U^2, \overline{U}^2)]$ in $\mathcal{T}$ and for a proper hypercovering $[(U^{2'}_\bullet, \overline{U}^{2'}_\bullet) \to (U^2_\bullet, \overline{U}^2_\bullet)]$ of $[(U^{2'}, \overline{U}^{2'}) \to (U^2, \overline{U}^2)]$ whose source and target satisfy* (9.2.1) $\sim$ (9.2.i) $(i = 1, \ldots, 4)$*, there*

exist a proper hypercovering $[(U^{1'}_{\bullet},\overline{U}^{1'}_{\bullet})\to(U^1_{\bullet},\overline{U}^1_{\bullet})]$ *of* $[(U^{1'},\overline{U}^{1'})\to(U^1,\overline{U}^1)]$ *whose source and target satisfy* (9.2.1) $\sim$ (9.2.i) *and a morphism*

$$\big[(U^{1'}_{\bullet},\overline{U}^{1'}_{\bullet})\to(U^1_{\bullet},\overline{U}^1_{\bullet})\big]\longrightarrow\big[(U^{2'}_{\bullet},\overline{U}^{2'}_{\bullet})\to(U^2_{\bullet},\overline{U}^2_{\bullet})\big]$$

over $[(U^{1'},\overline{U}^{1'})\to(U^1,\overline{U}^1)]\to[(U^{2'},\overline{U}^{2'})\to(U^2,\overline{U}^2)]$.

Definition 9.12. — We say that a proper hypercovering $\alpha_{\bullet}$ of α is *good* (resp. *split*, *gs*) if the source and the target of $\alpha_{\bullet}$ are good (resp. split, gs).

We also have the following by (9.9), 2):

Proposition 9.13. — *Let* α *be as in* (9.11). *If* $\alpha_{\bullet}$ *is a proper hypercovering of* α*, then there exists a gs proper hypercovering* $\beta_{\bullet}$ *of* α *factoring through the augmentation* $\alpha_{\bullet}\to\alpha$.

10. Truncated cosimplicial rigid cohomological complex

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with (not necessarily perfect) residue field κ. Let $\mathcal{W}$ be a Cohen ring of κ in $\mathcal{V}$. Let K (resp. K_0) be the fraction field of $\mathcal{V}$ (resp. $\mathcal{W}$). Let U be a separated scheme of finite type over κ. Let $j\colon U\hookrightarrow\overline{U}$ be an open immersion into a proper scheme over κ (see [**67**]). Let C be an overconvergent isocrystal on $(U,\overline{U})/K$.

In this section we construct a fundamental complex which we call a *truncated cosimplicial rigid cohomological complex*. This complex produces Tsuzuki's spectral sequence (see [**86**, (4.5.1)], [**87**, (7.1.2)]) of the rigid cohomology of C with respect to a proper hypercovering of $(U,\overline{U})$. We show that this complex is a coring object in Tsuzuki's theory in [**86**] and [**87**].

Let $\overline{Z}:=\coprod_{i=1}^m\overline{Z}_i$ ($m\in\mathbb{Z}_{>0}$) be the disjoint union of affine open subschemes which cover $\overline{U}$. Set $Z:=\overline{Z}\times_{\overline{U}}U$. The scheme $\overline{Z}$ can be embedded into a separated formally smooth p-adic formal $\mathcal{V}$-scheme $\mathcal{Z}$. Then we have a Čech diagram

$$\mathfrak{Z}:=(Z_{\bullet},\overline{Z}_{\bullet},\mathcal{Z}_{\bullet}):=\big(\mathrm{cosk}_0^U(Z),\mathrm{cosk}_0^{\overline{U}}(\overline{Z}),\mathrm{cosk}_0^{\mathcal{V}}(\mathcal{Z})\big).$$

Let $\pi\colon(Z_{\bullet},\overline{Z}_{\bullet})\to(U,\overline{U})$ be the natural augmentation. Let $j_{\bullet}\colon Z_{\bullet}\hookrightarrow\overline{Z}_{\bullet}$ be the open immersion. Then, by [**19**, (10.1.4)], the Čech diagram is a universally de Rham descendable hypercovering of $(U,\overline{U})$ over $\mathrm{Spf}(\mathcal{V})$. Hence we have, by definition,

$$R\Gamma_{\mathrm{rig}}(U/K,C)=R\Gamma\big(\,]\overline{Z}_{\bullet}[_{\mathcal{Z}_{\bullet}},\mathrm{DR}(\pi^*(C))\big)\tag{10.0.1}$$

(see [**19**, (10.4)]). Here $\mathrm{DR}(\pi^*(C))$ is the complex $\pi^*(C)_{\mathcal{Z}_\bullet}\otimes_{j^\dagger_\bullet\mathcal{O}_{]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}}} j^\dagger_\bullet\Omega^\bullet_{]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}}$ on $]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}$. For simplicity of notation, we often denote $\mathrm{DR}(\pi^*(C))$ only by $\mathrm{DR}(C)$.

We start with the following lemma:

Lemma 10.1. — *Let T be a closed subscheme of U over κ. Then there exists an integer c such that $H^h_{\mathrm{rig},T}(U/K,C)=0$ for all $h>c$.*

Proof. — (See also [**87**, (6.4.1)].) The proof consists of some steps. We may assume that U is reduced. Let V be the complement of T in U. Then we have the following exact sequence (cf. [**9**, (2.3.1)]):

(10.1.1) $$\cdots\to H^h_{\mathrm{rig},T}(U/K,C)\longrightarrow H^h_{\mathrm{rig}}(U/K,C)\longrightarrow H^h_{\mathrm{rig}}(V/K,C)\to\cdots.$$

Step 1. — Assume that U is affine. Then there exists a closed immersion $U\hookrightarrow\mathbb{A}^n_\kappa$ ($n\in\mathbb{N}$). Let $\overline{U}$ be the closure of U in $\mathbb{P}^n_\kappa\supset\mathbb{A}^n_\kappa$ with reduced subscheme structure. Let $j\colon U\hookrightarrow\overline{U}$ be the open immersion. Let $\mathcal{P}$ be the p-adic completion of $\mathbb{P}^n_{\mathcal{V}}$. Set $U_\lambda:=\,]\overline{U}[_{\mathcal{P}}\setminus](\overline{U}\setminus U)[_{\mathcal{P},\lambda}$ $(0<\lambda<1)$. Let Γ_0 be the set of absolute values of K^*. Set $\Gamma:=\Gamma_0\otimes_{\mathbb{Z}}\mathbb{Q}$. For an element $\eta\in\Gamma$ with $\eta<1$, let $[\overline{U}]_\eta$ be the closed tube of $\overline{U}$ with radius η defined in [**7**, (1.1.8)]. Then

$$\begin{aligned}R\Gamma_{\mathrm{rig}}(U/K,C)&=R\Gamma(\,]\overline{U}[_{\mathcal{P}},C_{\mathcal{P}}\otimes_{j^\dagger\mathcal{O}_{]\overline{U}[_{\mathcal{P}}}}j^\dagger\Omega^\bullet_{]\overline{U}[_{\mathcal{P}}})\\&=R\varprojlim_{\eta\to1^-}R\Gamma([\overline{U}]_{\mathcal{P},\eta},C_{\mathcal{P}}\otimes_{\mathcal{O}_{[\overline{U}]_{\mathcal{P},\eta}}}\Omega^\bullet_{[\overline{U}]_{\mathcal{P},\eta}}).\end{aligned}$$

As $R^i\varprojlim_{\eta\to1^-}R\Gamma([\overline{U}]_{\mathcal{P},\eta},C_{\mathcal{P}}\otimes_{\mathcal{O}_{[\overline{U}]_{\mathcal{P},\eta}}}\Omega^\bullet_{[\overline{U}]_{\mathcal{P},\eta}})=0$ for $i\geq2$, it suffices to prove that $H^h([\overline{U}]_{\mathcal{P},\eta},C_{\mathcal{P}}\otimes_{\mathcal{O}_{[\overline{U}]_{\mathcal{P},\eta}}}\Omega^i_{[\overline{U}]_{\mathcal{P},\eta}})=0$ for $h\gg0$ and $i\in\mathbb{N}$. [**7**, (1.2.2)] tells us that, on $[\overline{U}]_{\mathcal{P},\eta}$, we can replace the inductive system of strict neighborhoods of $]U[_{\mathcal{P}}$ in $]\overline{U}[_{\mathcal{P}}$ by the inductive system of $U_\lambda\cap W$ $(0<\lambda<1)$ for affinoid domains W's contained in $[\overline{U}]_{\mathcal{P},\eta}$. By using Tate's acyclicity theorem [**84**, (8.2), (8.7)], we have $H^h(U_\lambda\cap W,C_{\mathcal{P}}\otimes_{\mathcal{O}_W}\Omega^i_W)=0$ for $h\gg0$ and $i\in\mathbb{N}$. Because the cohomology commutes with direct limits on quasi-compact and quasi-separated rigid spaces over K, we have $H^h_{\mathrm{rig}}(U/K,C)=0$ for $h\gg0$.

Step 2. — Assume that U is an open subscheme of an affine scheme. Then U is separated and we have $H^h_{\mathrm{rig}}(U/K,C)=0$ for $h\gg0$ by [**7**, (2.1.8)] and the Step 1. By the exact sequence (10.1.1), we have $H^h_{\mathrm{rig},T}(U/K,C)=0$ for $h\gg0$.

Step 3. — Assume that T is a closed point x of U. Let U' be an affine open subscheme of U containing x. Then we have $H^h_{\mathrm{rig},T}(U/K,C)=$

$H^h_{\mathrm{rig},T}(U'/K,C)$ (cf. [**9**, (2.4) (i)]). By the Step 2, we have $H^h_{\mathrm{rig},T}(U'/K,C) = 0$ $(h \gg 0)$.

Step 4. — Now consider the general case. We may assume that T is reduced. Let x be the generic point of an irreducible component of T of dimension $\dim T$. Let U' be an affine open subscheme of U containing x. Let T' be the complement of U' in U. Then we have the exact sequence (cf. [**9**, (2.4) (i), (2.5)])

$$\cdots \longrightarrow H^h_{\mathrm{rig},T\cap T'}(U/K,C) \longrightarrow H^h_{\mathrm{rig},T}(U/K,C) \longrightarrow H^h_{\mathrm{rig},T\cap U'}(U'/K,C) \longrightarrow \cdots . \tag{10.1.2}$$

Since U' is affine, $H^h_{\mathrm{rig},T\cap U'}(U'/K,C) = 0$ for $h \gg 0$ by the Step 2. Thus we have only to prove that $H^h_{\mathrm{rig},T\cap T'}(U/K,C) = 0$ for $h \gg 0$. Use the argument above for $T \cap T'$, and continue this process. In the end, we may assume that T is a closed point x of U. Now we can finish the proof of (10.1) by the Step 3. □

Remark 10.2. — Let d be the dimension of U. By the argument in [**9**, (3.3)], it is easy to see that

$$H^h_{\mathrm{rig}}(U/K) = 0 \quad (h > 2d) \tag{10.2.1}$$

for a separated smooth scheme U of finite type κ.

We conjecture that $H^h_{\mathrm{rig}}(U/K) = 0$ for $h > 2d$ for a general separated scheme of finite type over κ of dimension d.

By [**8**, (1.1)] and [**6**, (3.1) (iii)], $H^h_{\mathrm{rig,c}}(U/K) = 0$ for $h > 2d$ for the general scheme above.

In the ℓ-adic case, $H^h_{\text{ét}}(U,\mathbb{Q}_\ell) = 0$ for $h > 2d$ for a scheme U of finite type over a separably closed field of dimension d (see [**36**, X (4.3)]).

Lemma 10.3. — *Let $f\colon Y_2 \longrightarrow Y_1$ be a quasi-projective morphism of quasi-projective schemes over κ. Then there exists a closed immersion $Y_i \hookrightarrow \mathcal{P}_i$ $(i = 1,2)$ into a formally smooth quasi-projective p-adic formal $\mathcal{V}$-scheme fitting* *into the commutative diagram*:

$$
\begin{array}{ccc}
Y_2 & \xrightarrow{\subset} & \mathcal{P}_2 \\
{\scriptstyle f}\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\subset} & \mathcal{P}_1 \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\kappa) & \longrightarrow & \mathrm{Spf}(\mathcal{V}).
\end{array}
\tag{10.3.1}
$$

Proof. — (The proof is due to A. Shiho.) Let P be a smooth quasi-projective scheme over $\mathbb{Z}$ such that there exists a closed immersion $Y_2 \hookrightarrow P_{Y_1} := P \times_{\mathrm{Spec}(\mathbb{Z})} Y_1$. Let $Y_1 \hookrightarrow \mathcal{P}_1$ be a closed immersion into a formally smooth quasi-projective p-adic formal $\mathcal{V}$-scheme. Then we have only to take the following composite morphism $Y_2 \hookrightarrow P_{Y_1} \hookrightarrow \widehat{P} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathcal{P}_1$ with a composite projection $\widehat{P} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathcal{P}_1 \longrightarrow \mathcal{P}_1 \longrightarrow \mathrm{Spf}(\mathcal{V})$. Here $\widehat{P}$ is the p-adic completion of P. □

LEMMA 10.4. — *Let T be a topological space. Let*

$$
\begin{array}{ccccc}
T_1 & \xrightarrow{\subset} & T_2 & \xrightarrow{\subset} & T_3 \\
\| & & \downarrow\cap & & \downarrow\cap \\
T_1 & \xrightarrow{\subset} & T_4 & \xrightarrow{\subset} & T
\end{array}
$$

be a commutative diagram of subspaces of T. Assume that T_1 is dense in T_4, that T_1 is closed in T_3 and that T_3 is open in T. Then $T_1 = T_2$.

Proof. — Assume that $T_1 \neq T_2$. Then there exists a point $t \in T_2 \setminus T_1$. Because T_1 is closed in T_3, there exists an open neighborhood V of t in T_3 such that $V \cap T_1 = \varnothing$. Because T_3 is open in T, V is also open in T. Hence $V \cap T_4$ is open in T_4. Since $(V \cap T_4) \cap T_1 = \varnothing$ and $t \in V \cap T_4$, $t \notin \overline{T}_1 = T_4$. This contradicts that $t \in T_2 \subset T_4$. □

LEMMA 10.5. — *Let N be a nonnegative integer. Let $U_{\bullet \leq N}$ be an N-truncated proper hypercovering of U. Assume that $U_{\bullet \leq N}$ is reduced. Then there exists an N-truncated proper hypercovering $(U_{\bullet \leq N}, \overline{U}_{\bullet \leq N})$ of $(U, \overline{U})$.*

Proof. — First we claim that, for a proper morphism $V \longrightarrow U$, there exists a strict pair $(V, \overline{V})$ over $(U, \overline{U})$ such that the morphism $\overline{V} \longrightarrow \overline{U}$ is proper. Indeed, as in [**9**, p. 340], we have an open immersion $V \hookrightarrow \overline{V}'$ into a proper scheme over κ fitting into the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\subset} & \overline{V}' \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\subset} & \overline{U}. \end{array}$$

Set $V' := \overline{V}' \times_{\overline{U}} U$. Since V and V' are open in $\overline{V}'$, the morphism $V \to V'$ is also an open immersion. Since the morphisms $V \longrightarrow U$ and $V' \to U$ are proper, the morphism $V \to V'$ is proper, in particular, closed. Let W be the complement of the image of V in V'. Set $\overline{V} := \overline{V}' \setminus W$. Since W is an open subscheme of V', W is also an open subscheme of $\overline{V}'$. Then $\overline{V} \times_{\overline{U}} U = V$. Since $\overline{V}$ is a closed subscheme of $\overline{V}'$, the morphism $\overline{V} \to \overline{U}$ is proper. Thus we have proved the claim. (The existence of $\overline{V}$ above is necessary for the papers [**19**], [**86**] and [**87**], which has not been pointed out in [*loc. cit.*].)

Let $(U_N, \overline{U}'_N)$ be a strict pair over $(U, \overline{U})$ such that the morphism $\overline{U}'_N \to \overline{U}$ is proper. Let $\Gamma^{\overline{U}}_N(\overline{U}'_N)_{\bullet \leq N}$ be the N-truncated simplicial scheme of the simplicial scheme $\Gamma^{\overline{U}}_N(\overline{U}'_N) := \Gamma^{\overline{U}}_N(\overline{U}'_N)^{\leq N}$ constructed in [**19**, §11]:

$$\Gamma^{\overline{U}}_N(\overline{U}'_N)_m := \prod_{\mathrm{Hom}_\Delta([N],[m])} \overline{U}'_N$$

with its natural structure as a simplicial object. Here the product is taken over $\overline{U}$. By (6.6), 1) we have a natural immersion $U_{\bullet \leq N} \hookrightarrow \Gamma^{\overline{U}}_N(\overline{U}'_N)_{\bullet \leq N}$. Let $\overline{U}_m$ $(0 \leq m \leq N)$ be the closure of U_m in $\Gamma^{\overline{U}}_N(\overline{U}'_N)_m$ with the reduced closed subscheme structure in $\Gamma^{\overline{U}}_N(\overline{U}'_N)_m$. (To consider the closure has been suggested by the referee.) Then $\overline{U}_{\bullet \leq N}$ is an N-truncated simplicial closed subscheme in $\Gamma^{\overline{U}}_N(\overline{U}'_N)_{\bullet \leq N}$, and we have a natural immersion $U_{\bullet \leq N} \hookrightarrow \overline{U}_{\bullet \leq N}$ over the open immersion $U \hookrightarrow \overline{U}$ by (6.6), 1). We claim that $U_{\bullet \leq N} = \overline{U}_{\bullet \leq N} \times_{\overline{U}} U$. (As a result, the immersion $U_{\bullet \leq N} \hookrightarrow \overline{U}_{\bullet \leq N}$ turns out to be an open immersion.) Indeed, it suffices to prove that $U_m = \overline{U}_m \times_{\overline{U}} U$ $(0 \leq m \leq N)$ as sets because U_m is reduced. Because $\overline{U}'_N \times_{\overline{U}} U = U_N$,

$$\overline{U}_m \times_{\overline{U}} U \subset \Gamma^{\overline{U}}_N(\overline{U}'_N)_m \times_{\overline{U}} U = \Gamma^{U}_N(U_N)_m.$$

Since U_m $(0 \leq m \leq N)$ and U_N are separated over κ, there exists an element $\gamma \in \mathrm{Hom}_\Delta([N],[m])$ such that $U(\gamma)\colon U_m \to U_N$ is a closed immersion by [**37**, (5.5.1) (v), (5.4.6)]. By [loc. cit., (5.4.5)] the image of U_m in $\Gamma^U_N(U_N)_m$ is closed. Using (10.4) for the commutative diagram below

$$\begin{array}{ccccc} U_m & \xrightarrow{\subset} & \overline{U}_m \times_{\overline{U}} U & \xrightarrow{\subset} & \Gamma^U_N(U_N)_m \\ \| & & \downarrow\cap & & \downarrow\cap \\ U_m & \xrightarrow{\subset} & \overline{U}_m & \xrightarrow{\subset} & \Gamma^{\overline{U}}_N(\overline{U}'_N)_m, \end{array}$$

we see that $U_m = \overline{U}_m \times_{\overline{U}} U$ $(0 \leq m \leq N)$ as sets, and, in fact, as schemes. The pair $(U_{\bullet \leq N}, \overline{U}_{\bullet \leq N})$ turns out to be an N-truncated proper hypercovering of $(U, \overline{U})$. □

Remark 10.6. — Let $U_\bullet$ be a proper hypercovering of U. In [**87**, (7.1.3)] we find that one does not know whether there exists a proper hypercovering $(U_\bullet, \overline{U}_\bullet)$ of $(U, \overline{U})$. (10.5) gives the affirmative answer for the N-truncated version of this question in the case where $U_{\bullet \leq N}$ is reduced. (10.5) is useful for simplifying the proof of [**87**, 7.5]. See (10.15), 2) below for details.

Though the following theorem is essentially contained in [**86**], we state the following to clarify our later argument.

Theorem 10.7. — *Let h be a nonnegative integer. Let N be a nonnegative integer satisfying the inequality* (2.2.1). *Let $(U_{\bullet \leq N}, \overline{U}_{\bullet \leq N})$ be an N-truncated proper hypercovering of $(U, \overline{U})$. Let C^t be the pull-back of C by the structural morphism $(U_t, \overline{U}_t) \longrightarrow (U, \overline{U})$ $(0 \leq t \leq N)$. Assume that $\overline{U}_N$ is a closed subscheme of a formally smooth p-adic formal $\mathcal{V}$-scheme $\mathcal{P}_N$. Let $\mathcal{Q}_\bullet$ be the formally smooth simplicial scheme $\Gamma^{\mathcal{V}}_N(\mathcal{P}_N) := \Gamma^{\mathcal{V}}_N(\mathcal{P}_N)^{\leq N}$ constructed in* [**19**, §11]. *Then there exists a canonical isomorphism*

$$H^h_{\rm rig}(U/K, C) \xrightarrow{\sim} H^h\big(R\Gamma(\,]\overline{U}_{\bullet \leq N}[_{\mathcal{Q}_{\bullet \leq N}}, {\rm DR}(C^{\bullet \leq N}))\big). \tag{10.7.1}$$

Proof. — (Cf. the proof of [**86**, (4.3.1) (1)]) By [**87**, (5.1.2)], the simplicial scheme

$$(U'_\bullet, \overline{U}'_\bullet) := \big({\rm cosk}^U_N(U_{\bullet \leq N}), {\rm cosk}^{\overline{U}}_N(\overline{U}_{\bullet \leq N})\big)$$

is a proper hypercovering of $(U, \overline{U})$. Let $C'^\bullet$ be the pull-back of C by the augmentation $(U'_\bullet, \overline{U}'_\bullet) \to (U, \overline{U})$. Then $C'^{\bullet \leq N} = C^{\bullet \leq N}$ since $(U'_{\bullet \leq N}, \overline{U}'_{\bullet \leq N}) = (U_{\bullet \leq N}, \overline{U}_{\bullet \leq N})$. By [**19**, (11.2.4), (11.2.5), (11.2.6)], $(U'_\bullet, \overline{U}'_\bullet, \mathcal{Q}_\bullet)$ is a proper hypercovering of $(U, \overline{U})$ over ${\rm Spf}(\mathcal{V})$ in the sense of [**86**, (2.2.1)]. Hence, by [**86**, (2.2.3)], we have an isomorphism

$$R\Gamma_{\rm rig}(U/K, C) \xrightarrow{\sim} R\Gamma\big(\,]\overline{U}'_\bullet[_{\mathcal{Q}_\bullet}, {\rm DR}(C'^\bullet)\big). \tag{10.7.2}$$

Furthermore, we obtain

$$H^h\big(R\Gamma(\,]\overline{U}'_\bullet[_{\mathcal{Q}_\bullet}, \mathrm{DR}(C'^\bullet))\big) = H^h\big(R\Gamma(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, \mathrm{DR}(C^{\bullet\leq N}))\big) \tag{10.7.3}$$

by using the following spectral sequence

$$E_1^{ij} = H^j_{\mathrm{rig}}(U'_i/K, C'^i) \Longrightarrow H^{i+j}\big(R\Gamma(\,]\overline{U}'_\bullet[_{\mathcal{Q}_\bullet}, \mathrm{DR}(C'^\bullet))\big) \tag{10.7.4}$$

as in the proof of (2.2). Thus we obtain (10.7). □

The following lemma 2) is similar to [**87**, (7.2.2) (3)]:

Lemma 10.8. — 1) *Let $(U', \overline{U}') \to (U, \overline{U})$ be a morphism of open immersions from separated schemes of finite type over κ into proper schemes over κ. Let $(U_\bullet, \overline{U}_\bullet)$ be a proper hypercovering of $(U, \overline{U})$. Then $(U_\bullet \times_U U', \overline{U}_\bullet \times_{\overline{U}} \overline{U}')$ is a proper hypercovering of $(U', \overline{U}')$.*

2) *Let*

$$\begin{array}{ccc} & & (U^1, \overline{U}^1) \\ & & \big\downarrow \\ (U^2, \overline{U}^2) & \longrightarrow & (U, \overline{U}) \end{array}$$

be a diagram of open immersions from separated schemes of finite type over κ into proper schemes over κ. Let $(U_\bullet, \overline{U}_\bullet)$ be a proper hypercovering of $(U, \overline{U})$. Set

$$(U^i_\bullet, \overline{U}^i_\bullet) := (U_\bullet \times_U U^i, \overline{U}_\bullet \times_{\overline{U}} \overline{U}^i) \quad (i = 1, 2).$$

Let $(V^i_\bullet, \overline{V}^i_\bullet)$ be a proper hypercovering of $(U^i, \overline{U}^i)$ such that the composite morphism $(V^i_\bullet, \overline{V}^i_\bullet) \to (U^i, \overline{U}^i) \to (U, \overline{U})$ factors through the morphism $(U_\bullet, \overline{U}_\bullet) \to (U, \overline{U})$. Assume that one of the morphisms $(V^i_\bullet, \overline{V}^i_\bullet) \to (U^i_\bullet, \overline{U}^i_\bullet)$ for $i = 1, 2$ is a refinement (see [**86**, (4.2.1)]) *of the proper hypercovering $(U^i_\bullet, \overline{U}^i_\bullet)$ of $(U^i, \overline{U}^i)$. Set $(U^3, \overline{U}^3) := (U^1 \times_U U^2, \overline{U}^1 \times_{\overline{U}} \overline{U}^2)$ and $(V^3_\bullet, \overline{V}^3_\bullet) := (V^1_\bullet \times_{U_\bullet} V^2_\bullet, \overline{V}^1_\bullet \times_{\overline{U}_\bullet} \overline{V}^2_\bullet)$. Then $(V^3_\bullet, \overline{V}^3_\bullet)$ is a proper hypercovering of $(U^3, \overline{U}^3)$.*

3) *Let N be a nonnegative integer. Then the N-truncated versions of* 1) *and* 2) *hold.*

Proof. — 1): We leave the proof to the reader since it is easy.

2): By the symmetry, we may assume that $(V^1_\bullet, \overline{V}^1_\bullet)$ is a refinement of the proper hypercovering $(U^1_\bullet, \overline{U}^1_\bullet)$ of $(U^1, \overline{U}^1)$. Since the morphism $\overline{V}^3_m \to \overline{U}^3$ ($m \in \mathbb{N}$) is the composite morphism

$$\overline{V}^1_m \times_{\overline{U}_m} \overline{V}^2_m \longrightarrow \overline{V}^1_m \times_{\overline{U}} \overline{V}^2_m \longrightarrow \overline{U}^1 \times_{\overline{U}} \overline{U}^2,$$

it is proper by [**37**, (5.4.2)] and [**38**, (5.4.2) (iv)].

The strictness of the morphism $(V^3_m, \overline{V}^3_m) \longrightarrow (U^3, \overline{U}^3)$ follows from

$$\begin{aligned}\overline{V}^3_m \times_{\overline{U}^3} U^3 &= (\overline{V}^1_m \times_{\overline{U}^1} \overline{U}^1) \times_{\overline{U}^1 \times_{\overline{U}} \overline{U}^2} \times (U^1 \times_U U^2) \times_{\overline{U}_m} (\overline{V}^2_m \times_{\overline{U}^2} \overline{U}^2) \\ &= (\overline{V}^1_m \times_{\overline{U}^1} \overline{U}^1) \times_{\overline{U}^1 \times_{\overline{U}} \overline{U}^2} \times (U^1 \times_{\overline{U}} U^2) \times_{\overline{U}_m} (\overline{V}^2_m \times_{\overline{U}^2} \overline{U}^2) \\ &= (\overline{V}^1_m \times_{\overline{U}^1} U^1) \times_{\overline{U}^1 \times_{\overline{U}} \overline{U}^2} \times (\overline{U}^1 \times_{\overline{U}} \overline{U}^2) \times_{\overline{U}_m} (\overline{V}^2_m \times_{\overline{U}^2} U^2) \\ &= V^1_m \times_{\overline{U}_m} V^2_m = V^1_m \times_{U_m} V^2_m = V^3_m.\end{aligned}$$

Now, for any integer $m \geq -1$, let us show that the morphism

$$\begin{aligned}(10.8.1) \quad V^3_{m+1} \longrightarrow &\operatorname{cosk}^{U^3}_m (V^3_{\bullet \leq m})_{m+1} \\ &= \operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^U_m (U_{\bullet \leq m})_{m+1}} \operatorname{cosk}^{U^2}_m (V^2_{\bullet \leq m})_{m+1}\end{aligned}$$

is proper and surjective. The morphism (10.8.1) factors through a proper surjective morphism

$$\begin{aligned}(10.8.2) \qquad &\operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^U_m (U_{\bullet \leq m})_{m+1}} V^2_{m+1} \\ &\longrightarrow \operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^U_m (U_{\bullet \leq m})_{m+1}} \operatorname{cosk}^{U^2}_m (V^2_{\bullet \leq m})_{m+1}.\end{aligned}$$

By the definition of the refinement in [**86**, (4.2.1)], the natural morphism

$$\begin{aligned}(10.8.3) \qquad & V^1_{m+1} \times_{U_{m+1}} V^2_{m+1} \\ \longrightarrow & \{\operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^{U^1}_m (U_{\bullet \leq m} \times_U U^1)_{m+1}} U_{m+1} \times_U U^1\} \times_{U_{m+1}} V^2_{m+1} \\ &= \operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^{U^1}_m (U_{\bullet \leq m} \times_U U^1)_{m+1}} (V^2_{m+1} \times_U U^1) \\ &= \operatorname{cosk}^{U^1}_m (V^1_{\bullet \leq m})_{m+1} \times_{\operatorname{cosk}^U_m (U_{\bullet \leq m})_{m+1}} V^2_{m+1}\end{aligned}$$

is proper and surjective. Since the morphism (10.8.1) is the composite morphism of (10.8.3) and (10.8.2), the morphism (10.8.1) is proper and surjective. Thus $(V^3_\bullet, \overline{V}^3_\bullet)$ is a proper hypercovering of $(U^3, \overline{U}^3)$.

3): The proof is similar to that of 1) and 2). □

The following is the main result in this section.

THEOREM 10.9. — *Let $U_\bullet$ be a proper hypercovering of U. Let C^t be the pull-back of C by the structural morphism $U_t \longrightarrow U$ ($t \in \mathbb{N}$). Let c be an integer in* (10.1) *for the case $T = \varnothing$. Let N be a positive integer satisfying the inequality* (2.2.1) *for $h = c$. Let $\mathrm{D}^+(K^{\bullet \leq N})$ be the derived category of N-truncated cosimplicial complexes of K-vector spaces* (*cf.* §2)*. Let* $\mathbf{s}$ (*resp.* e_t^{-1} $(0 \leq t \leq N)$) *be the N-truncated cosimplicial version of the functor* (*resp. the cosimplicial version of the functor*) *defined in* (2.2.16) (*resp.* (2.2.18)) *for* $\mathrm{D}^+(K^{\bullet\leq N})$ *with trivial filtration* ((2.3)) :

$$\mathrm{D}^+(K) \overset{\mathbf{s}}{\longleftarrow} \mathrm{D}^+(K^{\bullet\leq N}) \overset{e_t^{-1}}{\longrightarrow} \mathrm{D}^+(K).$$

Then there exists a complex $C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K, C)$ *in* $\mathrm{D}^+(K^{\bullet\leq N})$ *such that there exist canonical isomorphisms*

$$R\Gamma_{\mathrm{rig}}(U/K, C) \overset{\sim}{\longrightarrow} \tau_c \mathbf{s}\big(C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K, C)\big), \tag{10.9.1}$$

$$R\Gamma_{\mathrm{rig}}(U_t/K, C^t) \overset{\sim}{\longrightarrow} e_t^{-1}\big(C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K, C)\big) \quad (0 \leq \forall t \leq N). \tag{10.9.2}$$

The complex $C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K, C)$ *is functorial for a morphism of augmented simplicial schemes* $U_{\bullet\leq N} \to U$*'s and a morphism of overconvergent isocrystal* C*'s. For* $N \leq N'$*, there exists a canonical isomorphism*

$$\mathbf{s}\big((C_{\mathrm{rig}}^{\bullet\leq N'}((U_{\bullet\leq N'}/U)/K, C))^{\bullet\leq N}\big) \overset{\sim}{\longrightarrow} \mathbf{s}\big(C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K, C)\big). \tag{10.9.3}$$

Here $\bullet \leq N$ *on the left hand side of* (10.9.3) *means the* N*-truncation of the left cosimplicial degree.*

Proof. — Let the notations be before (10.1). Let N be a positive integer. Later we assume that N is a positive integer satisfying the inequality (2.2.1) for $h = c$ in (10.1) for the case $T = \varnothing$.

Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ (see [**67**]). Because we can replace $U_{\bullet\leq N}$ with $(U_{\bullet\mathrm{red}})_{\bullet\leq N}$ in the following argument, we may assume that $U_{\bullet\leq N}$ is reduced. By (10.5) there exists an N-truncated proper hypercovering $(U_{\bullet\leq N}, \overline{U}_{\bullet\leq N})$ of $(U, \overline{U})$. By (10.8), 3) the pair $(U_{\bullet\leq N} \times_U Z, \overline{U}_{\bullet\leq N} \times_{\overline{U}} \overline{Z})$ is an N-truncated proper hypercovering of $(Z, \overline{Z})$. Let $(V_{\bullet\leq N}, \overline{V}_{\bullet\leq N})$ be a refinement of the proper hypercovering $(U_{\bullet\leq N} \times_U Z, \overline{U}_{\bullet\leq N} \times_{\overline{U}} \overline{Z})$ of $(Z, \overline{Z})$ such that there exists a closed immersion $\overline{V}_N \hookrightarrow \mathcal{P}_N$ into a separated formally smooth p-adic formal $\mathcal{V}$-scheme (see [**86**, (4.2.3)]). Consider the N-truncated triple

$$\big(V_{\bullet\leq N}, \overline{V}_{\bullet\leq N}, \Gamma_N^{\mathcal{V}}(\mathcal{P}_N)_{\bullet\leq N}\big). \tag{10.9.4}$$

The simplicial formal $\mathcal{V}$-scheme $\Gamma_N^{\mathcal{V}}(\mathcal{P}_N)_{\bullet\leq N}$ contains $\overline{V}_{\bullet\leq N}$ as an N-truncated simplicial closed subscheme over $\mathcal{V}$ by (6.6), 1). Set

$$\mathcal{Q}_\bullet := \Gamma_N^{\mathcal{V}}(\mathcal{P}_N).$$

Following the idea in [**86**, (4.4)], we consider the triple $(V_{\bullet\leq N,\bullet}, \overline{V}_{\bullet\leq N,\bullet}, \mathcal{Q}_{\bullet\leq N,\bullet})$ of (N,∞)-truncated bisimplicial (formal) schemes defined by

$$
\begin{aligned}
&(10.9.5)\qquad (V_{mn}, \overline{V}_{mn}, \mathcal{Q}_{mn})\\
&:= \big(\mathrm{cosk}_0^{U_m}(V_m)_n, \mathrm{cosk}_0^{\overline{U}_m}(\overline{V}_m)_n, \mathrm{cosk}_0^{\mathcal{V}}(\mathcal{Q}_m\widehat{\times}_{\mathcal{V}}\mathcal{Z})_n\big) \quad (0\leq m\leq N, n\in\mathbb{N})
\end{aligned}
$$

with the natural morphisms which make $(V_{\bullet\leq N,\bullet}, \overline{V}_{\bullet\leq N,\bullet}, \mathcal{Q}_{\bullet\leq N,\bullet})$ a triple of (N,∞)-truncated bisimplicial schemes. Here, note that the triple (10.9.5) is different from the triple in [**86**, p. 125]; see (10.10) , 1) below for this. Since $\overline{V}_m \to \overline{Z}$ is proper, the morphism $\overline{V}_m \to \overline{V}_m \times_\kappa \overline{Z}$ of the graph of the morphism above is a closed immersion; hence we indeed have the triple (10.9.5) (cf. [**86**, p. 125]).

We have a natural morphism $(V_{\bullet\leq N,\bullet}, \overline{V}_{\bullet\leq N,\bullet}, \mathcal{Q}_{\bullet\leq N,\bullet}) \to (Z_\bullet, \overline{Z}_\bullet, \mathcal{Z}_\bullet)$ of triples and have the following commutative diagram of pairs of (N-truncated) (bi)simplicial schemes:

$$
(10.9.6)\qquad
\begin{array}{ccc}
(U_{\bullet\leq N}, \overline{U}_{\bullet\leq N}) & \longleftarrow & (V_{\bullet\leq N,\bullet}, \overline{V}_{\bullet\leq N,\bullet})\\
\big\downarrow & & \big\downarrow\\
(U, \overline{U}) & \longleftarrow & (Z_\bullet, \overline{Z}_\bullet).
\end{array}
$$

By the proof of [**86**, (4.4.1) (1)], the morphism

$$(V_{\bullet\leq N,n}, \overline{V}_{\bullet\leq N,n}) \longrightarrow (Z_n, \overline{Z}_n) \quad (n\in\mathbb{N})$$

is an N-truncated proper hypercovering of $(Z_n, \overline{Z}_n)$. In fact, the morphism

$$(V_{\bullet\leq N,n}, \overline{V}_{\bullet\leq N,n}, \mathcal{Q}_{\bullet\leq N,n}) \longrightarrow (Z_n, \overline{Z}_n, \mathcal{Z}_n) \quad (n\in\mathbb{N})$$

is an N-truncated proper hypercovering of $(Z_n, \overline{Z}_n, \mathcal{Z}_n)$ in the sense of [**86**, (2.2.1)]. Indeed, since

$$\mathrm{cosk}_\ell^{\mathcal{Z}_n}(\mathcal{Q}_{\bullet\leq\ell,n}) = \mathrm{cosk}_\ell^{\mathcal{V}}(\underbrace{\mathcal{Q}_{\bullet\leq\ell}\widehat{\times}_{\mathcal{V}}\cdots\widehat{\times}_{\mathcal{V}}\mathcal{Q}_{\bullet\leq\ell}}_{n\ \text{times}})\widehat{\times}_{\mathcal{V}}\mathcal{Z}_n \quad (0\leq\ell\leq N),$$

the morphism $\mathrm{cosk}_\ell^{\mathcal{Z}_n}(\mathcal{Q}_{\bullet\leq\ell,n})_m \to \mathrm{cosk}_{\ell-1}^{\mathcal{Z}_n}(\mathcal{Q}_{\bullet\leq\ell-1,n})_m$ $(m\in\mathbb{N})$ is formally smooth by [**19**, (11.2.6)].

Let h be a fixed arbitrary nonnegative integer. Let N be an integer satisfying the inequality (2.2.1). Let $C^{\bullet\leq N,\bullet}$ be the pull-back of C by the morphism $V_{\bullet\leq N,\bullet} \to U$. For simplicity of notation, we denote $\mathrm{DR}(C^{\bullet\leq N,\bullet})$ on $]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}}$ by $\mathrm{DR}(C)$. Then the morphism

$$(10.9.7)\qquad R\Gamma\big(]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}, \mathrm{DR}(C)\big) \longrightarrow R\Gamma\big(]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}}, \mathrm{DR}(C)\big)$$

induces an isomorphism

$$\begin{aligned}(10.9.8)\qquad H^h_{\mathrm{rig}}(U/K,C) &= H^h\big(R\Gamma(\,]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet},\mathrm{DR}(C))\big)\\ &\overset{\sim}{\longrightarrow} H^h\big(R\Gamma(\,]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}},\mathrm{DR}(C))\big).\end{aligned}$$

Indeed, we have the two spectral sequences:

$$(10.9.9)\qquad E_1^{ij} = H^j\big(\,]\overline{Z}_i[_{\mathcal{Z}_i},\mathrm{DR}(C)\big) \Longrightarrow H^{i+j}\big(\,]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet},\mathrm{DR}(C)\big),$$

$$\begin{aligned}(10.9.10)\qquad E_1^{ij} &= H^j\big(\,]\overline{V}_{\bullet\leq N,i}[_{\mathcal{Q}_{\bullet\leq N,i}},\mathrm{DR}(C)\big)\\ &\Longrightarrow H^{i+j}\big(\,]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}},\mathrm{DR}(C)\big).\end{aligned}$$

By [**86**, (2.1.3)] and by the proof of (10.7), the E_r-terms ($1\leq r\leq\infty$) of (10.9.9) are isomorphic to those of (10.9.10) for $i+j\leq h$. Hence we have an isomorphism (10.9.8).

So far we have proved that, for any integer h, there exists a sufficiently large integer N depending on h such that there exists an isomorphism

$$(10.9.11)\qquad \tau_h R\Gamma_{\mathrm{rig}}(U/K,C) \overset{\sim}{\longrightarrow} \tau_h R\Gamma\big(\,]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}},\mathrm{DR}(C)\big).$$

Let N be any integer satisfying the inequality (2.2.1) for $h=c$ in (10.1). Set

$$(10.9.12)\qquad C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K,C\big) := R\Gamma^{\bullet\leq N}\big(\,]\overline{V}_{\bullet\leq N,\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet}},\mathrm{DR}(C)\big).$$

Here the N-truncated cosimplicial degree in $R\Gamma^{\bullet\leq N}$ means the left N-truncated cosimplicial degree. Then, by (10.9.11), we have

$$(10.9.13)\qquad R\Gamma_{\mathrm{rig}}(U/K,C) \overset{\sim}{\longrightarrow} \tau_c\mathbf{s}\big(C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K,C)\big).$$

Moreover, for a nonnegative integer $t\leq N$, $(V_{t\bullet},\overline{V}_{t\bullet},\mathcal{Q}_{t\bullet})$ is a universally de Rham descendable covering of $(U_t,\overline{U}_t)$ by the proof of [**86**, (4.4.1) (2)] (however, see (10.10), 1) below). Hence

$$(10.9.14)\qquad R\Gamma_{\mathrm{rig}}(U_t/K,C^t) = R\Gamma\big(\,]\overline{V}_{t\bullet}[_{\mathcal{Q}_{t\bullet}},\mathrm{DR}(C^t)\big).$$

In other words, we have

$$(10.9.15)\quad R\Gamma_{\mathrm{rig}}(U_t/K,C) = e_t^{-1}\big(C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K,C)\big)\quad (0\leq\forall t\leq N).$$

Next we prove that $C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K,C)$ depends only on N, U, $U_{\bullet\leq N}$ and C.

(a) *Independence of the choice of the disjoint union of the open covering of $\overline{U}$ with a closed immersion into a separated formally smooth p-adic formal $\mathcal{V}$-scheme.*

Let $\overline{Z}^i$ $(i=1,2)$ be the disjoint union of affine open subschemes which cover $\overline{U}$. Let $(Z^i_\bullet, \overline{Z}^i_\bullet, \mathcal{Z}^i_\bullet)$ be the triple constructed as in the beginning of this section for $\overline{Z}^i$. Set

$$(Z^3, \overline{Z}^3, \mathcal{Z}^3) := (Z^1 \times_U Z^2, \overline{Z}^1 \times_{\overline{U}} \overline{Z}^2, \mathcal{Z}^1 \widehat{\times}_{\mathcal{V}} \mathcal{Z}^2),$$
$$(Z^3_\bullet, \overline{Z}^3_\bullet, \mathcal{Z}^3_\bullet) := (\mathrm{cosk}_0^U(Z^3), \mathrm{cosk}_0^{\overline{U}}(\overline{Z}^3), \mathrm{cosk}_0^{\mathcal{V}}(\mathcal{Z}^3)).$$

Since $\overline{U}/\kappa$ is separated, the immersion $\overline{Z}^3 \to \overline{Z}^1 \times_\kappa \overline{Z}^2$ is closed by [**37**, (5.4.2)]. Hence $\overline{Z}^3$ is affine.

Let $(V^i_{\bullet\leq N}, \overline{V}^i_{\bullet\leq N})$ be a refinement of $(U_{\bullet\leq N} \times_U Z^i, \overline{U}_{\bullet\leq N} \times_{\overline{U}} \overline{Z}^i)$ $(i=1,2)$ such that there exists a closed immersion $\overline{V}^i_N \hookrightarrow \mathcal{P}^i_N$ into a separated formally smooth p-adic formal $\mathcal{V}$-scheme. Set

$$(V^3_{\bullet\leq N}, \overline{V}^3_{\bullet\leq N}) := (V^1_{\bullet\leq N} \times_{U_{\bullet\leq N}} V^2_{\bullet\leq N}, \overline{V}^1_{\bullet\leq N} \times_{\overline{U}_{\bullet\leq N}} \overline{V}^2_{\bullet\leq N}),$$
$$\mathcal{P}^3_N := \mathcal{P}^1_N \widehat{\times}_{\mathcal{V}} \mathcal{P}^2_N.$$

Then we have a natural closed immersion $\overline{V}^3_N \hookrightarrow \mathcal{P}^3_N$ again by [**37**, (5.4.2)]. By (10.8), 3), $(V^3_{\bullet\leq N}, \overline{V}^3_{\bullet\leq N})$ is an N-truncated proper hypercovering of $(Z^3, \overline{Z}^3)$.

Let $(V^i_{\bullet\leq N,\bullet}, \overline{V}^i_{\bullet\leq N,\bullet}, \mathcal{Q}^i_{\bullet\leq N,\bullet})$ $(i=1,2,3)$ be the (N,∞)-truncated bisimplicial (formal) scheme constructed in (10.9.5) by the use of $\mathcal{P}^i_N$. Then $\mathcal{Q}^3_{\bullet\leq N,\bullet}$ is equal to the product $\mathcal{Q}^1_{\bullet\leq N,\bullet} \widehat{\times}_{\mathcal{V}} \mathcal{Q}^2_{\bullet\leq N,\bullet}$. We have the natural morphism

$$p_i \colon (V^3_{\bullet\leq N,\bullet}, \overline{V}^3_{\bullet\leq N,\bullet}, \mathcal{Q}^3_{\bullet\leq N,\bullet}) \longrightarrow (V^i_{\bullet\leq N,\bullet}, \overline{V}^i_{\bullet\leq N,\bullet}, \mathcal{Q}^i_{\bullet\leq N,\bullet}) \quad (i=1,2).$$

Let $C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K, C)_i$ be the N-truncated cosimplicial complex in (10.9.12) by using $(V^i_{\bullet\leq N,\bullet}, \overline{V}^i_{\bullet\leq N,\bullet}, \mathcal{Q}^i_{\bullet\leq N,\bullet})$ $(i=1,2,3)$. Then we have two morphisms:

(10.9.16)
$$C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big)_1 \xrightarrow{p_1^*} C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big)_3 \xleftarrow{p_2^*} C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big)_2.$$

By the proof of [**86**, (4.4.1) (2)], the product $(V^3_{t\bullet}, \overline{V}^3_{t\bullet}, \mathcal{Q}^3_{t\bullet})$ is a universally de Rham-descendable covering of $(U_t, \overline{U}_t)$. Hence

$$e_t^{-1} C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big)_3 = R\Gamma_{\mathrm{rig}}(U_t/K, C^t) \quad (0 \leq t \leq N).$$

By (10.9.15), p_i^* $(i=1,2)$ is an isomorphism. Consequently

$$C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big)$$

is independent of the choice of the disjoint union of the open covering of $\overline{U}$ with a closed immersion into a separated formally smooth p-adic formal $\mathcal{V}$-scheme; we have proved (a).

(b) *Independence of the choices of the refinement and the closed immersion of it into a separated formally smooth p-adic formal $\mathcal{V}$-scheme.*

Let $(V^i_{\bullet\leq N}, \overline{V}^i_{\bullet\leq N})$ $(i=1,2)$ be a refinement of $(U_{\bullet\leq N}\times_U Z, \overline{U}_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$. Then, by [**86**, (4.2.2) (2)], the fiber product $(V^3_{\bullet\leq N}, \overline{V}^3_{\bullet\leq N})$ of $(V^1_{\bullet\leq N}, \overline{V}^1_{\bullet\leq N})$ and $(V^2_{\bullet\leq N}, \overline{V}^2_{\bullet\leq N})$ over $(U_{\bullet\leq N}\times_U Z, \overline{U}_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$ is a refinement of $(U_{\bullet\leq N}\times_U Z, \overline{U}_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$ with projection $(V^3_{\bullet\leq N}, \overline{V}^3_{\bullet\leq N})\to(V^i_{\bullet\leq N}, \overline{V}^i_{\bullet\leq N})$. Let $\overline{V}^i_N\hookrightarrow\mathcal{P}^i_N$ $(i=1,2)$ be a closed immersion into a separated formally smooth p-adic formal $\mathcal{V}$-scheme. Set $\mathcal{P}^3_N:=\mathcal{P}^1_N\widehat{\times}_{\mathcal{V}}\mathcal{P}^2_N$. Let $C^{\bullet\leq N}_{\rm rig}((U_{\bullet\leq N}/U)/K, C)_i$ be the object constructed in (10.9.12) by using $(V^i_{\bullet\leq N,\bullet}, \overline{V}^i_{\bullet\leq N,\bullet}, \mathcal{Q}^i_{\bullet\leq N,\bullet})$ $(i=1,2,3)$. Then we have two morphisms:

$$C^{\bullet\leq N}_{\rm rig}\big((U_{\bullet\leq N}/U)/K, C\big)_1 \xrightarrow{p_1^*} C^{\bullet\leq N}_{\rm rig}\big((U_{\bullet\leq N}/U)/K, C\big)_3 \xleftarrow{p_2^*} C^{\bullet\leq N}_{\rm rig}\big((U_{\bullet\leq N}/U)/K, C\big)_2. \tag{10.9.17}$$

By (10.9.15), p_i^* $(i=1,2)$ is an isomorphism. We have proved (b).

(c) *Independence of the choice of $\overline{U}_{\bullet\leq N}$.*

Let $(U_{\bullet\leq N}, \overline{U}^i_{\bullet\leq N})$ $(i=1,2)$ be an N-truncated proper hypercovering of $(U,\overline{U})$. Let $\overline{U}^3_m$ $(0\leq m\leq N)$ be the closure of ${\rm Im}(U_m\xrightarrow{\rm diag}\overline{U}^1_m\times_{\overline{U}}\overline{U}^2_m)$ in $\overline{U}^1_m\times_{\overline{U}}\overline{U}^2_m$. Here we endow $\overline{U}^3_m$ with the reduced closed subscheme structure in $\overline{U}^1_m\times_{\overline{U}}\overline{U}^2_m$. Since U_m/U is separated, the diagonal image of U_m in $U_m\times_U U_m$ is closed. We claim that the pair $(U_{\bullet\leq N}, \overline{U}^3_{\bullet\leq N})$ is strict over $(U,\overline{U})$. Indeed, it suffices to prove that $U_m=\overline{U}^3_m\times_{\overline{U}}U$ as sets because U_m is reduced. This follows from (10.4) for the following commutative diagram

$$\begin{array}{ccccc}
U_m & \xrightarrow{\subset} & \overline{U}^3_m\times_{\overline{U}}U & \xrightarrow{\subset} & U_m\times_U U_m \\
\| & & \downarrow\cap & & \downarrow\cap \\
U_m & \xrightarrow{\subset} & \overline{U}^3_m & \xrightarrow{\subset} & \overline{U}^1_m\times_{\overline{U}}\overline{U}^2_m.
\end{array}$$

We also have a natural morphism $(U_{\bullet\leq N}, \overline{U}^3_{\bullet\leq N})\longrightarrow(U_{\bullet\leq N}, \overline{U}^i_{\bullet\leq N})$ $(i=1,2)$.

As in [**86**, (4.2.3)], there exist two N-truncated proper hypercoverings $(V^1_{\bullet\leq N}, \overline{V}^1_{\bullet\leq N})$ and $(V^3_{\bullet\leq N}, \overline{V}^3_{\bullet\leq N})$ of $(Z,\overline{Z})$ and quasi-projective refinements of $(U_{\bullet\leq N}\times_U Z,\overline{U}{}^1_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$ and $(U_{\bullet\leq N}\times_U Z,\overline{U}{}^3_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$, respectively, which fit into the commutative diagram:

$$
\begin{array}{ccc}
(V^3_{\bullet\leq N},\overline{V}{}^3_{\bullet\leq N}) & \longrightarrow & (V^1_{\bullet\leq N},\overline{V}{}^1_{\bullet\leq N}) \\
\big\downarrow & & \big\downarrow \\
(U_{\bullet\leq N}\times_U Z,\overline{U}{}^3_{\bullet\leq N}\times_{\overline{U}}\overline{Z}) & \longrightarrow & (U_{\bullet\leq N}\times_U Z,\overline{U}{}^1_{\bullet\leq N}\times_{\overline{U}}\overline{Z}).
\end{array}
\tag{10.9.18}
$$

We also have an analogous commutative diagram

$$
\begin{array}{ccc}
(V^4_{\bullet\leq N},\overline{V}{}^4_{\bullet\leq N}) & \longrightarrow & (V^2_{\bullet\leq N},\overline{V}{}^2_{\bullet\leq N}) \\
\big\downarrow & & \big\downarrow \\
(U_{\bullet\leq N}\times_U Z,\overline{U}{}^3_{\bullet\leq N}\times_{\overline{U}}\overline{Z}) & \longrightarrow & (U_{\bullet\leq N}\times_U Z,\overline{U}{}^2_{\bullet\leq N}\times_{\overline{U}}\overline{Z}).
\end{array}
\tag{10.9.19}
$$

By [**86**, (4.2.2) (2)] there exists a quasi-projective refinement $(V^5_{\bullet\leq N},\overline{V}{}^5_{\bullet\leq N})$ of $(V^3_{\bullet\leq N},\overline{V}{}^3_{\bullet\leq N})$ and $(V^4_{\bullet\leq N},\overline{V}{}^4_{\bullet\leq N})$ over $(Z,\overline{Z})$. By (10.3) there exists a commutative diagram

$$
\begin{array}{ccc}
\overline{V}{}^3_N & \overset{\subset}{\longrightarrow} & \mathcal{P}^3_N \\
\big\downarrow & & \big\downarrow \\
\overline{V}{}^1_N & \overset{\subset}{\longrightarrow} & \mathcal{P}^1_N \\
\big\downarrow & & \big\downarrow \\
\mathrm{Spec}(\kappa) & \longrightarrow & \mathrm{Spf}(\mathcal{V}),
\end{array}
\tag{10.9.20}
$$

where the horizontal morphisms are closed immersions into formally smooth quasi-projective schemes over $\mathcal{V}$. Then, as in (a), we see that two $C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K,C)$'s constructed by the use of $(\overline{V}{}^1_N,\mathcal{P}^1_N)$ and $(\overline{V}{}^3_N,\mathcal{P}^3_N)$ are naturally isomorphic. There also exists a natural isomorphism between two $C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K,C)$'s for $\overline{V}{}^4_N$ and $\overline{V}{}^2_N$. Since $(V^3_{\bullet\leq N},\overline{V}{}^3_{\bullet\leq N})$ and $(V^4_{\bullet\leq N},\overline{V}{}^4_{\bullet\leq N})$ are refinements of the same proper hypercovering $(U_{\bullet\leq N}\times_U Z,\overline{U}{}^3_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$, we see, by (b) and by using $(V^5_{\bullet\leq N},\overline{V}{}^5_{\bullet\leq N})$, that two $C_{\mathrm{rig}}^{\bullet\leq N}((U_{\bullet\leq N}/U)/K,C)$'s for $\overline{V}{}^3_N$ and $\overline{V}{}^4_N$ are naturally isomorphic. Therefore we have proved (c).

(d) *Independence of the open immersion of U into a proper scheme over κ.*

If we are given two open immersions $U\hookrightarrow\overline{U}{}^i$ $(i=1,2)$ into proper schemes over κ, then we may have a morphism $(\mathrm{id},f)\colon(U,\overline{U}{}^2)\to(U,\overline{U}{}^1)$. Let $(U_{\bullet\leq N},\overline{U}{}^1_{\bullet\leq N})$ be an N-truncated proper hypercovering of $(U,\overline{U}{}^1)$. Set

$$\overline{U}{}^2_{\bullet\leq N}:=\overline{U}{}^1_{\bullet\leq N}\times_{\overline{U}{}^1}\overline{U}{}^2.$$

Then $(U_{\bullet\leq N}, \overline{U}^2_{\bullet\leq N})$ is a proper hypercovering of $(U, \overline{U}^2)$. Obviously we have the following commutative diagram

$$\begin{array}{ccc} (U_{\bullet\leq N}, \overline{U}^2_{\bullet\leq N}) & \longrightarrow & (U_{\bullet\leq N}, \overline{U}^1_{\bullet\leq N}) \\ \downarrow & & \downarrow \\ (U, \overline{U}^2) & \longrightarrow & (U, \overline{U}^1). \end{array} \tag{10.9.21}$$

By (10.9.15), two $C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K, C)$'s by the use of $(U_{\bullet\leq N}, \overline{U}^1_{\bullet\leq N})$ and $(U_{\bullet\leq N}, \overline{U}^2_{\bullet\leq N})$ are naturally isomorphic. Hence we have proved (d).

Putting (a), (b), (c) and (d) together, we have proved that

$$C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K, C)$$

depends only on N, U, $U_{\bullet\leq N}$ and C.

The functoriality for a morphism of overconvergent isocrystals C's is clear by the construction of $C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K, C)$.

We prove that the complex $C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K, C)$ is functorial for a morphism of augmented simplicial schemes $U_{\bullet\leq N} \to U$'s as follows.

