Mémoires

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ONE-DIMENSIONAL GENERAL FOREST FIRE PROCESSES

Numéro 132

Nouvelle série

Xavier BRESSAUD

Nicolas FOURNIER

2 0 1 3

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre National de la Recherche Scientifique

Diffusion

Maison de la SMF
B.P. 67
13274 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs 2013

Vente au numéro : 35 € ($ 48)
Abonnement Europe : 262 €, hors Europe : 296 € ($ 444)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0249-633-X

ISBN 978-2-85629-765-0

Directeur de la publication : Marc PEIGNÉ

MÉMOIRES DE LA SMF 132

ONE-DIMENSIONAL GENERAL FOREST FIRE PROCESSES

Xavier Bressaud
Nicolas Fournier

Société Mathématique de France 2013
Publié avec le concours du Centre National de la Recherche Scientifique

Xavier Bressaud

Xavier Bressaud: Université P. Sabatier, Institut de Maths de Toulouse, F-31062 Toulouse Cedex, France.

E-mail : `bressaud@math.univ-toulouse.fr`

Nicolas Fournier

Nicolas Fournier: Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, Université Paris-Est, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France.

E-mail : `nicolas.fournier@univ-paris12.fr`

2000 *Mathematics Subject Classification*. — 60K35, 82C22.

***Key words and phrases*. —** Stochastic interacting particle systems, Self-organized criticality, Forest fire model.

The second author was supported during this work by the grant from the Agence Nationale de la Recherche with reference ANR-08-BLAN-0220-01.

ONE-DIMENSIONAL GENERAL FOREST FIRE PROCESSES

Xavier Bressaud, Nicolas Fournier

***Abstract.* —** We consider the one-dimensional generalized forest fire process: at each site of $\mathbb{Z}$, seeds and matches fall according to i.i.d. stationary renewal processes. When a seed falls on an empty site, a tree grows immediately. When a match falls on an occupied site, a fire starts and destroys immediately the corresponding connected component of occupied sites. Under some quite reasonable assumptions on the renewal processes, we show that when matches become less and less frequent, the process converges, with a correct normalization, to a limit forest fire model. According to the nature of the renewal processes governing seeds, there are four possible limit forest fire models. The four limit processes can be perfectly simulated. This study generalizes consequently previous results of **[15]** where seeds and matches were assumed to fall according to Poisson processes.

***Résumé* (Processus de feux de forêt généraux en dimension 1)**

Nous étudions le processus des feux de forêt généralisé en dimension 1 : sur chaque site de $\mathbb{Z}$, des graines et des allumettes tombent suivant des processus de renouvellement stationnaires i.i.d. Quand une graine tombe sur un site vide, un arbre pousse immédiatement. Quand une allumette tombe sur un site occupé, un feu démarre et brûle immédiatement la composante connexe occupée autour de ce site. Nous montrons — sous des hypothèses raisonnables sur les processus de renouvellement — que lorsque la fréquence des allumettes tend vers zéro, le processus converge, correctement renormalisé, vers un processus limite. Suivant la nature des processus de renouvellement gouvernant l'apparition des graines, quatre processus limites sont possibles. Les quatre modèles limites peuvent être simulés parfaitement. Cette étude généralise des résultats de **[15]**, où nous supposions que graines et allumettes tombaient suivant des processus de Poisson.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Introduction

Consider a graph $G = (S, A)$, S being the set of vertices and A the set of edges. Introduce the space of configurations $E = \{0, 1\}^S$. For $\eta \in E$, we say that $\eta(i) = 0$ if the site $i \in S$ is vacant and $\eta(i) = 1$ if i is occupied by a tree. Two sites are neighbors if there is an edge between them. We call forests the connected components of occupied sites. For $i \in S$ and $\eta \in E$, we denote by $C(\eta, i)$ the forest around i in the configuration η (with $C(\eta, i) = \varnothing$ if $\eta(i) = 0$). We consider the following (vague) rules:

- ▷ vacant sites become occupied (a seed falls and a tree immediately grows) at rate 1;
- ▷ occupied sites take fire (a match falls) at rate $\lambda > 0$;
- ▷ fires propagate to neighbors (inside the forest) at rate $\pi > 0$.

Such a model was introduced by Henley [**37**] and Drossel and Schwabl [**27**] as a toy model for forest fire propagation and as an example of a simple model intended to clarify the concept of *self-organized criticality.*

The order of magnitude of the rate of growth is much smaller than the propagation rate, $\pi \gg 1$. We will focus here on the limit case where the propagation is instantaneous: when a tree takes fire, the whole forest (to which it belongs) is destroyed immediately. The model is thus:

- ▷ vacant sites become occupied (a seed falls and a tree immediately grows) at rate 1;
- ▷ matches fall on occupied sites at rate λ and then burn instantaneously the corresponding forest.

The features of the model depend on the geometry of the graph; we only consider in this paper the case $S = \mathbb{Z}$ (with its natural set of edges). They also depend on the laws of the processes governing seeds and matches; the standard case is when these are Poisson processes so that the forest fire process is Markov. We deal here with the most general (stationary) case; Poisson processes are replaced by stationary renewal processes.

Our main preoccupation is the behavior of this model in the asymptotic of rare seeds, namely when $\lambda \to 0$. We present four possible limit processes (depending on the tail properties of the law of the stationary processes governing seeds) arising when we suitably rescale space and accelerate time while letting $\lambda \to 0$. This is a considerable generalization of the results obtained in [**15**].

This introduction consists of six subsections.

(i) In subsection 1.1.1, we briefly recall the concept of *self-organized criticality* and recall a certain number of models supposed to enjoy self-organized critical properties.

(ii) We present in subsection 1.1.2 a quick history of the forest-fire process, its other possible interpretations and its links with other models.

(iii) subsection 1.1.3 explains the importance of the geometry of the underlying graph G and the links of the forest-fire model with *percolation*.

(iv) In subsection 1.1.4, we recall what has been done for the (Markov) forest-fire process on $\mathbb{Z}$ from a rigorous mathematical point of view.

(v) subsection 1.1.5 is devoted to a brief exposition of the main ideas of the present paper.

(vi) Finally, we give the plan of the paper in subsection 1.1.6.

1.1.1. Self-organized criticality. —

One of the successes of statistical mechanics is to explain how local interactions generate macroscopic effects through simple models on lattices. Among the most striking phenomena are those observed around so-called *critical values* of the parameters of such models, such as scale-free patterns, power laws, conformal invariance, critical exponents or universality.

1.1.1.1. Paradigm. — The study of self-organized critical systems has become rather popular in physics since the end of the 80's. These are simple models supposed to clarify temporal and spatial randomness observed in a variety of natural phenomena showing *long range correlations*, like sand piles, avalanches, earthquakes, stock market crashes, forest fires, shapes of mountains, clouds, etc. It is remarkable that such phenomena reminiscent of critical behavior arise so frequently in nature where nobody is here to finely tune the parameters to critical values.

An idea proposed in 1987 by Bak-Tang-Wiesenfeld [**5**] to tackle this contradiction is, roughly, that of systems *growing* toward a *critical state* and relaxing through *catastrophic* events: avalanches, crashes, fires, etc. If the catastrophic events become more and more probable when approaching the critical state, the system spontaneously reaches an equilibrium *close* to the critical state. This idea was developed in [**5**] through the study of the *archetypical* sand pile model.

This paradigm was used to investigate various phenomena, from physics to sociology through biology, epidemiology or economics. The pertinence of the conclusions are not always convincing. Discussion to decide if whether or not there is self-organized criticality in nature or in one or another model, or even to decide what self-organized criticality should exactly be, is beyond our purpose. Anyhow let us summarize the usual features of these models:

▷ local dynamics but with possibly very long range effects (at high speed) through a simple mechanism;

▷ macroscopic states with scaling invariance properties, *a priori* related to the critical state of a well-known system;

▷ long range spatial correlations and power laws for natural observables at fixed times;

▷ presence of $1/f$ or $1/f^{\alpha}$-noise in the temporal fluctuations of natural observables. We are not experts on this topic, but it seems to be one of the main motivation of self-organized critical systems. It is the subject of the original article of Bak-Tang-Wiesenfield [**5**] and of considerably many physical papers.

One of the specificities of these models is that the interaction is formally non local; it is local in general, but may, when close to the *critical region* — whatever this means — have long range effects. This, together with a lack of monotonicity, yields mathematical difficulties that justify a careful treatment.

To understand, explain or illustrate these phenomena, a multitude of models have been proposed to explore various mechanisms that would produce these effects. Simple models, non necessarily realistic, are nice for they try to catch the underlying mechanisms. They have often been treated numerically, in the spirit of Bak-Tang-Wiesenfield [**5**]. Forest fire models are among them and still need a mathematical rigorous study. Sand pile models, while somehow more complicated, have been more studied.

1.1.1.2. Sand pile models. — Let us explain in a few words what a sand pile model is. First, we assume that we have a definition of what a *stable* sand pile is. Sand grains fall at random on sites. When a grain falls, if the new pile is *unstable*, it is immediately re-organized to become stable, through (possibly many) successive elementary steps. Such events are called *avalanches*. This model was introduced by Bak-Tang-Wiesenfeld [**5**] and studied by Dhar [**25**]. Since, there has been a huge amount of results and we will not try to be exhaustive; for surveys see for instance Holroyd-Levine-Meszaros-Peres-Propp-Wilson [**39**], Goles-Latapy-Magnien-Morvan-Phan [**34**] or Redig [**56**].

Let us give a slightly more precise description of the so-called *Abelian* sand pile model. The state of the system is described by $\eta \in \mathbb{Z}^S$, representing local *slopes* of the sand pile. For instance, when $S = \mathbb{Z}$, think that $\eta(i) = h(i+1) - h(i)$ where $h(i)$ is the height of the sand pile on the site i. A dynamic is defined on $\mathbb{Z}^S$ using a matrix Δ indexed by $S \times S$, called *toppling matrix*. It has positive entries on the diagonal (think of $\Delta_{i,i} = \gamma$ constant), negative entries when $i, j \in S$ are neighbors and null entries elsewhere. It is *dissipative* if $\Delta_{i,i} + \sum_{j \neq i} \Delta_{i,j} < 0$. Then define the toppling of a site i as the mapping $T_i : \mathbb{Z}^S \to \mathbb{Z}^S$ defined by

$$\begin{aligned} T_i(\eta)(j) &= \eta(j) - \Delta_{i,j}, \quad \forall j \in S \text{ if } \eta(i) > \Delta_{i,i}; \\ T_i(\eta) &= \eta \text{ otherwise.} \end{aligned}$$

Toppling at i consists, whenever the slope is *too big* at i, of spreading grains on neighboring sites (possibly in a non conservative way). A pile is stable if for all $i \in S$, $\eta(i) \leq \Delta_{i,i}$ (then, no toppling has any effect). Observe that successive topplings at different sites commute (which explains the term *Abelian*).

Now consider the situation where sand grains fall at random, on each site, at rate 1. Each time a grain falls, immediately topple (possibly many times) until stability is reached. Some dissipativity assumptions guarantee that this is always possible.

At first glance, arrival of a new sand grain on a site has only a local effect: a non trivial toppling at i may occur. But there can be a chain reaction creating an avalanche. And indeed, the action may, in general, have a long range effect.

These systems have a nice underlying group structure that depends on the size and geometry of the underlying lattice, see e.g. Le Borgne-Rossin [**13**] for such an algebraic point of view. The thermodynamic limits of the sand-pile models have been investigated. In particular, existence and uniqueness of a stationary measure have been proved. See for instance Maes-Redig-Saada [**47**] when $S = \mathbb{Z}$ and Járai [**42**] when $S = \mathbb{Z}^d$. Some features of self-organized criticality have been observed for $d > 1$, at least numerically, in the physical literature, see e.g. Lübeck-Usadel [**46**]. For instance, they have studied the *sizes of avalanches* (number of topplings necessary to stabilize after a grain has been added). A scaling limit was obtained recently by Dürre [**31**].

1.1.1.3. Other models.— The Abelian sand pile seems to be the most popular sand pile model. However it has a lot of variants: Zhang sand pile model (see Zhang [**65**], Pietronero-Tartaglia-Zhang [**51**]), Oslo model (see Christensen-Corral-Frette-Feder-Jossang [**20**], Amaral-Lauristsen [**4**]), Oslo rice pile model (see Brylawski [**18**]), chip firing game (see Tardos [**62**]), etc.

Moreover, various different models have been introduced and studied with the eyes of self-organized criticality. There is of course the forest fire model that we are going to discuss in this paper. Let us mention briefly some other models: rotor-router model (introduced by Priezzhev-Dhar-Dhar-Krishnamurthy [**52**] under the name *Eulerian walkers model*), loop-erased random walks (Majumdar [**48**]), diffusion/aggregation models (Cafiero-Pietronero-Vespignani [**19**]), Scheidegger's model of river basin

(Scheidegger [**57**]), models describing earthquakes (Olami-Feder-Christensen [**50**]) or crashes in stock markets (Staufer-Sornette [**61, 59**]), etc.

As we already mentioned those systems have often been subjected to numerical experimentations and studies. Of course this is a difficult task and it has sometimes been misleading: long range effects need huge simulations, the interpretation of which is not always meaningful.

For surveys on self-organized criticality, see Bak-Tang-Wiesenfeld [**6**], Dhar [**26**], Jensen [**43**] and the references therein.

1.1.2. Forest fire models. — Here we consider the classical forest fire model on $G = (S, A)$. Recall that on each site of S, seeds are falling at rate 1 and matches are falling at rate λ, according to some Poisson processes. A seed falling on a vacant site makes it immediately occupied, and a match falling on an occupied site makes instantaneously vacant the whole corresponding occupied connected component. Thus the forest fire process is Markov (at least if one is able to prove that it exists and is unique).

1.1.2.1. History and numerical studies. — The forest fire model was introduced independently by Henley [**37**] and Drossel-Schwabl [**27**]. In the literature, it is generally referred to as the Drossel-Schwabl forest fire model. In their original paper, they consider the case where S is a cube in $\mathbb{Z}^d$. They are interested in scaling laws and critical exponents for this model. Orders of magnitude of relevant quantities are derived by analytical computations using essentially mean field considerations. The results are *confirmed* by computer simulations. In Drossel-Clar-Schwabl [**28**], the asymptotic behavior of the density of vacant sites in the limit $\lambda \to 0$ is obtained when $S = \mathbb{Z}$ (using heuristic arguments, see subsubsection 1.1.4.3 below). After this work, numerous numerical or semi-analytical studies have been produced. Among others, let us mention Henecker-Peschel [**40**] and Pruessner-Jensen [**53**]. Numerical studies were handled again by Grassberger [**35**], who computes, when $S = \mathbb{Z}^2$, the density of occupied sites, the fractal dimension of fires and the distribution of the fire sizes, in the limit $\lambda \to 0$.

The first rigorous probabilistic treatment of this model is the paper by van den Berg and Járai [**9**]. They give a rigorous description of the asymptotic density of vacant sites in the limit $\lambda \to 0$ for the forest fire process on $\mathbb{Z}$. To our knowledge, all the rigorous results about the forest fire process concern the case where seeds and matches fall according to Poisson processes. See Dürre [**29**], [**30**], [**31**] (existence and uniqueness of the process on $\mathbb{Z}^d$ with $\lambda > 0$ fixed), van den Berg-Brouwer [**7**] (behavior of the process near the critical time in dimension 2, as $\lambda \to 0$) and Brouwer-Pennanen [**17**] (estimates on the cluster size distribution in the asymptotic $\lambda \to 0$, in dimension 1). See also the papers by the authors [**14**] (study of the invariant distribution when $\lambda = 1$ in dimension 1) and [**15**] (scaling limit of the one dimensional forest fire process in the asymptotic $\lambda \to 0$). We will discuss all these results more specifically in this introduction.

1.1.2.2. Real forest fires. — Real forest fires in nature are also a subject of preoccupation and of study from different point of views. In particular there are various statistical studies of sizes (and sometimes shapes) of real forest fires in different regions (see for instance Holmes-Hugget-Westerling [**38**]). One of the recurrent observations is that the distributions of those fires have heavy tails (power laws) and pleasant scale invariance properties. Another one is the tentative description of the (fractal) geometry of fires (see for instance Mangiavillano [**49**]). For references, connection with real life and practical interest of these studies, see Cui-Perera [**23**]. A few studies relate the dynamics of real fires in a given region with theoretical models. One natural task was to compare real data and numerical experiments done with the toy models we have. On this aspect, let us mention the recent (and encouraging) works by Zinck-Grimm-Johst [**66**], [**67**]. Other studies focus on the propagation of the fire itself, but this is not our main preoccupation here since we have assumed that the propagation is instantaneous.

A direction of study suggested by works on real forest fires is to consider fires in *inhomogeneous*, for instance *random*, media. To our knowledge, this aspect has not yet been investigated. Another one, that we address here, is to consider the non Markov case: seeds and matches may not (and actually should not) fall according to Poisson processes.

1.1.2.3. Other interpretations and variations. — The forest fire model has a very simple (and natural) dynamic. It may accept a variety of interpretations. And various modifications can make it fit the description of other phenomena. Indeed, we initially thought of it as a simplification of the avalanche process: snow flakes fall on each site, a snow flake falling on a vacant site makes it occupied, and a snow flake falling on an occupied site makes vacant the whole connected component of occupied sites (such an event being called *avalanche*). This is nothing but the forest fire process with $\lambda = 1$, see [**14**]. More generally, the forest fire process may be used to model phenomena involving geometric relations and a common behavior on connected components; natural examples arise e.g. in epidemiology (change *fire* by *virus*). From these points of view, some natural modifications could be explored such as making the growth process have effect only on sites which are neighbors of occupied sites (in the spirit of the so-called contact process). Such variants should be dominated by the standard contact process and by the forest fire process and may enjoy interesting features.

In a different spirit, a *directed* version of the forest fire model has been studied as a toy model for neural networks. Roughly, the idea is to think of growth as *activation* and of fire as *signal emission*. The signal is transmitted along the (directed) connected component which is at the same time deactivated. The difference is that the underlying graph is a directed graph (usually a tree) and that the signal is (instantaneously) sent according to the directed edge (instead of all the connected component). Let us mention the work of van den Berg-Brouwer [**7**], which include remarks about this model, and the work of van den Berg-Tóth [**10**].

1.1.2.4. Coagulation/Fragmentation. — A slight change of point of view about the forest fire model makes explicit a parallel with a class of coagulation/fragmentation processes. Assume e.g. that $S = \mathbb{Z}$. Say that each edge $(i, i+1)$ has mass 1, and that two neighbor edges $(i-1, i)$ and $(i, i+1)$ are connected (or belong to the same particle) if $\eta(i) = 1$. Then each time a seed falls on a vacant site, this glues two particles (preserving the total mass). And each time a match falls on a site (say, belonging to a forest containing $k \geq 1$ sites), this breaks a particle of mass $k+1$ into $k+1$ particles with mass 1.

We used this remark in [**14**] to study the evolution of the sizes of particles when neglecting correlation, using a deterministic coagulation-fragmentation equation. Of course, similar considerations can be handled on any graph G.

1.1.2.5. Recent results for related models in dimension 1. — Let us mention two recent results about one-dimensional forest fire processes with a somehow different flavor.

Volkov [**64**] considers a version of the forest fire process on $\mathbb{N}$ where ignition occurs only at 0. He studies the weak limit of the distribution of the (suitably normalized) delay between to fires involving n, as $n \to \infty$.

Bertoin [**12**] considers a modified version of Knuth's parking model where random fires burn connected components of cars. On a circle of size n, cars arrive at each site at rate 1. When a car arrives, it occupies the first vacant site (turning clockwise). Molotov cocktails fall on each site at rate $n^{-\alpha}$ where $0 < \alpha < 1$ is fixed. Bertoin studies the asymptotic behavior of the saturation time as $n \to \infty$ and observes a phase transition at $\alpha = \frac{2}{3}$.

1.1.2.6. Specific difficulties. — As we already mentioned, one of the difficulties with forest fire models (and with self-organized critical systems in general) is that the interaction is not local. The process, whenever it is Markov, is not Feller and some classical results fail. In dimension one, this difficulty does not yield real problems for the questions of existence and uniqueness of the process. This is essentially due to the fact that obviously, the sizes of the forests always remain finite (even when λ is very small). This difficulty is more important in higher dimensions, because in the absence of fires, clusters would become infinite in finite time (due to the fact that in dimension $d \geq 2$, percolation occurs). Fires prevent us from the existence of infinite clusters. But these arbitrarily huge clusters burning make difficult the control of the range of interactions. This difficulty also makes the usual proof of existence of stationary measures using compactness arguments fail (because indeed there is a lack of continuity).

The lack of monotonicity of these models, although not fundamental, makes the use of standard intuitions and techniques impossible. Monotonicity allows one to compare the processes started from two different ordered initial configurations (coupled in a suitable way). Monotonicity cannot hold here, because a configuration with more trees will burn sooner.

1.1.3. Geometry of the lattice. — The geometry of the underlying lattice is crucial in statistical mechanics. Recall for instance that *phase transition* for the Ising model on $\mathbb{Z}^d$ appears only for $d \geq 2$ (see Velenik [**63**]). For the forest fire models, the influence of the geometry clearly comes through the behavior of the lattice with respect to percolation. This geometrical influence was already striking in numerical studies. See Grimmett [**36**] for a very complete book on percolation.

1.1.3.1. Growth without fires/Percolation. — Consider a graph $G = (S, A)$. For all $0 \leq p \leq 1$ consider an i.i.d. family $\{\eta(i), i \in S\}$ of Bernoulli random variables with parameter p (a *percolation trial* with probability p). It is well known that there is $0 \leq p_c \leq 1$, depending on the graph, such that for all $p < p_c$, there are a.s. no infinite connected components of occupied sites, while for $p > p_c$, there is at least one infinite connected component with probability 1. The real number p_c is called percolation threshold of G. It is rather natural to consider (dynamical) *percolation processes* on G, that are couplings of percolation trials for all $0 \leq p \leq 1$. For instance, consider a family $\{T_i, i \in S\}$ of i.i.d. random variables on $\mathbb{R}_+$ with exponential distribution with parameter 1. Put $\eta_t(i) = 0$ if $t < T_i$ and $\eta_t(i) = 1$ if $t \geq T_i$. Then for all $t > 0$, $\{\eta_t(i), i \in S\}$ is a percolation trial with probability $P(T_i \leq t) = 1 - \mathrm{e}^{-t}$. Thus an infinite cluster appears at time t_c defined by $1 - \mathrm{e}^{-t_c} = p_c$.

It clearly appears that the percolation threshold plays a *crucial* role in understanding the behavior of the forest fire process on a given lattice. The simple observation is that the *growth process*, i.e. without fires ($\lambda = 0$), is exactly a percolation process on the lattice. For λ small, and *a fortiori* for $\lambda \to 0$ its study is a necessary preliminary. For instance, one aspect is the formation of infinite clusters (although in general those clusters will never appear since, taking fires into account, they must burn before they become infinite). Recall that the percolation threshold is 1 in dimension 1. It is $0 < p_c^{(d)} < 1$ on $\mathbb{Z}^d$ and once there is an infinite cluster, there is a unique one. While, for instance on a d-regular tree, just after the percolation threshold, there are infinitely many infinite clusters: these situations are rather different and should yield different behaviors for the corresponding forest fire processes. Observe that though, for all $\lambda > 0$, the forest fire process is easy to define for small times, things turn out to be more complicated when we reach the *critical time* t_c. Even in dimension 1 the separate study of the percolation process makes sense as we shall see further, subsection 1.1.4.4.

1.1.3.2. Modified percolation models. — It has also been fruitful to study modified (for instance dynamical) versions of percolation processes. Models like frozen percolation (Aldous [**3**], see also Brouwer [**16**]), invasion-percolation (see for instance Damron-Sapozhnikov-Vágvölgyi [**24**]), or self-destructive percolation (see van den Berg-Brouwer [**7**] and more recently van den Berg-Brouwer-Vágvölgyi [**8**]) are closely related to the forest fire processes. Let us focus one moment on this last example since it has direct implications on forest fire processes.

A typical configuration for the self-destructive percolation model on $\mathbb{Z}^2$ with parameter (p, δ) is generated in three steps: first generate a configuration for the ordinary percolation model with parameter p. Next, make all sites in the infinite occupied cluster vacant. Finally, make occupied each vacant site with probability δ. Let $\theta(p, \delta)$ be the probability that 0 belongs, in the final configuration, to an infinite occupied cluster. In a recent paper [**8**], van den Berg, Brouwer and Vágvölgyi prove that this function is continuous outside of a set of the form $\{(p_c, \delta) : \delta < \delta_0\}$. It is conjectured that this function has a discontinuity, roughly meaning that there is $\delta > 0$ such that for any $p > p_c$, the model with parameter (p, δ) is sub-critical (there a.s. is no infinite cluster).

In [**7**], van den Berg and Brouwer have proved that assumption of a strongly related conjecture yields a result for a 2-dimensional forest fire process after the critical time: there is $t > t_c$ such that for all $m \geq 1$,

$$\liminf_{\lambda \to 0} \liminf_{n \to \infty} \Pr \begin{bmatrix} \text{a tree in } [\![-m, m]\!]^2 \text{ burns before } t \\ \text{in the forest fire process on } S_n = [\![-n, n]\!]^2 \end{bmatrix} \leq \frac{1}{2}.$$

1.1.3.3. Thermodynamic limit. — The forest-fire process on a finite graph is a finite state space continuous time Markov chain (if matches and seeds fall according to Poisson processes). Existence and uniqueness of the process thus come for free. Existence of an invariant measure as well. A basic argument also yields uniqueness of the invariant measure (because the configuration with all sites vacant is recurrent). Hence interesting phenomena may arise only when we let the size of the lattice tend to infinity.

When $S = \mathbb{Z}$, it is not very expensive to go directly to the limit: the process is naturally uniquely defined on $\mathbb{Z}$. This is easily seen through a graphical construction of the process (see [**15**]), see also Proposition 2.1.4 below.

In dimension $d > 1$ the situation is more delicate. On $\mathbb{Z}^d$ (and actually on any graph with bounded vertex degree) existence has been proved recently by Dürre [**29**]. He also proved uniqueness, but in two steps: firstly, in [**30**], he shows that, for $\lambda > 0$ large enough (the bound is related to the percolation threshold), the forest-fire process is unique. Only very recently the same author, in [**31**], tackled the same question on a graph with bounded vertex degree and for all $\lambda > 0$. This is a much more subtle task. To prove this result he has to introduce the so-called blur processes, to show that the influence of matches falling far away from 0 is negligible.

1.1.3.4. Mean field model. — The mean field case is slightly different. Indeed, one has to adopt the dual point of view (on edges). Furthermore, the process cannot be defined directly on an infinite lattice since we consider the complete graph. The point of view developed by Ráth and Tóth in [**55**] is based on the Erdös-Rényi construction [**32**]. For all $n \geq 1$, let S_n be a set (of vertices) with $|S_n| = n$, and consider the complete graph $G_n = (S_n, A_n)$. Start initially with all edges vacant. Then edges appear independently at rate $1/n$. Matches fall at rate λ_n on each site and destroy instantaneously the whole corresponding occupied connected component. We consider the asymptotic $n \to \infty$. The various regimes (see Ráth-Tóth [**55**]) are quite illuminating.

(I) If $\lambda_n \ll 1/n$, then fires are (asymptotically) negligible. Thus we have the same asymptotics as in the Erdös-Réyni model: a giant component appears after some time T_{gel} (the critical time in this formalism).

(II) If $\lambda_n \simeq \lambda/n$, then a giant component appears, but is destroyed after some time. Only the giant component may burn: there are no matches enough to burn finite size forests.

(III) If $1/n \ll \lambda_n \ll 1$, there are not enough fires to burn finite size forests, but too many to let any infinite forest appear. Hence no giant component appears.

(IV) If $\lambda_n \simeq \lambda$, then matches may kill finite forests, so that of course, no giant component emerges.

To formalize these statements rigorously, Ráth-Tóth [**55**] consider the cluster size distributions: $\nu_{n,k}(t)$ is the number of vertices belonging to a connected component of size k at time t divided by n. Consider also the *concentrations* $c_{n,k}(t) := \nu_{n,k}(t)/k$. As $n \to \infty$, the limit concentrations $(c_k(t))_{k\geq 1}$ should satisfy a system of differential equations closely related to Smoluchowski's coagulation equations with multiplicative kernel and mono-disperse initial condition:

$$
\begin{aligned}
&c_1(0) = 1, \quad c_k(0) = 0, \quad k \geq 2,\\
&\frac{\mathrm{d}}{\mathrm{d}t} c_k(t) = \frac{1}{2} \sum_{i=1}^{k-1} i(k-i) c_i(t) c_{k-i}(t) - k c_k(t) \sum_{i=1}^{\infty} i c_i(t), \quad k \geq 1.
\end{aligned}
$$

Such equations, discussed in details in Aldous [**2**], have been introduced by Smoluchowski [**58**] in 1916. These equations are subjected to a *phase transition* known as *gelation*: some mass is lost at some positive finite instant T_{gel}, due to the emergence of a giant particle. For $t > T_{\mathrm{gel}}$, we have to decide what to do with the giant particle. It can e.g. interact with finite particles (Flory's equation) or be removed from the system (Smoluchowski's equation). See Aldous [**1**] and [**33**] for such considerations.

In the regime (I), the limit equations are the Flory equations: a giant particle appears at time T_{gel} and then coexists with other particles (finite particles do coalesce with the giant particle). In the regime (II), the limit equations are closer to the Smoluchowski equations: a giant particle appears at time T_{gel} (the same one as previously) but once it is giant, it is replaced by particles with mass 1 (in a conservative way). In the regimes (III) and (IV), some other modifications of the Smoluchowski equations appear.

The most interesting results obtained by Ráth-Tóth in [**55**] are that in the regime (III), the modified Smoluchowski coagulation system has a unique solution which is the classical one for all $t < T_{\text{gel}}$ and has a particular (critical-like) form for $t > T_{\text{gel}}$, and $(c_{n,k}(t))_{t\geq 0,k\geq 1}$ converges to this unique solution as $n \to \infty$. This shows that the complete graph exhibits self-organized criticality in the sense that beyond T_{gel}, it remains critical forever: no giant component appears but, after T_{gel}, the size-distribution is, in some sense, critical.

1.1.3.5. Stationary measures. — The existence of invariant measures for the forest-fire process in $\mathbb{Z}^d$ (with any $\lambda > 0$ fixed) has been proved by Stahl [**60**]. For the case of $\mathbb{Z}$ the situation is simpler, see the next subsection.

1.1.4. Forest fire on $\mathbb{Z}$. — Let us review in details known results about the forest fire processes in dimension 1. We still focus on the usual case where seeds and matches fall according to i.i.d. Poisson processes, with respective rates 1 and $\lambda > 0$. We denote $\eta_t^\lambda \in \{0,1\}^{\mathbb{Z}}$ the configuration at time t and, for $i \in \mathbb{Z}$, $C(\eta_t^\lambda, i)$ is the connected component of occupied sites around i. Observe that (possible) infinite clusters in the initial configuration would immediately disappear.

From the point of view of self-organized criticality, the interesting regime is the asymptotic behavior of the forest-fire process as $\lambda \to 0$: then fires are very rare, but concern huge occupied components.

1.1.4.1. Stationary measures. — Existence of a stationary measure does not immediately follow from standard compactness arguments since the process is not Feller. However, in [**17**], Brouwer and Pennanen prove the existence of a stationary measure for all fixed $\lambda > 0$. In [**14**], we proved the uniqueness of this invariant distribution, as well as the exponential convergence to equilibrium in the special case where $\lambda = 1$. We also proved that the invariant distribution is (spatially) exponentially mixing and can be graphically constructed. The methods in [**14**] should be easily extended to the case where $\lambda \geq 1$ (and actually to $\lambda > 1 - \varepsilon_0$ for some rather small $\varepsilon_0 > 0$) but our proof completely breaks down for small values of $\lambda > 0$.

1.1.4.2. Asymptotic density. — Van den Berg and Járai study in [**9**] the asymptotic density of vacant sites in the limit $\lambda \to 0$. Their result states that there are two constants $0 < c < C$ such that for any initial configuration, for any $\lambda > 0$ small enough, for t large enough (of order $\log(1/\lambda)$),

$$\frac{c}{\log(1/\lambda)} \leq \Pr\left[\eta_t^\lambda(0)=0\right] \leq \frac{C}{\log(1/\lambda)}.$$

This is coherent with the intuition that the rarer fires are, the more space is occupied by trees (although because of the lack of monotonicity, this is not straightforward). We mentioned that such result was stated in Drossel-Clar-Schwabl [**28**]. But the proof in [**28**] is not rigorous: it is based on the *ansatz* that the cluster sizes were following a cutoff power law, for cluster-sizes up to some $s^\lambda_{\max}$ defined by $s^\lambda_{\max}\log s^\lambda_{\max} = 1/\lambda$, i.e.

$$s^\lambda_{\max} \simeq \frac{1}{\lambda\log(1/\lambda)}.$$

In [**9**], van den Berg and Járai also show that the cluster sizes cannot follow the predicted power law.

1.1.4.3. Sizes of clusters, first results. — In [**17**], Brouwer and Pennanen show that this last *ansatz* holds true up to $s^{\frac{1}{3}}_{\max}$. More specifically, they show that there are constants $0<c<C$ such that for all $0<\lambda<1$ and all stationary measures μ_λ (invariant by translation) of the forest fire model on $\mathbb{Z}$ with parameter λ, for all $x<(s^\lambda_{\max})^{\frac{1}{3}}$,

$$\frac{c}{(1+x)\log(1/\lambda)} \leq \mu_\lambda\big(\left|C(\eta,0)\right|=x\big) \leq \frac{C}{(1+x)\log(1/\lambda)}.$$

Observe that this estimate is valid for relatively small clusters that will not be seen after rescaling (microscopic clusters).

1.1.4.4. Kingman's Process. — We detail a classical construction related to the Smoluchowski equation with constant kernel which is quite close to our point of view. Most ideas and references for proofs can be found in Aldous [**2**]. Let us consider the following percolation process on $\mathbb{Z}$. Starting from the vacant configuration, we let appear trees at each site at some rate $r(t)$, that allows us to control the *speed* of the process. Say that each edge $(i,i+1)$ has mass 1 (see subsubsection 1.1.2.4). Let a seed fall on each site i at some random time T_i with $P(T_i>t)=2/(t+2)$ independently (this corresponds to the rate $r(t)=1/(t+2)$, because then $\exp\big(-\int_0^t r(s)\,\mathrm{d}s\big)=2/(t+2)$). Call $D(t,i)$ the *particle* containing the edge $(i,i+1)$ at time t (say that two neighbor edges $(j-1,j)$ and $(j,j+1)$ are glued if $\eta_t(j)=1$). At time t, the particle containing a given edge (e.g. $(0,1)$) has mass m with probability

$$\rho_m(t) = m\Big(\frac{2}{2+t}\Big)^2\Big(\frac{t}{2+t}\Big)^{m-1}$$

and hence the concentration of clusters with mass m per unit length is nothing but

$$c_m(t) = \Big(\frac{2}{2+t}\Big)^2\Big(\frac{t}{2+t}\Big)^{m-1}.$$

We recognize the solution to Smoluchowski's equation with constant coagulation kernel and mono-disperse initial condition, see Aldous [**2**].

Now consider a standard construction of the so-called Kingman coalescent process. Take independent exponential random variables $\{\xi_k, k \geq 2\}$ of rates $\binom{2}{k}$. Since $E[\sum_{k=2}^{\infty} \xi_k] = 2$, we can define random times $0 < \cdots < \tau_3 < \tau_2 < \tau_1 < \infty$ by

$$\tau_i = \sum_{k=i+1}^{\infty} \xi_k.$$

Take $\{U_i, i \geq 1\}$ independent random variables uniformly distributed on $(0,1)$. For each i draw a vertical segment from (U_i, τ_i) to $(U_i, 0)$. At time t this construction splits $(0,1)$ into i intervals, where $\tau_i < t < \tau_{i-1}$. Write $X(t)$ for the list of the lengths of these subintervals. This is a version of the stochastic coalescent called *Kingman's coalescent*. Observe that we also could have put the marks $\{(U_i, \tau_i), i \geq 1\}$ using a Poisson measure on $[0,1] \times \mathbb{R}_+$ with a well-chosen intensity measure.

Straightforward computations show that Kingman's coalescent is a limit of the previously defined percolation process in the following sense: consider the list of (distinct) normalized clusters $\lambda D(t/\lambda, \lfloor x/\lambda \rfloor)$ when x runs along $[0,1]$ (cutoff the boundary clusters at 0 and 1) at time t. When $\lambda \to 0$, it converges to $X(t)$ in law (in an appropriate topology). This construction shows how the growth process behaves in the large scales. In some sense we have identified $\{0, \dots, \boldsymbol{n}_\lambda\} \subset \mathbb{Z}$ with $[0,1] \subset \mathbb{R}$ (here $\boldsymbol{n}_\lambda = 1/\lambda$) and obtained a limiting process for the rescaled percolation process.

We stress the fact that the convergence holds globally only for the specific speed $r(t) = 1/(t+2)$ of the percolation process. This fact is related to the self-similarity of the percolation (coalescent) process. In particular, for a constant rate (exponential times for seeds), there is no hope for such a convergence to Kingman's coalescent: after normalization, the size of clusters at time t is of order λ^{1-t} and converges to 0 or ∞ according to whether $t < 1$ or $t > 1$. Conversely, if the rate of growth has a polynomial decay, there is a hope to have a limit process.

1.1.4.5. Asymptotic regime: relevant space/time scales. — As already mentioned, we are interested in the behavior of the system in the large space and time scales in the limit $\lambda \to 0$. Hence the first difficulty is to decide what the relevant scales are. Let us recall the heuristic developed in [**15**]. We need a time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large just before they burn. We thus also have to rescale space, in order that just before burning, clusters have a size of order 1.

Consider the cluster $C(\eta_t^\lambda, 0)$ around the site 0 (for example) at time t. For $\lambda > 0$ very small and for t not too large, one might neglect fires and consider only the growth process; it follows that $|C(\eta_t^\lambda, 0)| \simeq \mathrm{e}^t$ for t not too large (because since seeds fall according to Poisson processes with rate 1, each site is vacant at time t with probability e^{-t}). Then the cluster $C(\eta_t^\lambda, 0)$ burns at rate $\lambda |C(\eta_t^\lambda, 0)| \simeq \lambda \mathrm{e}^t$, so that we decide to accelerate time by a factor $\boldsymbol{a}_\lambda := \log(1/\lambda)$. By this way, $\lambda |C(\eta_{\boldsymbol{a}_\lambda}^\lambda, 0)| \simeq 1$.

Now we rescale space in such a way that during a time interval of order $\boldsymbol{a}_\lambda$, something like one match falls per unit of (space) length. Since matches fall at rate λ on each site, our space scale has to be of order $\boldsymbol{n}_\lambda := 1/(\lambda \boldsymbol{a}_\lambda)$: this means that we will identify $\{0, \dots, \boldsymbol{n}_\lambda\} \subset \mathbb{Z}$ with $[0,1] \subset \mathbb{R}$. Observe that there holds $\boldsymbol{n}_\lambda \simeq s^\lambda_{\max}$, where $s^\lambda_{\max}$ was introduced in subsubsection 1.1.4.2.

Consider now the time/space rescaled cluster around 0

$$D^\lambda_t(0) = \frac{1}{\boldsymbol{n}_\lambda} C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0).$$

The same difficulty as in subsubsection 1.1.4.4 appears: neglecting fires (which is roughly valid for small values of t), we see that

$$\left|D^\lambda_t(0)\right| \simeq \boldsymbol{n}_\lambda^{-1} \mathrm{e}^{\boldsymbol{a}_\lambda t} = \lambda^{1-t} \log(1/\lambda),$$

which goes to 0 for $t < 1$ and to ∞ for $t \geq 1$. For $t \geq 1$, we hope that fires will be in effect, which will limit the size of clusters. But for $t < 1$, $|D^\lambda_t(0)|$ will indeed tend to 0. This means that we have lost some information. To describe the limit process, we have to keep in mind more information and thus introduce another quantity (a sort of *degree of smallness*) which measures the order of magnitude of the *microscopic* clusters, that is clusters that we can not see at macroscopic scales (of which the sizes are much smaller than $\boldsymbol{n}_\lambda$).

1.1.4.6. Limit processes. — We have proved in [**15**] that in the asymptotic of rare matches, the forest fire process converges, under the previously described normalization, to some limit forest fire process. We described precisely the dynamics of this limit process and have shown that it is unique, that it can be built by using a graphical construction and thus can be perfectly simulated. Using the limit process, we have also estimated the size of clusters. Very roughly, we have proved that in a very weak sense, for λ small enough and for t large enough (of order $\log(1/\lambda)$), the cluster-size distribution resembles

$$\Pr\left[C(\eta^\lambda_t, 0) = x\right] \simeq \frac{a}{(x+1)\log(1/\lambda)} \mathbb{1}_{\{x \ll \boldsymbol{n}_\lambda\}} + \frac{b\,\mathrm{e}^{-x/\boldsymbol{n}_\lambda}}{\boldsymbol{n}_\lambda},$$

where a, b are two positive constants. Very roughly, we are able to replace the condition $x < (s^\lambda_{\max})^{\frac{1}{3}}$ of [**17**] by the condition $x < (s^\lambda_{\max})^{1-\varepsilon}$ for any $\varepsilon \in (0,1)$ (but our result is weaker, in the sense that it holds when integrated in x, and we have to take the limit $\lambda \to 0$). This means that there are two types of clusters: *microscopic clusters*, described by a power-like law and *macroscopic* clusters, described by an exponential-like law. This shows a *phase transition* around the *critical size* $\boldsymbol{n}_\lambda$.

1.1.4.7. No self-organized criticality. — From the qualitative point of view the conclusion is rather different from that of Ráth and Tóth [**55**] (presented in subsubsection 1.1.3.4). Here, the (asymptotic) cluster-size distribution does not exhibit self-organized criticality features. We proved the presence of a power law, but this power law describes clusters which are much smaller than the critical size. Large clusters (clusters near the *critical size*) have a law with fast decay.

1.1.5. Main ideas of the present paper. — From the modelling point of view, the Poisson assumption is quite reasonnable for ignitions, but clearly not well justified for recoveries (seeds). Thus it seems interesting to study what happens when seeds and matches are driven by other renewal processes. The goal of this paper is to extend the previous study [**15**] described above to a more general class of renewal processes. We assume that the renewal processes are stationary for simplicity, but this can be more or less justified by the fact that it is the only way that time 0 does not play a special role.

We thus consider the case where seeds (respectively matches) fall on each site of $\mathbb{Z}$ independently, according to some stationary renewal processes, with *stationary delay* distributed according to some law ν_S (respectively ν_M^λ). This means that for any time $t \geq 0$ and on any site $i \in \mathbb{Z}$, the time we have to wait for the next seed is a ν_S-distributed random variable. We have an assumption saying that as $\lambda \to 0$, matches are rarer and rarer. We also assume that ν_S has a bounded support or a tail with fast or regular or slow variations. We prove that, after re-scaling, the corresponding forest fire process converges, as $\lambda \to 0$, to a limit process. And we show that there are four classes of limit processes, according to the fact that

- ▷ ν_S has a bounded support $(HS(BS))$,
- ▷ ν_S has a tail with fast decay $(HS(\infty))$,
- ▷ ν_S has a tail with polynomial decay $(HS(\beta))$,
- ▷ ν_S has a tail with logarithmic decay $(HS(0))$.

As we will see, the limit forest fire process built in [**15**] is quite universal: it describes the asymptotics of a large class (roughly exponential decay for ν_S) of forest fire processes. A similar limit process arises when ν_S has bounded support. But some quite different limit processes arise when ν_S has a heavy tail. We also develop the necessary tools to study the cluster size distributions. Let us mention at once that there is indeed presence of a *critical size* under $(HS(BS))$ and $(HS(\infty))$ but not under $(HS(\beta))$ or $(HS(0))$. In the latter situation, there are only macroscopic clusters. This is related to subsubsection 1.1.4.4.

It is striking that in [**15**] we made repeated use of the Markov property of Poisson processes while it turns out the result still holds without this assumption (and with no significant increase of the complexity). Indeed, proofs remain essentially elementary except maybe from the combinatorial and computational point of view.

From the qualitative point of view, the main novelty is the rise of a new class of processes (those corresponding to polynomial tails), reminiscent of the Kingman coalescent (with deaths). But for this case as for the others, the conclusion is that, as expected, self-organized criticality features do not show up for this model in dimension 1.

Let us finally insist on the fact that surprisingly (in view of the complexity and length of the proofs), our assumptions are really light. Consider e.g. the case where ν_S has an unbounded support and a fast decay, which means (for us) that for any $t > 0$,

$$\lim_{x\to\infty} \frac{\nu_S((x,\infty))}{\nu_S((tx,\infty))} = t^\infty,$$

where $t^\infty = 0$ if $t < 1$, $1^\infty = 1$, and $t^\infty = \infty$ if $t > 1$. We do not need the least additional condition.

1.1.6. Plan of the paper. — Chapter 2 is devoted to a complete exposition of our results. We start in section 2.1 with notation and with the definitions of the objects under study, and we state our assumptions. In section 2.2, we explain the heuristic scales and the relevant quantities (rescaled macroscopic clusters and measure of microscopic clusters). Then we describe precisely our results in sections 2.3 (case with fast decay), 2.4 (case with bounded support), 2.5 (case with polynomial decay) and 2.6 (case with logarithmic decay). We conclude this part with a quick discussion about our modeling choices and with a short list of open problems and perspectives. Chapter 3 (sections 3.1 to 3.11) contains all the proofs. In Chapter 4 we handle a few numerical simulations to illustrate our results. Finally, Chapter 5 contains an appendix about regularly varying functions and coupling.

CHAPTER 2

NOTATION AND RESULTS

2.1. Definitions, notation and assumptions

2.1.1. Stationary renewal processes. — We first fix notation about stationary renewal processes. We refer to Cocozza-Thivent [**21**] for a book on renewal processes.

Definition 2.1.1. — *For μ a probability measure on $(0,\infty)$ with finite expectation m_μ, set*

$$\nu_\mu(\mathrm{d}t) = m_\mu^{-1}\mu\big((t,\infty)\big)\,\mathrm{d}t,$$

which is also a probability measure on $(0,\infty)$. Let T_1 be a ν_μ-distributed random variable and let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. random variables with law μ, independent of T_1. Set

$$T_{k+1} = T_k + X_k \text{ for all } k \geq 1 \quad \text{and} \quad N_t = \sum_{k\geq 1} \mathbb{1}_{\{T_k \leq t\}} \text{ for all } t \geq 0.$$

We say that $(N_t)_{t\geq 0}$ is a stationary renewal process with parameter μ, *or a* SR(μ)-*process in short.*

It is well-known, see e.g. [**21**, Corollaire 6.19, p. 169], that for $(N_t)_{t\geq 0}$ a SR(μ)-process in the sense of Definition 2.1.1, the law of $T_{N_t+1} - t$ (i.e. the time we have to wait for the next mark at time t) is ν_μ for all $t \geq 0$. Another possible definition is the following.

Definition 2.1.2. — *For μ a probability measure on $(0,\infty)$ with finite expectation m_μ, set*

$$\nu_\mu(\mathrm{d}t) = m_\mu^{-1}\mu\big((t,\infty)\big)\,\mathrm{d}t \quad \text{and} \quad \zeta_\mu(\mathrm{d}t) = m_\mu^{-1} t\mu(\mathrm{d}t),$$

which are also probability measures on $(0,\infty)$. Consider a collection of random variables $(X_i)_{i\in\mathbb{Z}\setminus\{0\}}$ with law μ. Consider also X_0 with law ζ_μ and U uniformly distributed on $[0,1]$. Assume that all these random variables are independent. Define

$$T_0 = -(1-U)X_0, \quad T_1 = UX_0$$

and then, for $n \geq 1$,

$$T_{n+1} = T_n + X_n \quad \textit{and} \quad T_{-n} = T_{-(n-1)} - X_{-n}.$$

Then we say that $(T_n)_{n\in\mathbb{Z}}$ *is a* $\mathrm{SR}(\mu)$*-process.*

If $(T_n)_{n\in\mathbb{Z}}$ is a $\mathrm{SR}(\mu)$-process in the sense of Definition 2.1.2 and if one considers the associated counting process $N_t = \sum_{n\geq 1} \mathbb{1}_{\{T_n\leq t\}}$, it is indeed a $\mathrm{SR}(\mu)$-process in the sense of Definition 2.1.1. This can be checked immediately: it suffices to observe that the law of T_1 is ν_μ.

If we have a $\mathrm{SR}(\mu)$-process $(N_t)_{t\geq 0}$ as in Definition 2.1.1 and if we denote by $(T_n)_{n\geq 1}$ its successive instants of jump, one can easily build $(T_n)_{n\leq 0}$ in such a way that $(T_n)_{n\in\mathbb{Z}}$ is a $\mathrm{SR}(\mu)$-process as in Definition 2.1.2.

For $(T_n)_{n\in\mathbb{Z}}$ a $\mathrm{SR}(\mu)$-process as in Definition 2.1.2, for any $t \in \mathbb{R}$, the random sets

$$\bigcup_{n\in\mathbb{Z}} \{T_n\}, \quad \bigcup_{n\in\mathbb{Z}} \{-T_n\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} \{T_n + t\}$$

have the same law. Thus if we introduce n_t such that $T_{n_t} + t < 0 < T_{n_t+1} + t$, the process $(T_{n_t+n} + t)_{n\in\mathbb{Z}}$ is a $\mathrm{SR}(\mu)$-process. By the same way, the process $(-T_{1-n})_{n\in\mathbb{Z}}$ is a $\mathrm{SR}(\mu)$-process.

2.1.2. The discrete model. — Next, we introduce the forest fire model. For $a, b \in \mathbb{Z}$ with $a \leq b$, we set

$$[\![a, b]\!] = \{a, \dots, b\} \subset \mathbb{Z}.$$

For $\eta \in \{0,1\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we define the occupied connected component around i as

$$C(\eta, i) = \begin{cases} \varnothing & \text{if } \ \eta(i) = 0, \\ [\![\ell(\eta, i), r(\eta, i)]\!] & \text{if } \ \eta(i) = 1, \end{cases}$$

where $\ell(\eta, i) = \sup\{k < i : \ \eta(k) = 0\} + 1$ and $r(\eta, i) = \inf\{k > i : \ \eta(k) = 0\} - 1$.

Definition 2.1.3. — *Let* μ_S *and* μ_M *be two laws on* $(0, \infty)$ *with finite expectations. For each* $i \in \mathbb{Z}$, *we consider a* $\mathrm{SR}(\mu_S)$*-process* $(N_t^S(i))_{t\geq 0}$ *and a* $\mathrm{SR}(\mu_M)$*-process* $(N_t^M(i))_{t\geq 0}$, *all these processes being independent. A* $\{0,1\}$*-valued process* $(\eta_t(i))_{i\in\mathbb{Z},t\geq 0}$ *such that* $(\eta_t(i))_{t\geq 0}$ *is a.s. càdlàg for all* $i \in \mathbb{Z}$ *is said to be a* $\mathrm{FF}(\mu_S, \mu_M)$*-process if a.s., for all* $t \geq 0$, *all* $i \in \mathbb{Z}$,

$$\eta_t(i) = \int_0^t \mathbb{1}_{\{\eta_{s-}(i)=0\}} \, \mathrm{d}N_s^S(i) - \sum_{j\in\mathbb{Z}} \int_0^t \mathbb{1}_{\{j\in C(\eta_{s-}, i)\}} \, \mathrm{d}N_s^M(j).$$

Formally, we say that $\eta_t(i) = 0$ if there is no tree at site i at time t and $\eta_t(i) = 1$ else. Thus the forest fire process starts from an empty initial configuration, seeds fall according to i.i.d. $\mathrm{SR}(\mu_S)$-processes and matches fall according to i.i.d. $\mathrm{SR}(\mu_M)$-processes. When a seed falls on an empty site, a tree appears immediately. When a match falls on an occupied site, it burns immediately the corresponding connected

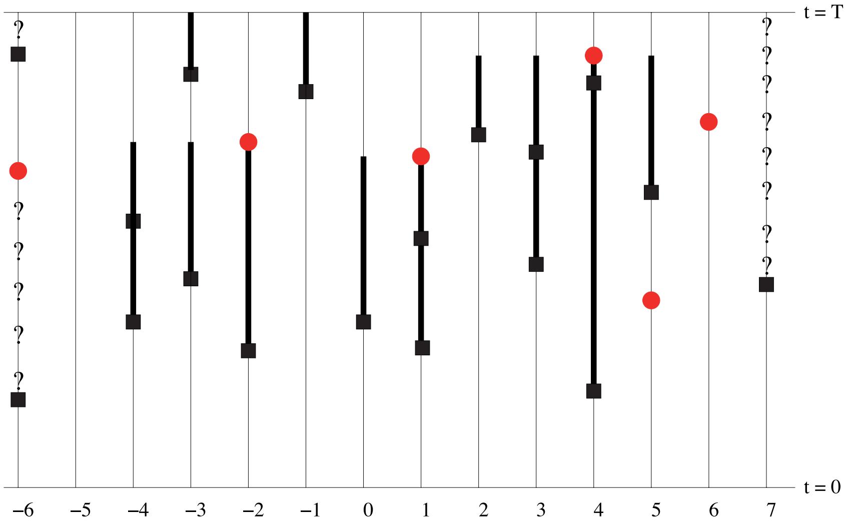

FIGURE 1. Graphical construction of the $\mathrm{FF}(\mu_S, \mu_M)$-process. Matches are represented as bullets and seeds as squares. On the sites -5 and 6, no seed fall during $[0, T]$, so that these sites remain vacant until T. One can thus clearly deduce the values of the process in $[\![-5, 6]\!]$ during $[0, T]$ using only the bullets and squares inside $[\![-5, 6]\!]$.

component of occupied sites. Seeds falling on occupied sites and matches falling on vacant sites have no effect.

Assume for a moment that the support of μ_S is unbounded (thus so is that of ν_{μ_S}). Then the $\mathrm{FF}(\mu_S, \mu_M)$-process can be shown to exist and to be unique (for almost every realization of $(N_t^S(i), N_t^M(i))_{i \in \mathbb{Z}, t \geq 0}$), by using a genuine *graphical construction*. Indeed, to build the process until a given time $T > 0$, it suffices to work between sites i which are vacant until time T (because $N_T^S(i) = 0$). Interaction cannot cross such sites. Since such sites are a.s. infinitely many (because $\Pr[N_T^S(i) = 0] = \nu_{\mu_S}((T, \infty)) > 0$ by assumption), this allows us to handle a graphical construction. This is illustrated by Figure 1. See Liggett [**45**] for many examples of graphical constructions.

We will also study the more complicated case where μ_S has a bounded support and this will lead to the following general result.

PROPOSITION 2.1.4. — *Let μ_S and μ_M be two laws on $(0, \infty)$ with some finite expectations. For each $i \in \mathbb{Z}$, we consider a $\mathrm{SR}(\mu_S)$-process $(N_t^S(i))_{t \geq 0}$ and a $\mathrm{SR}(\mu_M)$-process $(N_t^M(i))_{t \geq 0}$, all these processes being independent. Almost surely, there exists a unique $\mathrm{FF}(\mu_S, \mu_M)$-process.*

This proposition is proved in section 3.1.

2.1.3. Assumptions. — We now state the assumptions we will impose on the laws μ_S and μ_M. First, we want to express the fact that matches are less and less frequent. To do so, we consider a family of laws μ_M^λ, for $\lambda \in (0,1]$, as follows.

$$(H_M)\quad \begin{cases} \text{For each } \lambda \in (0,1],\ \mu_M^\lambda \text{ is the image measure of } \mu_M^1 \text{ by the map } t \mapsto t/\lambda \text{ and the probability measure } \mu_M^1 \text{ on } (0,\infty) \text{ satisfies } \int_0^\infty t\mu_M^1(\mathrm{d}t) = 1. \text{ We set} \\ \nu_M^\lambda(\mathrm{d}t) = \nu_{\mu_M^\lambda}(\mathrm{d}t) = \lambda\mu_M^\lambda((t,\infty))\,\mathrm{d}t = \lambda\mu_M^1((\lambda t,\infty))\,\mathrm{d}t. \end{cases}$$

The idea we have in mind is that we slow down matches: for $(N_t^M)_{t\geq 0}$ a SR(μ_M^1)-process, $(N_{\lambda t}^M)_{t\geq 0}$ is a SR(μ_M^λ)-process.

Assume that $\int_0^\infty t\mu_M^1(\mathrm{d}t) = \kappa \in (0,\infty)$. Then $\widetilde{\mu}_M^\lambda = \mu_M^{\kappa\lambda}$ satisfies (H_M). We thus may of course assume that $\kappa = 1$ without loss of generality.

Next, we put some conditions about μ_S.

$$(H_S)\quad \begin{cases} \text{The probability measure } \mu_S \text{ on } (0,\infty) \text{ has a finite mean } m_S = \int_0^\infty t\mu_S(\mathrm{d}t). \text{ We set} \\ \qquad \nu_S(\mathrm{d}t) = \nu_{\mu_S}(\mathrm{d}t) = m_S^{-1}\mu_S((t,\infty))\,\mathrm{d}t. \\ \text{Either } \mu_S \text{ has a bounded support or } \mu_S \text{ has an unbounded support and} \\ \qquad \forall\, t>0,\quad \lim_{x\to\infty} \dfrac{\nu_S((x,\infty))}{\nu_S((tx,\infty))} \in [0,\infty)\cup\{\infty\} \quad \text{exists.} \end{cases}$$

Surprisingly, we will consider these assumptions in full generality: no supplementary technical condition is needed. In the whole paper, we admit the following convention:

$$t^\infty = \begin{cases} 0 & \text{if } t\in(0,1) \\ 1 & \text{if } t = 1 \\ \infty & \text{if } t \in (1,\infty). \end{cases}$$

As proved in Lemma 5.1.1, (H_S) implies either

$$(H_S(BS))\quad \begin{cases} \text{The probability measure } \mu_S \text{ on } (0,\infty) \text{ has a bounded support. We denote by } m_S \text{ the expectation of } \mu_S \text{ and define } T_S = \max\ \mathrm{supp}\,\mu_S \text{ and } \nu_S(\mathrm{d}t) = m_S^{-1}\mu_S((t,\infty))\,\mathrm{d}t. \text{ Observe that } \mathrm{supp}\,\nu_S = [0,T_S]. \end{cases}$$

or, for some $\beta \in [0,\infty)\cup\{\infty\}$,

$$(H_S(\beta))\quad \begin{cases} \text{The probability measure } \mu_S \text{ on } (0,\infty) \text{ has an unbounded support, a finite mean } m_S \text{ and for } \nu_S(\mathrm{d}t) = m_S^{-1}\mu_S((t,\infty))\,\mathrm{d}t, \\ \qquad \forall\, t>0,\quad \lim_{x\to\infty} \dfrac{\nu_S((x,\infty))}{\nu_S((tx,\infty))} = t^\beta. \end{cases}$$

We finally introduce the following notation.

NOTATION 2.1.5. — (i) *Assume* $(H_S(\beta))$ *for some* $\beta \in [0,\infty)$*. We denote by* ϕ_S *the inverse function of* $t \mapsto t/\nu_S((t,\infty))$*. Note that* $\phi_S : (0,\infty) \mapsto (0,\infty)$ *is an increasing continuous bijection.*

(ii) *Assume* $(H_S(\infty))$*. We denote by* ϕ_S *the inverse function of* $t \mapsto t/\nu_S((t,\infty))$ *and by* ψ_S *the inverse function of* $t \mapsto \nu_S((0,t))$*. The functions* $\phi_S : (0,\infty) \mapsto (0,\infty)$ *and* $\psi_S : (0,1) \mapsto (0,\infty)$ *are increasing bijections.*

(iii) *Assume* $(H_S(BS))$*. We denote by* ψ_S *the inverse function of* $t \mapsto \nu_S((0,t))$*. The function* $\psi_S : (0,1) \mapsto (0,T_S)$ *is an increasing continuous bijection.*

2.1.4. Examples. — Concerning (H_M), the situation is clear. The Poisson case studied in [**15**] corresponds to $\mu_M^1(\mathrm{d}t) = \mathrm{e}^{-t}\mathbb{1}_{\{t>0\}}\,\mathrm{d}t$, whence

$$\mu_M^\lambda(\mathrm{d}t) = \nu_M^\lambda(\mathrm{d}t) = \lambda\mathrm{e}^{-\lambda t}\mathbb{1}_{\{t>0\}}\,\mathrm{d}t.$$

We study here a much more general case. However, this is not the main point of the paper, since it will not generate some very interesting behaviors. Concerning (H_S), we present here four classes of examples, that will lead to different behaviors.

Example 1. — If $\mu_S = \delta_{T_S}$, whence $\nu_S(\mathrm{d}t) = T_S^{-1}\mathbb{1}_{[0,T_S]}(t)\,\mathrm{d}t$, then $(H_S(BS))$ holds and $\psi_S(z) = T_S z$.

Example 2. — Assume that $\mu_S((t,\infty)) \overset{\infty}{\approx} \mathrm{e}^{-t^\alpha}$ for some $\alpha > 0$, so that $\nu_S((t,\infty)) \overset{\infty}{\approx} ct^{1-\alpha}\mathrm{e}^{-t^\alpha}$. Then $(H_S(\infty))$ holds. Furthermore, $\phi_S(z) \overset{\infty}{\approx} (\log z)^{1/\alpha}$ and $\psi_S(z) \overset{1}{\sim} [\log(1/(1-z))]^{1/\alpha}$.

Example 3. — Assume that $\mu_S((t,\infty)) \overset{\infty}{\approx} t^{-1-\beta}$ for some $\beta > 0$, whence $\nu_S((t,\infty)) \overset{\infty}{\approx} ct^{-\beta}$. Then $(H_S(\beta))$ holds and $\phi_S(z) \overset{\infty}{\approx} (cz)^{1/(\beta+1)}$.

Example 4. — If $\mu_S((t,\infty)) \overset{\infty}{\approx} t^{-1}(\log t)^{-1-\gamma}$ for some $\gamma > 0$, then $\nu_S((t,\infty)) \overset{\infty}{\approx} c(\log t)^{-\gamma}$, so that $(H_S(0))$ is satisfied and $\phi_S(z) \overset{\infty}{\approx} cz(\log z)^{-\gamma}$.

The Poisson case treated in [**15**], which corresponds to the case where $\mu_S((t,\infty)) = \mathrm{e}^{-t} = \nu_S((t,\infty))$, is thus included in Example 2. Example 1 might seem slightly strange from the modelling point of view, but it can happen e.g. if seeds are thrown by a machine.

Observe that (H_S) is not very restrictive, since it is satisfied by all reasonable laws. Anyway, our results (not only the proofs) clearly break down without such an assumption.

It is not so easy to build a law μ_S not meeting (H_S), because $t \mapsto \nu_S((t,\infty))$ is automatically quite smooth (Lipschitz continuous, decreasing and convex). One can however verify that (H_S) is not holding for

$$\mu_S(\mathrm{d}t) = \mathbb{1}_{\{t>0\}}[20 - 3\cos\log(1+t) + \sin\log(1+t)]/[9(1+t)^3)]\,\mathrm{d}t,$$

for which $\nu_S((t,\infty)) = [10 + \sin\log(1+t)]/[10(1+t)]$. One easily checks that $\nu_S((x,\infty))/\nu_S((x\,\mathrm{e}^{\pi/2},\infty))$ has no limit as $x \to \infty$, choosing e.g. the sequences $x_n = \mathrm{e}^{2n\pi}$ and $x_n = \mathrm{e}^{2n\pi+\pi/2}$.

2.1.5. Notation. — We denote:

- ▷ for $I \subset \mathbb{Z}$, by $|I| = \#I$ the number of elements in I;
- ▷ for $I = [\![a, b]\!] = \{a, \dots, b\} \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I := [\alpha a, \alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we of course take the convention that $\alpha\varnothing = \varnothing$;
- ▷ for $J = [a, b]$ an interval of $\mathbb{R}$, $|J| = b - a$ stands for the length of J and for $\alpha > 0$, we set $\alpha J = [\alpha a, \alpha b]$;
- ▷ for $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x;
- ▷ we denote by $\mathcal{I} = \{[a, b], a \le b\}$ the set of all closed finite intervals of $\mathbb{R}$;
- ▷ for two intervals $[a, b]$ and $[c, d]$, we set
$$\boldsymbol{\delta}\big([a, b], [c, d]\big) = |a - c| + |b - d|, \quad \boldsymbol{\delta}\big([a, b], \varnothing\big) = |b - a|;$$
- ▷ for two functions $I, J : [0, T] \to \mathcal{I} \cup \{\varnothing\}$, we set
$$\boldsymbol{\delta}_T(I, J) = \int_0^T \boldsymbol{\delta}(I_t, J_t)\,\mathrm{d}t;$$
- ▷ for $(x, I), (y, J)$ in $\mathbb{D}([0, T], \mathbb{R}_+ \times \mathcal{I} \cup \{\varnothing\})$, the set of càdlàg functions from $[0, T]$ into $\mathbb{R}_+ \times \mathcal{I} \cup \{\varnothing\}$, we define
$$\boldsymbol{d}_T\big((x, I), (y, J)\big) = \sup_{t\in[0,T]} \big|x(t) - y(t)\big| + \boldsymbol{\delta}_T(I, J).$$

2.2. Heuristic scales and relevant quantities

For μ_S, μ_M^λ satisfying (H_S) and (H_M), we consider the FF(μ_S, μ_M^λ)-process $(\eta_t^\lambda(i))_{t\ge 0, i\in\mathbb{Z}}$. We look for some time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large just before they burn. We thus also have to rescale space.

Time scale. — For $\lambda > 0$ very small and for t not too large, one might neglect fires, so that roughly, each site is vacant with probability $\nu_S((t, \infty))$. Indeed, the time we have to wait for the first seed follows, on each site, the law ν_S. Thus
$$C(\eta_t^\lambda, 0) \simeq [\![-X, Y]\!],$$
where X, Y are geometric random variables with parameter $\nu_S((t, \infty))$. Consequently, for t not too large,
$$\big|C(\eta_t^\lambda, 0)\big| \simeq 1/\nu_S((t, \infty)).$$
Under $(H_S(BS))$, $|C(\eta_t^\lambda, 0)|$ becomes infinite at time T_S, so there is no really need to accelerate time: we are sure that $|C(\eta_t^\lambda, 0)|$ will be involved in a fire before T_S. We will accelerate time by a factor T_S (in some sense, this allows us to assume that $T_S = 1$).

Next we assume $(H_S(\beta))$ for some $\beta \in [0, \infty) \cup \{\infty\}$. We observe that thanks to (H_M),
$$\nu_M^\lambda((t, \infty)) \simeq 1 - \lambda \int_0^t \mu_M^1\big((\lambda s, \infty)\big)\,\mathrm{d}s \simeq 1 - \lambda t.$$

Hence the probability that at least one match falls in the cluster $C(\eta^\lambda, 0)$ during $[0,t]$ is roughly similar, under (H_M), to

$$1-\big(\nu_M^\lambda\big((t,\infty)\big)\big)^{|C(\eta_t^\lambda,0)|}\simeq \lambda t\,\big|C(\eta_t^\lambda,0)\big|\simeq \lambda t/\nu_S((t,\infty)).$$

We decide to accelerate time by a factor $\boldsymbol{a}_\lambda$, where $\boldsymbol{a}_\lambda$ solves $\lambda\boldsymbol{a}_\lambda=\nu_S((\boldsymbol{a}_\lambda,\infty))$. By this way, the probability that a match falls in $C(\eta^\lambda,0)$ during $[0,\boldsymbol{a}_\lambda]$ should tend to some nontrivial value.

To summarize, we have set, recalling notation 2.1.5 for the definition of ϕ_S,

$$\begin{cases}\text{under } (H_S(BS)), \quad \boldsymbol{a}_\lambda=T_S,\\ \text{under } (H_S(\beta)) \text{ with } \beta\in[0,\infty)\cup\{\infty\}, \quad \boldsymbol{a}_\lambda=\phi_S(1/\lambda),\\ \text{which solves } \lambda\boldsymbol{a}_\lambda=\nu_S((\boldsymbol{a}_\lambda,\infty)).\end{cases}\tag{2.2.1}$$

Under $(H_S(\beta))$ for some $\beta\in[0,\infty)\cup\{\infty\}$, one easily checks that

$$\lim_{\lambda\to0}\boldsymbol{a}_\lambda=\infty \quad\text{and thus}\quad \lim_{\lambda\to0}\lambda\boldsymbol{a}_\lambda=\lim_{\lambda\to0}\nu_S\big((\boldsymbol{a}_\lambda,\infty)\big)=0.$$

Space scale. — Now we rescale space in such a way that during a time interval with length of order $\boldsymbol{a}_\lambda$, something like one fire starts per unit of (space) length. Since on each site, the probability that (at least) one match falls during $[0,\boldsymbol{a}_\lambda]$ equals

$$\nu_M\big((0,\boldsymbol{a}_\lambda)\big)=\lambda\int_0^{\boldsymbol{a}_\lambda}\mu_M^1\big((\lambda t,\infty)\big)\,\mathrm{d}t\simeq\lambda\boldsymbol{a}_\lambda,$$

our space scale has to be of order

$$\boldsymbol{n}_\lambda=\big\lfloor 1/(\lambda\boldsymbol{a}_\lambda)\big\rfloor.\tag{2.2.2}$$

This means that we will identify $[\![0,\boldsymbol{n}_\lambda]\!]\subset\mathbb{Z}$ with $[0,1]\subset\mathbb{R}$. We always have

$$\lim_{\lambda\to0}\boldsymbol{n}_\lambda=\infty.$$

Rescaled clusters. — We thus set, for $\lambda\in(0,1)$, $t\geq0$ and $x\in\mathbb{R}$, recalling subsection 2.1.5,

$$D_t^\lambda(x):=\frac{1}{\boldsymbol{n}_\lambda}C\big(\eta^\lambda_{\boldsymbol{a}_\lambda t},\lfloor\boldsymbol{n}_\lambda x\rfloor\big)\subset\mathbb{R}.\tag{2.2.3}$$

Using the computation handled in paragraph *Time scale*, we see that roughly, when neglecting fires,

$$\big|D_t^\lambda(x)\big|\simeq\frac{1}{\boldsymbol{n}_\lambda\nu_S((\boldsymbol{a}_\lambda t,\infty))}\simeq\frac{\lambda\boldsymbol{a}_\lambda}{\nu_S((\boldsymbol{a}_\lambda t,\infty))}.$$

Under $(H_S(\beta))$ for some $\beta\in[0,\infty)\cup\{\infty\}$, one gets

$$\big|D_t^\lambda(x)\big|\simeq\frac{\nu_S((\boldsymbol{a}_\lambda,\infty))}{\nu_S((\boldsymbol{a}_\lambda t,\infty))}\simeq t^\beta.$$

Under $(H_S(BS))$, we obtain roughly (assume that $t\neq1$)

$$\big|D_t^\lambda(x)\big|\simeq t^\infty.$$

Indeed, $\nu_S((\boldsymbol{a}_\lambda t,\infty))=\nu_S((T_S t,\infty))$ does not depend on λ and is positive if and only if $t<1$.

Case $\beta\in[0,\infty)$. — In this case, everything is fine: for all times of order $\boldsymbol{a}_\lambda t$, the good space scale is indeed $\boldsymbol{n}_\lambda$. Thus we will describe the $\mathrm{FF}(\mu_S,\mu_M^\lambda)$-process through $(D_t^\lambda(x))_{x\in\mathbb{R},t\geq 0}$.

Case $\beta\in\{\infty,BS\}$. — Then we have a difficulty as in [**15**]: the previous estimate (neglecting fires) suggests that for all $x\in\mathbb{R}$, for $t<1$, $|D_t^\lambda(x)|\to 0$ and for $t>1$, $|D_t^\lambda(x)|\to\infty$. For $t>1$, fires might be in effect and we hope that this will make finite the possible limit of $|D_t^\lambda(x)|$. But fires can only reduce the size of clusters, so that for $t<1$, the limit of $|D_t^\lambda(x)|$ will really be 0.

Since we would like to have an idea of the sizes of microscopic clusters, we have to keep some information about the *degree of smallness* of microscopic clusters. We adopt a different strategy than in [**15**], which is more adapted to the case where $\beta=BS$ and which leads us to a slightly more direct proof (even in the Poisson case).

We consider a function $\boldsymbol{m}_\lambda:(0,1]\mapsto\mathbb{N}$ satisfying

$$(2.2.4)\qquad \begin{cases}\lim_{\lambda\to 0}\boldsymbol{m}_\lambda=\infty,\ \lim_{\lambda\to 0}(\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda)=0,\\ \lambda\mapsto\boldsymbol{m}_\lambda \text{ is non-increasing}\\ \text{and additionally, under } (H_S(\infty)),\\ \forall z\in[0,1),\quad \lim_{\lambda\to 0}\boldsymbol{m}_\lambda\nu_S((\boldsymbol{a}_\lambda z,\infty))=\infty.\end{cases}$$

Such a function exists: under $(H_S(\infty))$, see Lemma 5.1.2 and under $(H_S(BS))$, choose for example $\boldsymbol{m}_\lambda=\lfloor\sqrt{1/\lambda}\rfloor$.

Of course, there is no uniqueness of $\boldsymbol{m}_\lambda$, but that does not matter: the only thing we need is that the scale $\boldsymbol{m}_\lambda$ is smaller than the macroscopic scale $\boldsymbol{n}_\lambda\simeq 1/\nu_S((\boldsymbol{a}_\lambda,\infty))$ and larger than all the microscopic scales $1/\nu_S((\boldsymbol{a}_\lambda z,\infty))$ (for all $z\in(0,1)$). Since only these scales will appear to be relevant, any choice of such a function $\boldsymbol{m}_\lambda$ will be suitable.

We introduce, for $\lambda>0$, $x\in\mathbb{R}$, $t>0$, recall subsection 2.1.5 and that by notation 2.1.5, ψ_s is the inverse of $t\mapsto\nu_S((0,t))$,

$$(2.2.5)\qquad \begin{cases} K_t^\lambda(x) := \dfrac{\big|\{i\in[\![\lfloor\boldsymbol{n}_\lambda x\rfloor-\boldsymbol{m}_\lambda,\lfloor\boldsymbol{n}_\lambda x\rfloor+\boldsymbol{m}_\lambda]\!]\ :\ \eta^\lambda_{\boldsymbol{a}_\lambda t}(i)=1\}\big|}{2\boldsymbol{m}_\lambda+1}\in[0,1],\\[2ex] Z_t^\lambda(x) := \dfrac{\psi_S(K_t^\lambda(x))}{\boldsymbol{a}_\lambda}\wedge 1\in[0,1].\end{cases}$$

Observe that $K_t^\lambda(x)$ stands for the *local density of occupied sites* around $\lfloor\boldsymbol{n}_\lambda x\rfloor$ at time $\boldsymbol{a}_\lambda t$. This density is *local* because $\boldsymbol{m}_\lambda\ll\boldsymbol{n}_\lambda$. We hope that for $t<1$, neglecting fires, $K_t^\lambda(x)\simeq\nu_S((0,\boldsymbol{a}_\lambda t))$, whence $Z_t^\lambda(x)\simeq t$.