Assume that we are given a commutative diagram

$$\begin{array}{ccc} U'_\bullet & \longrightarrow & U_\bullet \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

of proper hypercoverings of separated schemes of finite type over κ. We may assume that $U'_\bullet$ and $U_\bullet$ are reduced. Then, by (9.5.2), the functoriality of $\Gamma^\kappa(?)$ and by the proof of (10.5), for any nonnegative integer N, we have the following commutative diagram

$$\begin{array}{ccc} \overline{U}'_{\bullet\leq N} & \longrightarrow & \overline{U}_{\bullet\leq N} \\ \downarrow & & \downarrow \\ \overline{U}' & \longrightarrow & \overline{U} \end{array}$$

of proper (N-truncated simplicial) schemes over κ such that $\overline{U}'$ and $\overline{U}$ contain U' and U as open subschemes, respectively, such that $(U'_{\bullet\leq N}, \overline{U}'_{\bullet\leq N})$ and $(U_{\bullet\leq N}, \overline{U}_{\bullet\leq N})$ are N-truncated proper hypercoverings of $(U', \overline{U}')$ and $(U, \overline{U})$, respectively, and such that the resulting four diagrams are commutative. Then, as in (c), we have a morphism

$$C^{\bullet\leq N}_{\mathrm{rig}}\big((U_{\bullet\leq N}/U)/K, C\big) \longrightarrow C^{\bullet\leq N}_{\mathrm{rig}}\big((U'_{\bullet\leq N}/U')/K, C\big) \tag{10.9.22}$$

by the remark for a ringed space in the proof of (6.1), by [**86**, (4.2.3)], and by (10.3). This morphism is independent of the choice of the intermediate objects in the construction of the morphism (10.9.22) by the similar argument to (a), (b), (c) and (d), and by the property (10.9.2).

Let $N' \geq N$ be two positive integers satisfying the inequality (2.2.1) for $h = c$. Set $\mathcal{Q}'_{\bullet} := \Gamma^{\mathcal{V}}_{N'}(\mathcal{P}_{N'})$. Using the formula (10.9.5) for N' instead of N, we obtain the triple $(V_{\bullet \leq N', \bullet}, \overline{V}_{\bullet \leq N', \bullet}, \mathcal{Q}'_{\bullet \leq N', \bullet})$. This triple gives us the triple $(V_{\bullet \leq N, \bullet}, \overline{V}_{\bullet \leq N, \bullet}, \mathcal{Q}'_{\bullet \leq N, \bullet})$. Using the formula (10.9.5), we also obtain the triple $(V_{\bullet \leq N, \bullet}, \overline{V}_{\bullet \leq N, \bullet}, \mathcal{Q}_{\bullet \leq N, \bullet})$. By the proof of (a), we obtain (10.9.3).

Now we have finished the proof of (10.9). □

REMARKS 10.10. — 1) The turn of the simplicial degrees in (10.9.5) is different from that in the triple in [**86**, p. 125]; because we are careful about the morphisms between the E_1-terms of the spectral sequence (10.12.1) below, we do not follow [loc. cit.]. In addition, in (10.9.5), we do not consider the m-times fiber product $\mathcal{Z}_m$ of $\mathcal{Z}/\mathcal{V}$ (=$\mathcal{U}_n$ in [*loc. cit.*]) because it is not necessary to consider it.

2) The complex $C^{\bullet \leq N}_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K, C)$ depends only on N, U_{red}, $(U_{\bullet \mathrm{red}})_{\bullet \leq N}$, K and C.

DEFINITION 10.11. — We call $C^{\bullet \leq N}_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K, C)$ the *N-truncated cosimplicial rigid cohomological complex* of C with respect to $(U_{\bullet \leq N}/U)/K$. When C is the trivial coefficient, we call $C^{\bullet \leq N}_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K, C)$ the *N-truncated cosimplicial rigid cohomological complex* of $(U_{\bullet}/U)/K$ and we denote it by $C^{\bullet \leq N}_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K)$. We call the N-truncated cosimplicial rigid cohomological complex the N-truncated CRCC for short.

COROLLARY 10.12 (see [**86**, (4.5.1)], [**87**, (7.1.2)] with different signs from ours)

There exists the following spectral sequence

$$E_1^{ij} = H^j_{\mathrm{rig}}(U_i/K, C^i) \Longrightarrow H^{i+j}_{\mathrm{rig}}(U/K, C). \tag{10.12.1}$$

Proof. — Let N be any positive integer. Set

$$E_1^{ij}(N) := \begin{cases} H^{i+j}(e_i^{-1} C^{\bullet \leq N}_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K, C)[-i]) & (0 \leq i \leq N), \\ 0 & (\text{otherwise}). \end{cases}$$

Then, by the Convention (6) and (10.9), we have

$$E_1^{ij}(N) = H^j\big(C^i_{\mathrm{rig}}((U_{\bullet \leq N}/U)/K, C)\big) = H^j_{\mathrm{rig}}(U_i/K, C)$$

for $0\leq i\leq N$. Hence we have the following spectral sequence

$$(10.12.2)\quad E_1^{ij}=E_1^{ij}(N)=H^j_{\mathrm{rig}}(U_i/K,C^i)\Longrightarrow H^{i+j}\big(\mathbf{s}(C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K,C))\big)$$

for $0\leq i\leq N$. For any fixed $h\in\mathbb{N}$, let N be a positive integer satisfying the inequality (2.2.1). Then the E_r^{ij}-terms $(1\leq r\leq\infty, i+j=h)$ of the spectral sequence (10.12.2) are independent of the choice of N ((10.9.3)) and the graded objects of $H^h_{\mathrm{rig}}(U/K,C)=H^h(\mathbf{s}(C^{\bullet\leq N}_{\mathrm{rig}}((U_{\bullet\leq N}/U)/K,C)))$ by the filtration induced by the spectral sequence (10.12.2) are E_∞^{ij}-terms $(i+j=h)$ of (10.12.2). Hence we obtain (10.12). □

Proposition 10.13. — *Let i be a nonnegative integer. Let $\delta_k\colon U_{i+1}\to U_i$ $(0\leq k\leq i+1)$ be the standard face morphism. Then the boundary morphism $E_1^{ij}\to E_1^{i+1,j}$ of* (10.12.1) *is $\Sigma_{k=0}^{i+1}(-1)^k\delta_k^*$.*

Proof. — This follows from the convention on the signs in (2.0.1). □

Definition 10.14. — We call the spectral sequence (10.12.1) *Tsuzuki's spectral sequence.*

Remarks 10.15. — 1) The convention on signs in [**19**, (3.9)] are different from our convention in (2.0.1). Hence the boundary morphism between the E_1-terms of their spectral sequence is not equal to $\Sigma_{k=0}^{i+1}(-1)^k\delta_k^*$. Since each $(-1)^k\delta_k^*$ in [**19**, (3.9)] (resp. [**25**, (5.1.9.2)]) has different signs for the parity of the degrees of sheaves of differential forms, it seems impossible to express their boundary morphism between the E_1-terms of [**87**, (7.1.2)] by the induced morphism of morphisms of geometry. One should not make the convention in [**19**, (3.9)] (resp. [**25**, (5.1.9.2)]).

2) As pointed out in [**87**, (7.1.3)], [**86**, (4.5.2)] was not proved in the general case in [**86**]. [**86**, (4.5.2)] was used for the proof of [**86**, (4.5.1)]. [**87**, 7.5] gives a proof of [**86**, (4.5.1)] without using [**86**, (4.5.2)]. One can also give a shorter proof of [**86**, (4.5.1)] by the proof of [**86**, (4.5.1)] and by (10.5) instead of [**86**, (4.5.2)].

Theorem 10.16. — *Let $(U_\bullet,\overline{U}_\bullet)$ be a proper hypercovering of $(U,\overline{U})$. Let C be an overconvergent isocrystal on $(U,\overline{U})/K$. Let $C^\bullet$ be the image of C by the natural functor*

$$\mathrm{Isoc}^\dagger((U,\overline{U})/K)\longrightarrow\mathrm{Isoc}^\dagger\big((U_\bullet,\overline{U}_\bullet)/K\big),$$

where $\mathrm{Isoc}^\dagger((U_\bullet,\overline{U}_\bullet)/K)$ is the category defined in [**19**, (10.3.1)]. *Let $\overline{U}_\bullet\hookrightarrow\mathcal{P}_\bullet$ be a closed immersion into a simplicial p-adic formal $\mathcal{V}$-scheme. Assume that*

$\mathcal{P}_\bullet$ is formally smooth over $\mathrm{Spf}(\mathcal{V})$ around $U_\bullet$. Then there exists a canonical isomorphism

$$R\Gamma_{\mathrm{rig}}(U/K,C) \xrightarrow{\sim} R\Gamma\big(\,]\overline{U}_\bullet[_{\mathcal{P}_\bullet}, \mathrm{DR}(C^\bullet)\big). \tag{10.16.1}$$

Proof. — We fix a nonnegative integer h. Let N be an integer satisfying the inequality (2.2.1). Let

$$\Gamma^{\mathcal{V}}_N(\mathcal{P}_N) := \Gamma^{\mathcal{V}}_N(\mathcal{P}_N)^{\leq N}$$

be the simplicial formal scheme constructed in [**19**, §11]. Set

$$(U_{\bullet\leq N}, \overline{U}_{\bullet\leq N}, \mathcal{Q}_{\bullet\leq N}) := \big(U_{\bullet\leq N}, \overline{U}_{\bullet\leq N}, \Gamma^{\mathcal{V}}_N(\mathcal{P}_N)_{\bullet\leq N}\big).$$

Embed $\overline{U}_{\bullet\leq N}$ into the product $\mathcal{R}_{\bullet\leq N} := \mathcal{P}_{\bullet\leq N}\widehat{\times}_{\mathcal{V}}\mathcal{Q}_{\bullet\leq N}$. Then we have two natural morphisms

$$R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{P}_{\bullet\leq N}}, \mathrm{DR}(C)\big) \longrightarrow R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{R}_{\bullet\leq N}}, \mathrm{DR}(C)\big), \tag{10.16.2}$$

$$R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, \mathrm{DR}(C)\big) \longrightarrow R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{R}_{\bullet\leq N}}, \mathrm{DR}(C)\big). \tag{10.16.3}$$

The morphisms (10.16.2) and (10.16.3) are isomorphisms. Indeed, set $\mathcal{S} = \mathcal{P}$, $\mathcal{Q}$ or $\mathcal{R}$. By the standard spectral sequences of $H^j(R\Gamma(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{S}_{\bullet\leq N}}, \mathrm{DR}(C)))$ ($\mathcal{S} := \mathcal{P}, \mathcal{Q}, \mathcal{R}$), it suffices to prove that two morphisms

$$H^j(R\Gamma\big(\,]\overline{U}_t[_{\mathcal{P}_t}, \mathrm{DR}(C))\big) \longrightarrow H^j(R\Gamma\big(\,]\overline{U}_t[_{\mathcal{R}_t}, \mathrm{DR}(C))\big), \tag{10.16.4}$$

$$H^j(R\Gamma\big(\,]\overline{U}_t[_{\mathcal{Q}_t}, \mathrm{DR}(C))\big) \longrightarrow H^j(R\Gamma\big(\,]\overline{U}_t[_{\mathcal{R}_t}, \mathrm{DR}(C))\big) \tag{10.16.5}$$

are isomorphisms for all $t \leq N$. Since $\mathcal{P}_t$, $\mathcal{Q}_t$ and $\mathcal{R}_t$ are formally smooth over $\mathrm{Spf}(\mathcal{V})$ around U_t, both sides of (10.16.4) and (10.16.5) are equal to $H^j_{\mathrm{rig}}(U_t/K,C)$ (see [**9**, (1.4), (1.6)], [**19**, (10.6.1)]). Therefore we have an isomorphism

$$R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{P}_{\bullet\leq N}}, \mathrm{DR}(C)\big) \xrightarrow{\sim} R\Gamma\big(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, \mathrm{DR}(C)\big). \tag{10.16.6}$$

Next, set $(U'_\bullet, \overline{U}'_\bullet, \mathcal{P}'_\bullet) := (U_\bullet \times_U Z, \overline{U}_\bullet \times_{\overline{U}} \overline{Z}, \mathcal{P}_\bullet\widehat{\times}_{\mathcal{V}}\mathcal{Z})$ and

$$(U'_{\bullet\leq N}, \overline{U}'_{\bullet\leq N}, \mathcal{Q}'_{\bullet\leq N}) := (U'_{\bullet\leq N}, \overline{U}'_{\bullet\leq N}, \Gamma^{\mathcal{V}}_N(\mathcal{P}'_N)_{\bullet\leq N}).$$

We consider the triple $\mathfrak{V}' := (U'_{\bullet\leq N,\bullet}, \overline{U}'_{\bullet\leq N,\bullet}, \mathcal{Q}'_{\bullet\leq N,\bullet})$ of (N,∞)-truncated bisimplicial (formal) schemes defined by

$$(U'_{mn}, \overline{U}'_{mn}, \mathcal{Q}'_{mn}) := (\mathrm{cosk}_0^{U_m}(U'_m)_n, \mathrm{cosk}_0^{\overline{U}_m}(\overline{U}'_m)_n, \mathrm{cosk}_0^{\mathcal{V}}(\mathcal{Q}'_m\widehat{\times}_{\mathcal{V}}\mathcal{Z})_n)$$
$$(0 \leq m \leq N, n \in \mathbb{N})$$

with the natural morphisms which make $\mathfrak{V}'$ a triple of (N,∞)-truncated bisimplicial (formal) schemes. Then we have the following commutative diagram of triples of (N,∞)-truncated (bi)simplicial schemes:

$$\begin{array}{ccc} (U_{\bullet\leq N}, \overline{U}_{\bullet\leq N}, \mathcal{Q}_{\bullet\leq N}) & \longleftarrow & (U'_{\bullet\leq N,\bullet}, \overline{U}'_{\bullet\leq N,\bullet}, \mathcal{Q}'_{\bullet\leq N,\bullet}) \\ & & \big\downarrow \\ & & (Z_{\bullet}, \overline{Z}_{\bullet}, \mathcal{Z}_{\bullet}). \end{array} \tag{10.16.7}$$

Then, by the proof of (10.9), we have an isomorphism

$$R\Gamma(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, \mathrm{DR}(C)) \xrightarrow{\sim} R\Gamma(\,]\overline{U}'_{\bullet\leq N,\bullet}[_{\mathcal{Q}'_{\bullet\leq N,\bullet}}, \mathrm{DR}(C)) \tag{10.16.8}$$

and the natural morphism

$$R\Gamma(\,]\overline{U}'_{\bullet\leq N,\bullet}[_{\mathcal{Q}'_{\bullet\leq N,\bullet}}, \mathrm{DR}(C)) \longleftarrow R\Gamma(\,]\overline{Z}_{\bullet}[_{\mathcal{Z}_{\bullet}}, \mathrm{DR}(C)) \tag{10.16.9}$$

induces an isomorphism

$$H^h(R\Gamma(\,]\overline{U}'_{\bullet\leq N,\bullet}[_{\mathcal{Q}'_{\bullet\leq N,\bullet}}, \mathrm{DR}(C))) \xleftarrow{\sim} H^h(R\Gamma(\,]\overline{Z}_{\bullet}[_{\mathcal{Z}_{\bullet}}, \mathrm{DR}(C))). \tag{10.16.10}$$

By (10.16.9), (10.16.8) and (10.16.6), we have a morphism

$$R\Gamma_{\mathrm{rig}}(U/K, C) \longrightarrow R\Gamma(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{P}_{\bullet\leq N}}, \mathrm{DR}(C)) \tag{10.16.11}$$

which induces an isomorphism

$$\begin{aligned} H^h_{\mathrm{rig}}(U/K, C) &\xrightarrow{\sim} H^h\big(R\Gamma(\,]\overline{U}_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, \mathrm{DR}(C))\big) \\ &= H^h\big(R\Gamma(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C))\big). \end{aligned} \tag{10.16.12}$$

Let c be any integer in (10.1) for the case $T = \varnothing$ and let $h > c$ be any integer. Then, by (10.1) and (10.16.12), $H^h(R\Gamma(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C))) = 0$. Hence

$$\tau_c\big(R\Gamma(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C))\big) = R\Gamma\big(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C)\big). \tag{10.16.13}$$

Let N be an integer satisfying the inequality (2.2.1) for $h = c$. Then, by (10.16.11), (10.16.12), (10.16.13) and (10.1) again, we have an isomorphism

$$\tau_c\big(R\Gamma_{\mathrm{rig}}(U/K, C)\big) \xrightarrow{\sim} \tau_c\big(R\Gamma(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C))\big). \tag{10.16.14}$$

Therefore we have an isomorphism

$$R\Gamma_{\mathrm{rig}}(U/K, C) \xrightarrow{\sim} R\Gamma\big(\,]\overline{U}_{\bullet}[_{\mathcal{P}_{\bullet}}, \mathrm{DR}(C)\big). \tag{10.16.15}$$

We can prove that the isomorphism (10.16.15) is independent of the choice of c and the choice of $(Z, \overline{Z}, \mathcal{Z})$. Moreover, we can prove the functoriality of (10.16.15). (We omit the proof of the independence and the functoriality with respect to overconvergent isocrystals C's on $(U, \overline{U})$'s and with respect to a morphism of triples $(U_{\bullet}, \overline{U}_{\bullet}, \mathcal{P}_{\bullet})$ because the proof is easier than the proof of (10.9).)

Thus we have proved (10.16). □

PROPOSITION 10.17 (see [**87**, (8.1.1) (2)]). — *Let $\mathcal{V} \longrightarrow \mathcal{V}'$ be a ring homomorphism of complete discrete valuation rings of mixed characteristics. Let K' be the fraction field of $\mathcal{V}'$ and κ' the residue field of $\mathcal{V}'$. Set $U' := U \otimes_\kappa \kappa'$. Then the canonical morphism $R\Gamma_{\mathrm{rig}}(U/K)_{K'} \to R\Gamma_{\mathrm{rig}}(U'/K')$ is an isomorphism.*

Proof. — (Compare this proof with that in [**87**, (8.1.1) (2)].) Let h be a nonnegative integer. We have only to prove that the morphism $H^h_{\mathrm{rig}}(U/K)_{K'} \to H^h_{\mathrm{rig}}(U'/K')$ is an isomorphism. By the finite base change theorem (see [**9**, (1.8)]), we can make a finite extension of K. Let N be a positive integer satisfying the inequality (2.2.1). Let $(U_{\bullet \leq N}, \overline{U}_{\bullet \leq N}, \mathcal{P}_{\bullet \leq N})$ be an N-truncated triple such that $(U_{\bullet \leq N}, \overline{U}_{\bullet \leq N})$ is an N-truncated proper hypercovering of $(U, \overline{U})$, such that $U_{\bullet \leq N}$ is regular and such that $\mathcal{P}_{\bullet \leq N}$ is formally smooth over $\mathrm{Spf}(\mathcal{V})$ around $U_{\bullet \leq N}$: the existence of the triple above follows from (9.2) and from the existence of the simplicial formal scheme $\Gamma^{\mathcal{V}}_N(\mathcal{P}_N)$. Because we may make a finite extension of K, we may assume that $U_{\bullet \leq N}$ is smooth over κ. Set

$$(U'_{\bullet \leq N}, \overline{U}'_{\bullet \leq N}, \mathcal{P}'_{\bullet \leq N}) := (U_{\bullet \leq N} \otimes_\kappa \kappa', \overline{U}_{\bullet \leq N} \otimes_\kappa \kappa', \mathcal{P}_{\bullet \leq N} \hat{\otimes}_{\mathcal{V}} \mathcal{V}').$$

Let $j_{\bullet \leq N} \colon U_{\bullet \leq N} \hookrightarrow \overline{U}_{\bullet \leq N}$ be the open immersion. Set $j'_{\bullet \leq N} := j_{\bullet \leq N} \otimes_\kappa \kappa'$. Because $U_{\bullet \leq N}$ is smooth over κ and because $\mathcal{P}_{\bullet \leq N}$ is formally smooth over $\mathrm{Spf}(\mathcal{V})$ around $U_{\bullet \leq N}$, the canonical morphism $H^h_{\mathrm{rig}}(U_t/K)_{K'} \longrightarrow H^h_{\mathrm{rig}}(U'_t/K')$ $(0 \leq t \leq N)$ is an isomorphism by the final remark in [**8**]. Hence the canonical morphism

$$\begin{aligned}(10.17.1) \quad & H^h\big(R\Gamma(\,]\overline{U}_{\bullet \leq N}[_{\mathcal{P}_{\bullet \leq N}}, \mathrm{DR}(j^\dagger_{\bullet \leq N}\mathcal{O}_{]\overline{U}_{\bullet \leq N}[_{\mathcal{P}_{\bullet \leq N}}}))_{K'} \\ & \longrightarrow H^h\big(R\Gamma(\,]\overline{U}'_{\bullet \leq N}[_{\mathcal{P}'_{\bullet \leq N}}, \mathrm{DR}(j'^\dagger_{\bullet \leq N}\mathcal{O}_{]\overline{U}'_{\bullet \leq N}[_{\mathcal{P}'_{\bullet \leq N}}}))\end{aligned}$$

is an isomorphism by the standard spectral sequences of both sides of (10.17.1). By the proof of (10.7) and (10.16), we have two canonical isomorphisms

$$H^h_{\mathrm{rig}}(U/K) \xrightarrow{\sim} H^h\big(R\Gamma(\,]\overline{U}_{\bullet \leq N}[_{\mathcal{P}_{\bullet \leq N}}, \mathrm{DR}(j^\dagger_{\bullet \leq N}\mathcal{O}_{]\overline{U}_{\bullet \leq N}[_{\mathcal{P}_{\bullet \leq N}}}))\big),$$

$$H^h_{\mathrm{rig}}(U'/K') \xrightarrow{\sim} H^h\big(R\Gamma(\,]\overline{U}'_{\bullet \leq N}[_{\mathcal{P}'_{\bullet \leq N}}, \mathrm{DR}(j'^\dagger_{\bullet \leq N}\mathcal{O}_{]\overline{U}'_{\bullet \leq N}[_{\mathcal{P}'_{\bullet \leq N}}}))\big).$$

Hence we obtain (10.17). □

Let U^i $(i = 1, 2)$ be a separated smooth scheme over κ. Set $U^{12} := U^1 \times_\kappa U^2$. Let C_i $(i = 1, 2)$ be an overconvergent F-isocrystal on U^i/K. By Kedlaya's Künneth formula [**55**, (1.2.4)] which is a generalization of Berthelot's Künneth formula [**8**, (3.2)] for the trivial coefficient case (cf. the last sentence in (10.19) below), the following canonical morphism

$$R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \otimes_K R\Gamma_{\mathrm{rig}}(U^2/K, C_2) \longrightarrow R\Gamma_{\mathrm{rig}}(U^{12}/K, C_1 \boxtimes C_2) \tag{10.17.2}$$

is an isomorphism. This is generalized to the following as stated in the last sentence in [**55**, (1.5)]:

Theorem 10.18 (Künneth formula). — *Let U^i $(i = 1,2)$ be a separated scheme of finite type over κ. Let C_i $(i = 1,2)$ be an overconvergent F-isocrystal on U^i/K. Then the canonical morphism*

$$R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \otimes_K R\Gamma_{\mathrm{rig}}(U^2/K, C_2) \longrightarrow R\Gamma_{\mathrm{rig}}(U^{12}/K, C_1 \boxtimes C_2) \tag{10.18.1}$$

is an isomorphism.

Proof. — Let $j^i\colon U^i \hookrightarrow \overline{U}^i$ $(i = 1,2)$ be an open immersion into a proper scheme over κ. Set $\overline{U}^{12} := \overline{U}^1 \times_\kappa \overline{U}^2$ and $C_{12} := C_1 \boxtimes C_2$. Let c be an integer such that $H^h_{\mathrm{rig}}(U^i/K, C_i) = 0$ $(i = 1,2,12)$ for all $h > c$ ((10.1)). Let $N-1$ be an integer satisfying the inequality (2.2.1) for $h = c$. Let $(U^i_{\bullet\le N}, \overline{U}^i_{\bullet\le N}, \mathcal{P}^i_{\bullet\le N})$ $(i = 1,2)$ be an N-truncated triple in the proof of (10.17) for $(U^i, \overline{U}^i)$. Replacing κ with a sufficiently large finite extension of κ and using a finite flat base change theorem of Chiarellotto and Tsuzuki (see [**19**, (11.8.1)]), we may assume that $U^i_{\bullet\le N}$ is smooth over κ. Consider a triple

$$(U^{12}_{\bullet\le N}, \overline{U}^{12}_{\bullet\le N}, \mathcal{P}^{12}_{\bullet\le N}) := (U^1_{\bullet\le N} \times_\kappa U^2_{\bullet\le N}, \overline{U}^1_{\bullet\le N} \times_\kappa \overline{U}^2_{\bullet\le N}, \mathcal{P}^1_{\bullet\le N} \widehat{\times}_{\mathcal{V}} \mathcal{P}^2_{\bullet\le N})$$

of N-truncated simplicial (formal) schemes. Then $(U^{12}_{\bullet\le N}, \overline{U}^{12}_{\bullet\le N})$ is an N-truncated proper hypercovering of $(U^{12}, \overline{U}^{12})$ (cf. (9.4), 4)). Let $j^i_{\bullet\le N}\colon U^i_{\bullet\le N} \hookrightarrow \overline{U}^i_{\bullet\le N}$ $(i = 1,2,12)$ be the open immersion. Set

$$(V^i_\bullet, \overline{V}^i_\bullet, \mathcal{Q}^i_\bullet) := \big(\mathrm{cosk}_N^{U^i}(U^i_{\bullet\le N}), \mathrm{cosk}_N^{\overline{U}^i}(\overline{U}^i_{\bullet\le N}), \mathrm{cosk}_N^{\mathcal{V}}(\mathcal{P}^i_{\bullet\le N})\big) \quad (i = 1,2),$$
$$(V^{12}_\bullet, \overline{V}^{12}_\bullet, \mathcal{Q}^{12}_\bullet) := (V^1_\bullet \times_\kappa V^2_\bullet, \overline{V}^1_\bullet \times_\kappa \overline{V}^2_\bullet, \mathcal{Q}^1_\bullet \widehat{\times}_{\mathcal{V}} \mathcal{Q}^2_\bullet).$$

By abuse of notation, we denote the pull-backs of C_i to $U^i_{\bullet\le N}$ and $V^i_\bullet$ simply by C_i. Let $k^i_\bullet\colon V^i_\bullet \hookrightarrow \overline{V}^i_\bullet$ $(i = 1,2,12)$ be the open immersion. By a theorem of Eilenberg-Zilber (see [**89**, Theorem (8.5.1)], cf. [*loc. cit.*, Theorem (8.3.8)]) (cf. [**25**, (8.1.25)]), we have an isomorphism

$$
\begin{aligned}
(10.18.2)\qquad & \mathbf{s}\big[R\Gamma^{\bullet}(\,]\overline{V}^1_{\bullet}[_{\mathcal{Q}^1_{\bullet}}, \mathrm{DR}(k^{1\dagger}_{\bullet}((C_1)_{]\overline{V}^1_{\bullet}[_{\mathcal{Q}^1_{\bullet}}})))\\
&\qquad \otimes_K R\Gamma^{\bullet}(\,]\overline{V}^2_{\bullet}[_{\mathcal{Q}^2_{\bullet}}, \mathrm{DR}(k^{2\dagger}_{\bullet}((C_2)_{]\overline{V}^2_{\bullet}[_{\mathcal{Q}^2_{\bullet}}})))\big]\\
&\overset{\sim}{\longrightarrow} \mathbf{s}\big[R\Gamma^{\bullet}(\,]\overline{V}^1_{\bullet}[_{\mathcal{Q}^1_{\bullet}}, \mathrm{DR}(k^{1\dagger}_{\bullet}((C_1)_{]\overline{V}^1_{\bullet}[_{\mathcal{Q}^1_{\bullet}}})))\\
&\qquad \otimes_K R\Gamma^{*}(\,]\overline{V}^2_{*}[_{\mathcal{Q}^2_{*}}, \mathrm{DR}(k^{2\dagger}_{*}((C_2)_{]\overline{V}^2_{*}[_{\mathcal{Q}^2_{*}}})))\big].
\end{aligned}
$$

The canonical filtration τ_c on the left hand side of (10.18.2) is canonically isomorphic to

$$
\tau_c\big[R\Gamma(\,]\overline{U}^{12}_{\bullet\le N}[_{\mathcal{P}^{12}_{\bullet\le N}}, \mathrm{DR}(j^{12\dagger}_{\bullet\le N}((C_{12})_{]\overline{U}^{12}_{\bullet\le N}[_{\mathcal{P}^{12}_{\bullet\le N}}})))\big]
$$

by (10.17.2). Hence we have a canonical isomorphism

$$
\begin{aligned}
(10.18.3)\qquad & \tau_c\big[\mathbf{s}\{R\Gamma^{\bullet\le N}(\,]\overline{U}^1_{\bullet\le N}[_{\mathcal{P}^1_{\bullet\le N}}, \mathrm{DR}(j^{1\dagger}_{\bullet\le N}((C_1)_{]\overline{U}^1_{\bullet\le N}[_{\mathcal{P}^1_{\bullet\le N}}})))\\
&\qquad \otimes_K R\Gamma^{*\le N}(\,]\overline{U}^2_{*\le N}[_{\mathcal{P}^2_{*\le N}}, \mathrm{DR}(j^{2\dagger}_{*\le N}((C_2)_{]\overline{U}^2_{*\le N}[_{\mathcal{P}^2_{*\le N}}})))\}\big]\\
&\overset{\sim}{\longleftarrow} \tau_c[\mathbf{s}\{R\Gamma^{\bullet\le N}(\,]\overline{U}^{12}_{\bullet\le N}[_{\mathcal{P}^{12}_{\bullet\le N}}, \mathrm{DR}(j^{12\dagger}_{\bullet\le N}((C_{12})_{]\overline{U}^{12}_{\bullet\le N}[_{\mathcal{P}^{12}_{\bullet\le N}}})))\}].
\end{aligned}
$$

Since the left hand side of (10.18.3) is equal to

$$
\begin{aligned}
&\tau_c\mathbf{s}\big[\tau_c\mathbf{s}\big[R\Gamma^{\bullet\le N}(\,]\overline{U}^1_{\bullet\le N}[_{\mathcal{P}^1_{\bullet\le N}}, \mathrm{DR}(j^{1\dagger}_{\bullet\le N}((C_1)_{]\overline{U}^1_{\bullet\le N}[_{\mathcal{P}^1_{\bullet\le N}}})))\big]\\
&\qquad \otimes_K \tau_c\mathbf{s}[R\Gamma^{*\le N}(\,]\overline{U}^2_{*\le N}[_{\mathcal{P}^2_{*\le N}}, \mathrm{DR}(j^{2\dagger}_{*\le N}((C_2)_{]\overline{U}^2_{*\le N}[_{\mathcal{P}^2_{*\le N}}})))]\big],
\end{aligned}
$$

it is equal to

$$
\tau_c\mathbf{s}\big[\tau_c R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \otimes_K \tau_c R\Gamma_{\mathrm{rig}}(U^2/K, C_2)\big].
$$

Hence, by (10.18.3), we have an isomorphism

$$
\begin{aligned}
(10.18.4)\qquad & \tau_c\mathbf{s}\big[\tau_c R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \otimes_K \tau_c R\Gamma_{\mathrm{rig}}(U^2/K, C_2)\big]\\
&\qquad \overset{\sim}{\longrightarrow} \tau_c R\Gamma_{\mathrm{rig}}(U^{12}/K, C_{12}).
\end{aligned}
$$

Since $\tau_c R\Gamma_{\mathrm{rig}}(U^i/K, C_i) = R\Gamma_{\mathrm{rig}}(U^i/K, C_i)$ $(i = 1, 2, 12)$, we see that the morphism (10.18.1) is an isomorphism.

It is a routine work to verify that the morphism (10.18.1) is independent of the choice of c and N.

Now we finish the proof. □

REMARKS 10.19. — 1) In (16.15) below, we shall generalize (10.18) to the case of rigid cohomology with closed support as stated in the last sentence in [**55**, (1.5)].

2) The proof of the Poincaré duality and the Künneth formula with compact support (see [**8**, (2.4), (3.2) (ii)]) is not perfect: we have to make a finite extension of the base field κ in [loc. cit.] (cf. the proof of (10.17)). Strictly speaking, we first obtain the Poincaré duality and the Künneth formula with compact support over a finite extension of κ. Then we need the following two facts:

(a) The rigid cohomology with compact support of U commutes with the base change of a (finite) extension of K.

(b) The trace morphism constructed in [**8**] commutes with the base change of a (finite) extension of K.

Part (a) is stated in [**87**, (8.1.1)]; the proof is the same as that of [**86**, (5.1.1)].

Part (b) follows from a fact that the trace morphism in an algebraic situation is compatible with base change (see [**20**, Theorem 3.6.5]) (we need only the compatibility with the flat base change) and the construction of the trace morphism in [**8**].

In conclusion, [**8**, (2.4), (3.2)] follows. In fact, we need not assume the existence of an ambient formal scheme over $\mathcal{V}$ by the proof of [**8**, (2.4), (3.2)].

11. Comparison theorem between rigid cohomology and log crystalline cohomology

Let κ, $\mathcal{V}$, $\mathcal{W}$, K and K_0 be as in §10. In this section we prove a comparison theorem between the rigid cohomology of the trivial coefficient of a separated scheme U of finite type over κ and the log crystalline cohomology of the cosimplicial trivial coefficient of a gs proper hypercovering of U. More generally, we prove a comparison theorem for an F-isocrystal on U whose pull-back to a gs proper hypercovering of U has only logarithmic singularities along boundaries; Shiho has first considered this coefficient in a constant simplicial case (see [**82**]); he has conjectured that any F-isocrystal on a separated scheme of finite type over κ has only logarithmic singularities along boundaries if one makes a suitable alteration (see [*loc. cit.*]). Recently Kedlaya has proved Shiho's conjecture (see [**56**], [**57**], [**58**], [**59**, (2.4.4)]).

First we prove the functoriality of Shiho's comparison theorems between some cohomologies.

Let $f\colon Y \to (\mathrm{Spec}(\kappa), \kappa^*)$ be a proper log smooth morphism from an fs log scheme. Let $U := Y_{\mathrm{triv}}$ be the maximal log open subscheme of Y whose log structure is trivial.

Let $R\Gamma_{\log\text{-naiv}}(Y/K)$ be the complex of the log naive cohomologies of the trivial coefficient defined in [**82**, Definition 2.2.12] (the terminology "naive rigid cohomology" has been used in [**6**, p. 14]; the terminology "log analytic rigid cohomology" has been used in [**82**] instead of the terminology "log naive rigid cohomology"). Then, by [**82**, Theorem 2.4.4], there exists an isomorphism

$$R\Gamma_{\log\text{-naiv}}(Y/K) \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(U/K). \tag{11.0.1}$$

Let L be a fine log structure on $\mathrm{Spf}(\mathcal{V})$. Let L_1 be the pull-back of L to $\mathrm{Spec}(\kappa)$. Let $(\mathrm{Spec}(\kappa), L_1) \hookrightarrow (\mathrm{Spf}(\mathcal{V}), L)$ be the natural exact closed immersion. Let $Z \longrightarrow (\mathrm{Spec}(\kappa), L_1)$ be a morphism of fine log schemes. Assume that the underlying morphism $\overset{\circ}{Z} \to \mathrm{Spec}(\kappa)$ is of finite type. By [**82**, Corollary 2.3.9], there exists an isomorphism

$$R\Gamma((\widetilde{Z/\mathcal{V}})^{\log}_{\mathrm{conv}}, \mathcal{K}_{Z/\mathcal{V}}) \xrightarrow{\sim} R\Gamma_{\log\text{-naiv}}(Z/K), \tag{11.0.2}$$

where $(\widetilde{Z/\mathcal{V}})^{\log}_{\mathrm{conv}}$ is the log convergent topos of a log convergent site $(Z/\mathcal{V})^{\log}_{\mathrm{conv,zar}}$ in [**82**, Definition 2.1.3] and $\mathcal{K}_{Z/\mathcal{V}}$ is the trivial isostructure sheaf in $(\widetilde{Z/\mathcal{V}})^{\log}_{\mathrm{conv}}$.

Assume, furthermore, that $Z \to (\mathrm{Spec}(\kappa), L_1)$ is log smooth. By [**82**, Theorem 3.1.1], there exists an isomorphism

$$R\Gamma\big((\widetilde{Z/\mathcal{W}})^{\log}_{\mathrm{conv}}, \mathcal{K}_{Z/\mathcal{W}}\big) \xrightarrow{\sim} R\Gamma(Z/\mathcal{W})_{K_0}. \tag{11.0.3}$$

Here the right hand side on (11.0.3) is the complex of the log crystalline cohomologies of $Z/(\mathrm{Spf}(\mathcal{W}), L)$.

Before giving the proof of the functoriality of the three morphisms (11.0.1), (11.0.2) and (11.0.3), we prove the following:

PROPOSITION 11.1. — *Let $g\colon X' \longrightarrow X$ be a morphism of fine log (formal) schemes over a morphism $S' \longrightarrow S$ of fine log (formal) schemes such that the underlying schemes $\overset{\circ}{X}{}'$ and $\overset{\circ}{X}$ are locally of finite type over $\overset{\circ}{S}{}'$ and $\overset{\circ}{S}$, respectively. Then there exist a disjoint union X'_0 (resp. X_0) of log affine open subschemes covering X' (resp. X) and a (formal) embedding system $(\mathrm{cosk}_0^{X'}(X'_0), P'_\bullet)$ (resp. $(\mathrm{cosk}_0^{X}(X_0), P_\bullet)$) over S' (resp. S) fitting into the* *following commutative diagram*

$$
\begin{array}{ccc}
\mathrm{cosk}_0^{X'}(X'_0) & \xrightarrow{\subset} & P'_{\bullet} \\
\big\downarrow & & \big\downarrow \\
\mathrm{cosk}_0^{X}(X_0) & \xrightarrow{\subset} & P_{\bullet}
\end{array}
\tag{11.1.1}
$$

of simplicial fine log (formal) schemes over the morphism $S' \longrightarrow S$. (*As to the definition of the embedding system in* [**45**, p. 237], *we have a remark in* (11.2), 1) *and* 2) *below.*) *If* $\overset{\circ}{X}{}'$ *and* $\overset{\circ}{X}$ *are separated over* $\overset{\circ}{S}{}'$ *and* $\overset{\circ}{S}$, *respectively, then one can take the two horizontal morphisms in* (11.1.1) *as closed immersions.*

Proof. — (Cf. the proof of [**82**, Proposition 2.2.11].) We give the proof which is valid only for log schemes; in the case of fine log formal schemes, we have only to take suitable completions in the following.

First assume that $S' = S$. Let V' and V be log affine open subschemes of X' and X, respectively, such that g induces a morphism $V' \to V$. We assume that the structural morphism $V \to S$ factors through a log affine open subscheme T of S which has a global chart $P \to \Gamma(T, \mathcal{O}_T)$ and that the morphism $V' \longrightarrow V$ has a global chart

$$
\begin{array}{ccc}
Q' & \longrightarrow & \Gamma(V', \mathcal{O}_{V'}) \\
\big\uparrow & & \big\uparrow \\
Q & \longrightarrow & \Gamma(V, \mathcal{O}_V)
\end{array}
$$

over $P \to \Gamma(T, \mathcal{O}_T)$. Set $A := \Gamma(T, \mathcal{O}_T)$, $B := \Gamma(V, \mathcal{O}_V)$ and $B' := \Gamma(V', \mathcal{O}_{V'})$. Let $Q_1 \longrightarrow Q$ (resp. $B_1 \to B$) be a surjection from a commutative free finitely generated monoid (resp. a polynomial algebra over A with finite number of indeterminates). The morphisms $P \longrightarrow Q$ and $Q_1 \to Q$ induces a morphism $P \oplus Q_1 \to Q$. Since B_1 is an A-algebra, we also have a natural composite morphism $P \oplus Q_1 \to A \oplus Q_1 \to B_1[Q_1]$ of monoids (see [loc. cit.]). Then we have the commutative diagram

$$
\begin{array}{ccc}
P \oplus Q_1 & \longrightarrow & Q \\
\big\downarrow & & \big\downarrow \\
B_1[Q_1] & \longrightarrow & B.
\end{array}
$$

By the criterion of the log smoothness (see [**51**, (3.5)]), the morphism

$$(A, P) \longrightarrow \big(B_1[Q_1], P \oplus Q_1\big)$$

is log smooth. Hence we have a closed immersion $V \hookrightarrow W$ into a log smooth affine scheme over S.

Next we retake W as a base log scheme, that is, we consider the composite morphism $V' \to V \to W$. Then, by the argument above, we obtain a closed immersion $V' \hookrightarrow W'$ into a log smooth affine scheme over W. Since $W' \to W$ and $W \to S$ is log smooth, so is the composite morphism $W' \to S$.

Since the rest of the proof for the case $S' = S$ is routine (we have only to use the functor cosk_0), we leave the rest to the reader (cf. [**82**, Proposition 2.2.11], (11.4) below).

Finally we consider the case of a general morphism $S' \to S$ of fine log schemes. Then we first construct a closed immersion $V \hookrightarrow W$ as above. Now we have only to apply the argument before the previous paragraph for the composite morphism $V' \to V \times_S S' \to W \times_S S'$ over S'. □

REMARKS 11.2. — 1) The claim of the existence of the embedding system in [**45**, p. 237] is not perfect; we have to assume that X is separated over S in [*loc. cit.*]. Let X_i $(i = 1, 2)$ be an open subscheme of X. The condition on the separateness assures that the natural morphism $X_1 \times_X X_2 \hookrightarrow X_1 \times_S X_2$ of schemes is a closed immersion. By the same reason, the argument in [**47**, p. 602] is not perfect since $\mathrm{cosq}(U_0/X) \to \mathrm{cosq}(Y_0/W)$ is not necessarily a closed immersion.

2) As in [**82**, Definition 2.2.10], we allow the (not necessarily closed) immersion in the definition of the embedding system in [**45**, p. 237].

LEMMA 11.3. — *Let $g\colon \mathfrak{Y}' \longrightarrow \mathfrak{Y}$ be a morphism of log (formal) schemes. Set $\mathfrak{U}' := \mathfrak{Y}'_{\mathrm{triv}}$ and $\mathfrak{U} := \mathfrak{Y}_{\mathrm{triv}}$. Then $g|_{\mathfrak{U}'}$ factors through a morphism $\mathfrak{U}' \to \mathfrak{U}$.*

Proof. — Let y' be a point of $\mathfrak{U}'$. Set $y := g(y')$. Let (M, α) be the log structure of $\mathfrak{Y}$ and m an element of M_y. Then $g^*(\alpha(m)) = g^*(m)$ is an invertible element of $\mathcal{O}_{\mathfrak{Y}',y'}$. Since the morphism $g^*\colon \mathcal{O}_{\mathfrak{Y},y} \to \mathcal{O}_{\mathfrak{Y}',y'}$ is a local morphism, $\alpha(m) \in \mathcal{O}^*_{\mathfrak{Y},y}$. Hence $M_y = \mathcal{O}^*_{\mathfrak{Y},y}$ and $g\colon \mathfrak{Y}' \to \mathfrak{Y}$ induces a morphism $g|_{\mathfrak{U}'}\colon \mathfrak{U}' \to \mathfrak{U}$. □

PROPOSITION 11.4. — *Let $\mathcal{V} \longrightarrow \mathcal{V}'$ be a ring homomorphism of complete discrete valuation rings of mixed characteristics. Let κ' (resp. K') be the residue (resp. fraction) field of $\mathcal{V}'$. Let $(\mathrm{Spf}(\mathcal{V}'), L') \to (\mathrm{Spf}(\mathcal{V}), L)$ be a morphism of fine log schemes. Let L'_1 be the pull-back of L' to $\mathrm{Spec}(\kappa')$. Then the following hold:*

1) *The isomorphism* (11.0.1) *is functorial: for a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(\kappa'), \kappa'^*) & \longrightarrow & (\mathrm{Spec}(\kappa), \kappa^*), \end{array}$$

where the vertical morphisms from fs log schemes are proper and log smooth, the following diagram

$$\begin{array}{ccc} R\Gamma_{\text{log-naiv}}(Y'/K') & \xrightarrow{\sim} & R\Gamma_{\text{rig}}(U'/K') \\ \uparrow & & \uparrow \\ R\Gamma_{\text{log-naiv}}(Y/K) & \xrightarrow{\sim} & R\Gamma_{\text{rig}}(U/K) \end{array}$$

is commutative, where $U' := Y'_{\text{triv}}$.

2) *The isomorphism* (11.0.2) *is functorial in the obvious sense.*

3) *Let* $\mathcal{W}'$ *be a Cohen ring of* κ' *in* $\mathcal{V}'$*. Then the isomorphism* (11.0.3) *is functorial in the obvious sense.*

Proof. — We recall the constructions of (11.0.1), (11.0.2) and (11.0.3) in [**82**]. By (11.3), the morphism $g\colon Y' \to Y$ induces a natural morphism $g\colon U' \to U$.

1): Let Y_0 be the disjoint union of log affine open subschemes covering Y. Set $Y_\bullet := \mathrm{cosk}_0^Y(Y_0)$ and $U_\bullet := Y_{\bullet\mathrm{triv}}$. Let $j_\bullet\colon U_\bullet \hookrightarrow Y_\bullet$ be the open immersion. Let $\mathcal{Y}_0$ be a lift of Y_0 over $\mathrm{Spf}(\mathcal{V})$. (In fact, we can take a lift of Y_0 above over $\mathrm{Spf}(\mathcal{W})$.) Take the disjoint union Y'_0 of log affine open subschemes covering Y' such that g induces a morphism $g_0\colon Y'_0 \longrightarrow Y_0$. Then there exists a lift $\mathcal{Y}'_0$ of Y'_0 over $\mathrm{Spf}(\mathcal{V}')$ with a morphism $\mathcal{Y}'_0 \longrightarrow \mathcal{Y}_0$ over $\mathrm{Spf}(\mathcal{V}') \longrightarrow \mathrm{Spf}(\mathcal{V})$ fitting into the commutative diagram

$$\begin{array}{ccc} Y'_0 & \xrightarrow{\subset} & \mathcal{Y}'_0 \\ {\scriptstyle g_0}\downarrow & & \downarrow \\ Y_0 & \xrightarrow{\subset} & \mathcal{Y}_0. \end{array}$$

Set $\mathcal{Y}_\bullet := \mathrm{cosk}_0^{\mathcal{V}}(\mathcal{Y}_0)$ and $\mathcal{Y}'_\bullet := \mathrm{cosk}_0^{\mathcal{V}'}(\mathcal{Y}'_0)$. Then we have the commutative diagram

$$\begin{array}{ccc} Y'_\bullet & \xrightarrow{\subset} & \mathcal{Y}'_\bullet \\ {\scriptstyle g_\bullet}\downarrow & & \downarrow \\ Y_\bullet & \xrightarrow{\subset} & \mathcal{Y}_\bullet. \end{array} \tag{11.4.1}$$

Set $U'_\bullet := Y'_{\bullet\mathrm{triv}}$ and let $j'_\bullet \colon U'_\bullet \hookrightarrow Y'_\bullet$ be the natural open immersion. Let

$$\begin{array}{ccc}]Y'_\bullet[^{\log}_{\mathcal{Y}'_\bullet} & \xrightarrow{\varphi'} &]\overset{\circ}{Y}{}'_\bullet[_{\overset{\circ}{\mathcal{Y}}{}'_\bullet} \\ {\scriptstyle g^{\log}_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle \overset{\circ}{g}_\bullet} \\ ]Y_\bullet[^{\log}_{\mathcal{Y}_\bullet} & \xrightarrow{\varphi} &]\overset{\circ}{Y}_\bullet[_{\overset{\circ}{\mathcal{Y}}_\bullet} \end{array}$$

be the natural commutative diagram (cf. [**82**, pp. 59–61 and p. 114]). Then we have the commutative diagram (see [**82**, pp. 121-122, p. 114])

$$\begin{array}{ccc} R\varphi_* Rg^{\log}_{\bullet *}(\Lambda^\bullet_{]Y'_\bullet[^{\log}_{\mathcal{Y}'_\bullet}}) & \longrightarrow & R\varphi_* Rg^{\log}_{\bullet *}(j'^{\dagger}_\bullet\Lambda^\bullet_{]Y'_\bullet[^{\log}_{\mathcal{Y}'_\bullet}}) \\ \big\uparrow & & \big\uparrow \\ R\varphi_*(\Lambda^\bullet_{]Y_\bullet[^{\log}_{\mathcal{Y}_\bullet}}) & \longrightarrow & R\varphi_*(j^\dagger_\bullet\Lambda^\bullet_{]Y_\bullet[^{\log}_{\mathcal{Y}_\bullet}}) \end{array} \tag{11.4.2}$$

$$\begin{array}{ccccc} \overset{\sim}{\longleftarrow} & R\overset{\circ}{g}_{\bullet *} j'^{\dagger}_\bullet(\Lambda^\bullet_{]Y'_\bullet[^{\log}_{\mathcal{Y}'_\bullet}}) & \overset{\sim}{\longleftarrow} & R\overset{\circ}{g}_{\bullet *}(j'^{\dagger}_\bullet\Omega^\bullet_{]\overset{\circ}{Y}{}'_\bullet[_{\overset{\circ}{\mathcal{Y}}{}'_\bullet}}) \\ & \big\uparrow & & \big\uparrow \\ \overset{\sim}{\longleftarrow} & j^\dagger_\bullet\Lambda^\bullet_{]Y_\bullet[^{\log}_{\mathcal{Y}_\bullet}} & \overset{\sim}{\longleftarrow} & j^\dagger_\bullet\Omega^\bullet_{]\overset{\circ}{Y}_\bullet[_{\overset{\circ}{\mathcal{Y}}_\bullet}}. \end{array}$$

Here $\Lambda^\bullet$ means the sheaf of logarithmic differential forms. Because checking the independence of the choice of the commutative diagram (11.4.1) is a routine work, the functoriality of (11.0.1) is obvious.

2): Let $Z' \to Z$ be a morphism of fine log schemes over $(\mathrm{Spec}(\kappa'), L'_1) \longrightarrow (\mathrm{Spec}(\kappa), L_1)$. Assume that $\overset{\circ}{Z}$ (resp. $\overset{\circ}{Z}{}'$) is of finite type over κ (resp. κ'). Let

$$\begin{array}{ccc} Z'_\bullet & \overset{\subset}{\longrightarrow} & \mathcal{P}'_\bullet \\ \big\downarrow & & \big\downarrow \\ Z_\bullet & \overset{\subset}{\longrightarrow} & \mathcal{P}_\bullet \end{array} \tag{11.4.3}$$

be a commutative diagram over the morphism $(\mathrm{Spf}(\mathcal{V}'), L') \to (\mathrm{Spf}(\mathcal{V}), L)$ constructed in (11.1), where $\mathcal{P}'_\bullet \to (\mathrm{Spf}(\mathcal{V}'), L')$ and $\mathcal{P}_\bullet \to (\mathrm{Spf}(\mathcal{V}), L)$ are formally log smooth. Let π (resp. π') be a uniformizer of $\mathcal{V}$ (resp. $\mathcal{V}'$). Let

$$\{T_{\bullet,n}(i)\}_{n=1}^{\infty} \quad (\text{resp. } \{T'_{\bullet,n}(i)\}_{n=1}^{\infty}) \qquad (i = 0, 1, 2)$$

be the inductive system of the universal log enlargements of the immersion from $Z_\bullet$ (resp. $Z'_\bullet$) to the $(i+1)$-times fiber product of $\mathcal{P}_\bullet$ (resp. $\mathcal{P}'_\bullet$) over $(\mathrm{Spf}(\mathcal{V}), L)$ (resp. $(\mathrm{Spf}(\mathcal{V}'), L')$) (see [**82**, Definition 2.1.22]). Then we have the inductive systems of natural morphisms $T'_{\bullet,n}(i) \to T_{\bullet,n}(i)$ $(n \in \mathbb{Z}_{\geq 1})$. Because $T_{\bullet,n}(0)$ (resp. $T'_{\bullet,n}(0)$) is a formal model of the rigid spaces $]Z_\bullet[^{\log}_{\mathcal{P}_\bullet,|\pi|^{1/n}}$, $]Z'_\bullet[^{\log}_{\mathcal{P}'_\bullet,|\pi'|^{1/n}}$, respectively, the functoriality of (11.0.2) is clear by the proof of [**82**, Corollary 2.3.8]. Checking the choice of the independence of the commutative diagram (11.4.3) is a routine work.

3): Assume that $\mathcal{V} = \mathcal{W}$ and $\mathcal{V}' = \mathcal{W}'$ in 2). Assume that $Z' \to (\mathrm{Spec}(\kappa'), L'_1)$ and $Z \to (\mathrm{Spec}(\kappa), L_1)$ are log smooth. Let $\mathfrak{D}_\bullet(i)$ (resp. $\mathfrak{D}'_\bullet(i)$) be the log PD-envelope of the immersion from $Z_\bullet$ (resp. $Z'_\bullet$) to the $(i+1)$-times fiber product of $\mathcal{P}_\bullet$ (resp. $\mathcal{P}'_\bullet$) over $(\mathrm{Spf}(\mathcal{W}), p\mathcal{W}, [\ ], L)$ (resp. $(\mathrm{Spf}(\mathcal{W}'), p\mathcal{W}', [\ ], L')$) $(i = 0, 1, 2)$. Let $\{m(\ell)\}_{\ell \in \mathbb{N}}$ be a sequence of natural numbers such that the ideal sheaves $\mathrm{Ker}(\mathcal{O}^{\otimes i+1}_{\mathcal{P}_\ell} \to \mathcal{O}_{Z_\ell})$ and $\mathrm{Ker}(\mathcal{O}^{\otimes i+1}_{\mathcal{P}'_\ell} \to \mathcal{O}_{Z'_\ell})$ are generated by $m(\ell)$ local sections over $\mathcal{O}^{\otimes i+1}_{\mathcal{P}_\ell}$ and $\mathcal{O}^{\otimes i+1}_{\mathcal{P}'_\ell}$, respectively, for all $i = 0, 1, 2$. Set $n'(\ell) := (p-1)m(\ell) + 1$ and $n(\ell) := \max\{n'(\ell') \,|\, 0 \leq \ell' \leq \ell\}$. Then, by the universality of the universal enlargement, we have a natural morphism $\mathfrak{D}_\ell(i) \to T_{\ell,n(\ell)}(i)$. Set $T_\ell(i) := \varinjlim_n T_{\ell,n}(i)$. Then we have a natural morphism $\mathfrak{D}_\bullet(i) \to T_\bullet(i)$ of inductive systems of simplicial log formal schemes over $(\mathrm{Spf}(\mathcal{W}), p\mathcal{W}, [\ ], L)$. Set $T'_\ell(i) := \varinjlim_n T'_{\ell,n}(i)$ and let $\mathfrak{D}'_\bullet(i) \to T'_\bullet(i)$ be an analogous morphism. Then we have the following commutative diagram:

(11.4.4)
$$\begin{array}{ccc} \mathfrak{D}'_\bullet(i) & \longrightarrow & T'_\bullet(i) \\ \downarrow & & \downarrow \\ \mathfrak{D}_\bullet(i) & \longrightarrow & T_\bullet(i). \end{array}$$

The commutative diagram (11.4.4) is compatible with respect to i by natural projections (cf. [**81**, (5.3.1)]). Hence the functoriality of (11.0.3) is clear; checking the independence of the choice of the commutative diagram (11.4.3) is a routine work.

We finish the proof. ☐

Remarks 11.5. — 1) In the statements [**82**, Theorem 2.4.4], [**82**, Corollary 2.3.9] and [**82**, Theorem 3.1.1], there are only claims that there exist only isomorphisms between cohomologies. As in [**47**, II (1.1)], we have to check that the constructed isomorphisms do not depend on the choice of open coverings; this verification is not difficult.

By the same reason (as pointed out in [**70**]), Steenbrink's main theorem on the existence of the $\mathbb{Q}$-structure of the Steenbrink complex alone has no sense because some constructions in [**83**] obviously depend heavily on the choice of the charts (e.g., $L^1_{\underline{D}}$ in [loc. cit., (4.5)]); we cannot discuss, for example, the functoriality of his weight spectral sequence a priori; the $\mathbb{Q}$-structure of the Steenbrink complex might depend on the choice of the local chart. Fujisawa and Nakayama have saved his result (see [**29**]) by using the real blow up in [**53**]: the $\mathbb{Q}$-structure of the Steenbrink complex is independent of the choice of the local chart. However no one has proved the independence of the $\mathbb{Z}$-structure of the Steenbrink complex in [loc. cit.].

2) The use of the superscripts indicating simplicial degrees in [**45**] and [**82**] is wrong.

3) Let X be a proper smooth scheme over κ and D an (S)NCD on X/κ. Set $U := X \setminus D$. Then Shiho's isomorphism (= the composite isomorphism obtained from (11.0.1), (11.0.2) and (11.0.3))

$$R\Gamma_{\mathrm{rig}}(U/K_0) \overset{\sim}{\longrightarrow} R\Gamma\big((X,D)/\mathcal{W}\big)_{K_0} \tag{11.5.1}$$

is a log version of the isomorphism [**9**, (1.9.1)] without the assumption of a global embedding of U into a formal scheme which is formally smooth around U.