The quantity $Z_t^\lambda(x)$ has no physical interpretation. We use it to transform the local density $K_t^\lambda(x)$ (which depends on t in a complicated way involving ν_S) in a quantity of which the behavior does not depend too much on ν_S (at least for $t<1$

and neglecting fires). This will allow us to describe the limit process in an unified way (not depending on ν_S).

For all $\lambda > 0$ small enough (we need that $2\boldsymbol{m}_\lambda + 1 < \boldsymbol{n}_\lambda$), we have $Z_t^\lambda(x) = 1$ if and only if $K_t^\lambda(x) = 1$, i.e. if and only if all the sites are occupied around $\lfloor \boldsymbol{n}_\lambda x \rfloor$. Indeed, under $(H_S(BS))$, $Z_t^\lambda(x) = 1$ implies that $\psi_S(K_t^\lambda(x)) = T_S$, so that

$$K_t^\lambda(x) = \nu_S\big((0, T_S)\big) = 1.$$

Under $(H_S(\infty))$, $Z_t^\lambda(x) = 1$ implies that $\psi_S(K_t^\lambda(x)) \geq \boldsymbol{a}_\lambda$, so that

$$K_t^\lambda(x) \geq \nu_S\big((0, \boldsymbol{a}_\lambda)\big) = 1 - \nu_S\big((\boldsymbol{a}_\lambda, \infty)\big) = 1 - \lambda \boldsymbol{a}_\lambda \geq 1 - 1/\boldsymbol{n}_\lambda,$$

whence $K_t^\lambda(x) = 1$. This last assertion comes from the facts that $K_t^\lambda(x)$ takes its values in $\{k/(2\boldsymbol{m}_\lambda + 1) : k \in \{0, \dots, 2\boldsymbol{m}_\lambda + 1\}$ and that $2\boldsymbol{m}_\lambda + 1 < \boldsymbol{n}_\lambda$.

Since the scale $\boldsymbol{m}_\lambda$ is larger than all the microscopic scales, $Z_t^\lambda(x) = 1$ will imply, roughly, that the cluster containing $\lfloor \boldsymbol{n}_\lambda x \rfloor$ is macroscopic, i.e. has a length of order $\boldsymbol{n}_\lambda$.

We will study the FF(μ_S, μ_M^λ)-process through $(D_t^\lambda(x), Z_t^\lambda(x))_{x \in \mathbb{R}, t \geq 0}$. The main idea is that for $\lambda > 0$ very small:

▷ If $Z_t^\lambda(x) = z \in (0, 1)$, then $|D_t^\lambda(x)| \simeq 0$ and the (rescaled) cluster containing x is microscopic (in the sense that the non-rescaled cluster is small when compared to $\boldsymbol{n}_\lambda$), but we control the local density of occupied sites around x, which resembles $\nu_S((0, \boldsymbol{a}_\lambda z))$. Observe that this density tends to 1 as $\lambda \to 0$ for all $z \in (0, 1)$ under $(H_S(\infty))$, while it remains bounded as $\lambda \to 0$ for all $z \in (0, 1)$ under $(H_S(BS))$.

▷ If $Z_t^\lambda(x) = 1$ and $D_t^\lambda(x) = [a, b]$, then the (rescaled) cluster containing x is macroscopic and has a length equal to $b - a$, or

$$\big|C(\eta_{\boldsymbol{a}_\lambda t}^\lambda, \lfloor \boldsymbol{n}_\lambda x \rfloor)\big| \simeq (b - a)\boldsymbol{n}_\lambda$$

in the original scales.

Summary. — Assume $(H_S(\beta))$ for some $\beta \in [0, \infty) \cup \{\infty, BS\}$.

- ▷ We accelerate time by the factor $\boldsymbol{a}_\lambda$, defined by $\lambda \boldsymbol{a}_\lambda = \nu_S((\lambda \boldsymbol{a}_\lambda, \infty))$ if β belongs to $[0, \infty) \cup \{\infty\}$ and by $\boldsymbol{a}_\lambda = T_S$ if $\beta = BS$.
- ▷ Our space scale is $\boldsymbol{n}_\lambda = \lfloor 1/(\lambda \boldsymbol{a}_\lambda) \rfloor$.
- ▷ If $\beta \in [0, \infty)$, we will only study the rescaled clusters $(D_t^\lambda(x))_{t \geq 0, x \in \mathbb{R}}$, see (2.2.3).
- ▷ If $\beta \in \{\infty, BS\}$, we will study the rescaled clusters $(D_t^\lambda(x))_{t \geq 0, x \in \mathbb{R}}$, as well as the local densities of occupied sites $(Z_t^\lambda(x))_{t \geq 0, x \in \mathbb{R}}$, see (2.2.4)–(2.2.5).

2.3. Main result in the case $\beta = \infty$

2.3.1. Definition of the limit process. — We describe the limit process in the case where $\beta = \infty$. It is exactly the same process as in the Poisson case studied in [**15**]. We consider a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x)$ on $[0, \infty) \times \mathbb{R}$, with intensity measure $\mathrm{d}t\,\mathrm{d}x$, whose marks correspond to matches.

Before stating a precise definition, let us describe briefly the limit process. Initially, all the sites are vacant. Matches fall according to π_M. All the zones remain microscopic (meaning roughly that vacant sites are dense in $\mathbb{R}$) until time 1. When a match falls at some time $t \in (0,1)$ at some place $x \in \mathbb{R}$, it destroys a microscopic zone, that will be filled again after a delay t (at time $2t$). Hence there is a *barrier* at x during $(t, 2t)$. At time 1, all the sites become occupied, except sites where there is an active barrier. Hence if a fire falls, just after time 1, it destroys a macroscopic zone, delimited by some active barriers. Such a destroyed macroscopic zone will need a delay 1 to be completely filled again. During this delay, matches produce again some barriers. And so on. See Figure 2 next page for an illustration.

The precise definition of the limit process is as follows.

Definition 2.3.1. — *A process* $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ *with values in* $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ *such that a.s., for all* $x \in \mathbb{R}$, $(Z_t(x), H_t(x))_{t\geq 0}$ *is càdlàg, is said to be a* LFF(∞)*-process if a.s., for all* $t \geq 0$, *all* $x \in \mathbb{R}$,

$$\left\{\begin{aligned} Z_t(x) &= \int_0^t \mathbb{1}_{\{Z_s(x)<1\}}\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}}\pi_M(\mathrm{d}s,\mathrm{d}y),\\ H_t(x) &= \int_0^t Z_{s-}(x)\mathbb{1}_{\{Z_{s-}(x)<1\}}\pi_M(\mathrm{d}s\times\{x\}) - \int_0^t \mathbb{1}_{\{H_s(x)>0\}}\,\mathrm{d}s,\end{aligned}\right. \tag{2.3.1}$$

where $D_t(x) = [L_t(x), R_t(x)]$, *with*

$$L_t(x) = \sup\{y \leq x :\ Z_t(y) < 1 \text{ or } H_t(y) > 0\},$$
$$R_t(x) = \inf\{y \geq x :\ Z_t(y) < 1 \text{ or } H_t(y) > 0\}$$

and where $D_{t-}(x)$ *is defined in the same way.*

2.3.2. Formal dynamics. — Let us explain the dynamics of this process. We consider $T > 0$ fixed and set $\mathcal{A}_T = \{x \in \mathbb{R} :\ \pi_M([0,T]\times\{x\}) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster is *microscopic* if $D_t(x) = \{x\}$. We have $D_t(x) = D_t(y)$ for all $y \in D_t(x)$.

1. *Initial condition.* — We have $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.

2. *Occupation of vacant zones.* — We consider here $x \in \mathbb{R} \setminus \mathcal{A}_T$. Then we have $H_t(x) = 0$ for all $t \in [0,T]$. When $Z_t(x) < 1$, then $D_t(x) = \{x\}$ and $Z_t(x)$ stands for the *local density of occupied sites* around x (or rather for a suitable function of this local density). Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the RHS of the first equation in (2.3.1). When $Z_t(x) = 1$, the cluster containing x is macroscopic and is described by $D_t(x)$.

3. *Microscopic fires.* — Here we assume that $x \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x) < 1$. In such a case, the cluster containing x is microscopic. Then we set $H_t(x) = Z_{t-}(x)$, as described by the first term on the RHS of the second equation of (2.3.1) and we leave unchanged the value of $Z_t(x)$. We then let $H_t(x)$ decrease linearly until it reaches 0, see the second term

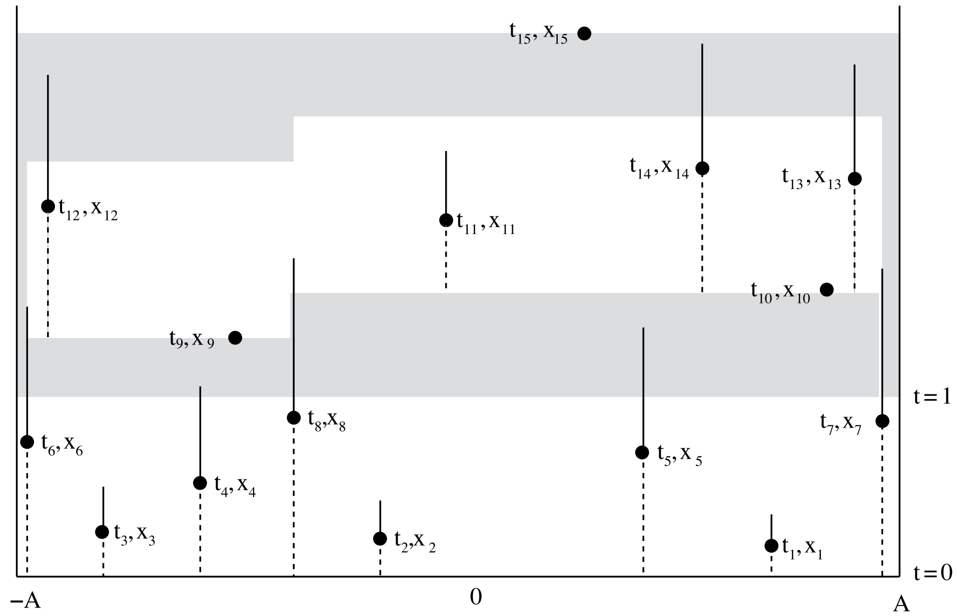

FIGURE 2. LFF(∞)-process in a finite box.

The marks of π_M (matches) are represented as •'s. The filled zones represent zones in which $Z_t^A(x) = 1$ and $H_t^A(x) = 0$, that is macroscopic clusters. The plain vertical segments represent the sites where $H_t^A(x) > 0$. In the rest of the space, we always have $Z_t^A(x) < 1$. Until time 1, all the clusters are microscopic. The 8 first matches fall in that zone. Thus at each of these marks, a process H^A starts and its life-time equals the instant where it has started. For example the segment above (t_1, x_1) ends at time $2t_1$: we draw a dotted segment from $(0, x_1)$ to (t_1, x_1) and then a plain vertical segment above (t_1, x_1) with the same length.

At time 1, all the clusters where there has been no mark become macroscopic and merge together. But this is limited by vertical segments. Here we have at time 1 the clusters $[-A, x_6]$, $[x_6, x_4]$, $[x_4, x_8]$, $[x_8, x_5]$, $[x_5, x_7]$ and $[x_7, A]$. The segment above (t_4, x_4) ends at time $2t_4$ and thus at this time the clusters $[x_6, x_4]$ and $[x_4, x_8]$ merge into $[x_6, x_8]$. The 9-th mark falls in the (macroscopic) zone $[x_6, x_8]$ and thus destroys it immediately.

This zone $[x_6, x_8]$ will become macroscopic again only at time $t_9 + 1$. A process H^A starts at x_{12} at time t_{12}: we draw a dotted segment from (t_9, x_{12}) to (t_{12}, x_{12}) and then a plain vertical segment above (t_{12}, x_{12}) with the same length ($Z^A_{t_{12}-}(x_{12}) = t_{12} - t_9$ because $Z^A_{t_9}(x_{12})$ has been set to 0). The segment $[x_8, x_7]$ has been destroyed at time t_{10} and thus will remain microscopic until $t_{10} + 1$. As a consequence, the only macroscopic clusters at time $t_9 + 1$ are $[-A, x_{12}]$, $[x_{12}, x_8]$ and $[x_7, A]$. Then the zone $[x_8, x_7]$ becomes macroscopic (but there have been marks at x_{13}, x_{14}), so that at time $t_{10} + 1$, we get the macroscopic clusters $[-A, x_{12}]$, $[x_{12}, x_{14}]$, $[x_{14}, x_{13}]$ and $[x_{13}, A]$. These clusters merge by pairs, at times $2t_{12} - t_9$, $2t_{13} - t_{10}$ and $2t_{14} - t_{10}$, etc.

Here we have $0 \in (x_{11}, x_{15})$ and thus $Z_t^A(0) = t$ for $t \in [0, 1]$, $Z_t^A(0) = 1$ for $t \in [1, t_{10})$, then $Z_t^A(0) = t - t_{10}$ for $t \in [t_{10}, t_{10} + 1)$, then $Z_t^A(0) = 1$ for $t \in [t_{10} + 1, t_{15})$,... We also see that $D_t^A(0) = \{0\}$ for $t \in [0, 1)$, $D_t^A(0) = [x_8, x_5]$ for $t \in [1, 2t_5)$, $D_t^A(0) = [x_8, x_7]$ for $t \in [2t_5, t_{10})$, $D_t^A(0) = \{0\}$ for $t \in [t_{10}, t_{10} + 1)$, $D_t^A(0) = [x_{12}, x_{14}]$ for $t \in [t_{10} + 1, 2t_{12} - t_9)$, $D_t^A(0) = [-A, x_{14}]$ for $t \in [2t_{12} - t_9, 2t_{14} - t_{10})$, ... Of course, $H_t^A(0) = 0$ for all $t \geq 0$, but for example $H_t^A(x_{11}) = 0$ for $t \in [0, t_{11})$, $H_t^A(x_{11}) = 2t_{11} - t_{10} - t$ for $t \in [t_{11}, 2t_{11} - t_{10})$ and then $H_t^A(x_{11}) = 0$ for $t \in [2t_{11} - t_{10}, \infty)$.

on the RHS of the second equation in (2.3.1). At all times where $H_t(x) > 0$, the site x acts like a barrier (see point 5 below).

4. Macroscopic fires. — Here we assume that $x \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x) = 1$. This means that the cluster containing x is macroscopic and thus this mark destroys the whole component $D_{t-}(x)$, that is for all $y \in D_{t-}(x)$, we set $D_t(y) = \{y\}$, $Z_t(y) = 0$. This is described by the second term on the RHS of the first equation in (2.3.1).

5. Clusters. — Finally the definition of the clusters $(D_t(x))_{x\in\mathbb{R}}$ becomes more clear: these clusters are delimited by zones with local density smaller than 1 (i.e. $Z_t(y) < 1$) or by sites where a microscopic fire has (recently) started (i.e. $H_t(y) > 0$).

For $A > 0$, we call $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ the finite box version of the LFF(∞)-process: it has the same dynamics as the true LFF(∞)-process, but we restrict the space of tree positions to $x \in [-A, A]$. See section 3.10 for a more precise definition. On Figure 2, a typical path of this finite box LFF(∞)-process is discussed. See also Algorithm 3.6.3 (with the function $F_S(z, v) = z$).

2.3.3. Well-posedness. — The existence and uniqueness of the LFF(∞)-process has been proved in [**15**, Theorem 3]. We will provide here a simpler proof, which also works for the case where $\beta = BS$.

THEOREM 2.3.2. — *For any Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x)$ on $[0,\infty) \times \mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$, there a.s. exists a unique* LFF(∞)*-process. Furthermore, it can be constructed graphically and its restriction to any finite box $[0,T] \times [-n,n]$ can be perfectly simulated.*

The LFF(∞)-process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ is furthermore Markov, since it solves a well-posed time homogeneous Poisson-driven S.D.E.

2.3.4. The convergence result. — Recall subsection 2.1.5.

THEOREM 2.3.3. — *Assume (H_M) and $(H_S(\infty))$. Recall that $\boldsymbol{a}_\lambda$, $\boldsymbol{n}_\lambda$ and $\boldsymbol{m}_\lambda$ were defined in* (2.2.1) (2.2.2)-(2.2.4). *Consider, for each $\lambda \in (0,1]$, the process $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the* FF(μ_S, μ_M^λ)*-process, see Definition* 2.1.3, (2.2.3) *and* (2.2.5). *Consider also the* LFF(∞)*-process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$.*

(a) *For any $T > 0$, any finite subset $\{x_1, \dots, x_p\} \subset \mathbb{R}$,*

$$\left(Z_t^\lambda(x_i), D_t^\lambda(x_i)\right)_{t\in[0,T], i=1,\dots,p}$$

goes in law to $(Z_t(x_i), D_t(x_i))_{t\in[0,T], i=1,\dots,p}$, in $\mathbb{D}([0,T], \mathbb{R} \times \mathcal{I} \cup \{\varnothing\})^p$, as λ tends to 0. Here $\mathbb{D}([0,\infty), \mathbb{R} \times \mathcal{I} \cup \{\varnothing\})$ is endowed with the distance $\boldsymbol{d}_T$.

(b) *For any finite subset $\{(t_1, x_1), \dots, (t_p, x_p)\} \subset [0,\infty) \times \mathbb{R}$, with $t_k \neq 1$ for $k = 1, \dots, p$, $(Z_{t_i}^\lambda(x_i), D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ goes in law to $(Z_{t_i}(x_i), D_{t_i}(x_i))_{i=1,\dots,p}$ in $(\mathbb{R} \times \mathcal{I} \cup \{\varnothing\})^p$. Here $\mathcal{I} \cup \{\varnothing\}$ is endowed with $\boldsymbol{\delta}$.*

(c) *Recall notation* 2.1.5 (ii). *For all* $t > 0$,

$$\Big(\frac{\psi_S\big(1 - 1/|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)|\big)}{\boldsymbol{a}_\lambda} \mathbb{1}_{\{|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)| \geq 1\}}\Big) \wedge 1$$

goes in law to $Z_t(0)$ *as* $\lambda \to 0$.

Point (c) will allow us to check some estimates on the cluster-size distribution. Since we deal with finite-dimensional marginals in space, it is quite clear that the process H does not appear in the limit, since for each $x \in \mathbb{R}$, a.s., for all $t \geq 0$, $H_t(x) = 0$. (Of course, it is false that a.s., for all $x \in \mathbb{R}$, all $t \geq 0$, $H_t(x) = 0$).

We cannot guarantee the convergence in law of $D^\lambda_t(0)$ to $D_t(0)$ at time $t = 1$. This is due to the fact that when neglecting fires, the probability that a macroscopic zone is completely occupied at time $\boldsymbol{a}_\lambda t$, tends to 1 if $t > 1$, but to a nontrivial value if $t = 1$.

For example, in the absence of fires, a zone with length $\boldsymbol{n}_\lambda$ is completely occupied at time $\boldsymbol{a}_\lambda t$ with probability $\nu_S((0, \boldsymbol{a}_\lambda t))^{\boldsymbol{n}_\lambda} \simeq \exp(-\boldsymbol{n}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty)))$, which tends to 1 if $t > 1$ and to $1/e$ if $t = 1$.

We believe that this is really not important and we decided to keep this definition of the LFF(∞)-process despite this light defect.

2.3.5. Heuristic arguments. — Let us explain here roughly the reasons why Theorem 2.3.3 holds true. We consider, for $\lambda > 0$ very small, a FF(μ_S, μ^λ_M)-process $(\eta^\lambda_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and the associated processes $(Z^\lambda_t(x), D^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$.

0. *Matches.* — The times and positions at which matches fall will tend, in our scales, to the marks of a Poisson measure with intensity measure 1. A hint for this is the following. Consider e.g. the domain $[0, T] \times [0, 1]$, which corresponds to $[0, \boldsymbol{a}_\lambda T] \times [\![0, \boldsymbol{n}_\lambda]\!]$. The probability that two matches fall on the same site during $[0, \boldsymbol{a}_\lambda T]$ is very small. Thus the number of matches falling in $[0, \boldsymbol{a}_\lambda T] \times [\![0, \boldsymbol{n}_\lambda]\!]$ has approximately a Binomial distribution with parameters $\boldsymbol{n}_\lambda$ and $\nu_M([0, \boldsymbol{a}_\lambda T])$. Since

$$\boldsymbol{n}_\lambda \nu^\lambda_M\big([0, \boldsymbol{a}_\lambda T]\big) \simeq \frac{1}{\lambda \boldsymbol{a}_\lambda}\Big[\int_0^{\boldsymbol{a}_\lambda T} \lambda \mu^1_M((\lambda \boldsymbol{a}_\lambda t, \infty))\,\mathrm{d}t\Big] \longrightarrow T$$

as $\lambda \to 0$, the asymptotic number of matches falling in $[0, T] \times [0, 1]$ should have a Poisson distribution with parameter T.

1. Initial condition. — For all $x \in \mathbb{R}$, $(Z^\lambda_0(x), D^\lambda_0(x)) = (0, \varnothing) \simeq (0, \{x\})$ (recall that $\psi_S(0) = 0$).

2. Occupation of vacant zones. — Assume that a zone $[a, b]$ becomes completely vacant at some time t (because it has been destroyed by a fire).

(i) For $s \in [0, 1)$ and if no fire starts on $[a, b]$ during $[t, t + s]$, we have

$$D^\lambda_{t+s}(x) \simeq \big[x \pm 1/(\boldsymbol{n}_\lambda \nu_S(\boldsymbol{a}_\lambda s, \infty))\big] \simeq \{x\}$$

and $Z^\lambda_{t+s}(x) \simeq s$ for all $x \in [a, b]$.

Indeed, $D^\lambda_{t+s}(x) \simeq [x - X/\boldsymbol{n}_\lambda, x + Y/\boldsymbol{n}_\lambda]$, where X and Y are approximately geometric random variables with parameter $\nu_S((\boldsymbol{a}_\lambda s, \infty))$. (Recall that for any $t \geq 0$ and for any site, ν_S is the law of the time we have to wait until the next seed falls). Thus

$$D^\lambda_{t+s}(x) \simeq \big[x \pm 1/(\boldsymbol{n}_\lambda \nu_S((\boldsymbol{a}_\lambda s, \infty))\big] \simeq \{x\}$$

due to $(H_S(\infty))$, since $\nu_S\big((\boldsymbol{a}_\lambda s, \infty)\big) \gg \nu_S\big((\boldsymbol{a}_\lambda, \infty)\big) \simeq 1/\boldsymbol{n}_\lambda$. For the same reasons,

$$K^\lambda_{t+s}(x) \simeq \nu_S\big((0, \boldsymbol{a}_\lambda s)\big),$$

whence $Z^\lambda_{t+s}(x) \simeq s$.

(ii) If no fire starts on $[a, b]$ during $[t, t+1]$, then $Z^\lambda_{t+1}(x) \simeq 1$ and all the sites in $[a, b]$ are occupied (with very high probability) just after time $t+1$.

Indeed, we have $(b-a)\boldsymbol{n}_\lambda$ sites and each of them is occupied at time $t + 1 + \varepsilon$ with approximate probability $\nu_S((0, \boldsymbol{a}_\lambda(1+\varepsilon)])$, so that all of them are occupied with approximate probability

$$\big(\nu_S\big((0, \boldsymbol{a}_\lambda(1+\varepsilon))\big)\big)^{(b-a)\boldsymbol{n}_\lambda} \simeq \exp\Big(-(b-a)\nu_S((\boldsymbol{a}_\lambda(1+\varepsilon), \infty))/\nu_S((\boldsymbol{a}_\lambda, \infty))\Big),$$

which tends to 1 as $\lambda \to 0$ for any $\varepsilon > 0$ by $(H_S(\infty))$.

3. Microscopic fires. — Assume that a fire starts at some place x at some time t, with $Z^\lambda_{t-}(x) = z \in (0, 1)$. Then the possible clusters on the left and right of x cannot be connected during (approximately) $[t, t+z]$, but can be connected after (approximately) $t + z$.

Indeed, the match falls in a zone with approximate density $\nu_S((0, \boldsymbol{a}_\lambda z))$, so that it should destroy a zone A of approximate length $1/\nu_S((\boldsymbol{a}_\lambda z, \infty)) \ll \boldsymbol{n}_\lambda$. The probability that a fire starts again in A after t is very small. Thus the probability that A is completely occupied at time $t+s$ is approximately

$$\big(\nu_S((0, \boldsymbol{a}_\lambda s])\big)^{1/\nu_S((\boldsymbol{a}_\lambda z, \infty))} \simeq \exp\big(-\nu_S((\boldsymbol{a}_\lambda s, \infty))/\nu_S((\boldsymbol{a}_\lambda z, \infty))\big).$$

When $\lambda \to 0$, this quantity tends to 0 if $s < z$ and to 1 if $s > z$ thanks to $(H_S(\infty))$.

4. Macroscopic fires. — Assume now that a fire starts at some place x, at some time t and that $Z^\lambda_{t-}(x) \simeq 1$, so that $D^\lambda_{t-}(x)$ is macroscopic (that is its length is of order 1 in our scales, or of order $\boldsymbol{n}_\lambda$ in the original process). This will thus make vacant the zone $D^\lambda_{t-}(x)$. Such a (macroscopic) zone needs a time of order 1 to be completely occupied, see point 2.

5. Clusters. — For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D^\lambda_t(x)$ resembles

$$\big[x \pm 1/(\boldsymbol{n}_\lambda \nu_S((\boldsymbol{a}_\lambda z, \infty)))\big] \simeq \{x\}$$

if $Z^\lambda_t(x) = z \in (0, 1)$. We then say that x is microscopic. Macroscopic clusters are delimited either by microscopic zones, or by sites where there has been recently a microscopic fire.

Even if the above arguments are (hopefully) quite convincing, the rigorous proof is long and tedious. The main idea is that even if each isolated event is easily treated (for example, the fact that a vacant macroscopic zone needs a delay 1 to be completely filled again relies on an immediate computation; estimating the delay needed to fill again the zone destroyed by a microscopic fire is not difficult, etc.), it is quite hard to follow the process during an arbitrary large time interval. Indeed, we have to check that the small errors due to one such event do not become large errors after some time. For example, if a macroscopic zone is not filled at time 1, but slightly after (say at time $t_0 > 1$), this could reduce consequently the impact of a match falling in this zone between 1 and t_0, etc. The main ideas of the proof are however quite simple and really rely on the above heuristic arguments.

2.3.6. Cluster-size distribution. — We will deduce from Theorem 2.3.3 the following estimates on the cluster-size distribution.

COROLLARY 2.3.4. — *Assume* (H_M) *and* $(H_S(\infty))$. *Recall that* $\boldsymbol{a}_\lambda$ *and* $\boldsymbol{n}_\lambda$ *were defined in* (2.2.1) *and* (2.2.2). *Let* $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ *be a* LFF(∞)*-process. For each* $\lambda \in (0,1]$, *let* $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ *be a* FF(μ_S, μ_M^λ)*-process.*

(i) *For some* $0 < c_1 < c_2$, *for all* $t \geq \frac{5}{2}$, *all* $0 < a < b < 1$,

$$\lim_{\lambda\to 0} \Pr\left(|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)| \in [1/\nu_S((\boldsymbol{a}_\lambda a, \infty)), 1/\nu_S((\boldsymbol{a}_\lambda b, \infty))]\right)$$
$$= \Pr\left(Z_t(0) \in [a,b]\right) \in \left[c_1(b-a), c_2(b-a)\right].$$

(ii) *For some* $0 < c_1 < c_2$ *and* $0 < \kappa_1 < \kappa_2$, *for all* $t \geq \frac{3}{2}$, *all* $B > 0$,

$$\lim_{\lambda\to 0} \Pr\left(|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)| \geq B\boldsymbol{n}_\lambda\right) = \Pr\left(|D_t(0)| \geq B\right) \in [c_1 \mathrm{e}^{-\kappa_2 B}, c_2 \mathrm{e}^{-\kappa_1 B}].$$

This results shows that there is a *phase transition* around the *critical size* $\boldsymbol{n}_\lambda$: the cluster-size distribution changes of shape at $\boldsymbol{n}_\lambda$.

Consider the case of Example 2, where $\mu_S((t,\infty)) \overset{\infty}{\approx} \mathrm{e}^{-t^\alpha}$. Then

$$\boldsymbol{a}_\lambda \sim (\log(1/\lambda))^{1/\alpha} \quad \text{and} \quad \boldsymbol{n}_\lambda \sim 1/[\lambda(\log(1/\lambda))^{1/\alpha}].$$

Very roughly, Corollary 2.3.4 proves that when $\lambda \to 0$, the law of $|C(\eta^\lambda, 0)|$, for large times, resembles

$$\frac{[\log(1+x)]^{1/\alpha - 1}}{(1+x)[\log(1/\lambda)]^{1/\alpha}} \mathbb{1}_{\{x\in[0,\boldsymbol{n}_\lambda]\}} \,\mathrm{d}x + (1/\boldsymbol{n}_\lambda)\,\mathrm{e}^{-x/\boldsymbol{n}_\lambda} \mathbb{1}_{\{x\geq 0\}} \,\mathrm{d}x.$$

The first term corresponds approximately to the law of $1/\nu_S((\boldsymbol{a}_\lambda U, \infty))$, for U uniformly distributed on $[0,1]$ and the second term is an exponential law with mean $\boldsymbol{n}_\lambda$.

The main idea is that two types of clusters are present: macroscopic clusters, of which the size is of order $\boldsymbol{n}_\lambda \sim \lambda^{-1}[\log(1/\lambda)]^{-1/\alpha}$, with an exponential-like distribution; and microscopic clusters, of which the size is smaller than $\boldsymbol{n}_\lambda$, with a law with shape $\log(1+x)^{1/\alpha - 1}/(1+x)$.

2.4. Main result in the case $\beta = BS$

This case is slightly more complicated than the case $\beta = \infty$. The limit process is essentially the same, except that the height of the barriers (vertical segments in Figure 2) are more random.

2.4.1. Law of the heights of the barriers. — Start at time 0 with all sites vacant. Let $u \in (0,1)$. Assume that a match falls at site 0 at time $T_S u$ and neglect all other fires. Call Θ_u the time needed for the destroyed zone to be completely regenerated and θ_u the law of Θ_u / T_S. Clearly, θ_u is supported by $[0,1]$. We will show in Lemma 3.9.1 below that θ_u can be defined as follows.

Definition 2.4.1. — *Assume $(H_S(BS))$. For $t, s \in [0,\infty)$, we denote by*

$$g_S(t,s) = \Pr\left[N^S_{T_S t} > 0, N^S_{T_S(t+s)} > N^S_{T_S t}\right],$$

where $(N^S_t)_{t\geq 0}$ is a $\mathrm{SR}(\mu_S)$-process. For $u \in (0,1)$, we consider the probability measure θ_u on $[0,1]$ defined by

$$\forall h \in [0,1], \quad \theta_u\big([0,h]\big) = \nu_S\big((T_S u, T_S)\big) + \Big(\frac{\nu_S((T_S u, T_S))}{1 - g_S(u,h)}\Big)^2 g_S(u,h).$$

Finally, we consider a function

$$F_S : [0,1] \times [0,1] \longrightarrow [0,1]$$

such that for each $u \in [0,1]$ and for V a uniformly distributed random variable on $[0,1]$, the law of $F_S(u,V)$ is θ_u. We can choose F_S in such a way that for each $u \in [0,1]$, $v \mapsto F_S(u,v)$ is nondecreasing.

Let $u \in [0,1]$ be fixed. Since $\mu_S([0,T_S]) = 1$, there holds $g_S(u,1) = \nu_S([0,T_S u])$, whence $\theta_u([0,1]) = 1$. To check that $h \mapsto \theta_u([0,h])$ is nondecreasing, it suffices to observe that $h \mapsto g(u,h)$ is nondecreasing. Notice that $\theta_u(\{0\}) = \nu_S((T_S u, T_S))$: this corresponds to the situation where nothing has been destroyed because the match has fallen on an empty site. For $F_S(u,.)$, one can e.g. use the generalized inverse function of $\theta_u([0,.])$.

2.4.2. Definition of the limit process. — Let $\pi_M(\mathrm{d}t, \mathrm{d}x)$ be a Poisson measure on $[0,\infty) \times \mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$, whose marks correspond to matches. We also consider an i.i.d. sequence $(V_k)_{k\geq 1}$ of uniformly distributed random variables on $[0,1]$, independent of π_M. If

$$\pi_M(\mathrm{d}t, \mathrm{d}x) = \sum_{k\geq 1} \delta_{(T_k, X_k)},$$

we (abusively) write

$$\pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = \sum_{k\geq 1} \delta_{(T_k, X_k, V_k)}.$$

Observe that $\pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$ is a Poisson measure on $[0,\infty)\times\mathbb{R}\times[0,1]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$.

Definition 2.4.2. — *A process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ with values in $\mathbb{R}_+\times\mathcal{I}\times\mathbb{R}_+$ such that a.s., for all $x\in\mathbb{R}$, $(Z_t(x), H_t(x))_{t\geq 0}$ is càdlàg, is said to be a* LFF(BS)*-process if a.s., for all $t\geq 0$, all $x\in\mathbb{R}$,*

$$(2.4.1)\quad \begin{cases} Z_t(x) = \displaystyle\int_0^t \mathbb{1}_{\{Z_s(x)<1\}}\,\mathrm{d}s - \int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}}\pi_M(\mathrm{d}s, \mathrm{d}y),\\ H_t(x) = \displaystyle\int_0^t\int_0^1 F_S(Z_{s-}(x), v)\mathbb{1}_{\{Z_{s-}(x)<1\}}\pi_M(\mathrm{d}s\times\{x\}\times\mathrm{d}v)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \displaystyle\int_0^t \mathbb{1}_{\{H_s(x)>0\}}\,\mathrm{d}s, \end{cases}$$

where $D_t(x) = [L_t(x), R_t(x)]$, with

$$L_t(x) = \sup\{y\leq x:\ Z_t(y)<1 \text{ or } H_t(y)>0\},$$
$$R_t(x) = \inf\{y\geq x:\ Z_t(y)<1 \text{ or } H_t(y)>0\}$$

and where $D_{t-}(x)$ is defined in the same way.

The difference with the LFF(∞)-process is that when a match falls at (t,x) with $Z_{t-}(x)<1$, we choose $H_t(x)$ according to the law $\theta_{Z_{t-}(x)}$, instead of simply setting $H_t(x) = Z_{t-}(x)$.

2.4.3. Formal dynamics. — Let us explain the dynamics of this process. We consider $T>0$ fixed and set

$$\mathcal{A}_T = \big\{x\in\mathbb{R}:\ \pi_M([0,T]\times\{x\})>0\big\}.$$

For each $t\geq 0$, $x\in\mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster is *microscopic* if $D_t(x)=\{x\}$. We have $D_t(x) = D_t(y)$ for all $y\in D_t(x)$.

1. Initial condition. — We have, for all $x\in\mathbb{R}$

$$Z_0(x) = H_0(x) = 0 \quad\text{and}\quad D_0(x) = \{x\}.$$

2. Occupation of vacant zones. — We consider here $x\in\mathbb{R}\setminus\mathcal{A}_T$. Then we have $H_t(x)=0$ for all $t\in[0,T]$. When $Z_t(x)<1$, then $D_t(x)=\{x\}$ and $Z_t(x)$ stands for the local density of occupied sites around x (or rather for a suitable function of this density) Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the RHS of the first equation in (2.4.1). When $Z_t(x)=1$, the cluster containing x is macroscopic and is described by $D_t(x)$.

3. Microscopic fires. — Here we assume that $x\in\mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x)<1$. In such a case, the cluster containing x is microscopic. Then we set $H_t(x) = F_S(Z_{t-}(x), V)$, for some uniformly distributed V on $[0,1]$ as described by the first term on the RHS of the second equation of (2.4.1). We then let $H_t(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the second equation in (2.4.1). At all times where $H_s(x)>0$, the site x

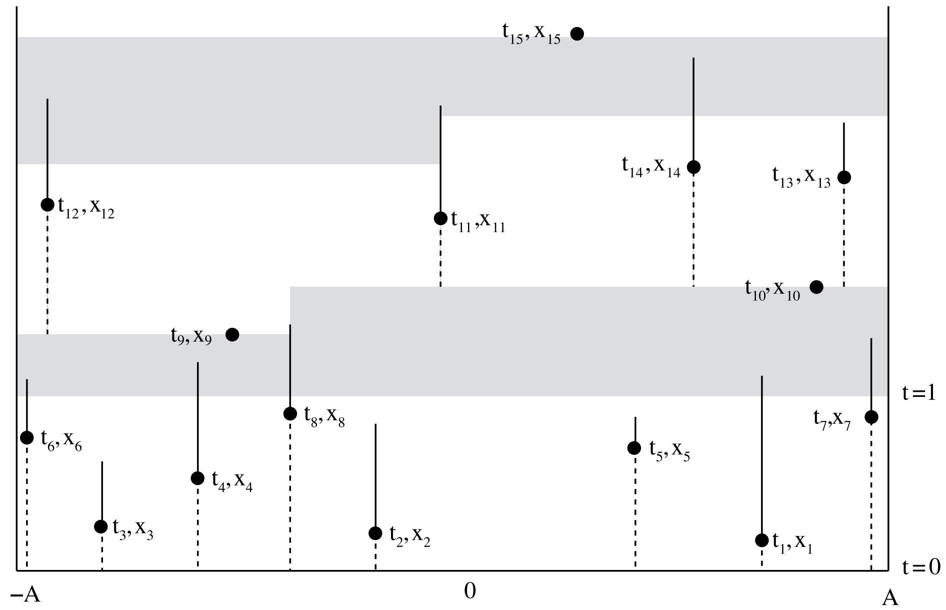

FIGURE 3. LFF(BS)-process in a finite box.

The marks of π_M (matches) are represented as •'s. The filled zones represent zones in which $Z_t^A(x) = 1$ and $H_t^A(x) = 0$, that is macroscopic clusters. The plain vertical segments represent the sites where $H_t^A(x) > 0$. In the rest of the space, we always have $Z_t^A(x) < 1$.

acts like a barrier (see point 5 below). All this means that at x, there is a barrier during $[t, t + H_t(x))$, where $H_t(x)$ is chosen at random, according to the law $\theta_{Z_{t-}(x)}$.

4. Macroscopic fires. — Here we assume that $x \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x) = 1$. This means that the cluster containing x is macroscopic and thus this mark destroys the whole component $D_{t-}(x)$, that is for all $y \in D_{t-}(x)$, we set $D_t(y) = \{y\}$, $Z_t(y) = 0$. This is described by the second term on the RHS of the first equation in (2.4.1).

5. Clusters. — Finally the clusters $(D_t(x))_{x\in\mathbb{R}}$ are delimited by zones with density smaller than 1 (i.e. $Z_t(y) < 1$) or by sites where a microscopic fire has (recently) started (i.e. $H_t(y) > 0$).

A typical path of a finite-box version $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ of the LFF(BS)-process is discussed on Figure 3. It is very similar to Figure 2: the only difference is that each time there is a bullet falling at some (t, x) in a white zone, the height of the segment above (t, x) is chosen at random, according to the law $\theta_{Z_{t-}(x)}$. And $Z_{t-}(x)$ equals the time passed since x was involved in a macroscopic fire (the case LFF(∞) corresponds to the law $\theta_z = \delta_z$). See also Algorithm 3.6.3 below.

2.4.4. Well-posedness. — We will prove the following result.

THEOREM 2.4.3. — *For any Poisson measure* $\pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$ *on* $[0,\infty)\times\mathbb{R}\times[0,1]$ *with intensity measure* $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$ *(and for* $\pi_M(\mathrm{d}t, \mathrm{d}x) = \int_{v\in[0,1]} \pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$*), there a.s. exists a unique* LFF(BS)*-process. Furthermore, it can be constructed graphically and its restriction to any finite box* $[0,T]\times[-n,n]$ *can be perfectly simulated.*

The LFF(BS)-process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ is furthermore Markov, since it solves a well-posed time homogeneous Poisson-driven S.D.E.

2.4.5. The convergence result. — We are now in a position to state the main result of this section. Recall subsection 2.1.5.

THEOREM 2.4.4. — *Assume* (H_M) *and* $(H_S(BS))$*. Recall that* $\boldsymbol{a}_\lambda = T_S$*,* $\boldsymbol{n}_\lambda = \lfloor 1/(\lambda T_S)\rfloor$ *and let* $\boldsymbol{m}_\lambda$ *satisfy* (2.2.4)*. Consider, for each* $\lambda\in(0,1]$*, the process* $(D_t^\lambda(x), Z_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ *associated with the* FF(μ_S, μ_M^λ)*-process* $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$*, see Definition* 2.1.3*,* (2.2.3) *and* (2.2.5)*. Consider also the* LFF(BS)*-process*

$$\big(Z_t(x), D_t(x), H_t(x)\big)_{t\geq 0, x\in\mathbb{R}}.$$

(a) *For any* $T>0$*, any finite subset* $\{x_1,\dots,x_p\}\subset\mathbb{R}$*,* $(Z_t^\lambda(x_i), D_t^\lambda(x_i))_{t\in[0,T], i=1,\dots,p}$ *goes in law to* $(Z_t(x_i), D_t(x_i))_{t\in[0,T], i=1,\dots,p}$*, in* $\mathbb{D}([0,T], \mathbb{R}\times\mathcal{I}\cup\{\varnothing\})^p$*, as* λ *tends to* 0*. Here* $\mathbb{D}([0,\infty), \mathbb{R}\times\mathcal{I}\cup\{\varnothing\})$ *is endowed with the distance* $\boldsymbol{d}_T$*.*

(b) *For any finite subset* $\{(t_1,x_1),\dots,(t_p,x_p)\}\subset[0,\infty)\times\mathbb{R}$*,* $(Z_{t_i}^\lambda(x_i), D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ *goes in law to* $(Z_{t_i}(x_i), D_{t_i}(x_i))_{i=1,\dots,p}$ *in* $(\mathbb{R}\times\mathcal{I}\cup\{\varnothing\})^p$*. Here* $\mathcal{I}\cup\{\varnothing\}$ *is endowed with* $\boldsymbol{\delta}$*.*

(c) *For any* $t\geq 0$*, any* $k\in\mathbb{N}$*,*

$$\lim_{\lambda\to 0}\Pr\left[|C(\eta^\lambda_{T_S t}, 0)| = k\right] = \mathbb{E}\left[q_k(Z_t(0))\right],$$

where, for $z\in[0,1]$*,*

$$\begin{cases} q_0(z) = \nu_S\big((zT_S, T_S)\big), \\ q_k(z) = k\big[\nu_S((zT_S, T_S))\big]^2\big[\nu_S((0, zT_S))\big]^k \quad \text{if} \quad k\geq 1. \end{cases} \tag{2.4.2}$$

Here we have no problem with $t=1$: for the discrete process (in the absence of fires), all the sites are occupied at time T_S (which corresponds to time 1 after normalization). Point (c) will be useful to prove some estimates about the cluster-size distribution. Observe that for $z\in(0,1)$, $q_k(z)$ is the probability that the cluster around 0 has the size k at time $T_S z$, in the absence of fires, if seeds fall according to i.i.d. SR(μ_S)-processes.

2.4.6. Heuristic arguments. — Let us explain roughly the reasons why Theorem 2.4.4 holds true. We consider a $FF(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ and the corresponding processes $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$. We assume below that λ is very small.

0. *Matches.* — As in the case $\beta = \infty$, the times and positions at which matches fall will tend, in our scales, to the marks of a Poisson measure with intensity measure 1.

1. Initial condition. — We have, for all $x \in \mathbb{R}$, $(Z_0^\lambda(x), D_0^\lambda(x)) = (0, \varnothing) \simeq (0, \{x\})$.

2. Occupation of vacant zones. — Assume that a zone $[a, b]$ becomes completely vacant at some time t (because it has been destroyed by a fire).

(i) For $s \in [0, 1)$ and if no fire starts on $[a, b]$ during $[t, t+s]$ (or $[T_S t, T_S(t+s)]$ in the original scales) the density of vacant sites in $[a, b]$ at time $t+s$ should clearly resemble $\nu_S((0, T_S s))$. Hence for $x \in [a, b]$, $Z_t^\lambda(x) \simeq \psi_S(\nu_S((0, T_S s))) = s$ and $D_{t+s}^\lambda(x) \simeq \{x\}$.

(ii) If no fire starts on $[a, b]$ during $[t, t+1]$ (or $[T_S t, T_S(t+1)]$ in the original scales), then all the sites of $[a, b]$ become occupied at time $t+1$ (recall that $\nu_S((0, T_S]) = 1$).

3. Microscopic fires. — Assume that a fire starts at some place x at some time t, with $Z_{t-}^\lambda(x) = z \in (0, 1)$. Then the possible clusters on the left and right of x cannot be connected during (approximately) $[t, t+\Theta_z T_S]$, but can be connected after (approximately) $t + \Theta_z T_S$, where Θ_z follows approximately the law θ_z. Indeed, θ_z is designed for that: consider a zone where the density of occupied sites is z and assume that the sites are exchangeable in this zone. Pick at random a cluster in this zone. The law of its size depends on z. Then θ_z is the law of the time needed for a seed to fall on each sites of this cluster (divided by T_S).

4. Macroscopic fires. — Assume now that a fire starts at some place x, at some time t and that $Z_{t-}^\lambda(x) \simeq 1$, so that $D_{t-}^\lambda(x)$ is macroscopic (that is its length is of order 1 in our scales, or of order $\boldsymbol{n}_\lambda$ in the original process). This will thus make vacant the zone $D_{t-}^\lambda(x)$. Such a (macroscopic) zone needs a time of order 1 to be completely occupied, see point 2.

5. Clusters. — For $t \geq 0$, $x \in \mathbb{R}$, there are some vacant sites in the neighborhood of x if $Z_t^\lambda(x) < 1$ (then we say that x is microscopic), or if there has been (recently) a microscopic fire at x (see point 3). Now macroscopic clusters are delimited either by microscopic zones, or by sites where there has been recently a microscopic fire.

To transform these heuristic arguments into a rigorous proof, we have essentially the same difficulties as when $\beta = \infty$ (see subsection 2.3.5): each isolated event is easily treated, but it is quite hard to put everything together.

2.4.7. Cluster-size distribution. — We will deduce from Theorem 2.4.4 the following estimates on the cluster-size distribution.

Corollary 2.4.5. — *Assume* (H_M) *and* $(H_S(BS))$. *Recall that* $\boldsymbol{a}_\lambda$ *and* $\boldsymbol{n}_\lambda$ *were defined in* (2.2.1) *and* (2.2.2). *Let* $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ *be a* LFF(BS)*-process. For each* $\lambda \in (0,1]$, *let* $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ *be a* FF(μ_S, μ_M^λ)*-process.*

(i) *For some* $0 < c_1 < c_2$, *for all* $t \geq \frac{5}{2}$, *all* $k \in \{0,1,\dots\}$,

$$\lim_{\lambda\to 0} \Pr\big(|C(\eta^\lambda_{T_S t}, 0)| = k\big) \in [c_1 q_k, c_2 q_k],$$

where, for $k \geq 1$

$$q_0 = \int_0^1 \nu_S\big((T_S z, T_S)\big)\,\mathrm{d}z \quad \textit{and} \quad q_k = k\int_0^1 \big[\nu_S((T_S z, T_S))]^2[\nu_S((0, T_S z))\big]^k\,\mathrm{d}z.$$

(ii) *For some* $0 < c_1 < c_2$ *and* $0 < \kappa_1 < \kappa_2$, *for all* $t \geq \frac{3}{2}$, *all* $B > 0$,

$$\lim_{\lambda\to 0} \Pr\big(|C(\eta^\lambda_{T_S t}, 0)| \geq B\boldsymbol{n}_\lambda\big) = \Pr\left(|D_t(0)| \geq B\right) \in [c_1\,\mathrm{e}^{-\kappa_2 B}, c_2\,\mathrm{e}^{-\kappa_1 B}].$$

Consider the case of Example 1, where $\mu_S = \delta_1$, $T_S = 1$ and $\nu_S(\mathrm{d}t) = \mathbb{1}_{[0,1]}(t)\,\mathrm{d}t$. Then $\boldsymbol{n}_\lambda \sim 1/\lambda$ and one can check that $q_0 = \frac{1}{2}$ and $q_k = 2k/[(k+1)(k+2)(k+3)]$ for $k \geq 1$.

Corollary 2.4.5 shows the presence of two regimes: for $\lambda > 0$ very small, there are some finite (uniformly in λ) clusters, as described in point (i) and some clusters of order $1/\lambda$, as described in point (ii). Roughly, for $\lambda > 0$ very small, the cluster-size distribution resembles, for large times,

$$\sum_{k\geq 0} q_k \delta_k(\mathrm{d}x) + \lambda\,\mathrm{e}^{-\lambda x}\mathbb{1}_{\{x\geq 0\}}\,\mathrm{d}x.$$

2.5. Main results when $\beta \in (0,\infty)$

2.5.1. Definition of the limit process. — Surprisingly, the limit process in this case is more natural than in the previous cases, in the sense that there are only macroscopic clusters and thus no microscopic fires: heavy tails can sometimes produce natural objects. This is due to the fact that for $\beta < \infty$, the scale space $\boldsymbol{n}_\lambda$ is correct for all times. We describe the limit forest fire process by a *graphical construction*. The limit forest fire process $(Y_t(x))_{x\in\mathbb{R}, t\geq 1}$ will take its values in $\{0,1\}$. In some sense, $Y_t(x) = 0$ means that there is no tree at x at time t.

For $(Y(x))_{x\in\mathbb{R}}$ with values in $\{0,1\}$, we define the occupied component around $x \in \mathbb{R}$ as

(2.5.1) $$C(Y,x) := \big[\ell(Y,x), r(Y,x)\big]$$

where $\ell(Y,x) = \sup\big\{y \leq x :\ Y(y) = 0\big\}$ and $r(Y,x) = \inf\big\{y \geq x :\ Y(y) = 0\big\}$.

If $Y(x) = 0$, this implies $C(Y,x) = \{x\}$.

We consider a Poisson measure

$$\pi_M(\mathrm{d}t, \mathrm{d}x)$$

on $[0,\infty)\times\mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$, whose marks correspond to matches. We also introduce a Poisson measure

$$\pi_S(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\ell)$$

on $[0,\infty)\times\mathbb{R}\times[0,\infty)$, independent of π_M, with intensity measure

$$\mathrm{d}t\,\mathrm{d}x\beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell.$$

Roughly, when π_S has mark (τ, X, L), this means that no seed fall on X during $[\tau-L,\tau]$. In all the other zones, seeds fall *continuously.*

Before handling the precise construction of the limit process, let us say roughly what happens. Matches fall according to π_M. Draw a vertical dotted segment at X between $\tau-L$ and τ for each mark (τ,X,L) of π_S. Start from time 0. All the sites become immediately occupied, except sites for which there is a dotted vertical segment crossing $t=0$. These sites remain vacant until the height of these segments. Thus we overwrite in plain the parts of these segments above zero. When there is a fire at some time t_0, it destroys a zone delimited by some active plain segments. But all the sites in this zone are immediately occupied again, except those for which there is a dotted vertical segment crossing $t=t_0$. Such sites will remain vacant until the height of these segments, so that we overwrite in plain the parts of these segments above t_0. And so on. Of course, plain segments represent vacant sites. See Figure 4 next page for an illustration.

We now handle the rigorous construction on a fixed time interval $[0,T]$. First, we set

$$Y_t^0(x) = \mathbb{1}_{\{\pi_S(\{(s,x,\ell)\;:\;s>t,s-\ell<0\})=0\}}$$

for all $t\in[0,T]$, all $x\in\mathbb{R}$. Observe that for all $x\in\mathbb{R}$, $t\mapsto Y_t^0(x)$ is non-decreasing on $[0,T]$. Since $\int_0^\infty\int_0^\infty \mathbb{1}_{\{s>T,s-\ell<0\}}\beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell\,\mathrm{d}s>0$, one can clearly find an unbounded family $\{\chi_i\}_{i\in\mathbb{Z}}\subset\mathbb{R}$ such that for all $t\in[0,T]$, all $i\in\mathbb{Z}$, $Y_t^0(\chi_i)=0$. We take the convention that for all $i\in\mathbb{Z}$,

$$\chi_i\le\chi_{i+1},\quad \chi_0\le 0<\chi_1,\quad \lim_{i\to-\infty}\chi_i=-\infty\quad\text{and}\quad\lim_{i\to\infty}\chi_i=\infty.$$

We now handle the construction on each box $[0,T]\times[\chi_i,\chi_{i+1}]$ separately. Let thus i be fixed. The Poisson measure π_M has a.s. a finite number n_i of marks $(\rho_1^i,\alpha_1^i),\dots,(\rho_{n_i}^i,\alpha_{n_i}^i)$ in $[0,T]\times[\chi_i,\chi_{i+1}]$, ordered in such a way that $0<\rho_1^i<\dots<\rho_{n_i}^i$.

We consider the occupied cluster $I_1^i = C(Y^0_{\rho_1^i-},\alpha_1^i)$ (which is included in $[\chi_i,\chi_{i+1}]$ by construction). For $(t,x)\in[0,T]\times[\chi_i,\chi_{i+1}]$, we set

$$Y_t^1(x) = \mathbb{1}_{\{\pi_S(\{(s,x,\ell)\,:\,s>t,s-\ell<\rho_1^i\})=0\}}$$

if $(t,x)\in[\rho_1^i,T]\times I_1^i$ and $Y_t^1(x)=Y_t^0(x)$ else.

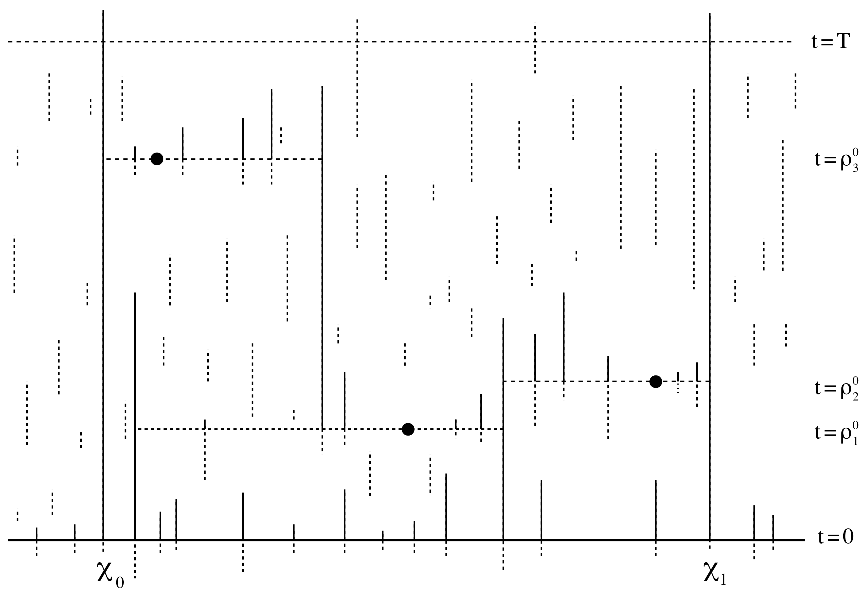

FIGURE 4. LFF(β)-process with $\beta \in (0, \infty)$.

The plain segments represent vacant sites and the occupied clusters are delimited by these segments. The marks of π_M (matches) are represented as $\bullet$'s.

Step 0. – First, we draw on the whole space $[0, \infty) \times \mathbb{R}$ all the $\bullet$'s and we draw a vertical dotted segment from $(\tau - L, X)$ to (τ, X) when π_S has a mark at (τ, X, L). Of course, such segments are infinitely many so that it is not possible to draw all of them on a figure.

Step 1. – For each of these dotted segments that encounter the axis $t = 0$, we overwrite in plain its part above $t = 0$. Then we denote by χ_0 and χ_1 the first places on the left and right of 0 such that plain segments go beyond T. At this stage, we have built $(Y_t^0(x))_{t \in [0,T], x \in \mathbb{R}}$.

Step 3. – At time ρ_1^0, we consider the component I_1^0 (between plain segments) where the match $\bullet$ falls. Then, for each dotted segment (lying in I_1^0) that encounters the axis $t = \rho_1^0$, we overwrite in plain its part above $t = \rho_1^0$. At this stage, we have built $(Y_t^1(x))_{t \in [0,T], x \in [\chi_0, \chi_1]}$.

Step 3. – At time ρ_2^0, we consider the component I_2^0 (between plain segments) where the match $\bullet$ falls. Then, for each dotted segment (lying in I_2^0) that encounters the axis $t = \rho_2^0$, we overwrite in plain its part above $t = \rho_2^0$. We have built $(Y_t^2(x))_{t \in [0,T], x \in [\chi_0, \chi_1]}$.

And so on...

Remark. If we draw a vertical dotted segment from $(\tau - L, X)$ to (τ, X) when π_S has a mark at (τ, X, L) only if $L > \delta$, and if $\delta > 0$ is smaller than $\min\{\rho_1^0, \rho_2^0 - \rho_1^0, \rho_3^0 - \rho_2^0\}$, then we get the exact values of $Y_t(x)$ for all $x \in [\chi_0, \chi_1]$ and all $t \in [0, T] \setminus ([0, \delta] \cup [\rho_1^0, \rho_1^0 + \delta] \cup [\rho_2^0, \rho_2^0 + \delta] \cup [\rho_3^0, \rho_3^0 + \delta])$.

Assume that for some $k = 2, \dots, n_i$, $(Y_t^{k-1}(x))_{t \in [0,T], x \in [\chi_i, \chi_{i+1}]}$ has been built and consider the occupied cluster $I_k^i = C(Y_{\rho_k^i-}^{k-1}, \alpha_k^i)$ (which is still included in $[\chi_i, \chi_{i+1}]$). For $(t, x) \in [0, T] \times [\chi_i, \chi_{i+1}]$, we define $Y_t^k(x)$ by setting

$$Y_t^k(x) = \mathbb{1}_{\{\pi_S(\{(s,x,\ell)\,:\, s>t, s-\ell<\rho_k^i\})=0\}}$$

if $(t, x) \in [\rho_k^i, T] \times I_k^i$ and $Y_t^k(x) = Y_t^{k-1}(x)$ else.

We finally set $Y_t(x) = Y_t^{n_i}(x)$ for all $t \in [0,T]$, all $x \in [\chi_i, \chi_{i+1}]$. Doing this for each i, this defines a process $(Y_t(x))_{t\in[0,T],x\in\mathbb{R}}$.

A typical path of the LFF(β)-process is drawn and discussed on Figure 4, from which the following remark is clear.

Remark 2.5.1. — (i) *If we build the process using some larger final time $T' > T$, this does not change the values of the process on $[0,T]\times\mathbb{R}$. Thus the process can be extended to $[0,\infty)\times\mathbb{R}$.*

(ii) *For $\delta > 0$, denote by π_S^δ the restriction of π_S to $[0,\infty)\times\mathbb{R}\times[\delta,\infty)$. The sequence $(\chi_i)_{i\in\mathbb{Z}}$ clearly depends only on π_S^T. Then for each $i\in\mathbb{Z}$, we denote*

$$\mathcal{T}_M^{i,T} = \big\{t\in[0,T] \,:\, \pi_M(\{t\}\times[\chi_i,\chi_{i+1}])>0\big\}\cup\{0\}, \quad \delta_{i,T} = \inf_{\substack{s,t\in\mathcal{T}_M^{i,T}\\ s\neq t}} |t-s|.$$

Then for all $\delta\in(0,\delta_{i,T}\wedge T)$, all $x\in[\chi_i,\chi_{i+1}]$ and all $t\in[0,T]\setminus\bigcup_{s\in\mathcal{T}_M^{i,T}}[s,s+\delta]$, the value of $Y_t(x)$ depends only on π_M, π_S^δ.

Observe that for all $t\geq 0$, $\{Y_t = 0\}$ is countable and for all $t>0$ such that $\pi_M(\{t\}\times\mathbb{R}) = 0$, $\{Y_t=0\}$ is discrete (it has no accumulation point).

Proposition 2.5.2. — *Let π_M, π_S be two independent Poisson measures on $[0,\infty)\times\mathbb{R}$ and $[0,\infty)\times\mathbb{R}\times[0,\infty)$ with intensity measures $\mathrm{d}t\,\mathrm{d}x$ and $\mathrm{d}t\,\mathrm{d}x\beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell$. There a.s. exists a unique LFF(β)-process $(Y_t(x))_{t\geq0,x\in\mathbb{R}}$. It can be simulated exactly on any finite box $[0,T]\times[-n,n]$. For each $t\geq0$ and $x\in\mathbb{R}$, we set $D_t(x) = C(Y_t,x)$, recall (2.5.1).*

This proposition is obvious from the previous construction. Of course, we can build *exactly* the process on any finite box, but we cannot draw it *exactly*: when a match falls in some occupied cluster I at some time t, the set $\{x\in I:\ Y_t(x)=0\}$ is dense in I (but $\{x\in I:\ Y_{t+\varepsilon}(x)=0\}$ is finite for all small $\varepsilon>0$).

Note that it would have been more natural to set $Y_t(x)=0$ for all $x\in I$ when a match falls in some occupied cluster I at some time t. However, since then I becomes occupied almost everywhere immediately after t, the present definition (which only implies that $\{x\in I:\ Y_t(x)=0\}$ is dense in I) is simpler for mathematical purpose.

2.5.2. On the Markov property. — The LFF(β)-process $(Y_t(x))_{t\geq0}$ is clearly not Markov, in particular because the heights of the barriers are not exponentially distributed. The aim of this subsection is to build a Markov process that contains more information than $(Y_t(x))_{t\geq0}$.

Let the Poisson measures π_M and π_S be given. Write $\pi_S = \sum_{k\geq1}\delta_{(t_k,x_k,\ell_k)}$ and introduce

$$\pi_S^1 = \sum_{k\geq1}\delta_{(t_k-\ell_k,x_k,\ell_k)}\mathbb{1}_{\{t_k-\ell_k>0\}}, \quad \pi_S^0 = \sum_{k\geq1}\delta_{(t_k,x_k,\ell_k)}\mathbb{1}_{\{t_k-\ell_k<0\}}.$$

Observe that π_S^0 and π_S^1 are independent. Furthermore, π_S^1 has a mark (τ, X, L) if and only if there is a dotted vertical segment from (τ, X) to $(\tau+L, X)$ (with $\tau>0$) and π_S^0 has a mark (τ, X, L) if and only if there is a dotted vertical segment from $(\tau-L, X)$ to (τ, X) (with $\tau-L<0<\tau$). One can easily check that π_S^1 is a Poisson measure on $[0,\infty)\times\mathbb{R}\times(0,\infty)$ with intensity measure $\mathrm{d}t\,\mathrm{d}x\beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell$. We set, for $x\in\mathbb{R}$,

$$\Gamma_0(x) = \int_0^\infty\int_0^\infty s\pi_S^0\big(\mathrm{d}s\times\{x\}\times\mathrm{d}\ell\big),$$

which represents the height above 0 of the dotted (or plain) vertical segment at x that crosses the axis $t=0$, with of course $\Gamma_0(x)=0$ if there is no such dotted segment. We then introduce, for $x\in\mathbb{R}$ and $t\geq 0$,

$$\Gamma_t(x) = \Gamma_0(x) + \int_0^t\int_0^\infty \max\big\{\ell-\Gamma_{s-}(x),0\big\}\pi_S^1(\mathrm{d}s\times\{x\}\times\mathrm{d}\ell) - \int_0^t \mathbb{1}_{\{\Gamma_s(x)>0\}}\,\mathrm{d}s,$$

which represents the height above t of the dotted (or plain) vertical segment at x that crosses the horizontal axis with ordinate t, with $\Gamma_t(x)=0$ if there is no such dotted segment. Indeed, $\Gamma_t(x)$ clearly decreases linearly when it is positive, and jumps from $\Gamma_{s-}(x)$ to $\max\{\Gamma_{s-}(x),\ell\}$ when π_S^1 has a mark at (s,x,ℓ). Using the fact that a.s., for all $x\in\mathbb{R}$, there is at most one dotted segment at x, it is possible to replace $\max\{\ell-\Gamma_{s-}(x),0\}$ by ℓ. Finally, we define, for $x\in\mathbb{R}$ and $t\geq 0$,

$$H_t(x) = \Gamma_0(x) + \int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{y\in\mathring{C}(Y_{s-},x)\}}\Gamma_{s-}(x)\pi_M(\mathrm{d}s,\mathrm{d}y) - \int_0^t \mathbb{1}_{\{H_s(x)>0\}}\,\mathrm{d}s,$$
$$Y_t(x) = \mathbb{1}_{\{H_t(x)=0\}},$$

where $\mathring{C}(Y_{s-},x)$ stands for the interior of $C(Y_{s-},x)$. Then $H_t(x)$ is the height above t of the plain segment at x that crosses the horizontal axis with ordinate t (with $H_t(x)=0$ if there is no such plain segment), and thus $(Y_t(x))_{t\geq 0}$ is the LFF(β)-process. Indeed, since we overwrite in plain all the dotted segments that cross the axis $t=0$, we clearly have $H_0(x)=\Gamma_0(x)$. Then $H_t(x)$ decreases linearly when it is positive, and jumps to $\Gamma_{s-}(x)$ when x is involved in a fire at some time s (whence necessarily $H_{s-}(x)=0$): recall that we then overwrite in plain the dotted segment at x that crosses the horizontal axis with ordinate s, of which the height above s it given by $\Gamma_{s-}(x)$.

The process $(\Gamma_t(x), H_t(x), Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ is Markov, since it solves a well-posed homogeneous Poisson-driven S.D.E.

2.5.3. The convergence result. — We now state our main result in the case $\beta\in(0,\infty)$. We use subsection 2.1.5.

Theorem 2.5.3. — *Assume (H_M) and $(H_S(\beta))$ for some $\beta\in(0,\infty)$. Consider, for each $\lambda\in(0,1]$, the process $(D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the* FF(μ_S,μ_M^λ)*-process,* *see Definition* 2.1.3 *and* (2.2.3). *Consider also a* LFF(β)*-process* $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ *and set* $D_t(x) = C(Y_t, x)$ *for all* $t \geq 0$, *all* $x \in \mathbb{R}$ *as in Proposition* 2.5.2.

(a) *For any* $T > 0$, *any finite subset* $\{x_1, \dots, x_p\} \subset \mathbb{R}$, $(D^\lambda_t(x_i))_{t\in[0,T],i=1,\dots,p}$ *goes in law to* $(D_t(x_i))_{t\in[0,T],i=1,\dots,p}$ *in* $\mathbb{D}([0,T],\mathcal{I})^p$, *as* $\lambda \to 0$. *Here* $\mathbb{D}([0,T],\mathcal{I})$ *is endowed with* $\boldsymbol{\delta}_T$.

(b) *For any finite subset* $\{(t_1, x_1), \dots, (t_p, x_p)\} \subset (0,\infty)\times\mathbb{R}$, $(D^\lambda_{t_i}(x_i))_{i=1,\dots,p}$ *goes in law to* $(D_{t_i}(x_i))_{i=1,\dots,p}$ *in* $\mathcal{I}^p$, $\mathcal{I}$ *being endowed with* $\boldsymbol{\delta}$.

2.5.4. Heuristic arguments. — We assume below that $\lambda > 0$ is very small.

0. Matches. — Exactly as in the case $\beta = \infty$, we hope that matches will fall, in our scales, according to a Poisson measure with intensity 1 (in mean, 1 match per unit of time per unit of space, which corresponds to 1 match per $\boldsymbol{n}_\lambda$ sites during $[0, \boldsymbol{a}_\lambda]$ in the original scales).

1. Occupation of vacant zones. — Consider a zone $[a, b]$ (or $[\![\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor b\boldsymbol{n}_\lambda\rfloor]\!]$ in the original scales). At time 0, this zone is completely empty. In this zone, each site will be empty at time t if no seed has fallen during $[0, t]$ (or $[0, \boldsymbol{a}_\lambda t]$ in the original scale). This occurs with probability $\nu_S((\boldsymbol{a}_\lambda t, \infty))$. Thus in the absence of fires, the number of empty sites in $[a, b]$ at time t follows a binomial distribution with parameters $(b-a)\boldsymbol{n}_\lambda$ and $\nu_S((\boldsymbol{a}_\lambda t, \infty))$. Recalling (2.2.1), (2.2.2) and $(H_S(\beta))$, we see that

$$(b-a)\boldsymbol{n}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty)) \simeq (b-a)\nu_S((\boldsymbol{a}_\lambda t, \infty))/\nu_S((\boldsymbol{a}_\lambda, \infty)) \longrightarrow (b-a)t^{-\beta}.$$

Hence the number of empty sites in $[a, b]$ at time t follows approximately a Poisson law with parameter $(b-a)t^{-\beta}$ (when neglecting fires).

The link with the LFF(β)-process is simple: for any $a < b$ and any $t > 0$, the random variable $\pi_S(\{(s, x, \ell) \,:\, x \in [a, b], s > t, s - \ell < 0)\})$ follows a Poisson law with parameter

$$\int_t^\infty \mathrm{d}s \int_a^b \mathrm{d}x \int_s^\infty \beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell = (b-a)t^{-\beta}.$$

2. Fires. — Now when a match falls at some place, this destroys the whole occupied cluster. The destroyed cluster is then treated as in point 1.

The rigorous proof is, as usual, not so easy. The first step is to find a suitable coupling between the seed processes $(N^S_t(i))_{t\geq 0}$ and the Poisson measure π_S describing times/places where seeds do not fall in the limit process. Next, we have to find a (necessarily complicated) event on which the normalized discrete process resembles the limit process and to show that this event occurs with high probability. For example, this event has to guarantee us that for sites on which seeds fall *continuously* in the limit process, seeds fall sufficiently often in the discrete process. We also need that a small error in the time/place where a fire starts (or where a seed falls) does not produce large errors after some time, etc.

2.5.5. Cluster-size distribution. — We aim here to estimate the law of the occupied cluster around 0. No phase transition occurs here.

Corollary 2.5.4. — *Let $\beta \in (0,\infty)$. Assume (H_M) and $(H_S(\beta))$. Recall that $\boldsymbol{a}_\lambda$ and $\boldsymbol{n}_\lambda$ were defined in (2.2.1) and (2.2.2). Consider the* LFF(β)*-process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0,x\in\mathbb{R}}$. For each $\lambda \in (0,1]$, let $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ be a* FF(μ_S,μ_M^λ)*-process. There are constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ such that for all $t \geq 1$ and all $B > 0$,*

$$\lim_{\lambda\to 0} \Pr\left[|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)| \geq B\boldsymbol{n}_\lambda\right] = \Pr\left[|D_t(0)| \geq B\right] \in [c_1\,\mathrm{e}^{-\kappa_2 B}, c_2\,\mathrm{e}^{-\kappa_1 B}].$$

2.6. Main results when $\beta = 0$

2.6.1. Definition of the limit process. — In this case, the limiting process is trivial: we consider a Poisson measure π_S on $\mathbb{R}$ with intensity measure $\mathrm{d}x$ and we put, for all $t \geq 0$, all $x \in \mathbb{R}$,

$$Y_t(x) = \mathbb{1}_{\{\pi_S(x)=0\}}.$$

Denote by $\{\chi_i\}_{i\in\mathbb{Z}}$ the marks of π_S with the convention that $\cdots < \chi_{-1} < \chi_0 < 0 < \chi_1 < \chi_2 < \ldots$. Then for all $t \geq 0$, all $i \in \mathbb{Z}$, recalling (2.5.1), $C(Y_t, x) = [\chi_i, \chi_{i+1}]$ for all $x \in (\chi_i, \chi_{i+1})$ and $C(Y_t, \chi_i) = \{\chi_i\}$. Matches fall according to a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x)$ on $[0,\infty) \times \mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$.

The LFF(0)-process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ is obviously Markov and the following statement is trivial.

Proposition 2.6.1. — *Let π_S be a Poisson measure on $\mathbb{R}$ with intensity measure* $\mathrm{d}x$. *There a.s. exists a unique* LFF(0)*-process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$. It can be simulated exactly on any finite box $[0,T] \times [-n,n]$. For each $t \geq 0$ and $x \in \mathbb{R}$, we will denote by $D_t(x) = C(Y_t, x)$ the occupied cluster around x (see* (2.5.1)*).*

Of course, fires do not appear in the construction. Hence it is not necessary to introduce π_M. However, it allows us to keep in mind that fires do occur. But these fires generate empty zones that are immediately regenerated. The main idea is that in our scales: on the great majority of sites, seeds fall almost continuously for all times; but there are *rare* sites where the first seed will never fall. Hence when there is a fire, this always concerns a zone where seeds fall *continuously*, so that one does not observe the fire at the limit. A typical path of the LFF(0)-process is commented on Figure 5 next page.

2.6.2. The convergence result. — We now state our last main result, using subsection 2.1.5.

Theorem 2.6.2. — *Assume (H_M) and $(H_S(0))$. Consider, for each $\lambda \in (0,1]$, the process $(D_t^\lambda(x))_{t\geq 0,x\in\mathbb{R}}$ associated to the* FF(μ_S,μ_M^λ)*-process, see Definition* 2.1.3

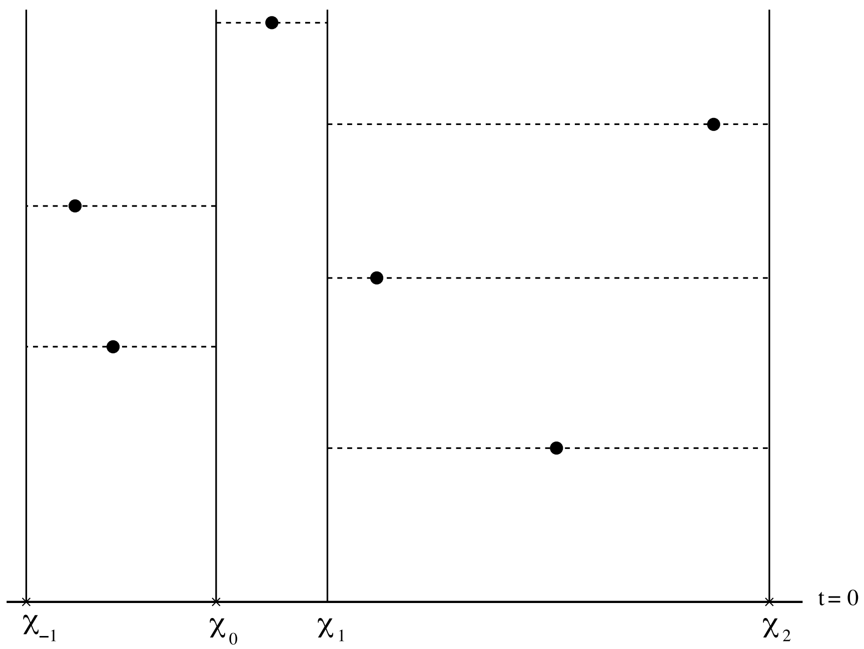

Figure 5. LFF(0)-process.