4) In Shiho's articles [**81**] and [**82**] and in our proof of (11.4), we use the disjoint unions $\mathcal{Y}_0$ and $\mathcal{Y}'_0$ of local lifts of Y and Y', respectively. More generally, all we need is the following commutative diagram

$$\begin{array}{ccc} Y'_0 & \overset{\subset}{\longrightarrow} & \mathcal{Y}' \\ \big\downarrow & & \big\downarrow \\ Y_0 & \overset{\subset}{\longrightarrow} & \mathcal{Y}, \end{array}$$

where the two horizontal morphisms are closed immersions into fine formally log smooth log p-adic formal $\mathcal{V}$-schemes whose log structures are trivial around $Y'_{0\mathrm{triv}}$ and $Y_{0\mathrm{triv}}$ (cf. [**82**, Proposition 2.2.4]); the assumption on the triviality of the log structures implies that the underlying schemes of the exactifications are formally smooth around $Y'_{0\mathrm{triv}}$ and $Y_{0\mathrm{triv}}$ since a log etale morphism over a trivial log scheme is etale (cf. [**82**, p. 122]). The proofs of the comparison theorems for the generalized case are the same as Shiho's proofs for the case of local lifts.

This remark is important for the proof of (11.6) below.

Now we prove a generalization of the comparison theorem (1.0.17): the base field κ is not necessarily perfect and we do not assume the splitness.

Let $(U, \overline{U})$ be as in §10. Let $(U_\bullet, X_\bullet)$ be a good proper hypercovering of $(U, \overline{U})$ which has the disjoint union $(U'_\bullet, X'_\bullet)$ of the members of an affine simplicial open covering of $(U_\bullet, X_\bullet)$ over $(U, \overline{U})$ ((9.7), 1)). Assume, furthermore, that $X_\bullet$ is smooth over κ. Set $D_\bullet := X_\bullet \setminus U_\bullet$.

Theorem 11.6 (Comparison theorems). — 1) *There exists an isomorphism*

$$R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\rm conv}, \mathcal{K}_{(X_\bullet, D_\bullet)/\mathcal{V}}\big) \overset{\sim}{\longrightarrow} R\Gamma_{\rm rig}(U/K). \tag{11.6.1}$$

The isomorphism (11.6.1) *is functorial: for a morphism* $(V, \overline{V}) \to (U, \overline{U})$ *of pairs and for good proper hypercoverings* $(U_\bullet, X_\bullet)$ *of* $(U, \overline{U})$ *and* $(V_\bullet, Y_\bullet)$ *of* $(V, \overline{V})$ *which have the disjoint unions of the members of affine simplicial open coverings of* $(U_\bullet, X_\bullet)$ *and* $(V_\bullet, Y_\bullet)$ *over* $(U, \overline{U})$ *and* $(V, \overline{V})$, *respectively, and which fit into the commutative diagram*

$$\begin{array}{ccc} (V_\bullet, Y_\bullet) & \longrightarrow & (U_\bullet, X_\bullet) \\ \big\downarrow & & \big\downarrow \\ (V, \overline{V}) & \longrightarrow & (U, \overline{U}), \end{array}$$

the following diagram

(11.6.2)

$$\begin{array}{ccc} R\Gamma(((\widetilde{Y_\bullet, E_\bullet})/\mathcal{V})^{\log}_{\rm conv}, \mathcal{K}_{(Y_\bullet, E_\bullet)/\mathcal{V}}) & \longleftarrow & R\Gamma(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\rm conv}, \mathcal{K}_{(X_\bullet, D_\bullet)/\mathcal{V}}) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ R\Gamma_{\rm rig}(V/K) & \longleftarrow & R\Gamma_{\rm rig}(U/K) \end{array}$$

is commutative, where $E_\bullet := Y_\bullet \setminus V_\bullet$. *In particular,*

$$R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\rm conv}, \mathcal{K}_{(X_\bullet, D_\bullet)/\mathcal{V}}\big)$$

depends only on U/κ *and* K.

2) *There exists a functorial isomorphism*

$$R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\rm conv}, \mathcal{K}_{(X_\bullet, D_\bullet)/\mathcal{W}}\big) \overset{\sim}{\longrightarrow} R\Gamma\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_{K_0}. \tag{11.6.3}$$

3) *There exists a functorial isomorphism*

$$R\Gamma_{\rm rig}(U/K_0) \overset{\sim}{\longrightarrow} R\Gamma\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_{K_0}. \tag{11.6.4}$$

4) *Let* c *be an integer such that* $H^h_{\rm rig}(U/K_0) = 0$ *for all* $h > c$. *Let* N *be an integer satisfying the inequality* (2.2.1) *for* $h = c$ *in* (2.2.1). *Let* $(U_{\bullet \leq N}, X_{\bullet \leq N})$ *be an* N*-truncated good proper hypercovering of* $(U, \overline{U})$. *Set* $D_t := X_t \setminus U_t$ $(0 \leq t \leq N)$. *Assume that* $X_{\bullet \leq N}$ *is smooth over* κ *and that there exists a closed immersion* $(X_N, D_N) \hookrightarrow (\mathcal{R}, \mathcal{M})$ *into a fine log formally smooth scheme*

over $\mathrm{Spf}(\mathcal{W})$ *such that the underlying p-adic formal* $\mathcal{W}$*-scheme* $\mathcal{R}$ *is formally smooth over* $\mathrm{Spf}(\mathcal{W})$ *around* $X_N \setminus D_N$. *Then there exists an isomorphism*

$$R\Gamma_{\mathrm{rig}}(U/K_0) \xrightarrow{\sim} \tau_c R\Gamma\big((X_{\bullet\leq N}, D_{\bullet\leq N})/\mathcal{W}\big)_{K_0}. \tag{11.6.5}$$

The isomorphism (11.6.5) *is independent of the choice of* N *satisfying the inequality* (2.2.1) *for* $h = c$ *in* (2.2.1). *The isomorphism* (11.6.5) *is functorial.*

Proof. — 1): Let h be a fixed nonnegative integer. Let N be an integer satisfying the inequality (2.2.1). Let $(U_{\bullet\bullet}, X_{\bullet\bullet}) \to (U_\bullet, X_\bullet)$ be the Čech diagram of $(U'_\bullet, X'_\bullet)$ over $(U_\bullet, X_\bullet)/(U, \overline{U})$ ((9.6), (9.7), 2)). Let $(Z, \overline{Z}, \mathcal{Z})$ be a triple as in the beginning of §10. Set $D_{\bullet\bullet} := X_{\bullet\bullet} \setminus U_{\bullet\bullet}$. Consider the fiber product $(U_{\bullet\bullet} \times_U Z, X_{\bullet\bullet} \times_{\overline{U}} \overline{Z})$ and the pair of schemes defined by

$$(X_{\ell mn}, D_{\ell mn}) := (\mathrm{cosk}_0^{X_{\ell m}}(X_{\ell m} \times_{\overline{U}} \overline{Z})_n, \mathrm{cosk}_0^{D_{\ell m}}(D_{\ell m} \times_{\overline{U}} \overline{Z})_n)$$

with $\ell, m, n \in \mathbb{N}$. Then we have a pair $(X_{\bullet\bullet\bullet}, D_{\bullet\bullet\bullet})$ of a smooth trisimplicial scheme with trisimplicial SNCD over κ. Set $U_{\bullet\bullet\bullet} := X_{\bullet\bullet\bullet} \setminus D_{\bullet\bullet\bullet}$. We claim that there exists a closed immersion

$$(X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet}) \hookrightarrow (\mathcal{R}_{\bullet\leq N,\bullet\bullet}, \mathcal{M}_{\bullet\leq N,\bullet\bullet}) \tag{11.6.6}$$

into a formally log smooth (N, ∞, ∞)-truncated trisimplicial log p-adic formal $\mathcal{V}$-scheme such that the underlying trisimplicial formal scheme $\mathcal{R}_{\bullet\leq N,\bullet\bullet}$ is formally smooth over $\mathrm{Spf}(\mathcal{V})$; moreover there exists a morphism $\mathcal{R}_{\bullet\leq N,\bullet\bullet} \to \mathcal{Z}_\bullet$ over $\mathcal{V}$ fitting into the commutative diagram

$$\begin{array}{ccc} X_{\bullet\leq N,\bullet\bullet} & \xrightarrow{\subset} & \mathcal{R}_{\bullet\leq N,\bullet\bullet} \\ \big\downarrow & & \big\downarrow \\ \overline{Z}_\bullet & \xrightarrow{\subset} & \mathcal{Z}_\bullet. \end{array}$$

Indeed, since X_{N0} is the disjoint union of affine open subschemes of X_N, there exists a lift $(\mathcal{X}_{N0}, \mathcal{D}_{N0})$ of (X_{N0}, D_{N0}) over $\mathrm{Spf}(\mathcal{V})$ such that $\mathcal{X}_{N0}$ is a formally smooth scheme over $\mathrm{Spf}(\mathcal{V})$ and such that $\mathcal{D}_{N0}$ is a relative SNCD on $\mathcal{X}_{N0}/\mathrm{Spf}(\mathcal{V})$. The morphism $(X_{\bullet\leq N,0}, D_{\bullet\leq N,0}) \longrightarrow \Gamma_N^{\mathcal{V}}((\mathcal{X}_{N0}, \mathcal{D}_{N0}))_{\bullet\leq N}$ in (6.6), 1) is a closed immersion into a formally log smooth N-truncated simplicial log scheme over $\mathrm{Spf}(\mathcal{V})$. Then we have a closed immersion

$$\begin{aligned}(X_{\bullet\leq N,\bullet}, D_{\bullet\leq N,\bullet}) &:= \big(\mathrm{cosk}_0^{X_{\bullet\leq N}}(X_{\bullet\leq N,0}), \mathrm{cosk}_0^{D_{\bullet\leq N}}(D_{\bullet\leq N,0})\big) \\ &\longrightarrow \mathrm{cosk}_0^{\mathcal{V}}\big(\Gamma_N^{\mathcal{V}}((\mathcal{X}_{N0}, \mathcal{D}_{N0}))_{\bullet\leq N}\big) =: (\mathcal{R}_{\bullet\leq N,\bullet}, \mathcal{M}_{\bullet\leq N,\bullet})\end{aligned} \tag{11.6.7}$$

into a formally log smooth (N,∞)-truncated bisimplicial log p-adic formal $\mathcal{V}$-scheme. Hence we have a closed immersion

(11.6.8)
$$\begin{aligned}(X_{\bullet\le N,\bullet\bullet}, D_{\bullet\le N,\bullet\bullet}) \\ := (\mathrm{cosk}_0^{X_{\bullet\le N,\bullet}}(X_{\bullet\le N,\bullet}\times_{\overline{U}}\overline{Z}), \mathrm{cosk}_0^{D_{\bullet\le N,\bullet}}(D_{\bullet\le N\bullet}\times_{\overline{U}}\overline{Z})) \\ \hookrightarrow \mathrm{cosk}_0^{\mathcal{V}}((\mathcal{R}_{\bullet\le N,\bullet},\mathcal{M}_{\bullet\le N,\bullet})\widehat{\times}_{\mathcal{V}}\mathcal{Z}) =: (\mathcal{R}_{\bullet\le N,\bullet\bullet},\mathcal{M}_{\bullet\le N,\bullet\bullet})\end{aligned}$$

into a formally log smooth (N,∞,∞)-truncated trisimplicial log formal scheme over $\mathrm{Spf}(\mathcal{V})$ such the underlying formal scheme of the log formal scheme on the right hand side is formally smooth over $\mathrm{Spf}(\mathcal{V})$. Thus the claim follows.

For clarity, we summarize the construction above as follows: we have the following commutative diagram of pairs of (tri)simplicial schemes

(11.6.9)
$$\begin{array}{ccccc}(U_{\bullet\le N}, X_{\bullet\le N}) & \longleftarrow & (U_{\bullet\le N,\bullet}, X_{\bullet\le N,\bullet}) & \longleftarrow & (U_{\bullet\le N,\bullet\bullet}, X_{\bullet\le N,\bullet\bullet}) \\ \big\downarrow & & & & \big\downarrow \\ (U,\overline{U}) & & \longleftarrow & & (Z_\bullet,\overline{Z}_\bullet)\end{array}$$

and the following morphism of triples:

(11.6.10)
$$(U_{\bullet\le N,\bullet\bullet}, X_{\bullet\le N,\bullet\bullet}, \mathcal{R}_{\bullet\le N,\bullet\bullet}) \longrightarrow (Z_\bullet,\overline{Z}_\bullet,\mathcal{Z}_\bullet).$$

Let C be an overconvergent isocrystal on $(U,\overline{U})/K$. We claim that the induced morphism

(11.6.11)
$$\begin{aligned}H^h\big(R\Gamma_{\mathrm{rig}}(U/K,C)\big) = H^h\big(R\Gamma(\,]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet},\mathrm{DR}(C))\big) \\ \longrightarrow H^h\big(R\Gamma(\,]X_{\bullet\le N,\bullet\bullet}[_{\mathcal{R}_{\bullet\le N,\bullet\bullet}},\mathrm{DR}(C))\big)\end{aligned}$$

by (11.6.10) is an isomorphism. We prove this claim in the following.

Let $(V_\bullet,\overline{V}_\bullet)$ be a refinement of the proper hypercovering

$$(U_\bullet\times_U Z, X_\bullet\times_{\overline{U}}\overline{Z})$$

of $(Z,\overline{Z})$ such that there exists a closed immersion $\overline{V}_N\hookrightarrow\mathcal{P}_N$ into a separated formally smooth p-adic formal $\mathcal{V}$-scheme (see [**86**, (4.2.2)]). Let $(V_{\bullet\le N,\square\bullet},\overline{V}_{\bullet\le N,\square\bullet},\mathcal{Q}_{\bullet\le N,\square\bullet})$ be the (N,∞)-truncated bisimplicial (formal) schemes defined in (10.9.5) for $\overline{U}_{\bullet\le N}:=X_{\bullet\le N}$. Here $\square$ means a blank. Let $(X_{\bullet\square\bullet},D_{\bullet\square\bullet})$ be a bisimplicial scheme defined by

(11.6.12)
$$\begin{aligned}(X_{\ell\square n}, D_{\ell\square n}) \\ := (\mathrm{cosk}_0^{X_\ell}(X_\ell\times_{\overline{U}}\overline{Z})_n, \mathrm{cosk}_0^{D_\ell}(D_\ell\times_{\overline{U}}\overline{Z})_n) \quad (\ell,n\in\mathbb{N}).\end{aligned}$$

Set $U_{\ell\square n} := X_{\ell\square n} \setminus D_{\ell\square n}$. Since there exists a natural morphism $(V_{\bullet\leq N}, \overline{V}_{\bullet\leq N}) \to (U_{\bullet\leq N} \times_U Z, X_{\bullet\leq N} \times_{\overline{U}} \overline{Z})$, we have a natural morphism

$$(V_{\bullet\leq N,\square\bullet}, \overline{V}_{\bullet\leq N,\square\bullet}) \longrightarrow (U_{\bullet\leq N,\square\bullet}, X_{\bullet\leq N,\square\bullet}). \tag{11.6.13}$$

It is straightforward to check that

$$\big(\mathrm{cosk}_0^{U_{\bullet\square\bullet}}(U_{\bullet\square\bullet} \times_{U_\bullet} U_{\bullet 0}), \mathrm{cosk}_0^{X_{\bullet\square\bullet}}(X_{\bullet\square\bullet} \times_{X_\bullet} X_{\bullet 0})\big) = (U_{\bullet\bullet\bullet}, X_{\bullet\bullet\bullet}). \tag{11.6.14}$$

Set

$$\begin{aligned}(V_{\bullet\leq N,\bullet\bullet}, \overline{V}_{\bullet\leq N,\bullet\bullet}) := \big(&\mathrm{cosk}_0^{V_{\bullet\leq N,\square\bullet}}(V_{\bullet\leq N,\square\bullet} \times_{U_{\bullet\leq N}} U_{\bullet\leq N,0}),\\ &\mathrm{cosk}_0^{\overline{V}_{\bullet\leq N,\square\bullet}}(\overline{V}_{\bullet\leq N,\square\bullet} \times_{X_{\bullet\leq N}} X_{\bullet\leq N,0})\big).\end{aligned} \tag{11.6.15}$$

Then, by (11.6.14), we have the following cartesian diagram

$$\begin{array}{ccc}(V_{\bullet\leq N,\square\bullet}, \overline{V}_{\bullet\leq N,\square\bullet}) & \longleftarrow & (V_{\bullet\leq N,\bullet\bullet}, \overline{V}_{\bullet\leq N,\bullet\bullet})\\ \big\downarrow & & \big\downarrow\\ (U_{\bullet\leq N,\square\bullet}, X_{\bullet\leq N,\square\bullet}) & \longleftarrow & (U_{\bullet\leq N,\bullet\bullet}, X_{\bullet\leq N,\bullet\bullet})\end{array} \tag{11.6.16}$$

of trisimplicial and bisimplicial schemes over κ. Let

$$\overline{V}_{N\square\bullet} \times_{X_N} X_{N0} = \coprod_\lambda \overline{V}_{N\square\bullet\lambda}$$

be the disjoint union induced by the disjoint union $X_{N0} = \coprod_\lambda X_{N0\lambda}$, where $X_{N0\lambda}$ is an open subscheme of X_N. Set

$$\mathcal{Q}'_{N\square\bullet\lambda} := \mathcal{Q}_{N\square\bullet} \setminus (\overline{V}_{N\square\bullet} \setminus \overline{V}_{N\square\bullet\lambda}), \quad \mathcal{Q}_{\ell\square\bullet\lambda} := \Gamma_N^{\mathcal{V}}(\mathcal{Q}'_{N\square\bullet\lambda})_\ell$$

$(0 \leq \ell \leq N)$ and

$$\mathcal{Q}_{\bullet\leq N,\bullet\bullet} := \mathrm{cosk}_0^{\mathcal{V}}\Big(\coprod_\lambda \mathcal{Q}_{\bullet\leq N,\square\bullet\lambda}\Big). \tag{11.6.17}$$

Then we have the closed immersion

$$\overline{V}_{\bullet\leq N,\bullet\bullet} \hookrightarrow \mathcal{Q}_{\bullet\leq N,\bullet\bullet}. \tag{11.6.18}$$

Set $\mathcal{S}_{\bullet\leq N,\bullet\bullet} := \mathcal{Q}_{\bullet\leq N,\bullet\bullet} \widehat{\times}_{\mathcal{V}} \mathcal{R}_{\bullet\leq N,\bullet\bullet}$. Then we have natural closed immersions

$$\overline{V}_{\bullet\leq N,\bullet\bullet} \hookrightarrow \overline{V}_{\bullet\leq N,\bullet\bullet} \times_\kappa X_{\bullet\leq N,\bullet\bullet} \hookrightarrow \mathcal{S}_{\bullet\leq N,\bullet\bullet}.$$

Hence we have the following natural morphisms of triples:

$$\begin{array}{ccc}(V_{\bullet\leq N,\bullet\bullet}, \overline{V}_{\bullet\leq N,\bullet\bullet}, \mathcal{Q}_{\bullet\leq N,\bullet\bullet}) & \longleftarrow & (V_{\bullet\leq N,\bullet\bullet}, \overline{V}_{\bullet\leq N,\bullet\bullet}, \mathcal{S}_{\bullet\leq N,\bullet\bullet})\\ & & \big\downarrow\\ & & (U_{\bullet\leq N,\bullet\bullet}, X_{\bullet\leq N,\bullet\bullet}, \mathcal{R}_{\bullet\leq N,\bullet\bullet}).\end{array} \tag{11.6.19}$$

Since $\mathcal{Q}_{\ell mn}$ and $\mathcal{S}_{\ell mn}$ $(0 \leq \ell \leq N, m, n \in \mathbb{N})$ are formally smooth over $\mathrm{Spf}(\mathcal{V})$, we have a natural isomorphism

$$R\Gamma\big(]\overline{V}_{\bullet \leq N,\bullet\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big) \xrightarrow{\sim} R\Gamma\big(]\overline{V}_{\bullet \leq N,\bullet\bullet}[_{\mathcal{S}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big). \tag{11.6.20}$$

We also have a natural isomorphism

$$R\Gamma\big(]\overline{V}_{\ell\bullet\bullet}[_{\mathcal{Q}_{\ell\bullet\bullet}}, \mathrm{DR}(C)\big) \xrightarrow{\sim} R\Gamma\big(]\overline{V}_{\ell\bullet\bullet}[_{\mathcal{S}_{\ell\bullet\bullet}}, \mathrm{DR}(C)\big) \tag{11.6.21}$$

for each $0 \leq \ell \leq N$. By using the morphism

$$R\Gamma\big(]\overline{V}_{\bullet \leq N,\bullet\bullet}[_{\mathcal{S}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big) \longleftarrow R\Gamma\big(]X_{\bullet \leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big)$$

and by (11.6.20), we have a morphism

$$R\Gamma\big(]\overline{V}_{\bullet \leq N,\bullet\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big) \longleftarrow R\Gamma\big(]X_{\bullet \leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big). \tag{11.6.22}$$

We claim that the morphism (11.6.22) is an isomorphism. Indeed, by the spectral sequence

$$E_1^{ij} := H^j\big(R\Gamma(]\overline{V}_{i\bullet\bullet}[_{\mathcal{Q}_{i\bullet\bullet}}, \mathrm{DR}(C))\big) \Longrightarrow H^{i+j}\big(R\Gamma(]\overline{V}_{\bullet\leq N,\bullet\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C))\big)$$

$(0 \leq i \leq N)$ and the similar spectral sequence of

$$H^h\big(R\Gamma(]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C))\big),$$

we have only to prove that the induced morphism

$$\begin{aligned} &H^j\big(R\Gamma(]\overline{V}_{i\bullet\bullet}[_{\mathcal{Q}_{i\bullet\bullet}}, \mathrm{DR}(C))\big) \\ &\qquad \longleftarrow H^j\big(R\Gamma(]X_{i\bullet\bullet}[_{\mathcal{R}_{i\bullet\bullet}}, \mathrm{DR}(C))\big) \quad (j \in \mathbb{N}) \end{aligned} \tag{11.6.23}$$

is an isomorphism $(0 \leq i \leq N)$. To prove this, let $X_{i0} \hookrightarrow \mathcal{T}_{i0}$ be a closed immersion into a separated formally smooth scheme over $\mathrm{Spf}(\mathcal{V})$. Set $\mathcal{T}_{in} := \mathrm{cosk}_0^{\mathcal{V}}(\mathcal{T}_{i0})_n$. Then we have a closed immersion $X_{i\bullet} \hookrightarrow \mathcal{T}_{i\bullet}$ and

$$R\Gamma_{\mathrm{rig}}(U_i/K, C) = R\Gamma\big(]X_{i\bullet}[_{\mathcal{T}_{i\bullet}}, \mathrm{DR}(C)\big).$$

As we have obtained $\mathcal{Q}_{\bullet\leq N,\bullet\bullet}$, we have a bisimplicial formally smooth formal scheme $\mathcal{T}_{i\bullet\bullet}$ over $\mathrm{Spf}(\mathcal{V})$. Using the fiber product $\mathcal{T}_{i\bullet\bullet}\widehat{\times}_{\mathcal{V}}\mathcal{R}_{i\bullet\bullet}$, we have a natural isomorphism

$$R\Gamma\big(]X_{i\bullet\bullet}[_{\mathcal{T}_{i\bullet\bullet}}, \mathrm{DR}(C)\big) \xrightarrow{\sim} R\Gamma\big(]X_{i\bullet\bullet}[_{\mathcal{R}_{i\bullet\bullet}}, \mathrm{DR}(C)\big).$$

Since $R\Gamma(]X_{i\bullet}[_{\mathcal{T}_{i\bullet}}, \mathrm{DR}(C)) = R\Gamma(]X_{i\bullet\bullet}[_{\mathcal{T}_{i\bullet\bullet}}, \mathrm{DR}(C))$, we have an isomorphism

$$R\Gamma_{\mathrm{rig}}(U_i/K, C) \xrightarrow{\sim} R\Gamma\big(]X_{i\bullet\bullet}[_{\mathcal{R}_{i\bullet\bullet}}, \mathrm{DR}(C)\big).$$

On the other hand, since $(V_{i\square\bullet}, \overline{V}_{i\square\bullet}, \mathcal{Q}_{i\square\bullet})$ is a universally de Rham descendable covering of (U_i, X_i) as in the proof of (10.9), we have an equality

$$R\Gamma\big(]X_{i\bullet}[_{\mathcal{T}_{i\bullet}}, \mathrm{DR}(C)\big) = R\Gamma_{\mathrm{rig}}(U_i/K, C) = R\Gamma\big(]\overline{V}_{i\square\bullet}[_{\mathcal{Q}_{i\square\bullet}}, \mathrm{DR}(C)\big).$$

Using the standard spectral sequence and using the de Rham-descendability of an affine open covering, the right hand side is isomorphic to $R\Gamma\big(]\overline{V}_{i\bullet\bullet}[_{\mathcal{Q}_{i\bullet\bullet}}, \mathrm{DR}(C)\big) = R\Gamma\big(]\overline{V}_{i\bullet\bullet}[_{\mathcal{S}_{i\bullet\bullet}}, \mathrm{DR}(C)\big)$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} R\Gamma_{\mathrm{rig}}(U_i/K, C) & = & R\Gamma(]\overline{V}_{i\square\bullet}[_{\mathcal{Q}_{i\square\bullet}}, \mathrm{DR}(C)) & \xrightarrow{\sim} & R\Gamma(]\overline{V}_{i\bullet\bullet}[_{\mathcal{S}_{i\bullet\bullet}}, \mathrm{DR}(C)) \\ \| & & & & \uparrow \\ R\Gamma_{\mathrm{rig}}(U_i/K, C) & = & R\Gamma(]X_{i\bullet}[_{\mathcal{T}_{i\bullet}}, \mathrm{DR}(C)) & \xrightarrow{\sim} & R\Gamma(]X_{i\bullet\bullet}[_{\mathcal{R}_{i\bullet\bullet}}, \mathrm{DR}(C)). \end{array}$$

Therefore we see that the morphism (11.6.23) is an isomorphism.

By using two morphisms $\mathcal{Q}_{\bullet\leq N,\bullet\bullet} \to \mathcal{Z}_\bullet$ and $\mathcal{R}_{\bullet\leq N,\bullet\bullet} \to \mathcal{Z}_\bullet$, we have two composite morphisms

$$\mathcal{S}_{\bullet\leq N,\bullet\bullet} \longrightarrow \mathcal{Q}_{\bullet\leq N,\bullet\bullet} \longrightarrow \mathcal{Z}_\bullet \quad \text{and} \quad \mathcal{S}_{\bullet\leq N,\bullet\bullet} \longrightarrow \mathcal{R}_{\bullet\leq N,\bullet\bullet} \longrightarrow \mathcal{Z}_\bullet. \tag{11.6.24}$$

Hence we have a natural morphism

$$\mathcal{S}_{\bullet\leq N,\bullet\bullet} \widehat{\times}_{\mathcal{V}} \mathcal{S}_{\bullet\leq N,\bullet\bullet} \longrightarrow \mathcal{Z}_\bullet \widehat{\times}_{\mathcal{V}} \mathcal{Z}_\bullet. \tag{11.6.25}$$

Set

$$\begin{aligned} R\Gamma_1 &:= R\Gamma\big(]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}, \mathrm{DR}(C)\big), \\ R\Gamma_2 &:= R\Gamma\big(]\overline{V}_{\bullet\leq N,\square\bullet}[_{\mathcal{Q}_{\bullet\leq N,\square\bullet}}, \mathrm{DR}(C)\big), \\ R\Gamma_3 &:= R\Gamma\big(]\overline{V}_{\bullet\leq N,\bullet\bullet}[_{\mathcal{Q}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big), \\ R\Gamma_4 &:= R\Gamma\big(]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big), \\ R\Gamma_5 &:= R\Gamma\big(]\overline{V}_{\bullet\leq N,\bullet\bullet}[_{\mathcal{S}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big), \\ R\Gamma_6 &:= R\Gamma\big(]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet \widehat{\times}_{\mathcal{V}} \mathcal{Z}_\bullet}, \mathrm{DR}(C)\big), \\ R\Gamma_7 &:= R\Gamma\big(]\overline{V}_{\bullet\leq N,\bullet\bullet}[_{\mathcal{S}_{\bullet\leq N,\bullet\bullet} \widehat{\times}_{\mathcal{V}} \mathcal{S}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(C)\big). \end{aligned}$$

Here $\overline{Z}_\bullet$ (resp. $\overline{V}_{\bullet\leq N,\bullet\bullet}$) is embedded into $\mathcal{Z}_\bullet \widehat{\times}_{\mathcal{V}} \mathcal{Z}_\bullet$ (resp. $\mathcal{S}_{\bullet\leq N,\bullet\bullet} \widehat{\times}_{\mathcal{V}} \mathcal{S}_{\bullet\leq N,\bullet\bullet}$) diagonally. Then we have the following commutative diagram:

$$\begin{array}{ccccc} R\Gamma_5 & \xrightarrow{\sim} & R\Gamma_7 & \xleftarrow{\sim} & R\Gamma_5 \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma_1 & \xrightarrow{\sim} & R\Gamma_6 & \xleftarrow{\sim} & R\Gamma_1. \end{array} \tag{11.6.26}$$

Here we obtain the left (resp. right) vertical morphism in (11.6.26) by using the first (resp. second) morphism in (11.6.24). We also have the following commutative diagram

$$\begin{array}{ccc} \overline{V}_{\bullet\leq N,\bullet\bullet} & \longrightarrow & \overline{V}_{\bullet\leq N,\square\bullet} \\ \big\downarrow & & \big\downarrow \\ X_{\bullet\leq N,\bullet\bullet} & \longrightarrow & \overline{Z}_{\bullet} \end{array} \tag{11.6.27}$$

and the following (not necessarily commutative) diagram

$$\begin{array}{ccc} \mathcal{S}_{\bullet\leq N,\bullet\bullet} & \longrightarrow & \mathcal{Q}_{\bullet\leq N,\square\bullet} \\ \big\downarrow & & \big\downarrow \\ \mathcal{R}_{\bullet\leq N,\bullet\bullet} & \longrightarrow & \mathcal{Z}_{\bullet}. \end{array} \tag{11.6.28}$$

Then, using (11.6.27), (11.6.28), (11.6.23) and (11.6.26), we have the following commutative diagram

$$\begin{array}{ccccccc} R\Gamma_4 & \overset{\sim}{\longrightarrow} & R\Gamma_5 & \overset{\sim}{\longrightarrow} & R\Gamma_7 & \overset{\sim}{\longleftarrow} & R\Gamma_5 \\ \big\uparrow & & & & & & \big\uparrow \simeq \\ R\Gamma_1 & \longrightarrow & & R\Gamma_2 & & \overset{\sim}{\longrightarrow} & R\Gamma_3. \end{array}$$

Because $R\Gamma_1 \to R\Gamma_2$ induces an isomorphism $H^h(R\Gamma_1) \overset{\sim}{\longrightarrow} H^h(R\Gamma_2)$, which has been proved in the proof of (10.9), the left vertical morphism $R\Gamma_1 \to R\Gamma_4$ induces an isomorphism $H^h(R\Gamma_1) \overset{\sim}{\longrightarrow} H^h(R\Gamma_4)$. In conclusion, the morphism (11.6.11) is an isomorphism.

Let $R\Gamma((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}}))$ be the complex producing log naive cohomologies which are the trisimplicial versions of the log naive cohomologies defined in [**82**, Definition 2.2.12]. Let $j_{\bullet\leq N,\bullet\bullet} \colon U_{\bullet\leq N,\bullet\bullet} \hookrightarrow X_{\bullet\leq N,\bullet\bullet}$ be the open immersion. Then, by the cohomological descent and by the proof of [**82**, Corollary 2.3.9] and [**82**, Theorem 2.4.4], there exists a natural morphism (cf. (11.4.2)):

$$\begin{aligned} & R\Gamma\big(((\widetilde{X_{\bullet\leq N}, D_{\bullet\leq N}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N},D_{\bullet\leq N})/\mathcal{V}}\big) \\ & = R\Gamma\big(((\widetilde{X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N,\bullet\bullet},D_{\bullet\leq N,\bullet\bullet})/\mathcal{V}}\big) \\ & = R\Gamma\big((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}})\big) \\ & \longrightarrow R\Gamma\big((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(j^{\dagger}_{\bullet\leq N,\bullet\bullet}\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}})\big) \\ & = R\Gamma\big(]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(j^{\dagger}_{\bullet\leq N,\bullet\bullet}\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}})\big). \end{aligned} \tag{11.6.29}$$

The induced morphism $H^h((11.6.29))$ is an isomorphism. Indeed, set $E^{ts}_{\mathrm{conv},1} := H^s(((\widetilde{X_t, D_t})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_t,D_t)/\mathcal{V}})$ for $t \leq N$ and $E^{ts}_{\mathrm{conv},1} := 0$ for

$t > N$. Set also $E^{ts}_{\mathrm{rig},1} := H^s_{\mathrm{rig}}(U_t/K)$ for $t \leq N$ and $E^{ts}_{\mathrm{rig},1} := 0$ for $t > N$. Then we have the following two spectral sequences

$$(11.6.30)\qquad E^{ts}_{\mathrm{conv},1} \Longrightarrow H^{s+t}\big(R\Gamma((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}})))\big)$$

$$(11.6.31)\qquad E^{ts}_{\mathrm{rig},1} \Longrightarrow H^{s+t}(R\Gamma(]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(j^{\dagger}_{\bullet\leq N,\bullet}\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}}))).$$

By Shiho's comparison theorems (see [**82**, Corollary 2.3.9, Theorem 2.4.4]) and (11.5), 4), the induced morphism

$$H^s\big(((\widetilde{X_t, D_t})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_t,D_t)/\mathcal{V}}\big) \longrightarrow H^s_{\mathrm{rig}}(U_t/K)$$

between the E_1-terms of (11.6.31) and (11.6.30) for $t \leq N$ by the morphism (11.6.29) is an isomorphism. Hence $H^h((11.6.29))$ is an isomorphism.

So far we have proved that, for any nonnegative integer h, there exists a positive integer N satisfying the inequality (2.2.1) such that there exists an isomorphism

$$(11.6.32)\qquad \tau_h R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}\big) = \tau_h R\Gamma\big(((\widetilde{X_{\bullet\leq N}, D_{\bullet\leq N}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N},D_{\bullet\leq N})/\mathcal{V}}\big) \xrightarrow{\sim} \tau_h R\Gamma_{\mathrm{rig}}(U/K).$$

By (10.1), there exists an integer c such that $H^i_{\mathrm{rig}}(U/K) = 0$ for $i > c$. Hence $R\Gamma_{\mathrm{rig}}(U/K) = \tau_c R\Gamma_{\mathrm{rig}}(U/K)$. By (11.6.32), for $i > c$,

$$H^i\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}\big) = 0.$$

Hence

$$R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}\big) = \tau_c R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}\big).$$

Using (11.6.32) for $h = c$, we have the isomorphism (11.6.1).

The isomorphism (11.6.1) is independent of the choice of the lift $(\mathcal{X}_{N0}, \mathcal{D}_{N0})$ by using the product of two log formal schemes and by (11.5), 4) (we may use the construction of a standard exactification in [**44**, 2]). By the proof of (9.8), 2) and by using the product of log formal schemes and by (11.5), 4) again, the isomorphism (11.6.1) is independent of the choice of the disjoint union of the members of an affine simplicial open covering of $(U_\bullet, X_\bullet)$. Since we need only the isomorphisms (11.6.11) and $H^h((11.6.29))$, the isomorphism (11.6.1) is independent of the choice of the closed immersion $V_N \hookrightarrow \mathcal{P}_N$ and the refinement $(V_\bullet, \overline{V}_\bullet)$. Moreover, as in the proof of (10.9), we can see that the isomorphism (11.6.1) is independent of the choice of $(Z_\bullet, \overline{Z}_\bullet, \mathcal{Z}_\bullet)$. Note also that the isomorphism (11.6.1) is independent of the choice of N satisfying (2.2.1) for $h=c$. Indeed, if we take two integers $N'\geq N$, we have two isomorphisms

$$i_N, i_{N'}\colon \tau_c R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}\big) \xrightarrow{\sim} \tau_c R\Gamma_{\mathrm{rig}}(U/K),$$

where i_N and $i_{N'}$ are constructed by the use of lifts $(\mathcal{X}_{N0},\mathcal{D}_{N0})$ and $(\mathcal{X}_{N'0},\mathcal{D}_{N'0})$ of (X_{N0},D_{N0}) and $(X_{N'0},D_{N'0})$, respectively. By (11.5), 4) and by using the natural closed immersion

$$(X_{\bullet\leq N,0}, D_{\bullet\leq N,0}) \hookrightarrow \Gamma^{\mathcal{V}}_N\big((\mathcal{X}_{N0},\mathcal{D}_{N0})\big)_{\bullet\leq N}\widehat{\times}_{\mathcal{V}}\Gamma^{\mathcal{V}}_{N'}\big((\mathcal{X}'_{N'0},\mathcal{D}'_{N'0})\big)_{\bullet\leq N},$$

we obtain the following commutative diagram:

$$\begin{array}{ccc}
\tau_c R\Gamma(((\widetilde{X_{\bullet\leq N'}, D_{\bullet\leq N'}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N'},D_{\bullet\leq N'})/\mathcal{V}}) & \overset{i_{N'}}{\underset{\sim}{\longrightarrow}} & \tau_c R\Gamma_{\mathrm{rig}}(U/K) \\
\big\downarrow & & \big\| \\
\tau_c R\Gamma(((\widetilde{X_{\bullet\leq N}, D_{\bullet\leq N}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N},D_{\bullet\leq N})/\mathcal{V}}) & \overset{i_{N}}{\underset{\sim}{\longrightarrow}} & \tau_c R\Gamma_{\mathrm{rig}}(U/K).
\end{array}$$

Thus we have proved the existence of the isomorphism (11.6.1).

Next we prove the functoriality.

By the remark for a ringed space in the proof of (6.1), there exists the following commutative diagram

$$\begin{array}{ccc}
\overline{W} & \longrightarrow & \overline{Z} \\
\big\downarrow & & \big\downarrow \\
\overline{V} & \longrightarrow & \overline{U},
\end{array} \tag{11.6.33}$$

where $\overline{Z}$ and $\overline{W}$ are disjoint unions of the members of affine open coverings of $\overline{U}$ and $\overline{V}$, respectively. Set $Z:=\overline{Z}\times_{\overline{U}}U$ and $W:=\overline{W}\times_{\overline{V}}V$. By (10.3), there exists the following commutative diagram

$$\begin{array}{ccc}
\overline{W} & \overset{\subset}{\longrightarrow} & \mathcal{W} \\
\big\downarrow & & \big\downarrow \\
\overline{Z} & \overset{\subset}{\longrightarrow} & \mathcal{Z},
\end{array}$$

where the horizontal morphisms are closed immersions into formally smooth p-adic formal $\mathcal{V}$-schemes and $\mathcal{W}$ is not the Witt ring until the end of the proof of 1). Set $\mathcal{W}_\bullet:=\mathrm{cosk}^{\mathcal{V}}_0(\mathcal{W})$. Using (9.8), 1), the proof of (11.4), 1) and the construction of the closed immersion (11.6.7), we have the following commutative diagram

$$
\begin{array}{ccc}
(Y_{\bullet\leq N,\bullet\bullet}, E_{\bullet\leq N,\bullet\bullet}) & \xrightarrow{\subset} & (\mathcal{T}_{\bullet\leq N,\bullet\bullet}, \mathcal{N}_{\bullet\leq N,\bullet\bullet}) \\
\big\downarrow & & \big\downarrow \\
(X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet}) & \xrightarrow{\subset} & (\mathcal{R}_{\bullet\leq N,\bullet\bullet}, \mathcal{M}_{\bullet\leq N,\bullet\bullet}),
\end{array}
\tag{11.6.34}
$$

where the upper horizontal morphism is a closed immersion into a formally log smooth (N,∞,∞)-truncated trisimplicial log scheme whose underlying (N,∞,∞)-truncated trisimplicial formal scheme is formally smooth over $\mathcal{V}$. Here we can assume that the right vertical morphism in (11.6.34) is a morphism over the morphism $\mathcal{W}_\bullet \longrightarrow \mathcal{Z}_\bullet$. Let c be a positive integer such that $H^h_{\mathrm{rig}}(U/K) = 0 = H^h_{\mathrm{rig}}(V/K)$ for all $h > c$. Then we have the following commutative diagram

$$
\begin{array}{cc}
\tau_c R\Gamma(((Y_{\bullet\leq N,\bullet\bullet}, \widetilde{E_{\bullet\leq N,\bullet\bullet}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(Y_{\bullet\leq N,\bullet\bullet},E_{\bullet\leq N,\bullet\bullet})/\mathcal{V}})_K & \xrightarrow{\sim} \\
\big\uparrow & \\
\tau_c R\Gamma(((X_{\bullet\leq N,\bullet\bullet}, \widetilde{D_{\bullet\leq N,\bullet\bullet}})/\mathcal{V})^{\log}_{\mathrm{conv}}, \mathcal{K}_{(X_{\bullet\leq N,\bullet\bullet},D_{\bullet\leq N,\bullet\bullet})/\mathcal{V}})_K & \xrightarrow{\sim} \\
& \\
\tau_c R\Gamma((Y_{\bullet\leq N,\bullet\bullet}, E_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(\mathcal{O}_{]Y_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{T}_{\bullet\leq N,\bullet\bullet},\mathcal{N}_{\bullet\leq N,\bullet\bullet})}})) & \xrightarrow{\sim} \\
\big\uparrow & \\
\tau_c R\Gamma((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K, \mathrm{LDR}(\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}})) & \xrightarrow{\sim} \\
& \\
\tau_c R\Gamma(\,]Y_{\bullet\leq N,\bullet\bullet}[_{\mathcal{T}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(j^\dagger_{\bullet\leq N,\bullet\bullet}\mathcal{O}_{]Y_{\bullet\leq N,\bullet\bullet}[_{\mathcal{T}_{\bullet\leq N,\bullet\bullet}}}))) & \xleftarrow{\sim} \\
\big\uparrow & \\
\tau_c R\Gamma(\,]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}, \mathrm{DR}(j^\dagger_{\bullet\leq N,\bullet\bullet}\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}})) & \xleftarrow{\sim} \\
& \\
R\Gamma(\,]\overline{W}_\bullet[_{\mathcal{W}_\bullet}, \mathrm{DR}(j^\dagger_\bullet\mathcal{O}_{]\overline{W}_\bullet[_{\mathcal{W}_\bullet}})) & \\
\big\uparrow & \\
R\Gamma(\,]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}, \mathrm{DR}(j^\dagger_\bullet\mathcal{O}_{]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}})). &
\end{array}
\tag{11.6.35}
$$

Thus we obtain the functoriality of the isomorphism (11.6.1).

2): Let $\mathcal{A}_{\bullet\leq N,\bullet\bullet}$ be the structure sheaf of the log PD-envelope of the closed immersion $(X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet}) \hookrightarrow (\mathcal{R}_{\bullet\leq N,\bullet\bullet}, \mathcal{M}_{\bullet\leq N,\bullet\bullet})$ over $(\mathrm{Spf}(\mathcal{W}), p\mathcal{W}, [\ ])$.

By the proof of [**82**, Theorem 3.1.1] (cf. the proof of (11.4)) and by (11.5), 4), we have a morphism

$$
\begin{aligned}
(11.6.36)\qquad & R\Gamma\big((X_{\bullet\leq N,\bullet\bullet}, D_{\bullet\leq N,\bullet\bullet})/K_0, \mathrm{LDR}(\mathcal{O}_{]X_{\bullet\leq N,\bullet\bullet}[^{\log}_{(\mathcal{R}_{\bullet\leq N,\bullet\bullet},\mathcal{M}_{\bullet\leq N,\bullet\bullet})}})\big) \\
& \longrightarrow R\Gamma\big(X_{\bullet\leq N,\bullet\bullet}, \mathcal{A}_{\bullet\leq N,\bullet\bullet}\otimes_{\mathcal{O}_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}}}\Omega^{\bullet}_{\mathcal{R}_{\bullet\leq N,\bullet\bullet}/\mathcal{W}}(\log\mathcal{M}_{\bullet\leq N,\bullet\bullet})\big)_{K_0}.
\end{aligned}
$$

By the usual cohomological descent, the canonical filtration τ_h on the target is nothing but $\tau_h R\Gamma((X_\bullet, D_\bullet)/\mathcal{W})_{K_0}$.

As in the case of (11.6.29), we see that the induced morphism $H^h((11.6.36))$ is an isomorphism. Indeed, this follows from standard spectral sequences and from Shiho's comparison theorem (see [**82**, Theorem 3.1.1]).

As in the proof of (11.6.1), we have the isomorphism (11.6.3).

As in the proof of (11.6.1), we can prove that the isomorphism (11.6.3) is independent of the choice of the lift $(\mathcal{X}_{N0}, \mathcal{D}_{N0})$. Note also that the isomorphism (11.6.3) is independent of the choice of N satisfying (2.2.1) for $h = c$ as in the proof of (11.6.1).

By the same proof as that of 1), we obtain the functoriality of (11.6.3). Thus we have proved 2).

3): Follows from 1) and 2).

4): By the proof of 1) and 2), we obtain 4); the proof of 4) is easier than that of 1) + 2) because we can take $(V_{\bullet\leq N}, \overline{V}_{\bullet\leq N})$ (resp. $\mathcal{P}_N$) as

$$(U_{\bullet\leq N}\times_U Z, X_{\bullet\leq N}\times_{\overline{U}}\overline{Z})$$

(resp. the disjoint union of open subschemes of $\mathcal{R}$: if $\overline{V}_N\times_{\overline{U}}\overline{Z} = \coprod_{i=1}^m \overline{V}_{Ni}$ is the disjoint union induced by the disjoint union $\overline{Z} = \coprod_{i=1}^m \overline{Z}_i$, then we have only to set $\mathcal{P}_N := \coprod_{i=1}^m \mathcal{R}\setminus(\overline{V}_N\setminus\overline{V}_{Ni})$.) □

Corollary 11.7. — *The following hold*:

1) *Let the notations and the assumptions be as in* (11.6), 1). *Then there exists a canonical isomorphism*

$$(11.7.1)\qquad R\Gamma_{\mathrm{rig}}(U/K)\xrightarrow{\sim} R\Gamma\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_K.$$

In particular, $R\Gamma((X_\bullet, D_\bullet)/\mathcal{W})_K$ *depends only on* U *and* K, *and there exists canonical isomorphisms*

$$(11.7.2)\qquad H^h_{\mathrm{rig}}(U/K)\xrightarrow{\sim} H^h\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_K\quad (h\in\mathbb{Z}).$$

In particular, there exists the following spectral sequence

$$(11.7.3)\qquad E_1^{ts} = H^s\big((X_t, D_t)/\mathcal{W}\big)_K \Longrightarrow H^{t+s}_{\mathrm{rig}}(U/K).$$

2) *Assume that* κ *is perfect. Let* $(U^i_\bullet, X^i_\bullet)$ $(i = 1, 2)$ *be two good proper hypercoverings of* $(U, \overline{U})$ *which have the disjoint unions of the members of affine simplicial open coverings of* $(U^i_\bullet, X^i_\bullet)$. *Set* $D^i_\bullet := X^i_\bullet \setminus U^i_\bullet$. *Then there exists an isomorphism*

$$\rho_{12}\colon R\Gamma\big((X^2_\bullet, D^2_\bullet)/\mathcal{W}\big)_K \xrightarrow{\sim} R\Gamma\big((X^1_\bullet, D^1_\bullet)/\mathcal{W}\big)_K \tag{11.7.4}$$

satisfying the cocycle condition; that is, for another good proper hypercovering $(U^3_\bullet, X^3_\bullet)$ *of* $(U, \overline{U})$ *which has the disjoint union of the members of an affine simplicial open covering of* $(U^3_\bullet, X^3_\bullet)$, *the formula* $\rho_{12} \circ \rho_{23} = \rho_{13}$ *holds.*

3) *Let the assumptions be as in* (11.6), 4). *Then there exists a canonical isomorphism*

$$R\Gamma_{\mathrm{rig}}(U/K) \xrightarrow{\sim} \tau_c R\Gamma\big((X_{\bullet \le N}, D_{\bullet \le N})/\mathcal{W}\big)_K. \tag{11.7.5}$$

In particular, $\tau_c R\Gamma((X_{\bullet \le N}, D_{\bullet \le N})/\mathcal{W})_K$ *depends only on* U *and* K.

Proof. — 1): By (10.17) we may assume that $\mathcal{V} = \mathcal{W}$. Hence 1) follows from (11.6), 3).

2): Set $\mathfrak{U}_i = (U^i_\bullet, X^i_\bullet)$ $(i = 1, 2, 3)$. Note that $X^i_\bullet$ is smooth over κ since κ is perfect. By (9.9), 2) and 1), we may assume that $\mathfrak{U}_i$ is split and there exists a canonical isomorphism

$$\rho_i\colon R\Gamma_{\mathrm{rig}}(U/K) \xrightarrow{\sim} R\Gamma\big((X^i_\bullet, D^i_\bullet)/\mathcal{W}\big)_K. \tag{11.7.6}$$

By (9.4), 1), there exists a gs proper hypercovering $\mathfrak{U}_{12} = (U^{12}_\bullet, X^{12}_\bullet)$ of $(U, \overline{U})$ which covers $(U^i_\bullet, X^i_\bullet)$ $(i = 1, 2)$. Set $D^{12}_\bullet := X^{12}_\bullet \setminus U^{12}_\bullet$. Let $\rho'_{12}\colon R\Gamma_{\mathrm{rig}}(U/K) \xrightarrow{\sim} R\Gamma((X^{12}_\bullet, D^{12}_\bullet)/\mathcal{W})_K$ be the isomorphism in 1). By the functoriality in (11.6), 3), we have the commutative diagram

$$\begin{array}{ccc} R\Gamma_{\mathrm{rig}}(U/K) & \xrightarrow[\sim]{\rho_1} & R\Gamma((X^1_\bullet, D^1_\bullet)/\mathcal{W})_K \\ \| & & \downarrow \\ R\Gamma_{\mathrm{rig}}(U/K) & \xrightarrow[\sim]{\rho'_{12}} & R\Gamma((X^{12}_\bullet, D^{12}_\bullet)/\mathcal{W})_K \\ \| & & \uparrow \\ R\Gamma_{\mathrm{rig}}(U/K) & \xrightarrow[\sim]{\rho_2} & R\Gamma((X^2_\bullet, D^2_\bullet)/\mathcal{W})_K. \end{array} \tag{11.7.7}$$

Thus we have an isomorphism ρ_{12}. Furthermore, we have a pair $\mathfrak{U}_{23}$ which covers $\mathfrak{U}_2$ and $\mathfrak{U}_3$, and then we have a pair $\mathfrak{U}_{123}$ which covers $\mathfrak{U}_{12}$ and $\mathfrak{U}_{23}$. Thus we have 2).

3): Follows immediately from (11.6), 4) and (10.17). □

Corollary 11.8. — *Let κ be a field of characteristic $p>0$. Let $\sigma \in \mathrm{Aut}(\mathcal{V})$ be a lift of the p-th power endomorphism of κ. Let C be a nilpotent F-isocrystal on U/κ. Then the rigid cohomology $H^h_{\mathrm{rig}}(U/K, C)$ $(h \in \mathbb{Z})$ is an F-isocrystal over K.*

Proof. — (Compare this proof with that of [**86**, (5.1.1) (2)].) By the five lemma, we may assume that C is trivial. By the base change of rigid cohomologies of trivial coefficients on U ((10.17)), we may assume that $\mathcal{V} = \mathcal{W}$. Then (11.8) immediately follows from (11.7), from the following spectral sequence (cf. [**65**, (3.1)], [**69**, (9.3.2)])

$$(11.8.1) \quad E_1^{-k,h+k} = H^{h-k}\big((\widetilde{D_i^{(k)}/\mathcal{W}})_{\mathrm{crys}}, \mathcal{O}_{D_i^{(k)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D_i/\mathcal{W})\big)(-k)_{K_0} \Longrightarrow H^h\big((X_i, D_i)/\mathcal{W}\big)_{K_0}$$

and from the Poincaré duality of the crystalline cohomology of a proper smooth scheme over κ. (Instead of using (11.8.1), we may use the Poincaré duality of Tsuji [**85**].) □

Remark 11.9. — The right hand side of (11.7.1) depends only on U/κ and K; this solves a problem raised in [**50**, Introduction] for the case where the pair of the simplicial schemes has the disjoint union of the members of an affine simplicial open covering (especially the split case) in a stronger form. Moreover, if κ is perfect, then (12.5) (resp. (13.2)) below tells us that the weight filtration on $H^h((X_\bullet, D_\bullet)/\mathcal{W})_K$ (resp. the log Hodge-Witt cohomology $H^j(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))_{K_0}$ $(i, j \in \mathbb{N})$) depends only on U/κ and K (resp. U/κ).

The generalization (11.6) of (1.0.17) is necessary for the following:

Corollary 11.10. — *Let $j^i \colon U^i \hookrightarrow \overline{U}^i$ $(i=1,2)$ be an open immersion from a separated scheme of finite type over κ into a proper scheme over κ. Let $(U^i_\bullet, X^i_\bullet)$ be a good proper hypercovering of $(U^i, \overline{U}^i)$ which has the disjoint union of the members of an affine simplicial open covering of $(U^i_\bullet, X^i_\bullet)$ over $(U^i, \overline{U}^i)$. Set $D^i_\bullet := X^i_\bullet \setminus U^i_\bullet$. Then there exist the canonical isomorphisms* (11.6.1), (11.6.3) *and* (11.7.1) *for $U^1 \times_\kappa U^2$ and $(X^1_\bullet \times_\kappa X^2_\bullet, (D^1_\bullet \times_\kappa X^2_\bullet) \cup (X^1_\bullet \times_\kappa D^2_\bullet))$.*

Proof. — Set $(U^{12}_\bullet, X^{12}_\bullet) := (U^1_\bullet \times_\kappa U^2_\bullet, X^1_\bullet \times_\kappa X^2_\bullet)$. Then, by (9.4), 4), $(U^{12}_\bullet, X^{12}_\bullet)$ is a good proper hypercovering of $(U^1 \times_\kappa U^2, \overline{U}^1 \times_\kappa \overline{U}^2)$. Let $(U^{i\prime}_\bullet, X^{i\prime}_\bullet)$ be the disjoint union of the members of an affine simplicial open covering of $(U^i_\bullet, X^i_\bullet)$ over $(U^i, \overline{U}^i)$. Then the pair $(U^{1\prime}_\bullet \times_\kappa U^{2\prime}_\bullet, X^{1\prime}_\bullet \times_\kappa X^{2\prime}_\bullet)$ is the disjoint union of the members of an affine simplicial open covering of $(U^{12}_\bullet, X^{12}_\bullet)$. Hence (11.10) follows. □

I do not know whether the assumption on the embedding in (11.6), 4) is mild:

Problem 11.11. — Let (X, D) be a projective smooth scheme with an SNCD over κ. Does there exist a closed immersion $(X, D) \hookrightarrow (\mathcal{P}, \mathcal{M})$ into a formally log smooth log p-adic formal $\mathcal{V}$-scheme whose underlying formal scheme $\mathcal{P}$ is also formally smooth over $\mathrm{Spf}(\mathcal{V})$?

The following is interesting:

Problem 11.12. — Does there exist a condition under which the induced $\mathcal{W}$-integral structure on $H^h_{\mathrm{rig}}(U/K)$ by the isomorphism (11.7.2) is independent of the choice of the good proper hypercovering of $(U, \overline{U})$ which has the disjoint union of the members of an affine simplicial open covering? For the case $h = 1$ and $p \geq 3$ and under a condition that the augmentation morphism $X_0 \setminus D_0 \to U$ is generically etale, the $\mathcal{W}$-integral structure is independent of the choice by [**1**, (7.5.4)] and the functoriality of the isomorphism (11.7.2).