The marks of π_M (matches) are represented as $\bullet$'s. We draw a plain vertical segment above each mark of π_S. For all times, the occupied clusters are delimited by these vertical segments. In some sense, fires have an instantaneous effect, represented as dotted horizontal segments, that we decided to neglect for obvious practical reasons.

and (2.2.3). *Consider also the* LFF(0)*-process* $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ *and the associated* $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$.

(a) *For any* $T > 0$, *any finite subset* $\{x_1, \dots, x_p\} \subset \mathbb{R}$, $(D_t^\lambda(x_i))_{t\in[0,T], i=1,\dots,p}$ *goes in law to* $(D_t(x_i))_{t\in[0,T], i=1,\dots,p}$ *in* $\mathbb{D}([0,T], \mathcal{I})^p$ *as* $\lambda \to 0$. *Here* $\mathbb{D}([0,\infty), \mathcal{I})$ *is endowed with* $\boldsymbol{\delta}_T$.

(b) *For any finite subset* $\{(t_1, x_1), \dots, (t_p, x_p)\} \subset (0,\infty)\times\mathbb{R}$, $(D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ *goes in law to* $(D_{t_i}(x_i))_{i=1,\dots,p}$ *in* $\mathcal{I}^p$, $\mathcal{I}$ *being endowed with* $\boldsymbol{\delta}$.

2.6.3. Heuristic arguments. — The only difference with the case $\beta \in (0,\infty)$ is the following. In some sense, for each site i, in our scales, either seeds fall *continuously* on i, or the first seed never falls on i. A first hint for this is the following.

Consider a zone $[a,b]$. At time 0, this zone is completely vacant. Fix $T > 0$. Then in the absence of fires, the number of vacant sites in $[a,b]$ at time T (or in $[\![\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor b\boldsymbol{n}_\lambda\rfloor]\!]$ at time $\boldsymbol{a}_\lambda T$ in the original scales) follows a binomial distribution with parameters $(b-a)\boldsymbol{n}_\lambda$ and $\nu_S((\boldsymbol{a}_\lambda T, \infty))$. Observe now that for any value of $T > 0$, using $(H_S(0))$, (2.2.1) and (2.2.2),

$$(b-a)\boldsymbol{n}_\lambda \nu_S\big((\boldsymbol{a}_\lambda T,\infty)\big) \simeq (b-a)\nu_S\big((\boldsymbol{a}_\lambda T,\infty)\big)/\nu_S\big((\boldsymbol{a}_\lambda,\infty)\big) \longrightarrow (b-a).$$

Hence the number of sites that are still vacant at time T follows approximately a Poisson distribution with parameter $(b-a)$. Since this parameter does not decrease with T, this means that in our scales, sites are either immediately occupied or vacant forever.

Here the rigorous proof is rather simple, but it still needs some care. We have essentially the same difficulties as in the case where $\beta \in (0,\infty)$ (see subsection 2.5.4), but they are more easily treated.

2.6.4. Cluster-size distribution. — Since the LFF(0)-process is very simple, we obtain of course some more precise information on the asymptotic cluster-size distribution.

REMARK 2.6.3. — *Assume* (H_M) *and* $(H_S(0))$. *For each* $\lambda \in (0,1]$, *let a* $\mathrm{FF}(\mu_S,\mu_M^\lambda)$*-process* $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ *be given, see Definition* 2.1.3. *Consider the* LFF(0)*-process* $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ *and the associated* $(D_t(x))_{t\geq 0,x\in\mathbb{R}}$. *Then for* $t>0$ *and* $B>0$,

$$\lim_{\lambda\to 0} \Pr\left[|C(\eta^\lambda_{\boldsymbol{a}_\lambda t},0)| \geq B\boldsymbol{n}_\lambda\right] = \Pr\left[|D_t(0)|\geq B\right] = \int_B^\infty x\,\mathrm{e}^{-x}\,\mathrm{d}x = (B+1)\,\mathrm{e}^{-B}.$$

No proof is needed here: $x\,\mathrm{e}^{-x}\mathbb{1}_{\{x>0\}}$ is just the density of $|D_t(0)| = \chi_1-\chi_0$. The convergence in law of $|C(\eta^\lambda_{\boldsymbol{a}_\lambda t},0)|/\boldsymbol{n}_\lambda = |D_t^\lambda(0)|$ to $|D_t(0)|$ follows from Theorem 2.6.2.

2.7. On some other modelling choices

For μ a probability law on $(0,\infty)$, we say that $N_t = \sum_{n\geq 1}\mathbb{1}_{\{X_1+\dots+X_n\leq t\}}$ is a natural renewal process with parameter μ, or a $NR(\mu)$-process in short, if the random variables X_i are i.i.d. with law μ. When extending the traditional forest fire model (where all the renewal processes are Poisson), we had to make some choices.

1. Matches can fall according to i.i.d. (i) $\mathrm{SR}(\mu_M^\lambda)$-processes, (ii) $NR(\mu_M^\lambda)$-processes.

2. Seeds can fall according to i.i.d. (i) $\mathrm{SR}(\mu_S)$-processes, (ii) $NR(\mu_S)$-processes.

3. When a fire destroys an occupied component $[\![a,b]\!]$, we can (i) keep the i.i.d. renewal processes governing seeds as they are, (ii) forget everything and make start some new i.i.d. renewal processes governing seeds in the zone $[\![a,b]\!]$.

Recall that when dealing with Poisson processes, choosing (i) or (ii) in points 1, 2, 3 does not change the law of the $\mathrm{FF}(\mu_S,\mu_M^\lambda)$-process.

From the point of view of modelling, it seemed more natural to choose (i) in points 1 and 2: this is the only way that time 0 does not play a special role. We also decided to choose (i) in point 3, because its seems more close to applications. Let us discuss briefly what could happen with other choices.

First, for matches (point 1), choosing (i) does not play a fundamental role. Indeed, in our scales, only 0 or 1 match can fall on each site. Thus our results should extend, without difficulty, to the choice 1 (ii), replacing (H_M) by the assumption $\mu_M^\lambda((0,t)) \simeq \lambda t$ as $\lambda \to 0$ (together with some additional regularity conditions if we want a strong coupling as in Proposition 3.2.1).

Next, we believe that in point 2, our results should still hold if choosing (ii) when $\beta = \infty$. In the case where μ_S has a bounded support, one would have to assume some regularity on μ_S (the case $\mu_S = \delta_1$ is trivial) and to modify the dynamics of the LFF(BS)-process (the law θ_u should also depend on time). Our study would completely break down when $\beta \in [0,\infty)$. In the latter case, the situation would be quite intricate and we are not able to predict scales (and, *a fortiori*, to predict some limit process). Let us explain briefly the situation. If $\beta = \infty$, then ν_S and μ_S have a similar tail (see example 2). Thus the time and space scales we have considered will fit both ν_S and μ_S. On the contrary, if $\beta \in [0,\infty)$, the tails of μ_S and ν_S are really different. Consequently, if we accelerate time according to μ_S (in order that for a $NR(\mu_S)$-process, the cluster containing the site 0 burns before time 1 with a positive probability), then this will be too slow for larger times (when a fire destroys a cluster (a,b), this zone (a,b) will never regenerate).

Finally, in point 3, we also believe that choosing (ii) would not change our results when $\beta = \infty$ and not change too much the situation when μ_S has a bounded support. When $\beta \in [0,\infty)$, we expect that this would not change time/space scales, but we would have to modify the limit processes. For example if $\beta = 0$, we expect that each time a fire burns a zone (a,b), this zone would regenerate immediately, *except* in a random number of sites, that follows a Poisson distribution with parameter $(b-a)$. Next if $\beta \in (0,\infty)$, then when a fire burns a zone (a,b) at some time t, we would have to pick another Poisson measure $\pi_S^{(a,b),t}(ds, \mathrm{d}x, \mathrm{d}\ell)$ on $[t,\infty) \times (a,b) \times (0,\infty)$, independent of everything else and use this Poisson measure above (a,b) instead of the original π_S.

2.8. Open problems and perspectives

Of course, the main interesting problem is to find a scaling limit of the forest-fire process, e.g. when seeds and matches fall according to Poisson processes, in dimension 2 or more. We believe that the 2-dimensional limit process should enjoy self-organized criticality. However, it is highly probable that our work, though quite complete in dimension 1, does not give the least hint of what could happen in dimension 2. Indeed, all our study is based on the fact that connectedness is very simple in dimension 1: a vacant site is sufficient to stop a fire. One immediately gets convinced that the situation is much more complicated in higher dimension. A possible intermediary step, that we investigate, is to study the case where the underlying lattice is a tree, in which connectedness is much simpler than in $\mathbb{Z}^2$.

A much easier problem, on which we also work, is to study (e.g. in the Poisson case) the possible scaling limits of the forest-fire process, in dimension 1, when fires propagate at finite speed. We then expect that several limit processes should arise: (i) if fires propagate sufficiently fast, then we should recover the same limit process as when fires propagate at infinite speed, (ii) when fires propagate at some precise speed (to be determined as a function of λ), then we should find a modified limit process, in which the microscopic fires are unchanged, but in which the macroscopic fires propagate at finite speed, (iii) when fires propagate slowly, a quite different limit process should arise.

Other possible variants could be studied. First, one could consider the case where the processes governing seeds are not independent. It should not be too difficult to get some results (probably with the same scaling limits as in the present paper), under a suitable ergodicity assumption. We could also study the case where seeds fall in a random media. For example, choose (independently) for each site some parameter $\lambda_i > 0$ at random, and assume that seeds fall on this site according to a Poisson process with rate λ_i. In the case where the support of the law governing the λ_i's is bounded from below, a scaling limit could reasonably be found and should not differ much from the LFF(∞)-process. More subtle phenomena could occur if there are some sites with arbitrarily small rate (on which seeds will fall very rarely). And so on.

It also would be very interesting to study the existence and uniqueness of invariant probability measures for the four limit processes, as well as their convergence to equilibrium. The case $\beta = 0$ is obvious, since the limit process LFF(0) is stationary. But the three other cases seem quite intricate. Finite-box versions of these processes obviously converge in law to a unique invariant probability measure. However, we have no idea of how to check that correlations do not become longer and longer when time increases for the true limit processes. Although this problem seems hard, it is probably less difficult to study invariant distributions for the limit processes than for the original forest-fire processes.

Finally, it might be possible (possibly using the ideas of the present paper), to give much stronger versions of Corollaries 2.3.4, 2.4.5, 2.5.4 and Remark 2.6.3 concerning the asymptotics of the cluster-size distribution. For example in the Poisson case (use Corollary 2.3.4 with $\mu_S((t,\infty)) = \mathrm{e}^{-t}$), we deduce from our convergence result that the probability that the cluster containing 0 is of size x, in the original scales and for sufficiently large times, resembles

$$\frac{1}{(1+x)[\log(1/\lambda)]}\mathbb{1}_{\{x\in[0,1/(\lambda\log(1/\lambda))]\}} + \lambda\log(1/\lambda)\,\mathrm{e}^{-x\lambda\log(1/\lambda)}\mathbb{1}_{\{x\geq 0\}}$$

in a very weak sense. It would be interesting to prove a stronger version of this claim. For example, it was proved rigorously in Brouwer-Pennanen [**17**] that there are constants $0 < c < C$ such that for all $0 < \lambda < 1$ and all stationary measures μ_λ (invariant by translation) of the forest fire model on $\mathbb{Z}$ with parameter λ, for all $x < (1/[\lambda \log(1/\lambda)])^{\frac{1}{3}}$,

$$\frac{c}{(1+x)\log(1/\lambda)} \le \mu_\lambda\big(|C(\eta,0)| = x\big) \le \frac{C}{(1+x)\log(1/\lambda)}.$$

Our result shows that at least a weakened version of such inequalities extends to much higher values of x, possibly to all $x < 1/[\lambda \log^2(1/\lambda)]$. It would be very interesting to prove that these inequalities *really* hold true for such values of x.

CHAPTER 3

PROOFS

3.1. Graphical construction of the discrete process

The goal of this section is to prove Proposition 2.1.4 by using a *graphical construction.*

Proof of Proposition 2.1.4. — Our aim is to prove that for any $T > 0$, a.s., the values of the $\mathrm{FF}(\mu_S, \mu_M)$-process $(\eta_t(i))_{t\in[0,T],i\in\mathbb{Z}}$ are uniquely determined by $(N_t^S(i), N_t^M(i))_{t\geq 0,i\in\mathbb{Z}}$. Recall that

$$\nu_S(\mathrm{d}x) = m_S^{-1}\mu_S\big((x,\infty)\big)\,\mathrm{d}x \quad \text{and} \quad \nu_M(\mathrm{d}x) = m_M^{-1}\mu_M\big((x,\infty)\big)\,\mathrm{d}x,$$

where m_S and m_M are the expectations of μ_S and μ_M. We consider $h_0 > 0$ such that $\nu_S([2h_0,\infty)) > 0$ (if ν_S has an unbounded support, any value of h_0 is possible) and we put $c_0 = \nu_S((2h_0,\infty))\nu_M((0,h_0)) > 0$. We also set $K = \lfloor T/h_0 \rfloor$.

For $n \in \mathbb{Z}$, we consider the event $\Omega_{n,T}$, on which the following conditions are satisfied:

(i) $N_{h_0}^S(n) = 0$;

(ii) $\forall i \in [\![1,K]\!],\ N_{(i+1)h_0}^S(n+i) = N_{(i-1)h_0}^S(n+i)$;

(iii) $\forall i \in [\![1,K]\!],\ N_{ih_0}^M(n+i) > N_{(i-1)h_0}^M(n+i)$.

We first observe that for any $n \in \mathbb{Z}$, using the stationarity of the renewal processes,

$$\Pr[\Omega_{n,T}] = \nu_S\big((h_0,\infty)\big)c_0^K =: c_T > 0.$$

Next we prove that necessarily,

$$\Omega_{n,T} \subset \big\{\forall t \in [0,T], \exists\, i \in [\![n,\ n+K]\!], \eta_t(i) = 0\big\}.$$

This is not hard:

(i) implies that $\eta_t(n) = 0$ for $t \in [0,h_0]$, since no seed falls on n during this interval;

(iii) implies that for $i \in [\![1,K]\!]$, a match falls on $n+i$ during $((i-1)h_0, ih_0]$;

(ii) guarantees us that no seed falls on $n+i$ during $((i-1)h_0,(i+1)h_0]$, whence the site $n+i$ is necessarily vacant during (at least) $(ih_0,(i+1)h_0]$.

Consequently, on $\Omega_{n,T}$, there is always at least one vacant site in $[\![n,n+K]\!]$ during

$$[0,h_0]\cup\Big(\bigcup_{i=1,\dots,K}(ih_0,(i+1)h_0]\Big)\supset[0,T]$$

(with our choice for K, we have $(K+1)h_0\geq T$).

Hence conditionally on $\Omega_{n,T}$, during $[0,T]$, the fires starting on the right of $n+K$ do not affect the values of the forest fire process on the left of n; and the fires starting on the left of n do not affect the values of the forest fire process on the right of $n+K$.

Since $\Pr[\Omega_{n,T}]=c_T>0$, we can find $\dots<n_{-1}<n_0<0<n_1<n_2\dots$ such that $\bigcap_{\ell\in\mathbb{Z}}\Omega_{n_\ell,T}$ is realized (use that $\Omega_{n,T}$ is independent of $\Omega_{m,T}$ if $|m-n|>K$).

We deduce that for any $i\in\mathbb{Z}$, the values of $(\eta_t(i))_{t\in[0,T]}$ are entirely determined by the values of $(N_t^S(j),N_t^M(j))_{t\in[0,(K+1)h_0]}$ for a finite number of j's, namely (at most) $j\in[\![n_k,n_\ell+K]\!]$, where $k<\ell$ satisfy $n_k+K<i<n_\ell$.

We have shown that for any $T>0$, $(\eta_t(i))_{t\geq 0,i\in\mathbb{Z}}$ is entirely and uniquely defined by the values of $(N_t^S(i),N_t^M(i))_{t\in[0,(K+1)h_0],i\in\mathbb{Z}}$. □

3.2. Convergence of matches

In this section, we consider any function $\lambda\mapsto\boldsymbol{a}_\lambda$ bounded from below and such that $\boldsymbol{n}_\lambda=\lfloor 1/(\lambda\boldsymbol{a}_\lambda)\rfloor\to\infty$. For $A>0$ and $i\in\mathbb{Z}$, we set

$$A_\lambda=\lfloor A\boldsymbol{n}_\lambda\rfloor,\quad I_A^\lambda=[\![-A_\lambda,A_\lambda]\!],\quad i_\lambda=[i/\boldsymbol{n}_\lambda,(i+1)/\boldsymbol{n}_\lambda).$$

The following result will be used to prove our four main theorems.

Proposition 3.2.1. — *Assume* (H_M). *Let* $A>0$ *and* $T>0$ *be fixed. We can find, for any* $\lambda\in(0,1]$, *a coupling between a Poisson measure* $\pi_M(\mathrm{d}t,\mathrm{d}x)$ *on* $[0,\infty)\times\mathbb{R}$ *with intensity measure* $\mathrm{d}t\,\mathrm{d}x$ *and a family of i.i.d.* $\mathrm{SR}(\mu_M^\lambda)$*-processes* $(N_t^{M,\lambda}(i))_{i\in\mathbb{Z},t\geq 0}$ *such that for*

$$\Omega_{A,T}^M(\lambda):=\big\{\forall t\in[0,T],\ \forall i\in I_A^\lambda,\ \Delta N_{\boldsymbol{a}_\lambda t}^{M,\lambda}(i)\neq 0\ \textit{ iff }\ \pi_M(\{t\}\times i_\lambda)\neq 0\big\},$$

one has $\lim_{\lambda\to 0}\Pr[\Omega_{A,T}^M(\lambda)]=1$.

This means that in our scales, with a high probability, the matches used in the discrete processes can be prescribed by a Poisson measure, as in the limit processes.

Proof. — We divide the proof into several steps. Observe that

$$B_\lambda:=\bigcup_{i\in I_A^\lambda}i_\lambda=\big[-A_\lambda/\boldsymbol{n}_\lambda,(A_\lambda+1)/\boldsymbol{n}_\lambda\big)$$

(which is approximately $[-A,A]$). It of course suffices to build π_M restricted to $[0,T]\times B_\lambda$ and the family $(N_t^{M,\lambda}(i))$ for $i\in I_A^\lambda$ and $t\in[0,\boldsymbol{a}_\lambda T]$.

Step 1. — We observe that a possible way to build π_M (restricted to $[0,T]\times B_\lambda$) is the following:

(i) Consider a family of i.i.d. r.v. $(Z_i^\lambda)_{i\in I_A^\lambda}$ following a Poisson distribution with parameter $T|i_\lambda| = T/\boldsymbol{n}_\lambda$.

(ii) For each i with $Z_i^\lambda > 0$, pick some i.i.d. r.v. $(T_1^{i,\lambda}, X_1^{i,\lambda}), \dots, (T_{Z_i^\lambda}^{i,\lambda}, X_{Z_i^\lambda}^{i,\lambda})$ with uniform law on $[0,T]\times i_\lambda$ (conditionally on Z_i^λ). Set finally

$$\pi_M = \sum_{i\in I_A^\lambda}\sum_{k=1}^{Z_i^\lambda} \delta_{(T_k^{i,\lambda}, X_k^{i,\lambda})}.$$

Step 2. — Next, we note it is possible to build the family $(N_t^{M,\lambda}(i))_{i\in I_A^\lambda, t\in[0,\boldsymbol{a}_\lambda T]}$ as follows: introduce

$$q_k(\lambda,T) = \Pr[N_{\boldsymbol{a}_\lambda T}^{M,\lambda}(i) = k] \quad \text{and} \quad \zeta_k^{\lambda,T}(\mathrm{d}t_1,\dots,\mathrm{d}t_k)$$

the law of the k jump instants of $N^{M,\lambda}(i)$ in $[0,\boldsymbol{a}_\lambda T]$ conditionally on $\{N_{\boldsymbol{a}_\lambda T}^{M,\lambda}(i) = k\}$.

(i) Consider a family of i.i.d. r.v. $(\widetilde{Z}_i^\lambda)_{i\in I_A^\lambda}$ with law $(q_k(\lambda,T))_{k\geq 0}$.

(ii) For each i with $\widetilde{Z}_i^\lambda > 0$, pick $(\widetilde{T}_1^{i,\lambda},\dots,\widetilde{T}_{\widetilde{Z}_i^\lambda}^{i,\lambda})$ according to $\zeta_{\widetilde{Z}_i^\lambda}^{\lambda,T}(\mathrm{d}t_1,\dots,\mathrm{d}t_{\widetilde{Z}_i^\lambda})$ (conditionally on $\widetilde{Z}_i^\lambda$).

Set finally $N_t^{M,\lambda}(i) = \sum_{k=1}^{\widetilde{Z}_i^\lambda} \mathbb{1}_{\{t\geq \widetilde{T}_k^{i,\lambda}\}}$ for $t\in[0,\boldsymbol{a}_\lambda T]$, $i\in I_A^\lambda$.

Step 3. — We show in this step that for each $i\in I_A^\lambda$, one can couple Z_i^λ (as in step 1 (i)) and $\widetilde{Z}_i^\lambda$ (as in step 2 (i)) in such a way that

$$\begin{aligned}
&\Pr[Z_i^\lambda = \widetilde{Z}_i^\lambda = 0] \geq 1 - \lambda \boldsymbol{a}_\lambda T\big(1+\varepsilon_T(\lambda)\big),\\
&\Pr[Z_i^\lambda = \widetilde{Z}_i^\lambda = 1] \geq \lambda \boldsymbol{a}_\lambda T\big(1-\varepsilon_T(\lambda)\big),
\end{aligned}$$

where $\lim_{\lambda\to 0}\varepsilon_T(\lambda) = 0$. Below, the function ε_T may change from line to line.

It is classically possible (see Lemma 5.1.3 (i)) to build a coupling in such a way that

$$\begin{aligned}
&\Pr[Z_i^\lambda = \widetilde{Z}_i^\lambda = 0] \geq \Pr[Z_i^\lambda = 0]\wedge \Pr[\widetilde{Z}_i^\lambda = 0],\\
&\Pr[Z_i^\lambda = \widetilde{Z}_i^\lambda = 1] \geq \Pr[Z_i^\lambda = 1]\wedge \Pr[\widetilde{Z}_i^\lambda = 1].
\end{aligned}$$

We now use (H_M): recalling that $\int_0^\infty \mu_M^1((s,\infty))\,\mathrm{d}s = 1$,

$$\begin{aligned}
\Pr[\widetilde{Z}_i^\lambda = 0] &= \nu_M^\lambda\big((\boldsymbol{a}_\lambda T,\infty)\big) = \lambda\int_{\boldsymbol{a}_\lambda T}^\infty \mu_M^1\big((\lambda s,\infty)\big)\,\mathrm{d}s\\
&= \int_{\lambda \boldsymbol{a}_\lambda T}^\infty \mu_M^1\big((u,\infty)\big)\,\mathrm{d}u = 1 - \int_0^{\lambda\boldsymbol{a}_\lambda T}\mu_M^1\big((u,\infty)\big)\,\mathrm{d}u \geq 1-\lambda\boldsymbol{a}_\lambda T.
\end{aligned}$$

Since $\Pr[Z_i^\lambda = 0] = \mathrm{e}^{-T/\boldsymbol{n}_\lambda} = \mathrm{e}^{-T/\lfloor 1/(\lambda \boldsymbol{a}_\lambda)\rfloor} = 1 - \lambda \boldsymbol{a}_\lambda T(1+\varepsilon_T(\lambda))$, this concludes the proof of the first lower-bound. Next, recalling Definition 2.1.1 and (H_M),

$$\begin{aligned}\Pr[\widetilde{Z}_i^\lambda = 1] &= \int_0^{\boldsymbol{a}_\lambda T} \mu_M^\lambda\big((\boldsymbol{a}_\lambda T - s,\infty)\big)\nu_M^\lambda(\mathrm{d}s)\\ &= \int_0^{\boldsymbol{a}_\lambda T} \mu_M^1\big((\lambda(\boldsymbol{a}_\lambda T - s),\infty)\big)\lambda\mu_M^1\big((\lambda s,\infty)\big)\,\mathrm{d}s\\ &= \int_0^{\lambda\boldsymbol{a}_\lambda T} \mu_M^1\big((\lambda\boldsymbol{a}_\lambda T - u,\infty)\big)\mu_M^1\big((u,\infty)\big)\,\mathrm{d}u\\ &= \lambda\boldsymbol{a}_\lambda T\big(1-\varepsilon_T(\lambda)\big),\end{aligned}$$

since $\lambda\boldsymbol{a}_\lambda \to 0$ as $\lambda \to 0$. But now

$$\Pr[Z_i^\lambda = 1] = (T/\boldsymbol{n}_\lambda)\mathrm{e}^{-T/\boldsymbol{n}_\lambda} = \lambda\boldsymbol{a}_\lambda T\big(1-\varepsilon_T(\lambda)\big),$$

because $\boldsymbol{n}_\lambda = \lfloor 1/(\lambda\boldsymbol{a}_\lambda)\rfloor$ and this concludes the step.

Step 4. — We now check that for each $i \in I_A^\lambda$, conditionally on $\{Z_i^\lambda = \widetilde{Z}_i^\lambda = 1\}$, we can couple $T_1^{i,\lambda}$ and $\widetilde{T}_1^{i,\lambda}$ (see steps 1 (ii) and 2 (ii)) in such a way that for

$$r_T(\lambda) = \Pr\big[T_1^{i,\lambda} = \widetilde{T}_1^{i,\lambda}/\boldsymbol{a}_\lambda \mid Z_i^\lambda = \widetilde{Z}_i^\lambda = 1\big],$$

$\lim_{\lambda\to 0} r_T(\lambda) = 1$. We first recall that $T_1^{i,\lambda}$ is uniformly distributed on $[0,T]$ (conditionally on $\{Z_i^\lambda = 1\}$). We next remark that the conditional law of $\widetilde{T}_1^{i,\lambda}$ knowing $\{\widetilde{Z}_i^\lambda = 1\}$ (which we called $\zeta_1^{\lambda,T}$) is nothing but

$$\begin{aligned}\zeta_1^{\lambda,T}(\mathrm{d}t) &= \frac{\nu_M^\lambda(\mathrm{d}t)\mu_M^\lambda((\boldsymbol{a}_\lambda T - t,\infty))\mathbb{1}_{\{t\in[0,\boldsymbol{a}_\lambda T]\}}}{\int_0^{\boldsymbol{a}_\lambda T}\mu_M^\lambda((\boldsymbol{a}_\lambda T - s,\infty))\nu_M^\lambda(\mathrm{d}s)}\\ &= \frac{\mu_M^1((\lambda(\boldsymbol{a}_\lambda T - t),\infty))\lambda\mu_M^1((\lambda t,\infty))\mathbb{1}_{\{t\in[0,\boldsymbol{a}_\lambda T]\}}}{\lambda\boldsymbol{a}_\lambda T(1-\varepsilon_T(\lambda))}\,\mathrm{d}t,\end{aligned}$$

where we used the same computations as in step 3. Consequently, the conditional law of $\widetilde{T}_1^{i,\lambda}/\boldsymbol{a}_\lambda$ knowing $\{\widetilde{Z}_i^\lambda = 1\}$ has a density $g_{\lambda,T}$ of the form

$$g_{\lambda,T}(t) = \frac{1+\varepsilon_T(\lambda)}{T}\mu_M^1\big((\lambda\boldsymbol{a}_\lambda(T-t),\infty)\big)\mu_M^1\big((\lambda\boldsymbol{a}_\lambda t,\infty)\big)\mathbb{1}_{\{t\in[0,T]\}}.$$

Observe that $\lim_{\lambda\to 0} g_{\lambda,T}(t) = T^{-1}\mathbb{1}_{\{t\in[0,T]\}}$, since $\lambda\boldsymbol{a}_\lambda \to 0$. Hence, classical arguments (see Lemma 5.1.3 (ii)) show that conditionally on $\{Z_i^\lambda = \widetilde{Z}_i^\lambda = 1\}$, we can couple $T_1^{i,\lambda}$ and $\widetilde{T}_1^{i,\lambda}$ in such a way that

$$\Pr\big[T_1^{i,\lambda} = \widetilde{T}_1^{i,\lambda}/\boldsymbol{a}_\lambda \mid Z_i^\lambda = \widetilde{Z}_i^\lambda = 1\big] \geq \int_0^T \min\Big(\frac{1}{T}, g_{\lambda,T}(t)\Big)\,\mathrm{d}t,$$

which tends to 1 as $\lambda \to 0$ by dominated convergence.

Step 5. — We finally may build the complete coupling.

(i) For each $i \in I_A^\lambda$, we consider some coupled random variables $(Z_i^\lambda, \widetilde{Z}_i^\lambda)$ as in step 3.

(ii) For $i \in I_A^\lambda$ such that $Z_i^\lambda = \widetilde{Z}_i^\lambda = 0$, there is nothing to do.

(iii) For $i \in I_A^\lambda$ such that $Z_i^\lambda = \widetilde{Z}_i^\lambda = 1$, couple $T_1^{i,\lambda}$ and $\widetilde{T}_1^{i,\lambda}$ as in step 4 and pick $X_1^{i,\lambda}$ uniformly in i_λ.

(iv) If $i \in I_A^\lambda$ does not meet one of the two above conditions (ii) and (iii), then we build $(T_1^{i,\lambda}, X_1^{i,\lambda}), \dots, (T_{Z_i^\lambda}^{i,\lambda}, X_{Z_i^\lambda}^{i,\lambda})$ and $\widetilde{T}_1^{i,\lambda}, \dots, \widetilde{T}_{\widetilde{Z}_i^\lambda}^{i,\lambda}$ in any way (e.g., follow the rules of step 1 (ii) and step 2 (ii) independently).

(v) For $i \in I_A^\lambda$, $t \in [0, T\boldsymbol{a}_\lambda]$, set

$$\pi_M = \sum_{i \in I_A^\lambda} \sum_{k=1}^{Z_i^\lambda} \delta_{(T_k^{i,\lambda}, X_k^{i,\lambda})} \quad \text{and} \quad N_t^{M,\lambda}(i) = \sum_{k=1}^{\widetilde{Z}_i^\lambda} \mathbb{1}_{\{t \geq \widetilde{T}_k^{i,\lambda}\}}.$$

Step 6. — Define the event

$$\widetilde{\Omega}_{A,T}^M(\lambda) = \bigcap_{i \in I_A^\lambda} \Big(\{Z_i^\lambda = \widetilde{Z}_i^\lambda = 0\} \cup \{Z_i^\lambda = \widetilde{Z}_i^\lambda = 1,\ T_1^{i,\lambda} = \widetilde{T}_1^{i,\lambda}/\boldsymbol{a}_\lambda\} \Big).$$

Then we have $\widetilde{\Omega}_{A,T}^M(\lambda) \subset \Omega_{A,T}^M(\lambda)$ (where $\Omega_{A,T}^M(\lambda)$ was defined in the statement). Indeed, on $\widetilde{\Omega}_{A,T}^M(\lambda)$, for any $i \in I_A^\lambda$, $t \in [0,T]$, we have $\Delta N_{\boldsymbol{a}_\lambda t}^{M,\lambda}(i) \neq 0$ iff ($\widetilde{Z}_i^\lambda = 1$ and $\boldsymbol{a}_\lambda t = \widetilde{T}_1^{i,\lambda}$) iff ($Z_i^\lambda = 1$ and $t = T_1^{i,\lambda}$) iff $\pi_M(\{t\} \times i_\lambda) > 0$.

Finally, using steps 3 and 4 and that $|I_A^\lambda| = 2A_\lambda + 1$,

$$\begin{aligned}\Pr[\Omega_{A,T}^M(\lambda)] &\geq \Big(\Pr[Z_0^\lambda = \widetilde{Z}_0^\lambda = 0] + \Pr\Big[Z_0^\lambda = \widetilde{Z}_0^\lambda = 1,\ \ T_1^{0,\lambda} = \widetilde{T}_1^{0,\lambda}/\boldsymbol{a}_\lambda\Big]\Big)^{2A_\lambda+1} \\ &\geq \big(1 - \lambda \boldsymbol{a}_\lambda T(1+\varepsilon_T(\lambda)) + \lambda \boldsymbol{a}_\lambda T(1-\varepsilon_T(\lambda)) r_T(\lambda)\big)^{2A_\lambda+1}.\end{aligned}$$

Recall that $\lim_{\lambda\to 0} \varepsilon_T(\lambda) = 0$, that $\lim_{\lambda \to 0} r_T(\lambda) = 1$ and that $A_\lambda \leq A/(\lambda \boldsymbol{a}_\lambda)$. Hence for some (other) function ε_T with limit 0 at 0,

$$\Pr[\Omega_{A,T}^M(\lambda)] \geq (1 - \lambda \boldsymbol{a}_\lambda T \varepsilon_T(\lambda))^{2A/(\lambda \boldsymbol{a}_\lambda)+1}.$$

This last quantity tends to 1 as $\lambda \to 0$, which concludes the proof. □

3.3. Convergence proof when $\beta \in (0,\infty)$

We split this section into three parts. First, we handle some preliminary computations on SR(μ_S)-processes. Next, we show how to couple the set of times/locations where no seed fall (in the discrete model) with the Poisson measure π_S. Then we conclude the convergence proof. In the whole section, we assume (H_M) and $(H_S(\beta))$ for some $\beta \in (0,\infty)$. We recall that $\boldsymbol{a}_\lambda$ and $\boldsymbol{n}_\lambda$ are defined in (2.2.1) and (2.2.2). For $A > 0$ and $i \in \mathbb{Z}$, we set

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda \rfloor, \quad I_A^\lambda = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = \big[i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda\big).$$

3.3.1. Preliminary computations. — First, we will need the following estimate.

LEMMA 3.3.1. — *For any* $\ell \in (0,\infty)$ *fixed,*

$$\mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big) \sim m_S \beta \ell^{-\beta-1}\lambda \quad \text{as } \lambda \to 0.$$

Proof. — Recall that $\mu_S((t,\infty))\,\mathrm{d}t = m_S \nu_S(\mathrm{d}t)$. For $\alpha > 0$, one may write, using the monotonicity of $x \mapsto \mu_S((x,\infty))$,

$$\begin{aligned}
\frac{\mu_S((\boldsymbol{a}_\lambda \ell,\infty))}{\lambda} &\geq \frac{1}{\alpha\lambda\boldsymbol{a}_\lambda}\int_{\boldsymbol{a}_\lambda \ell}^{\boldsymbol{a}_\lambda(\ell+\alpha)} \mu_S\big((x,\infty)\big)\,\mathrm{d}x \\
&= \frac{m_S}{\alpha\lambda\boldsymbol{a}_\lambda}\big[\nu_S((\boldsymbol{a}_\lambda\ell,\infty)) - \nu_S((\boldsymbol{a}_\lambda(\ell+\alpha),\infty))\big] \\
&= \frac{m_S}{\alpha}\left[\frac{\nu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\nu_S((\boldsymbol{a}_\lambda,\infty))} - \frac{\nu_S((\boldsymbol{a}_\lambda(\ell+\alpha),\infty))}{\nu_S((\boldsymbol{a}_\lambda,\infty))}\right].
\end{aligned}$$

For the last equality, we used that by definition, $\nu_S((\boldsymbol{a}_\lambda,\infty)) = \lambda\boldsymbol{a}_\lambda$. Due to $(H_S(\beta))$, we deduce that for any $\alpha > 0$,

$$\liminf_{\lambda\to 0} \frac{\mu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\lambda} \geq \frac{m_S}{\alpha}\big[\ell^{-\beta} - (\ell+\alpha)^{-\beta}\big] \geq m_S\beta(\ell+\alpha)^{-\beta-1}.$$

One gets an upper bound by the same way: for any $\alpha \in (0,\ell)$,

$$\limsup_{\lambda\to 0} \frac{\mu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\lambda} \leq \limsup_{\lambda\to 0} \frac{1}{\alpha\lambda\boldsymbol{a}_\lambda}\int_{\boldsymbol{a}_\lambda(\ell-\alpha)}^{\boldsymbol{a}_\lambda\ell} \mu_S\big((x,\infty)\big)\,\mathrm{d}x \leq m_S\beta(\ell-\alpha)^{-\beta-1}.$$

We have proved that for any $\alpha \in (0,\ell)$,

$$m_S\beta(\ell+\alpha)^{-\beta-1} \leq \liminf_{\lambda\to 0}\frac{\mu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\lambda} \leq \limsup_{\lambda\to 0}\frac{\mu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\lambda} \leq m_S\beta(\ell-\alpha)^{-\beta-1}.$$

Making α tend to 0 allows us to conclude. □

Next, we compute the asymptotic probability that on a given site, no seed fall during some *large* time interval. By large, we mean with a length of order $\boldsymbol{a}_\lambda$.

LEMMA 3.3.2. — *Let* $(T_n)_{n\in\mathbb{Z}}$ *be a* SR(μ_S)*-process* (*see subsection* 2.1.1). *For* $\lambda > 0$, $t \geq 0$ *and* $\ell > 0$, *we set*

$$S_t^\lambda(\ell) = \#\big\{n \in \mathbb{Z} : T_n \in [0,\boldsymbol{a}_\lambda t],\ T_n - T_{n-1} \geq \boldsymbol{a}_\lambda \ell\big\},$$

which represents the number of delays with length greater than $\boldsymbol{a}_\lambda\ell$ *that end in* $[0,\boldsymbol{a}_\lambda t]$.

(i) *For* $t > 0$ *and* $\ell > 0$ *fixed, as* $\lambda \to 0$, $\Pr\big[S_t^\lambda(\ell) = 1\big] \sim t\lambda\boldsymbol{a}_\lambda\beta\ell^{-\beta-1}$.

(ii) *For* $t > 0$ *and* $\ell > 0$ *fixed,* $\limsup_{\lambda\to 0}(\lambda\boldsymbol{a}_\lambda)^{-2}\Pr\big[S_t^\lambda(\ell) \geq 2\big] < \infty$.

(iii) *On the event* $\{S_t^\lambda(\ell) = 1\}$, *we put*

$$\tau := T_n \quad \text{and} \quad L = T_n - T_{n-1},$$

where n *is the unique index such that* $T_n \in [0,t]$ *and* $T_n - T_{n-1} \geq \boldsymbol{a}_\lambda\ell$. *For all* $s \in [0,t]$, *all* $x \in (\ell,\infty)$,

$$\lim_{\lambda\to 0}\Pr[\tau/\boldsymbol{a}_\lambda \leq s,\ L/\boldsymbol{a}_\lambda \geq x \mid S_t^\lambda(\ell) = 1] = (s/t)(x/\ell)^{-\beta-1}.$$

Proof. — Let us recall that the SR(μ_S)-process $(T_n)_{n\in\mathbb{Z}}$ is built as follows: one considers an i.i.d. sequence $(X_i)_{i\in\mathbb{Z}\setminus\{0\}}$ of μ_S-distributed r.v., X_0 a $x\mu_S(\mathrm{d}x)/m_S$-distributed r.v. and U uniformly distributed on $[0,1]$. Then we set

$$T_0 = -(1-U)X_0, \quad T_1 = UX_0$$

and for all $n \geq 1$,

$$T_{n+1} = T_n + X_n \quad \text{and} \quad T_{-n} = T_{-n+1} - X_{-n}.$$

We also introduce, for $\lambda > 0$, $\ell > 0$ and $0 \leq s \leq t$

$$S^{\lambda}_{s,t}(\ell) = \#\big\{n \in \mathbb{Z} \,:\, T_n \in [\boldsymbol{a}_\lambda s, \boldsymbol{a}_\lambda t],\ T_n - T_{n-1} \geq \boldsymbol{a}_\lambda \ell\big\}.$$

Step 1. — First assume that $\ell \geq t$. Then by construction, $S^{\lambda}_t(\ell) \in \{0,1\}$ and

$$\big\{S^{\lambda}_t(\ell) = 1\big\} = \big\{T_1 \leq \boldsymbol{a}_\lambda t,\ T_1 - T_0 \geq \boldsymbol{a}_\lambda \ell\big\} = \big\{UX_0 \leq \boldsymbol{a}_\lambda t,\ X_0 \geq \boldsymbol{a}_\lambda \ell\big\}.$$

Hence

$$\begin{aligned}\mathbb{E}\big[S^{\lambda}_t(\ell)\big] &= \Pr\big[UX_0 \leq \boldsymbol{a}_\lambda t,\ X_0 \geq \boldsymbol{a}_\lambda \ell\big] \\ &= \int_{\boldsymbol{a}_\lambda \ell}^{\infty} \frac{x\mu_S(\mathrm{d}x)}{m_S} \int_0^1 \mathrm{d}u \mathbf{1}_{\{ux \leq \boldsymbol{a}_\lambda t\}} \\ &= \int_{\boldsymbol{a}_\lambda \ell}^{\infty} \frac{x\mu_S(\mathrm{d}x)}{m_S} \frac{\boldsymbol{a}_\lambda t}{x} = \frac{\boldsymbol{a}_\lambda t}{m_S} \mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big).\end{aligned}$$

We used here that since $\ell \geq t$, for $x \geq \boldsymbol{a}_\lambda \ell$, there holds $\boldsymbol{a}_\lambda t/x \leq 1$.

Step 2. — We now show that for any $\ell > 0$, any $t \geq 0$,

$$\mathbb{E}[S^{\lambda}_t(\ell)] = \frac{\boldsymbol{a}_\lambda t}{m_S} \mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big).$$

Consider $n \geq 1$ such that $t/n \leq \ell$ and observe that

$$S^{\lambda}_t(\ell) = \sum_{i=0}^{n-1} S^{\lambda}_{it/n,(i+1)t/n}(\ell).$$

By stationarity, we have $\mathbb{E}[S^{\lambda}_{it/n,(i+1)t/n}(\ell)] = \mathbb{E}[S^{\lambda}_{t/n}(\ell)]$ for $i = 0, \ldots, n-1$, which is nothing but $\frac{\boldsymbol{a}_\lambda t}{n m_S} \mu_S((\boldsymbol{a}_\lambda \ell, \infty))$ by step 1. The conclusion follows by linearity of expectation.

Step 3. — We now check point (ii). Let

$$\begin{aligned}\rho_1 &= \inf\big\{T_n :\ n \in \mathbb{N}, T_n - T_{n-1} \geq \boldsymbol{a}_\lambda \ell, T_n > 0\big\}, \\ \rho_2 &= \inf\big\{T_n :\ n \in \mathbb{N}, T_n - T_{n-1} \geq \boldsymbol{a}_\lambda \ell, T_n > \rho_1\big\}.\end{aligned}$$

Then $\Pr[S^{\lambda}_t(\ell) \geq 2] = \Pr[\rho_2 \leq \boldsymbol{a}_\lambda t]$. We also observe that

$$\Pr[\rho_1 \leq \boldsymbol{a}_\lambda t] = \Pr\big[S^{\lambda}_t(\ell) \geq 1\big] \leq \mathbb{E}\big[S^{\lambda}_t(\ell)\big] = \boldsymbol{a}_\lambda t \mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big)/m_S.$$

Denote by $\zeta_{\lambda,\ell}$ the law of $\rho_1/\boldsymbol{a}_\lambda$. Then a renewal argument shows that

$$\Pr\big[S_t^\lambda(\ell)\geq 2\big]=\int_0^t \zeta_{\lambda,\ell}(\mathrm{d}r)f(\lambda,\ell,t-r),$$

where

$$f(\lambda,\ell,s)=\Pr\big[\exists\, n\geq 1; X_n\geq \boldsymbol{a}_\lambda\ell; X_1+\cdots+X_n\leq \boldsymbol{a}_\lambda s\big].$$

We can rewrite this as (recall that $T_1=UX_0\sim\nu_S$)

$$\begin{aligned}
&f(\lambda,\ell,s)\\
&\quad=\Pr[\exists\, n\geq 1; X_n\geq \boldsymbol{a}_\lambda\ell; UX_0+X_1+\cdots+X_n\leq \boldsymbol{a}_\lambda s+UX_0]\\
&\quad\leq \Pr\big[\exists\, n\geq 0; X_n\geq \boldsymbol{a}_\lambda\ell; UX_0+X_1+\cdots+X_n\leq \boldsymbol{a}_\lambda(s+1)\big]+\Pr[UX_0\geq \boldsymbol{a}_\lambda]\\
&\quad=\Pr\big[S_{s+1}^\lambda(\ell)\geq 1\big]+\nu_S\big((\boldsymbol{a}_\lambda,\infty)\big)\\
&\quad=\frac{\boldsymbol{a}_\lambda(s+1)}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)+\lambda\boldsymbol{a}_\lambda
\end{aligned}$$

thanks to step 2. As a conclusion,

$$\begin{aligned}
\Pr[S_t^\lambda(\ell)\geq 2]&\leq\Big(\frac{\boldsymbol{a}_\lambda(t+1)}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)+\lambda\boldsymbol{a}_\lambda\Big)\int_0^t\zeta_{\lambda,\ell}(\mathrm{d}r)\\
&=\Big(\frac{\boldsymbol{a}_\lambda(t+1)}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)+\lambda\boldsymbol{a}_\lambda\Big)\Pr[\rho_1\leq \boldsymbol{a}_\lambda T]\\
&\leq\Big(\frac{\boldsymbol{a}_\lambda(t+1)}{m_S}\mu_S((\boldsymbol{a}_\lambda\ell,\infty))+\lambda\boldsymbol{a}_\lambda\Big)\frac{\boldsymbol{a}_\lambda t}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big).
\end{aligned}$$

Due to Lemma 3.3.1, this last term is equivalent to $(\lambda\boldsymbol{a}_\lambda)^2[(t+1)\beta\ell^{-\beta-1}+1]t\beta\ell^{-\beta-1}$, from which point (ii) follows.

Step 4. — Steps 2 and 3 imply point (i). Indeed, we clearly have

$$\Pr\big[S_t^\lambda(\ell)=1\big]\leq\mathbb{E}\big[S_t^\lambda(\ell)\big]=\frac{\boldsymbol{a}_\lambda t}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)\sim t\lambda\boldsymbol{a}_\lambda\beta\ell^{-\beta-1}$$

by Lemma 3.3.1. Next, using that $S_t^\lambda(\ell)\leq 1+t/\ell$ by construction,

$$\begin{aligned}
\Pr\big[S_t^\lambda(\ell)=1\big]&=\mathbb{E}\big[S_t^\lambda(\ell)\mathbb{1}_{\{S_t^\lambda(\ell)=1\}}\big]=\mathbb{E}\big[S_t^\lambda(\ell)\big]-\mathbb{E}\big[S_t^\lambda(\ell)\mathbb{1}_{\{S_t^\lambda(\ell)\geq 2\}}\big]\\
&\geq\frac{\boldsymbol{a}_\lambda t}{m_S}\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)-(1+t/\ell)\Pr[S_t^\lambda(\ell)\geq 2].
\end{aligned}$$

Point (ii) allows us to conclude easily.

Step 5. — It remains to check (iii). We thus fix $0\leq s\leq t$ and $0<\ell<x$. Then as $\lambda\to 0$, and due to point (i)

$$\begin{aligned}
\Pr\big[\tau/\boldsymbol{a}_\lambda<s,\ L/\boldsymbol{a}_\lambda>x\mid S_t^\lambda(\ell)=1\big]&=\frac{\Pr[S_s^\lambda(x)=1]}{\Pr[S_t^\lambda(\ell)=1]}\\
&\sim\frac{s\lambda\boldsymbol{a}_\lambda\beta x^{-\beta-1}}{t\lambda\boldsymbol{a}_\lambda\beta\ell^{-\beta-1}}=(s/t)(x/\ell)^{-\beta-1}.\qquad\square
\end{aligned}$$

3.3.2. Coupling of seeds. — We aim to couple the Poisson measure $\pi_S(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\ell)$ used to define the LFF(β)-process with times/places where seeds do not fall in the FF(μ_S, μ_M^λ)-process. We would like that roughly, $\pi_S(\{t\} \times i_\lambda \times \{\ell\}) = 1$ if and only if no seed falls on i during $[\boldsymbol{a}_\lambda(t-\ell), \boldsymbol{a}_\lambda t]$ (and if this is the maximal interval, that is seeds fall in i at times $\boldsymbol{a}_\lambda(t-\ell)$ and $\boldsymbol{a}_\lambda t$). We have to consider the finite Poisson measure π_S restricted to the set $\{\ell > \delta\}$, for some arbitrarily small $\delta > 0$.

Proposition 3.3.3. — *Let $A > 0$, $T > 0$, $\alpha > 0$ and $\delta > 0$ be fixed. For any $\lambda \in (0,1]$, it is possible to find a coupling between a Poisson measure $\pi_S(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\ell)$ on $[0,\infty) \times \mathbb{R} \times [0,\infty)$ with intensity measure $\beta(\beta+1)\ell^{-\beta-2}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\ell$ and an i.i.d. family of* SR(μ_S)*-processes $(\widetilde{T}_n^i)_{i\in\mathbb{Z},n\in\mathbb{Z}}$ (recall subsection* 2.1.1) *in such a way that for*

$$S_T^\lambda(\delta,i) = \pi_S\big([0,T] \times i_\lambda \times [\delta,\infty)\big),$$
$$\widetilde{S}_T^\lambda(\delta,i) = \#\big\{n \geq 1 \,:\, \widetilde{T}_n^i \in [0,\boldsymbol{a}_\lambda T],\ \widetilde{T}_n^i - \widetilde{T}_{n-1}^i \geq \boldsymbol{a}_\lambda \delta\big\},$$

setting

$$\Omega_{A,T,\delta,\alpha}^S(\lambda) := \bigcap_{i\in I_A^\lambda} \Big(\big\{S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 0\big\} \cup \Big\{S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1, \\ \Big|\tau_T^\lambda(\delta,i) - \frac{\widetilde{\tau}_T^\lambda(\delta,i)}{\boldsymbol{a}_\lambda}\Big| + \Big|L_T^\lambda(\delta,i) - \frac{\widetilde{L}_T^\lambda(\delta,i)}{\boldsymbol{a}_\lambda}\Big| < \alpha\Big\}\Big),$$

there holds

$$\lim_{\lambda\to 0} \Pr\big(\Omega_{A,T,\delta,\alpha}^S(\lambda)\big) = 1.$$

On the event $\{S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1\}$, we have denoted by $(\tau_T^\lambda(\delta,i), L_T^\lambda(\delta,i))$ the unique element $(t,\ell) \in [0,T] \times [\delta,\infty)$ such that $\pi_S(\{t\} \times i_\lambda \times \{\ell\}) = 1$ and we have put $\widetilde{\tau}_T^\lambda(\delta,i) = \widetilde{T}_n^i$ and $\widetilde{L}_T^\lambda(\delta,i) = \widetilde{T}_n^i - \widetilde{T}_{n-1}^i$, where $n \geq 1$ is the unique element of $\mathbb{N}$ such that $\widetilde{T}_n^i \in [0,\boldsymbol{a}_\lambda T]$ and $\widetilde{T}_n^i - \widetilde{T}_{n-1}^i \geq \boldsymbol{a}_\lambda \delta$.

Proof. — We fix $T > 0$, $A > 0$, $\delta > 0$ and $\alpha > 0$. We divide the proof into several steps. Observe that

$$B_\lambda := \bigcup_{i\in I_A^\lambda} i_\lambda = \big[-A_\lambda/\boldsymbol{n}_\lambda, (A_\lambda+1)/\boldsymbol{n}_\lambda\big)$$

(which is approximately $[-A,A]$). It of course suffices to build π_S restricted to $[0,T] \times B_\lambda \times [\delta,\infty)$ (we abusively still denote by π_S this restriction) and the family $(\widetilde{T}_n^i)$ for $i \in I_A^\lambda$ and $n \geq 0$ (with our notation, we have $\widetilde{T}_0^i \leq 0 \leq \widetilde{T}_1^i$).

Step 1. — A possible way to build π_S (restricted to $[0,T] \times B_\lambda \times [\delta,\infty)$) is the following.

(i) Consider a family of i.i.d. r.v. $(S_T^\lambda(\delta,i))_{i\in I_A^\lambda}$ following a Poisson distribution with parameter

$$T|i_\lambda| \int_\delta^\infty \beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell = \beta\delta^{-\beta-1}T/\boldsymbol{n}_\lambda.$$

(ii) For each $i \in I_A^\lambda$ with $S_T^\lambda(\delta, i) > 0$, pick some i.i.d. r.v.

$$\big\{(T_k^{i,\lambda}, X_k^{i,\lambda}, L_k^{i,\lambda})\big\}_{k=1,\dots,S_T^\lambda(\delta,i)}$$

with density $\mathbb{1}_{\{t\in[0,T], x\in i_\lambda, \ell>\delta\}}(\beta+1)\boldsymbol{n}_\lambda \delta^{\beta+1}\ell^{-\beta-2}/T$. Put

$$\pi_S = \sum_{i\in I_A^\lambda} \sum_{k=1}^{S_T^\lambda(\delta,i)} \delta_{(T_k^{i,\lambda}, X_k^{i,\lambda}, L_k^{i,\lambda})}.$$

Step 2. — Next, we note it is possible to build the family $(\widetilde{T}_n^i)_{i\in I_A^\lambda, n\geq 0}$ as follows: denote by $q_k(\lambda) = \Pr[\widetilde{S}_T^\lambda(\delta, i) = k]$ and by Λ_k^λ the law of $(\widetilde{T}_n^i)_{n\geq 0}$ conditionally on $\{\widetilde{S}_T^\lambda(\delta, i) = k\}$.

(i) Consider a family of i.i.d. r.v. $(\widetilde{S}_T^\lambda(\delta, i))_{i\in I_A^\lambda}$ with law $(q_k(\lambda))_{k\geq 0}$.

(ii) For each $i \in I_A^\lambda$, pick $(\widetilde{T}_n^i)_{n\geq 0}$ according to $\Lambda^\lambda_{\widetilde{S}_T^\lambda(\delta,i)}$ (conditionally on $\widetilde{S}_T^\lambda(\delta, i)$).

Step 3. — For each i, it is possible to couple $S_T^\lambda(\delta, i)$ and $\widetilde{S}_T^\lambda(\delta, i)$, distributed as in step 1 (i) and step 2 (i), in such a way that

$$\Pr\big[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 0\big] \geq 1 - \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1+\varepsilon_{T,\delta}(\lambda)\big),$$
$$\Pr\big[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1\big] \geq \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1-\varepsilon_{T,\delta}(\lambda)\big),$$

where $\lim_{\lambda\to 0}\varepsilon_{T,\delta}(\lambda) = 0$. It is classically possible (see Lemma 5.1.3 (i)) to build a coupling in such a way that

$$\Pr\big[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 0\big] \geq \Pr\big(S_T^\lambda(\delta,i) = 0\big) \wedge \Pr\big(\widetilde{S}_T^\lambda(\delta,i) = 0\big),$$
$$\Pr\big[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1\big] \geq \Pr\big(S_T^\lambda(\delta,i) = 1\big) \wedge \Pr\big(\widetilde{S}_T^\lambda(\delta,i) = 1\big).$$

First, we infer from Lemma 3.3.2 that

$$\Pr\big(\widetilde{S}_T^\lambda(\delta,i) = 0\big) \geq 1 - \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1+\varepsilon_{T,\delta}(\lambda)\big),$$
$$\Pr\big(\widetilde{S}_T^\lambda(\delta,i) = 1\big) \geq \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1-\varepsilon_{T,\delta}(\lambda)\big).$$

Next, since $S_T^\lambda(\delta, i)$ follows a Poisson distribution with parameter $\beta\delta^{-\beta-1}T/\boldsymbol{n}_\lambda \sim \lambda\boldsymbol{a}_\lambda\beta\delta^{-\beta-1}T$, we have

$$\Pr\big(S_T^\lambda(\delta,i) = 0\big) = \mathrm{e}^{-\beta\delta^{-\beta-1}T/\boldsymbol{n}_\lambda} \geq 1 - \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1+\varepsilon_{T,\delta}(\lambda)\big)$$

and there holds

$$\Pr\big(S_T^\lambda(\delta,i) = 1\big) = [\beta\delta^{-\beta-1}T/\boldsymbol{n}_\lambda]\,\mathrm{e}^{-\beta\delta^{-\beta-1}T/\boldsymbol{n}_\lambda} \geq \lambda \boldsymbol{a}_\lambda \beta\delta^{-\beta-1}T\big(1-\varepsilon_{T,\delta}(\lambda)\big).$$

This concludes the step.

Step 4. — We now check that for each $i \in I_A^\lambda$, conditionally on

$$\big\{S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1\big\},$$

we can couple $(T_1^{i,\lambda}, L_1^{i,\lambda}, X^{i,\lambda,1})$ and $(\widetilde{T}_n^i)_{n\geq 0}$ in such a way that $\lim_{\lambda\to 0} r_{T,\delta,\alpha}(\lambda) = 1$ for (see the statement)

$$r_{T,\delta,\alpha}(\lambda) = \Pr\Big[\ \Big|\tau_T^\lambda(\delta,i) - \frac{\widetilde{\tau}_T^\lambda(\delta,i)}{\boldsymbol{a}_\lambda}\Big| + \Big|L_T^\lambda(\delta,i) - \frac{\widetilde{L}_T^\lambda(\delta,i)}{\boldsymbol{a}_\lambda}\Big| < \alpha \mid Z_i^\lambda = \widetilde{Z}_i^\lambda = 1\Big].$$

To this end, consider $(\widetilde{T}_n^i)_{n\geq 0}$ with law Λ_1^λ (recall step 2). Denote by

$$p_{\delta,T}^\lambda(\mathrm{d}t, \mathrm{d}\ell)$$

the law of $(\widetilde{\tau}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda, \widetilde{L}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda)$ (under Λ_1^λ). We know from Lemma 3.3.2 (iii) that $p_{\delta,T}^\lambda(\mathrm{d}t, \mathrm{d}\ell)$ goes weakly, as $\lambda \to 0$, to

$$p_{\delta,T}(\mathrm{d}t, \mathrm{d}\ell) := T^{-1}(\beta+1)\delta^{\beta+1}\ell^{-\beta-2}\mathbb{1}_{\{t\in[0,T], l\geq\delta\}}\,\mathrm{d}t\,\mathrm{d}\ell.$$

Indeed, observe that $p_{\delta,T}([0,s]\times[x,\infty)) = (s/T)(x/\delta)^{-\beta-1}$ for $s\in[0,T]$ and $x>\delta$.

But $p_{\delta,T}(\mathrm{d}t, \mathrm{d}\ell)$ is nothing but the law of $(\tau_T^\lambda(\delta,i), L_T^\lambda(\delta,i)) = (T_1^{i,\lambda}, L_1^{i,\lambda})$ conditionally on $\{S_T^\lambda(\delta,i) = 1\}$ (recall step 1 (ii)). We easily conclude: first, we couple $(\widetilde{\tau}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda, \widetilde{L}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda)$ and $(\tau_T^\lambda(\delta,i), L_T^\lambda(\delta,i))$ in such a way that they are close to each other (with a distance smaller than α) with high probability (tending to 1 when $\lambda\to 0$), using Lemma 5.1.3 (iii). Then we choose $X_1^{i,\lambda}$ at random, uniformly in i_λ, independently of everything else and finally, we pick $(\widetilde{T}_n^i)_{n\geq 0}$ conditionally on $\{\widetilde{S}_T^\lambda(\delta,i) = 1\}$ and $(\widetilde{\tau}_T^\lambda(\delta,i), \widetilde{L}_T^\lambda(\delta,i))$.

Step 5. — We finally may build the complete coupling.

(i) For each $i\in I_A^\lambda$, consider some coupled r.v. $(S_T^\lambda(\delta,i), \widetilde{S}_T^\lambda(\delta,i))$ as in step 3.

(ii) For $i\in I_A^\lambda$ such that $S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1$, couple $(T_1^{i,\lambda}, L_1^{i,\lambda}, X_1^{i,\lambda})$ and $(\widetilde{T}_n^i)_{n\geq 0}$ as in step 4.

(iii) For $i\in I_A^\lambda$ not meeting the above condition (ii), follow the rules of step 1 (ii) to build $(T_k^{i,\lambda}, X_k^{i,\lambda}, L_k^{i,\lambda})_{1\leq k\leq S_T^\lambda(\delta,i)}$ and the rules of step 2 (ii) to build $\{\widetilde{T}_n^i\}_{n\geq 0}$ (e.g. independently).

This defines $\{\widetilde{T}_n^i\}_{n\geq 0, i\in I_A^\lambda}$ and $\pi_S := \sum_{i\in I_A^\lambda}\sum_{k=1}^{S_T^\lambda(\delta,i)}\delta_{(T_k^{i,\lambda}, X_k^{i,\lambda}, L_k^{i,\lambda})}$.

Step 6. — With this coupling, using steps 3 and 4 and that $|I_A^\lambda| = 2A_\lambda+1$,

$$\begin{aligned}\Pr[\Omega^S_{A,T,\delta,\alpha}]&\\ \geq \Big(&\Pr[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 0] + \Pr[S_T^\lambda(\delta,i) = \widetilde{S}_T^\lambda(\delta,i) = 1,\\ &\qquad |\tau_T^\lambda(\delta,i) - \widetilde{\tau}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda| + L_T^\lambda(\delta,i) - \widetilde{L}_T^\lambda(\delta,i)/\boldsymbol{a}_\lambda| < \alpha]\Big)^{2A_\lambda+1}\\ \geq \big(&1 - \lambda\boldsymbol{a}_\lambda\beta\delta^{\beta-1}T(1+\varepsilon_{T,\delta}(\lambda)) + \lambda\boldsymbol{a}_\lambda\beta\delta^{\beta-1}T(1-\varepsilon_{T,\delta}(\lambda))r_{T,\delta,\alpha}(\lambda)\big)^{2A_\lambda+1}.\end{aligned}$$

Recall that $\lim_{\lambda\to 0}\varepsilon_{T,\delta}(\lambda) = 0$, that $\lim_{\lambda\to 0} r_{T,\delta,\alpha}(\lambda) = 1$ and that $A_\lambda \leq A/(\lambda\boldsymbol{a}_\lambda)$. Hence for some function $\varepsilon_{T,\delta,\alpha}$ with limit 0 at 0,

$$\Pr[\Omega^S_{A,T,\delta,\alpha}] \geq \big(1 - \lambda\boldsymbol{a}_\lambda\beta\delta^{\beta-1}T\varepsilon_{T,\delta,\alpha}(\lambda)\big)^{2A/(\lambda\boldsymbol{a}_\lambda)+1}.$$

This last quantity tends to 1 as $\lambda\to 0$, which concludes the proof. □

3.3.3. Convergence. — We are now able to conclude. Intuitively, the situation is clear: using Proposition 3.2.1, we couple the time/positions at which matches fall in the LFF(β)-process with those of the FF(μ_S, μ_M^λ)-process; and using Proposition 3.3.3, we couple the time/positions at which no seed fall in the LFF(β)-process with time/positions at which no seed fall during a time interval of length of order $\boldsymbol{a}_\lambda$ in the FF(μ_S, μ_M^λ)-process. Then we only have to show carefully that this is sufficient to couple the FF(μ_S, μ_M^λ)-process and the LFF(β)-process in such a way that they remain close to each other. But there are many technical problems: our couplings concern only finite boxes $[0,T] \times [-A,A]$, do not allow to treat *small* time intervals with no seed falling, etc. We thus have to localize the processes in space and time and to work on an event (with high probability) on which everything works as desired.

Proof of Theorem 2.5.3. — We fix $T > 0$, $x_1 < \cdots < x_p$ and $t_1, \dots, t_p \in [0,T]$. We introduce $B > 0$ such that $-B < x_1 < x_p < B$. We fix $\varepsilon > 0$ and $a > 0$. Our aim is to check that for all $\lambda > 0$ small enough, there exists a coupling between a FF(μ_S, μ_M^λ)-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ and a LFF(β)-process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ such that, recalling (2.2.3) and Proposition 2.5.2,

$$\Pr\Big[\sum_{k=1}^p \boldsymbol{\delta}_T(D^\lambda(x_k), D(x_k)) + \sum_{k=1}^p \boldsymbol{\delta}(D_{t_k}^\lambda(x_k), D_{t_k}(x_k)) \geq a\Big] \leq \varepsilon. \tag{3.3.1}$$

This will of course conclude the proof.

Step 1. — Consider two independent Poisson measures $\pi_S(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\ell)$ with intensity measure $\beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\ell$ and $\pi_M(\mathrm{d}t, \mathrm{d}x)$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$. Set, for $A > B$,

$$\begin{aligned}\Omega_{A,T}^{S,1} := &\big\{\pi_S(\{(t,x,\ell)\,:\, x\in[B,A], t > T+1, \ell > t+1\}) > 0\big\}\\ &\cap \big\{\pi_S(\{(t,x,\ell)\,:\, x\in[-A,-B], t > T+1, \ell > t+1\big\}) > 0\}.\end{aligned}$$

A simple computation shows that

$$\Pr[\Omega_{A,T}^{S,1}] \geq 1 - 2\exp\Big(-\int_B^A \mathrm{d}x \int_{T+1}^\infty \mathrm{d}t \int_{t+1}^\infty \beta(\beta+1)\ell^{-\beta-2}\Big),$$

so that we can choose A large enough in such a way that $\Pr[\Omega_{A,T}^{S,1}] \geq 1 - \frac{1}{6}\varepsilon$. This will ensure us that there are $\chi_g \in [-A,-B]$ and $\chi_d \in [A,B]$ with $Y_t(\chi_g) = Y_t(\chi_d) = 0$ for all $t \in [0, T+1]$ (recall Figure 4). This fixes the value of A for the whole proof.

Next we consider $T_0 > T+1$ large enough, so that for

$$\Omega_{A,T,T_0}^{S,2} = \big\{\pi_S(\{(t,x,\ell)\,:\, t > T_0, t-\ell < T+1, x\in[-A,A]\}) = 0\big\},$$

$\Pr[\Omega_{A,T,T_0}^{S,2}] \geq 1 - \frac{1}{6}\varepsilon$. This is possible, because

$$\Pr[\Omega_{A,T,T_0}^{S,2}] = \exp\Big(-\int_{-A}^A \mathrm{d}x \int_{T_0}^\infty \mathrm{d}t \int_{t-(T+1)}^\infty \beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell\Big),$$

which clearly tends to 1 as T_0 increases to infinity. This will ensure us that all the dotted vertical segments in $[-A, A]$ that intersect $[0, T+1]$ end before T_0 (see Figure 4). This fixes the value of T_0 for the whole proof. Next we call

$$\mathcal{X}_M = \big\{x \in [-A, A] \,:\, \pi_M([0,T] \times \{x\}) > 0\big\},$$
$$\mathcal{T}_M = \big\{t \in [0,T] \,:\, \pi_M(\{t\} \times [-A, A]) > 0\big\} \cup \{0\}.$$

Classical results about Poisson measures allow us to choose $K_M > 0$ (large) and $c_M > 0$ (small) in such a way that $\Pr[\Omega^{M,1}_{K_M,c_M}] \geq 1 - \frac{1}{6}\varepsilon$ for

$$\Omega^{M,1}_{K_M,c_M} = \Big\{|\mathcal{T}_M| \leq K_M,\ \min_{\substack{t,s\in\mathcal{T}_M\\ s\neq t}} |t-s| > c_M,\ \min_{\substack{t\in\mathcal{T}_M\\ k=1,\dots,p}} |t-t_k| > c_M,$$
$$\min_{\substack{x\in\mathcal{X}_M\\ k=1,\dots,p}} |x - x_k| > c_M\Big\}.$$

We can now fix $\delta > 0$ for the whole proof, in such a way that

$$\delta < \tfrac{1}{4} c_M \quad \text{and} \quad \delta < \frac{a}{8ApK_M}\cdot$$

We use this δ to cutoff the Poisson measure π_S (in order that it has only a finite number of marks) without affecting the values of the LFF(β)-process in the zone under study.

Next, we consider the finite Poisson measure π_S^{A,δ,T_0} defined as the restriction of π_S to the set $[0, T_0] \times [-A, A] \times [\delta, \infty)$. We define

$$\mathcal{X}^\delta_S = \big\{x \in [-A, A] \,:\, \pi_S([0, T_0] \times \{x\} \times [\delta, \infty)) > 0\big\},$$
$$\mathcal{T}^\delta_S = \Big(\bigcup_{(t,x,\ell)\in \text{ supp } \pi_S^{A,\delta,T_0}} \{t, t-\ell\}\Big) \cap [0, T].$$

Then for $K_S > 0$ large enough and $c_S > 0$ small enough, the event

$$\Omega^{S,3}_{K_S,c_S,\delta} = \Big\{|\mathcal{T}^\delta_S| \leq K_S,\ \min_{\substack{t\in\mathcal{T}_M\\ s\in\mathcal{T}^\delta_S}} |t-s| > c_S,\ \min_{\substack{t\in\mathcal{T}^\delta_S\\ k=1,\dots,p}} |t-t_k| > c_S,$$
$$\min_{\substack{x,y\in\mathcal{X}^\delta_S\\ x\neq y}} |x-y| > c_S,\ \min_{\substack{x\in\mathcal{X}^\delta_S\\ y\in\mathcal{X}_M}} |x-y| > c_S,\ \min_{\substack{x\in\mathcal{X}^\delta_S\\ k=1,\dots,p}} |x-x_k| > c_S\Big\}$$

satisfies $\Pr[\Omega^{S,3}_{K_S,c_S,\delta}] \geq 1 - \frac{1}{6}\varepsilon$. Finally, we fix $\alpha > 0$ in such a way that

$$\alpha < \tfrac{1}{4} c_S, \quad \alpha < \tfrac{1}{4} c_M, \quad \alpha < \tfrac{1}{2} \quad \text{and} \quad \alpha < a/(8Ap(2K_S + K_M)).$$

Step 2. — Using Proposition 3.2.1, we know that for all $\lambda > 0$ small enough, it is possible to couple a family of i.i.d. SR(μ^λ_M)-processes $(N^{M,\lambda}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ with π_M in such a way that

$$\Omega^M_{A,T}(\lambda) := \big\{\forall t \in [0,T],\ \forall i \in I^\lambda_A,\ \Delta N^{M,\lambda}_{\boldsymbol{a}_\lambda t}(i) \neq 0 \ \text{ iff } \ \pi_M(\{t\} \times i_\lambda) \neq 0\big\}$$

satisfies $\Pr[\Omega^{M}_{A,T}(\lambda)] \geq 1 - \frac{1}{6}\varepsilon$. We infer from Proposition 3.3.3 that for all $\lambda > 0$ small enough, it is possible to couple an i.i.d. family of SR(μ_S)-processes $(\widetilde{T}^i_n)_{i\in\mathbb{Z},n\geq 0}$ with π_S in such a way that for

$$S^{\lambda}_{T_0}(\delta, i) = \pi_S\big([0, T_0] \times \{i_\lambda\} \times [\delta, \infty)\big),$$

$$\widetilde{S}^{\lambda}_{T_0}(\delta, i) = \#\big\{n \geq 1 \,:\, \widetilde{T}^i_n \in [0, \boldsymbol{a}_\lambda T_0],\ \widetilde{T}^i_n - \widetilde{T}^i_{n-1} \geq \boldsymbol{a}_\lambda \delta\big\},$$

setting

$$\Omega^{S}_{A,T_0,\delta,\alpha}(\lambda) := \bigcap_{i\in I^\lambda_A} \Big(\big\{S^{\lambda}_{T_0}(\delta, i) = \widetilde{S}^{\lambda}_{T_0}(\delta, i) = 0\big\} \cup \Big\{S^{\lambda}_{T_0}(\delta, i) = \widetilde{S}^{\lambda}_{T_0}(\delta, i) = 1, \Big|\tau^\lambda(\delta, i) - \frac{\widetilde{\tau}^\lambda(\delta, i)}{\boldsymbol{a}_\lambda}\Big| + \Big|L^\lambda(\delta, i) - \frac{\widetilde{L}^\lambda(\delta, i)}{\boldsymbol{a}_\lambda}\Big| < \alpha\Big\}\Big),$$

$\Pr(\Omega^{S}_{A,T_0,\delta,\alpha}(\lambda)) \geq 1 - \frac{1}{6}\varepsilon$. On the event $\{S^{\lambda}_{T_0}(\delta, i) = \widetilde{S}^{\lambda}_{T_0}(\delta, i) = 1\}$, we have denoted by $(\tau^\lambda(\delta, i), L^\lambda(\delta, i))$ the unique element $(t, \ell) \in [0, T_0] \times [\delta, \infty)$ such that $\pi_S(\{t\} \times i_\lambda \times \{\ell\}) = 1$ and we have put

$$\widetilde{\tau}^\lambda(\delta, i) = \widetilde{T}^i_n \quad \text{and} \quad \widetilde{L}^\lambda(\delta, i) = \widetilde{T}^i_n - \widetilde{T}^i_{n-1},$$

where $n \geq 1$ is the unique element of $\mathbb{N}$ such that $\widetilde{T}^i_n \in [0, \boldsymbol{a}_\lambda T_0]$ and $\widetilde{T}^i_n - \widetilde{T}^i_{n-1} \geq \boldsymbol{a}_\lambda \delta$. We put

$$N^S_t(i) = \sum_{n\geq 1} \mathbb{1}_{\{\widetilde{T}^i_n \geq t\}}$$

for all $i \in \mathbb{Z}$, all $t \geq 0$, which is a family of i.i.d. SR(μ_S)-processes in the sense of Definition 2.1.1, see subsection 2.1.1.