More generally, the comparison theorems in (11.6) holds for a certain F-isocrystal by using Shiho's comparison theorems. Before stating the generalization, we prove the following:

Lemma 11.13. — 1) *Let $f\colon X \to (\mathrm{Spec}(\kappa), L_1)$ be a morphism of fine (not necessarily fs) log schemes. Assume that $\overset{\circ}{X} \to \mathrm{Spec}(\kappa)$ is of finite type. Let $(\mathrm{Spec}(\kappa), L_1) \hookrightarrow (\mathrm{Spf}(\mathcal{V}), L)$ be an exact closed immersion of fine log schemes. Let $U := X_{f\text{-triv}}$ be the maximal log open subscheme of X whose log structure is the pull-back of L_1* (see [**81**, 2.3]). *Let*

$$j^\dagger_X\colon \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}\big(X/(\mathcal{V}, L)\big) := \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv,zar}}\big(X/(\mathcal{V}, L)\big) \longrightarrow \mathrm{Isoc}^\dagger\big((U, X)/K\big)$$

be the functor defined in [**82**, Proposition 2.4.1]. *Then the following hold*:

(a) *$j^\dagger_X$ is compatible with the operations $\oplus$, $\otimes$, $\mathcal{H}om$: $j^\dagger_X$ is a morphism of rigid abelian K-linear tensor categories.*

(b) *Let*

$$\begin{array}{ccc} (\mathrm{Spec}(\kappa'), L'_1) & \longrightarrow & (\mathrm{Spec}(\kappa), L_1) \\ \cap\big\downarrow & & \big\downarrow\cap \\ (\mathrm{Spf}(\mathcal{V}'), L') & \longrightarrow & (\mathrm{Spf}(\mathcal{V}), L) \end{array}$$

be a commutative diagram of fine log schemes, where the left vertical morphism is an exact closed immersion and $\mathcal{V}'$ is a complete discrete *valuation ring of mixed characteristics with residue field* κ'. *Set* $K' := \operatorname{Frac}\mathcal{V}'$. *For a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (\operatorname{Spec}(\kappa'), L'_1) & \longrightarrow & (\operatorname{Spec}(\kappa), L_1) \end{array}$$

of fine log schemes, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}(X'/(\mathcal{V}', L')) & \xrightarrow{j^{\dagger}_{X'}} & \mathrm{Isoc}^{\dagger}((U', X')/K') \\ {\scriptstyle g^*}\big\uparrow & & \big\uparrow{\scriptstyle g^*} \\ \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}(X/(\mathcal{V}, L)) & \xrightarrow{j^{\dagger}_{X}} & \mathrm{Isoc}^{\dagger}((U, X)/K) \end{array}$$

Here $U' := X'_{f'\text{-triv}}$ *and* $\overset{\circ}{f'}$ *is assumed to be of finite type.*

(c) *Let* $f^i \colon X^i \longrightarrow (\operatorname{Spec}(\kappa), L_1)$ $(i = 1, 2)$ *be a morphism of fine* (*not necessarily fs*) *log schemes. Set* $U^i := X^i_{f^i\text{-triv}}$. *Let* $f^3 \colon X^3 := X^1 \times_{(\operatorname{Spec}(\kappa), L_1)} X^2 \to (\operatorname{Spec}(\kappa), L_1)$ *be the structural morphism and set* $U^3 := X^3_{f^3\text{-triv}}$. *Then the following diagram is commutative:*

$$\begin{array}{ccc} \Pi_{i=1,2}\mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}(X^i/(\mathcal{V}, L)) & \xrightarrow{j^{\dagger}_{X^1}\times j^{\dagger}_{X^2}} & \Pi_{i=1,2}\mathrm{Isoc}^{\dagger}((U^i, X^i)/K) \\ {\scriptstyle \boxtimes}\big\downarrow & & \big\downarrow{\scriptstyle \boxtimes} \\ \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}(X^3/(\mathcal{V}, L)) & \xrightarrow{j^{\dagger}_{X^3}} & \mathrm{Isoc}^{\dagger}((U^3, X^3)/K). \end{array}$$

2) *Let* $(\operatorname{Spec}(\kappa), L_1) \hookrightarrow (\operatorname{Spf}(\mathcal{W}), L)$ *be an exact closed immersion of fine log schemes. Let* $X \to (\operatorname{Spec}(\kappa), L_1)$ *be a morphism of fine log schemes. Let*

$$\Xi_X \colon \mathrm{Isoc}_{\mathrm{conv}}\big(X/(\mathcal{W}, L)\big) := \mathrm{Isoc}_{\mathrm{conv,zar}}\big(X/(\mathcal{W}, L)\big) \longrightarrow \mathrm{Isoc}_{\mathrm{crys}}\big(X/(\mathcal{W}, L)\big)$$

be the functor defined in [**81**, Theorem 5.3.1] *and denoted by* Φ *in* [loc. cit.] (*cf.* [**82**, 3.1]). *Then the obvious analogues of* (a), (b) and (c) *in* 1) *hold.*

3) *Let* N *be a nonnegative integer. The* N*-truncated simplicial versions of* 1) *and* 2) *hold.*

Proof. — 1): Follows from the construction of the functor $j^{\dagger}_X$; $j^{\dagger}_X$ is locally defined by the equivalence of the functor [**82**, Proposition 2.2.7], by restricting an object of $\mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}(X/(\mathcal{V}, L))$ on a log tube to a strict neighborhood of the tube of U in the log tube of X and by taking the direct image of the restricted object to the tube of X (see [**82**, Proposition 2.4.1]).

2): Follows from the construction of the functor Ξ_X; essentially Ξ_X is defined by the pull-back of the local morphism from a log divided power scheme to a universal log enlargement (see [**81**, Theorem 5.3.1]).

3): Follows immediately from 1) and 2). □

Shiho's conjecture [**82**, Conjecture 3.1.8] (see also (11.17) below) motivates us to consider the following statement:

THEOREM 11.14 (Comparison theorems). — 1) *Let the notations be as in* (11.6), 1)*. Let C be an overconvergent F-isocrystal on $(U,\overline{U})/K$. Let*

$$\pi^*_{(U,U_\bullet)}\colon \mathrm{Isoc}^\dagger\big((U,\overline{U})/K\big)\longrightarrow \mathrm{Isoc}^\dagger\big((U_\bullet,X_\bullet)/K\big)$$

be the natural pull-back functor. Let $C^\bullet_{\mathrm{conv}}$ be a (locally free) F-isocrystal in the log convergent topoi $((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}=((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv,zar}}$, which is the simplicial version of the topos in [**82**, Definition 2.1.6]*. Let*

$$j^\dagger_{(U_\bullet,X_\bullet)}\colon \mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}\big(((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})\big)\longrightarrow \mathrm{Isoc}^\dagger\big((U_\bullet,X_\bullet)/K\big)$$

be the simplicial version of the functor defined in [**82**, Proposition 2.4.1]*. Assume that*

$$\pi^*_{(U,U_\bullet)}(C)=j^\dagger_{(U_\bullet,X_\bullet)}(C^\bullet_{\mathrm{conv}}). \tag{11.14.1}$$

Then there exists a functorial isomorphism

$$R\Gamma\big(((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}},C^\bullet_{\mathrm{conv}}\big)\xrightarrow{\ \sim\ } R\Gamma_{\mathrm{rig}}(U/K,C). \tag{11.14.2}$$

Here the functoriality means the functoriality for the commutative diagram (11.6), 1) *and morphisms of C's and $C^\bullet_{\mathrm{conv}}$'s satisfying the obvious commutative diagram. The isomorphism* (11.14.2) *is compatible with the operations $\oplus$, $\otimes$ and $\mathcal{H}om$ for C's and $C^\bullet_{\mathrm{conv}}$'s. For example, there exists the following commutative diagram:*

(11.14.3)

$$\begin{array}{ccc}
R\Gamma(((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}},C^\bullet_{1\mathrm{conv}})\otimes_K R\Gamma(((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}},C^\bullet_{2\mathrm{conv}}) & \xrightarrow{\ \sim\ } & R\Gamma_{\mathrm{rig}}(U/K,C_1)\otimes_K R\Gamma_{\mathrm{rig}}(U/K,C_2)\\
\big\downarrow & & \big\downarrow{\cup}\\
R\Gamma(((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}},C^\bullet_{1\mathrm{conv}}\otimes_{\mathcal{K}_{(X_\bullet,D_\bullet)/\mathcal{V}}}C^\bullet_{2\mathrm{conv}}) & \xrightarrow{\ \sim\ } & R\Gamma_{\mathrm{rig}}(U/K,C_1\otimes C_2).
\end{array}$$

2) *Let* $U^i \hookrightarrow \overline{U}^i$ $(i=1,2)$ *be an open immersion from a separated scheme of finite type over* κ *into a proper scheme over* κ. *Let* $(U^i_\bullet, X^i_\bullet)$ *be a good proper hypercovering of* $(U^i, \overline{U}^i)$ *which has the disjoint union of the members of an affine simplicial open covering of* $(U^i_\bullet, X^i_\bullet)$ *over* $(U^i, \overline{U}^i)$. *Let* C_i *be an overconvergent* F*-isocrystal on* $(U^i, \overline{U}^i)$. *Let* $C^\bullet_{i\mathrm{conv}}$ *be a (locally free) log convergent* F*-isocrystal satisfying* (11.14.1) *for* C_i, U^i *and* $(U^i_\bullet, X^i_\bullet)$. *Set* $U^{12} := U^1 \times_\kappa U^2$ *and* $(U^{12}_\bullet, X^{12}_\bullet) := (U^1_\bullet \times_\kappa U^2_\bullet, X^1_\bullet \times_\kappa X^2_\bullet)$. *Set also* $D^i_\bullet := X^i_\bullet \setminus U^i_\bullet$ $(i=1,2,12)$. *Then the following diagram is commutative:*

(11.14.4)
$$\begin{array}{ccc}
\otimes^2_{K,i=1} R\Gamma(((\widetilde{X^i_\bullet, D^i_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, C^\bullet_{i\mathrm{conv}}) & \xrightarrow{\sim} & \otimes^2_{K,i=1} R\Gamma_{\mathrm{rig}}(U^i/K, C_i) \\
\cup\big\downarrow & & \big\downarrow\cup \\
R\Gamma(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, C^\bullet_{1\mathrm{conv}} \boxtimes C^\bullet_{2\mathrm{conv}}) & \xrightarrow{\sim} & R\Gamma_{\mathrm{rig}}(U^{12}/K, C_1 \boxtimes C_2).
\end{array}$$

3) *Let* $(X_\bullet, D_\bullet)$ *be a proper smooth simplicial scheme with a simplicial SNCD over* κ *such that* $(X_\bullet \setminus D_\bullet, X_\bullet)$ *has the disjoint union of the members of an affine simplicial open covering of* $(X_\bullet \setminus D_\bullet, X_\bullet)$ *over* κ. *Let* $C^\bullet_{\mathrm{conv}}$ *be a (locally free)* F*-isocrystal in the log convergent topoi* $((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}} = ((\widetilde{X_\bullet, D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv,zar}}$. *Let*

$$\Xi_{(X_\bullet, D_\bullet)} \colon \mathrm{Isoc}\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{conv}}\big) \longrightarrow \mathrm{Isoc}\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}\big)$$

be the simplicial version of the functor defined in [**81**, Theorem 5.3.1] *and denoted by* Φ. *Assume that there exists an integer* c *such that for all* $h > c$

$$H^h\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{conv}}, C^\bullet_{\mathrm{conv}}\big) = 0$$

or

$$H^h\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet, D_\bullet)}(C^\bullet_{\mathrm{conv}})\big) = 0.$$

Then there exists a functorial isomorphism

$$(11.14.5)\qquad R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{conv}}, C^\bullet_{\mathrm{conv}}\big) \xrightarrow{\sim} R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet, D_\bullet)}(C^\bullet_{\mathrm{conv}})\big).$$

The isomorphism (11.14.5) *is compatible with* $\oplus$, $\otimes$, $\mathcal{H}om$ *and* $\boxtimes$.

4) *Let* $(X^i_\bullet, D^i_\bullet)$ *and* $C^\bullet_{i\mathrm{conv}}$ $(i=1,2)$ *be the log scheme and the (locally free)* F*-isocrystal in* 3)*, respectively. Set* $(X^{12}_\bullet, D^{12}_\bullet) := (X^1_\bullet, D^1_\bullet) \times_\kappa (X^2_\bullet, D^2_\bullet)$. *Then*

the following diagram is commutative:

$$\begin{array}{ccc} \otimes^2_{K,i=1} R\Gamma(((\widetilde{X^i_\bullet, D^i_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, C^\bullet_{i\mathrm{conv}}) & \overset{\sim}{\longrightarrow} \\ \cup\big\downarrow & \\ R\Gamma(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, C^\bullet_{1\mathrm{conv}} \boxtimes C^\bullet_{2\mathrm{conv}}) & \overset{\sim}{\longrightarrow} \end{array} \tag{11.14.6}$$

$$\begin{array}{c} \otimes^2_{K,i=1} R\Gamma(((\widetilde{X^i_\bullet, D^i_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X^i_\bullet, D^i_\bullet)}(C^\bullet_{i\mathrm{conv}})) \\ \big\downarrow\cup \\ R\Gamma(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X^1_\bullet, D^1_\bullet)}(C^\bullet_{1\mathrm{conv}}) \boxtimes \Xi_{(X^2_\bullet, D^2_\bullet)}(C^\bullet_{2\mathrm{conv}})). \end{array}$$

Proof. — 1), 2), 3), 4): The proof of the existence of the isomorphisms is the same as that of (11.6) by using Shiho's comparison theorems [**82**, Corollary 2.3.9, Theorem 2.4.4, Theorem 3.1.1] and Tsuzuki's functor (6.4) as in the proof of (10.7) and (11.6). The compatibility of the isomorphisms with $\oplus$, $\otimes$ and $\mathcal{H}om$ immediately follows from (11.13), 3).

As in (11.10), we have isomorphisms

$$R\Gamma\big(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}, C^\bullet_{1\mathrm{conv}} \boxtimes C^\bullet_{2\mathrm{conv}}\big) \overset{\sim}{\longrightarrow} R\Gamma_{\mathrm{rig}}(U^{12}/K, C_1 \boxtimes C_2),$$

$$R\Gamma\big(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{W})^{\log}_{\mathrm{conv}}, C^\bullet_{1\mathrm{conv}} \boxtimes C^\bullet_{2\mathrm{conv}}\big)$$
$$\overset{\sim}{\longrightarrow} R\Gamma\big(((\widetilde{X^{12}_\bullet, D^{12}_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X^1_\bullet, D^1_\bullet)}(C^\bullet_{1\mathrm{conv}}) \boxtimes \Xi_{(X^2_\bullet, D^2_\bullet)}(C^\bullet_{2\mathrm{conv}})\big)$$

which fit into the commutative diagrams (11.14.4) and (11.14.6), respectively. □

Corollary 11.15. — *Let the notations and the assumptions be as in* (11.14), 1) *and* 3). (*The vanishing of the higher cohomologies in* (11.14), 3) *can be shown to hold for the simplicial log scheme* $(X_\bullet, D_\bullet)$ *in* (11.6), 1) *by* (10.1) *and by the proof of* (11.6).). *Then there exists a canonical isomorphism*

$$R\Gamma_{\mathrm{rig}}(U/K_0, C) \overset{\sim}{\longrightarrow} R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet, D_\bullet)}(C^\bullet_{\mathrm{conv}})\big). \tag{11.15.1}$$

In particular, $R\Gamma(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet, D_\bullet)}(C^\bullet_{\mathrm{conv}}))$ *depends only on* U/κ *and* C. *The isomorphism* (11.15.1) *is compatible with* $\oplus$, $\otimes$, $\mathcal{H}om$ *and* $\boxtimes$.

Remark 11.16. — Though we have already proved in (10.18) that the Künneth morphism is an isomorphism, we can prove this by the log Künneth formula for the constant case (see [**51**, (6.12)]) and from the Eilenberg-Zilber's theorem as in the proof of (5.10), 2) if the assumption in (11.14), 1) holds.

The following is Shiho's conjecture on semistable reduction for overconvergent F-isocrystals (see [**82**, Conjecture 3.1.8]):

Conjecture 11.17 ([**82**, Conjecture 3.1.8]). — Let the notations be as in §10. Let C be an overconvergent F-isocrystal on $(U,\overline{U})/K$. Assume that κ is perfect and that $\overline{U}$ is a compactification of U. Then there exists a morphism $\pi\colon (U_0,X_0)\to(U,\overline{U})$ of pairs such that X_0 is a proper smooth scheme over κ, such that the complement $D_0:=X_0\setminus U_0$ is an SNCD on X_0, such that $\pi\colon U_0\to U$ is proper, surjective and generically etale, and such that there exists an F-isocrystal C_0 in $\mathrm{Isoc}^{\mathrm{lf}}_{\mathrm{conv}}((X_0,D_0)/\mathcal{V})$ whose image $j^{\dagger}(C_0)$ is equal to $\pi^*(C)$.

Recently Kedlaya has proved Shiho's conjecture (see [**56**], [**57**], [**58**], [**59**]):

Theorem 11.18 (see [**59**, (2.4.4)]). — (11.17) *holds.*

Theorem 11.19 (Comparison theorem). — *Let the notations be as in* (11.17)*. Then there exist a gs proper hypercovering* $(U_\bullet,X_\bullet)$ *of* $(U,\overline{U})$ *and an F-isocrystal* $C^{\bullet}_{\mathrm{conv}}$ *in* $((\widetilde{X_\bullet,D_\bullet})/\mathcal{V})^{\log}_{\mathrm{conv}}$ *such that* (11.14), 1) *holds. Assume, furthermore, that $\mathcal{V}$ is the Witt ring of κ. Then there exists the canonical isomorphism* (11.15.1)*.*

Proof. — By using (11.18) and the standard argument in [**25**, (6.2.5)] ((9.2)), there exists a gs proper hypercovering $(U_\bullet,X_\bullet)$ such that the formula (11.14.1) holds. Hence (11.19) follows. □

Corollary 11.20. — *Let the notations be as in* (11.17)*. Then* $H^h_{\mathrm{rig}}(U/K,C)$ $(h\in\mathbb{Z})$ *is an F-isocrystal over κ.*

Proof. — By (11.15) we have

$$H^h_{\mathrm{rig}}(U/K,C)=H^h\big(((\widetilde{X_\bullet,D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}},\Xi_{(X_\bullet,D_\bullet)}(C^{\bullet}_{\mathrm{conv}})\big).$$

By the log version of [**12**, 7.24 Theorem],

$$H^h\big(((\widetilde{X_\bullet,D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}},\Xi_{(X_\bullet,D_\bullet)}(C^{\bullet}_{\mathrm{conv}})\big)$$

is a finite dimensional K-vector space, and so is $H^h_{\mathrm{rig}}(U/K,C)$. By the log Poincaré duality with the Frobenius compatibility (cf. [**85**]), the Frobenius on

$$H^h\big(((\widetilde{X_\bullet,D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}},\Xi_{(X_\bullet,D_\bullet)}(C^{\bullet}_{\mathrm{conv}})\big)_{K_0}$$

is bijective, and so is on $H^h_{\mathrm{rig}}(U/K,C)$. □

REMARK 11.21. — The finite dimensionality of $H^h_{\mathrm{rig}}(U/K, C)$ for an overconvergent F-isocrystal C on U/K has already been proved by Kedlaya (see [**55**, (1.2.1)]).

12. Weight filtration on rigid cohomology

Unless stated otherwise, we assume that the base field κ is perfect in this section. Let $\mathcal{V}$, $\mathcal{W}$, K and K_0 be as in §10. We fix an automorphisms σ of $\mathcal{V}$ which is a lift of the p-th power endomorphism of κ. For a $\mathcal{V}$-module M with σ-linear action F_M and for a nonnegative integer k, $M(-k)$ denotes the usual Tate twist of M: $M(-k) := M$, $F_{M(-k)} := p^k F_M$.

Let U be a separated scheme of finite type over κ. Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Let $(U_\bullet, X_\bullet)$ be a good proper hypercovering of $(U, \overline{U})$ which has the disjoint union of the members of an affine simplicial open covering of $(U_\bullet, X_\bullet)$ over $(U, \overline{U})$. Set $D_\bullet := X_\bullet \setminus U_\bullet$. Set also $\mathfrak{U} := (U_\bullet, X_\bullet)$.

THEOREM-DEFINITION 12.1. — *There exists the following spectral sequence*
(12.1.1)

$$\begin{aligned} E_1^{-k,h+k}\big((X_\bullet, D_\bullet)/K\big) = \bigoplus_{t\geq 0} H^{h-2t-k}\big(&(\widetilde{D_t^{(t+k)}/\mathcal{W}})_{\mathrm{crys}}, \mathcal{O}_{D_t^{(t+k)}/\mathcal{W}} \\ &\otimes_{\mathbb{Z}} \varpi^{(t+k)}_{\mathrm{crys}}(D_t/\mathcal{W})\big)\big(-(t+k)\big)_K \\ \Longrightarrow H^h_{\mathrm{rig}}(U/K).& \end{aligned}$$

The spectral sequence (12.1.1) *degenerates at E_2. We call* (12.1.1) *the* weight spectral sequence *of $H^h_{\mathrm{rig}}(U/K)$ with respect to $\mathfrak{U}$. We denote by $P^{\mathfrak{U}}$ the induced filtration on $H^h_{\mathrm{rig}}(U/K)$, and we call this filtration the* weight filtration *on $H^h_{\mathrm{rig}}(U/K)$ with respect to $\mathfrak{U}$ provisionally* (cf. (12.6) below).

Proof. — By (11.7.1) and (5.1.3), we have (12.1.1). The E_2-degeneration follows from (5.6), 1). □

REMARK 12.2. — (12.1.1) is a generalization of the spectral sequence in [**86**, (5.2.4)], though the boundary morphisms between the E_1-terms of (12.1.1) are different from those in [*loc. cit.*] ((10.15), 1)).

Next, we prove that the weight filtration $P^{\mathfrak{U}}$ is independent of the choice of the good proper hypercovering of $(U, \overline{U})$ which has the disjoint union of the members of an affine simplicial open covering over $(U, \overline{U})$. To prove it, we use Grothendieck's idea for the reduction of geometric problems to arithmetic problems (see [**40**, §8], [**33**, Partie II, VII 5], cf. [**2**, §6]): a standard deformation theory, the specialization argument of Deligne-Illusie (see [**46**, (3.10)]) and the E_2-degeneration of the weight spectral sequence ((5.6), 1)).

Let $(U_\bullet, X_\bullet)$ and $(U'_\bullet, X'_\bullet)$ be two good proper hypercoverings of $(U, \overline{U})$ which have the disjoint unions of the members of affine simplicial open coverings of them over $(U, \overline{U})$. By (9.9), 2), we may assume that $(U_\bullet, X_\bullet)$ and $(U'_\bullet, X'_\bullet)$ are gs proper hypercoverings of $(U, \overline{U})$. Set $D_\bullet := X_\bullet \setminus U_\bullet$ and $D'_\bullet := X'_\bullet \setminus U'_\bullet$. Because two gs proper hypercoverings of $(U, \overline{U})$ are covered by another gs proper hypercovering ((9.4), 1)), we may assume that there exists a morphism $(X'_\bullet, D'_\bullet) \to (X_\bullet, D_\bullet)$ of simplicial log schemes. Fix a nonnegative integer h. Fix a positive integer N satisfying the inequality (2.2.1). By [**40**, (8.9.1) (iii)], there exists a subring A_1 of κ which is smooth over a finite field $\mathbb{F}_q$ such that the morphism

$$(X'_{\bullet \leq N}, D'_{\bullet \leq N}) \longrightarrow (X_{\bullet \leq N}, D_{\bullet \leq N})$$

of N-truncated simplicial log schemes has a model

$$(\mathcal{X}'_{\bullet \leq N}, \mathcal{D}'_{\bullet \leq N}) \longrightarrow (\mathcal{X}_{\bullet \leq N}, \mathcal{D}_{\bullet \leq N})$$

over A_1 since the morphism of N-truncated simplicial log schemes is given by a finite number of morphisms of log schemes which satisfy a finite number of equations of morphisms. Here, note that the extension $\kappa/\mathrm{Frac}\, A_1$ may be infinite and transcendental. Set

$$S_1 := \mathrm{Spec}(A_1), \quad \mathcal{U}_{\bullet \leq N} := \mathcal{X}_{\bullet \leq N} \setminus \mathcal{D}_{\bullet \leq N}, \quad \mathcal{U}'_{\bullet \leq N} := \mathcal{X}'_{\bullet \leq N} \setminus \mathcal{D}'_{\bullet \leq N}.$$

We can also assume that there exists a model $(\mathcal{U}, \overline{\mathcal{U}})$ of $(U, \overline{U})$ over S_1 such that $(\mathcal{U}_{\bullet \leq N}, \mathcal{X}_{\bullet \leq N})$ and $(\mathcal{U}'_{\bullet \leq N}, \mathcal{X}'_{\bullet \leq N})$ are N-truncated gs proper hypercoverings of $(\mathcal{U}, \overline{\mathcal{U}})$. By a standard deformation theory (see [**41**, III (6.10)]), there exists a formally smooth scheme $S = \mathrm{Spf}(A)$ over $\mathrm{Spf}(\mathcal{W}(\mathbb{F}_q))$ such that

$$S \otimes_{\mathcal{W}(\mathbb{F}_q)} \mathbb{F}_q = S_1.$$

We fix a lift $F\colon S \to S$ of the Frobenius of S_1. Then, as is well-known, we have a natural morphism $A \to \mathcal{W}(A)$ which is a section of the projection $\mathcal{W}(A) \to A$ (see [**61**, VII (4.12)], [**47**, 0. (1.3.16)]). Consequently, for an A_1-algebra B_1, $\mathcal{W}(B_1)$ becomes an A-algebra by the composite morphism

$$A \longrightarrow \mathcal{W}(A) \longrightarrow \mathcal{W}(A_1) \longrightarrow \mathcal{W}(B_1).$$

By the same proof as that of [**69**, (3.5)] (by using the specialization argument of Deligne-Illusie and by using the base change of the Gysin morphism and that of the face morphisms δ_i^*), we have the following:

PROPOSITION 12.3. — *Let k and h be integers. There exists an affine open formal subscheme* $\mathrm{Spf}(B)$ *of S such that, for any closed point $x \in \mathrm{Spec}(B/p)$, there exists an isomorphism*

$$E_2^{-k,h+k}\big((X_\bullet, D_\bullet)/\mathcal{W}\big)\otimes_{\mathcal{W}(\mathbb{F}_q)}\mathcal{W}\big(\kappa(x)\big) \overset{\sim}{\longrightarrow} E_2^{-k,h+k}\big((\mathcal{X}_{\bullet\leq N}(x), \mathcal{D}_{\bullet\leq N}(x))/\mathcal{W}(\kappa(x))\big)\otimes_{\mathcal{W}(\mathbb{F}_q)}\mathcal{W}$$

of $\mathcal{W}\otimes_{\mathcal{W}(\mathbb{F}_q)}\mathcal{W}(\kappa(x)) \simeq \mathcal{W}(\kappa(x))\otimes_{\mathcal{W}(\mathbb{F}_q)}\mathcal{W}$*-modules. Here* $(\mathcal{X}_{\bullet\leq N}(x), \mathcal{D}_{\bullet\leq N}(x))$ *is the fiber of* $(\mathcal{X}_{\bullet\leq N}, \mathcal{D}_{\bullet\leq N})$ *at x and $\kappa(x)$ is the residue field at x.*

COROLLARY 12.4. — *Set* $\mathcal{U}(x) := \mathcal{U} \otimes_{A_1} \kappa(x)$ *for a closed point* $x \in \mathrm{Spec}(B/p)$. *Let* $K_0(\kappa(x))$ *be the fraction field of* $\mathcal{W}(\kappa(x))$. *Then*

(12.4.1) $$\dim_K H^h_{\mathrm{rig}}(U/K) = \dim_{K_0(\kappa(x))} H^h_{\mathrm{rig}}\big(\mathcal{U}(x)/K_0(\kappa(x))\big).$$

Proof. — (12.1) and (12.3) show (12.4). □

THEOREM 12.5. — *Let* $\mathfrak{U} = (U_\bullet, X_\bullet)$ *and* $\mathfrak{U}' = (U'_\bullet, X'_\bullet)$ *be two good proper hypercoverings of* $(U, \overline{U})$ *which have the disjoint unions of the members of affine simplicial open coverings of them over* $(U, \overline{U})$. *Set* $D_\bullet := X_\bullet \setminus U_\bullet$ *and* $D'_\bullet := X'_\bullet \setminus U'_\bullet$. *Then the following hold*:

1) *There exists a canonical isomorphism*

(12.5.1) $$E_2^{-k,h+k}((X'_\bullet, D'_\bullet)/\mathcal{W})_K \overset{\sim}{\longrightarrow} E_2^{-k,h+k}((X_\bullet, D_\bullet)/\mathcal{W})_K \quad (k, h \in \mathbb{Z}).$$

2) $P^{\mathfrak{U}'} = P^{\mathfrak{U}}$.

Proof. — By (11.7.1) and (9.9), 2), we may assume that $(U_\bullet, X_\bullet)$ and $(U'_\bullet, X'_\bullet)$ are gs proper hypercoverings of $(U, \overline{U})$. By (9.4), 1) we may assume that there exists a morphism $\mathfrak{U}' = (U'_\bullet, X'_\bullet) \to \mathfrak{U} = (U_\bullet, X_\bullet)$ of pairs of simplicial schemes over $(U, \overline{U})$. By (11.7.1) we have the following commutative diagram:

(12.5.2) $$\begin{array}{ccc} H^h_{\mathrm{rig}}(U/K) & \overset{\sim}{\longrightarrow} & H^h((X'_\bullet, D'_\bullet)/\mathcal{W})_K \\ \| & & \uparrow \\ H^h_{\mathrm{rig}}(U/K) & \overset{\sim}{\longrightarrow} & H^h((X_\bullet, D_\bullet)/\mathcal{W})_K. \end{array}$$

The right vertical morphism preserves the weight filtration with respect to $P^{\mathfrak{U}}$ and $P^{\mathfrak{U}'}$.

We prove 1) and 2) at the same time.

For simplicity of notation, set

$$E_r^{**}(X_\bullet) := E_r^{**}((X_\bullet, D_\bullet)/\mathcal{W})_K \quad (1 \leq r \leq \infty),$$
$$E_r^{**}(X'_\bullet) := E_r^{**}((X'_\bullet, D'_\bullet)/\mathcal{W})_K.$$

First, assume that $k=0$. By the commutative diagram

$$(12.5.3)\qquad \begin{array}{ccc} P_0^{\mathfrak{U}'}H^h_{\rm rig}(U/K)=E_\infty^{h0}(X'_\bullet)=E_2^{h0}(X'_\bullet) & \overset{\subset}{\longrightarrow} & H^h_{\rm rig}(U/K) \\ \uparrow & & \| \\ P_0^{\mathfrak{U}}H^h_{\rm rig}(U/K)=E_\infty^{h0}(X_\bullet)=E_2^{h0}(X_\bullet) & \overset{\subset}{\longrightarrow} & H^h_{\rm rig}(U/K), \end{array}$$

the morphism $E_2^{h0}(X_\bullet)\hookrightarrow E_2^{h0}(X'_\bullet)$ is injective.

Let the notations be as in (12.3). Then,

$$\dim_K E_2^{h0}(X_\bullet)=\dim_{K_0(\kappa(x))}E_2^{h0}\big(\mathcal{X}_\bullet(x)\big)$$

and we may assume that $\dim_K E_2^{h0}(X'_\bullet)=\dim_{K_0(\kappa(x))}E_2^{h0}(\mathcal{X}'_\bullet(x))$. Since $E_2^{h0}(\mathcal{X}_\bullet(x))$ and $E_2^{h0}(\mathcal{X}'_\bullet(x))$ are the weight 0-part of $H^h_{\rm rig}(\mathcal{U}(x)/K_0(\kappa(x)))$ by the purity of weight of the crystalline cohomology of a proper smooth variety over a finite field (see [**54**], [**16**, (1.2)] (cf. [**69**, (2.2) (4)])), we have $E_2^{h0}(\mathcal{X}_\bullet(x))=E_2^{h0}(\mathcal{X}'_\bullet(x))$. Hence, by (12.3), $\dim_K E_2^{h0}(X_\bullet)=\dim_K E_2^{h0}(X'_\bullet)$. Thus we have equalities

$$E_2^{h0}(X_\bullet)=E_2^{h0}(X'_\bullet)\quad\text{and}\quad P_0^{\mathfrak{U}'}H^h_{\rm rig}(U/K)=P_0^{\mathfrak{U}}H^h_{\rm rig}(U/K).$$

Next, consider the commutative diagram with exact rows:

(12.5.4)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0^{\mathfrak{U}'}H^h_{\rm rig}(U/K) & \longrightarrow & P_1^{\mathfrak{U}'}H^h_{\rm rig}(U/K) & \longrightarrow & E_2^{h-1,1}(X'_\bullet) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & P_0^{\mathfrak{U}}H^h_{\rm rig}(U/K) & \longrightarrow & P_1^{\mathfrak{U}}H^h_{\rm rig}(U/K) & \longrightarrow & E_2^{h-1,1}(X_\bullet) & \longrightarrow & 0. \end{array}$$

We obviously see that the morphism $P_1^{\mathfrak{U}}H^h_{\rm rig}(U/K)\longrightarrow P_1^{\mathfrak{U}'}H^h_{\rm rig}(U/K)$ is injective. Hence the morphism $E_2^{h-1,1}(X_\bullet)\longrightarrow E_2^{h-1,1}(X'_\bullet)$ is injective, and, in fact, an isomorphism by (12.3) as above. By the diagram (12.5.4) again, the morphism $P_1^{\mathfrak{U}}H^h_{\rm rig}(U/K)\longrightarrow P_1^{\mathfrak{U}'}H^h_{\rm rig}(U/K)$ turns out to be an isomorphism. Repeating this argument, we obtain $P_k^{\mathfrak{U}}=P_k^{\mathfrak{U}'}$ $(\forall k\in\mathbb{Z})$. ☐

Definition 12.6. — We call the filtration on $H^h_{\rm rig}(U/K)$ induced by the spectral sequence (12.1.1) the *weight filtration* on $H^h_{\rm rig}(U/K)$. We denote it by $\{P_{k,K}\}$ or simply by $\{P_k\}$.

Problem 12.7. — Does there exist a condition of $(X_\bullet,D_\bullet)$ which assures that the submodule $P_kH^h_{\text{log-crys}}((X_\bullet,D_\bullet)/\mathcal{W})$ of $H^h_{\text{log-crys}}((X_\bullet,D_\bullet)/\mathcal{W})$ depends only on U/κ? For the case $q=1$ and $p\geq 3$, the answer is yes by [**1**].

Next, we prove some fundamental properties of the weight filtration on rigid cohomology.

PROPOSITION 12.8. — *Let $\mathcal{V} \longrightarrow \mathcal{V}'$ be a ring homomorphism of complete discrete valuation rings with perfect residue fields of mixed characteristics. Let K' be the fraction field of $\mathcal{V}'$. Then $P_{k,K} \otimes_K K' = P_{k,K'}$ $(k \in \mathbb{Z})$.*

Proof. — Let $\mathcal{W}'$ be the Witt ring of the residue field of $\mathcal{V}'$. Let K'_0 be the fraction field of $\mathcal{W}'$. By the construction of the weight filtration ((12.1.1)), it suffices to prove that $P_{k,K_0} \otimes_{K_0} K'_0 = P_{k,K'_0}$. This equality follows from (12.1.1) and the base change theorem of crystalline cohomologies (see [**5**, V Proposition 3.5.2]). □

COROLLARY 12.9. — 1) *The filtration on $H^h_{\mathrm{rig}}(U/K)$ induced by the canonical identifications*

$$H^h_{\mathrm{rig}}(U/K) = H^h\big((X_\bullet, D_\bullet)/\mathcal{W}\big)_K = H^h\big(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)\big)_K \tag{12.9.1}$$

and by the filtered complex $(\mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet), \{P_k\mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)\}_{k\in\mathbb{Z}})$ on $X_\bullet$ coincides with P.

2) *If U is the complement of an SNCD D on a proper smooth scheme X over κ, then the weight filtration P on $H^h_{\mathrm{rig}}(U/K) = H^h((X,D)/\mathcal{W})_K$ coincides with the weight filtration defined by the filtered zariskian complex $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/\mathcal{W}}), P)$ (see* (3.3), 3)*).*

Proof. — 1): Follows immediately from (7.7), (11.7), (12.1), (12.5) and (12.6).

2): Consider the constant proper hypercovering $(U_\bullet, X_\bullet) = (U, X)$ of (U, X). Then it is split. By (11.7) we have an isomorphism $H^h_{\mathrm{rig}}(U/K) \xrightarrow{\sim} H^h((X,D)/\mathcal{W})_K$. (This isomorphism is nothing but Shiho's comparison theorem [**82**, Corollary 2.3.9, Theorem 2.4.4, Theorem 3.1.1].) Thus 2) follows from the definition of the weight filtration on $H^h_{\mathrm{rig}}(U/K)$. □

THEOREM 12.10 (Strict compatibility). — *Let $f\colon U' \longrightarrow U$ be a morphism of separated schemes of finite type over κ. Then the induced morphism*

$$f^*\colon H^h_{\mathrm{rig}}(U/K) \longrightarrow H^h_{\mathrm{rig}}(U'/K) \quad (h \in \mathbb{Z})$$

by f is strictly compatible with the weight filtration.

Proof. — (The proof is similar to [**72**, (2.18.2)] and slightly easier than that of [*loc. cit.*].) By (12.8) we may assume that $\mathcal{V} = \mathcal{W}$.

Fix a nonnegative integer h and a positive integer N satisfying the inequality (2.2.1). Let $\overline{U}$ and $\overline{U}'$ are proper schemes over κ fitting into the commutative diagram (9.5.1). Let

$$\begin{array}{ccc} (U'_\bullet, X'_\bullet) & \longrightarrow & (U_\bullet, X_\bullet) \\ \big\downarrow & & \big\downarrow \\ (U', \overline{U}') & \longrightarrow & (U, \overline{U}) \end{array}$$

be a commutative diagram such that the vertical morphisms above are gs proper hypercoverings. Set $D_\bullet := X_\bullet \setminus U_\bullet$ and $D'_\bullet := X'_\bullet \setminus U'_\bullet$. Let $f_\bullet \colon (X'_\bullet, D'_\bullet) \longrightarrow (X_\bullet, D_\bullet)$ be the natural morphism of simplicial log schemes over κ. As in the explanation between (12.2) and (12.3), there exists a subring A_1 of κ which is smooth over a finite field $\mathbb{F}_q$ and there exist a model

$$g_{\bullet\leq N} \colon (\mathcal{X}'_{\bullet\leq N}, \mathcal{D}'_{\bullet\leq N}) \longrightarrow (\mathcal{X}_{\bullet\leq N}, \mathcal{D}_{\bullet\leq N})$$

of $f_{\bullet\leq N}$ over the spectrum $S_1 = \mathrm{Spec}(A_1)$. Furthermore, there exists a formally smooth scheme $S = \mathrm{Spf}(A)$ over $\mathrm{Spf}(\mathcal{W}(\mathbb{F}_q))$ such that $S \otimes_{\mathcal{W}(\mathbb{F}_q)} \mathbb{F}_q = S_1$. We fix a lift $F \colon S \longrightarrow S$ of the Frobenius of S_1. Then $\mathcal{W}$ can become an A-algebra. By (8.1) it suffices to prove that the induced morphism is strict:

$$\begin{aligned} g^*_{\bullet\leq N} \colon R^h f_{(\mathcal{X}_{\bullet\leq N}, \mathcal{D}_{\bullet\leq N})/S*}(\mathcal{O}_{(\mathcal{X}_{\bullet\leq N}, \mathcal{D}_{\bullet\leq N})/S})_K \\ \longrightarrow R^h f_{(\mathcal{X}'_{\bullet\leq N}, \mathcal{D}'_{\bullet\leq N})/S*}(\mathcal{O}_{(\mathcal{X}'_{\bullet\leq N}, \mathcal{D}'_{\bullet\leq N})/S})_K. \end{aligned}$$

Hence (12.10) follows from (8.5) and (8.6). □

Theorem 12.11. — *Let U be a separated scheme of finite type over κ. Let h be a nonnegative integer. Then the following hold*:

1) $P_{-1}H^h_{\mathrm{rig}}(U/K) = 0$ *and* $P_{2h}H^h_{\mathrm{rig}}(U/K) = H^h_{\mathrm{rig}}(U/K)$.

2) *If U is proper over κ, then* $P_h H^h_{\mathrm{rig}}(U/K) = H^h_{\mathrm{rig}}(U/K)$.

Proof. — 1): Follows from the spectral sequence (12.1.1).

2): We can take $D_\bullet = \phi$ in the notation (12.1.1). □

The following is a generalization of [**15**, (2.2)]:

Theorem 12.12. — *Let U be a separated scheme of finite type over κ. If U is smooth over κ, then* $P_{h-1}H^h_{\mathrm{rig}}(U/K) = 0$.

Proof. — If κ is a finite field, then (12.12) follows from [**15**, (2.2)].

We reduce (12.12) in the case of a general perfect field of characteristic $p > 0$ to the case above as follows. Let $\overline{U}$ be a compactification of U/κ. Fix a nonnegative integer h and fix an integer N satisfying the inequality (2.2.1).

Let $(U_\bullet, X_\bullet)$ be a gs proper hypercovering of $(U, \overline{U})$. Set $D_\bullet := X_\bullet \setminus U_\bullet$. Let the notations be between (12.2) and (12.3) and as in the proof of (12.10). Let $\{P_{k,S}\}_{k\in\mathbb{Z}}$ be the weight filtration on

$$R^h f_{(\mathcal{X}_{\bullet\leq N},\mathcal{D}_{\bullet\leq N})/S*}(\mathcal{O}_{(\mathcal{X}_{\bullet\leq N},\mathcal{D}_{\bullet\leq N})/S}) \otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q),$$

where $K_0(\mathbb{F}_q)$ is the fraction field of $\mathcal{W}(\mathbb{F}_q)$. By (8.1) it suffices to prove that $P_{h-1,S} = 0$ by shrinking S. Because $P_{k,S}$ ($k \in \mathbb{Z}$) is a convergent isocrystal ((5.7), 1)), it suffices to prove that the stalk $(P_{h-1,S})_x$ for any closed point x is equal to 0 (see [**74**, (3.17)]). Since $\kappa(x)$ is a finite field, $(P_{h-1,S})_x = 0$ by [**15**, (2.2)]. ☐

Theorem 12.13 (cf. [**25**, (8.1.25)]). — *Let U^i $(i = 1, 2)$ be a separated scheme of finite type over κ. Then the induced isomorphism*

$$\bigoplus_{i+j=h} H^i_{\mathrm{rig}}(U^1/K) \otimes_K H^j_{\mathrm{rig}}(U^2/K) \overset{\sim}{\longrightarrow} H^h_{\mathrm{rig}}(U^1 \times_\kappa U^2/K) \tag{12.13.1}$$

by the canonical isomorphism (10.18.1) *is compatible with the weight filtrations.*

Proof. — We may assume that $\mathcal{V} = \mathcal{W}$.

Let $j^i : U^i \hookrightarrow \overline{U}^i$ $(i = 1, 2)$ be an open immersion into a proper scheme over κ. Set $U^{12} := U^1 \times_\kappa U^2$ and $\overline{U}^{12} := \overline{U}^1 \times_\kappa \overline{U}^2$. Let $(U^i_\bullet, X^i_\bullet)$ $(i = 1, 2)$ be a good proper hypercovering of $(U^i, \overline{U}^i)$ which has the disjoint union of the members of an affine simplicial open covering of $(U^i_\bullet, X^i_\bullet)$ over $(U^i, \overline{U}^i)$. Set

$$D^i_\bullet := X^i_\bullet \setminus U^i_\bullet, \quad (i = 1, 2),$$
$$X^{12}_\bullet := X^1_\bullet \times_\kappa X^2_\bullet, \quad U^{12}_\bullet := U^1_\bullet \times_\kappa U^2_\bullet,$$
$$D^{12}_\bullet := X^{12}_\bullet \setminus U^{12}_\bullet.$$

By (11.10) we have a canonical isomorphism

$$R\Gamma_{\mathrm{rig}}(U^{12}/K) \overset{\sim}{\longrightarrow} R\Gamma((X^{12}_\bullet, D^{12}_\bullet)/\mathcal{W})_K.$$

By Eilenberg-Zilber's theorem and by (11.15), we have the commutative diagram:

(12.13.2)

$$\begin{array}{ccc} R\Gamma((X^1_\bullet, D^1_\bullet)/\mathcal{W})_K \otimes_K R\Gamma((X^2_\bullet, D^2_\bullet)/\mathcal{W})_K & \overset{\sim}{\longrightarrow} & R\Gamma((X^{12}_\bullet, D^{12}_\bullet)/\mathcal{W})_K \\ \simeq\uparrow & & \uparrow\simeq \\ R\Gamma_{\mathrm{rig}}(U^1/K) \otimes_K R\Gamma_{\mathrm{rig}}(U^2/K) & \overset{\sim}{\longrightarrow} & R\Gamma_{\mathrm{rig}}(U^{12}/K). \end{array}$$

Hence we can identify the Künneth isomorphism (10.18.1) with the Künneth isomorphism in cosimplicial log crystalline cohomologies in the upper horizontal isomorphism in (12.13.2). Therefore (12.13) follows from (2.2), (5.10), 1), (11.6) and (10.1). □

Corollary 12.14. — *Let U be a separated scheme of finite type over κ. The cup product*

$$H^h_{\mathrm{rig}}(U/K) \otimes_K H^{h'}_{\mathrm{rig}}(U/K) \longrightarrow H^{h+h'}_{\mathrm{rig}}(U/K) \tag{12.14.1}$$

is strictly compatible with the weight filtration on $H^{\bullet}_{\mathrm{rig}}(U/K)$.

Proof. — The proof is the same as that of [**25**, (8.2.11)]: (12.14) immediately follows from (12.13) and (12.10) for the diagonal closed immersion $U \hookrightarrow U \times_\kappa U$. □

In the rest of this section, we prove the analogues of propositions in [**25**, 8.2].

Proposition 12.15. — *Let X be a proper scheme over κ. Let $\pi\colon Y \longrightarrow X$ be a surjective morphism over κ from a proper smooth scheme over κ. Let h be an integer. Then the following sequence is exact:*

$$\begin{aligned} 0 \to H^h_{\mathrm{rig}}(X/K)/P_{h-1}H^h_{\mathrm{rig}}(X/K) &\xrightarrow{\pi^*} H^h_{\mathrm{rig}}(Y/K) \\ &\xrightarrow{p_2^*-p_1^*} H^h_{\mathrm{rig}}(Y\times_X Y/K)/P_{h-1}H^h_{\mathrm{rig}}(Y\times_X Y/K). \end{aligned} \tag{12.15.1}$$

Proof. — (The proof is almost the same as that of [**25**, (8.2.5)].) Let $Y_\bullet$ be a split proper hypercovering of X such that $Y_0 = Y$ and such that $Y_\bullet$ is smooth over κ. Let $\delta_i\colon Y_1 \longrightarrow Y_0$ $(i = 0, 1)$ be the standard face morphism. Using (12.11), 2) and the E_2-degeneration of the spectral sequence (12.1.1), we have the exact sequence

$$\begin{aligned} 0 \to H^h_{\mathrm{rig}}(X/K)/P_{h-1}H^h_{\mathrm{rig}}(X/K)& \\ \xrightarrow{\pi^*} H^h_{\mathrm{rig}}(Y/K) &\xrightarrow{\delta_0^*-\delta_1^*} H^h_{\mathrm{rig}}(Y_1/K). \end{aligned} \tag{12.15.2}$$

The morphism $\delta_0^* - \delta_1^*$ factors through the morphism

$$p_2^* - p_1^*\colon H^h_{\mathrm{rig}}(Y/K) \longrightarrow H^h_{\mathrm{rig}}(Y\times_X Y/K)/P_{h-1}H^h_{\mathrm{rig}}(Y\times_X Y/K)$$

since $P_{h-1}H^h_{\mathrm{rig}}(Y_1/K) = 0$ ((12.12)). Now we see that (12.15.1) is exact. □

PROPOSITION 12.16. — *Let $\pi\colon Y \to U$ be a morphism over κ from a proper scheme over κ to a separated smooth scheme of finite type over κ. Let $j\colon U \to X$ be an open immersion into a proper smooth scheme over κ such that the image $j(U)$ is the complement of an SNCD on X. Then the images of $H^h_{\mathrm{rig}}(U/K)$ and $H^h_{\mathrm{rig}}(X/K)$ have the same image in $H^h_{\mathrm{rig}}(Y/K)$ $(h \in \mathbb{Z})$.*

Proof. — By (1.0.11), there exists the following spectral sequence (cf. [**65**, (3.1)], [**69**, (8.7), (9.3)]):

$$E_1^{-k,h+k} = H^{h-k}\big((\widetilde{D^{(k)}/\mathcal{W}})_{\mathrm{crys}}, \mathcal{O}_{D^{(k)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D/\kappa)\big)(-k) \Longrightarrow H^h\big((X,D)/\mathcal{W}\big).$$

Hence the natural morphism $H^h_{\mathrm{crys}}(X/\mathcal{W}) \to \mathrm{gr}^P_h H^h((X,D)/\mathcal{W})$ is surjective, and by (12.9), 2), the morphism $H^h_{\mathrm{crys}}(X/\mathcal{W})_K \to \mathrm{gr}^P_h H^h_{\mathrm{rig}}(U/K)$ is surjective. By using (12.10), (12.11), 2) and (12.12), the rest of the proof is the same as that of [**25**, (8.2.6)]. □

PROPOSITION 12.17. — *Let $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{f} Y$ be a composite morphism of separated schemes of finite type over κ. Assume that $\widetilde{X}$ is proper and smooth over κ, that X is proper over κ, that Y is smooth over κ, and that π is surjective. Then*

$$\mathrm{Ker}\big(f^*\colon H^h_{\mathrm{rig}}(Y/K) \longrightarrow H^h_{\mathrm{rig}}(X/K)\big) = \mathrm{Ker}\big((f\pi)^*\colon H^h_{\mathrm{rig}}(Y/K) \longrightarrow H^h_{\mathrm{rig}}(\widetilde{X}/K)\big).$$

Proof. — By using (12.10), (12.11) , 2), (12.12) and (12.15.2), the proof is the same as that of [**25**, (8.2.7)]. □

In §17 below, we shall endow the rigid cohomology with compact support with the weight filtration by using Berthelot's Poincaré duality (see [**8**, (2.4)]). Though the following, which is an analogue of [**25**, (8.2.8)], logically depends on this fact, we include it here for convenience:

PROPOSITION 12.18. — *Let $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{\iota} Y$ be a composite morphism of separated schemes of finite type over κ. Assume that Y is proper and smooth over κ, that ι is a closed immersion of pure codimension c, and that $\widetilde{X}$ is a resolution of X. Set $q := \iota \circ \pi$ and $U := Y \setminus X$. Then the sequence*

$$H^{h-2c}_{\mathrm{rig}}(\widetilde{X}/K)(-c) \xrightarrow{q_*} H^h_{\mathrm{rig}}(Y/K) \longrightarrow H^h_{\mathrm{rig}}(U/K) \qquad (12.18.1)$$

is exact. Here $(-c)$ means the usual Tate twist, which shifts the weight filtration by $2c$. Moreover, the exact sequence (12.18.1) *is strictly compatible with the weight filtration.*

Proof. — By (17.13.1) in §17 below, the excision exact sequence

$$\cdots \longrightarrow H^h_{\mathrm{rig,c}}(U/K) \longrightarrow H^h_{\mathrm{rig}}(Y/K) \longrightarrow H^h_{\mathrm{rig}}(X/K) \longrightarrow \cdots \tag{12.18.2}$$

(see [**6**, (3.1) (iii)]) is strictly compatible with the weight filtration. By the same argument of the proof of [**25**, (8.2.8)], (12.18) follows from (12.17), (12.18.2) and Berthelot's Poincaré duality (see [**8**, (2.4)]). □

Remark 12.19. — [**25**, (8.2.8.1)] is mistaken in the cohomological degree of $H^\bullet(\widetilde{X},\mathbb{Q})$ and the Tate twist is forgotten in [*loc. cit.*]. Because the definition of the weight filtration on the singular cohomology with compact support is missing in [*loc. cit.*]; we cannot use the duality of the weight filtration in [*loc. cit.*].

Remark 12.20. — Let Y be a smooth scheme over a field κ of characteristic $p \geq 0$. Let E be an SNCD on Y. Set $V := Y \setminus E$. Let $j\colon V \hookrightarrow Y$ be the natural open immersion. Let $\ell \neq p$ be a prime number. Recall the étale orientation sheaf $\varpi^{(k)}_{\text{ét}}(E/\kappa)(-k)$ ($k \in \mathbb{N}$) of $E^{(k)}$ (see [**72**, (2.20)]):

$$\varpi^{(k)}_{\text{ét}}(E/\kappa)(-k) := u^{-1}\Big(\bigwedge^k (M(E)/\mathcal{O}^*_Y)\Big)|_{E^{(k)}_{\text{ét}}},$$

where u is the canonical morphism $\widetilde{Y}_{\text{ét}} \to \widetilde{Y}_{\text{zar}}$. Let $b^{(k)}\colon E^{(k)} \longrightarrow Y$ be the natural morphism. By Grothendieck's absolute purity, which has been solved by O. Gabber (see [**31**]), we obtain

$$R^k j_*(\mathbb{Z}/\ell^n) = b^{(k)}_*\big((\mathbb{Z}/\ell^n)_{E^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\text{ét}}(E/\kappa)(-k)\big) \quad (n \in \mathbb{N}).$$

Let κ be a field of characteristic $p \geq 0$. Let U be a separated scheme of finite type over κ. Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Then, by using a good proper hypercovering $(U_\bullet, X_\bullet)$ of $(U, \overline{U})$, by using Gabber's purity and by using the standard specialization argument in ℓ-adic cohomologies (cf. the proof of (12.5)), we can endow $H^h_{\text{ét}}(U \otimes_\kappa \kappa_{\mathrm{sep}}, \mathbb{Q}_\ell)$ ($h \in \mathbb{Z}$) with a weight filtration which is shown to depend only on $U \otimes_\kappa \kappa_{\mathrm{sep}}$ and ℓ. Note that the analogous explanation in [**50**, Introduction] is not perfect since Gabber's purity is not mentioned. Statements in the ℓ-adic case which are analogous to those about weights in §12 and later sections hold by a standard specialization argument.

13. Slope filtration on rigid cohomology

Let U be a separated scheme of finite type over a perfect field κ of characteristic $p > 0$. Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ.

Let $\mathcal{W}$ be the Witt ring of κ and K_0 the fraction field of $\mathcal{W}$. By [**86**, (5.1.1)] or (11.8), $H^h_{\mathrm{rig}}(U/K_0)$ is an F-isocrystal; hence it has the slope filtration. In this section we construct the slope spectral sequence of $H^h_{\mathrm{rig}}(U/K_0)$.

Let $(U_\bullet, X_\bullet)$ be a good proper hypercovering of $(U, \overline{U})$ which has the disjoint union of the members of an affine simplicial open covering of $(U_\bullet, X_\bullet)$ over $(U, \overline{U})$. Set $D_\bullet := X_\bullet \setminus U_\bullet$. By (12.9.1) we have

$$H^h_{\mathrm{rig}}(U/K_0) = H^h\big(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet)\big)_{K_0}. \tag{13.0.1}$$

Hence we have the following spectral sequence

$$E_1^{ij} = H^j\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0} \Longrightarrow H^{i+j}_{\mathrm{rig}}(U/K_0). \tag{13.0.2}$$

Definition 13.1. — We call (13.0.2) the *slope spectral sequence* of $H^{i+j}_{\mathrm{rig}}(U/K_0)$.

Theorem 13.2 (Slope decomposition). — *There exists the following canonical isomorphism*:

$$H^j\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0} \xrightarrow{\sim} H^{i+j}_{\mathrm{rig}}(U/K_0)_{[i,i+1)}. \tag{13.2.1}$$

In particular, $H^j(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))_{K_0}$ depends only on U/κ. Moreover, there exists the following canonical decomposition

$$H^h_{\mathrm{rig}}(U/K_0) = \bigoplus_{i=0}^{h} H^{h-i}\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0}. \tag{13.2.2}$$

Proof. — (13.2) immediately follows from (7.9) and (11.7). □

The following is the compatibility of the weight filtration on $H^h_{\mathrm{rig}}(U/K_0)$ with the slope filtration on it.

Corollary 13.3. — *One has the equality*

$$\begin{aligned} P_k H^{h-i}\big(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet)\big)_{K_0} \\ = H^h_{\mathrm{rig}}(U/K_0)_{[i,i+1)} \cap P_k H^h_{\mathrm{rig}}(U/K_0). \end{aligned} \tag{13.3.1}$$

In particular, $P_k H^{h-i}(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))_{K_0}$ depends only on U/κ.

Proof. — (13.3) follows from (5.6), 1), (7.10) and (12.9), 1). □

Remark 13.4. — We shall give the range of the slopes of $H^h_{\mathrm{rig}}(U/K_0)$ in §16 below.

Problem 13.5. — Does there exist a condition of $(X_\bullet, D_\bullet)$ which assures that the submodule $P_k H^h(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))$ of $H^h(X_\bullet, \mathcal{W}\Omega^i_{X_\bullet}(\log D_\bullet))$ depends only on U/κ?

P. Berthelot, S. Bloch and H. Esnault have proved the following impressive theorem:

Theorem 13.6 ([**10**]). — *Let X be a proper scheme over κ. Then there exists a functorial morphism*

$$H^h_{\mathrm{rig}}(X/K_0) \longrightarrow H^h\big(X, \mathcal{W}(\mathcal{O}_X)\big)_{K_0} \quad (h \in \mathbb{N})$$

which induces an isomorphism $H^h_{\mathrm{rig}}(X/K_0)_{[0,1)} \xrightarrow{\sim} H^h(X, \mathcal{W}(\mathcal{O}_X))_{K_0}$.

Combining (13.6) with (13.2), we have the following mysterious corollary:

Corollary 13.7. — *Let X be a proper scheme over κ. Let $X_\bullet \to X$ be a good proper hypercovering of X which has the disjoint union of the members of an affine simplicial open covering of $X_\bullet$ over X. Then the natural morphism*

$$H^h\big(X, \mathcal{W}(\mathcal{O}_X)\big)_{K_0} \longrightarrow H^h\big(X_\bullet, \mathcal{W}(\mathcal{O}_{X_\bullet})\big)_{K_0} \quad (h \in \mathbb{N})$$

is a functorial isomorphism.

The aim in the rest of this section is to give an interesting example (13.12) below: an example of a good proper hypercovering $X_\bullet \to X$ of a proper scheme over κ such that $H^1(X, \mathcal{W}(\mathcal{O}_X)) = \kappa^{\mathbb{N}}$ but $H^1(X_\bullet, \mathcal{W}(\mathcal{O}_{X_\bullet})) = 0$ in the case where κ is algebraically closed.

Let κ be a perfect field of characteristic $p > 0$. Let $\mathcal{W}_n$ be the Witt ring of κ of length $n > 0$. Let X be a proper scheme over κ and let $X_\bullet$ be a good proper hypercovering of X. Then we have two spectral sequences

$$E_1^{ij} = H^j_{\mathrm{crys}}(X_i/\mathcal{W}_n) \Longrightarrow H^{i+j}_{\mathrm{crys}}(X_\bullet/\mathcal{W}_n), \tag{13.7.1}$$

$$E_1^{ij} = H^j\big(X_i, \mathcal{W}_n(\mathcal{O}_{X_i})\big) \Longrightarrow H^{i+j}\big(X_\bullet, \mathcal{W}_n(\mathcal{O}_{X_\bullet})\big). \tag{13.7.2}$$

Consider the complexes

$$(13.7.3) \qquad 0 \to H^0_{\mathrm{crys}}(X_0/\mathcal{W}_n) \xrightarrow{\partial^0-\partial^1} H^0_{\mathrm{crys}}(X_1/\mathcal{W}_n) \xrightarrow{\partial^0-\partial^1+\partial^2} H^0_{\mathrm{crys}}(X_2/\mathcal{W}_n),$$

$$(13.7.4) \qquad 0 \to H^0\big(X_0, \mathcal{W}_n(\mathcal{O}_{X_0})\big) \xrightarrow{\partial^0-\partial^1} H^0\big(X_1, \mathcal{W}_n(\mathcal{O}_{X_1})\big) \xrightarrow{\partial^0-\partial^1+\partial^2} H^0\big(X_2, \mathcal{W}_n(\mathcal{O}_{X_2})\big).$$

Here $\partial^i := \delta_i^*$, where $\delta_i \colon X_{\ell+1} \to X_\ell$ $(0 \leq i \leq \ell+1)$ is a standard face morphism. Assume that $X_\bullet$ is split. Then there exists a family $\{N_m\}_{m\in\mathbb{N}}$ of closed and open subschemes of X_m's ($N_m \subset X_m$) such that

$$X_m = \coprod_{0\le \ell\le m} \coprod_{[m]\twoheadrightarrow[\ell]} N_\ell,$$

where the subscripts $[m] \twoheadrightarrow [\ell]$ means a surjective non-decreasing morphism $[m] \to [\ell]$. We have the natural morphism

$$\beta_m \colon N_m \to \mathrm{cosk}_{m-1}^X(X_{\bullet\le m-1})_m \quad (m \in \mathbb{N}).$$

For a nonnegative integer m and an m-truncated simplicial scheme $Y_{\bullet\le m}$ over X, we have a standard face morphism $\delta_i' \colon \mathrm{cosk}_m^X(Y_{\bullet\le m})_{m+1} \to \mathrm{cosk}_m^X(Y_{\bullet\le m})_m = Y_m$ $(0 \le i \le m+1)$. In the projective limit

$$\mathrm{cosk}_m^X(Y_{\bullet\le m})_r = \varprojlim_{\Delta^+_{m[r]}} Y_q$$

over X (see (6.1.3)), we often denote the corresponding scheme to a morphism $\gamma \colon [q] \to [r]$ $(q \le m, r \in \mathbb{N})$ by $Y_q(\gamma)$.

Let $\gamma_0 \colon [2] \to [1]$ and $\gamma_1 \colon [2] \to [1]$ be the surjective non-decreasing functions determined by the equations $\gamma_0(1) = 0$ and $\gamma_1(1) = 1$, respectively. Set $N_1(\gamma_0) = N_1(\gamma_1) = N_1$. Then we have the direct sum

$$X_2 = N_0 \coprod N_1(\gamma_0) \coprod N_1(\gamma_1) \coprod N_2.$$

The complexes (13.7.3) and (13.7.4) are equal to

$$\begin{aligned}(13.7.5)\quad 0 &\to H^0_{\mathrm{crys}}(N_0/\mathcal{W}_n)\\ &\xrightarrow{\partial^0-\partial^1} H^0_{\mathrm{crys}}(N_0/\mathcal{W}_n) \oplus H^0_{\mathrm{crys}}(N_1/\mathcal{W}_n)\\ &\xrightarrow{\partial^0-\partial^1+\partial^2} H^0_{\mathrm{crys}}(N_0/\mathcal{W}_n) \oplus H^0_{\mathrm{crys}}(N_1(\gamma_0)/\mathcal{W}_n)\\ &\qquad\qquad \oplus H^0_{\mathrm{crys}}(N_1(\gamma_1)/\mathcal{W}_n) \oplus H^0_{\mathrm{crys}}(N_2/\mathcal{W}_n),\end{aligned}$$

$$\begin{aligned}(13.7.6)\quad 0 &\to H^0\big(N_0, \mathcal{W}_n(\mathcal{O}_{N_0})\big)\\ &\xrightarrow{\partial^0-\partial^1} H^0\big(N_0, \mathcal{W}_n(\mathcal{O}_{N_0})\big) \oplus H^0\big(N_1, \mathcal{W}_n(\mathcal{O}_{N_1})\big)\\ &\xrightarrow{\partial^0-\partial^1+\partial^2} H^0\big(N_0, \mathcal{W}_n(\mathcal{O}_{N_0})\big) \oplus H^0\big(N_1(\gamma_0), \mathcal{W}_n(\mathcal{O}_{N_1(\gamma_0)})\big)\\ &\qquad \oplus H^0\big(N_1(\gamma_1), \mathcal{W}_n(\mathcal{O}_{N_1(\gamma_1)})\big) \oplus H^0\big(N_2, \mathcal{W}_n(\mathcal{O}_{N_2})\big),\end{aligned}$$

respectively. Then the following holds:

LEMMA 13.8. — 1) *The morphisms* $\partial^0 - \partial^1$ *in* (13.7.5) *and* (13.7.6) *are equal to*

$$a \longmapsto \big(0, (\delta_0'\beta_1)^*(a) - (\delta_1'\beta_1)^*(a)\big)$$

for $a \in H^0_{\mathrm{crys}}(N_0/\mathcal{W}_n)$ *and* $a \in H^0(N_0, \mathcal{W}_n(\mathcal{O}_{N_0}))$*, respectively.*

2) *Let* ∂'^i $(i = 0, 1)$ *be the pull-back of the composite morphism*

$$N_1 \hookrightarrow X_1 \xrightarrow{\delta_i} N_0.$$

Then the morphisms $\partial^0 - \partial^1 + \partial^2$ *in* (13.7.5) *and* (13.7.6) *are equal to*

$$(a, b) \longmapsto \big(a, \partial'^1(a), \partial'^0(a), c\big),$$

where

$$c = \sum_{i=0}^{2} (-1)^i \big\{ (\delta'_i \beta_2)^*(a) + (\delta'_i \beta_2)^*(b) \big\}$$

for $a \in H^0_{\mathrm{crys}}(N_0/\mathcal{W}_n)$*,* $b \in H^0_{\mathrm{crys}}(N_1/\mathcal{W}_n)$ *and* $a \in H^0(N_0, \mathcal{W}_n(\mathcal{O}_{N_0}))$*,* $b \in H^0(N_1, \mathcal{W}_n(\mathcal{O}_{N_1}))$*, respectively.*

Proof. — Because the proof of 1) is simpler than that of 2), we give only the proof of 2). Recall the definitions of the standard degeneracy morphism $s_i \colon X_m \to X_{m+1}$ $(0 \leq i \leq m)$ and the standard face morphism $\delta_i \colon X_{m+1} \to X_m$ $(0 \leq i \leq m+1)$ as in the proof of [**35**, V$^{\mathrm{bis}}$ (5.1.3)]) (and (6.9)).

The degeneracy morphism

$$s_0 \colon N_0 = X_0 \longrightarrow X_1 = N_0 \coprod N_1$$

is equal to the natural inclusion map $N_0 \hookrightarrow N_0 \coprod N_1$.

Let $\mathrm{cosk}_1^X(X_{\bullet \leq 1})_0 \to \mathrm{cosk}_1^X(X_{\bullet \leq 1})_2$ be the corresponding morphism to the map $[2] \to [0]$. Then the restriction of $\delta_i \colon X_2 \longrightarrow X_1$ to N_0 $(\subset X_2)$ is equal to the composite morphism

$$N_0 = X_0 = \mathrm{cosk}_1^X(X_{\bullet \leq 1})_0 \longrightarrow \mathrm{cosk}_1^X(X_{\bullet \leq 1})_2 \xrightarrow{\delta'_i} \mathrm{cosk}_1^X(X_{\bullet \leq 1})_1 = X_1.$$

For any morphism $[1] \to [2]$, the composite morphism $[1] \to [2] \to [0]$ is the unique morphism. Hence the morphism $N_0 \to X_1$ corresponding to the map $[1] \to [0]$ is equal to $s_0 \colon N_0 \to X_1 = N_0 \coprod N_1$.

Next we make the restrictions

$$\delta_0|_{N_1(\gamma_j)}, \delta_1|_{N_1(\gamma_j)}, \delta_2|_{N_1(\gamma_j)} \colon N_1(\gamma_j) \longrightarrow X_1 \quad (j = 0, 1)$$

to $N_1(\gamma_j)$ explicit. For simplicity of notation, we denote $\delta_i|_{N_1(\gamma_j)}$ $(i = 0, 1, 2)$ simply by δ_i. Let $X_1 = \mathrm{cosk}_1^X(X_{\bullet \leq 1})_1 \to \mathrm{cosk}_1^X(X_{\bullet \leq 1})_2$ be the corresponding morphism to the map $\gamma_0 \colon [2] \to [1]$. Set

$$\sigma_i := \sigma_i^2 \colon [1] \longrightarrow [2] \quad (0 \leq i \leq 2)$$

in the proof of (6.9). Then the composite morphisms $[1] \xrightarrow{\sigma_i} [2] \xrightarrow{\gamma_0} [1]$ $(i = 0, 1, 2)$ are the following morphisms

$$\gamma_0\sigma_0 : 0 \longmapsto 0, 1 \longmapsto 1, \quad \gamma_0\sigma_1 : 0 \longmapsto 0, 1 \longmapsto 1, \quad \gamma_0\sigma_2 : 0 \longmapsto 0, 1 \longmapsto 0.$$

Hence $\delta_0, \delta_1 \colon N_1(\gamma_0) \to X_1$ are the natural inclusion $N_1(\gamma_0) = N_1 \hookrightarrow N_0 \coprod N_1$. Set also $\sigma_i := \sigma_i^1 \colon [0] \to [1]$ $(0 \leq i \leq 1)$ in the proof of (6.9). Since $\gamma_0\sigma_2 = ([1] \to [0] \xrightarrow{\sigma_1} [1])$, $\delta_2 \colon N_1(\gamma_0) \to X_1$ is equal to the composite morphism

$$N_1 \subset X_1 \xrightarrow{\delta_1} N_0 \xrightarrow{s_0} X_1.$$

Similarly, we see that $\delta_1, \delta_2 \colon N_1(\gamma_1) \to X_1$ are the natural inclusion $N_1 \hookrightarrow N_0 \coprod N_1$ and $\delta_0 \colon N_1(\gamma_1) \longrightarrow X_1$ is equal to the composite morphism

$$N_1 \subset X_1 \xrightarrow{\delta_0} N_0 \xrightarrow{s_0} X_1.$$

Therefore the morphisms (13.7.5) and (13.7.6) are equal to

$$\begin{aligned}(a, b) \longmapsto \big(a - a + a, (b - b + \partial'^1(a)), (\partial'^0(a) - b + b), c\big)\\ = \big(a, \partial'^1(a), \partial'^0(a), c\big). \end{aligned} \qquad \square$$

Proposition 13.9. — *Assume that κ is algebraically closed and that $\dim X = 1$. Assume also that the augmentation morphism $\beta_0 \colon X_0 \longrightarrow X$ and the morphism $\beta_1 \colon N_1 \longrightarrow X_0 \times_X X_0$ are homeomorphisms as topological spaces. Assume, furthermore, that β_0 and β_1 are birational. Let E_2^{ij} and $(E_2^{ij})_{<1}$ be the E_2^{ij}-terms of the spectral sequences* (13.7.1) *and* (13.7.2), *respectively. Then $E_2^{10} = 0 = (E_2^{10})_{<1}$. Especially*

$$H^1_{\mathrm{crys}}(X_\bullet/\mathcal{W}_n) = \mathrm{Ker}(E_2^{01} \to E_2^{20}),$$
$$H^1(X_\bullet, \mathcal{W}_n(\mathcal{O}_{X_\bullet})) = \mathrm{Ker}\big((E_2^{01})_{<1} \to (E_2^{20})_{<1}\big).$$

Proof. — Since β_0 is homeomorphic and birational and since κ is algebraically closed, we can identify $X_0 \times_X X_0$ with X_0 as topological spaces by the diagonal morphism $X_0 \hookrightarrow X_0 \times_X X_0$ (see [**37**, (3.4.9)]). Then the morphism

$$\delta_i' \colon X_0 \approx X_0 \times_X X_0 \longrightarrow X_0 \quad (i = 0, 1)$$

is the identity of the topological space, and hence we can identify $\delta_i \colon N_1 \hookrightarrow X_1 \to X_0$ with $\beta_1 \colon N_1 \to X_0 \times_X X_0 = X_0$ as a morphism of topological spaces. Let $\gamma_i \colon [0] \to [2]$ be a map defined by $\gamma_i(0) = i$ $(0 \leq i \leq 2)$. Then, by (6.1.3), we have

(13.9.1) $$\begin{aligned}&\mathrm{cosk}_1^X(X_{\bullet \leq 1})_2\\ &\quad = \varprojlim \big(X_0(\gamma_0), X_0(\gamma_1), X_0(\gamma_2), X_1(\delta_0), X_1(\delta_1), X_1(\delta_2)\big).\end{aligned}$$

Because $\beta_1\colon N_1 \to X_0 \times_X X_0$ is homeomorphic and birational, one has $\operatorname{cosk}_1^X(X_{\bullet\leq 1})_2 = X_1$ as topological spaces and because the morphisms $\delta'_0, \delta'_1\colon X_0 = X_0 \times_X X_0 \to X_0$ are the same as a morphism of topological spaces, the three morphisms $\operatorname{cosk}_1^X(X_{\bullet\leq 1})_2 \xrightarrow{\delta'_i} X_1$ $(i = 0, 1, 2)$ are the same. Therefore $c = (\delta'_0\beta_2)^*(a + b)$ in the notation of (13.8), 2). Because $\beta_2\colon N_2 \to \operatorname{cosk}_1^X(X_{\bullet\leq 1})_2$ is surjective, $\delta'_0\beta_2\colon N_2 \to X_1$ is also surjective. Hence

$$\operatorname{Ker}\big(\partial^0 - \partial^1 + \partial^2\colon H^0_{\mathrm{crys}}(X_1/\mathcal{W}_n) \to H^0_{\mathrm{crys}}(X_2/\mathcal{W}_n)\big) = 0.$$

By the same argument, we see that

$$\operatorname{Ker}(\partial^0 - \partial^1 + \partial^2\colon H^0\big(X_1, \mathcal{W}_n(\mathcal{O}_{X_1})\big) \to H^0\big(X_2, \mathcal{W}_n(\mathcal{O}_{X_2})\big) = 0. \qquad \square$$

Lemma 13.10. — *Let Y be a scheme over a perfect field κ of characteristic $p > 0$. Then the following hold*:

1) *Assume that Y is quasi-compact, quasi-separated and smooth over κ and that $H^1_{\mathrm{crys}}(Y/\mathcal{W}_{n_0}) = 0$ for a positive integer n_0. Then $H^1_{\mathrm{crys}}(Y/\mathcal{W}_{nn_0}) = 0$ $(n \in \mathbb{Z}_{\geq 1})$.*

2) *Assume that $H^1(Y, \mathcal{W}_{n_0}(\mathcal{O}_Y)) = 0$ for a positive integer n_0. Then $H^1(Y, \mathcal{W}_{nn_0}(\mathcal{O}_Y)) = 0$ $(n \in \mathbb{Z}_{\geq 1})$.*

Proof. — 1): Follows immediately from the following triangle (the proof of [**12**, (7.16)]) and the induction on n:

$$R\Gamma_{\mathrm{crys}}(Y/\mathcal{W}_{n_0}) \longrightarrow R\Gamma_{\mathrm{crys}}(Y/\mathcal{W}_{nn_0}) \longrightarrow R\Gamma_{\mathrm{crys}}(Y/\mathcal{W}_{(n-1)n_0}) \xrightarrow{+1}.$$

2): Follows immediately follows from the following exact sequence and the induction on n:

$$0 \to \mathcal{W}_{n_0}(\mathcal{O}_Y) \xrightarrow{V^{(n-1)n_0}} \mathcal{W}_{nn_0}(\mathcal{O}_Y) \longrightarrow \mathcal{W}_{(n-1)n_0}(\mathcal{O}_Y) \to 0. \qquad \square$$

Corollary 13.11. — *Let the notations and the assumptions be as in* (13.9). *Then the following hold*:

1) *Assume that X_0 is smooth over κ and that $H^1_{\mathrm{crys}}(X_0/\mathcal{W}_{n_0}) = 0$ for a positive integer n_0. Then $H^1_{\mathrm{crys}}(X_\bullet/\mathcal{W}_{nn_0}) = 0$ $(n \in \mathbb{Z}_{\geq 1})$.*

2) *Assume that $H^1(X_0, \mathcal{W}_{n_0}(\mathcal{O}_{X_0})) = 0$ for a positive integer n_0. Then $H^1(X_\bullet, \mathcal{W}_{nn_0}(\mathcal{O}_{X_\bullet})) = 0$ $(n \in \mathbb{Z}_{\geq 1})$.*

Proof. — The Corollary follows immediately from (13.9) and (13.10). $\square$

Example 13.12. — Let κ be an algebraically closed field of characteristic $p > 0$. Let X be a proper integral curve over κ of genus 0. Let X_0 be the normalization of X. Assume that the morphism $X_0 \to X$ is a homeomorphism.

We identify the points of X_0 with those of X. Then, for a point Q of X, we obtain a natural inclusion $\mathcal{O}_{X,Q} \hookrightarrow \mathcal{O}_{X_0,Q}$. For a closed point Q of X, let t_Q be a uniformizer of the discrete valuation ring $\mathcal{O}_{X_0,Q}$. Recall that X has only a closed point P "de rebroussement ordinaire" (see [**79**, §6]) if $\mathcal{O}_{X_0,Q} = \mathcal{O}_{X,Q}$ for $Q \neq P$ and $\mathcal{O}_{X,P} = \{f \in \mathcal{O}_{X_0,P} \mid f = c_0 + c_2 t_P^2 + c_3 t_P^3 + \cdots \ (c_i \in \kappa)\}$. Let n be a positive integer. Assume that X has only a closed point P de rebroussement ordinaire. Then, by [*loc. cit.*], $H^1(X, \mathcal{W}_n(\mathcal{O}_X)) \simeq \kappa^n$ and $H^1(X, \mathcal{W}(\mathcal{O}_X)) \simeq \kappa^{\mathbb{N}}$. On the other hand, let $X_\bullet$ be a split proper hypercovering of X such that N_1 is the normalization of $X_0 \times_X X_0$. Then N_1 is homeomorphic to $X_0 \times_X X_0$. Indeed, the localization ring of $X_0 \times_X X_0$ at (P, P) is isomorphic to $\kappa[t, t']_{(t,t')}/(t^2 - t'^2, t^3 - t'^3)$. Blow up $X_0 \times_X X_0$ at (P, P) and let T be the strict transform of $X_0 \times_X X_0$ by this blow up. Then the relations $t' = ut$, $t^2 - t'^2 = 0$ and $t^3 - t'^3 = 0$ means that $t = 0$ or $u = 1$. Hence the fiber of T at (P, P) consists of one point, at which it is nonsingular because the localization ring is isomorphic to $\kappa[t]_{(t)}$. Furthermore, it is clear that the morphism $N_1 \to X_0 \times_X X_0$ is birational. By (13.11), $H^1(X_\bullet, \mathcal{W}_n(\mathcal{O}_{X_\bullet})) = 0$ and consequently $H^1(X_\bullet, \mathcal{W}(\mathcal{O}_{X_\bullet})) = 0$.

PART III. WEIGHT FILTRATIONS AND SLOPE FILTRATIONS ON RIGID COHOMOLOGIES WITH CLOSED SUPPORT AND WITH COMPACT SUPPORT

14. Mapping fiber and mapping cone in log crystalline cohomology

In this section we generalize results in §2, §3, §5 and §7 to results on the mapping fibers and the mapping cones of the induced morphisms of log crystalline cohomologies and cohomologies of log de Rham-Witt complexes by a morphism of smooth multisimplicial schemes with multisimplicial relative SNCD's.

First we work in homological algebra. Let the notations be as in §2. Let $\rho\colon (M,P)\to(N,P)$ be a morphism in $\mathrm{CF}(\mathcal{A}^{\underline{\bullet}}) = \mathrm{CF}(\mathcal{A}^{\bullet\cdots\bullet})$ (r-points). Here P's are increasing filtrations. Let $\mathrm{MF}(\rho) := (M,P)\oplus(N,P)[-1]$ be the mapping fiber of ρ. Then

$$\mathbf{s}(\mathrm{MF}(\rho)) = \mathbf{s}(M,P)\oplus(\mathbf{s}(N,P)[-1]).$$

We define the diagonal filtration $\delta_{\mathrm{MF}}(\underline{L},P)$ on $\mathbf{s}(\mathrm{MF}(\rho))$ as follows:

$$\delta_{\mathrm{MF}}(\underline{L},P)_k\mathbf{s}\big(\mathrm{MF}(\rho)\big) := \bigoplus_{\underline{t}\geq 0,s} P_{\underline{t}_r+k}M^{\underline{t}s}\oplus\bigoplus_{\underline{t}\geq 0,s} P_{\underline{t}_r+k+1}N^{\underline{t},s-1}. \tag{14.0.1}$$

Denote by $\langle n\rangle$ ($n\in\mathbb{Z}$) the shift of a filtration in [**24**, (1.1)]:

$$P_\ell\big((M,P)\langle n\rangle\big) := P_{\ell+n}M \tag{14.0.2}$$

(in [*loc. cit.*], it is denoted by $[n]$). The morphism ρ induces the following morphism

$$\rho\langle 1\rangle\colon (M,P)\longrightarrow (N,P)\langle 1\rangle. \tag{14.0.3}$$

By (14.0.1) we have

$$\delta_{\mathrm{MF}}(\underline{L},P)_k\mathbf{s}\big(\mathrm{MF}(\rho)\big) = \delta(\underline{L},P)_k\mathbf{s}\big(\mathrm{MF}(\rho\langle 1\rangle)\big). \tag{14.0.4}$$

Hence we have the following triangle:

$$\cdots \longrightarrow \big(\mathbf{s}(\mathrm{MF}(\rho)), \delta_{\mathrm{MF}}(\underline{L}, P)\big) \longrightarrow \big(\mathbf{s}(M), \delta(\underline{L}, P)\big) \to \big(\mathbf{s}(N), \delta(\underline{L}, P)\big)\langle 1\rangle \xrightarrow{+1} \cdots . \tag{14.0.5}$$

By the definition of $\delta_{\mathrm{MF}}(\underline{L}, P)$, we have

$$\mathrm{gr}_k^{\delta_{\mathrm{MF}}(\underline{L},P)}\mathbf{s}\big(\mathrm{MF}(\rho)\big) = \bigoplus_{\underline{t}\geq 0, s} \mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}s} \oplus \bigoplus_{\underline{t}\geq 0, s} \mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}, s-1}. \tag{14.0.6}$$

Since the morphism $\rho\colon \mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet} \to \mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}\bullet}$ is zero,

(14.0.7)
$$\mathrm{gr}_k^{\delta_{\mathrm{MF}}(\underline{L},P)}\mathbf{s}(\mathrm{MF}(\rho)) = \bigoplus_{\underline{t}\geq 0} \mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}[-\underline{t}_r] \oplus \bigoplus_{\underline{t}\geq 0} \mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}\bullet}[-\underline{t}_r - 1].$$

Assume that the two filtrations $\delta(\underline{L}, P)$'s on $\mathbf{s}(M)$ and $\mathbf{s}(N)$ are exhaustive and complete, that $\mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}$ and $\mathrm{gr}^P_{\underline{t}_r+k} N^{\underline{t}\bullet}$ are quasi-isomorphic to objects of $\mathrm{C}^+(\mathcal{A}^{\underline{\bullet}})$ and that the spectral sequence arising from the two filtrations $\delta(\underline{L}, P)$'s on $\mathbf{s}(M)$ and $\mathbf{s}(N)$ are regular and bounded below. Then we have the convergent spectral sequence:

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \oplus \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r-1}(\mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}\bullet}) \Longrightarrow \mathcal{H}^h\big(\mathbf{s}(\mathrm{MF}(\rho))\big). \tag{14.0.8}$$

Let

$$d(M)\colon \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \longrightarrow \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r+1}(\mathrm{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet})$$

be the boundary morphism arising from the exact sequence

$$0 \to \mathrm{gr}_{k-1}^{\delta(\underline{L},P)}\mathbf{s}(M) \longrightarrow \delta(\underline{L}, P)_k/\delta(\underline{L}, P)_{k-2}\mathbf{s}(M) \longrightarrow \mathrm{gr}_k^{\delta(\underline{L},P)}\mathbf{s}(M) \to 0.$$

Let

$$d(N)\colon \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r-1}(\mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}\bullet}) \longrightarrow \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} N^{\underline{t}\bullet})$$

be the analogous boundary morphism for (N, P). Then the boundary morphism

$$d\colon E_1^{-k,h+k} \longrightarrow E_1^{-k+1,h+k}$$

between the E_1-terms in (14.0.8) is equal to

$$(14.0.9)\quad \begin{pmatrix} d(M) & \rho \\ 0 & -d(N) \end{pmatrix} : \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} M^{\underline{t}\bullet}) \oplus \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r-1}(\mathrm{gr}^P_{\underline{t}_r+k+1} N^{\underline{t}\bullet}) \longrightarrow \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r+1}(\mathrm{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet}) \oplus \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} N^{\underline{t}\bullet}).$$

Next we give the analogues of the above for the mapping cone.

Let $\mathrm{MC}(\rho) := (M,P)[1] \oplus (N,P)$ be the mapping cone of ρ. We define the diagonal filtration $\delta_{\mathrm{MC}}(\underline{L},P)$ on $\mathbf{s}(\mathrm{MC}(\rho))$ as follows:

$$(14.0.10)\quad \delta_{\mathrm{MC}}(\underline{L},P)_k \mathbf{s}\big(\mathrm{MC}(\rho)\big) := \bigoplus_{\underline{t}\geq 0,s} P_{\underline{t}_r+k-1} M^{\underline{t},s+1} \oplus \bigoplus_{\underline{t}\geq 0,s} P_{\underline{t}_r+k} N^{\underline{t}s}.$$

The morphism ρ induces the following morphism

$$(14.0.11)\quad \rho\langle -1\rangle \colon (M,P)\langle -1\rangle \longrightarrow (N,P).$$

By (14.0.10) we have

$$(14.0.12)\quad \delta_{\mathrm{MC}}(\underline{L},P)_k \mathbf{s}\big(\mathrm{MC}(\rho)\big) = \delta(\underline{L},P)_k \mathbf{s}\big(\mathrm{MC}(\rho\langle -1\rangle)\big).$$

Hence we have the following triangle:

$$(14.0.13)\quad \cdots \longrightarrow \big(\mathbf{s}(M), \delta(\underline{L},P)\big)\langle -1\rangle \longrightarrow \big(\mathbf{s}(N), \delta(\underline{L},P)\big) \longrightarrow \big(\mathbf{s}\big(\mathrm{MC}(\rho)\big), \delta_{\mathrm{MC}}(\underline{L},P)\big) \longrightarrow \cdots .$$

By (14.0.10) we have

$$(14.0.14)\quad \mathrm{gr}_k^{\delta_{\mathrm{MC}}(\underline{L},P)} \mathbf{s}\big(\mathrm{MC}(\rho)\big) = \bigoplus_{\underline{t}\geq 0} \mathrm{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet}[-\underline{t}_r+1] \oplus \bigoplus_{\underline{t}\geq 0} \mathrm{gr}^P_{\underline{t}_r+k} N^{\underline{t}\bullet}[-\underline{t}_r].$$

Hence, under the same assumptions in the case of the mapping fiber, we have the convergent spectral sequence:

$$(14.0.15)\quad E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r+1}(\mathrm{gr}^P_{\underline{t}_r+k-1} M^{\underline{t}\bullet}) \oplus \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k} N^{\underline{t}\bullet}) \Longrightarrow \mathcal{H}^h\big(\mathbf{s}(\mathrm{MC}(\rho))\big).$$

The boundary morphism

$$d\colon E_1^{-k,h+k} \longrightarrow E_1^{-k+1,h+k}$$

between the E_1-terms in (14.0.15) is equal to

$$\begin{pmatrix} -d(M) & \rho \\ 0 & d(N) \end{pmatrix}: \tag{14.0.16}$$
$$\bigoplus_{\underline{t}\geq 0}\mathcal{H}^{h-\underline{t}_r+1}(\mathrm{gr}^P_{\underline{t}_r+k-1}M^{\underline{t}\bullet})\oplus\bigoplus_{\underline{t}\geq 0}\mathcal{H}^{h-\underline{t}_r}(\mathrm{gr}^P_{\underline{t}_r+k}N^{\underline{t}\bullet})$$
$$\longrightarrow\bigoplus_{\underline{t}\geq 0}\mathcal{H}^{h-\underline{t}_r+2}(\mathrm{gr}^P_{\underline{t}_r+k-2}M^{\underline{t}\bullet})\oplus\bigoplus_{\underline{t}\geq 0}\mathcal{H}^{h-\underline{t}_r+1}(\mathrm{gr}^P_{\underline{t}_r+k-1}N^{\underline{t}\bullet}).$$

The filtrations $\delta_{\mathrm{MF}}(\underline{L},P)$ and $\delta_{\mathrm{MC}}(\underline{L},P)$ are well-defined for the morphisms of objects of $\mathrm{DF}(\mathcal{A}^{\underline{\bullet}})$, that is, we have the functor

$$\mathrm{Mor}\big(\mathrm{DF}(\mathcal{A}^{\underline{\bullet}})\big)\ni[\rho]\longmapsto\big[(\mathbf{s}(\mathrm{M}\mathfrak{FC}(\rho)),\delta_{\mathrm{M}\mathfrak{FC}}(\underline{L},P))\big]\in\mathrm{DF}(\mathcal{A}),$$

where $\mathfrak{FC}$ is a letter F or C.

Next we consider the decreasing stupid filtration on the mapping fiber and the mapping cone. Let $\rho\colon M\to N$ be a morphism in $\mathrm{C}^+(\mathcal{A}^{\underline{\bullet}})$. Then ρ induces a morphism

$$\rho\colon(M,\sigma)\longrightarrow(N,\sigma)$$

of filtered complexes of $\mathcal{A}^{\underline{\bullet}}$-modules, where σ's are decreasing stupid filtrations. Then we define decreasing stupid filtrations on $\mathbf{s}(\mathrm{MF}(\rho))$ and $\mathbf{s}(\mathrm{MC}(\rho))$ as follows:

$$\sigma^i\mathbf{s}\big(\mathrm{MF}(\rho)\big)=\bigoplus_{\underline{t}\geq 0,s\geq i}M^{\underline{t}s}\oplus\bigoplus_{\underline{t}\geq 0,s-1\geq i}N^{\underline{t},s-1}\quad(i\in\mathbb{N}), \tag{14.0.17}$$

$$\sigma^i\mathbf{s}\big(\mathrm{MC}(\rho)\big)=\bigoplus_{\underline{t}\geq 0,s+1\geq i}M^{\underline{t},s+1}\oplus\bigoplus_{\underline{t}\geq 0,s\geq i}N^{\underline{t}s}\quad(i\in\mathbb{N}). \tag{14.0.18}$$

Let $\rho^i\colon M^i\longrightarrow N^i$ $(i\in\mathbb{Z})$ be the degree i-part of ρ. Then

$$\begin{aligned}\mathrm{gr}^i_\sigma\mathbf{s}\big(\mathrm{MF}(\rho)\big)&=\bigoplus_{\underline{t}\geq 0}M^{\underline{t}i}[-\underline{t}_r]\{-i\}\oplus\bigoplus_{\underline{t}\geq 0}N^{\underline{t}i}[-\underline{t}_r-1]\{-i\}\\&=\mathbf{s}\big(\mathrm{MF}(\rho^i\{-i\}\colon M^i\{-i\}\to N^i\{-i\})\big),\end{aligned} \tag{14.0.19}$$

$$\begin{aligned}\mathrm{gr}^i_\sigma\mathbf{s}\big(\mathrm{MC}(\rho)\big)&=\bigoplus_{\underline{t}\geq 0}M^{\underline{t}i}[-\underline{t}_r+1]\{-i\}\oplus\bigoplus_{\underline{t}\geq 0}N^{\underline{t}i}[-\underline{t}_r]\{-i\}\\&=\mathbf{s}\big(\mathrm{MC}(\rho^i\{-i\}\colon M^i\{-i\}\to N^i\{-i\})\big).\end{aligned} \tag{14.0.20}$$

Hence we have the spectral sequences:

$$E_1^{i,h-i}=\mathcal{H}^{h-i}\big(\mathbf{s}(\mathrm{MF}(\rho^i))\big)\Longrightarrow\mathcal{H}^h\big(\mathbf{s}(\mathrm{MF}(\rho))\big), \tag{14.0.21}$$

$$E_1^{i,h-i}=\mathcal{H}^{h-i}\big(\mathbf{s}(\mathrm{MC}(\rho^i))\big)\Longrightarrow\mathcal{H}^h\big(\mathbf{s}(\mathrm{MC}(\rho))\big). \tag{14.0.22}$$

The E_1-terms of (14.0.21) and (14.0.22) fit respectively into the exact sequences

(14.0.23) $$\cdots \longrightarrow E_1^{i,h-i} \longrightarrow \mathcal{H}^{h-i}\big(\mathbf{s}(M^i)\big) \longrightarrow \mathcal{H}^{h-i}\big(\mathbf{s}(N^i)\big) \longrightarrow \cdots$$

(14.0.24) $$\cdots \longrightarrow \mathcal{H}^{h-i}\big(\mathbf{s}(M^i)\big) \longrightarrow \mathcal{H}^{h-i}\big(\mathbf{s}(N^i)\big) \longrightarrow E_1^{i,h-i} \longrightarrow \cdots .$$

Now we consider the mapping fiber and the mapping cone in log crystalline cohomologies. Let $(S,\mathcal{I},\gamma)$ and S_0 be as in §3. Let

(14.0.25) $$\rho_{\underline{\bullet}} \colon (Y_{\underline{\bullet}}, E_{\underline{\bullet}} \cup W_{\underline{\bullet}}) \longrightarrow (X_{\underline{\bullet}}, D_{\underline{\bullet}} \cup Z_{\underline{\bullet}})$$

be a morphism of smooth r-simplicial schemes with transversal r-simplicial relative SNCD's over S_0 which induces two morphisms

$$\rho_{(Y_{\underline{\bullet}},E_{\underline{\bullet}})} \colon (Y_{\underline{\bullet}}, E_{\underline{\bullet}}) \longrightarrow (X_{\underline{\bullet}}, D_{\underline{\bullet}}), \quad \rho_{(Y_{\underline{\bullet}},W_{\underline{\bullet}})} \colon (Y_{\underline{\bullet}}, W_{\underline{\bullet}}) \longrightarrow (X_{\underline{\bullet}}, Z_{\underline{\bullet}}).$$

Let $f_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S}$ and $f_{(Y_{\underline{\bullet}},W_{\underline{\bullet}})/S}$ be the structural morphisms of $(Y_{\underline{\bullet}}, E_{\underline{\bullet}} \cup W_{\underline{\bullet}})/S$ and $(Y_{\underline{\bullet}}, W_{\underline{\bullet}})/S$, respectively. Let $f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}$ and $f_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S}$ be the analogous structural morphisms. Then $\rho_{\underline{\bullet}}$ induces two morphisms

(14.0.26)
$$\begin{aligned}\rho^{\log *}_{\underline{\bullet}\mathrm{crys}} \colon [Rf_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}(E^{\log, Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{D_{\underline{\bullet}}})] \\ \longrightarrow [Rf_{(Y_{\underline{\bullet}},W_{\underline{\bullet}})/S*}(E^{\log, W_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{E_{\underline{\bullet}}})]\end{aligned}$$

(14.0.27)
$$\begin{aligned}\rho^{\log *}_{\underline{\bullet}\mathrm{crys}\underline{\bullet}} \colon [Rf_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S_{\underline{\bullet}}*}(E^{\log, Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})), P^{D_{\underline{\bullet}}}] \\ \longrightarrow [Rf_{(Y_{\underline{\bullet}},W_{\underline{\bullet}})/S_{\underline{\bullet}}*}(E^{\log, W_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})), P^{E_{\underline{\bullet}}}].\end{aligned}$$

Then

(14.0.28) $$\rho^{\log *}_{\underline{\bullet}\mathrm{crys}} = (\mathbf{s}, \delta)(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}\underline{\bullet}}).$$

By (14.0.5) and (14.0.13), we have the following:

PROPOSITION 14.1. — *There exist the following triangles*:

$$(14.1.1)\quad \longrightarrow \big(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}), \delta_{\mathrm{MF}}(\underline{L}, P)\big) \longrightarrow \big[Rf_{(X_{\underline{\bullet}}, Z_{\underline{\bullet}})/S*}(E^{\log, Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{D_{\underline{\bullet}}})\big] \longrightarrow \big[Rf_{(Y_{\underline{\bullet}}, W_{\underline{\bullet}})/S*}(E^{\log, W_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(Y_{\underline{\bullet}}, E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{E_{\underline{\bullet}}})\big]\langle 1\rangle \xrightarrow{+1},$$

$$(14.1.2)\quad \longrightarrow \big[Rf_{(X_{\underline{\bullet}}, Z_{\underline{\bullet}})/S*}(E^{\log, Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}}, D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{D_{\underline{\bullet}}})\big]\langle -1\rangle \longrightarrow \big[Rf_{(Y_{\underline{\bullet}}, W_{\underline{\bullet}})/S*}(E^{\log, W_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(Y_{\underline{\bullet}}, E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{E_{\underline{\bullet}}})\big] \longrightarrow \big(\mathrm{MC}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}), \delta_{\mathrm{MC}}(\underline{L}, P)\big) \xrightarrow{+1}.$$

By (14.0.8), (14.0.15) and (3.1.2), we have the following spectral sequences:

THEOREM 14.2. — *There exist the following spectral sequences*:

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k} f_{(D^{(\underline{t}_r+k)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S*} \big(\mathcal{O}_{(D^{(\underline{t}_r+k)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k)}_{\underline{t}}})/S} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}})\big)\big(-(\underline{t}_r+k)\big) \oplus \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k-2} f_{(E^{(\underline{t}_r+k+1)}_{\underline{t}}, W|_{E^{(\underline{t}_r+k+1)}_{\underline{t}}})/S*} \big(\mathcal{O}_{(E^{(\underline{t}_r+k+1)}_{\underline{t}}, W|_{E^{(\underline{t}_r+k+1)}_{\underline{t}}})/S} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k+1)\log}_{\mathrm{crys}}(E_{\underline{t}}/S; W_{\underline{t}})\big)\big(-(\underline{t}_r+k+1)\big)$$

$$(14.2.1)\qquad \Longrightarrow \mathcal{H}^h\big(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}})\big),$$

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k+2} f_{(D^{(\underline{t}_r+k-1)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k-1)}_{\underline{t}}})/S*} \big(\mathcal{O}_{(D^{(\underline{t}_r+k-1)}_{\underline{t}}, Z|_{D^{(\underline{t}_r+k-1)}_{\underline{t}}})/S} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k-1)\log}_{\mathrm{crys}}(D_{\underline{t}}/S; Z_{\underline{t}})\big)\big(-(\underline{t}_r+k-1)\big) \oplus \bigoplus_{\underline{t}\geq 0} R^{h-2\underline{t}_r-k} f_{(E^{(\underline{t}_r+k)}_{\underline{t}}, W|_{E^{(\underline{t}_r+k)}_{\underline{t}}})/S*} \big(\mathcal{O}_{(E^{(\underline{t}_r+k)}_{\underline{t}}, W|_{E^{(\underline{t}_r+k)}_{\underline{t}}})/S} \otimes_{\mathbb{Z}} \varpi^{(\underline{t}_r+k)\log}_{\mathrm{crys}}(E_{\underline{t}}/S; W_{\underline{t}})\big)\big(-(\underline{t}_r+k)\big)$$

$$(14.2.2)\qquad \Longrightarrow \mathcal{H}^h\big(\mathrm{MC}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}})\big).$$

DEFINITION 14.3. — We call (14.2.1) (resp. (14.2.2)) the *preweight spectral sequence* of the mapping fiber (resp. mapping cone) of $\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}$ with respect to $D_{\underline{\bullet}}$ and $E_{\underline{\bullet}}$. If $Z_{\underline{\bullet}} = W_{\underline{\bullet}} = \phi$, then we call (14.2.1) (resp. (14.2.2)) the *preweight spectral sequence* of the mapping fiber (resp. mapping cone) of $\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}$. If S

is a p-adic formal $\mathcal{V}$-scheme in the sense of [**74**, §1], if $S_0 = \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$ and if $X_{\underline{\bullet}}$ and $Y_{\underline{\bullet}}$ are proper over S_0, then we call them the *weight spectral sequences* of the mapping fiber and the mapping cone of $\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}$ with respect to $D_{\underline{\bullet}}$ and $E_{\underline{\bullet}}$, respectively. Furthermore, if $Z_{\underline{\bullet}} = W_{\underline{\bullet}} = \phi$, then we call them the *weight spectral sequence* of the mapping fiber and the mapping cone of $\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}$, respectively

Proposition 14.4. — 1) *The natural exact sequence*

$$(14.4.1)\qquad \cdots \longrightarrow \mathcal{H}^h(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}})) \longrightarrow R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}) \longrightarrow R^h f_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S*}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S}) \longrightarrow \cdots$$

is compatible with the preweight filtration with respect to $\delta_{\mathrm{MF}}(\underline{L},P)$, $D_{\underline{\bullet}}$ *and* $E_{\underline{\bullet}}$.

2) *The natural exact sequence*

$$(14.4.2)\qquad \cdots \longrightarrow R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}) \longrightarrow R^h f_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S*}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S}) \longrightarrow \mathcal{H}^h\big(\mathrm{MC}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}})\big) \longrightarrow \cdots$$

is compatible with the preweight filtration with respect to $D_{\underline{\bullet}}$, $E_{\underline{\bullet}}$ *and* $\delta_{\mathrm{MC}}(\underline{L},P)$.

Proof. — We give the proof only for the mapping fiber. (Note that the triangles in (14.1) do not show (14.4) because the preweight filtration depends on the degrees of cohomologies.)

First the natural morphism

$$\big[Rf_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})),\delta(\underline{L},P^{D_{\underline{\bullet}}})\big] \longrightarrow \big[Rf_{(Y_{\underline{\bullet}},W_{\underline{\bullet}})/S*}(E^{\log,W_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})),\delta(\underline{L},P^{E_{\underline{\bullet}}})\big]$$

induces the morphism

$$R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S}) \longrightarrow R^h f_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S*}(\mathcal{O}_{(Y_{\underline{\bullet}},E_{\underline{\bullet}}\cup W_{\underline{\bullet}})/S})$$

which is compatible with the preweight filtration with respect to $D_{\underline{\bullet}}$ and $E_{\underline{\bullet}}$.

By the definition of $\delta_{\mathrm{MF}}(\underline{L},P)$ in (14.0.1), the natural morphism

$$(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}}),\delta_{\mathrm{MF}}\big(\underline{L},P)\big) \longrightarrow \big[Rf_{(X_{\underline{\bullet}},Z_{\underline{\bullet}})/S*}(E^{\log,Z_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})),\delta(\underline{L},P^{D_{\underline{\bullet}}})\big]$$

induces the morphism

$$\mathcal{H}^h\big(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{crys}})\big) \longrightarrow R^h f_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S*}(\mathcal{O}_{(X_{\underline{\bullet}},D_{\underline{\bullet}}\cup Z_{\underline{\bullet}})/S})$$

which is compatible with the preweight filtration with respect to $\delta_{\mathrm{MF}}(\underline{L}, P)$ and $D_{\underline{\bullet}}$.

Finally, by the definition of $\delta_{\mathrm{MF}}(\underline{L}, P)$ in (14.0.1), the natural morphism

$$\big[Rf_{(Y_{\underline{\bullet}}, W_{\underline{\bullet}})/S*}(E_{\mathrm{crys}}^{\log, W_{\underline{\bullet}}}(\mathcal{O}_{(Y_{\underline{\bullet}}, E_{\underline{\bullet}} \cup W_{\underline{\bullet}})/S})), \delta(\underline{L}, P^{E_{\underline{\bullet}}})\big]\langle 1\rangle \longrightarrow \big(\mathrm{MF}(\rho_{\underline{\bullet}\mathrm{crys}}^{\log *}), \delta_{\mathrm{MF}}(\underline{L}, P)\big)[1]$$

induces the morphism

$$R^h f_{(Y_{\underline{\bullet}}, E_{\underline{\bullet}} \cup W_{\underline{\bullet}})/S*}\big(\mathcal{O}_{(Y_{\underline{\bullet}}, E_{\underline{\bullet}} \cup W_{\underline{\bullet}})/S}\big) \longrightarrow \mathcal{H}^{h+1}\big(\mathrm{MF}(\rho_{\underline{\bullet}\mathrm{crys}}^{\log *})\big)$$

which is compatible with the preweight filtration with respect to $E_{\underline{\bullet}}$ and $\delta_{\mathrm{MF}}(\underline{L}, P)$. □

Theorem 14.5. — *Let $N_1, \ldots, N_r$ be nonnegative integers. Set*

$$\underline{N} := (N_1, \ldots, N_r).$$

Let u, S_0 and S_0' be as in (3.8). *Let us mean by $'$ the base change with respect to u. Let $\mathfrak{FC}$ be a letter* F *or* C. *Then the base change morphism*

$$Lu^*\big[\mathrm{M}\mathfrak{FC}(\rho_{\underline{\bullet}\leq \underline{N},\mathrm{crys}}^{\log *}), \delta_{\mathrm{M}\mathfrak{FC}}(\underline{L}, P)\big] \to \big[\mathrm{M}\mathfrak{FC}(\rho'^{\log *}_{\underline{\bullet}\leq \underline{N},\mathrm{crys}}), \delta_{\mathrm{M}\mathfrak{FC}}(\underline{L}, P')\big] \tag{14.5.1}$$

is an isomorphism.

Proof. — The Theorem immediately follows from (5.9) and (14.1). □

Let κ be a perfect field of characteristic $p > 0$. Let $\mathcal{V}$, K, $\mathcal{W}$ and K_0 be as in §10.

Corollary 14.6. — *Let S be a p-adic formal $\mathcal{V}$-scheme. Let $\mathfrak{FC}$ be a letter* F *or* C. *Assume that $X_{\underline{\bullet}}$ and $Y_{\underline{\bullet}}$ are proper over $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. Then the image*

$$\mathrm{Im}\big(\mathcal{H}^h\big(\delta_{\mathrm{M}\mathfrak{FC}}(\underline{L}, P)_k \mathrm{M}\mathfrak{FC}(\rho_{\underline{\bullet}\mathrm{crys}}^{\log *})\big) \to \mathcal{H}^h\big(\mathrm{M}\mathfrak{FC}(\rho_{\underline{\bullet}\mathrm{crys}}^{\log *})\big)\big)$$

prolongs to a convergent F-isocrystal on $S/\mathcal{V}$. (We denote the image by $P_k[\mathcal{H}^h(\mathrm{M}\mathfrak{FC}(\rho_{\underline{\bullet}\mathrm{crys}}^{\log *}))]$).

Proof. — The proof is the same as that of [**72**, (2.13.3)] by using (14.5). □

Proposition 14.7. — *Let S be a p-adic formal $\mathcal{V}$-scheme. Assume that $X_{\underline{\bullet}}$ and $Y_{\underline{\bullet}}$ are proper over $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. Then the E_2-terms of* (14.2.1) *and* (14.2.2) *prolong to convergent F-isocrystals on $S/\mathcal{V}$.*

Proof. — By [**72**, (2.13.10)], the Gysin morphism in (5.3.3) extends to a morphism of convergent F-isocrystals on $S/\mathcal{V}$. By (14.0.9), (14.0.16) and (5.3.3), the boundary morphisms between the E_1-terms are morphisms of convergent F-isocrystals on $S/\mathcal{V}$. Hence we obtain (14.7) by [**74**, (3.7), (2.10)]. □

THEOREM 14.8. — *Assume that $Z_{\underline{\bullet}} = W_{\underline{\bullet}} = \phi$ and that $X_{\underline{\bullet}}$ and $Y_{\underline{\bullet}}$ are proper over $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. Let S be a p-adic formal $\mathcal{V}$-scheme. Then the spectral sequences* (14.2.1) *and* (14.2.2) *degenerate at E_2 modulo torsion.*

Proof. — By (8.3), (14.1.1) and (14.1.2), we have the nilpotent deformation invariance of the mapping fiber and the mapping cone of log crystalline cohomologies tensorized with $\mathbb{Q}$. By using (14.5) and (14.7) and by the same proof of [**72**, (2.17.2)], we obtain (14.8). □

THEOREM 14.9. — *Let S be a p-adic formal $\mathcal{V}$-scheme. Set*

$$S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p).$$

Let

$$\begin{array}{ccc} (Y^2_{\underline{\bullet}}, E^2_{\underline{\bullet}}) & \xrightarrow{\rho^2_{\underline{\bullet}}} & (X^2_{\underline{\bullet}}, D^2_{\underline{\bullet}}) \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ (Y^1_{\underline{\bullet}}, E^1_{\underline{\bullet}}) & \xrightarrow{\rho^1_{\underline{\bullet}}} & (X^1_{\underline{\bullet}}, D^1_{\underline{\bullet}}) \end{array}$$

be a commutative diagram of proper smooth r-simplicial schemes over S_0 with r-simplicial relative SNCD's over S_0. Let $\mathfrak{FC}$ be a letter F *or* C. *Then the induced morphism*

$$(u^*, v^*)\colon \mathrm{M}\mathfrak{FC}(\rho^{1\log *}_{\underline{\bullet}\mathrm{crys}})_K \longrightarrow \mathrm{M}\mathfrak{FC}(\rho^{2\log *}_{\underline{\bullet}\mathrm{crys}})_K$$

is strictly compatible with the weight filtration.

Proof. — The proof is the same as that of [**72**, (2.18.2)]. □

THEOREM 14.10. — *Let the notations and the assumptions be as in* (14.8). *The exact sequences* (14.4.1) *and* (14.4.2) *modulo torsions are strictly exact with respect to the weight filtration.*

Proof. — If κ is a finite field, (14.10) is obvious. In the general case, we reduce it to the case of finite fields as in [**72**, (2.18.2) (1)]. □

Now assume that $S = \mathrm{Spf}(\mathcal{W})$, where $\mathcal{W}$ be the Witt ring of a perfect field κ of characteristic $p > 0$ and that $S_0 = \mathrm{Spec}(\kappa)$. Then, for each nonnegative integer i, we have the following morphism by the functoriality of log Hodge-Witt sheaves with weight filtrations (see [**69**, (9.3) (2)]):

$$\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}}\colon \big[R\Gamma(X_{\underline{\bullet}},\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))),\delta(\underline{L},P^{D_{\underline{\bullet}}})\big] \longrightarrow \big[R\Gamma(Y_{\underline{\bullet}},\mathcal{W}\Omega^i_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))),\delta(\underline{L},P^{E_{\underline{\bullet}}})\big]. \tag{14.10.1}$$

Hence we have the following morphism

$$\mathrm{gr}_k(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\colon \mathrm{gr}_k^{\delta(\underline{L},P^{D_{\underline{\bullet}}})}R\Gamma\big(X_{\underline{\bullet}},\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))\big) \longrightarrow \mathrm{gr}_k^{\delta(\underline{L},P^{E_{\underline{\bullet}}})}R\Gamma\big(Y_{\underline{\bullet}},\mathcal{W}\Omega^i_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))\big). \tag{14.10.2}$$

Using the Godement resolutions of $\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))$ and $\mathcal{W}\Omega^i_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))$, we see that the morphism $\rho_{\underline{\bullet}}$ in (14.0.25) induces a morphism

$$\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}}\colon \big[R\Gamma(X_{\underline{\bullet}},\mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))),\delta(\underline{L},P^{D_{\underline{\bullet}}})\big] \longrightarrow \big[R\Gamma(Y_{\underline{\bullet}},\mathcal{W}\Omega^{\bullet}_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))),\delta(\underline{L},P^{E_{\underline{\bullet}}})\big]. \tag{14.10.3}$$

Theorem 14.11. — *Let $\mathfrak{FC}$ be a letter* F *or* C. *Then the following hold*:

1) *There exists the following spectral sequence*

$$E_1^{i,h-i} = H^{h-i}\big(\mathrm{M}\mathfrak{FC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big) \Longrightarrow H^h\big(\mathrm{M}\mathfrak{FC}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}})\big). \tag{14.11.1}$$

2) *The E_1-term $H^{h-i}(\mathrm{MF}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}}))$ fits into the following exact sequence*:

$$\cdots \longrightarrow H^{h-i}\big(\mathrm{MF}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big) \longrightarrow H^{h-i}(X_{\underline{\bullet}},\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))) \longrightarrow H^{h-i}(Y_{\underline{\bullet}},\mathcal{W}\Omega^i_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))) \longrightarrow \cdots. \tag{14.11.2}$$

3) *The E_1-term $H^{h-i}(\mathrm{MC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}}))$ fits into the exact sequence*:

$$\cdots \longrightarrow H^{h-i}\big(X_{\underline{\bullet}},\mathcal{W}\Omega^i_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))\big) \longrightarrow H^{h-i}\big(Y_{\underline{\bullet}},\mathcal{W}\Omega^i_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))\big) \longrightarrow H^{h-i}\big(\mathrm{MC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big) \longrightarrow \cdots. \tag{14.11.3}$$

4) *Assume that $X_{\underline{\bullet}}$ and $Y_{\underline{\bullet}}$ are proper over κ. Then the spectral sequence* (14.11.1) *degenerates at E_1 modulo torsion. There exists the following decomposition*:

$$H^h\big(\mathrm{M}\mathfrak{FC}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}})\big) = \bigoplus_{i=0}^{h} H^{h-i}\big(\mathrm{M}\mathfrak{FC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big). \tag{14.11.4}$$

5) *There exist the following spectral sequences*

$$
\begin{aligned}
E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} & H^{h-i-2\underline{t}_r-k}\big(D_{\underline{t}}^{(\underline{t}_r+k)}, \mathcal{W}\Omega^i_{D_{\underline{t}}^{(\underline{t}_r+k)}}(\log Z|_{D_{\underline{t}}^{(\underline{t}_r+k)}}) \\
& \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(\underline{t}_r+k)}(D_{\underline{t}}/\kappa)\big)(-(\underline{t}_r+k)) \\
\oplus \bigoplus_{\underline{t}\geq 0} & H^{h-i-2\underline{t}_r-k-2}\big(E_{\underline{t}}^{(\underline{t}_r+k+1)}, \mathcal{W}\Omega^i_{E_{\underline{t}}^{(\underline{t}_r+k+1)}}(\log W|_{E_{\underline{t}}^{(\underline{t}_r+k+1)}}) \\
& \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(\underline{t}_r+k+1)}(E_{\underline{t}}/\kappa)\big)(-(\underline{t}_r+k+1)) \\
& \Longrightarrow H^{h-i}\big(\mathrm{MF}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big),
\end{aligned}
\tag{14.11.5}
$$

$$
\begin{aligned}
E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} & H^{h-i-2\underline{t}_r-k+2}\big(D_{\underline{t}}^{(\underline{t}_r+k-1)}, \mathcal{W}\Omega^i_{D_{\underline{t}}^{(\underline{t}_r+k-1)}}(\log Z|_{D_{\underline{t}}^{(\underline{t}_r+k-1)}}) \\
& \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(\underline{t}_r+k-1)}(D_{\underline{t}}/\kappa)\big)(-(\underline{t}_r+k-1)) \\
\oplus \bigoplus_{t\geq 0} & H^{h-i-2\underline{t}_r-k}\big(E_{\underline{t}}^{(\underline{t}_r+k)}, \mathcal{W}\Omega^i_{E_{\underline{t}}^{(\underline{t}_r+k)}}(\log W|_{E_{\underline{t}}^{(\underline{t}_r+k)}}) \\
& \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(\underline{t}_r+k)}(E_{\underline{t}}/\kappa)\big)(-(\underline{t}_r+k)) \\
& \Longrightarrow H^{h-i}\big(\mathrm{MC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}})\big).
\end{aligned}
\tag{14.11.6}
$$

6) *Assume that* $Z_{\underline{\bullet}} = W_{\underline{\bullet}} = \phi$ *and that* $X_{\underline{\bullet}}$ *and* $Y_{\underline{\bullet}}$ *are proper over* κ. *Then the spectral sequences* (14.11.5) *and* (14.11.6) *degenerate at* E_2 *modulo torsion.*

Proof. — 1): The stupid filtration on $\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}})$ immediately gives the spectral sequence (14.11.1). Here we have calculated $R\Gamma$'s on the right hand side of (14.10.3) by using the Godement resolutions of $\mathcal{W}\Omega^{\bullet}_{X_{\underline{\bullet}}}(\log(D_{\underline{\bullet}}\cup Z_{\underline{\bullet}}))$ and $\mathcal{W}\Omega^{\bullet}_{Y_{\underline{\bullet}}}(\log(E_{\underline{\bullet}}\cup W_{\underline{\bullet}}))$.