Step 3. — We work with the FF(μ_S, μ^λ_M)-process $(\eta^\lambda_t(i))_{t\geq 0,i\in\mathbb{Z}}$ built from $(N^S_t(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N^{M,\lambda}_t(i))_{t\geq 0,i\in\mathbb{Z}}$ and the LFF(β)-process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ built from π_S and π_M, all these processes being coupled as in step 2. We consider the associated clusters $(D^\lambda_t(x))_{t\geq 0,x\in\mathbb{R}}$ and $(D_t(x))_{t\geq 0,x\in\mathbb{R}}$, see (2.2.3) and Proposition 2.5.2. We will work on the event

$$\Omega_\lambda = \Omega^{S,1}_{A,T} \cap \Omega^{S,2}_{A,T,T_0} \cap \Omega^{M,1}_{K_M,c_M} \cap \Omega^{S,3}_{K_S,c_S,\delta} \cap \Omega^{M}_{A,T}(\lambda) \cap \Omega^{S}_{A,T_0,\delta,\alpha}(\lambda).$$

Thanks to the previous steps, we know that $\Pr[\Omega_\lambda] \geq 1 - \varepsilon$ for all $\lambda > 0$ small enough. We introduce

$$\mathcal{S} = (\cup_{t\in\mathcal{T}_M}[t, t + \delta + \alpha]) \cup (\cup_{t\in\mathcal{T}^\delta_S}[t - \alpha, t + \alpha]).$$

We will prove in the next steps that for $\lambda > 0$ small enough, on Ω_λ, for all $k = 1, \ldots, p$, for all $t \in [0, T]$,

$$\boldsymbol{\delta}(D^\lambda_t(x_k), D_t(x_k)) \leq 4/\boldsymbol{n}_\lambda + 2A\mathbb{1}_{\{t\in\mathcal{S}\}}, \tag{3.3.2}$$

which will imply that

$$\boldsymbol{\delta}_T(D^\lambda(x_k), D(x_k)) \leq 4T/\boldsymbol{n}_\lambda + 2A|\mathcal{S}|.$$

This will conclude the proof. Indeed, on Ω_λ, we know that $t_1, \ldots, t_p$ do not belong to $\mathcal{S}$ (thanks to $\Omega^{S,3}_{K_S,c_S,\delta}$ and $\Omega^{M,1}_{K_M,c_M}$ and since $c_S > \alpha$ and $c_M > \delta + \alpha$) and that the Lebesgue measure of $\mathcal{S}$ is smaller than $K_M\delta + (2K_S + K_M)\alpha$. Thus on Ω_λ, since $\delta < a/(8ApK_M)$ and $\alpha < a/(8Ap(2K_S + K_M))$,

$$\sum_{k=1}^{p} \boldsymbol{\delta}_T\big(D^\lambda(x_k), D(x_k)\big) + \sum_{k=1}^{p} \boldsymbol{\delta}\big(D^\lambda_{t_k}(x_k), D_{t_k}(x_k)\big) \\ \leq p\big[2A(K_M\delta + (2K_S + K_M)\alpha) + 4T/\boldsymbol{n}_\lambda + 4/\boldsymbol{n}_\lambda\big] \leq a/2 + (4T+4)p/\boldsymbol{n}_\lambda,$$

which is smaller than a for all $\lambda > 0$ small enough. This implies (3.3.1) for all $\lambda > 0$ small enough.

Step 4. — Here we localize the processes, on the event Ω_λ. Due to $\Omega^{S,1}_{A,T}$, we know that π_S has some marks (τ_g, χ_g, L_g) and (τ_d, χ_d, L_d) such that $-A < \chi_g < -B$, $B < \chi_d < A$, $\tau_g > T+1$, $\tau_d > T+1$, $L_g > \tau_g + 1$ and $L_d > \tau_d + 1$. This implies, by definition of the LFF(β)-process, that $Y_t(\chi_g) = Y_t(\chi_d) = 0$ for all $t \in [0, T+1]$. Consequently, for all $t \in [0,T]$ and all $x \in [\chi_g, \chi_d] \supset [-B, B]$, we have $D_t(x) \subset [\chi_g, \chi_d]$.

Set now $g_\lambda = \lfloor \boldsymbol{n}_\lambda \chi_g \rfloor$ and $d_\lambda = \lfloor \boldsymbol{n}_\lambda \chi_d \rfloor$. These are those sites of $I^\lambda_A \subset \mathbb{Z}$ such that $\chi_g \in (g_\lambda)_\lambda$ and $\chi_d \in (d_\lambda)_\lambda$. We claim that on Ω_λ, for all $t \in [0, \boldsymbol{a}_\lambda T]$,

$$\eta^\lambda_t(g_\lambda) = \eta^\lambda_t(d_\lambda) = 0.$$

Consequently on Ω_λ, we clearly have $C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, i) \subset [\![g_\lambda + 1, d_\lambda - 1]\!]$ for all $t \in [0,T]$ and all $i \in [\![g_\lambda + 1, d_\lambda - 1]\!]$.

Indeed, consider e.g. the case of d_λ. Due to $\Omega^S_{A,T_0,\delta,\alpha}(\lambda)$ and since $S^\lambda_{T_0}(\delta, d_\lambda) > 0$ (because π_S has the mark (τ_d, χ_d, L_d) that falls in $[0,T_0] \times (d_\lambda)_\lambda \times [\delta, \infty)$), we deduce that $S^\lambda_{T_0}(\delta, d_\lambda) = \widetilde{S}^\lambda_{T_0}(\delta, d_\lambda) = 1$ and that

$$\big|\widetilde{\tau}^\lambda(\delta, d_\lambda)/\boldsymbol{a}_\lambda - \tau_d\big| + \big|\widetilde{L}^\lambda(\delta, d_\lambda)/\boldsymbol{a}_\lambda - L_d\big| < \alpha < 1.$$

But no seed falls on d_λ, by definition, during $(\widetilde{\tau}^\lambda(\delta, d_\lambda) - \widetilde{L}^\lambda(\delta, d_\lambda), \widetilde{\tau}^\lambda(\delta, d_\lambda))$. This last interval contains $[0, \boldsymbol{a}_\lambda T]$: since $\alpha < \frac{1}{2}$, $\widetilde{\tau}^\lambda(\delta, d_\lambda) \geq \boldsymbol{a}_\lambda(\tau_d - \alpha) \geq \boldsymbol{a}_\lambda(T + 1 - \alpha) > \boldsymbol{a}_\lambda T$ and $\widetilde{\tau}^\lambda(\delta, d_\lambda) - \widetilde{L}^\lambda(\delta, d_\lambda) \leq \boldsymbol{a}_\lambda(\tau_d - L_d + 2\alpha) \leq \boldsymbol{a}_\lambda(-1 + 2\alpha) < 0$. This proves that $\eta^\lambda_t(d_\lambda) = 0$ for all $t \in [0, \boldsymbol{a}_\lambda T]$.

Using furthermore $\Omega^{S,2}_{A,T,T_0}(0)$, we deduce that on Ω_λ, $(Y_t(x), D_t(x))_{t \in [0,T], x \in [\chi_g, \chi_d]}$ is completely determined by the values of π_S and π_M restricted to the boxes $[0,T_0] \times [\chi_g, \chi_d] \times (0, \infty)$ and $[0,T] \times [\chi_g, \chi_d]$. By the same way, $(\eta^\lambda_t(i))_{t \in [0, \boldsymbol{a}_\lambda T], i \in [\![g_\lambda, d_\lambda]\!]}$ is completely determined by $(N^S_t(i), N^{M,\lambda}_t(i))_{t \in [0, \boldsymbol{a}_\lambda T], i \in [\![g_\lambda, d_\lambda]\!]}$. And we recall that $[-B, B] \subset [\chi_g, \chi_d] \subset [-A, A]$.

Step 5. — In this whole step, we work on Ω_λ. We denote by $(\rho_i, \alpha_i)_{i=1,\dots,n}$ the marks of π_M in $[0,T] \times [\chi_g, \chi_d]$, ordered chronologically ($0 = \rho_0 < \rho_1 < \cdots < \rho_n < T$). For each k, we recall that in the FF(μ_S, μ^λ_M)-process, there is match falling at time $\boldsymbol{a}_\lambda \rho_k$ on the site $\lfloor \boldsymbol{n}_\lambda \alpha_k \rfloor$ (recall $\Omega^M_{A,T}(\lambda)$ and that $x \in i_\lambda$ iff $i = \lfloor \boldsymbol{n}_\lambda x \rfloor$). Furthermore, these are the only fires in $[0, \boldsymbol{a}_\lambda T] \times [\![g_\lambda, d_\lambda]\!]$. For $k = 0, \dots, n$, let us consider the properties

$$(H_k): \quad \forall\, i \in [\![g_\lambda, d_\lambda]\!], \quad \eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = \inf_{x \in i_\lambda} Y_{\rho_k}(x);$$

$$(H^*_k): \quad \forall\, i \in [\![g_\lambda, d_\lambda]\!],\ \forall\, t \in [\rho_k, \rho_{k+1}) \setminus \mathcal{S}, \quad \eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x \in i_\lambda} Y_t(x).$$

We observe that (H_0) holds: for any $i \in \mathbb{Z}$, $\eta^\lambda_0(i) = 0$ and $\inf_{x \in i_\lambda} Y_0(x) = 0$ because the set $\{x \in \mathbb{R} : Y_0(x) = 0\}$ is a.s. dense in $\mathbb{R}$. Indeed, recall that $Y_0(x) = 0$ as soon as $\pi_S(\{(t, x, \ell) : l > t\}) > 0$ and that $\int_0^\infty \mathrm{d}t \int_t^\infty \beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell = \infty$.

We are going to prove that for $k \in \{0, \dots, n-1\}$, (H_k) implies (H^*_k) and (H_{k+1}). Assume thus that (H_k) holds for some $k \in \{0, \dots, n-1\}$.

We first prove that (H^*_k) holds.

We recall that for all $i \in [\![g_\lambda, d_\lambda]\!]$, $S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i)$ is either 0 or 1. On $\{S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i) = 1\}$, we have

$$\left|\tau^\lambda(\delta, i) - \widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda\right| < \alpha \quad \text{and} \quad \left|L^\lambda(\delta, i) - \widetilde{L}^\lambda(\delta, i)/\boldsymbol{a}_\lambda\right| < \alpha.$$

Recalling furthermore $\Omega^{M,1}_{K_M, c_M}$ and $\Omega^{S,3}_{K_S, c_S, \delta}$, using that $\alpha < \frac{1}{4} c_M$, we deduce that:

▷ either $\tau^\lambda(\delta, i)$ and $\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda$ both belong to the same interval $(\rho_{q(i)}, \rho_{q(i)+1})$ for some $q(i) \in \{0, \dots, n-1\}$ or are both greater than ρ_n (then we say that $q(i) = n$);

▷ either $\tau^\lambda(\delta, i) - L^\lambda(\delta, i)$ and $(\widetilde{\tau}^\lambda(\delta, i) - \widetilde{L}^\lambda(\delta, i))/\boldsymbol{a}_\lambda$ both belong to the same interval $(\rho_{q'(i)}, \rho_{q'(i)+1})$ for some $q'(i) \in \{0, \dots, n-1\}$ or are both greater than ρ_n (then we adopt the convention that $q'(i) = n$), or are both smaller than 0 (then we say that $q'(i) = -1$).

We next observe that since $\delta < c_M$, the values of

$$\big(Y_t(x), D_t(x)\big)_{t \in [0,T] \setminus \cup_{s \in \mathcal{T}_M}[s, s+\delta],\, x \in [\chi_g, \chi_d]}$$

depends on π_S only through its restriction to $[0, T_0] \times [\chi_g, \chi_d] \times [\delta, \infty)$. Furthermore, for any $t \in [0, T] \setminus \cup_{s \in \mathcal{T}_M}[s, s+\delta]$ and any $x \in [\chi_g, \chi_d]$, $D_t(x)$ has its extremities in $\mathcal{X}^\delta_S$. Have a look at Figure 4 and use the fact that all the dotted segments with length smaller than δ cannot concern two fires. See also Remark 2.5.1 (ii).

We now distinguish several situations to prove (H^*_k). We use, in all the cases below, that there are no fires in the time interval (ρ_k, ρ_{k+1}) in the LFF(β)-process in the box $[\chi_g, \chi_d]$ and no fire during $(\boldsymbol{a}_\lambda \rho_k, \boldsymbol{a}_\lambda \rho_{k+1})$ for the FF(μ_S, μ^λ_M)-process in the box $[\![g_\lambda, d_\lambda]\!]$, recall $\Omega^M_{A,T}(\lambda)$. Let $i \in [\![g_\lambda, d_\lambda]\!]$.

Case (a): $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = 1$. Then by (H_k), $\inf_{x \in i_\lambda} Y_{\rho_k}(x) = 1$. An obvious monotonicity argument shows that for all $t \in (\rho_k, \rho_{k+1})$, $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x \in i_\lambda} Y_t(x) = 1$.

Case (b): $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = 0$ and $S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i) = 0$. Then $\inf_{x \in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1})$, because in i_λ, there is no dotted segment with length greater than δ that intersect $[0, T]$ (see Figure 4). Next, $\widetilde{S}^\lambda_{T_0}(\delta, i) = 0$ means that all the delays we wait for a seed (on the site i during $[0, \boldsymbol{a}_\lambda T_0]$) are smaller than $\boldsymbol{a}_\lambda \delta$. Consequently,

$\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1})$. Hence $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x\in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1}) \supset (\rho_k, \rho_{k+1}) \setminus \mathcal{S}$.

Case (c): $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = 0$ and $S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i) = 1$ and $q(i) < k$. Then $\inf_{x\in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k+\delta, \rho_{k+1})$, because the only dotted segment in i_λ with length greater than δ that intersects $[0, T]$ has ended before ρ_k (because $q(i) < k$). Next, the only delay (between two seeds on i during $[0, \boldsymbol{a}_\lambda T]$) greater than $\boldsymbol{a}_\lambda\delta$ is ended before $\boldsymbol{a}_\lambda\rho_k$ (because $q(i) < k$), so that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1})$. Hence $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x\in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1}) \supset (\rho_k, \rho_{k+1}) \setminus \mathcal{S}$.

Case (d): $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = 0$, $S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i) = 1$ and $q'(i) \geq k$. Then $\inf_{x\in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k+\delta, \rho_{k+1})$. Indeed, the only dotted segment in i_λ with length greater than δ that intersects $[0, T]$ starts (strictly) after ρ_k (because $q'(i) \geq k$). Next, the only delay (between two seeds on i during $[0, \boldsymbol{a}_\lambda T]$) greater than $\boldsymbol{a}_\lambda\delta$ will start strictly after $\boldsymbol{a}_\lambda\rho_k$ (because $q'(i) \geq k$), so that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for all $t \in [\rho_k+\delta, \rho_{k+1})$. Hence $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x\in i_\lambda} Y_t(x) = 1$ for all $t \in [\rho_k + \delta, \rho_{k+1}) \supset (\rho_k, \rho_{k+1}) \setminus \mathcal{S}$.

Case (e): $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_k}(i) = 0$ and $S^\lambda_{T_0}(\delta, i) = \widetilde{S}^\lambda_{T_0}(\delta, i) = 1$ and $q'(i) < k \leq q(i)$. Then

$$
\begin{aligned}
&\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 0 \ \text{ for all } t \in \big[\rho_k, (\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda) \wedge \rho_{k+1}\big),\\
&\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1 \ \text{ for all } t \in \big[(\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda) \wedge \rho_{k+1}, \rho_{k+1}\big)
\end{aligned}
$$

(because no seed fall on i during $[\widetilde{\tau}^\lambda(\delta, i) - \widetilde{L}^\lambda(\delta, i), \widetilde{\tau}^\lambda(\delta, i)) \ni \rho_k$ and a seed falls on i at time $\widetilde{\tau}^\lambda(\delta, i)$). By (H_k), we also know that $\inf_{x\in i_\lambda} Y_{\rho_k}(x) = 0$. Calling $(\tau^\lambda(\delta, i), x_0, L^\lambda(\delta, i))$ the only mark of π_S that falls in $[0, T_0] \times i_\lambda \times [\delta, \infty)$, we claim that necessarily, $Y_{\rho_k}(x_0) = 0$. Indeed, all the other dotted segments in i_λ that intersect $[0, T]$ have a length smaller than $\delta < c_M \leq \rho_k - \rho_{k-1}$. Thus if $\inf_{x\in i_\lambda} Y_{\rho_k-}(x) = 0$, necessarily, $Y_{\rho_k-}(x_0) = 0$ and thus $Y_{\rho_k}(x_0) = 0$. If now $\inf_{x\in i_\lambda} Y_{\rho_k-}(x) = 1$, then i_λ is connected at time time ρ_k-, whence the fire at time ρ_k burns completely i_λ (because $\inf_{x\in i_\lambda} Y_{\rho_k}(x) = 0$ by assumption), so that in particular, $Y_{\rho_k}(x_0) = 0$. Then we have to separate two situations.

▷ If $\tau^\lambda(\delta, i) < \rho_k + \delta$, then we easily deduce that $\inf_{x\in i_\lambda} Y_t(x) = 1$ for $t \in [\rho_k + \delta, \rho_{k+1})$. Recalling that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for all $t \in [(\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda) \wedge \rho_{k+1}, \rho_{k+1})$ and that $|\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda - \tau^\lambda(\delta, i)| < \alpha$, we easily conclude that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for $t \in [\rho_k+\alpha+\delta, \rho_{k+1})$. Thus $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x\in i_\lambda} Y_t(x)$ for $t \in [\rho_k + \delta + \alpha, \rho_{k+1}) \supset [\rho_k, \rho_{k+1}) \setminus \mathcal{S}$.

▷ If now $\tau^\lambda(\delta, i) \geq \rho_k + \delta$, then we have, by construction, $\inf_{x\in i_\lambda} Y_t(x) = 0$ for $t \in [\rho_k, \tau^\lambda(\delta, i) \wedge \rho_{k+1})$ and $\inf_{x\in i_\lambda} Y_t(x) = 1$ for $t \in [\tau^\lambda(\delta, i) \wedge \rho_{k+1}, \rho_{k+1})$. Recalling the values of $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i)$ and that $|\widetilde{\tau}^\lambda(\delta, i)/\boldsymbol{a}_\lambda - \tau^\lambda(\delta, i)| < \alpha$, one easily concludes that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x\in i_\lambda} Y_t(x)$ for $t \in [\rho_k, \rho_{k+1}) \setminus \mathcal{S}$ (because $\tau^\lambda(\delta, i) \in \mathcal{T}^\delta_S$ whence $[\tau^\lambda(\delta, i) - \alpha, \tau^\lambda(\delta, i) + \alpha] \subset \mathcal{S}$).

We have proved (H^*_k) and this implies that

$$
\forall\, i \in [\![g_\lambda, d_\lambda]\!], \quad \eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}-}(i) = \inf_{x\in i_\lambda} Y_{\rho_{k+1}-}(x).
$$

It remains to prove (H_{k+1}).

Consider the ignited cluster $[a,b] = D_{\rho_{k+1}-}(\alpha_{k+1})$ in the LFF(β)-process. Then the ignited cluster in the FF(μ_S, μ_M^λ)-process at time $\boldsymbol{a}_\lambda \rho_{k+1}$ (due to a match falling on the site $\lfloor \boldsymbol{n}_\lambda \alpha_{k+1} \rfloor$) is nothing but $I_{k+1}^\lambda := \{i \in [\![g_\lambda, d_\lambda]\!] \; : \; i_\lambda \subset D_{\rho_{k+1}-}(\alpha_{k+1})\}$, at least if λ is small enough (such that $1/\boldsymbol{n}_\lambda < c_S$). Indeed, we have $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}-}(i) = \inf_{x \in i_\lambda} Y_{\rho_{k+1}-}(x) = 1$ for all i such that $i_\lambda \subset D_{\rho_{k+1}-}(\alpha_{k+1})$ and (on the two boundary sites) $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}-}(i) = \inf_{x \in i_\lambda} Y_{\rho_{k+1}-}(x) = 0$ for i such that $i_\lambda \not\subset D_{\rho_{k+1}-}(\alpha_{k+1})$ with $i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}) \neq \varnothing$. And for $\lambda > 0$ small enough (such that $1/\boldsymbol{n}_\lambda < c_S$), $\lfloor \boldsymbol{n}_\lambda \alpha_{k+1} \rfloor \in I_{k+1}^\lambda$ (because $[a + 1/\boldsymbol{n}_\lambda, b - 1/\boldsymbol{n}_\lambda] \subset I_{k+1}^\lambda$ by the previous study, because $D_{\rho_{k+1}-}(\alpha_{k+1}) = [a,b]$ has its extremities a, b in $\mathcal{X}_S^\delta$, because $\alpha_{k+1} \in \mathcal{X}_M$ and because the distance between $\mathcal{X}_S^\delta$ and $\mathcal{X}_M$ is greater than c_S, recall $\Omega^{S,3}_{K_S,c_S,\delta}$, so that actually, $\alpha_{k+1} \in [a + c_S, b - c_S]$).

Then on the one hand, for all $i \in [\![g_\lambda, d_\lambda]\!]$, we have

$$\inf_{x \in i_\lambda} Y_{\rho_{k+1}}(x) = \inf_{x \in i_\lambda} Y_{\rho_{k+1}-}(x) \;\text{ if }\; i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}) = \varnothing,$$
$$\inf_{x \in i_\lambda} Y_{\rho_{k+1}}(x) = 0 \;\text{ if }\; i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}) \neq \varnothing.$$

The first case is obvious and the second one follows from the fact that a.s.,

$$\pi_S\big(\{(t,x,\ell) \; : \; t \geq \rho_{k+1}, x \in i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}), t - \ell < \rho_{k+1}\}\big) = \infty$$

(but this concerns marks (t,x,ℓ) with a very small length $\ell > 0$).

On the other hand, for all $i \in [\![g_\lambda, d_\lambda]\!]$, we have $\eta^\lambda_{\rho_{k+1}}(i) = \eta^\lambda_{\rho_{k+1}-}(i)$ if $i \notin I^\lambda_{k+1}$ and $\eta^\lambda_{\rho_{k+1}}(i) = 0$ if $i \in I^\lambda_{k+1}$.

As a conclusion, for all $i \in [\![g_\lambda, d_\lambda]\!]$,

- ▷ if $i_\lambda \subset D_{\rho_{k+1}-}(\alpha_{k+1})$, i.e. if $i \in I^\lambda_{k+1}$, then we have seen that $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}}(i) = 0 = \inf_{x \in i_\lambda} Y_{\rho_{k+1}}(x)$;
- ▷ if $i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}) = \varnothing$ (hence $i \notin I^\lambda_{k+1}$), then we have seen that $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}}(i) = \eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}-}(i) = \inf_{x \in i_\lambda} Y_{\rho_{k+1}-}(x) = \inf_{x \in i_\lambda} Y_{\rho_{k+1}}(x)$;
- ▷ if $i \notin I^\lambda_{k+1}$ but $i_\lambda \cap D_{\rho_{k+1}-}(\alpha_{k+1}) \neq \varnothing$, then we have seen that $\inf_{x \in i_\lambda} Y_{\rho_{k+1}}(x) = 0$ and $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}}(i) = 0$ because $\eta^\lambda_{\boldsymbol{a}_\lambda \rho_{k+1}-}(i) = 0$ (since then i lies at the boundary of I^λ_{k+1}).

Hence (H_{k+1}) holds.

Step 6. — We finally can prove (3.3.2) on Ω_λ and this will conclude the proof. First, we know from step 4 that for all $t \in [0,T]$, all $k = 1, \dots, p$, $D_t(x_k) \subset [\chi_g, \chi_d] \subset [-A, A]$ and that $C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, \lfloor \boldsymbol{a}_\lambda x_k \rfloor) \subset [\![g_\lambda + 1, d_\lambda - 1]\!]$ whence $D^\lambda_t(x_k) \subset [-A, A]$ (because $(g_\lambda + 1)/\boldsymbol{n}_\lambda \geq \chi_g \geq -A$ and $(d_\lambda - 1)/\boldsymbol{n}_\lambda \leq \chi_d \leq A$). This obviously implies that $\boldsymbol{\delta}(D_t(x_k), D^\lambda_t(x_k)) \leq 2A$.

Next, step 5 implies that for all $t \in [0,T] \setminus \mathcal{S}$ (or rather for all $t \in [0, \rho_n) \setminus \mathcal{S}$, but the extension is straightforward), for all $i \in [\![g_\lambda, d_\lambda]\!]$, $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = \inf_{x \in i_\lambda}(Y_t(x))$. This implies that for all $t \in [0,T] \setminus \mathcal{S}$, for all $k = 1,\dots,p$, $\boldsymbol{\delta}(D_t^\lambda(x_k), D_t(x_k)) \leq 4/\boldsymbol{n}_\lambda$ as desired.

Indeed, assume that $D_t(x_k) = [a,b] \subset [\chi_g, \chi_d]$ for some $t \in [0,T] \setminus \mathcal{S}$. Recall that $a, b \in \mathcal{X}_S^\delta$. We have $Y_t(y) = 1$ for all $y \in (a,b)$ and $Y_t(a) = Y_t(b) = 0$. Hence we deduce that

$$\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1 \text{ for all } i \in [\![\lfloor a\boldsymbol{n}_\lambda \rfloor + 1, \lfloor b\boldsymbol{n}_\lambda \rfloor - 1]\!] \quad \text{and} \quad \eta^\lambda_{\boldsymbol{a}_\lambda t}\big(\lfloor a\boldsymbol{n}_\lambda \rfloor\big) = \eta^\lambda_{\boldsymbol{a}_\lambda t}\big(\lfloor b\boldsymbol{n}_\lambda \rfloor\big) = 0.$$

Next, we observe that for $\lambda > 0$ small enough, $\lfloor a\boldsymbol{n}_\lambda \rfloor < \lfloor x_k \boldsymbol{n}_\lambda \rfloor < \lfloor b\boldsymbol{n}_\lambda \rfloor$. Indeed, on Ω_λ, we have, since $a, b \in \mathcal{X}_S^\delta$, $|x_k - a| > c_S$ and $|b - x_k| > c_S$. We finally obtain

$$C\big(\eta^\lambda_{\boldsymbol{a}_\lambda t}, \lfloor x_k \boldsymbol{n}_\lambda \rfloor\big) = [\![\lfloor a\boldsymbol{n}_\lambda \rfloor + 1, \lfloor b\boldsymbol{n}_\lambda \rfloor - 1]\!],$$

whence $D_t^\lambda(x_k) = [(\lfloor a\boldsymbol{n}_\lambda \rfloor + 1)/\boldsymbol{n}_\lambda, (\lfloor b\boldsymbol{n}_\lambda \rfloor - 1)/\boldsymbol{n}_\lambda]$. Recalling that $D_t(x) = [a,b]$, one easily deduces that $\boldsymbol{\delta}(D_t^\lambda(x_k), D_t(x_k)) \leq 4/\boldsymbol{n}_\lambda$. □

3.4. Cluster-size distribution when $\beta \in (0,\infty)$

This section is entirely devoted to the

Proof of Corollary 2.5.4. — We thus fix $\beta \in (0,\infty)$ and assume (H_M) and $(H_S(\beta))$. For each $\lambda > 0$, we consider a $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t \geq 0, i \in \mathbb{Z}}$. Let also $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFF(β)-process. We know from Theorem 2.5.3 that $|C(\eta_t^\lambda, 0)|/\boldsymbol{n}_\lambda$ goes in law to $|D_t(0)|$, for any $t > 0$. In step 1 below, we will check that for $t > 0$, the law of $|D_t(0)|$ does not charge points. Thus for any $B \geq 0$, $t > 0$, we will have

$$\lim_{\lambda \to 0} \Pr\big[|C(\eta_t^\lambda, 0)| \geq \boldsymbol{n}_\lambda B\big] = \Pr\big[|D_t(0)| \geq B\big].$$

In steps 2 to 6, we will check that there are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ such that if $t > 1$, for any $B \geq 2$, $\Pr[|D_t(0)| > B] \in [c_1 \,\mathrm{e}^{-\kappa_2 B}, c_2 \,\mathrm{e}^{-\kappa_1 B}]$. One immediately checks that this implies

$$\Pr\big[|D_t(0)| > B\big] \in [c_1 \,\mathrm{e}^{-2\kappa_2} \,\mathrm{e}^{-\kappa_2 B}, (c_2 \vee \mathrm{e}^{2\kappa_1}) \,\mathrm{e}^{-\kappa_1 B}]$$

for all $t > 1$, $B > 0$ and this will conclude the proof.

Step 1. — The goal of this step is to check that for any $t > 0$ fixed, the law of $|D_t(0)|$ does not charge points.

Consider the first mark (T_d, χ_d, L_d) of π_S on the right of 0 ($\chi_d > 0$) such that $[0,t] \subset [T_d - L_d, T_d]$. Consider a similar mark (T_g, χ_g, L_g) of π_S with $\chi_g < 0$.

Then $Y_s(\chi_g) = Y_s(\chi_d) = 0$ for all $s \in [0,t]$, so that fires falling outside $[\chi_g, \chi_d]$ cannot affect 0 during $[0,t]$.

Next, denote by (T_M, X_M) the instant/position of the last match falling before t in $[\chi_g, \chi_d]$. Then a.s., $t - T_M > 0$, and $D_t(0)$ is of the form $[a,b]$, for some marks (T_a, a, L_a) and (T_b, b, L_b) of π_S satisfying $\chi_g \leq a < 0 < b < \chi_d$, $T_a - L_a < T_M$, $T_b - L_b < T_M$, $T_a > t$ and $T_b > t$. There are a.s. a finite number of such marks (because a.s., $\int_t^\infty \mathrm{d}s \int_{s-T_M}^\infty \beta(\beta+1)\ell^{-\beta-2}\,\mathrm{d}\ell = (t-t_M)^{-\beta} < \infty$), and their (spatial) positions clearly have densities, whence the result.

Step 2. — For $t > 1$, $a \in \mathbb{R}$, we consider the event $\Omega_{t,a}$ defined as follows, see Figure 6 next page:

(i) π_M has exactly one mark (T_M, X_M) in $[t-1,t] \times [a, a+1]$ and there holds $(T_M, X_M) \in [t-\frac{2}{3}, t-\frac{1}{2}] \times [a+\frac{1}{4}, a+\frac{3}{4}]$;

(ii) π_S has one mark (T_g, X_g, L_g) such that $T_g - L_g < t-1 < t < T_g$ and $X_g \in [a, a+\frac{1}{4}]$ and one mark (T_d, X_d, L_d) such that $T_d - L_d < t-1 < t < T_d$ and $X_d \in [a+\frac{3}{4}, a+1]$ (recalling Figure 4, there are dotted vertical segments in $[a, a+\frac{1}{4}]$ and in $[a+\frac{3}{4}, a+1]$ that run across $[t-1,t]$);

(iii) all the other marks (T, X, L) of π_S with $X \in [a, a+1]$ and $[T-L, T] \cap [t-1, t]$ not empty satisfy $L < \frac{1}{4}$ (recalling Figure 4, all the other vertical dotted segments in $[a, a+1]$ that intersect $[t-1,t]$ have a length smaller than $\frac{1}{4}$).

Step 3. — In this step, we prove that on $\Omega_{t,a}$, we have either $Y_s(X_g) = 0$ for all $s \in [t-\frac{1}{2}, t]$ or $Y_s(X_d) = 0$ for all $s \in [t-\frac{1}{2}, t]$. We distinguish two situations.

- ▷ First assume that $[X_g, X_d]$ is connected at time T_M- (that is $Y_{T_M-}(x) = 1$ for all $x \in [X_g, X_d]$). Since $X_M \in [X_g, X_d]$, the fire destroys the cluster and thus we deduce that $Y_s(X_g) = 0$ for all $s \in [T_M, T_g) \supset [t-\frac{1}{2}, t]$ and that $Y_s(X_d) = 0$ for all $s \in [T_M, T_d) \supset [t-\frac{1}{2}, t]$.
- ▷ Next assume that $[X_g, X_d] \subset [a, a+1]$ is not connected at time T_M- (that is, there is some $x_0 \in [X_g, X_d]$ such that $Y_{T_M-}(x_0) = 0$). Then we claim that either $Y_{T_M-}(X_g) = 0$ (then $Y_s(X_g) = 0$ for all $s \in [T_M, T_g) \supset [t-\frac{1}{2}, t]$) or $Y_{T_M-}(X_d) = 0$ (then $Y_s(X_d) = 0$ for all $s \in [T_M, T_d) \supset [t-\frac{1}{2}, t]$). Indeed, recall that all the dotted segments that intersect $[t-1,t]$ in (X_g, X_d) have a length smaller than $\frac{1}{4}$. Thus if $[X_g, X_d]$ is disconnected at time T_M- due to a fire that started before $t-1$, it can be only with $x_0 = X_g$ or $x_0 = X_d$, whence the conclusion. But if now $[X_g, X_d]$ is disconnected at time T_M- due to a fire that started at some time $\tau \in [t-1, T_M)$ at some place $\chi \notin [a, a+1]$ (since there are no fires in $[a, a+1]$ during $[t-1, T_M)$), this necessarily also concerns one of the extremities X_g or X_d of $[X_g, X_d]$. Thus in any case, we obtain either $Y_{T_M-}(X_g) = 0$ or $Y_{T_M-}(X_d) = 0$ as desired.

Step 4. — Let us prove that $p := \Pr[\Omega_{t,a}] > 0$. This value will obviously does not depend on $a \in \mathbb{R}$, $t \geq 1$, by homogeneity in (s, x) of the Poisson measures $\pi_M(\mathrm{d}s, \mathrm{d}x)$ and $\pi_S(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\ell)$. Define the zones

$$
\begin{aligned}
A_M &= \big(t-\tfrac{2}{3}, t-\tfrac{1}{2}\big) \times \big(a+\tfrac{1}{4}, a+\tfrac{3}{4}\big),\\
B_M &= \big((t-1,t) \times (a, a+1)\big) \setminus A_M,\\
A_S &= \big\{(s, x, \ell),\ x \in (a, a+\tfrac{1}{4}), s > t > t-1 > s-\ell\big\},
\end{aligned}
$$

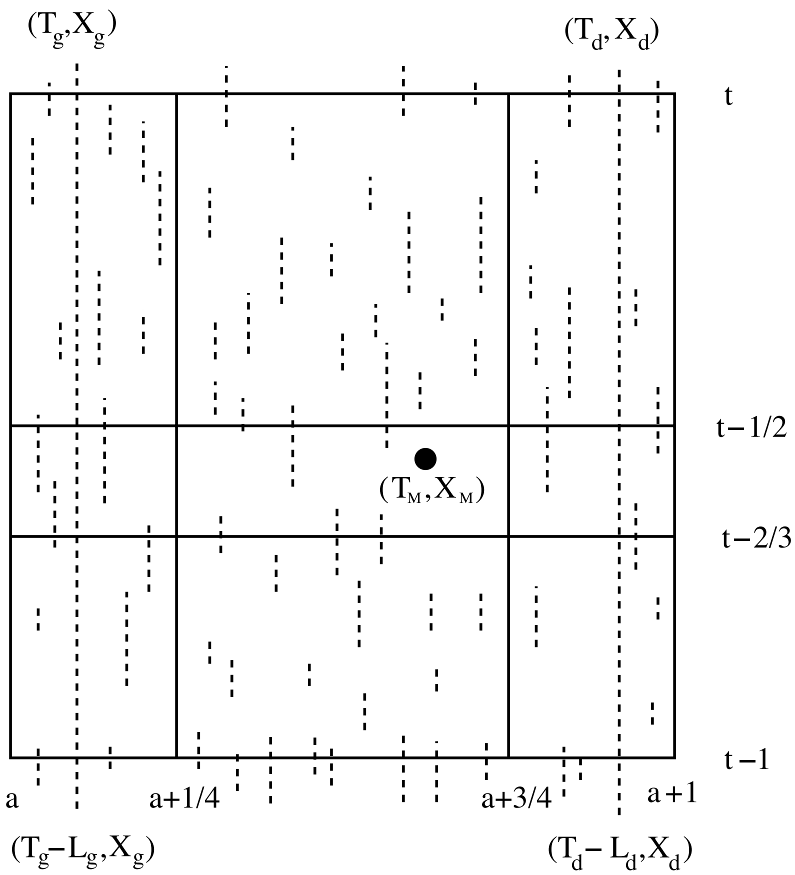

FIGURE 6. The event $\Omega_{t,A}$.

$$B_S = \{(s, x, \ell),\ x \in (a + \tfrac{3}{4}, a + 1), s > t > t - 1 > s - \ell\},$$
$$C_S = \{(s, x, \ell),\ x \in (a + \tfrac{1}{4}, a + \tfrac{3}{4}), s > t > t - 1 > s - \ell\},$$
$$D_S = \{(s, x, \ell),\ x \in (a, a + 1), [s - \ell, s] \cap [t - 1, t] \neq \varnothing, \ell > \tfrac{1}{4}\} \setminus (A_S \cup B_S \cup C_S).$$

The zones A_M and B_M are disjoint and for $\zeta_M(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\,\mathrm{d}x$, $\zeta_M(A_M) = \frac{1}{12}$ and $\zeta_M(B_M) = \frac{11}{12}$. The zones A_S, B_S, C_S, D_S are also disjoint and simple computations show that, for $\zeta_S(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\ell) = \beta(\beta + 1)\ell^{-\beta-2}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\ell$, $\zeta_S(A_S) = \zeta_S(B_S) = \frac{1}{4}$, $\zeta_S(C_S) = \frac{1}{2}$ and $\zeta_S(D_S) = 4^\beta(5\beta + 1) - 1$. Consequently, recalling that π_M and π_S are independent Poisson measures with intensity measures ζ_M and ζ_S,

$$\Pr[\Omega_{t,a,\delta}] = \Pr\big(\pi_M(A_M) = 1, \pi_M(B_M) = 0, \pi_S(A_S) = \pi_S(B_S) = 1, \\ \pi_S(C_S) = \pi_S(D_S) = 0\big)$$

$$= \zeta(A_M)\,\mathrm{e}^{-\zeta_M(A_M)}\,\mathrm{e}^{-\zeta_M(B_M)}\zeta_S(A_S)\,\mathrm{e}^{-\zeta_S(A_S)}$$
$$\zeta_S(B_S)\,\mathrm{e}^{-\zeta_S(B_S)}\,\mathrm{e}^{-\zeta_S(C_S)}\,\mathrm{e}^{-\zeta_S(D_S)}$$
$$= \tfrac{1}{12}\,\mathrm{e}^{-\frac{1}{1}2}\,\mathrm{e}^{-\frac{11}{12}}(\tfrac{1}{4})^2\,\mathrm{e}^{-\frac{1}{2}}\,\mathrm{e}^{-4^\beta(5\beta+1)+1} =: p > 0.$$

Step 5. — We clearly have, for any $t \geq 1$, any $B \geq 2$,

$$\big\{|D_t(0)| \geq B\big\} \subset \{\forall\, x \in [0, \tfrac{1}{2}B],\ Y_t(x) = 1\big\} \cup \big\{\forall\, x \in [-\tfrac{1}{2}B, 0],\ Y_t(x) = 1\big\},$$

whence $\Pr[|D_t(0)| \geq B] \leq 2\Pr[\forall\, x \in [0, \frac{1}{2}B],\ Y_t(x) = 1]$ by symmetry. Furthermore, step 3 implies that

$$\{\forall\, x \in [0, \tfrac{1}{2}B],\ Y_t(x) = 1\} \subset \Omega^c_{t,0} \cap \Omega^c_{t,1} \cap \cdots \cap \Omega^c_{t,\lfloor \frac{1}{2}B-1\rfloor}.$$

Using then step 4 (and some obvious independence arguments), we get

$$\Pr\big[|D_t(0)| \geq B\big] \leq 2(1-p)^{\lfloor \frac{1}{2}B-1\rfloor+1} \leq 2(1-p)^{\frac{1}{2}B-1}.$$

Consequently, for all $t \geq 1$, all $B \geq 2$, $\Pr[|D_t(0)| \geq B] \leq c_2\,\mathrm{e}^{-\kappa_1 B}$, with $c_2 = 2/(1-p)$ and $\kappa_1 = -\frac{1}{2}[\log(1-p)]$.