2), 3): Follow immediately from the definition of the mapping fiber and the mapping cone, respectively.

4): The proof is standard: see the proof of (7.3).

5): Follows from (14.0.8), (14.0.15) and the Poincaré residue isomorphism

$$
\begin{aligned}
\mathrm{Res}^{D_{\underline{t}}} \colon \mathrm{gr}_{\ell}^{P^{D_{\underline{t}}}} & \mathcal{W}_n\Omega^{\bullet}_{X_{\underline{t}}}(\log(D_{\underline{t}}\cup Z_{\underline{t}})) \\
& \xrightarrow{\sim} \mathcal{W}_n\Omega^{\bullet-\ell}_{D_{\underline{t}}^{(\ell)}}(\log Z_{\underline{t}}|_{D_{\underline{t}}^{(\ell)}}) \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(\ell)}(D_{\underline{t}}/\kappa)(-\ell) \quad (\ell\in\mathbb{N})
\end{aligned}
$$

in [**72**, (2.12.4.2)] (cf. [**66**, 1.4.5], [**69**, (9.3) (1)]).

6): We obtain this assertion as in (7.10) by using (14.8). □

Corollary 14.12. — *The slopes of* $H^h(\mathrm{MF}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}}))_{K_0}$ (*resp.* $H^h(\mathrm{MC}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}}))_{K_0}$) *lie in* $[0,h]$ (*resp.* $[0,h+1]$).

Proof. — The Corollary follows immediately from (14.11.1), (14.11.2) and (14.11.3). ☐

Definition 14.13. — We call the spectral sequence (14.11.1) the *slope spectral sequence* of $H^h(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}}))$. We call the direct sum decomposition (14.11.4) the *slope decomposition* of $H^h(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *}_{\underline{\bullet}\mathrm{dRW}}))$. We call the spectral sequences (14.11.5) and (14.11.6) the *weight spectral sequences* of $H^{h-i}(\mathrm{MF}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}}))$ and $H^{h-i}(\mathrm{MC}(\rho^{\log *i}_{\underline{\bullet}\mathrm{dRW}}))$, respectively, and we denote by $P := \{P_k\}_{k\in\mathbb{Z}}$ the induced filtrations.

Next, we prove the Künneth formula of the mapping fiber and the mapping cone of the induced morphism of log crystalline cohomologies by a morphism of smooth multi-truncated multisimplicial schemes with multi-truncated multisimplicial relative SNCD's.

Let $(S,\mathcal{I},\gamma)$ and S_0 be as in §3. Let Y be a quasi-compact smooth scheme over S_0. Assume that S is quasi-compact. Let $(X^i_{\underline{\bullet}}, D^i_{\underline{\bullet}}\cup Z^i_{\underline{\bullet}})$ and $(Y^i_{\underline{\bullet}}, E^i_{\underline{\bullet}}\cup W^i_{\underline{\bullet}})$ $(i=1,2)$ be smooth r-simplicial schemes with transversal r-simplicial relative SNCD's over S_0. Let $\rho^i\colon (Y^i_{\underline{\bullet}}, E^i_{\underline{\bullet}}\cup W^i_{\underline{\bullet}}) \to (X^i_{\underline{\bullet}}, D^i_{\underline{\bullet}}\cup Z^i_{\underline{\bullet}})$ be a morphism of r-simplicial log schemes which induces two morphisms $\rho^i_{E^i_{\underline{\bullet}}}\colon (Y^i_{\underline{\bullet}}, E^i_{\underline{\bullet}}) \to (X^i_{\underline{\bullet}}, D^i_{\underline{\bullet}})$ and $\rho^i_{W^i_{\underline{\bullet}}}\colon (Y^i_{\underline{\bullet}}, W^i_{\underline{\bullet}}) \to (X^i_{\underline{\bullet}}, Z^i_{\underline{\bullet}})$ of log schemes. We assume that the underlying structural morphisms $X^i_{\underline{\bullet}} \to S_0$ and $Y^i_{\underline{\bullet}} \to S_0$ $(i=1,2)$ are quasi-compact and quasi-separated. Set

$$
\begin{aligned}
(Y^{12}_{\underline{\bullet}}, E^{12}_{\underline{\bullet}}\cup W^{12}_{\underline{\bullet}}) &:= (Y^1_{\underline{\bullet}}, E^1_{\underline{\bullet}}\cup W^1_{\underline{\bullet}}) \times_{S_0} (Y^2_{\underline{\bullet}}, E^2_{\underline{\bullet}}\cup W^2_{\underline{\bullet}}),\\
(XY^{12}_{\underline{\bullet}}, DE^{12}_{\underline{\bullet}}\cup ZW^{12}_{\underline{\bullet}}) &:= (X^1_{\underline{\bullet}}, D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}}) \times_{S_0} (Y^2_{\underline{\bullet}}, E^2_{\underline{\bullet}}\cup W^2_{\underline{\bullet}}),\\
(YX^{12}_{\underline{\bullet}}, ED^{12}_{\underline{\bullet}}\cup WZ^{12}_{\underline{\bullet}}) &:= (Y^1_{\underline{\bullet}}, E^1_{\underline{\bullet}}\cup W^1_{\underline{\bullet}}) \times_{S_0} (X^2_{\underline{\bullet}}, D^2_{\underline{\bullet}}\cup Z^2_{\underline{\bullet}}),\\
(X^{12}_{\underline{\bullet}}, D^{12}_{\underline{\bullet}}\cup Z^{12}_{\underline{\bullet}}) &:= (X^1_{\underline{\bullet}}, D^1_{\underline{\bullet}}\cup Z^1_{\underline{\bullet}}) \times_{S_0} (X^2_{\underline{\bullet}}, D^2_{\underline{\bullet}}\cup Z^2_{\underline{\bullet}}),\\
(Y^{12}_{\underline{\bullet}}, W^{12}_{\underline{\bullet}}) &:= (Y^1_{\underline{\bullet}}, W^1_{\underline{\bullet}}) \times_{S_0} (Y^2_{\underline{\bullet}}, W^2_{\underline{\bullet}}),\\
(XY^{12}_{\underline{\bullet}}, ZW^{12}_{\underline{\bullet}}) &:= (X^1_{\underline{\bullet}}, Z^1_{\underline{\bullet}}) \times_{S_0} (Y^2_{\underline{\bullet}}, W^2_{\underline{\bullet}}),\\
(YX^{12}_{\underline{\bullet}}, WZ^{12}_{\underline{\bullet}}) &:= (Y^1_{\underline{\bullet}}, W^1_{\underline{\bullet}}) \times_{S_0} (X^2_{\underline{\bullet}}, Z^2_{\underline{\bullet}}),\\
(X^{12}_{\underline{\bullet}}, Z^{12}_{\underline{\bullet}}) &:= (X^1_{\underline{\bullet}}, Z^1_{\underline{\bullet}}) \times_{S_0} (X^2_{\underline{\bullet}}, Z^2_{\underline{\bullet}}).
\end{aligned}
\tag{14.13.1}
$$

Here $E^{12}_{\underline{\bullet}} := (E^1_{\underline{\bullet}} \times_{S_0} Y^2_{\underline{\bullet}}) \cup (Y^1_{\underline{\bullet}} \times_{S_0} E^2_{\underline{\bullet}})$, $DE^{12}_{\underline{\bullet}} := (D^1_{\underline{\bullet}} \times_{S_0} Y^2_{\underline{\bullet}}) \cup (X^1_{\underline{\bullet}} \times_{S_0} E^2_{\underline{\bullet}})$ and so on are analogous unions of direct products.

We have the following four morphisms of log schemes:

$$(\rho^1,\mathrm{id})\colon (Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})\longrightarrow (XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet}), \tag{14.13.2}$$

$$(\mathrm{id},\rho^2)\colon (Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})\longrightarrow (YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet}), \tag{14.13.3}$$

$$(\rho^1,\mathrm{id})\colon (YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})\longrightarrow (X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet}), \tag{14.13.4}$$

$$(\mathrm{id},\rho^2)\colon (XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})\longrightarrow (X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet}). \tag{14.13.5}$$

Hence we have the four morphisms:

$$\begin{aligned}(\rho^1,\mathrm{id})^*\colon \big[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}\big(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{DE^{12}_{\bullet}})\big]\\ \longrightarrow \big[Rf_{(Y^{12}_{\bullet},W^{12}_{\bullet})/S*}\big(E^{\log,W^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{E^{12}_{\bullet}})\big],\end{aligned} \tag{14.13.6}$$

$$\begin{aligned}(\mathrm{id},\rho^2)^*\colon \big[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}\big(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{ED^{12}_{\bullet}})\big]\\ \longrightarrow \big[Rf_{(Y^{12}_{\bullet},W^{12}_{\bullet})/S*}\big(E^{\log,W^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{E^{12}_{\bullet}})\big],\end{aligned} \tag{14.13.7}$$

$$\begin{aligned}(\rho^1,\mathrm{id})^*\colon \big[Rf_{(X^{12}_{\bullet},Z^{12}_{\bullet})/S*}\big(E^{\log,Z^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{D^{12}_{\bullet}})\big]\\ \longrightarrow \big[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}\big(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{ED^{12}_{\bullet}})\big],\end{aligned} \tag{14.13.8}$$

$$\begin{aligned}(\mathrm{id},\rho^2)^*\colon \big[Rf_{(X^{12}_{\bullet},Z^{12}_{\bullet})/S*}\big(E^{\log,Z^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{D^{12}_{\bullet}})\big]\\ \longrightarrow \big[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}\big(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})\big),\\ \delta(\underline{L},P^{DE^{12}_{\bullet}})\big].\end{aligned} \tag{14.13.9}$$

Consider the two morphisms:

$$(14.13.10)\qquad (\rho^1,\mathrm{id})^*\oplus(\mathrm{id},\rho^2)^*$$
$$: \bigl[Rf_{(X^{12}_{\bullet},Z^{12}_{\bullet})/S*}(E^{\log,Z^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet})/S})),\delta(\underline{L},P^{D^{12}_{\bullet}})\bigr]$$
$$\longrightarrow \bigl[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})),\delta(\underline{L},P^{ED^{12}_{\bullet}})\bigr]$$
$$\oplus[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})),\delta(\underline{L},P^{DE^{12}_{\bullet}})],$$

$$(14.13.11)\qquad (\mathrm{id},\rho^2)^*+(\rho^1,\mathrm{id})^*$$
$$:[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})),\delta(\underline{L},P^{ED^{12}_{\bullet}})]$$
$$\oplus[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})),\delta(\underline{L},P^{DE^{12}_{\bullet}})]$$
$$\longrightarrow[Rf_{(Y^{12}_{\bullet},W^{12}_{\bullet})/S*}(E^{\log,W^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})/S})),\delta(\underline{L},P^{E^{12}_{\bullet}})].$$

First we take the mapping fiber of (14.13.11):

$$[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})),\delta(\underline{L},P^{ED^{12}_{\bullet}})]$$
$$\oplus[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})),\delta(\underline{L},P^{DE^{12}_{\bullet}})]$$
$$\oplus[Rf_{(Y^{12}_{\bullet},W^{12}_{\bullet})/S*}(E^{\log,W^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})/S})),\delta(\underline{L},P^{E^{12}_{\bullet}})]\langle 1\rangle[-1].$$

Next, by using the morphism (14.13.10), we have the complex

(14.13.12)

$$[Rf_{(X^{12}_{\bullet},Z^{12}_{\bullet})/S*}(E^{\log,Z^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},D^{12}_{\bullet}\cup Z^{12}_{\bullet})/S})),\delta(\underline{L},P^{D^{12}_{\bullet}})]$$
$$\oplus[Rf_{(YX^{12}_{\bullet},WZ^{12}_{\bullet})/S*}(E^{\log,WZ^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\bullet},ED^{12}_{\bullet}\cup WZ^{12}_{\bullet})/S})),\delta(\underline{L},P^{ED^{12}_{\bullet}})]\langle 1\rangle[-1]$$
$$\oplus[Rf_{(XY^{12}_{\bullet},ZW^{12}_{\bullet})/S*}(E^{\log,ZW^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(XY^{12}_{\bullet},DE^{12}_{\bullet}\cup ZW^{12}_{\bullet})/S})),\delta(\underline{L},P^{DE^{12}_{\bullet}})]\langle 1\rangle[-1]$$
$$\oplus[Rf_{(Y^{12}_{\bullet},W^{12}_{\bullet})/S*}(E^{\log,W^{12}_{\bullet}}_{\mathrm{crys}}(\mathcal{O}_{(Y^{12}_{\bullet},E^{12}_{\bullet}\cup W^{12}_{\bullet})/S})),\delta(\underline{L},P^{E^{12}_{\bullet}})]\langle 2\rangle[-2].$$

Definition 14.14. — We denote the complex (14.13.12) by

$$(14.14.1)\quad \bigl[Rf_{(X^{12}_{\bullet},(XY^{12}_{\bullet}\cup YX^{12}_{\bullet}))/S*}(E^{\log}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},(XY^{12}_{\bullet}\cup YX^{12}_{\bullet}))/S})),\delta(\underline{L},P^{\log})\bigr].$$

If $Z_{\bullet}=W_{\bullet}=\phi$, then we denote this complex by

$$(14.14.2)\quad [Rf_{(X^{12}_{\bullet},(XY^{12}_{\bullet}\cup YX^{12}_{\bullet}))/S*}(E_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\bullet},(XY^{12}_{\bullet}\cup YX^{12}_{\bullet}))/S})),\delta(\underline{L},P)].$$

Theorem 14.15. — *Let* p_1 (*resp.* p_2) *be the first* (*resp. second*) *projections for the last four objects in* (14.13.1). *Then the following hold*:

1) *Let h be a nonnegative integer. Let N be a nonnegative integer satisfying* (2.2.1)*. Then the cohomology $\mathcal{H}^h$ of the N-truncated simplicial version of the complex* (14.14.1)*, that is, in the case $\underline{\bullet} = \bullet$ and $\bullet \leq N$, is canonically isomorphic to*

$$\mathcal{H}^h\big(Lp_1^*\big(\mathrm{MF}(\rho^{\log *}_{1\bullet \leq N,\mathrm{crys}}), \delta_{\mathrm{MF}}(L,P)\big) \otimes^L_{\mathcal{O}_S} Lp_2^*(\mathrm{MF}(\rho^{\log *}_{2\bullet \leq N,\mathrm{crys}}), \delta_{\mathrm{MF}}(L,P))\big).$$

2) *Assume that the four complexes appearing in* (14.13.12) *as direct factors are filteredly quasi-isomorphic to filtered bounded complexes. Then the complex* (14.13.12) *is canonically isomorphic to*

$$Lp_1^*\big(\mathrm{MF}(\rho^{\log *}_{1\underline{\bullet}\mathrm{crys}}), \delta_{\mathrm{MF}}(\underline{L},P)\big) \otimes^L_{\mathcal{O}_S} Lp_2^*\big(\mathrm{MF}(\rho^{\log *}_{2\underline{\bullet}\mathrm{crys}}), \delta_{\mathrm{MF}}(\underline{L},P)\big).$$

This isomorphism is compatible with the base change of $(S,\mathcal{I},\gamma)$.

Proof. — (14.15) follows from the filtered log Künneth formulas (5.10.1), (5.10.2) and Eilenberg-Zilber's theorem. □

Theorem 14.16. — *Let $\mathcal{V}$, K, S and S_0 be as in* (5.4)*. Let k and h be two integers. Let $\mathcal{R}$ be the complex* (14.14.1)*. Assume that $X^i_{\underline{\bullet}}$ and $Y^i_{\underline{\bullet}}$ $(i=1,2)$ are proper over S_0. Then the following hold*:

1) *The weight filtration*

$$P^{\log}_k\mathcal{H}^h(\mathcal{R})_K := \mathrm{Im}\big(\mathcal{H}^h(\delta(\underline{L},P^{\log})_k\mathcal{R}) \to \mathcal{H}^h(\mathcal{R})\big)_K \quad (k\in\mathbb{Z})$$

prolongs to a convergent F-isocrystal on $S/\mathcal{V}$.

2) *Assume that $Z_{\underline{\bullet}} = W_{\underline{\bullet}} = \phi$. Then the following spectral sequence*

$$E_1^{-k,h+k} = \bigoplus_{\underline{t}\geq 0} \mathcal{H}^{h-\underline{t}_r-k} f_*(\mathrm{gr}^P_{\underline{t}_r+k}\mathcal{R}^{\underline{t}\bullet}) \Longrightarrow \mathcal{H}^h(\mathcal{R}) \tag{14.16.1}$$

degenerates at E_2. This spectral sequence prolongs to that of convergent F-isocrystals on $S/\mathcal{V}$.

Proof. — 1): The proof is the same as that of [**72**, (2.13.3)] by using the base change theorem for the $\underline{N} = (N_1,\ldots,N_r)$-truncated version of the complex (14.13.12) $(N_1,\ldots,N_r\in\mathbb{N})$.

2): The proof is the same as that of [**72**, (2.17.2)]. □

Remark 14.17. — The corresponding object to (14.14.1) for the mapping cone is:

$$[Rf_{(X^{12}_{\underline{\bullet}},Z^{12}_{\underline{\bullet}})/S*}(E^{\log,Z^{12}_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(X^{12}_{\underline{\bullet}},D^{12}_{\underline{\bullet}}\cup Z^{12}_{\underline{\bullet}})/S})),\delta(\underline{L},P^{D^{12}_{\underline{\bullet}}})]\langle -2\rangle[2] \tag{14.17.1}$$

$$\oplus[Rf_{(YX^{12}_{\underline{\bullet}},WZ^{12}_{\underline{\bullet}})/S*}(E^{\log,WZ^{12}_{\underline{\bullet}}}_{\mathrm{crys}}(\mathcal{O}_{(YX^{12}_{\underline{\bullet}},ED^{12}_{\underline{\bullet}}\cup WZ^{12}_{\underline{\bullet}})/S})),\delta(\underline{L},P^{ED^{12}_{\underline{\bullet}}})]\langle -1\rangle[1]$$

$$\oplus[Rf_{(XY^{12}_{\underline{\bullet}},ZW^{12}_{\underline{\bullet}})/S*}(E^{\log,ZW^{12}_{\underline{\bullet}}}_{\text{crys}}(\mathcal{O}_{(XY^{12}_{\underline{\bullet}},DE^{12}_{\underline{\bullet}}\cup ZW^{12}_{\underline{\bullet}})/S})),\delta(\underline{L},P^{DE^{12}_{\underline{\bullet}}})]\langle -1\rangle[1]$$

$$\oplus[Rf_{(Y^{12}_{\underline{\bullet}},W^{12}_{\underline{\bullet}})/S*}(E^{\log,W^{12}_{\underline{\bullet}}}_{\text{crys}}(\mathcal{O}_{(Y^{12}_{\underline{\bullet}},E^{12}_{\underline{\bullet}}\cup W^{12}_{\underline{\bullet}})/S})),\delta(\underline{L},P^{E^{12}_{\underline{\bullet}}})].$$

The obvious analogues of (14.15) (for N such that $N-2$ satisfies the inequality (2.2.1)) and (14.16) for the complex (14.17.1) hold.

We conclude this section by stating the following Künneth formula of coefficients:

THEOREM 14.18. — *Let $(T,\mathcal{J},\delta)$ be a fine log PD-scheme. Let Z be a quasi-compact fine log scheme over $\underline{\operatorname{Spec}}_T(\mathcal{O}_T/\mathcal{J})$. Let h be a nonnegative integer. Then the following hold*:

1) *Let N be a nonnegative integer satisfying the inequality* (2.2.1). *Let*

$$f^i\colon X^i_{\bullet\leq N}\longrightarrow Z \quad \textit{and} \quad g^i\colon Y^i_{\bullet\leq N}\longrightarrow Z \quad (i=1,2)$$

be log smooth and integral morphisms from fine N-truncated simplicial log schemes. Assume that $\overset{\circ}{f}{}^i$ and $\overset{\circ}{g}{}^i$ are quasi-compact and quasi-separated. Let $\rho^i\colon Y^i_{\bullet\leq N}\longrightarrow X^i_{\bullet\leq N}$ be a morphism of fine N-truncated simplicial log schemes over Z. Set

$$Y^{12}_{\bullet\leq N}:=Y^1_{\bullet\leq N}\times_Z Y^2_{\bullet\leq N},\quad YX^{12}_{\bullet\leq N}:=Y^1_{\bullet\leq N}\times_Z X^2_{\bullet\leq N},$$

$$XY^{12}_{\bullet\leq N}:=X^1_{\bullet\leq N}\times_Z Y^2_{\bullet\leq N},\quad X^{12}_{\bullet\leq N}:=X^1_{\bullet\leq N}\times_Z X^2_{\bullet\leq N}.$$

Let $f_{Y^{12}_{\bullet\leq N}}$, $f_{YX^{12}_{\bullet\leq N}}$, $f_{XY^{12}_{\bullet\leq N}}$ and $f_{X^{12}_{\bullet\leq N}}$ be the structural morphisms of $Y^{12}_{\bullet\leq N}$, $YX^{12}_{\bullet\leq N}$, $XY^{12}_{\bullet\leq N}$ and $X^{12}_{\bullet\leq N}$ to Z, respectively. Let $B^{\bullet\leq N}_i$ (resp. $C^{\bullet\leq N}_i$) be a flat quasi-coherent crystal of $\mathcal{O}_{X^i_{\bullet\leq N}/T}$-modules (resp. $\mathcal{O}_{Y^i_{\bullet\leq N}/T}$-modules). Set

$$C^{\bullet\leq N}_{12}:=C^{\bullet\leq N}_1\boxtimes C^{\bullet\leq N}_2,\quad CB^{\bullet\leq N}_{12}:=C^{\bullet\leq N}_1\boxtimes B^{\bullet\leq N}_2,$$

$$BC^{\bullet\leq N}_{12}:=B^{\bullet\leq N}_1\boxtimes C^{\bullet\leq N}_2,\quad B^{\bullet\leq N}_{12}:=B^{\bullet\leq N}_1\boxtimes B^{\bullet\leq N}_2.$$

Let $\lambda^{\bullet\leq N}_i\colon B^{\bullet\leq N}_i\longrightarrow R\rho^{i\log}_{\text{crys}*}(C^{\bullet\leq N}_i)$ be a morphism in $\mathrm{D}^+(\mathcal{O}_{X^i_{\bullet\leq N}/T})$. Then

$$\mathcal{H}^h(Rf^{1\log}_{X^1_{\bullet\leq N}\text{crys}*}\mathrm{MF}(\lambda^{\bullet\leq N}_1)\otimes^L_{\mathcal{O}_{Z/T}}Rf^{2\log}_{X^2_{\bullet\leq N}\text{crys}*}\mathrm{MF}(\lambda^{\bullet\leq N}_2))$$

is canonically isomorphic to the following cohomology:

$$\text{(14.18.1)}\quad \mathcal{H}^h\{Rf^{\log}_{X^{12}_{\bullet\leq N}\text{crys}*}(B^{\bullet\leq N}_{12})\oplus Rf^{\log}_{YX^{12}_{\bullet\leq N}\text{crys}*}(CB^{\bullet\leq N}_{12})[-1]$$

$$\oplus Rf^{\log}_{XY^{12}_{\bullet\leq N}\text{crys}*}(BC^{\bullet\leq N}_{12})[-1]\oplus Rf^{\log}_{Y^{12}_{\bullet\leq N}\text{crys}*}(C^{\bullet\leq N}_{12})[-2]\}.$$

2) *Let $N-2$ be a nonnegative integer satisfying the inequality* (2.2.1). *Let the notations and the assumptions be as in* 1). *Then*

$$\mathcal{H}^h\{Rf^{1\log}_{X^1_{\bullet\leq N}\mathrm{crys}*}\mathrm{MC}(\lambda_1^{\bullet\leq N})\otimes^L_{\mathcal{O}_{Z/T}} Rf^{2\log}_{X^2_{\bullet\leq N}\mathrm{crys}*}\mathrm{MC}(\lambda_2^{\bullet\leq N})\}$$

is canonically isomorphic to the following cohomology:

$$\mathcal{H}^h\big\{Rf^{\log}_{X^{12}_{\bullet\leq N}\mathrm{crys}*}(B^{\bullet\leq N}_{12})[2]\oplus Rf^{\log}_{YX^{12}_{\bullet\leq N}\mathrm{crys}*}(CB^{\bullet\leq N}_{12})[1] \oplus Rf^{\log}_{XY^{12}_{\bullet\leq N}\mathrm{crys}*}(BC^{\bullet\leq N}_{12})[1]\oplus Rf^{\log}_{Y^{12}_{\bullet\leq N}\mathrm{crys}*}(C^{\bullet\leq N}_{12})\big\}. \tag{14.18.2}$$

3) *Let r be a positive integer. Let $f^i\colon X^i_{\underline{\bullet}}\longrightarrow Z$ and $g^i\colon Y^i_{\underline{\bullet}}\longrightarrow Z$ $(i=1,2)$ be log smooth, integral morphisms from fine r-simplicial log schemes. Assume that $\overset{\circ}{f}{}^i$ and $\overset{\circ}{g}{}^i$ are quasi-compact and quasi-separated. Let $\rho^i\colon Y^i_{\underline{\bullet}}\longrightarrow X^i_{\underline{\bullet}}$ be a morphism of fine r-simplicial log schemes over Z. Let $B^{\underline{\bullet}}_i$, $C^{\underline{\bullet}}_i$, $\lambda^{\underline{\bullet}}_i$ and so on are the obvious analogues of* 1). *Assume that $Rf^{\log}_{X^{12}_{\underline{\bullet}}\mathrm{crys}*}(B^{\underline{\bullet}}_{12})$, $Rf^{\log}_{YX^{12}_{\underline{\bullet}}\mathrm{crys}*}(CB^{\underline{\bullet}}_{12})$, $Rf^{\log}_{XY^{12}_{\underline{\bullet}}\mathrm{crys}*}(BC^{\underline{\bullet}}_{12})$ and $Rf^{\log}_{Y^{12}_{\underline{\bullet}}\mathrm{crys}*}(C^{\underline{\bullet}}_{12})$ are bounded above. Then the complex*

$$Rf^{1\log}_{X^1_{\underline{\bullet}}\mathrm{crys}*}\mathrm{MF}(\lambda^{\underline{\bullet}}_1)\otimes^L_{\mathcal{O}_{Z/T}} Rf^{2\log}_{X^2_{\underline{\bullet}}\mathrm{crys}*}\mathrm{MF}(\lambda^{\underline{\bullet}}_2)$$

is canonically isomorphic to the following complex:

$$Rf^{\log}_{X^{12}_{\underline{\bullet}}\mathrm{crys}*}(B^{\underline{\bullet}}_{12})\oplus Rf^{\log}_{YX^{12}_{\underline{\bullet}}\mathrm{crys}*}(CB^{\underline{\bullet}}_{12})[-1] \oplus Rf^{\log}_{XY^{12}_{\underline{\bullet}}\mathrm{crys}*}(BC^{\underline{\bullet}}_{12})[-1]\oplus Rf^{\log}_{Y^{12}_{\underline{\bullet}}\mathrm{crys}*}(C^{\underline{\bullet}}_{12})[-2].$$

The complex

$$Rf^{1\log}_{X^1_{\underline{\bullet}}\mathrm{crys}*}\mathrm{MC}(\lambda^{\underline{\bullet}}_1)\otimes^L_{\mathcal{O}_{Z/T}} Rf^{2\log}_{X^2_{\underline{\bullet}}\mathrm{crys}*}\mathrm{MC}(\lambda^{\underline{\bullet}}_2)$$

is canonically isomorphic to the following complex:

$$Rf^{\log}_{X^{12}_{\underline{\bullet}}\mathrm{crys}*}(B^{\underline{\bullet}}_{12})[2]\oplus Rf^{\log}_{YX^{12}_{\underline{\bullet}}\mathrm{crys}*}(CB^{\underline{\bullet}}_{12})[1] \oplus Rf^{\log}_{XY^{12}_{\underline{\bullet}}\mathrm{crys}*}(BC^{\underline{\bullet}}_{12})[1]\oplus Rf^{\log}_{Y^{12}_{\underline{\bullet}}\mathrm{crys}*}(C^{\underline{\bullet}}_{12}).$$

Proof. — 1), 2): Follow from Kato's log Künneth formula (see [**51**, (6.12)]) and Eilenberg-Zilber's theorem as in (5.10), 1).

3): Follows from Kato's log Künneth formula (see [**51**, (6.12)]) and Eilenberg-Zilber's theorem as in (5.10), 2). □

15. Mapping fiber and mapping cone in rigid cohomology

In this section we define the weight filtration on the mapping fiber and the mapping cone of the induced morphism of rigid cohomologies by a morphism of separated schemes of finite type over a perfect field κ of characteristic $p > 0$. We also calculate the slope filtrations on the mapping fiber and the mapping cone by the mapping fiber and the mapping cone of the cohomologies of the log de Rham-Witt complexes of proper smooth simplicial schemes with simplicial SNCD's over κ, respectively.

Unless stated otherwise, the base field κ is a perfect field of characteristic $p > 0$. Let $\mathcal{V}$, $\mathcal{W}$, K and K_0 be as in §10.

Let $\rho\colon V \to U$ be a morphism of separated schemes of finite type over κ. Then we have a morphism $(V, \overline{V}) \to (U, \overline{U})$ of pairs over κ ((9.5.1)). By (9.2) and (9.4), 2), there exist good proper hypercoverings $(U_\bullet, X_\bullet)$ and $(V_\bullet, Y_\bullet)$ of $(U, \overline{U})$ and $(V, \overline{V})$, respectively, which have the disjoint unions of the members of affine simplicial open coverings of $(U_\bullet, X_\bullet)$ and $(V_\bullet, Y_\bullet)$ over $(U, \overline{U})$ and $(V, \overline{V})$, respectively, and which fit into the following commutative diagram:

(15.0.1)
$$\begin{array}{ccc} (V_\bullet, Y_\bullet) & \xrightarrow{\rho_\bullet} & (U_\bullet, X_\bullet) \\ \big\downarrow & & \big\downarrow \\ (V, \overline{V}) & \xrightarrow{\rho} & (U, \overline{U}). \end{array}$$

Set $D_\bullet := X_\bullet \setminus U_\bullet$ and $E_\bullet := Y_\bullet \setminus V_\bullet$. Let $\mathfrak{FC}$ be a letter F or C. Then, by the functoriality in (11.6), 3), we have the isomorphism

(15.0.2)
$$\begin{aligned} \mathrm{M}\mathfrak{FC}(\rho^*_{\mathrm{rig}}) &:= \mathrm{M}\mathfrak{FC}\big(\rho^*_{\mathrm{rig}}\colon R\Gamma_{\mathrm{rig}}(U/K) \to R\Gamma_{\mathrm{rig}}(V/K)\big) \\ &\overset{\sim}{\longrightarrow} \mathrm{M}\mathfrak{FC}\big(\rho^{\log *}_{\bullet\mathrm{crys}}\colon R\Gamma\big(X_\bullet, D_\bullet)/\mathcal{W})_K \to R\Gamma\big((Y_\bullet, E_\bullet)/\mathcal{W})_K\big). \end{aligned}$$

In fact, by (9.11), 1), we see that the isomorphism (15.0.2) is independent of the choice of the commutative diagram (15.0.1).

By (15.0.2), (14.2) and (14.8), we have :

Theorem-Definition 15.1. — *There exist the following spectral sequences*:

$$E_1^{-k,h+k} = \bigoplus_{t\geq 0} H^{h-2t-k}\big((\widetilde{D_t^{(t+k)}}/\mathcal{W})_{\mathrm{crys}}, \mathcal{O}_{D_t^{(t+k)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(t+k)}(D_t/\mathcal{W})\big)\big(-(t+k)\big)_K \oplus \bigoplus_{t\geq 0} H^{h-2t-k-2}\big((\widetilde{E_t^{(t+k+1)}}/\mathcal{W})_{\mathrm{crys}}, \mathcal{O}_{E_t^{(t+k+1)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(t+k+1)}(E_t/\mathcal{W})\big)\big(-(t+k+1)\big)_K \Longrightarrow H^h\big(\mathrm{MF}(\rho^*_{\mathrm{rig}})\big), \tag{15.1.1}$$

$$E_1^{-k,h+k} = \bigoplus_{t\geq 0} H^{h-2t-k+2}\big((\widetilde{D_t^{(t+k-1)}}/\mathcal{W})_{\mathrm{crys}}, \mathcal{O}_{D_t^{(t+k-1)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(t+k-1)}(D_t/\mathcal{W})\big)\big(-(t+k-1)\big)_K \oplus \bigoplus_{t\geq 0} H^{h-2t-k}\big((\widetilde{E_t^{(t+k)}}/\mathcal{W})_{\mathrm{crys}}, \mathcal{O}_{E_t^{(t+k)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(t+k)}(E_t/\mathcal{W})\big)\big(-(t+k)\big)_K \Longrightarrow H^h\big(\mathrm{MC}(\rho^*_{\mathrm{rig}})\big). \tag{15.1.2}$$

The spectral sequences (15.1.1) *and* (15.1.2) *degenerate at* E_2.

We call the spectral sequences (15.1.1) *and* (15.1.2) *the weight spectral sequences of* $H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}}))$ *and* $H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))$ *with respect to* $\rho_\bullet$, *respectively.*

Theorem 15.2. — *The weight filtrations on* $H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}}))$ *and* $H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))$ *with respect to* $\rho_\bullet$ *are independent of the choice of* $\rho_\bullet$.

Proof. — Using (9.11), 1), (15.1) and variants of (12.3) for the mapping fiber and the mapping cone, we can prove (15.2) as in (12.5). □

Definition 15.3. — We call the well-defined filtration in (15.2) the *weight filtration* on $H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}}))$ and $H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))$, and we denote them by

$$P = \{P_k\}_{k\in\mathbb{Z}}.$$

Proposition 15.4. — *The weight filtrations on*

$$H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}})) \quad \textit{and} \quad H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))$$

are compatible with the extension of complete discrete valuation rings of mixed characteristics with perfect residue fields.

Proof. — As in (12.8), (15.4) immediately follows from (15.1.1) and (15.1.2) and the base change theorem of crystalline cohomology (see [**5**, V Proposition 3.5.2]). □

THEOREM 15.5 (Strict compatibility). — *Let*

$$\begin{array}{ccc} V' & \xrightarrow{\rho'} & U' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ V & \xrightarrow{\rho} & U \end{array}$$

be a commutative diagram of separated schemes of finite type over κ. *Let* $\mathfrak{FC}$ *be a letter* F *or* C. *Then the induced morphism*

$$(u^*, v^*)\colon H^h\big(\mathrm{M}\mathfrak{FC}(\rho^*_{\mathrm{rig}})\big) \longrightarrow H^h\big(\mathrm{M}\mathfrak{FC}({\rho'}^*_{\mathrm{rig}})\big) \quad (h \in \mathbb{Z})$$

is strictly compatible with the weight filtration.

Proof. — By using (9.11), the proof is similar to (12.10). □

THEOREM 15.6. — *The exact sequences*

$$(15.6.1) \qquad \cdots \longrightarrow H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}})) \longrightarrow H^h_{\mathrm{rig}}(U/K) \longrightarrow H^h_{\mathrm{rig}}(V/K) \longrightarrow \cdots$$

and

$$(15.6.2) \qquad \cdots \longrightarrow H^h_{\mathrm{rig}}(U/K) \longrightarrow H^h_{\mathrm{rig}}(V/K) \longrightarrow H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}})) \longrightarrow \cdots$$

are strictly exact with respect to the weight filtration.

Proof. — The Theorem follows from (14.10) as in (12.10). □

Next, we consider the slope filtration on the mapping fiber and the mapping cone. Assume that $\mathcal{V} = \mathcal{W}$. We have the isomorphism

$$(15.6.3) \qquad \begin{aligned} \mathrm{M}\mathfrak{FC}(\rho^*_{\mathrm{rig}}) \xrightarrow{\sim} \mathrm{M}\mathfrak{FC}(\rho^{\log *}_{\bullet\mathrm{dRW}}\colon R\Gamma(X_\bullet, \mathcal{W}\Omega^\bullet_{X_\bullet}(\log D_\bullet))_{K_0} \\ \longrightarrow R\Gamma(Y_\bullet, \mathcal{W}\Omega^\bullet_{Y_\bullet}(\log E_\bullet))_{K_0}) \end{aligned}$$

by (15.0.2) and (7.7.1).

The following theorem is an immediate consequence of results in (14.11).

THEOREM 15.7. — *Let* $\mathfrak{FC}$ *be a letter* F *or* C. *Assume that* $\mathcal{V} = \mathcal{W}$. *Then the following hold*:

1) *There exists the following spectral sequence*

$$(15.7.1) \qquad E_1^{i,h-i} = H^{h-i}\big(\mathrm{M}\mathfrak{FC}(\rho^{\log *i}_{\bullet\mathrm{dRW}})\big)_{K_0} \Longrightarrow H^h\big(\mathrm{M}\mathfrak{FC}(\rho^*_{\mathrm{rig}})\big).$$

This spectral sequence degenerates at E_1.

2) *There exists the following isomorphism*:

$$H^{h-i}\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *i}_{\bullet \mathrm{dRW}})\big)_{K_0} \simeq \big(H^h(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^*_{\mathrm{rig}}))\big)_{[i,i+1)}. \tag{15.7.2}$$

3) *There exists the following slope decomposition*

$$H^h\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^*_{\mathrm{rig}})\big) = \bigoplus_{i=0}^{h} H^{h-i}\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *i}_{\bullet \mathrm{dRW}})\big)_{K_0}. \tag{15.7.3}$$

4) *There exist the following spectral sequences*

$$\begin{aligned} &E_1^{-k,h+k} \\ &= \bigoplus_{t\geq 0} H^{h-i-2t-k}(D_t^{(t+k)}, \mathcal{W}\Omega^i_{D_t^{(t+k)}} \otimes_{\mathbb{Z}} \varpi^{(t+k)}_{\mathrm{zar}}(D_t/\kappa))(-(t+k))_{K_0} \\ &\quad \oplus \bigoplus_{t\geq 0} H^{h-i-2t-k-2}(E_t^{(t+k+1)}, \mathcal{W}\Omega^i_{E_t^{(t+k+1)}} \\ &\qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(t+k+1)}_{\mathrm{zar}}(E_t/\kappa))(-(t+k+1))_{K_0} \\ &\qquad\qquad \Longrightarrow H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}}))_{[i,i+1)}, \end{aligned} \tag{15.7.4}$$

$$\begin{aligned} E_1^{-k,h+k} &= \bigoplus_{t\geq 0} H^{h-i-2t-k+2}(D_t^{(t+k-1)}, \mathcal{W}\Omega^i_{D_t^{(t+k-1)}} \\ &\qquad \otimes_{\mathbb{Z}} \varpi^{(t+k-1)}_{\mathrm{zar}}(D_t/\kappa))(-(t+k-1))_{K_0} \\ &\quad \oplus \bigoplus_{t\geq 0} H^{h-i-2t-k}(E_t^{(t+k)}, \mathcal{W}\Omega^i_{E_t^{(t+k)}} \otimes_{\mathbb{Z}} \varpi^{(t+k)}_{\mathrm{zar}}(E_t/\kappa))(-(t+k))_{K_0} \\ &\qquad \Longrightarrow H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))_{[i,i+1)}. \end{aligned} \tag{15.7.5}$$

These spectral sequences degenerate at E_2.

5) *The following equality holds*:

$$\begin{aligned} &P_k H^{h-i}\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^{\log *i}_{\bullet \mathrm{dRW}})\big)_{K_0} \\ &\qquad = P_k H^h\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^*_{\mathrm{rig}})\big) \cap H^h\big(\mathrm{M}\mathfrak{F}\mathfrak{C}(\rho^*_{\mathrm{rig}})\big)_{[i,i+1)} \end{aligned} \tag{15.7.6}$$

PROPOSITION 15.8. — *The slopes of* $H^h(\mathrm{MF}(\rho^*_{\mathrm{rig}}))$ (*resp.* $H^h(\mathrm{MC}(\rho^*_{\mathrm{rig}}))$) *lie in* $[0,h]$ (*resp.* $[0,h+1]$).

Proof. — The Proposition immediately follows from (14.12). □

16. Rigid cohomology with closed support

In this section we endow the rigid cohomology with closed support with the weight filtration. We prove that the generalized Künneth isomorphism of the rigid cohomology ((16.15), 1) below) is strictly compatible with the weight filtration and that the Gysin morphism in rigid cohomology is strictly compatible with the weight filtration. We determine the possible range of weights of the rigid cohomology with closed support. We also determine the possible range of slopes of it. These determinations are generalizations of works of Chiarellotto (see [**15**]) and Chiarellotto-Le Stum (see [**17**]), respectively. As a corollary of the existence of the weight filtration and the calculation of the slope filtration on the rigid cohomology with closed support, we prove the variant of Serre-Grothendieck's conjecture as to the existence of the desired functions in the Introduction.

Unless stated otherwise, the base field κ is a perfect field of characteristic $p>0$ in this section. Let U be a separated scheme of finite type over κ and Z a closed subscheme of U. Let V be the complement of Z in U and let $\rho\colon V\hookrightarrow U$ be the open immersion. Set $d:=\dim U$, $c:=\operatorname{codim}(Z,U)$ and $d_Z:=\dim Z$. If U is smooth over κ, then $d=c+d_Z$ (see [**43**, II Exercise, (3.20) (d)]). Assume that U is of pure dimension. Let us recall the definition of $R\Gamma_{\mathrm{rig},Z}(U/K)$:

$$R\Gamma_{\mathrm{rig},Z}(U/K):=\mathrm{MF}\big(\rho^*_{\mathrm{rig}}\colon R\Gamma_{\mathrm{rig}}(U/K)\to R\Gamma_{\mathrm{rig}}(V/K)\big). \tag{16.0.1}$$

As an immediate application of (15.1), (15.2), (15.4) and (15.7), we obtain the following:

Theorem 16.1. — *Let $U\hookrightarrow\overline{U}$ be an open immersion into a proper scheme over κ. Let $\overline{V}$ be the closure of V in $\overline{U}$. Let the notations be as in* (15.0.1). *Then the following hold*:

1) *There exists the following spectral sequence*

$$\begin{aligned}E_1^{-k,h+k}=\bigoplus_{t\geq 0}&H^{h-2t-k}\big((\widetilde{D_t^{(t+k)}}/\mathcal{W})_{\mathrm{crys}},\mathcal{O}_{D_t^{(t+k)}/\mathcal{W}}\\&\qquad\otimes_{\mathbb{Z}}\varpi^{(t+k)}_{\mathrm{crys}}(D_t/\mathcal{W})\big)\big(-(t+k)\big)_K\\&\oplus\bigoplus_{t\geq 0}H^{h-2t-k-2}\big((\widetilde{E_t^{(t+k+1)}}/\mathcal{W})_{\mathrm{crys}},\mathcal{O}_{E_t^{(t+k+1)}/\mathcal{W}}\\&\qquad\otimes_{\mathbb{Z}}\varpi^{(t+k+1)}_{\mathrm{crys}}(E_t/\mathcal{W})\big)\big(-(t+k+1)\big)_K\\&\qquad\Longrightarrow H^h_{\mathrm{rig},Z}(U/K).\end{aligned}\tag{16.1.1}$$

The filtration $P:=\{P_k\}_{k\in\mathbb{Z}}$ on $H^h_{\mathrm{rig},Z}(U/K)$ induced by the spectral sequence above is well-defined. The filtration P is compatible with the base change of complete discrete valuation rings of mixed characteristics. The spectral sequence (16.1.1) *degenerates at E_2.*

2) *There exists the following spectral sequence*

$$E_1^{-k,h+k} = \bigoplus_{t\geq 0} H^{h-i-2t-k}\big(D_t^{(t+k)}, \mathcal{W}\Omega^i_{D_t^{(t+k)}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(t+k)}(D_t/\kappa)\big)\big(-(t+k)\big)_{K_0} \oplus \bigoplus_{t\geq 0} H^{h-i-2t-k-2}\big(E_t^{(t+k+1)}, \mathcal{W}\Omega^i_{E_t^{(t+k+1)}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(t+k+1)}(E_t/\kappa)\big)(-(t+k+1))_{K_0} \Longrightarrow H^h_{\mathrm{rig},Z}(U/K_0)_{[i,i+1)}. \tag{16.1.2}$$

This spectral sequence degenerates at E_2.

3) *Let* $P := \{P_k\}_{k\in\mathbb{Z}}$ *be the filtration on* $H^h_{\mathrm{rig},Z}(U/K_0)_{[i,i+1)}$ *induced by the spectral sequence* (16.1.2). *Then the following formula holds*:

$$P_k H^{h-i}(\mathrm{MF}(\rho^{\log *i}_{\bullet \mathrm{dRW}}))_{K_0} = H^h_{\mathrm{rig},Z}(U/K_0)_{[i,i+1)} \cap P_k H^h_{\mathrm{rig},Z}(U/K_0).$$

Definition 16.2. — 1) We call the filtration P on

$$H^h_{\mathrm{rig},Z}(U/K) \text{ (resp. } H^h_{\mathrm{rig},Z}(U/K)_{[i,i+1)})$$

the *weight filtration* on $H^h_{\mathrm{rig},Z}(U/K)$ (resp. $H^h_{\mathrm{rig},Z}(U/K)_{[i,i+1)}$).

2) Let k be an integer. We say that $H^h_{\mathrm{rig},Z}(U/K)$ is *of weight* $\geq k$ (resp. $\leq k$) if $P_{k-1}H^h_{\mathrm{rig},Z}(U/K) = 0$ (resp. $H^h_{\mathrm{rig},Z}(U/K) = P_k H^h_{\mathrm{rig},Z}(U/K)$). We say that $H^h_{\mathrm{rig},Z}(U/K)$ is *of pure weight* k if $H^h_{\mathrm{rig},Z}(U/K)$ is *of weight* $\geq k$ and $\leq k$. We call a vector v of $H^h_{\mathrm{rig},Z}(U/K)$ is *of weight* $\leq k$ if $v \in P_k H^h_{\mathrm{rig},Z}(U/K)$.

By the spectral sequence (16.1.1) and by the purity of the weight (see [**54**], [**16**, (1.2)] (cf. [**69**, (2.2) (4)])), we see that, if κ is a finite field, then the definitions in (16.2) about $H^h_{\mathrm{rig},Z}(U/K)$ are usual ones using the eigenvalues of the Frobenius endomorphism.

Let h be a nonnegative integer. Let X be a proper smooth scheme over a perfect field κ of characteristic $p > 0$. By considering the constant simplicial scheme X as a gs proper hypercovering of X, we see that $H^h_{\mathrm{rig}}(X/K) = H^h_{\mathrm{crys}}(X/\mathcal{W})_K$ is of pure weight h (this purity also immediately follows from (12.11), 2) and (12.12)).

Theorem 16.3. — *Let* $u\colon (V,W) \longrightarrow (U,Z)$ *be a morphism of separated schemes of finite type over* κ *with closed subschemes. Then the induced morphism*

$$u^*_{\mathrm{rig}}\colon H^h_{\mathrm{rig},Z}(U/K) \longrightarrow H^h_{\mathrm{rig},W}(V/K) \tag{16.3.1}$$

is strictly compatible with the weight filtration.

Proof. — By (9.5.1) there exists a morphism $(V,\overline{V}) \to (U,\overline{U})$ over κ, where $\overline{V}$ and $\overline{U}$ are proper schemes over κ including V and U as open subschemes, respectively. Let $\overline{W}$ and $\overline{Z}$ be the closures of W and Z in $\overline{V}$ and $\overline{U}$, respectively. By (14.9) and (15.0.2), we obtain (16.3) (cf. the proof of (12.10)). ☐

LEMMA 16.4. — *Let $\mathcal{C}$ be an additive category. Let*

$$f\colon (E^\bullet,\{E^\bullet_k\}) \longrightarrow (F^\bullet,\{F^\bullet_k\}) \text{ and } g\colon (F^\bullet,\{F^\bullet_k\}) \longrightarrow (G^\bullet,\{G^\bullet_k\})$$

be filtered morphisms of filtered complexes of objects of $\mathcal{C}$. Set $h := g\circ f$. Then the following hold:

1) *The morphism*

$$(\mathrm{id},g)\colon \mathrm{MF}(f) = \big(E^\bullet,\{E^\bullet_k\}\big)\oplus\big(F^\bullet,\{F^\bullet_k\}\big)[-1] \longrightarrow \big(E^\bullet,\{E^\bullet_k\}\big)\oplus\big(G^\bullet,\{G^\bullet_k\}\big)[-1] = \mathrm{MF}(h)$$

is a morphism of filtered complexes of objects of $\mathcal{C}$.

2) *The mapping cone* $\mathrm{MC}((\mathrm{id},g))$ *is canonically isomorphic to* $\mathrm{MF}(g)$ *in* $\mathrm{KF}(\mathcal{C})$. *In particular, there exists the following triangle*

$$(16.4.1)\qquad \longrightarrow \mathrm{MF}(f) \longrightarrow \mathrm{MF}(h) \longrightarrow \mathrm{MF}(g) \xrightarrow{+1}.$$

3) *Let the notations be as in the beginning of* §14. *Let $\rho\colon M \longrightarrow N$ and $\sigma\colon N \longrightarrow L$ be a morphism in* $\mathrm{C}(\mathcal{A}^\bullet)$. *Then the morphism*

$$\mathbf{s}\big((\mathrm{id},\sigma)\big)\colon \mathbf{s}\big(\mathrm{MF}(\rho)\big) \longrightarrow \mathbf{s}\big(\mathrm{MF}(\sigma\circ\rho)\big)$$

is a morphism of complexes of $\mathcal{A}$-modules, and the mapping cone $\mathrm{MC}(\mathbf{s}((\mathrm{id},\sigma)))$ *is canonically isomorphic to* $\mathbf{s}(\mathrm{MF}(\sigma))$ *in* $\mathrm{KF}(\mathcal{A})$.

Proof. — 1): It is immediate to check.

2): For simplicity of notation, we omit to write the filtrations. There exists a natural injection

$$\iota\colon \mathrm{MF}(g) = F^\bullet\oplus G^\bullet[-1] \longrightarrow \big(E^\bullet\oplus F^\bullet[-1]\big)[1]\oplus\big(E^\bullet\oplus G^\bullet[-1]\big) = \mathrm{MC}((\mathrm{id},g))$$

of complexes of objects of $\mathcal{C}$. It is straightforward to check that ι is indeed a morphism of complexes of objects of $\mathcal{C}$. Furthermore, there exists a natural morphism $\pi\colon \mathrm{MC}((\mathrm{id},g)) \to \mathrm{MF}(g)$ of complexes of objects of $\mathcal{C}$ defined by the following formula

$$E^{q+1}\oplus F^q\oplus E^q\oplus G^{q-1}\text{“} \ni (x,y,z,w) \longmapsto \big(y+f(z),w\big) \in \text{”}F^q\oplus G^{q-1} \quad (q\in\mathbb{Z}).$$

It is straightforward to check that π is indeed a morphism of complexes (the boundary morphism $d\colon E^{q+1}\oplus F^q\oplus E^q\oplus G^{q-1} \longrightarrow E^{q+2}\oplus F^{q+1}\oplus E^{q+1}\oplus G^q$

is given by the formula "$d(x,y,z,w)=(-dx,dy-f(x),dz+x,-dw+g(y)+h(z))$". Since $\pi\circ\iota=\mathrm{id}$, it suffices to prove that the composite morphism $\mathrm{id}-\iota\circ\pi$ is homotopic to zero. A family

$$E^{q+1}\oplus F^q\oplus E^q\oplus G^{q-1}$$
$$\text{“}\ni(x,y,z,w)\longmapsto(z,0,0,0)\in\text{”}\,E^q\oplus F^{q-1}\oplus E^{q-1}\oplus G^{q-2}\quad(q\in\mathbb{Z})$$

of morphisms gives an homotopy from $\mathrm{id}-\iota\circ\pi$ to the zero morphism.

3): Follows from 1) and the proof of 2). □

The following is a slight generalization of [**9**, (2.5)] and another proof of [*loc. cit.*] itself.

Corollary 16.5. — *Let T be a closed subscheme of Z. Set $U':=U\setminus T$ and $Z':=Z\setminus T$. Then there exists a triangle*

$$(16.5.1)\qquad \longrightarrow R\Gamma_{\mathrm{rig},T}(U/K)\longrightarrow R\Gamma_{\mathrm{rig},Z}(U/K)\longrightarrow R\Gamma_{\mathrm{rig},Z'}(U'/K)\xrightarrow{+1}.$$

In particular, there exists the following exact sequence

$$(16.5.2)\qquad \cdots\longrightarrow H^h_{\mathrm{rig},T}(U/K)\longrightarrow H^h_{\mathrm{rig},Z}(U/K)\longrightarrow H^h_{\mathrm{rig},Z'}(U'/K)\longrightarrow\cdots.$$

Proof. — By considering the following two morphisms

$$f\colon R\Gamma_{\mathrm{rig}}(U/K)\longrightarrow R\Gamma_{\mathrm{rig}}(U'/K),$$
$$g: R\Gamma_{\mathrm{rig}}(U'/K)\longrightarrow R\Gamma_{\mathrm{rig}}\big((U'\setminus Z')/K\big)=R\Gamma_{\mathrm{rig}}\big((U\setminus Z)/K\big),$$

(16.5) immediately follows from (16.4.1). □

Theorem 16.6. — *The following hold*:

1) *The exact sequence*

$$(16.6.1)\qquad \cdots\longrightarrow H^h_{\mathrm{rig},Z}(U/K)\longrightarrow H^h_{\mathrm{rig}}(U/K)\longrightarrow H^h_{\mathrm{rig}}(V/K)\longrightarrow\cdots$$

(cf. [**9**, (2.3.1)]) *is strictly exact with respect to the weight filtration. More generally, the exact sequence* (16.5.2) *is strictly exact with respect to the weight filtration.*

2) *Let U' be an open subscheme of U which contains Z as a closed subscheme. Then the isomorphism*

$$(16.6.2)\qquad H^h_{\mathrm{rig},Z}(U/K)\xrightarrow{\sim}H^h_{\mathrm{rig},Z}(U'/K)$$

(cf. [**9**, (2.4.1)]) *is an isomorphism of weight-filtered K-vector spaces.*

3) *If $Z=Z_1\sqcup Z_2$, then the isomorphism*

$$(16.6.3)\qquad H^h_{\mathrm{rig},Z_1}(U/K)\oplus H^h_{\mathrm{rig},Z_2}(U/K)\xrightarrow{\sim}H^h_{\mathrm{rig},Z}(U/K)$$

(cf. [**9**, (2.4.2)]) *is an isomorphism of weight-filtered K-vector spaces.*

Proof. — 1): The former statement of 1) is a special case of the strict exactness of (15.6.1). We prove the latter statement of 1) as follows.

Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Set $U'' := U \setminus Z$. Let $\overline{U}''$ and $\overline{U}'$ be the closures of U'' and U' in $\overline{U}$, respectively. By (9.2) and (9.4), 2), there exist gs proper hypercoverings $(U_\bullet, X_\bullet)$, $(U'_\bullet, X'_\bullet)$ and $(U''_\bullet, X''_\bullet)$ of $(U, \overline{U})$, $(U', \overline{U}')$ and $(U'', \overline{U}'')$, respectively, which fit into the commutative diagram

$$(16.6.4)\qquad \begin{array}{ccccc} (U''_\bullet, X''_\bullet) & \xrightarrow{\sigma_\bullet} & (U'_\bullet, X'_\bullet) & \xrightarrow{\rho_\bullet} & (U_\bullet, X_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ (U'', \overline{U}'') & \xrightarrow{\sigma} & (U', \overline{U}') & \xrightarrow{\rho} & (U, \overline{U}). \end{array}$$

Set $D_\bullet := X_\bullet \setminus U_\bullet$, $D'_\bullet := X'_\bullet \setminus U'_\bullet$ and $D''_\bullet := X''_\bullet \setminus U''_\bullet$. Then we have the commutative diagram

$$\begin{array}{ccccc} R\Gamma((X_\bullet, D_\bullet)/\mathcal{W})_K & \xrightarrow{\rho_\bullet^*} & R\Gamma((X'_\bullet, D'_\bullet)/\mathcal{W})_K & \xrightarrow{\sigma_\bullet^*} & R\Gamma((X''_\bullet, D''_\bullet)/\mathcal{W})_K \\ \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\ R\Gamma_{\mathrm{rig}}(U/K) & \xrightarrow{\rho^*} & R\Gamma_{\mathrm{rig}}(U'/K) & \xrightarrow{\sigma^*} & R\Gamma_{\mathrm{rig}}((U' \setminus Z')/K) \\ & & & & = R\Gamma_{\mathrm{rig}}(U''/K). \end{array}$$

Consider the morphism $(\mathrm{id}, \sigma_\bullet^*) \colon \mathrm{MF}(\rho_\bullet^*) \longrightarrow \mathrm{MF}(\sigma_\bullet^* \circ \rho_\bullet^*)$. Then we have the weight filtration on $\mathrm{MC}((\mathrm{id}, \sigma_\bullet^*))$ by (14.0.10). By the obvious generalization of (14.10) (by the proof of [**72**, (2.18.2) (1)]), the exact sequence

$$\cdots \longrightarrow H^h\big(\mathrm{MF}(\rho_\bullet^*)\big) \longrightarrow H^h\big(\mathrm{MF}(\sigma_\bullet^* \circ \rho_\bullet^*)\big) \longrightarrow H^h\big(\mathrm{MC}((\mathrm{id}, \sigma_\bullet^*))\big) \longrightarrow \cdots$$

is strictly compatible with the weight filtration. Again, by the proof of [**72**, (2.18.2) (1)], the canonical isomorphism

$$H^h\big(\mathrm{MF}(\sigma_\bullet^*)\big) \xrightarrow{\sim} H^h\big(\mathrm{MC}((\mathrm{id}, \sigma_\bullet^*))\big) \quad (h \in \mathbb{Z})$$

is an isomorphism of weight-filtered vector spaces. Now we have proved the latter statement of 1).

2): Consider the commutative diagram:

$$(16.6.5)\qquad \begin{array}{ccc} U' \setminus Z & \xrightarrow{\subset} & U' \\ \cap\downarrow & & \downarrow\cap \\ U \setminus Z & \xrightarrow{\subset} & U. \end{array}$$

The assertion 2) immediately follows from (15.5).

3): Let U_1 (resp. U_2) be the complement of Z_2 (resp. Z_1) in U. Then we have natural morphisms $(U_i, Z_i) \to (U, Z_i)$ and $(U_i, Z_i) \to (U, Z)$ $(i = 1, 2)$. Hence we have the following isomorphism by (16.6.2) and the following morphism:

$$H^h_{\mathrm{rig},Z_i}(U/K) \xrightarrow{\sim} H^h_{\mathrm{rig},Z_i}(U_i/K) \longleftarrow H^h_{\mathrm{rig},Z}(U/K). \tag{16.6.6}$$

By (15.5) the two morphisms in (16.6.6) are strictly compatible with the weight filtration; by (15.5) the natural morphism

$$H^h_{\mathrm{rig},Z_i}(U/K) \longrightarrow H^h_{\mathrm{rig},Z}(U/K)$$

is also strictly compatible with the weight filtration. Thus 3) follows. □

The following is a generalization of [**15**, (2.3)] and (12.12).

THEOREM 16.7. — *Assume that U is smooth over κ. Then the following hold*:

1) *The weights of $H^h_{\mathrm{rig},Z}(U/K)$ lie in $[h, 2(h-c)]$.*

2) (*cf.* [**25**, (8.2.9)]) *The weights of $H^h_{\mathrm{rig},Z}(U/K)$ is less than or equal to $2d$.*

Proof. — 1): By (15.0.2) and by the proof of (12.12), we can reduce 1) to [**15**, (2.3)].

2): If κ is a finite field, the same proof as that of [**15**, (2.3)] works; we leave the detail to the reader. In the general case, we have only to use (15.0.2) and the proof of (12.12). □

REMARKS 16.8. — 1) The third term of the following exact sequence

$$\cdots \longrightarrow H^i_{\mathrm{rig}}(X'/K) \longrightarrow H^i_{\mathrm{rig}}(U_1/K) \longrightarrow H^i_{X'\setminus U_1,\mathrm{rig}}(X'/K) \longrightarrow \cdots$$

in [**15**, p. 690] have to be replaced by $H^{i+1}_{X'\setminus U_1,\mathrm{rig}}(X'/K)$.

2) By using recent results (17.5) below, we have another proof of (16.7) by (16.6), (17.8) below and the proof of [**15**, (2.3)].

The following is a slight correction of [**17**, (2.1.1)] since "une variété algébrique" in [**17**] is not necessarily connected (see [**17**, Convention]):

LEMMA 16.9 (cf. [**62**, (8.3.13)]). — *Let e be the cardinality of the geometric irreducible components of Z. Assume that Z is of pure dimension d_Z. Then there exists the following canonical isomorphism*

$$H^{2d_Z}_{\mathrm{rig,c}}(Z/K) \xrightarrow{\sim} K^e(-d_Z). \tag{16.9.1}$$

Hence the trace morphism $\mathrm{Tr}_Z\colon H^{2d_Z}_{\mathrm{rig,c}}(Z/K) \longrightarrow K$ *(cf.* [**8**], [**6**, pp. 21–23]) *is an underlying morphism of the following morphism*

$$\mathrm{Tr}_Z\colon H^{2d_Z}_{\mathrm{rig,c}}(Z/K) \longrightarrow K(-d_Z). \tag{16.9.2}$$

Proof. — We may assume that Z is reduced. Let T be the set of the singular points of Z. Set $Z^{\mathrm{o}} := Z \setminus T$. Then, by the exact sequence

$$\cdots \longrightarrow H^h_{\mathrm{rig,c}}(Z^{\mathrm{o}}/K) \longrightarrow H^h_{\mathrm{rig,c}}(Z/K) \longrightarrow H^h_{\mathrm{rig,c}}(T/K) \longrightarrow \cdots \tag{16.9.3}$$

[**6**, (3.1) (iii)] and by the vanishing of the rigid cohomology with compact support [**8**, (1.1)], we have a canonical isomorphism

$$H^{2d_Z}_{\mathrm{rig,c}}(Z^{\mathrm{o}}/K) \xrightarrow{\ \sim\ } H^{2d_Z}_{\mathrm{rig,c}}(Z/K). \tag{16.9.4}$$

Hence we may assume that Z is smooth and affine by taking a dense affine open set of Z. By [**9**, (5.7) (ii)], $H^0_{\mathrm{rig}}(Z/K) = K^e$. Hence, by the Poincaré duality [**8**, (2.4)], $H^{2d_Z}_{\mathrm{rig,c}}(Z/K) = K^e$. The compatibility of the Frobenius in (16.9.1) is obtained by the proof of [**17**, (2.1.1)]. □

Remark 16.10. — The trace morphism is defined for a separated scheme Z of finite type over κ because the rigid cohomology with compact support of Z is defined (see [**6**, pp. 21–23]), because there exists an exact sequence [**6**, (3.1) (iii)] and because the vanishing theorem for the rigid cohomology with compact support holds (see [**8**, (1.1)]).

By (11.8) and the exact sequence (16.6.1), we see that $H^h_{\mathrm{rig},Z}(U/K)$ is an F-isocrystal over κ (see [**87**, (8.1.1) (3)]). The following 3) is a generalization of the latter part of [**17**, (3.1.2)]:

Theorem 16.11. — *Let κ be a perfect field of characteristic $p > 0$. Then the following hold*:

1) *The slopes of $H^h_{\mathrm{rig},Z}(U/K_0)$ lie in $[0, h]$.*
2) *If U is smooth over κ, then the slopes of $H^h_{\mathrm{rig},Z}(U/K_0)$ lie in $[c, h-c]$.*
3) *Under the assumption in 2), the slopes of $H^h_{\mathrm{rig},Z}(U/K_0)$ lie in $[h-d, d]$.*

Consequently, the slopes of $H^h_{\mathrm{rig},Z}(U/K_0)$ lie in $[\max\{c, h-d\}, \min\{h-c, d\}]$.

Proof. — 1): This is a special case of (15.8).

2): If Z is smooth over κ, then we may assume that U and Z are of pure dimensions. In this case, 2) immediately follows from 1) and the following Gysin isomorphism

$$G_{Z/U}\colon H^h_{\mathrm{rig},Z}(U/K_0) \xrightarrow{\ \sim\ } H^{h-2c}_{\mathrm{rig}}(Z/K_0)(-c). \tag{16.11.1}$$

Note that we do not need to assume that Z and U are affine by [**8**, (2.4)] (see also (10.19), 2)) and (16.9) (cf. [**17**, (2.1.3)]). In the general case, we have only to consider the reduced scheme Z_{red}, remove the singular locus of Z_{red} and to proceed by induction on the dimension of Z as in [**15**, p. 691].

3): We give only a sketch of the proof of 3) because the following argument is well-known (see [**9**, (3.3)] (cf. [**15**, I (2.5)], [**17**, (3.1.2)])). Follow the argument of the inductions of two types in [**9**, (3.3)]: consider the following statements for any natural number h:

(a)$_d$ $\left\{\right.$ For any perfect field κ of characteristic $p>0$ and for any separated smooth scheme U of finite type over κ of dimension d, the slopes of $H^h_{\mathrm{rig}}(U/K_0)$ lie in $[h-d,d]$.

(b)$_d$ $\left\{\right.$ For any perfect field κ of characteristic $p>0$ and for any closed immersion $Z\hookrightarrow U$ from a separated scheme of finite type over κ of dimension $\leq d$ to a separated smooth scheme of finite type over κ, the slopes of $H^h_{\mathrm{rig},Z}(U/K_0)$ lie in $[h-d,d]$.

(a)$_0$ is clear as in [**15**, I (2.5)]. (b)$_0$ is also clear as in [**15**, I (2.5)]. The proof of the implication (b)$_{d-1}\Longrightarrow$ (a)$_d$ is obtained by a standard method using de Jong's alteration theorem (see [**50**, (4.1)]), the base change theorem in [**87**, (8.1.1) (2)] or (16.1) and the slope decomposition of the crystalline cohomology of a proper smooth scheme over κ (see [**47**, II (3.5.4)]).

Next, let us prove the implication (b)$_{d-1}$, (a)$_d \Longrightarrow$ (b)$_d$. Let d_Z (resp. d) be the dimension of Z (resp. U). The key point for the proof of this implication is the Gysin isomorphism (16.11.1); as in 2), we may assume that Z is smooth. Then, by the inductive hypothesis, the slopes of $H^{h-2c}_{\mathrm{rig}}(Z/K_0)(-c)$ lie in $[h-2c-d_Z+c,d_Z+c]$. Hence the slopes of $H^h_{\mathrm{rig},Z}(U/K)$ lie in $[h-d,d]$. □

Remarks 16.12. — 1) The proof of (16.11) shows a generalization of [**25**, (8.2.4) (i), (ii)] by an easy way.

2) In the statement (b)$_n$ in [**15**, I (2.5)], we have to assume that $\dim Z\leq n$.

Let us also consider the case of a certain coefficient.

Proposition 16.13. — *Let the notations be as in* (15.0.1) *and* (11.14). *Let* $B^{\bullet}_{\mathrm{conv}}$ *and* $C^{\bullet}_{\mathrm{conv}}$ *be locally free F-isocrystals in the log convergent topoi* $((\widetilde{X_{\bullet},D_{\bullet}})/\mathcal{V})^{\log}_{\mathrm{conv}}$ *and* $((\widetilde{Y_{\bullet},E_{\bullet}})/\mathcal{V})^{\log}_{\mathrm{conv}}$*, respectively. Let B and C be overconvergent F-isocrystals on U/K_0 and V/K_0 satisfying the condition in* (11.14). *Let* $\lambda^{\log}_{\mathrm{conv}}\colon B^{\bullet}_{\mathrm{conv}}\to R\rho^{\log}_{\bullet\mathrm{conv}*}(C^{\bullet}_{\mathrm{conv}})$ *be a morphism of F-isocrystals and let*

$\lambda\colon B \to R\rho_{\mathrm{rig}*}(C)$ *be a morphism of overconvergent isocrystals. Assume that the following diagram*

$$\begin{array}{ccc} R\Gamma_{\mathrm{rig}}(U/K_0, B) & \longleftarrow & R\Gamma(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet,D_\bullet)}(B^\bullet_{\mathrm{conv}})) \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \Xi(\lambda^{\log}_{\mathrm{conv}})} \\ R\Gamma_{\mathrm{rig}}(V/K_0, C) & \longleftarrow & R\Gamma(((\widetilde{Y_\bullet, E_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(Y_\bullet,E_\bullet)}(C^\bullet_{\mathrm{conv}})) \end{array}$$

is commutative. Then the following morphism is an isomorphism:

$$\begin{aligned} &\mathrm{M}\mathfrak{F}\mathfrak{C}\big(\Xi(\lambda^{\log}_{\mathrm{conv}})\colon R\Gamma\big(((\widetilde{X_\bullet, D_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(X_\bullet,D_\bullet)}(B^\bullet_{\mathrm{conv}})\big) \\ &\qquad \longrightarrow R\Gamma(((\widetilde{Y_\bullet, E_\bullet})/\mathcal{W})^{\log}_{\mathrm{crys}}, \Xi_{(Y_\bullet,E_\bullet)}(C^\bullet_{\mathrm{conv}}))\big) \\ &\qquad\qquad \longrightarrow \mathrm{M}\mathfrak{F}\mathfrak{C}\big(\lambda\colon R\Gamma_{\mathrm{rig}}(U/K_0, B) \longrightarrow R\Gamma_{\mathrm{rig}}(V/K_0, C)\big). \end{aligned} \tag{16.13.1}$$

Proof. — The Proposition immediately follows from (11.15). □

Next we generalize Kedlaya's Künneth formula in [**55**, (1.2.4)] as stated in the last sentence in [**55**, (1.5)] (Kedlaya's Künneth formula is a generalization of Berthelot's Künneth formula in [**8**, (3.2) (i)]). To generalize Kedlaya's Künneth formula above, we need the following lemma.

Lemma 16.14. — *Let $U = U^1 \cup U^2$ be an open covering of a separated scheme of finite type over κ. Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Set $U^{12} := U^1 \cap U^2$. Let $\overline{U}^1, \overline{U}^2$ and $\overline{U}^{12}$ be the closures of U^1, U^2 and U^{12} in $\overline{U}$, respectively. Let C be an overconvergent isocrystal on $(U, \overline{U})/K$. For $i = 1, 2$, let C_i (resp. C_{12}) be the restriction of C to $(U^i, \overline{U}^i)/K$ (resp. $(U^{12}, \overline{U}^{12})/K$). Let*

$$\sigma\colon R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \oplus R\Gamma_{\mathrm{rig}}(U^2/K, C_2) \longrightarrow R\Gamma_{\mathrm{rig}}(U^{12}/K, C_{12})$$

be the natural morphism. Then the natural morphism

$$R\Gamma_{\mathrm{rig}}(U/K, C) \longrightarrow R\Gamma_{\mathrm{rig}}(U^1/K, C_1) \oplus R\Gamma_{\mathrm{rig}}(U^2/K, C_2)$$

defined by "$x \longmapsto (-x|_{U^1}, x|_{U^2})$" induces an isomorphism

$$R\Gamma_{\mathrm{rig}}(U/K, C) \xrightarrow{\sim} \mathrm{MF}(\sigma). \tag{16.14.1}$$

Proof. — It suffices to prove that the morphism (16.14.1) induces an isomorphism

$$H^h_{\mathrm{rig}}(U/K, C) \xrightarrow{\sim} H^h\big(\mathrm{MF}(\sigma)\big).$$

By [**87**, (6.3.2)] we have the Mayer-Vietoris exact sequence:

$$\cdots \longrightarrow H^h_{\mathrm{rig}}(U/K, C) \longrightarrow H^h_{\mathrm{rig}}(U^1/K, C_1) \oplus H^h_{\mathrm{rig}}(U^2/K, C_2) \tag{16.14.2}$$

$$\longrightarrow H^h_{\mathrm{rig}}(U^{12}/K, C_{12}) \longrightarrow \cdots$$

Hence we obtain the isomorphism $H^h_{\mathrm{rig}}(U/K, C) \overset{\sim}{\longrightarrow} H^h(\mathrm{MF}(\sigma))$ by the five lemma. □

Theorem 16.15 (Künneth formula). — *Let U^i $(i = 1, 2)$ be a separated scheme of finite type over a not necessarily perfect field κ of characteristic $p > 0$. Let Z^i be a closed subscheme of U^i. Let $U^i \hookrightarrow \overline{U}^i$ be an open immersion into a proper scheme over κ. Let C_i be an overconvergent F-isocrystal on $(U^i, \overline{U}^i)/K_0$. Set*

$$U^{12} := U^1 \times_\kappa U^2 \quad \textit{and} \quad Z^{12} := Z^1 \times_\kappa Z^2.$$

Then the following hold:

1) *The following canonical morphism is an isomorphism:*

$$R\Gamma_{\mathrm{rig},Z^1}(U^1/K_0, C_1) \otimes_{K_0} R\Gamma_{\mathrm{rig},Z^2}(U^2/K_0, C_2) \longrightarrow R\Gamma_{\mathrm{rig},Z^{12}}(U^{12}/K_0, C_1 \boxtimes C_2). \tag{16.15.1}$$

2) *Assume that κ is perfect. Then the induced isomorphism on the cohomologies by the isomorphism* (16.15.1) *for the trivial coefficient is compatible with the weight filtration.*

Proof. — 1): We may assume that κ is perfect by the proof of [**86**, (5.1.1)]. Let V^i be the complement of Z^i in U^i. Let $\overline{V}^i$ be the closure of V^i in $\overline{U}^i$. Set

$$VU^{12} := V^1 \times_\kappa U^2, \quad UV^{12} := U^1 \times_\kappa V^2, \quad V^{12} := V^1 \times_\kappa V^2.$$

Then $(VU^{12}) \cap (UV^{12}) = V^{12}$ and the complement of Z^{12} in U^{12} is $(VU^{12}) \cup (UV^{12})$. Let B_i be the restriction of C_i to $(V^i, \overline{V}^i)$. Set

$$C_{12} := C_1 \boxtimes C_2, \quad BC_{12} := B_1 \boxtimes C_2, \quad CB_{12} := C_1 \boxtimes B_2, \quad B_{12} := B_1 \boxtimes B_2.$$

By (16.14) we have the canonical isomorphism

$$\begin{aligned} R\Gamma_{\mathrm{rig}}((VU^{12} \cup UV^{12})/K_0, C_{12}) &= \bigl\{R\Gamma_{\mathrm{rig}}(VU^{12}/K_0, BC_{12}) \oplus R\Gamma_{\mathrm{rig}}(UV^{12}/K_0, CB_{12})\bigr\} \\ &\quad \oplus R\Gamma_{\mathrm{rig}}(V^{12}/K_0, B_{12})[-1]. \end{aligned} \tag{16.15.2}$$

Hence we have the following isomorphism

$$\begin{aligned} &R\Gamma_{\mathrm{rig},Z^{12}}(U^{12}/K_0, C_{12}) \\ &= R\Gamma_{\mathrm{rig}}(U^{12}/K_0, C_1 \boxtimes C_2) \oplus \bigl\{R\Gamma_{\mathrm{rig}}(VU^{12}/K_0, BC_{12})[-1] \\ &\qquad \oplus R\Gamma_{\mathrm{rig}}(UV^{12}/K_0, CB_{12})[-1] \oplus R\Gamma_{\mathrm{rig}}(V^{12}/K_0, B_{12})[-2]\bigr\}. \end{aligned} \tag{16.15.3}$$

By the Künneth formula ((10.18)), this complex is isomorphic to

$$\begin{aligned}&\mathrm{MF}\big(R\Gamma_{\mathrm{rig}}(U^1/K_0,C_1)\longrightarrow R\Gamma_{\mathrm{rig}}(V^1/K_0,B_1)\big)\\&\qquad\otimes_{K_0}\mathrm{MF}\big(R\Gamma_{\mathrm{rig}}(U^2/K_0,C_2)\longrightarrow R\Gamma_{\mathrm{rig}}(V^2/K_0,B_2)\big),\end{aligned}$$

which is isomorphic to $R\Gamma_{\mathrm{rig},Z^1}(U^1/K_0,C_1)\otimes_{K_0}R\Gamma_{\mathrm{rig},Z^2}(U^2/K_0,C_2)$.

2) Let $(U^i_\bullet,X^i_\bullet)$ and $(V^i_\bullet,Y^i_\bullet)$ be good proper hypercoverings of $(U^i,\overline{U}^i)$ and $(V^i,\overline{V}^i)$ which have the disjoint unions of the members of affine simplicial open coverings of $(U^i_\bullet,X^i_\bullet)$ and $(V^i_\bullet,Y^i_\bullet)$ over $(U^i,\overline{U}^i)$ and $(V^i,\overline{V}^i)$, respectively, fitting into the commutative diagram (15.0.1). Set $D^i_\bullet:=X^i_\bullet\setminus U^i_\bullet$ and $E^i_\bullet:=Y^i_\bullet\setminus V^i_\bullet$. Let the notations be as in (14.13.1) for the case $Z^i_{\underline{\bullet}}=W^i_{\underline{\bullet}}=\phi$. Then, by (11.7.1), the right hand side on (16.15.3) for the trivial coefficient is equal to

$$\begin{aligned}&R\Gamma\big((X^{12}_\bullet,D^{12}_\bullet)/\mathcal{W}\big)_{K_0}\oplus\{R\Gamma\big((YX^{12}_\bullet,ED^{12}_\bullet)/\mathcal{W}\big)_{K_0}[-1]\\&\qquad\oplus R\Gamma\big((XY^{12}_\bullet,DE^{12}_\bullet)/\mathcal{W}\big)_{K_0}[-1]\oplus R\Gamma\big((Y^{12}_\bullet,E^{12}_\bullet)/\mathcal{W}\big)_{K_0}[-2]\}.\end{aligned}$$

By (10.1), (11.15) and (14.18), 3), this complex is isomorphic to

$$\begin{aligned}&\mathrm{MF}\big(R\Gamma\big((X^1_\bullet,D^1_\bullet)/\mathcal{W}\big)_{K_0}\longrightarrow R\Gamma\big((Y^1_\bullet,E^1_\bullet)/\mathcal{W}\big)\big)_{K_0}\\&\qquad\otimes_{K_0}\mathrm{MF}\big(R\Gamma((X^2_\bullet,D^2_\bullet)/\mathcal{W})_{K_0}\longrightarrow R\Gamma\big((Y^2_\bullet,E^2_\bullet)/\mathcal{W}\big)_{K_0}\big).\end{aligned}$$

Now, as in the proof of (12.13), 2) follows from (14.15). □

Next we prove that the Gysin morphism with closed support in rigid cohomology is strictly compatible with the weight filtration ((16.19) below).

Let Z be a separated scheme of finite type over κ. Until (16.19), assume that there exists a closed immersion

$$Z\hookrightarrow U \tag{16.15.4}$$

into a separated smooth scheme of finite type over κ of pure dimension d and that Z is of pure codimension c in U.

The following is a slight generalization of [**9**, (5.7)]:

Proposition 16.16 (**cf.** [**9**, (5.7)]). — *The following hold*:

1)

$$H^h_{\mathrm{rig},Z}(U/K)=0\quad(h<2c). \tag{16.16.1}$$

2) *Let e be the cardinality of the geometric irreducible components of Z. Then*

$$H^{2c}_{\mathrm{rig},Z}(U/K)=K^e. \tag{16.16.2}$$

Proof. — 1): We proceed by the descending induction on the codimension of Z in U. If Z is a 0-dimensional scheme, then 1) follows from (16.6.2) and [**9**, (5.7) (i)]. Let U' be an open subscheme of U. Set $T := U'^{c} \cap Z$ and $Z' := Z \setminus T$. By (16.5.2) and (16.6.2), we have the following exact sequence

$$H^h_{\mathrm{rig},T}(U/K) \longrightarrow H^h_{\mathrm{rig},Z}(U/K) \longrightarrow H^h_{\mathrm{rig},Z'}(U'/K). \tag{16.16.3}$$

By the same argument as that in the proof of (10.1) and by [**9**, (5.7) (i)], we may assume that $H^h_{\mathrm{rig},Z'}(U'/K) = 0$ for $h < 2c$. By the inductive hypothesis, $H^h_{\mathrm{rig},T}(U/K) = 0$ for $h < 2c$. Hence $H^h_{\mathrm{rig},Z}(U/K) = 0$ for $h < 2c$.

2): By a standard argument of e.g., [**64**, VI (9.1)] (cf. [**9**, (5.7)]), by (16.16.1) and by (16.5.2), we may assume that U is an open subscheme of $\overline{U}$ which is a closed formal $\mathcal{V}$-scheme which is formally smooth around U. In this case, 2) is nothing but [**9**, (5.7)]. □

Lemma 16.17 (**cf.** [**76**, (7.13)]). — *Let e be the cardinality of the geometric irreducible components of Z. Then the canonical isomorphism*

$$K^e \xrightarrow{\sim} H^{2c}_{\mathrm{rig},Z}(U/K)$$

in (16.16.2) *is an underlying isomorphism of the following isomorphism*

$$K^e(-c) \xrightarrow{\sim} H^{2c}_{\mathrm{rig},Z}(U/K) \tag{16.17.1}$$

as F-isocrystals.

Proof. — By the exact sequence (16.5.2) and by a standard argument of e.g., [**64**, VI (9.1)] (cf. [**9**, (5.7)]), we may assume that U and Z are smooth and affine, and, furthermore, they are liftable by the argument in [**15**, p. 691]. In this case, by the proof of [**15**, I (2.4)], we have (16.17). □

Theorem 16.18. — *Assume that κ is perfect. Then the rigid cohomology $H^{2c}_{\mathrm{rig},Z}(U/K)$ is pure of weight $2c$.*

Proof. — We may assume that $K = K_0$. Let $j\colon U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Let V be the complement of Z in U. Let $\overline{V}$ be the closure of V in $\overline{U}$. Then, by (9.2) and (9.4), 2), there exist a morphism $\rho_\bullet\colon (V_\bullet, Y_\bullet) \to (U_\bullet, X_\bullet)$ from a gs proper hypercovering of $(V, \overline{V})$ to that of $(U, \overline{U})$ fitting into the commutative diagram (15.0.1) for the morphism $(V, \overline{V}) \hookrightarrow (U, \overline{U})$. Set $D_\bullet := X_\bullet \setminus U_\bullet$ and $E_\bullet := Y_\bullet \setminus V_\bullet$. Let

$$\rho^{\log *}_{\bullet\mathrm{crys}}\colon R\Gamma\big((X_\bullet, D_\bullet)/\mathcal{W}\big) \longrightarrow R\Gamma\big((Y_\bullet, E_\bullet)/\mathcal{W}\big)$$

be the induced morphism by $\rho_\bullet$. Then $H^h_{\mathrm{rig},Z}(U/K_0) = H^h(\mathrm{MF}(\rho^{\log *}_{\bullet\mathrm{crys}}))_{K_0}$ (see (15.0.2)).

Let N be an integer satisfying the inequality (2.2.1) for $h = 2c$. As in §12, let A_1 be a smooth subring of κ over a finite field $\mathbb{F}_q$ such that the truncated morphism $\rho_{\bullet\leq N}\colon (Y_{\bullet\leq N}, E_{\bullet\leq N}) \to (X_{\bullet\leq N}, D_{\bullet\leq N})$ is defined over A_1. Let $\rho^{A_1}_{\bullet\leq N}$ be a model of $\rho_{\bullet\leq N}$ over A_1. Let A be a formally smooth lift of A_1 over $\mathcal{W}(\mathbb{F}_q)$. Fix an endomorphism F_A of A which is a lift of the Frobenius endomorphism of A_1. Then we have a natural ring morphism $A \to \mathcal{W}(A_1)$ (see [**61**, VII (4.12)]). Thus $\mathcal{W}(= \mathcal{W}(\kappa))$ becomes an A-algebra. Let x be a closed point of $\operatorname{Spec}(A_1)$. Let $K_0(\kappa(x))$ be the fraction field of the Witt ring $\mathcal{W}(\kappa(x))$ of the residue field $\kappa(x)$ of x. Let $\rho^{A_1}_{\bullet\leq N}(x)$ be the reduction of $\rho^{A_1}_{\bullet\leq N}$ at x. If $\operatorname{Spec}(A_1)$ is small enough, then the natural morphisms

$$\begin{aligned}(16.18.1)\quad & P_kH^h\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}})\bigr)\otimes_A K_0\\ &\longrightarrow P_kH^h\bigl(\mathrm{MF}(\rho^{\log *}_{\bullet\leq N,\mathrm{crys}})\bigr)\otimes_{\mathcal{W}} K_0 = P_kH^h(\mathrm{MF}(\rho^{\log *}_{\bullet\mathrm{crys}}))\otimes_{\mathcal{W}} K_0,\end{aligned}$$

$$\begin{aligned}(16.18.2)\quad & P_kH^h\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}})\bigr)\otimes_A K_0(\kappa(x))\\ &\longrightarrow P_kH^h\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}(x))\bigr)\otimes_{\mathcal{W}(\kappa(x))} K_0\bigl(\kappa(x)\bigr)\end{aligned}$$

are isomorphisms for all $h \leq 2c$ and for all $k \in \mathbb{Z}$ ((8.1)). The isomorphisms (16.18.1) and (16.18.2) are compatible with the Frobenius action. Since A_1 is reduced, the morphism $A \to \mathcal{W}(A_1)$ is injective by [**72**, (2.15.1)]. Consequently the composite map $A \to \mathcal{W}$ is injective. Hence the Frobenius on $H^{2c}(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}))\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)$ acts on p^cF_A by (16.17). Because

$$P_kH^h\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}})\bigr)\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)$$

is a convergent isocrystal ((14.6)), it suffices to prove that

$$(16.18.3)\qquad P_{2c-1}H^{2c}\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}(x))\bigr)\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q) = 0$$

and

$$\begin{aligned}& P_{2c}H^{2c}\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}(x))\bigr)\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)\\ (16.18.4)\qquad &= H^{2c}\bigl(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}(x))\bigr)\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)\end{aligned}$$

by [**74**, (3.17)]. Let $\sigma_x \in \operatorname{Aut}(\mathcal{W}(\kappa(x)))$ be the lift of the p-th power isomorphism of $\kappa(x)$. Then the Frobenius action on $H^{2c}(\mathrm{MF}(\rho^{A_1\log *}_{\bullet\leq N,\mathrm{crys}}(x)))\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)$ is $p^c\sigma_x$ by (16.17). Hence we have (16.18.3) and (16.18.4).

Now we finish the proof of (16.18). □

PROPOSITION 16.19. — *The Gysin morphism*

$$(16.19.1)\qquad H^h_{\mathrm{rig},Z}(U/K)(-c) \longrightarrow H^{h+2c}_{\mathrm{rig},Z}(U/K)$$

is strictly compatible with the weight filtration.

Proof. — Let $c_U(Z) \in H^{2c}_{\mathrm{rig},Z}(U/K)$ be the fundamental class of Z. Let $\Delta\colon (Z,U) \to (Z \times_\kappa Z, U \times_\kappa U)$ be the diagonal closed immersion. Because the Gysin morphism (16.19.1) is equal to the following composite morphism

$$(16.19.2) \qquad H^h_{\mathrm{rig},Z}(U/K)(-c) \xrightarrow{\mathrm{id}\otimes c_U(Z)} H^h_{\mathrm{rig},Z}(U/K) \otimes_K H^{2c}_{\mathrm{rig},Z}(U/K) \\ \subset H^{h+2c}_{\mathrm{rig},Z\times_\kappa Z}(U \times_\kappa U/K) \xrightarrow{\Delta^*} H^{h+2c}_{\mathrm{rig},Z}(U/K).$$

Hence (16.19) follows from (16.18), the Künneth formula (16.15), 2) and (16.3). □

Next we prove the variant of the Serre-Grothendieck's conjecture of virtual Betti numbers of a separated scheme of finite type over κ as to the existence of the desired functions in the Introduction (see [**33**, Partie II, A, II (46_9)]).

Let us consider the following Grothendieck group $\mathrm{GF}_2(K_0)$ of bifiltered finite dimensional K_0-vector spaces:

▷ The generators of $\mathrm{GF}_2(K_0)$ are classes

$$[H,P,F] := \big[H, \{P_k\}_{k\in\mathbb{Z}}, \{F^i\}_{i\in\mathbb{Z}}\big]\text{'s}$$

of triples $(H, \{P_k\}_{k\in\mathbb{Z}}, \{F^i\}_{i\in\mathbb{Z}})$'s, where $\{P_k\}_{k\in\mathbb{Z}}$ (resp. $\{F^i\}_{i\in\mathbb{Z}}$) is an increasing (resp. decreasing) filtration on a finite dimensional K_0-vector space H.

▷ The relation in $\mathrm{GF}_2(K_0)$ is

$$[H_3,P_3,F_3] = [H_1,P_1,F_1] + [H_2,P_2,F_2]$$

if the following sequence is exact for all $k \in \mathbb{Z}$ and all $i \in \mathbb{Z}$:

$$(16.19.3) \qquad 0 \to (P_{1,k} \cap F^i_1)H_1 \longrightarrow (P_{3,k} \cap F^i_3)H_3 \longrightarrow (P_{2,k} \cap F^i_2)H_2 \to 0.$$

Proposition 16.20. — *The sequence* (16.19.3) *is exact if and only if the following sequence is exact for all $k \in \mathbb{Z}$ and all $i \in \mathbb{Z}$:*

$$(16.20.1) \qquad 0 \to \mathrm{gr}^i_{F_1}\mathrm{gr}^{P_1}_k H_1 \longrightarrow \mathrm{gr}^i_{F_3}\mathrm{gr}^{P_3}_k H_3 \longrightarrow \mathrm{gr}^i_{F_2}\mathrm{gr}^{P_2}_k H_2 \to 0.$$

Proof. — More generally, consider a complex $(E^\bullet, \{P_kE^\bullet\}_{k\in\mathbb{Z}}, \{Q_kE^\bullet\}_{k\in\mathbb{Z}})$ with two increasing filtrations of K_0-vector spaces such that there exist integers k_0 and k_1 such that $P_{k_0}E^\bullet = Q_{k_0}E^\bullet = E^\bullet$ and $P_{k_1}E^\bullet = Q_{k_1}E^\bullet = 0$. We have only to prove that $(P_k \cap Q_i)E^\bullet$ is exact for all $k, i \in \mathbb{Z}$ if and only if $\mathrm{gr}^Q_i\mathrm{gr}^P_k E^\bullet$ is exact for all $k, i \in \mathbb{Z}$. Since

$$\mathrm{gr}^Q_i\mathrm{gr}^P_k E^\bullet = (P_k \cap Q_i)E^\bullet/((P_{k-1} \cap Q_i)E^\bullet + (P_k \cap Q_{i-1})E^\bullet)$$

and since the following sequence

$$0 \to (P_{k-1} \cap Q_{i-1})E^\bullet \longrightarrow (P_{k-1} \cap Q_i)E^\bullet \oplus (P_k \cap Q_{i-1})E^\bullet \longrightarrow (P_{k-1} \cap Q_i)E^\bullet + (P_k \cap Q_{i-1})E^\bullet \to 0$$

is exact, the implication $\Rightarrow$ is clear. We obtain the converse implication without difficulty by the double induction. □

Set

$$B_p^k(H, P, F) := \mathrm{gr}_k^P H, \quad \mathrm{gr}_{F,p}^i(H, P, F) := \mathrm{gr}_F^i H, \quad H_p^{i,k-i} := \mathrm{gr}_F^i \mathrm{gr}_k^P H.$$

By (16.20) we have well-defined morphisms of Grothendieck's groups:

$$B_p^k\colon \mathrm{GF}_2(K_0) \longrightarrow \mathrm{G}(K_0) \quad (k \in \mathbb{Z}), \tag{16.20.2}$$

$$\mathrm{gr}_{F,p}^i\colon \mathrm{GF}_2(K_0) \longrightarrow \mathrm{G}(K_0) \quad (i \in \mathbb{Z}), \tag{16.20.3}$$

$$H_p^{i,k-i}\colon \mathrm{GF}_2(K_0) \longrightarrow \mathrm{G}(K_0) \quad (k, i \in \mathbb{Z}). \tag{16.20.4}$$

The dimension $\dim_{K_0}$ from finite dimensional vector spaces over K_0 to natural numbers extends to the isomorphism

$$\dim_{K_0}\colon \mathrm{G}(K_0) \xrightarrow{\sim} \mathbb{Z}. \tag{16.20.5}$$

Set

$$h_p^k := \dim_{K_0} \circ B_p^k, \quad f_p^i := \dim_{K_0} \circ \mathrm{gr}_{F,p}^i \quad h_p^{i,k-i} := \dim_{K_0} \circ H_p^{i,k-i}. \tag{16.20.6}$$

Next let us consider a geometric case.

Let U be a separated scheme of finite type over κ. Let Z be a closed subscheme of U over κ. Then $H^m_{\mathrm{rig},Z}(U/K_0)$ $(m \in \mathbb{Z})$ has the weight filtration P (see (16.1)) and the slope filtration F (see (15.1)). Henceforth we omit to write the weight filtration and the slope filtration for an element of $\mathrm{GF}_2(K_0)$. We have integers $h_p^k(H^m_{\mathrm{rig},Z}(U/K_0))$, $f_p^i(H^m_{\mathrm{rig},Z}(U/K_0))$ and $h_p^{i,k-i}(H^m_{\mathrm{rig},Z}(U/K_0))$. Moreover, let us consider an element

$$\bigl[H_{\mathrm{rig},Z}(U/K_0)\bigr] := \sum_{m \in \mathbb{N}} (-1)^m \bigl[H^m_{\mathrm{rig},Z}(U/K_0)\bigr].$$

Definition 16.21. — We call the integers

$$h_p^k\bigl([H_{\mathrm{rig},Z}(U/K_0)]\bigr), \quad f_p^i\bigl([H_{\mathrm{rig},Z}(U/K_0)]\bigr), \quad h_p^{i,k-i}\bigl([H_{\mathrm{rig},Z}(U/K_0)]\bigr)$$

the *virtual Betti number*, the *virtual slope number* and the *virtual slope-Betti number* of $(Z, U)/\kappa$. We denote them by $h^k_{p,Z}(U)$, $f^i_{p,Z}(U)$ and $h^{i,k-i}_{p,Z}(U)$, respectively. If $Z = U$, then we denote them simply by $h_p^k(U)$, $f_p^i(U)$ and $h_p^{i,k-i}(U)$, respectively.

THEOREM 16.22 (cf. [**33**, Partie II, A, II (**46₉**)]). — *Let* $\mathbf{CS}(\kappa)$ *be the set of isomorphism classes of separated schemes of finite type over* κ *with closed subschemes over* κ. *Then the maps*

$$h_p^k\colon \mathbf{CS}(\kappa) \longrightarrow \mathbb{Z}, \tag{16.22.1}$$

$$h_p^{i,k-i}\colon \mathbf{CS}(\kappa) \longrightarrow \mathbb{Z} \tag{16.22.2}$$

satisfy the following equalities:

1) *Let* U *be a separated scheme of finite type over* κ. *For a closed subscheme* Z *of* U,

$$h_p^{i,k-i}(U) = h_{p,Z}^{i,k-i}(U) + h_p^{i,k-i}(U \setminus Z). \tag{16.22.3}$$

More generally, let T *be a closed subscheme of* Z. *Set* $U' := U \setminus T$ *and* $Z' := Z \setminus T$. *Then*

$$h_{p,Z}^{i,k-i}(U) = h_{p,T}^{i,k-i}(U) + h_{p,Z'}^{i,k-i}(U'). \tag{16.22.4}$$

2) *For a proper smooth scheme* X *over* κ,

$$\begin{aligned} h_p^{i,k-i}(X) &= (-1)^k \dim_{K_0} H^{k-i}(X, \mathcal{W}\Omega_X^i)_{K_0}, \\ h_p^k(X) &= (-1)^k \dim_{K_0} H^k_{\mathrm{crys}}(X/\mathcal{W})_{K_0}. \end{aligned} \tag{16.22.5}$$

3) *Let* U *be a separated scheme of finite type over* κ *and let* Z *be a closed subscheme of* U. *Let* U' *be an open subscheme of* U *which contains* Z *as a closed subscheme. Then*

$$h_{p,Z}^{i,k-i}(U) = h_{p,Z}^{i,k-i}(U'). \tag{16.22.6}$$

4)

$$h_{p,Z_1\times_\kappa Z_2}^{i,k-i}(U_1 \times_\kappa U_2) = \sum_{\substack{i_1+i_2=i \\ k_1+k_2=k}} h_{p,Z_1}^{i_1,k_1-i_1}(U_1) h_{p,Z_2}^{i_2,k_2-i_2}(U_2). \tag{16.22.7}$$

5)

$$h_{p,Z}^k(U) = \sum_{i\in\mathbb{Z}} h_{p,Z}^{i,k-i}(U). \tag{16.22.8}$$

Proof. — 1): By (16.6), 1) and the slope decomposition (15.7.3), the exact sequence (16.6.1) is strictly exact with respect to the weight filtration and the slope filtration. The graded objects of the weight filtrations of the graded objects of the slope filtrations of the exact sequence (16.6.1) are also exact (cf. (16.1.2)). Hence the first formula in 1) follows. Analogously we obtain the second formula in 1) by using the exact sequence (16.5.2).

2): The second formula in 2) immediately follows from the simple remark after (16.2). The first formula in 2) follows from the slope decomposition of the crystalline cohomology (see [**47**, II (3.5.4)]).

3): The formula (16.22.6) immediately follows from (16.6.2).

4): The formula (16.22.7) immediately follows from (16.15), 2).

5): The formula (16.22.8) follows from the definition of $h^{i,k-i}_{p,Z}(U)$. □

Let ℓ be a prime number which is prime to p. Denote $U\otimes_\kappa \kappa_{\rm sep}$ by $U_{\kappa_{\rm sep}}$. By (12.20) we can construct an analogue $h^k_{\ell,Z}(U)$ of $h^k_{p,Z}(U)$ for the ℓ-adic cohomology $H^\bullet_{\text{ét},Z_{\kappa_{\rm sep}}}(U_{\kappa_{\rm sep}},\mathbb{Q}_\ell)$ $((\ell,p)=1)$.

THEOREM 16.23. — *Let κ be a perfect field of characteristic $p>0$. Then*

$$h^k_{\ell,Z}(U)=h^k_{p,Z}(U). \tag{16.23.1}$$

Proof. — If κ is a finite field, then (16.23.1) is obvious by the Weil conjecture for the ℓ-adic cohomology and the crystalline cohomology of a proper smooth scheme over κ (see [**26**, (3.3.9)], [**54**], [**16**, (1.2)] (cf. [**69**, (2.2) (4)])).

In the general case, we can reduce (16.23) to the case above by using the spectral sequence (16.1.1), the analogue of (16.1.1) for the ℓ-adic cohomology, the specialization argument of Deligne-Illusie (see [**46**, (3.10)], [**69**, §3]) and the standard specialization argument in the ℓ-adic cohomology (cf. (12.3) and (12.4)). □

Consider the finite field case $\kappa=\mathbb{F}_q$, where q is a power of p. Let α be an algebraic number of pure weight i $(i\in\mathbb{N})$ with respect to q. Let v be a p-adic discrete valuation of $\mathbb{Q}_p(\alpha)$ normalized as $v(q)=1$. The algebraic number α is of slope $\geq i$ if $v(\alpha)\geq i$. The Frobenius F acts on $H^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}(U_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)$ and $H^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}(U_{\overline{\mathbb{F}}_q},\overline{\mathbb{Q}}_\ell)$. Set

$$\begin{aligned}\mathrm{Fil}^iH^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}&(U_{\overline{\mathbb{F}}_q},\overline{\mathbb{Q}}_\ell)\\ &=\big\{\text{the principal subspace of } H^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}(U_{\overline{\mathbb{F}}_q},\overline{\mathbb{Q}}_\ell)\\ &\qquad\text{where the eigenvalues } \alpha\text{'s of } F \text{ are of slope} \geq i\big\}.\end{aligned} \tag{16.23.2}$$

Using the filtration Fil and the weight filtration on

$$H^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}(U_{\overline{\mathbb{F}}_q},\overline{\mathbb{Q}}_\ell)=H^h_{\text{ét},Z_{\overline{\mathbb{F}}_q}}(U_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell,$$

we can define an analogue $h^{i,k-i}_{\ell,Z}(U)$ of the virtual slope-Betti number $h^{i,k-i}_{p,Z}(U)$ as in (16.20.6) (cf. [**26**, III (3.3.7)]). Then the following holds:

THEOREM 16.24. — *If κ is a finite field, then $h^{i,k-i}_{\ell,Z}(U)=h^{i,k-i}_{p,Z}(U)$.*

Proof. — The Theorem follows from (16.1.1), from the ℓ-adic analogue of (16.1.1), from [**16**, (1.3)] (cf. [**69**, (2.2.6)]). □

We would like to conjecture the following:

CONJECTURE 16.25. — Let κ be a not necessarily perfect field of characteristic $p > 0$. Let $f\colon V \to U$ be a morphism of separated schemes of finite type over κ. Let $\mathfrak{FC}$ be a letter F or C. Let $\ell \neq p$ be a prime number. Then

$$\begin{aligned}(16.25.1)\qquad &\dim_K H^h\big(\mathrm{M}\mathfrak{FC}(f^*\colon R\Gamma_{\mathrm{rig}}(U/K) \to R\Gamma_{\mathrm{rig}}(V/K))\big)\\ &= \dim_{\mathbb{Q}_\ell} H^h\big(\mathrm{M}\mathfrak{FC}(f^*\colon R\Gamma_{\text{ét}}(U \otimes_\kappa \kappa_{\mathrm{sep}}, \mathbb{Q}_\ell) \to R\Gamma_{\text{ét}}(V \otimes_\kappa \kappa_{\mathrm{sep}}, \mathbb{Q}_\ell))\big).\end{aligned}$$

If κ is perfect, then

$$\begin{aligned}(16.25.2)\qquad &\dim_K P_k H^h\big(\mathrm{M}\mathfrak{FC}(f^*\colon R\Gamma_{\mathrm{rig}}(U/K) \to R\Gamma_{\mathrm{rig}}(V/K))\big)\\ &= \dim_{\mathbb{Q}_\ell} P_k H^h\big(\mathrm{M}\mathfrak{FC}(f^*\colon R\Gamma_{\text{ét}}(U \otimes_\kappa \kappa_{\mathrm{sep}}, \mathbb{Q}_\ell) \to R\Gamma_{\text{ét}}(V \otimes_\kappa \kappa_{\mathrm{sep}}, \mathbb{Q}_\ell))\big)\end{aligned}$$

with $k \in \mathbb{Z}$.

PROPOSITION 16.26. — *The following hold*:

1) *Let κ be a not necessarily perfect field of characteristic $p > 0$. Let the notations be as in* (16.25). *The formula* (16.25.1) *holds for $\mathfrak{FC} = \mathrm{F}$ and $h = 0, 1$. The formula* (16.25.1) *holds for $\mathfrak{FC} = \mathrm{C}$ and $h = 0$.*

2) *If κ is perfect, then the formula* (16.25.2) *holds for $\mathfrak{FC} = \mathrm{F}$ and $h = 0, 1$. The formula* (16.25.2) *holds for $\mathfrak{FC} = \mathrm{C}$ and $h = 0$.*

Proof. — Here we prove only 2) for the case $h = 0, 1$ and $\mathfrak{FC} = \mathrm{F}$. We leave the rest to the reader.

By (11.6), by (14.2) and by the base change theorems of crystalline and ℓ-adic cohomologies, we may assume that κ is algebraically closed. Let the notations be as in (15.0.1). Consider the spectral sequence (15.1.1). In this proof we denote simply by $H^*(D_t^{(t+k)})(-t-k)$ the crystalline cohomology group

$$H^*\big((\widetilde{D_t^{(t+k)}}/\mathcal{W})_{\mathrm{crys}}, \mathcal{O}_{D_t^{(t+k)}/\mathcal{W}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(t+k)}(D_t/\mathcal{W})\big)\big(-(t+k)\big)_K.$$

Set $h = 0$. Then the nontrivial term of the E_1-terms of (15.1.1) is only $E_1^{00} = H^0(D_0^{(0)})$. Set $h = 1$. Then the nontrivial terms of the E_1-terms of (15.1.1) are only $E_1^{-12} = H^0(D_0^{(1)})(-1)$, $E_1^{01} = H^1(D_0^{(0)})$, $E_1^{10} = H^0(D_1^{(0)}) \oplus H^0(E_0^{(0)})$. The boundary morphisms $d_1^{00}\colon E_1^{00} \to E_1^{10}$ and $d_1^{10}\colon E_1^{10} \to E_1^{20}$ are defined over $\mathbb{Q}$. The boundary morphism $d_1^{01}\colon E_1^{01} \to E_1^{11}$ is induced from the morphisms of Albanese varieties; hence the dimensions of E_2^{01} in the p-adic case and the ℓ-adic case are equal. Finally, consider the part $E_1^{-12} = H^0(D_0^{(1)})(-1) \longrightarrow H^2(D_0^{(0)})$ of the boundary morphism

$$d_1^{-12} \colon E_1^{-12} \longrightarrow E_1^{02} = H^2(D_0^{(0)}) \oplus H^0(D_1^{(1)})(-1) \oplus H^0(E_0^{(1)})(-1)$$

in the p-adic case. The crystalline cohomology $H^0(D_0^{(1)})$ has a natural $\mathbb{Q}$-structure $H_{\mathbb{Q}}$ and the most nontrivial part of d_1^{-12} is induced by the following composite morphism

$$H_{\mathbb{Q}} \longrightarrow \mathrm{NS}(D_0^{(0)}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \mathrm{NS}(D_0^{(0)}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow H^2(D_0^{(0)}).$$

By the argument in the proof of [**70**, (8.3)], we see that the natural morphism

$$(\mathrm{NS}(D_0^{(0)}) \otimes_{\mathbb{Z}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \longrightarrow H^2(D_0^{(0)})$$

is injective. Therefore E_1^{02} has a natural $\mathbb{Q}$-structure $E_{\mathbb{Q}}$ and $E_2^{-12} = \mathrm{Ker}(H_{\mathbb{Q}} \longrightarrow E_{\mathbb{Q}}) \otimes_{\mathbb{Q}} K$. We also obtain an analogous much simpler fact in the ℓ-adic case.

Thus we obtain 2) for the case $\mathfrak{FC} = \mathrm{F}$ and $h = 0, 1$. □

17. Rigid cohomology with compact support

In [**80**] Shiho has recently proved the relative versions of his comparison theorems in [**82**] (see (17.5) below). In this section, by using the relative versions, we endow the rigid cohomology with compact support with the weight filtration and prove several basic properties of the weight filtration.

Unless stated otherwise, the base field κ is a perfect field of characteristic $p > 0$ in this section.

First we note that there is an announcement in [**6**, pp. 21–23] that the rigid cohomology with compact support of a separated scheme of finite type over κ is defined.

Let U be a separated smooth scheme of finite type over κ of pure dimension d. Then, by the same proof as that of [**8**, (2.4)], we have a canonical isomorphism

(17.0.1) $$H^h_{\mathrm{rig,c}}(U/K) \xrightarrow{\sim} \mathrm{Hom}_K(H^{2d-h}_{\mathrm{rig}}(U/K), K).$$

Here note that U is not necessarily an open subscheme of a proper scheme $\overline{U}$ which is a closed subscheme of a p-adic formal $\mathcal{V}$-scheme which is formally smooth around U.

By (16.9.2), the trace morphism $\mathrm{Tr}_U\colon H^{2d}_{\mathrm{rig,c}}(U/K)\to K(-d)$ is compatible with the Frobenius. Endow $K(-d)$ with an increasing filtration P defined by the formulas

$$P_{2d}K(-d)=K(-d),\quad P_{2d-1}K(-d)=0.$$

Because $H^{2d-h}_{\mathrm{rig}}(U/K)$ has the weight filtration, $\mathrm{Hom}_K(H^{2d-h}_{\mathrm{rig}}(U/K),K(-d))$ also has the weight filtration.

Thus, identifying $H^h_{\mathrm{rig,c}}(U/K)$ with $\mathrm{Hom}_K(H^{2d-h}_{\mathrm{rig}}(U/K),K(-d))$, we have the weight filtration P on $H^h_{\mathrm{rig,c}}(U/K)$.

Proposition 17.1 (**cf.** [**26**, III (3.3.4)]). — *Let U be a separated smooth scheme of finite type over κ of dimension d. Let $h\in[0,2d]$ be an integer. Then the weights of $H^h_{\mathrm{rig,c}}(U/K)$ lie in $[0,h]$.*

Proof. — The Proposition immediately follows from (16.7). □

Using the range of the slopes of the rigid cohomology of a separated smooth scheme of finite type over κ ((16.11)), we can reprove [**17**, (3.1.2)] (= the following) for the rigid cohomology with compact support of the scheme; see also [**26**, III (3.3.8)].

Proposition 17.2 (see [**17**]). — *Let U be a separated scheme of finite type over κ of dimension d. Let $h\in[0,2d]$ be an integer. Then the slopes of $H^h_{\mathrm{rig,c}}(U/K)$ lie in $[\max\{0,h-d\},\min\{h,d\}]$.*

Proof. — Let Z be a closed subscheme of U and V the complement of Z in U. Then there exists the following exact sequence (see [**6**, (3.1) (iii)]):

$$\cdots\longrightarrow H^h_{\mathrm{rig,c}}(V/K)\longrightarrow H^h_{\mathrm{rig,c}}(U/K)\longrightarrow H^h_{\mathrm{rig,c}}(Z/K)\longrightarrow\cdots. \tag{17.2.1}$$

Since $\dim Z\leq d$, we may assume that U is smooth over κ by a standard argument of the induction on $\dim U$. We may also assume that U is of pure dimension. Then (17.2) follows from the duality

$$H^h_{\mathrm{rig,c}}(U/K)=\mathrm{Hom}_K(H^{2d-h}_{\mathrm{rig}}(U/K),K(-d))$$

and from (16.11), 3). □

Proposition 17.3. — *Let $f\colon V\longrightarrow V'$ be a finite étale morphism of smooth affine schemes over κ. Then the trace morphism*

$$\mathrm{Tr}_f\colon H^h_{\mathrm{rig,c}}(V/K)\longrightarrow H^h_{\mathrm{rig,c}}(V'/K)$$

[**8**, (1.4)] *is strictly compatible with the weight filtration.*

Proof. — The Proposition immediately follows from (12.10) and the formula $\mathrm{Tr}_f = f_*$ (see [**8**, (2.3) (ii)]). □

Let (X, D) be a proper smooth scheme with an SNCD over κ. Let $H^h_{\mathrm{c}}((X, D)/\mathcal{W})$ ($h \in \mathbb{N}$) be the log crystalline cohomology with compact support of (X, D) over $\mathcal{W}$. Set $U := X \setminus D$. By [**85**], [**82**, Theorem 2.4.4, Corollary 2.3.9, Theorem 3.1.1] and [**8**, (2.4)], we obtain

$$H^h_{\mathrm{c}}\big((X, D)/\mathcal{W}\big)_K = H^h_{\mathrm{rig,c}}(U/K). \tag{17.3.1}$$

Proposition 17.4. — *The weight filtration on $H^h_{\mathrm{rig,c}}(U/K)$ in this book is equal to the weight filtration on $H^h_{\mathrm{c}}((X, D)/\mathcal{W})_K$ in* [**72**, (2.11.15.1)] *under the canonical isomorphism* (17.3.1)*.*

Proof. — We may assume that X is of pure dimension d. (17.4) immediately follows from the definition of the weight filtration on $H^h_{\mathrm{rig,c}}(U/K)$, from (12.9), 2) and from the fact that the Poincaré duality

$$H^h_{\mathrm{c}}\big((X, D)/\mathcal{W}\big)_K = \mathrm{Hom}_K\big(H^{2d-h}((X, D)/\mathcal{W})_K, K(-d)\big)$$

is compatible with the weight filtration (see [**72**, (2.19.1)]). □

The following are relative versions of Shiho's comparison theorems (see [**82**, Theorem 2.4.4, Corollary 2.3.9, Theorem 3.1.1]) for the trivial coefficients. In [**71**] Shiho and I have recently proved that the morphism (17.5.1) below is a functorial isomorphism (see [**71**]). In [**80**] Shiho has also proved that the morphism (17.5.2) below is a functorial isomorphism.

Theorem 17.5. — *Let S be a formally smooth p-adic formal $\mathcal{W}$-scheme. Set $S_1 := S \otimes_{\mathcal{W}} \kappa$. Let $f \colon (X, D) \to S_1$ be a proper smooth scheme with a relative SNCD over S_1. Set $U := X \setminus D$. By abuse of notation, denote also by f the structural morphism $U \to S_1$. Let $\mathrm{sp} \colon S_{K_0} \to S$ be the specialization map defined in* [**4**, (0.2.3)]*. Let $R^h f_{(X,D)/S_{K_0}*}(\mathcal{K}_{(X,D)/S_{K_0}})$ be the relative log naive convergent cohomology of $(X, D)/S_{K_0}$ and let $R^h f_{\mathrm{rig}*}(U/S_{K_0})$ be the relative rigid cohomology of U/S_{K_0}* (see [**19**, (10.6)])*. Then the following hold*:

1) (see [**71**], [**80**]) *The canonical morphism*

$$R^h f_{(X,D)/S_{K_0}*}(\mathcal{K}_{(X,D)/S_{K_0}}) \longrightarrow R^h f_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})_{K_0} \quad (h \in \mathbb{Z}) \tag{17.5.1}$$

is a functorial isomorphism.

2) (see [**80**]) *The canonical morphism*

$$R^h f_{(X,D)/S_{K_0}*}(\mathcal{K}_{(X,D)/S_{K_0}}) \longrightarrow \mathrm{sp}_* R^h f_{\mathrm{rig}*}(U/S_{K_0}) \quad (h \in \mathbb{Z}) \tag{17.5.2}$$

(cf. (11.4)) *is a functorial isomorphism.*

For the proof of the theorem (17.7) below, we recall the following GAGA in the rigid analytic geometry: we need only (17.6), 1) in this book; we state (17.6), 2) only for our memory. (Though (17.6) is contained in [**3**, (3.3.4) (ii), (3.4.9), (3.4.11)] (see also [**27**, (4.10.5)]), we give the proof of the GAGA for the completeness of this book.)

Proposition 17.6 (GAGA). — *Let $\mathcal{K}$ be a complete field with respect to a nontrivial non-archimedean absolute value. Then the following hold*:

1) *Let $f\colon X \longrightarrow Y$ be a proper morphism of schemes of locally of finite type over $\mathcal{K}$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $i\colon X_{\rm an} \longrightarrow X$ be the canonical morphism of ringed spaces defined in* [**7**, 0.3.3]. *Set $\mathcal{F}_{\rm an} := \mathcal{F} \otimes_{i^{-1}(\mathcal{O}_X)} \mathcal{O}_{X_{\rm an}}$. Then the canonical morphism*

$$(R^h f_*(\mathcal{F}))_{\rm an} \longrightarrow R^h f_{{\rm an}*}(\mathcal{F}_{\rm an}) \quad (h \in \mathbb{Z})$$

is an isomorphism.

2) *Let X be a proper scheme over $\mathcal{K}$. Then the functor*

$$\big\{\textit{coherent } \mathcal{O}_X\textit{-modules}\big\} \ni \mathcal{F} \longmapsto \mathcal{F}_{\rm an} \in \big\{\textit{coherent } \mathcal{O}_{X_{\rm an}}\textit{-modules}\big\}$$

gives the categories of equivalence.

Proof. — 1): The proof of is the obvious analogue of the proof of [**41**, XII (4.2)] and [**78**, 13].

In the case where f is projective, we may assume that X is the projective space $\mathbb{P}^r_Y$ ($r \in \mathbb{N}$). Set $\mathfrak{X} := X_{\rm an}$ and $\mathfrak{Y} := Y_{\rm an}$. We may assume that $\mathfrak{Y}$ is an affinoid space. As in [*loc. cit.*], we have to prove that $f_{{\rm an}*}(\mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_{\mathfrak{Y}}$ and $R^h f_{{\rm an}*}(\mathcal{O}_{\mathfrak{X}}) = 0$ ($h \in \mathbb{N}$). Consider the usual covering of $\mathfrak{X}$ consisting of $(r+1)$-pieces $\mathfrak{U}_0, \ldots, \mathfrak{U}_r$ of r-dimensional unit balls over $\mathfrak{Y}$ (see [**14**, (9.3.4) Example 3]). Then the covering $\{\mathfrak{U}_j\}_{j=0}^r$ is a Leray covering of $\mathfrak{X}$, that is,

$$H^h(\mathfrak{U}_{j_1} \cap \cdots \cap \mathfrak{U}_{j_s}, \mathcal{O}_{\mathfrak{X}}) = 0$$

for any $h \in \mathbb{Z}_{\geq 1}$ ($0 \leq j_1, \ldots, j_s \leq r$) by Tate's acyclicity theorem (see [**84**, (8.2), (8.7)]). Hence $H^h(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is isomorphic to the Čech cohomology $\check{H}^h(\{\mathfrak{U}_j\}_{j=0}^r, \mathcal{O}_{\mathfrak{X}})$. Since $(\mathbb{P}^r_Y)_{\rm an} := (\mathbb{P}^r_K)_{\rm an} \times_{{\rm Spm}(\mathcal{K})} Y_{\rm an}$ (see [**7**, (0.3.4)]) and since the completed tensor product is an exact functor for Banach modules over $\mathcal{K}$ (see [**27**, p. 8]), we may assume that $\mathfrak{Y} = {\rm Spm}(\mathcal{K})$. Let R be the valuation ring of $\mathcal{K}$. Let π be a nonzero element of the maximal ideal of R. Set $R_n := R/\pi^{n+1}$ ($n \in \mathbb{N}$). Then $(\mathbb{P}^r_K)_{\rm an}$ is the Raynaud generic fiber of $(\mathbb{P}^r_R)^\wedge$ (see [**7**, (0.3.5)]), where $^\wedge$ means the π-adic completion. For $\star := n$ or nothing, let $U_{0\star}, \ldots, U_{r\star}$ be the usual covering of $\mathbb{P}^r_{R_\star}$.

Then $H^h(\mathbb{P}^r_{R_\star}, \mathcal{O}_{\mathbb{P}^r_{R_\star}}) = \check{H}^h(\{(U_{j\star})\}_{j=0}^r, \mathcal{O}_{\mathbb{P}^r_{R_\star}}) = R_\star$ if $h = 0$ and $= 0$ if $h > 0$ by [**39**, (2.1.12), (2.1.13)]. Hence, by [**39**, 0, (13.2.3)],

$$\check{H}^h\big(\{(U_j)^\wedge\}_{j=0}^r, \mathcal{O}_{(\mathbb{P}^r_R)^\wedge}\big) = \varprojlim_n \check{H}^h\big(\{(U_{jn})\}_{j=0}^r, \mathcal{O}_{\mathbb{P}^r_{Rn}}\big) = R$$

if $h = 0$ and $= 0$ if $h > 0$. Because

$$H^h(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \check{H}^h\big(\{\mathfrak{U}_j\}_{j=0}^r, \mathcal{O}_{\mathfrak{X}}\big) = \check{H}^h\big(\{(U_j)^\wedge\}_{j=0}^r, \mathcal{O}_{(\mathbb{P}^r_R)^\wedge}\big) \otimes_R \mathcal{K},$$

we see that $f_{\mathrm{an}*}(\mathcal{O}_{\mathfrak{X}}) = \mathcal{K}$ and $R^h f_{\mathrm{an}*}(\mathcal{O}_{\mathfrak{X}}) = 0$ $(h \in \mathbb{N})$. (Instead of using [**39**, 0, (13.2.3)], one may use a more difficult fact "Note added in proof" in [**88**, p. 104] and [**30**, (1.2.6)] as a very special case: $H^h((\mathbb{P}^r_R)^\wedge, \mathcal{O}_{(\mathbb{P}^r_R)^\wedge}) = H^h(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r_R})$ (see [**39**, (4.1.7)] for the case where R is noetherian).)

The rest of the proof in the projective case is the same as that of [**78**, 13, Lemme 5].

In the case where f is proper, the proof is the same as that of [**41**, XII (4.2)]: by using Chow's lemma, we can reduce the proof to the projective case.

2): As in the proof of [**41**, XII (4.4)], we have only to prove 2) in the projective case by using Chow's lemma and 1). In the projective case, the proof is the same as that of [**78**, 16, Lemme 8]: we have only to use Kiehl's finiteness theorem (see [**60**]) (cf. [**88**, p. 103]) for the cohomology of a coherent sheaf on a rigid analytic projective space. □

THEOREM 17.7. — *Let U be a separated smooth scheme of finite type over κ of pure dimension d. Let Z be a smooth closed subscheme of U. Then the following composite morphism of the cup product with the trace morphism*

$$H^h_{\mathrm{rig,c}}(Z/K) \otimes_K H^{2d-h}_{\mathrm{rig},Z}(U/K) \longrightarrow H^{2d}_{\mathrm{rig,c}}(U/K) \xrightarrow{\mathrm{Tr}_U} K(-d) \tag{17.7.1}$$

induces an isomorphism

$$H^{2d-h}_{\mathrm{rig},Z}(U/K) \xrightarrow{\sim} \mathrm{Hom}_K(H^h_{\mathrm{rig,c}}(Z/K), K(-d)) \tag{17.7.2}$$

of weight-filtered vector spaces over K.

Proof. — By the same proof as that of [**8**, (2.4)], the pairing (17.7.1) is perfect. Hence we have the canonical isomorphism:

$$H^{2d-h}_{\mathrm{rig},Z}(U/K) \xrightarrow{\sim} \mathrm{Hom}_K\big(H^h_{\mathrm{rig,c}}(Z/K), K(-d)\big).$$

First we prove that this isomorphism induces the following morphism

$$P_k H^{2d-h}_{\mathrm{rig},Z}(U/K) \longrightarrow P_k \mathrm{Hom}_K\big(H^h_{\mathrm{rig,c}}(Z/K), K(-d)\big) \quad (k \in \mathbb{Z}). \tag{17.7.3}$$

Let c be the codimension of Z in U. We may assume that Z is of pure dimension d_Z. Then $d_Z = d - c$.

For the time being, we do not need to assume that Z and U are smooth over κ. Let $U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over κ. Let $\overline{Z}$ be the closure of Z in $\overline{U}$.

There exists a smooth subalgebra A_1 of κ over a finite field $\mathbb{F}_q$ such that the closed immersion $Z \hookrightarrow U$ is defined over A_1. Let $\mathcal{Z} \hookrightarrow \mathcal{U}$ be a model of $Z \hookrightarrow U$ over A_1. We may assume that the compactification $Z \hookrightarrow \overline{Z}$ has a model $\mathcal{Z} \hookrightarrow \overline{\mathcal{Z}}$ over A_1. Let $\mathrm{Spec}(A)$ be a smooth lift of $\mathrm{Spec}(A_1)$ over $\mathrm{Spec}(\mathcal{W}(\mathbb{F}_q))$. Let $\widehat{A}$ be the p-adic completion of A. Let $K_0(\mathbb{F}_q)$ be the fraction field of the Witt ring $\mathcal{W}(\mathbb{F}_q)$ of $\mathbb{F}_q$. Set

$$A_{K_0(\mathbb{F}_q)} := A \otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q) \quad \text{and} \quad \widehat{A}_{K_0(\mathbb{F}_q)} := \widehat{A} \otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q).$$

Fix a lift $\varphi\colon \widehat{A} \longrightarrow \widehat{A}$ of the Frobenius endomorphism of A_1. Then we have a natural morphism $\widehat{A} \to \mathcal{W}(A_1)$ (see [**61**, VII (4.12)]), and $\mathcal{W}$ becomes an $\widehat{A}$-algebra by the composite morphism $\widehat{A} \to \mathcal{W}(A_1) \to \mathcal{W}$. This morphism is injective as noted in the proof of (16.18).

We have the rigid cohomology $H^h_{\mathrm{rig}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ (see [**19**, (10.6)]) and, in fact, the rigid cohomology $H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ with support on $\mathcal{Z}$. We also have the rigid cohomology $H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)})$ with compact support.

Let h be a fixed nonnegative integer. Let N be a positive integer satisfying (2.2.1). Let

$$\begin{array}{ccc} (Z_{\bullet\leq N}, \overline{Z}_{\bullet\leq N}) & \longrightarrow & (U_{\bullet\leq N}, X_{\bullet\leq N}) \\ \downarrow & & \downarrow \\ (Z, \overline{Z}) & \xrightarrow{\subset} & (U, \overline{U}) \end{array} \tag{17.7.4}$$

be a commutative diagram such that the vertical morphisms are N-truncated gs proper hypercoverings. Let V be the complement of Z in U and $\overline{V}$ the closure of V in $\overline{U}$. Let

$$\begin{array}{ccc} (V_{\bullet\leq N}, Y_{\bullet\leq N}) & \longrightarrow & (U_{\bullet\leq N}, X_{\bullet\leq N}) \\ \downarrow & & \downarrow \\ (V, \overline{V}) & \xrightarrow{\subset} & (U, \overline{U}) \end{array} \tag{17.7.5}$$

be a commutative diagram such that the left vertical morphism is an N-truncated gs proper hypercovering. Then we may assume that the diagrams (17.7.4) and (17.7.5) are defined over A_1. Here we can take the same model of $(U_{\bullet\leq N}, X_{\bullet\leq N})$ in (17.7.4) and (17.7.5) over A_1. Set $D_{\bullet\leq N} := X_{\bullet\leq N} \setminus U_{\bullet\leq N}$ and $E_{\bullet\leq N} := Y_{\bullet\leq N} \setminus V_{\bullet\leq N}$. Let us mean the models of (N-truncated) schemes in (17.7.4) and (17.7.5) over A_1 by calligraphic letters. Then, by using the models of schemes, (17.5), Tsuzuki's proper descent (see [**86**, (2.1.3)]) and the argument in the proof of (11.6), we have the following canonical isomorphism

$$\begin{aligned}H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) &\xrightarrow{\sim} H^h\big(\mathrm{MF}(R\Gamma_{\mathrm{rig}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) \to R\Gamma_{\mathrm{rig}}((\mathcal{U}\setminus\mathcal{Z})/\widehat{A}_{K_0(\mathbb{F}_q)}))\big)\\ &\xrightarrow{\sim} H^h\big(\mathrm{MF}(R\Gamma((\mathcal{X}_{\bullet\leq N},\mathcal{D}_{\bullet\leq N})/\widehat{A})\otimes^L_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q)\\ &\qquad \to R\Gamma((\mathcal{Y}_{\bullet\leq N},\mathcal{E}_{\bullet\leq N})/\widehat{A})\otimes^L_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q))\big).\end{aligned}$$

Hence we have the following spectral sequence by (14.2.1):

$$\begin{aligned}E_1^{-k,h+k} = &\bigoplus_{t\geq 0} H^{h-2t-k}\big((\widetilde{\mathcal{D}_t^{(t+k)}}/\widehat{A})_{\mathrm{crys}}, \mathcal{O}_{\mathcal{D}_t^{(t+k)}/\widehat{A}}\\ &\qquad\otimes_{\mathbb{Z}} \varpi^{(t+k)}_{\mathrm{crys}}(\mathcal{D}_t/\widehat{A})\big)\big(-(t+k)\big)_{K_0(\mathbb{F}_q)}\\ &\oplus\bigoplus_{t\geq 0} H^{h-2t-k-2}\big((\widetilde{\mathcal{E}_t^{(t+k+1)}}/\widehat{A})_{\mathrm{crys}}, \mathcal{O}_{\mathcal{E}_t^{(t+k+1)}/\widehat{A}}\oplus\\ &\qquad\otimes_{\mathbb{Z}} \varpi^{(t+k+1)}_{\mathrm{crys}}(\mathcal{E}_t/\widehat{A})\big)\big(-(t+k+1)\big)_{K_0(\mathbb{F}_q)}\\ &\qquad\qquad\Longrightarrow H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}).\end{aligned} \tag{17.7.6}$$

Consequently, we have a weight filtration P on $H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ and the cohomology $H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ prolongs to a convergent F-isocrystal on $\mathrm{Spf}(\widehat{A})/\mathcal{W}(\mathbb{F}_q)$ by [**74**, (2.10), (3.7)]. If we take a small spectrum $\mathrm{Spec}(A_1)$, then we see that the base change morphism

$$P_kH^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})\otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K_0 \longrightarrow P_kH^h_{\mathrm{rig},Z}(U/K_0) \tag{17.7.7}$$

is an isomorphism by using the weight spectral sequences of $H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ ((17.7.6)) and $H^h_{\mathrm{rig},Z}(U/K_0)$ ((16.1.1)) and by Deligne's remark (see [**46**, (3.10)], (cf. [**69**, §3], [**72**, (2.14), (2.15)])). In particular, the base change morphism

$$H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})\otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K_0 \longrightarrow H^h_{\mathrm{rig},Z}(U/K_0)$$

is an isomorphism. By (16.1), 1) the base change morphism

$$H^h_{\mathrm{rig},Z}(U/K_0)_K \longrightarrow H^h_{\mathrm{rig},Z}(U/K)$$

is also an isomorphism. Hence the base change morphism

$$H^h_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})\otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K \longrightarrow H^h_{\mathrm{rig},Z}(U/K) \tag{17.7.8}$$

is an isomorphism.

By the exact sequence

$$(17.7.9)\qquad \cdots \longrightarrow H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) \longrightarrow H^h_{\mathrm{rig}}(\overline{\mathcal{Z}}/\widehat{A}_{K_0(\mathbb{F}_q)}) \longrightarrow H^h_{\mathrm{rig}}\big((\overline{\mathcal{Z}}\setminus\mathcal{Z})/\widehat{A}_{K_0(\mathbb{F}_q)}\big) \longrightarrow \cdots$$

and by the base change of rigid cohomologies in (17.7.8) for special cases $\mathcal{U}=\overline{\mathcal{Z}}$ and $\mathcal{U}=\overline{\mathcal{Z}}\setminus\mathcal{Z}$, the base change morphism

$$(17.7.10)\qquad H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)})\otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K \longrightarrow H^h_{\mathrm{rig,c}}(Z/K)$$

is an isomorphism. By (17.7.9) and by [**74**, (2.10)], we see that $H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)})$ prolongs to a convergent F-isocrystal on $\mathrm{Spf}(\widehat{A})/\mathcal{W}(\mathbb{F}_q)$.

From now on, we use the smoothness of Z, U, $\mathcal{Z}$ and $\mathcal{U}$. Let x be a closed point of $\mathrm{Spec}(A_1)$. Let $\mathcal{Z}(x)$ and $\mathcal{U}(x)$ be the fibers of $\mathcal{Z}$ and $\mathcal{U}$ at x, respectively. Set

$$\kappa(x) := A_1/\mathfrak{m}.$$

Let $\mathcal{W}(x)$ be the Witt ring of $\kappa(x)$ and $K_0(x)$ the fraction field of $\mathcal{W}(x)$. Because $H^h_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ is a convergent F-isocrystal on $\mathrm{Spf}(\widehat{A})/\mathcal{W}(\mathbb{F}_q)$ and because $H^h_{\mathrm{rig,c}}(\mathcal{U}(x)/K_0(\kappa(x)))=0$ for $h>2d$ and for any closed point x of $\mathrm{Spec}(A_1)$ (see [**8**, (1.1)]), we see that

$$H^h_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})=0$$

for $h>2d$ as in the proof of (16.18). Hence there exists a smooth affine subscheme $\mathcal{U}'$ of $\mathcal{U}$ such that $H^{2d}_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) = H^{2d}_{\mathrm{rig,c}}(\mathcal{U}'/\widehat{A}_{K_0(\mathbb{F}_q)})$ by the following exact sequence for an open subscheme $\mathcal{U}''$ of $\mathcal{U}$:

$$\cdots \longrightarrow H^h_{\mathrm{rig,c}}(\mathcal{U}''/\widehat{A}_{K_0(\mathbb{F}_q)}) \longrightarrow H^h_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) \longrightarrow H^h_{\mathrm{rig,c}}\big((\mathcal{U}\setminus\mathcal{U}'')/\widehat{A}_{K_0(\mathbb{F}_q)}\big) \longrightarrow \cdots.$$

Now we follow the argument in [**8**, (1.2)]. Let $\widetilde{\mathcal{U}}'$ be a smooth lift of $\mathcal{U}'$ over $\mathrm{Spec}(A)$ and let $\overline{\widetilde{\mathcal{U}}'}$ be a compactification of $\widetilde{\mathcal{U}}'$ over $\mathrm{Spec}(A)$. Let

$$\widetilde{\mathcal{U}}'_{K_0(\mathbb{F}_q)} := \widetilde{\mathcal{U}}'\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q) \quad\text{and}\quad \overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)} := \overline{\widetilde{\mathcal{U}}'}\otimes_{\mathcal{W}(\mathbb{F}_q)} K_0(\mathbb{F}_q).$$

Let ω be the dualizing sheaf of $\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)}$ over $\mathrm{Spec}(A_{K_0(\mathbb{F}_q)})$. Because ω has coherent cohomologies (see [**42**, VII (3.4) (a)] and [**20**, (3.3.1)]), the natural morphism $R\Gamma(\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)},\omega)_{\mathrm{an}} \to R\Gamma((\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)})_{\mathrm{an}},\omega_{\mathrm{an}})$ is an isomorphism by GAGA ((17.6), 1)).

As in [**8**, (1.2)], by using the trace morphism in [**42**, VII (3.4)] and [**20**, (3.6.13)], we have the following composite morphism

$$(17.7.11)\quad \begin{aligned} &H^d_{]\mathcal{U}'[}\big((\widetilde{\mathcal{U}}'_{K_0(\mathbb{F}_q)})_{\mathrm{an}}, \Omega^d_{(\widetilde{\mathcal{U}}'_{K_0(\mathbb{F}_q)})_{\mathrm{an}}/(A_{K_0(\mathbb{F}_q)})_{\mathrm{an}}}\big) \\ &\quad \overset{\sim}{\longrightarrow} H^0_{]\mathcal{U}'[}\big((\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)})_{\mathrm{an}}, \omega_{\mathrm{an}}\big) \\ &\qquad \longrightarrow H^0\big((\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)})_{\mathrm{an}}, \omega_{\mathrm{an}}\big) \overset{\sim}{\longleftarrow} H^0(\overline{\widetilde{\mathcal{U}}'}_{K_0(\mathbb{F}_q)}, \omega)_{\mathrm{an}} \\ &\qquad\quad \xrightarrow{\mathrm{Tr}_{\mathrm{an}}} (A_{K_0(\mathbb{F}_q)})_{\mathrm{an}}. \end{aligned}$$

Let $\widehat{\widetilde{\mathcal{U}}}'$ be the p-adic completion of $\widetilde{\mathcal{U}}'$. Restricting the morphism (17.7.11) to the open analytic space $\mathrm{Spf}(\widehat{A})_{K_0(\mathbb{F}_q)}$ of $\mathrm{Spec}(A_{K_0(\mathbb{F}_q)})_{\mathrm{an}}$ and noting that

$$\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)} = (\widetilde{\mathcal{U}}'_{K_0(\mathbb{F}_q)})_{\mathrm{an}} \times_{\mathrm{Spec}(A_{K_0(\mathbb{F}_q)})_{\mathrm{an}}} \mathrm{Spf}(\widehat{A})_{K_0(\mathbb{F}_q)}$$

(see [**7**, (0.3.5)]), we have the morphism

$$(17.7.12)\qquad \mathrm{Tr}_{\mathcal{U}}\colon H^d_{]\mathcal{U}'[}(\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}, \Omega^d_{\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}/\widehat{A}_{K_0(\mathbb{F}_q)}}) \longrightarrow \widehat{A}_{K_0(\mathbb{F}_q)}.$$

Set $\mathcal{U}'_K := \widetilde{\mathcal{U}}' \otimes_A K$ and $U' := \mathcal{U}' \otimes_A \kappa$. Because the composite morphism

$$H^d_{]U'[}\big((\mathcal{U}'_K)_{\mathrm{an}}, \Omega^{d-1}_{(\mathcal{U}'_K)_{\mathrm{an}}/K}\big) \longrightarrow H^d_{]U'[}\big((\mathcal{U}'_K)_{\mathrm{an}}, \Omega^{d}_{(\mathcal{U}'_K)_{\mathrm{an}}/K}\big) \longrightarrow K$$

is zero (see [**8**, (1.2)]) and because we have the commutative diagram

(17.7.13)

$$\begin{array}{ccccc} H^d_{]\mathcal{U}'[}(\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}, \Omega^{d-1}_{\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}}) & \longrightarrow & H^d_{]\mathcal{U}'[}(\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}, \Omega^{d}_{\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}}) & \longrightarrow & \widehat{A}_{K_0(\mathbb{F}_q)} \\ \downarrow & & \downarrow & & \downarrow \cap \\ H^d_{]U'[}((\mathcal{U}'_K)_{\mathrm{an}}, \Omega^{d-1}_{(\mathcal{U}'_K)_{\mathrm{an}}/K}) & \longrightarrow & H^d_{]U'[}((\mathcal{U}'_K)_{\mathrm{an}}, \Omega^{d}_{(\mathcal{U}'_K)_{\mathrm{an}}/K}) & \longrightarrow & K, \end{array}$$

the upper horizontal composite morphism is zero. Hence we have the following trace morphism

$$\begin{aligned} H^{2d}_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) &= H^{2d}_{\mathrm{rig,c}}(\mathcal{U}'/\widehat{A}_{K_0(\mathbb{F}_q)}) \\ &= H^{2d}_{]\mathcal{U}'[}(\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}, \Omega^{\bullet}_{\widehat{\widetilde{\mathcal{U}}}'_{K_0(\mathbb{F}_q)}}) \xrightarrow{\mathrm{Tr}_{\mathcal{U}}} \widehat{A}_{K_0(\mathbb{F}_q)}. \end{aligned}$$

By the Grothendieck base change of the trace morphism (see [**20**, Theorem 3.6.5]), the morphism

$$H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} H^{2d-h}_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) \xrightarrow{\mathrm{Tr}_{\mathcal{U}}(?\cup?)} \widehat{A}_{K_0(\mathbb{F}_q)}$$

is a morphism of convergent F-isocrystals. As in the proof of [**74**, (3.12)], we have

$$H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) = \mathrm{Hom}_{\widehat{A}_{K_0(\mathbb{F}_q)}}\big(H^{2d-h}_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}), \widehat{A}_{K_0(\mathbb{F}_q)}(-d)\big) \tag{17.7.14}$$

by Berthelot's Poincaré duality in rigid cohomology (see [**8**, (2.4)]).

By the Grothendieck base change of the trace morphism again, we have the following commutative diagram

$$\begin{array}{ccc} H^{2d}_{\mathrm{rig,c}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}} K_0 & \xrightarrow{\mathrm{Tr}_{\mathcal{U}}} & \widehat{A}_{K_0(\mathbb{F}_q)}(-d) \otimes_{\widehat{A}} K_0 \\ \simeq\big\downarrow & & \big\| \\ H^{2d}_{\mathrm{rig,c}}(U/K_0) & \xrightarrow{\mathrm{Tr}_U} & K_0(-d). \end{array} \tag{17.7.15}$$

Hence we have the following commutative diagram

(17.7.16)

$$\begin{array}{ccc} (H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} H^{2d-h}_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})) \otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K_0 & \longrightarrow & K_0(-d) \\ \simeq\big\downarrow & & \big\| \\ H^h_{\mathrm{rig,c}}(Z/K_0) \otimes_{K_0} H^{2d-h}_{\mathrm{rig},Z}(U/K_0) & \longrightarrow & K_0(-d). \end{array}$$

By the formula (17.7.14) for the special case $\mathcal{U} = \mathcal{Z}$, we have a weight filtration on $H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)})$. Because the morphism (17.7.7) is an isomorphism, we see that the base change morphism

$$P_\ell H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}} K_0 \longrightarrow P_\ell H^h_{\mathrm{rig,c}}(Z/K_0)$$

is an isomorphism by (17.7.15) for $\mathcal{U} = \mathcal{Z}$ and $U = Z$.

Now, to prove the existence of the morphism (17.7.3), it suffices to prove that, if $k + \ell < 2d$, then $\mathrm{Tr}_{\mathcal{U}}(a \cup b) = 0$ for $a \in P_k H^{2d-h}_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)})$ and $b \in P_\ell H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)})$.

Let $\mathcal{M}(A_1)$ be the set of the maximal ideals of A_1. Then, by [**72**, (2.15.3)], the natural morphism $\widehat{A} \to \Pi_{\mathfrak{m}\in\mathcal{M}(A_1)}\mathcal{W}(A_1/\mathfrak{m})$ is injective. Hence it suffices to prove that the image of $\mathrm{Tr}_{\mathcal{U}}(a \cup b)$ in $\mathcal{W}(A_1/\mathfrak{m}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is zero. Let $x \in \mathrm{Spec}(A_1)$ be the closed point corresponding to $\mathfrak{m}$. We may assume that the base change morphisms

$$P_k H^{2d-h}_{\mathrm{rig},\mathcal{Z}}(\mathcal{U}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K_0(x) \longrightarrow P_k H^{2d-h}_{\mathrm{rig},\mathcal{Z}(x)}\big(\mathcal{U}(x)/K_0(x)\big),$$

$$P_\ell H^h_{\mathrm{rig,c}}(\mathcal{Z}/\widehat{A}_{K_0(\mathbb{F}_q)}) \otimes_{\widehat{A}_{K_0(\mathbb{F}_q)}} K_0(x) \longrightarrow P_\ell H^h_{\mathrm{rig,c}}\big(\mathcal{Z}(x)/K_0(x)\big)$$

are isomorphisms. Because the base change isomorphisms above are compatible with the Frobenius, we see that $\mathrm{Tr}_{\mathcal{U}}(a \cup b) = 0$ in $K_0(x)$ by the yoga of weight. Hence we obtain the morphism (17.7.3).

If κ is a finite field, then (17.7.3) is an isomorphism since (17.7.2) is compatible with the Frobenius.

Now we consider the general case. We have only to prove that the dimensions of both sides of (17.7.3) are equal since the morphism (17.7.3) is injective.

Since $H^h_{\mathrm{rig,c}}(Z/K) = \mathrm{Hom}_K(H^{2d_Z-h}_{\mathrm{rig}}(Z/K), K(-d_Z))$, we have only to prove that

$$\dim_K P_k H^{2d-h}_{\mathrm{rig},Z}(U/K) = \dim_K P_{k-2c} H^{2d_Z-h}_{\mathrm{rig}}(Z/K). \tag{17.7.17}$$

For any closed point $x \in \mathrm{Spec}(A_1)$ and for all $h \in \mathbb{Z}$, we may assume that

$$\begin{aligned}\dim_K P_k H^h_{\mathrm{rig},Z}(U/K) \\ = \dim_{K_0(\kappa(x))} P_k H^h_{\mathrm{rig},\mathcal{Z}(x)}\big(\mathcal{U}(x)/K_0(\kappa(x))\big),\end{aligned} \tag{17.7.18}$$

$$\begin{aligned}\dim_K P_\ell H^h_{\mathrm{rig}}(Z/K) \\ = \dim_{K_0(\kappa(x))} P_\ell H^h_{\mathrm{rig}}\big(\mathcal{Z}(x)/K_0(\kappa(x))\big) \quad (\ell \in \mathbb{Z})\end{aligned} \tag{17.7.19}$$

as in (12.4). Since $\kappa(x)$ is a finite field, we have

$$\begin{aligned}\dim_{K_0(\kappa(x))} P_k H^{2d-h}_{\mathrm{rig},\mathcal{Z}(x)}\big(\mathcal{U}(x)/K_0(\kappa(x))\big) \\ = \dim_{K_0(\kappa(x))} P_{k-2c} H^{2d_Z-h}_{\mathrm{rig}}\big(\mathcal{Z}(x)/K_0(\kappa(x))\big).\end{aligned} \tag{17.7.20}$$

Therefore the morphism (17.7.3) is an isomorphism. We finish the proof. □

Corollary 17.8. — *Assume that Z is of pure codimension c. Then the Gysin isomorphism*

$$G_{Z/U}\colon H^{h-2c}_{\mathrm{rig}}(Z/K)(-c) \xrightarrow{\sim} H^h_{\mathrm{rig},Z}(U/K) \tag{17.8.1}$$

is an isomorphism of weight-filtered K-vector spaces.

Proof. — Because

$$H^{h-2c}_{\mathrm{rig}}(Z/K)(-c) = \mathrm{Hom}_K\big(H^{2d-h}_{\mathrm{rig,c}}(Z/K), K(-d)\big)$$

and

$$H^h_{\mathrm{rig},Z}(U/K) = \mathrm{Hom}_K\big(H^{2d-h}_{\mathrm{rig,c}}(Z/K), K(-d)\big),$$

(17.8) is clear. □

COROLLARY 17.9. — 1) *The function* $h_p^{i,k-i}\colon \mathbf{CS}(\kappa) \to \mathbb{Z}$ *in* (16.22.2) *satisfies the following equality*

$$h_{p,Z}^{i,k-i}(U) = h_p^{i-c,k-i-c}(Z) \tag{17.9.1}$$

if U *and* Z *are smooth over* κ *and if* Z *is of pure codimension* c ($c \in \mathbb{N}$) *in* U*. Consequently the function* $h_p^k\colon \mathbf{CS}(\kappa) \to \mathbb{Z}$ *in* (16.22.1) *satisfies the equality*

$$h_{p,Z}^{k}(U) = h_p^{k-2c}(Z) \tag{17.9.2}$$

under the same assumption.

2) *Let* $\mathbf{SCS}(\kappa)$ *be the set of isomorphism classes of separated smooth schemes of finite type over* κ *with closed subschemes over* κ*. If the embedded resolution of singularities holds for any variety with any closed subscheme over* κ*, then* (16.22.4)*, the first formula in* (16.22.5), (16.22.6) *and* (17.9.1) *characterize the function* $h_p^{i,k-i}|_{\mathbf{SCS}(\kappa)}$*. An obvious analogous characterization for the function* $h_p^k|_{\mathbf{SCS}(\kappa)}$ *holds.*

Proof. — 1): This immediately follows from (17.8.1).

2): We prove the assertion only for $h_p^{i,k-i}|_{\mathbf{SCS}(\kappa)}$. Because $h_{p,Z}^{i,k-i}(U) = h_p^{i,k-i}(U) - h_p^{i,k-i}(U \setminus Z)$ by (16.22.3), we have only to characterize $h_p^{i,k-i}(U)$ for a smooth scheme U over κ. Embed U into a proper scheme $\overline{U}$ such that $\dim(\overline{U} \setminus U) < \dim \overline{U}$. Because we assume that the embedded resolution of singularities holds for any variety over κ, we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\subset} & X \\ \| & & \downarrow \\ U & \xrightarrow{\subset} & \overline{U}, \end{array}$$

where X is a proper smooth scheme over κ and the complement $D := X \setminus U$ is an SNCD on X. Then $h_p^{i,k-i}(U) = h_p^{i,k-i}(X) - h_{p,D}^{i,k-i}(X)$, and because $h_p^{i,k-i}(X)$ is characterized by the first formula in (16.22.5), we have only to characterize $h_{p,D}^{i,k-i}(X)$. If D is smooth over κ, then

$$h_{p,D}^{i,k-i}(X) = h_p^{i-1,k-i-1}(D)$$

by (17.9.1). By the first formula in (16.22.5), $h_p^{i-1,k-i-1}(D)$ is characterized. Hence $h_{p,D}^{i,k-i}(X)$ is characterized. Consider the general SNCD D again. Let D_1 be a smooth irreducible component of D. Set $E' := D \setminus D_1$ and $U' := X \setminus D_1$. Then

$$h_{p,D}^{i,k-i}(X) = h_{p,D_1}^{i,k-i}(X) + h_{p,E'}^{i,k-i}(U')$$

by (16.22.4); we have only to characterize $h^{i,k-i}_{p,E'}(U')$. Let D' be the closure of E' in X. Set $Z' := D' \setminus E'$. Then

$$h^{i,k-i}_{p,D'}(X) = h^{i,k-i}_{p,Z'}(X) + h^{i,k-i}_{p,E'}(X \setminus Z') = h^{i,k-i}_{p,Z'}(X) + h^{i,k-i}_{p,E'}(U')$$

by (16.22.4) and (16.22.6). By the induction on the number of the irreducible components of D' and Z' (or by the induction on the dimension $\dim Z'$), $h^{i,k-i}_{p,D'}(X)$ and $h^{i,k-i}_{p,Z'}(X)$ are characterized. Therefore $h^{i,k-i}_{p,E'}(U')$ is characterized. □

Corollary 17.10 (see cf. [**21**, Exemple 3.3]). — *Let X be a proper smooth scheme over κ and let Y be a smooth closed subscheme of X. Then $\mathrm{gr}^P_j H^h_{\mathrm{rig}}((X \setminus Y)/K) = 0$ for $j \neq h,\ h+1$ and $P_h H^h_{\mathrm{rig}}((X \setminus Y)/K) = \mathrm{Im}(H^h_{\mathrm{rig}}(X/K) \to H^h_{\mathrm{rig}}((X \setminus Y)/K))$.*

Proof. — (17.10) immediately follows from the strict exactness of the Gysin exact sequence (16.6.1), (17.8) and the purity. □

Let Z be a separated scheme of finite type over κ. Assume that Z is a closed subscheme of a separated smooth scheme U of finite type over κ. Assume that U is of pure dimension d (we may assume this for the definition of the weight filtration on the rigid cohomology with compact support below). By the proof of [**8**, (2.4)], we have an isomorphism

(17.10.1) $$H^h_{\mathrm{rig,c}}(Z/K) = \mathrm{Hom}_K\big(H^{2d-h}_{\mathrm{rig},Z}(U/K), K\big).$$

Because $H^{2d-h}_{\mathrm{rig},Z}(U/K)$ has the weight filtration, $\mathrm{Hom}_K(H^{2d-h}_{\mathrm{rig},Z}(U/K), K(-d))$ has the weight filtration. Thus, under the following identification

(17.10.2) $$H^h_{\mathrm{rig,c}}(Z/K) = \mathrm{Hom}_K\big(H^{2d-h}_{\mathrm{rig},Z}(U/K), K(-d)\big),$$

we have a weight filtration P on $H^h_{\mathrm{rig,c}}(Z/K)$.

Theorem 17.11. — *The weight filtration P on $H^h_{\mathrm{rig,c}}(Z/K)$ by the formula (17.10.2) is independent of the choice of U.*

Proof. — Let $Z \hookrightarrow U'$ be another closed immersion into a separated smooth scheme of finite type over κ of pure dimension d'. Then, by using the diagonal closed immersion $Z \hookrightarrow U \times_\kappa U'$, we may assume that we have a morphism $u\colon U \to U'$ such that the composite morphism $Z \hookrightarrow U \to U'$ is also the given closed immersion. By the same proof as that of [**8**, (2.4)], we see that the natural morphisms

$$H^{2d-h}_{\mathrm{rig},Z}(U/K)(d) \longrightarrow \mathrm{Hom}_K\big(H^h_{\mathrm{rig,c}}(Z/K), K\big),$$
$$H^{2d'-h}_{\mathrm{rig},Z}(U'/K)(d') \longrightarrow \mathrm{Hom}_K\big(H^h_{\mathrm{rig,c}}(Z/K), K\big)$$

are isomorphisms. Because the cohomology $H^h_{\mathrm{rig,c}}(Z/K)$ is contravariant for the proper morphism id_Z, we have a natural isomorphism

$$(17.11.1) \qquad H^{2d-h}_{\mathrm{rig},Z}(U/K)(d) \xrightarrow{\sim} H^{2d'-h}_{\mathrm{rig},Z}(U'/K)(d').$$

It suffices to prove that the isomorphism (17.11.1) is an isomorphism of weight-filtered K-vector spaces.

First we consider the case where Z is smooth over κ of pure dimension. Let d_Z be the dimension of Z, and let c and c' be the codimensions of Z in U and U', respectively. By the obvious commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\subset} & U \\ \| & & \uparrow \cup \\ Z & = & Z \\ \| & & \downarrow \cap \\ Z & \xrightarrow{\subset} & U' \end{array}$$

and the contravariance of the rigid cohomology $H^h_{\mathrm{rig,c}}(Z/K)$ with respect to id_Z, we see that the morphism (17.11.1) is the following composite morphism

$$(17.11.2) \qquad H^{2d-h}_{\mathrm{rig},Z}(U/K)(d) \xrightarrow[\sim]{G^{-1}_{Z/U}} H^{2d_Z-h}_{\mathrm{rig}}(Z/K)(d_Z) \xrightarrow[\sim]{G_{Z/U'}} H^{2d'-h}_{\mathrm{rig},Z}(U'/K)(d').$$

By (17.8), the morphism (17.11.2) is an isomorphism of weight-filtered K-vector spaces.

By induction on $\dim Z$ for a separated scheme Z of finite type over κ, we prove that (17.11.1) is an isomorphism of weight-filtered K-vector spaces.

We may assume that Z is reduced and then that there exists a non-empty dense open smooth subscheme Z^o of Z.

If $\dim Z = 0$, then Z is smooth over κ. Hence (17.11.1) is an isomorphism of weight-filtered K-vector spaces by (17.8).

Set $T := Z \setminus Z^o$, $U^o := U \setminus T$ and $U'^o := U' \setminus T$. Then, by the contravariance of the rigid cohomology with compact support with respect to a proper morphism, we have the following commutative diagram

$$\begin{array}{ccccc} \cdots \longrightarrow & H^{2d-h}_{\mathrm{rig},T}(U/K)(d) & \longrightarrow & H^{2d-h}_{\mathrm{rig},Z}(U/K)(d) \\ & \simeq\downarrow & & \simeq\downarrow \\ \cdots \longrightarrow & H^{2d'-h}_{\mathrm{rig},T}(U'/K)(d') & \longrightarrow & H^{2d'-h}_{\mathrm{rig},Z}(U'/K)(d') \end{array}$$

$$\begin{array}{ccc} \longrightarrow & H^{2d-h}_{\mathrm{rig},Z^o}(U^o/K)(d) & \longrightarrow \cdots \\ & \simeq\downarrow & \\ \longrightarrow & H^{2d'-h}_{\mathrm{rig},Z^o}(U'^o/K)(d') & \longrightarrow \cdots \end{array}$$

whose horizontal lines are strictly exact with respect to the weight filtration ((16.6), 1)). Since $\dim T < \dim Z$, the morphism

$$H^{2d-h}_{\mathrm{rig},T}(U/K)(d) \xrightarrow{\sim} H^{2d'-h}_{\mathrm{rig},T}(U'/K)(d')$$

is an isomorphism of weight-filtered K-vector spaces. Hence, by the five lemma, we see that (17.11.1) is an isomorphism of weight-filtered K-vector spaces. □

Definition 17.12. — We call the well-defined weight filtration on $H^h_{\mathrm{rig,c}}(Z/K)$ given by the formula (17.10.2) the *weight filtration* on $H^h_{\mathrm{rig,c}}(Z/K)$.

Proposition 17.13. — *Assume that Z is a closed subscheme of a separated smooth scheme of finite type over κ. Set $V := U \setminus Z$. Then the sequence*

(17.13.1) $$\cdots \longrightarrow H^h_{\mathrm{rig,c}}(V/K) \longrightarrow H^h_{\mathrm{rig,c}}(U/K) \longrightarrow H^h_{\mathrm{rig,c}}(Z/K) \longrightarrow \cdots$$

(see [**6**, (3.1) (iii)]) *is strictly exact with respect to the weight filtration.*

Proof. — By the duality (17.10.2), the exact sequence (17.13.1) is equal to

(17.13.2) $$\cdots \longrightarrow H^{2d-h}_{\mathrm{rig}}(V/K)^*(-d) \longrightarrow H^{2d-h}_{\mathrm{rig}}(U/K)^*(-d) \longrightarrow H^{2d-h}_{\mathrm{rig},Z}(U/K)^*(-d) \longrightarrow \cdots .$$

The exact sequence (17.13.2) is strictly exact with respect to the weight filtration by (16.6), 1). □

The following is a p-adic version of a very special case of [**26**, III (3.3.4)]; see also [**33**, Partie II, III 3].

Proposition 17.14. — *Let Z be a separated scheme of finite type over κ of dimension d_Z. Assume that Z is a closed subscheme of a separated smooth scheme of finite type over κ. Let $h \in [0, 2d_Z]$ be an integer. Then the weights of $H^h_{\mathrm{rig,c}}(Z/K)$ lie in $[0, h]$.*

Proof. — The upper bound immediately follows from (16.7), 1). If Z is smooth over κ, the lower bound follows from the duality

$$H^h_{\mathrm{rig,c}}(Z/K) \xrightarrow{\sim} \mathrm{Hom}_K(H^{2d_Z-h}_{\mathrm{rig}}(Z/K), K(-d_Z)).$$

and from (16.7), 2). The lower bound in the general case follows from the induction on the dimension d_Z and from the exact sequence (17.13.1). □

In [**8**, (3.2) (ii)] we need not assume that X_1 and X_2 are smooth, which is a special case of Kedlaya's Künneth formula (see [**55**, Theorem 1.2.4]):

PROPOSITION 17.15. — *Let Z_i $(i = 1, 2)$ be a separated scheme of finite type over κ. Assume that Z_i is a closed subscheme of a separated smooth scheme U_i over κ. Then the following hold*:

1) *The Künneth morphism*

$$R\Gamma_{\mathrm{rig,c}}(Z_1/K) \otimes_K R\Gamma_{\mathrm{rig,c}}(Z_2/K) \longrightarrow R\Gamma_{\mathrm{rig,c}}(Z_1 \times_\kappa Z_2/K) \tag{17.15.1}$$

is an isomorphism.

2) *The induced isomorphism by* (17.15.1)

$$\bigoplus_{h_1+h_2=h} H^{h_1}_{\mathrm{rig,c}}(Z_1/K) \otimes_K H^{h_2}_{\mathrm{rig,c}}(Z_2/K) \xrightarrow{\sim} H^h_{\mathrm{rig,c}}(Z_1 \times_\kappa Z_2/K) \tag{17.15.2}$$

is compatible with the weight filtration.

Proof. — 1): Set $Z_{12} := Z_1 \times_\kappa Z_2$ and $U_{12} := U_1 \times_\kappa U_2$. We have only to prove that the morphism

$$\bigoplus_{h_1+h_2=h} \big(H^{h_1}_{\mathrm{rig,c}}(Z_1/K) \otimes_K H^{h_2}_{\mathrm{rig,c}}(Z_2/K)\big) \longrightarrow H^h_{\mathrm{rig,c}}(Z_1 \times_\kappa Z_2/K) \tag{17.15.3}$$

is an isomorphism. We may assume that U_i $(i = 1, 2)$ is of pure dimension d_i. By the construction of the trace morphism in rigid cohomology (see [**8**, (1.2)]) and the commutative diagram in [**5**, VII Proposition 2.4.1], we have the following commutative diagram

$$\begin{array}{ccc} H^{2d_1}_{\mathrm{rig,c}}(U_1/K) \otimes_K H^{2d_2}_{\mathrm{rig,c}}(U_2/K) & \xrightarrow{\sim} & H^{2(d_1+d_2)}_{\mathrm{rig,c}}(U_{12}/K) \\ \downarrow{\scriptstyle \mathrm{Tr}_{U_1}\otimes \mathrm{Tr}_{U_2}} & & \downarrow{\scriptstyle \mathrm{Tr}_{U_{12}}} \\ K(-d_1) \otimes_K K(-d_2) & = & K(-d_1-d_2). \end{array} \tag{17.15.4}$$

Hence the following diagram

(17.15.5)

$$\begin{array}{ccc}
\bigoplus_{h_1+h_2=h} H^{h_1}_{\mathrm{rig,c}}(Z_1/K)\otimes_K H^{h_2}_{\mathrm{rig,c}}(Z_2/K) & \longrightarrow & H^h_{\mathrm{rig,c}}(Z_{12}/K) \\
\| & & \| \\
\bigoplus_{h_1+h_2=h} H^{2d_1-h_1}_{\mathrm{rig},Z_1}(U_1/K)^*(-d_1)\otimes_K H^{2d_2-h_2}_{\mathrm{rig},Z_2}(U_2/K)^*(-d_2) & \longleftarrow & H^{2(d_1+d_2)-h}_{\mathrm{rig},Z_{12}}(U_{12}/K)^*(-d_1-d_2).
\end{array}$$

is commutative. The lower horizontal morphism in (17.15.5) is an isomorphism by (16.15), 1). Hence the upper horizontal morphism in (17.15.5) is an isomorphism.

2): This follows from the definition of the weight filtration on the rigid cohomology with compact support, the commutative diagram (17.15.5) and (16.15), 2). □

As in the rigid cohomology with closed support in §16, we obtain the Serre-Grothendieck formula of the virtual Betti numbers of a separated scheme of finite type over κ which is embeddable into a separated smooth scheme over κ as follows (see [**33**, Partie II, A, II (46_9)]).

Let Z be a separated scheme of finite type over κ. Then $H^h_{\mathrm{rig,c}}(Z/K_0)$ has the slope filtration F. Assume that Z is a closed subscheme of a separated smooth scheme of finite type over κ. Then $H^h_{\mathrm{rig,c}}(Z/K_0)$ has the weight filtration P (see (17.11)). As in §16, for an element $[H,P,F]$ of $\mathrm{GF}_2(K_0)$, we omit to write P and F. Consider

$$\big[H_{\mathrm{rig,c}}(Z/K_0)\big] := \sum_{h\in\mathbb{N}}(-1)^h\big[H^h_{\mathrm{rig,c}}(Z/K_0)\big].$$

Definition 17.16. — We call the integers

$$h^k_p\big([H_{\mathrm{rig,c}}(Z/K_0)]\big),\quad f^i_p\big([H_{\mathrm{rig,c}}(Z/K_0)]\big),\quad h^{i,k-i}_p\big([H_{\mathrm{rig,c}}(Z/K_0)]\big)$$

the *virtual Betti number*, the *virtual slope number* and the *virtual slope-Betti number* of Z/κ for the rigid cohomology with compact support. We denote them by $h^k_{p,\mathrm{c}}(Z)$, $f^i_{p,\mathrm{c}}(Z)$ and $h^{i,k-i}_{p,\mathrm{c}}(Z)$, respectively.

For a variety V over κ, we say that the resolution of singularities in the strong sense holds for V if, for any open smooth subscheme V^o of V, there exists a proper morphism $g\colon \widetilde{V} \longrightarrow V$ such that $\widetilde{V}$ is smooth over κ and such that g induces an isomorphism $g^{-1}(V^o) \xrightarrow{\sim} V^o$.

THEOREM 17.17 (cf. [**33**, Partie II, A, II (**46**$_{\mathbf{9}}$)]). — *Let* $\mathbf{ES}(\kappa)$ *be the set of isomorphism classes of separated schemes of finite type over* κ *which are closed subschemes of separated smooth schemes over* κ. *Then the maps*

$$h^k_{p,\mathrm{c}}\colon \mathbf{ES}(\kappa) \longrightarrow \mathbb{Z}, \tag{17.17.1}$$

$$h^{i,k-i}_{p,\mathrm{c}}\colon \mathbf{ES}(\kappa) \longrightarrow \mathbb{Z} \tag{17.17.2}$$

satisfy the following equalities:

1) *Let* Z *be a separated scheme of finite type over* κ *which is a closed subscheme of a separated smooth scheme over* κ. *For a closed subscheme* Z' *of* Z,

$$h^{i,k-i}_{p,\mathrm{c}}(Z) = h^{i,k-i}_{p,\mathrm{c}}(Z') + h^{i,k-i}_{p,\mathrm{c}}(Z \setminus Z'). \tag{17.17.3}$$

2) *For a proper smooth scheme* X *over* κ,

$$\begin{aligned} h^{i,k-i}_{p,\mathrm{c}}(X) &= (-1)^k \dim_{K_0} H^{k-i}(X, \mathcal{W}\Omega^i_X)_{K_0} \\ h^k_{p,\mathrm{c}}(X) &= (-1)^k \dim_{K_0} H^k_{\mathrm{crys}}(X/\mathcal{W})_{K_0}. \end{aligned} \tag{17.17.4}$$

3)

$$h^k_{p,\mathrm{c}}(Z) = \sum_{i\in\mathbb{Z}} h^{i,k-i}_{p,\mathrm{c}}(Z). \tag{17.17.5}$$

4) *Let* Z_i $(i = 1, 2)$ *be a separated scheme of finite type over* κ *which is a closed subscheme of a separated smooth scheme over* κ. *Then*

$$h^{i,k-i}_{p,\mathrm{c}}(Z_1 \times_\kappa Z_2) = \sum_{\substack{i_1+i_2=i\\ k_1+k_2=k}} h^{i_1,k_1-i_1}_{p,\mathrm{c}}(Z_1) h^{i_2,k_2-i_2}_{p,\mathrm{c}}(Z_2). \tag{17.17.6}$$

5) *If the resolution of singularities in the strong sense holds for any variety over* κ, *then* (17.17.3) *and the first formula in* (17.17.4) *characterize the function* $h^{i,k-i}_{p,\mathrm{c}}$. *An obvious analogous characterization for* $h^k_{p,\mathrm{c}}$ *holds.*

Proof. — Except 5), the proof is similar to that of (16.22): to obtain 1), we have only to use the strict exactness of the excision exact sequence (17.13.1) instead of (16.6.1). The proofs of 2) and 3) are similar to (16.22), 2) and 5), respectively. The proof of 4) is given by using (17.15), 2) instead of (16.15), 2). We prove 5) as follows.

Let Z be a separated scheme of finite type over κ. Let $g^{i,k-i}_{p,\mathrm{c}}\colon \mathbf{ES}(\kappa) \to \mathbb{Z}$ be a function satisfying the equations (17.17.3) and the first formula in (17.17.4). Then, by (17.17.3), we may assume that Z is reduced. Let Z^o be an open smooth subscheme of Z such that $\dim(Z \setminus Z^o) < \dim Z$. Because

$$g^{i,k-i}_{p,\mathrm{c}}(Z) = g^{i,k-i}_{p,\mathrm{c}}(Z^o) + g^{i,k-i}_{p,\mathrm{c}}(Z \setminus Z^o),$$

we have only to characterize $g^{i,k-i}_{p,\mathrm{c}}(Z^o)$ by induction on $\dim Z$. By Nagata's embedding theorem (see [**67**]) and the assumption, we may assume that Z^o is an open subscheme of a proper smooth scheme Y over κ such that $\dim(Y \setminus Z^o) < \dim Y$. Then, because

$$\begin{aligned} g^{i,k-i}_{p,\mathrm{c}}(Z^o) &= g^{i,k-i}_{p,\mathrm{c}}(Y) - g^{i,k-i}_{p,\mathrm{c}}(Y \setminus Z^o) \\ &= (-1)^k \dim_{K_0} H^{k-i}(Y, \mathcal{W}\Omega^i_Y)_{K_0} - g^{i,k-i}_{p,\mathrm{c}}(Y \setminus Z^o), \end{aligned}$$

the induction on $\dim Z$ characterizes the function $g^{i,k-i}_{p,\mathrm{c}}$. □

In an obvious way, we define an analogue $h^k_{\ell,\mathrm{c}}(Z)$ of $h^k_{p,\mathrm{c}}(Z)$ for the ℓ-adic cohomology $H^\bullet_{\text{ét},\mathrm{c}}(Z_{\kappa_{\mathrm{sep}}}, \mathbb{Q}_\ell)$ ($(\ell, p) = 1$) for any base field κ.

Theorem 17.18. — *Let κ be a perfect field of characteristic $p > 0$. Then*

$$h^k_{\ell,\mathrm{c}}(Z) = h^k_{p,\mathrm{c}}(Z) \quad \big(Z \in \mathbf{ES}(\kappa)\big). \tag{17.18.1}$$

Proof. — The Theorem is reduced to (16.23) by the strict exactness of (17.13.1) with respect to the weight filtration, by that of the ℓ-adic analogue of (17.13.1) and by the duality of the rigid cohomology and the duality of the ℓ-adic cohomology of a separated smooth scheme of finite type. □

If the base field κ is finite, then we can define an analogue $h^{i,k-i}_{\ell,\mathrm{c}}$ of the virtual slope-Betti number $h^{i,k-i}_{p,\mathrm{c}}$ by measuring the slopes of the eigenvalues of the Frobenius on the ℓ-adic cohomologies with compact supports as in (16.23.2) (cf. [**26**, III (3.3.7)]).

Theorem 17.19. — *If κ is a finite field, then $h^{i,k-i}_{\ell,\mathrm{c}} = h^{i,k-i}_{p,\mathrm{c}}$.*

Proof. — The Theorem is reduced to (16.24) as in (17.18). □

Proposition 17.20. — *Let $f\colon U \longrightarrow V$ be a proper morphism of separated schemes of finite type over κ. Assume that U and V are closed subschemes of separated smooth schemes of finite type over κ. Then the pull-back*

$$f^*\colon H^h_{\mathrm{rig,c}}(V/K) \longrightarrow H^h_{\mathrm{rig,c}}(U/K)$$

is strictly compatible with the weight filtration.

Proof. — By the proof of (17.7), we can reduce (17.20) to the case where the base field κ is a finite field. In this case, (17.20) is clear. □

COROLLARY 17.21. — *Let $f\colon U \longrightarrow V$ be a finite étale morphism of affine schemes over κ. Assume that U and V are smooth over κ. Then the trace morphism*

$$\mathrm{Tr}_f\colon H^h_{\mathrm{rig}}(U/K) \longrightarrow H^h_{\mathrm{rig}}(V/K)$$

(see [**8**, (1.4)]) *is strictly compatible with the weight filtration.*

Proof. — By [**8**, (2.3) (ii)], $\mathrm{Tr}_f = f_*$. Hence (17.21) immediately follows from (17.20). □

PROPOSITION 17.22. — *Let Z be a separated scheme of finite type over κ which is a closed subscheme of a separated smooth scheme over κ. The canonical morphism*

$$H^h_{\mathrm{rig,c}}(Z/K) \longrightarrow H^h_{\mathrm{rig}}(Z/K) \tag{17.22.1}$$

(see [**6**, (3.1) (i)]) *is strictly compatible with the weight filtration.*

Proof. — The proof is the same as that of (17.20). □

REMARK 17.23. — We leave the reader to the analogues of results in §§14–17 over the complex number field.

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