Step 6. — Next, we consider the event $\widetilde{\Omega}_{t,B}$ on which:

(i) $\pi_M([t-\frac{1}{2}, t] \times [0, B]) = 0$;

(ii) all the marks (T, X, L) of π_S with $X \in [0, B]$ satisfy either $T < t$ or $T - L > t - \frac{1}{2}$) (this means that there is no dotted vertical segment running across $[t-\frac{1}{2}, t]$ in $[0, B]$).

An easy computation as in step 4 implies that

$$\Pr[\widetilde{\Omega}_{t,B}] = \exp\Big(-\int_{t-\frac{1}{2}}^{t}\int_0^B \mathrm{d}s\,\mathrm{d}x - \int_t^\infty \mathrm{d}s\int_0^B \mathrm{d}x\int_{s-t+\frac{1}{2}}^\infty \mathrm{d}\ell\beta(\beta+1)\ell^{-\beta-2}\Big)$$
$$= \exp\big(-\tfrac{1}{2}B - 2^\beta B\big).$$

We claim that on $\Omega_{t,-1} \cap \widetilde{\Omega}_{t,B} \cap \Omega_{t,B}$, we have $[0, B] \subset D_t(0)$, whence $|D_t(0)| \geq B$. Indeed, we know from step 3 that there is $\chi_0 \in [-1, 0]$ and $\chi_1 \in [B, B+1]$ such that $Y_s(\chi_0) = Y_s(\chi_1) = 0$ for all $s \in [t-\frac{1}{2}, 1]$. Thus the fires starting outside $[\chi_0, \chi_1]$ do not affect the zone $[\chi_0, \chi_1]$ during $[t-\frac{1}{2}, t]$. Furthermore, there are no fires starting in $[\chi_0, \chi_1]$ during $[t-\frac{1}{2}, t]$. At last, since all the dotted segments in $[0, B]$ intersecting $\{t\}$ have started after $t-\frac{1}{2}$. We easily conclude that $Y_t(x) = 1$ for all $x \in [0, B]$.

Using finally some obvious independence arguments, we get

$$\Pr\big[|D_t(0)| \geq B\big] \geq \Pr[\Omega_{t,-1} \cap \widetilde{\Omega}_{t,B} \cap \Omega_{t,B}] \geq p^2 \exp\big(-\tfrac{1}{2}B - 2^\beta B\big) = c_1\,\mathrm{e}^{-\kappa_2 B},$$

with $c_1 = p^2$ and $\kappa_2 = \frac{1}{2} + 2^\beta$. □

3.5. Convergence proof when $\beta = 0$

This case is simpler than the case $\beta \in (0,\infty)$, but a little work is however needed. We also divide the section into three parts: preliminaries, coupling of seeds and convergence proof. In the whole section, we assume (H_M) and $(H_S(0))$. We recall that $\boldsymbol{a}_\lambda$ and $\boldsymbol{n}_\lambda$ are defined in (2.2.1) and (2.2.2). For $A > 0$ and $i \in \mathbb{Z}$, we set

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda \rfloor, \quad I_A^\lambda = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = [i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda).$$

3.5.1. Preliminaries.

— The proof will use the following estimate.

Lemma 3.5.1. — *For any $\ell \in (0,\infty)$ fixed, we have $\lim_{\lambda\to 0} \lambda^{-1}\mu_S((\boldsymbol{a}_\lambda \ell, \infty)) = 0$.*

Proof. — Using the monotonicity of $\mu_S((x,\infty))$ and since $\mu_S((x,\infty))\,\mathrm{d}x = m_S\nu_S(\mathrm{d}x)$,

$$\begin{aligned}\frac{\mu_S((\boldsymbol{a}_\lambda\ell,\infty))}{\lambda} &\le \frac{2}{\lambda\boldsymbol{a}_\lambda\ell}\int_{\boldsymbol{a}_\lambda\ell/2}^{\boldsymbol{a}_\lambda\ell} \mu_S\big((x,\infty)\big)\,\mathrm{d}x \\ &= \frac{2m_S}{\lambda\boldsymbol{a}_\lambda\ell}\big[\nu_S((\boldsymbol{a}_\lambda\ell/2,\infty)) - \nu_S((\boldsymbol{a}_\lambda\ell,\infty))\big] \\ &= \frac{2m_S}{\ell}\big[\nu_S((\boldsymbol{a}_\lambda\ell/2,\infty))/\nu_S((\boldsymbol{a}_\lambda,\infty)) - \nu_S((\boldsymbol{a}_\lambda\ell,\infty))/\nu_S((\boldsymbol{a}_\lambda,\infty))\big].\end{aligned}$$

For the last equality, we used that by definition, $\nu_S((\boldsymbol{a}_\lambda,\infty)) = \lambda\boldsymbol{a}_\lambda$. Using $(H_S(0))$, we easily conclude. □

The following statement contains some crucial facts about accelerated $\mathrm{SR}(\mu_S)$-processes under $(H_S(0))$.

Lemma 3.5.2. — *Let $(T_n)_{n\ge1}$ be a $\mathrm{SR}(\mu_S)$-process (see subsection 2.1.1). For $\lambda > 0$, $t \ge 0$ and $\ell > 0$, we set*

$$R_t^\lambda(\ell) = \#\big\{n \ge 1 \,:\, T_n \in [0,\boldsymbol{a}_\lambda t],\ T_{n+1} - T_n \ge \boldsymbol{a}_\lambda\ell\big\},$$

which represents the number of delays with length greater than $\boldsymbol{a}_\lambda\ell$ that start in $[0,\boldsymbol{a}_\lambda t]$.

(i) *For any $T > 0$, $\Pr[T_1 \ge \boldsymbol{a}_\lambda T] = \nu_S((\boldsymbol{a}_\lambda T,\infty)) \sim \lambda\boldsymbol{a}_\lambda$ as $\lambda \to 0$.*

(ii) *For any $T > 0$, any $\ell > 0$, as $\lambda \to 0$*

$$\mathbb{E}\big[R_T^\lambda(\ell)\big] = \boldsymbol{a}_\lambda T\mu_S\big((\boldsymbol{a}_\lambda\ell,\infty)\big)/m_S = o(\lambda\boldsymbol{a}_\lambda).$$

Proof. — Point (i) is immediate: ν_S is the law of T_1 and since $\lambda\boldsymbol{a}_\lambda = \nu_S((\boldsymbol{a}_\lambda,\infty))$ by definition, one has $\nu_S((\boldsymbol{a}_\lambda T,\infty)) = \lambda\boldsymbol{a}_\lambda\nu_S((\boldsymbol{a}_\lambda T,\infty))/\nu_S((\boldsymbol{a}_\lambda,\infty))$. One concludes using $(H_S(0))$. Point (ii) is slightly more delicate. First, we complete the $\mathrm{SR}(\mu_S)$-process $(T_n)_{n\ge1}$ in $(T_n)_{n\in\mathbb{Z}}$, see subsection 2.1.1. Then we observe that since $T_0 < 0 < T_1$,

$$R_T^\lambda(\ell) = \#\big\{n \in \mathbb{Z} \,:\, T_n \in [0,\boldsymbol{a}_\lambda T],\ T_{n+1} - T_n \ge \boldsymbol{a}_\lambda\ell\big\}.$$

Next, we set

$$\tau_n = \boldsymbol{a}_\lambda T - T_{-n}$$

and we introduce n_0 such that $\tau_{n_0} < 0 < \tau_{n_0+1}$. We put

$$\widetilde{T}_n = \tau_{n_0+n}.$$

Then $(\widetilde{T}_n)_{n\in\mathbb{Z}}$ is also a $\mathrm{SR}(\mu_S)$-process (see subsection 2.1.1). We have

$$\begin{aligned} R_T^\lambda(\ell) &= \#\big\{n \in \mathbb{Z} \, : \, \boldsymbol{a}_\lambda T - T_n \in [0, \boldsymbol{a}_\lambda T], \, (\boldsymbol{a}_\lambda T - T_n) - (\boldsymbol{a}_\lambda T - T_{n+1}) \geq \boldsymbol{a}_\lambda \ell\big\} \\ &= \#\big\{n \in \mathbb{Z} \, : \, \widetilde{T}_{-n-n_0} \in [0, \boldsymbol{a}_\lambda T], \, \widetilde{T}_{-n-n_0} - \widetilde{T}_{-n-1-n_0} \geq \boldsymbol{a}_\lambda \ell\big\} \\ &= \#\big\{n \in \mathbb{Z} \, : \, \widetilde{T}_n \in [0, \boldsymbol{a}_\lambda T], \, \widetilde{T}_n - \widetilde{T}_{n-1} \geq \boldsymbol{a}_\lambda T\big\} \\ &= \#\big\{n \geq 1 \, : \, \widetilde{T}_n \in [0, \boldsymbol{a}_\lambda T], \, \widetilde{T}_n - \widetilde{T}_{n-1} \geq \boldsymbol{a}_\lambda \ell\big\} =: \widetilde{S}_T^\lambda(\ell). \end{aligned}$$

We used that $\widetilde{T}_0 < 0 < \widetilde{T}_1$ by construction. But $\widetilde{S}_T^\lambda(\ell)$ is the number of delays with length greater than $\boldsymbol{a}_\lambda \ell$ that end in $[0, \boldsymbol{a}_\lambda T]$, for the $\mathrm{SR}(\mu_S)$-process $(\widetilde{T}_n)_{n\in\mathbb{Z}}$. Thus exactly as in the proof of Lemma 3.3.2 (steps 1 and 2), we get

$$\mathbb{E}\big[\widetilde{S}_T^\lambda(\ell)\big] = m_S^{-1} \boldsymbol{a}_\lambda T \mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big),$$

so that

$$\mathbb{E}\big[R_T^\lambda(\ell)\big] = m_S^{-1} \boldsymbol{a}_\lambda T \mu_S\big((\boldsymbol{a}_\lambda \ell, \infty)\big).$$

Finally, Lemma 3.5.1 implies that $\mathbb{E}[R_T^\lambda(\ell)] = o(\lambda \boldsymbol{a}_\lambda)$. □

3.5.2. Coupling of seeds. — We aim here to couple the Poisson measure $\pi_S(\mathrm{d}x)$ used to build the LFF(0)-process with a family of $\mathrm{SR}(\mu_S)$-processes, in such a way that roughly:

- ▷ if $\pi_S(i_\lambda) > 0$, then the first seed never falls on i;
- ▷ if $\pi_S(i_\lambda) = 0$, then seeds fall almost continuously on i.

The precise statement is as follows.

Proposition 3.5.3. — *Let $A > 0$, $T > 0$, $\delta > 0$ be fixed. For any $\lambda \in (0, 1]$, it is possible to find a coupling between a Poisson measure π_S on $\mathbb{R}$ with intensity measure $\mathrm{d}x$ and a family $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ of $\mathrm{SR}(\mu_S)$-processes in such a way that for*

$$\Omega_{A,T,\delta}^S(\lambda) = \bigcap_{i\in I_A^\lambda} \Big(\Big\{\pi_S(i_\lambda) = 0, \inf_{t\in[0,T-\delta]} [N^S_{\boldsymbol{a}_\lambda(t+\delta)}(i) - N^S_{\boldsymbol{a}_\lambda t}(i)] > 0\Big\} \\ \cup \big\{\pi_S(i_\lambda) = 1, N^S_{\boldsymbol{a}_\lambda T}(i) = 0\big\}\Big),$$

one has $\lim_{\lambda\to 0} \Pr[\Omega_{A,T,\delta}^S(\lambda)] = 1$.

Proof. — We split the proof in several steps. As usual, it suffices to build π_S on $B_\lambda = \bigcup_{i\in I_A^\lambda} i_\lambda \simeq [-A, A]$ and to build $N_t^S(i)$ for $t \in [0, \boldsymbol{a}_\lambda T]$ and $i \in I_A^\lambda$.

Step 1. — Denote by $(N_t^S)_{t\geq 0}$ a SR(μ_S)-process and by $(T_n)_{n\geq 1}$ its jump instants. Recall the notation of Lemma 3.5.2. Then we observe that

$$\begin{aligned}&\big\{N^S_{\boldsymbol{a}_\lambda T} = 0\big\} = \big\{T_1 > \boldsymbol{a}_\lambda T\big\},\\ &\Big\{\inf_{t\in[0,T-\delta]}[N^S_{\boldsymbol{a}_\lambda(t+\delta)} - N^S_{\boldsymbol{a}_\lambda t}] > 0\Big\} = \big\{T_1 < \boldsymbol{a}_\lambda \delta, R_T^\lambda(\delta) = 0\big\}.\end{aligned}$$

These two events are furthermore disjoint. By Lemma 3.5.2, we deduce that for some functions $\varepsilon_T(\lambda)$ and $\varepsilon_{T,\delta}(\lambda)$ tending to 0 when $\lambda \to 0$

$$\begin{aligned}\Pr\Big[\inf_{t\in[0,T-\delta]}[N^S_{\boldsymbol{a}_\lambda(t+\delta)} - N^S_{\boldsymbol{a}_\lambda t}] > 0\Big] &\geq 1 - \Pr\big[T_1 > \boldsymbol{a}_\lambda \delta\big] - \mathbb{E}\big[R_T^\lambda(\delta)\big]\\ &\geq 1 - \lambda \boldsymbol{a}_\lambda\big(1 + \varepsilon_{T,\delta}(\lambda)\big)\end{aligned}$$

and $p_T(\lambda) := \Pr[N^S_{\boldsymbol{a}_\lambda T} = 0] = \Pr[T_1 > \boldsymbol{a}_\lambda T] = \lambda \boldsymbol{a}_\lambda(1 + \varepsilon_T(\lambda))$.

Step 2. — Next, we prove that it is possible to couple a family $(Z_i^\lambda)_{i\in I_A^\lambda}$ of i.i.d. Poisson-distributed random variables with parameter $|i_\lambda| = 1/\boldsymbol{n}_\lambda$ and a family of $(\widetilde{Z}_i^\lambda)_{i\in I_A^\lambda}$ of i.i.d. Bernoulli random variables with parameter $p_T(\lambda)$ (see step 1) in such a way that for

$$\widetilde{\Omega}_{T,A}(\lambda) = \big\{\forall i \in I_A^\lambda, Z_i^\lambda = \widetilde{Z}_i^\lambda \in \{0,1\}\big\},$$

there holds $\lim_{\lambda\to 0}\Pr[\widetilde{\Omega}_{T,A}(\lambda)] = 1$. As usual, this follows from Lemma 5.1.3 (ii) and relies on the straightforward computations (here the function ε_T changes from line to line)

$$\Pr[Z_i^\lambda = 0] \wedge \Pr[\widetilde{Z}_i^\lambda = 0] = (\mathrm{e}^{-1/\boldsymbol{n}_\lambda}) \wedge \big(1 - p_T(\lambda)\big) \geq 1 - \lambda \boldsymbol{a}_\lambda\big(1 + \varepsilon_T(\lambda)\big),$$

recall that $\boldsymbol{n}_\lambda \sim 1/(\lambda \boldsymbol{a}_\lambda)$, and

$$\Pr[Z_i^\lambda = 1] \wedge \Pr[\widetilde{Z}_i^\lambda = 1] = (\mathrm{e}^{-1/\boldsymbol{n}_\lambda}/\boldsymbol{n}_\lambda) \wedge p_T(\lambda) \geq \lambda \boldsymbol{a}_\lambda\big(1 - \varepsilon_T(\lambda)\big)$$

from which

$$\Pr\big[\widetilde{\Omega}_{T,A}(\lambda)\big] \geq \big[1 - \lambda \boldsymbol{a}_\lambda(1+\varepsilon_T(\lambda)) + \lambda \boldsymbol{a}_\lambda(1 - \varepsilon_T(\lambda))\big]^{|I_A^\lambda|} \geq \big[1 - \lambda \boldsymbol{a}_\lambda \varepsilon_T(\lambda)\big]^{|I_A^\lambda|}.$$

This last quantity tends to 1 as $\lambda \to 0$, because $|I_A^\lambda| \sim 2A/(\lambda \boldsymbol{a}_\lambda)$.

Step 3. — We finally build the complete coupling.

(a) Consider $(Z_i^\lambda, \widetilde{Z}_i^\lambda)_{i\in I_A^\lambda}$ as in step 2.

(b) For each $i \in I_A^\lambda$ such that $Z_i^\lambda > 0$, pick some i.i.d. random variables $(X_1^{i,\lambda}, \dots, X_{Z_i^\lambda}^{i,\lambda})$ uniformly distributed in i_λ. Then $\pi_S = \sum_{i\in I_A^\lambda}\sum_{k=1}^{Z_i^\lambda}\delta_{X_k^{i,\lambda}}$ is a Poisson measure with intensity measure $\mathrm{d}x$ on $B_\lambda = \bigcup_{i\in I_A^\lambda} i_\lambda$.

(c) For each $i \in I_A^\lambda$ such that $\widetilde{Z}_i^\lambda = 1$, set $N^S_{\boldsymbol{a}_\lambda T}(i) = 0$. For each $i \in I_A^\lambda$ such that $\widetilde{Z}_i^\lambda = 0$, pick $(N_t^S(i))_{t\in[0,\boldsymbol{a}_\lambda T]}$ conditionally on $N^S_{\boldsymbol{a}_\lambda T}(i) \neq 0$. This defines a family of i.i.d. SR(μ_S) processes on $[0, \boldsymbol{a}_\lambda T]$ (because $\Pr[\widetilde{Z}_i^\lambda = 1] = p_T(\lambda) = \Pr[N^S_{\boldsymbol{a}_\lambda T}(i) = 0]$).

Step 4. — With this coupling, we have $\widetilde{\Omega}_{T,A}(\lambda)\cap\overline{\Omega}^{S}_{A,T,\delta}(\lambda)\subset\Omega^{S}_{A,T,\delta}(\lambda)$, where

$$\overline{\Omega}^{S}_{A,T,\delta}(\lambda)=\bigcap_{i\in I^\lambda_A}\big(\inf_{t\in[0,T-\delta]}[N^S_{\boldsymbol{a}_\lambda(t+\delta)}(i)-N^S_{\boldsymbol{a}_\lambda t}(i)]>0 \text{ or } N^S_{\boldsymbol{a}_\lambda T}(i)=0\big).$$

It thus only remains to check that $\lim_{\lambda\to 0}\Pr[\overline{\Omega}^{S}_{A,T,\delta}(\lambda)]=1$. But using step 1 and recalling that $|I^\lambda_A|\sim 2A/(\lambda\boldsymbol{a}_\lambda)$, we get

$$\Pr\big[\overline{\Omega}^{S}_{A,T,\delta}(\lambda)\big]\geq\big[1-\lambda\boldsymbol{a}_\lambda(1+\varepsilon_{T,\delta}(\lambda))+\lambda\boldsymbol{a}_\lambda(1+\varepsilon_T(\lambda))\big]^{|I^\lambda_A|},$$

which tends to 1 as $\lambda\to 0$, as usual, since $|I^\lambda_A|\sim 2A/(\lambda\boldsymbol{a}_\lambda)$. □

3.5.3. Convergence. — We now prove the convergence result in the case $\beta=0$.

Proof of Theorem 2.6.2. — We fix $T>0$, $x_1<\cdots<x_p$ and $t_1,\dots,t_p\in(0,T]$. We introduce $B>0$ such that $-B<x_1<x_p<B$. We fix $\varepsilon>0$ and $a>0$. Our aim is to check that for $\lambda>0$ small enough, there exists a coupling between a FF(μ_S,μ^λ_M)-process $(\eta^\lambda_t(i))_{t\geq 0,i\in\mathbb{Z}}$ and a LFF(0)-process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ such that, recalling (2.2.3) and Proposition 2.6.1, there holds

$$\Pr\Big[\sum_{k=1}^p\boldsymbol{\delta}_T\big(D^\lambda(x_k),D(x_k)\big)+\sum_{k=1}^p\boldsymbol{\delta}\big(D^\lambda_{t_k}(x_k),D_{t_k}(x_k)\big)\geq a\Big]\leq\varepsilon. \tag{3.5.1}$$

This will conclude the proof.

Step 1. — Consider two independent Poisson measures $\pi_S(\mathrm{d}x)$ and $\pi_M(\mathrm{d}t,\mathrm{d}x)$ with intensity measures $\mathrm{d}x$ and $\mathrm{d}t\,\mathrm{d}x$. First, we consider $A>B$ large enough, in such a way that for

$$\Omega^{S,1}_A=\big\{\pi_S([-A,-B])>0,\pi_S([B,A])>0\big\},$$

there holds $\Pr(\Omega^{S,1}_A)\geq 1-\frac14\varepsilon$. This fixes the value of A. Next we call

$$\begin{aligned}\mathcal{X}_S&=\big\{x\in[-A,A],\pi_S(\{x\})>0\big\},\\ \mathcal{T}_M&=\big\{t\in[0,T]:\pi_M(\{t\}\times[-A,A])>0\big\}\cup\{0\},\\ \mathcal{X}_M&=\big\{x\in[-A,A],\pi_M([0,T]\times\{x\})>0\big\}.\end{aligned}$$

Classical results about Poisson measures allow us to choose $K>0$ (large) and $c>0$ (small) in such a way that for

$$\Omega_{K,c}=\Big\{|\mathcal{T}_M|+|\mathcal{X}_S|\leq K,\ \min_{\substack{t\in\mathcal{T}_M\\k=1,\dots,p}}|t-t_k|>c,\ \min_{\substack{x,y\in\mathcal{X}_S\cup\mathcal{X}_M\\x\neq y}}|x-y|>c,\ \min_{\substack{x\in\mathcal{X}_S\cup\mathcal{X}_M\\k=1,\dots,p}}|x-x_k|>c\Big\},$$

there holds $\Pr[\Omega_{K,c}]\geq 1-\frac14\varepsilon$.

Step 2. — Next, we know from Proposition 3.2.1 that for all $\lambda > 0$ small enough, it is possible to couple a family of i.i.d. $\mathrm{SR}(\mu_M^\lambda)$-processes $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}\}}$ with π_M in such a way that for

$$\Omega_{A,T}^M(\lambda) := \big\{\forall t \in [0,T],\ \forall i \in I_A^\lambda,\ \Delta N_{\boldsymbol{a}_\lambda t}^{M,\lambda}(i) \neq 0 \text{ iff } \pi_M(\{t\} \times i_\lambda) \neq 0\big\},$$

there holds $\Pr[\Omega_{A,T}^M(\lambda)] \geq 1 - \frac{1}{4}\varepsilon$. We now fix $\delta > 0$ such that

$$\delta < c/4 \quad \text{and} \quad \delta < a/(4AKp).$$

Proposition 3.5.3 tells us how to couple, for all $\lambda > 0$ small enough, a family of i.i.d. $\mathrm{SR}(\mu_S)$-processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ with π_S in such a way that for

$$\Omega_{A,T,\delta}^S(\lambda) = \bigcap_{i\in I_A^\lambda} \Big(\big\{\pi_S(i_\lambda) = 0, \inf_{t\in[0,T-\delta]} [N_{\boldsymbol{a}_\lambda(t+\delta)}^S(i) - N_{\boldsymbol{a}_\lambda t}^S(i)] > 0\big\} \\ \cup \big\{\pi_S(i_\lambda) = 1, N_{\boldsymbol{a}_\lambda T}^S(i) = 0\big\}\Big),$$

there holds $\Pr[\Omega_{A,T,\delta}^S(\lambda)] \geq 1 - \frac{1}{4}\varepsilon$.

Step 3. — We consider π_M, π_S, $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}\}}$ and $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}\}}$ coupled as in step 2. Then we build the corresponding $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ and the associated rescaled clusters $(D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$, see (2.2.3) and we build the LFF(0)-process associated to π_S and the corresponding clusters $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$. We will work on the event

$$\Omega_\lambda = \Omega_A^{S,1} \cap \Omega_{K,c} \cap \Omega_{A,T}^M(\lambda) \cap \Omega_{A,T,\delta}^S(\lambda).$$

We know that for all $\lambda > 0$ small enough, $\Pr[\Omega_\lambda] \geq 1 - \varepsilon$. We introduce

$$\mathcal{S} = \bigcup_{t\in\mathcal{T}_M} [t, t+\delta].$$

We will prove in the next step that on Ω_λ, for all $\lambda > 0$ small enough, for all $k \in \{1, \dots, p\}$, for all $t \in [0,T] \setminus \mathcal{S}$,

$$\boldsymbol{\delta}\big(D_t^\lambda(x_k), D_t(x_k)\big) \leq 4/\boldsymbol{n}_\lambda + 2A\mathbb{1}_{\{t\in\mathcal{S}\}}, \tag{3.5.2}$$

which will imply that

$$\boldsymbol{\delta}_T\big(D^\lambda(x_k), D(x_k)\big) \leq 4T/\boldsymbol{n}_\lambda + 2A|\mathcal{S}|.$$

This will conclude the proof, since for $k = 1, \dots, p$, $t_k \notin \mathcal{S}$ (recall $\Omega_{K,c}$ and that $\delta < c$) and since the Lebesgue measure of $\mathcal{S}$ is smaller than $K\delta$ (recall $\Omega_{K,c}$ and that $\delta < c$). Thus (3.5.2) implies, since $\delta < a/(4AKp)$,

$$\sum_{k=1}^p \boldsymbol{\delta}_T\big(D^\lambda(x_k), D(x_k)\big) + \sum_{k=1}^p \boldsymbol{\delta}\big(D_{t_k}^\lambda(x_k), D_{t_k}(x_k)\big) \leq p\,[4T/\boldsymbol{n}_\lambda + 2AK\delta + 4/\boldsymbol{n}_\lambda] \\ \leq a/2 + 4p(T+1)/\boldsymbol{n}_\lambda,$$

which is smaller than a for all $\lambda > 0$ small enough. Thus (3.5.1) holds for all $\lambda > 0$ small enough.

Step 4. — It remains to check (3.5.2). In the whole step, we work on Ω_λ. Let thus $k \in \{1, \dots, p\}$ be fixed. Consider the first marks χ_g, χ_d of π_S on the left and right of x_k. Then by definition, we have $D_t(x_k) = [\chi_g, \chi_d]$ for all $t \in [0, T]$. By $\Omega_A^{S,1}$ and since $x_k \in (-B, B)$, we know that $-A < \chi_g < \chi_d < A$. Define

$$g_\lambda = \lfloor \boldsymbol{n}_\lambda \chi_g \rfloor \quad \text{and} \quad d_\lambda = \lfloor \boldsymbol{n}_\lambda \chi_d \rfloor.$$

Due to $\Omega_{A,T,\delta}^S(\lambda)$ and since $\pi_S(\{\chi_g\}) = \pi_S(\{\chi_d\}) = 1$ and $\pi_S((\chi_g, \chi_d)) = 0$ by construction, we know that

(i) $N^S_{\boldsymbol{a}_\lambda T}(g_\lambda) = N^S_{\boldsymbol{a}_\lambda T}(d_\lambda) = 0$ (because $\chi_g \in (g_\lambda)_\lambda$ and $\chi_d \in (d_\lambda)_\lambda$);

(ii) for all $i \in [\![g_\lambda + 1, d_\lambda - 1]\!]$, $\inf_{t \in [0, T-\delta]} [N^S_{\boldsymbol{a}_\lambda (t+\delta)}(i) - N^S_{\boldsymbol{a}_\lambda t}(i)] > 0$ (because $i_\lambda \subset (\chi_g, \chi_d)$).

Observe now that for $\lambda > 0$ small enough (it suffices that $1/\boldsymbol{n}_\lambda < c$), there holds $g_\lambda < \lfloor x_k \boldsymbol{n}_\lambda \rfloor < d_\lambda$ (use that $\chi_g, \chi_d \in \mathcal{X}_S$ and that $\chi_g < x_k < \chi_d$ so that due to $\Omega_{K,c}$, $\chi_g + c < x_k < \chi_d - c$).

Point (i) implies that $\eta^\lambda_{\boldsymbol{a}_\lambda t}(g_\lambda) = \eta^\lambda_{\boldsymbol{a}_\lambda t}(d_\lambda) = 0$ for all $t \in [0, \boldsymbol{a}_\lambda T]$. Consequently, for all $t \in [0, T]$, there holds $C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, \lfloor x_k \boldsymbol{n}_\lambda \rfloor) \subset [\![g_\lambda + 1, d_\lambda - 1]\!]$. This implies that $D_t^\lambda(x_k) \subset [(g_\lambda + 1)/\boldsymbol{n}_\lambda, (d_\lambda - 1)/\boldsymbol{n}_\lambda] \subset [\chi_g, \chi_d]$. Recalling that $D_t(x_k) = [\chi_g, \chi_d]$ and that $-A < \chi_g < \chi_d < A$, we deduce that

$$\boldsymbol{\delta}\big(D_t(x_k), D_t^\lambda(x_k)\big) \leq 2A \quad \text{for all } t \in [0, T].$$

Another consequence is that the matches falling outside $[\![g_\lambda, d_\lambda]\!]$ (and *a fortiori* outside I_A^λ) have no influence on $\lfloor x_k \boldsymbol{n}_\lambda \rfloor$ during $[0, \boldsymbol{a}_\lambda T]$.

It only remains to check that for $t \in [0, T] \setminus \mathcal{S}$, if $\lambda > 0$ is small enough, $\boldsymbol{\delta}(D_t(x_k), D_t^\lambda(x_k)) \leq 4/\boldsymbol{n}_\lambda$. We thus fix $t \in [0, T] \setminus \mathcal{S}$ and consider

$$t_0 = \max\{s \in \mathcal{T}_M \,:\, s < t\}.$$

Then by definition of $\mathcal{S}$, $t - t_0 > \delta$. Consequently, point (ii) guarantees us that

$$\forall i \in [\![g_\lambda + 1, d_\lambda - 1]\!], \quad N^S_{\boldsymbol{a}_\lambda t} - N^S_{\boldsymbol{a}_\lambda t_0} > 0.$$

A seed falls on each of these sites during $[\boldsymbol{a}_\lambda t_0, \boldsymbol{a}_\lambda t]$. Furthermore, there are no matches falling on $[\![g_\lambda + 1, d_\lambda - 1]\!]$ during $[\boldsymbol{a}_\lambda t_0, \boldsymbol{a}_\lambda t]$, by definition of t_0 and due to $\Omega^M_{A,T}(\lambda)$. Consequently, we have $\eta^\lambda_{\boldsymbol{a}_\lambda t}(i) = 1$ for all $i \in [\![g_\lambda + 1, d_\lambda - 1]\!]$. All this implies that $C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, \lfloor x_k \boldsymbol{n}_\lambda \rfloor) = [\![g_\lambda + 1, d_\lambda - 1]\!]$, whence

$$D_t^\lambda(x_k) = \big[(g_\lambda + 1)/\boldsymbol{n}_\lambda, (d_\lambda - 1)/\boldsymbol{n}_\lambda\big] = \big[(\lfloor \boldsymbol{n}_\lambda \chi_g \rfloor + 1)/\boldsymbol{n}_\lambda, (\lfloor \boldsymbol{n}_\lambda \chi_d \rfloor - 1)/\boldsymbol{n}_\lambda\big].$$

Recalling that $D_t(x_k) = [\chi_g, \chi_d]$, we easily conclude. □

3.6. Well-posedness of the limit process when $\beta \in \{\infty, BS\}$

The aim of this section is to prove Theorems 2.3.2 and 2.4.3, and to localize the limit processes. All the results below have already been proved in [**15**] for the LFF(∞)-process. We provide here a consequently simpler proof, that allows us to treat simultaneously the cases $\beta = BS$ and $\beta = \infty$.

REMARK 3.6.1. — *Under $(H_S(\infty))$, we put $\theta_u = \delta_u$ and $F_S(u, v) = u$ for all $u \in [0, 1]$, all $v \in [0, 1]$. Using this function F_S, the* LFF(BS)*-process is nothing but the* LFF(∞)*-process.*

We consider a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$ on $[0, \infty) \times \mathbb{R} \times [0, 1]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$ and abusively write $\pi_M(\mathrm{d}t, \mathrm{d}x) = \int_{v \in [0,1]} \pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$, which is a Poisson measure on $[0, \infty) \times \mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$.

DEFINITION 3.6.2. — *Let $\beta \in \{\infty, BS\}$. If $\beta = \infty$, consider F_S as in Remark* 3.6.1. *If $\beta = BS$, consider F_S as in Definition* 2.4.1. *Let $A > 0$ be fixed. A $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$-valued process $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t \geq 0, x \in [-A, A]}$ such that a.s., for all $x \in [-A, A]$, $(Z_t^A(x), H_t^A(x))_{t \geq 0}$ is càdlàg, is called a* LFF$_A(\beta)$*-process if a.s., for all $t \geq 0$, all $x \in [-A, A]$,*

$$Z_t^A(x) = \int_0^t \mathbb{1}_{\{Z_s^A(x) < 1\}}\,\mathrm{d}s - \int_0^t \int_{[-A,A]} \mathbb{1}_{\{Z_{s-}^A(x) = 1, y \in D_{s-}^A(x)\}} \pi_M(\mathrm{d}s, \mathrm{d}y),$$

$$H_t^A(x) = \int_0^t \int_0^1 F_S\big(Z_{s-}^A(x), v\big) \mathbb{1}_{\{Z_{s-}^A(x) < 1\}} \pi_M(\mathrm{d}s \times \{x\} \times \mathrm{d}v) - \int_0^t \mathbb{1}_{\{H_s^A(x) > 0\}}\,\mathrm{d}s,$$

where $D_t^A(x) = [L_t^A(x), R_t^A(x)]$, with

$$\begin{cases} L_t^A(x) = (-A) \vee \sup\{y \in [-A, x];\ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\} \\ R_t^A(x) = A \wedge \inf\{y \in [x, A];\ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\} \end{cases} \tag{3.6.1}$$

and where $D_{t-}^A(x)$ is defined similarly.

Observe that for $\beta \in \{\infty, BS\}$, for any $A > 0$, the LFF$_A(\beta)$-process is obviously well and uniquely defined and can be built as follows.

ALGORITHM 3.6.3. — Consider the marks $(T_k, X_k, V_k)_{k=1,\dots,n}$ of π_M in $[0, T] \times [-A, A] \times [0, 1]$, ordered chronologically and set $T_0 = 0$.

Step 0. — Put $Z_0^A(x) = H_0^A(x) = 0$ and $D_0^A(x) = \{x\}$ for all $x \in [-A, A]$. Assume that for some $k \in \{0, \dots, n-1\}$,

$$\big(Z_t^A(x), D_t^A(x), H_t^A(x)\big)_{t \in [0, T_k], x \in [-A, A]}$$

has been built.

Step $k+1$. — Then for $t \in (T_k, T_{k+1})$ and $x \in [-A, A]$, put

$$Z_t^A(x) = \min(1, Z_{T_k}^A(x) + t - T_k),$$

set $H_t^A(x) = \max(0, H_{T_k}^A(x) - t + T_k)$ and then define $D_t^A(x)$ as in (3.6.1). Finally, build $(Z_{T_{k+1}}^A(x), D_{T_{k+1}}^A(x), H_{T_{k+1}}^A(x))$ as follows.

- ▷ If $Z_{T_{k+1}-}^A(X_{k+1}) = 1$, set $H_{T_{k+1}}^A(x) = H_{T_{k+1}-}^A(x)$ for all $x \in [-A, A]$ and consider $[a, b] := D_{T_{k+1}-}^A(X_{k+1})$. Set

$$Z_{T_{k+1}}^A(x) = \begin{cases} 0 & \text{for all } x \in (a, b), \\ Z_{T_{k+1}-}^A(x) & \text{for all } x \in [-A, A] \setminus [a, b]. \end{cases}$$

$$Z_{T_{k+1}}^A(a) = \begin{cases} 0 & \text{if } Z_{T_{k+1}-}^A(a) = 1, \\ Z_{T_{k+1}-}^A(a) & \text{if } Z_{T_{k+1}-}^A(a) < 1, \end{cases}$$

$$Z_{T_{k+1}}^A(b) = \begin{cases} 0 & \text{if } Z_{T_{k+1}-}^A(b) = 1, \\ Z_{T_{k+1}-}^A(b) & \text{if } Z_{T_{k+1}-}^A(b) < 1. \end{cases}$$

- ▷ If $Z_{T_{k+1}-}^A(X_{k+1}) < 1$, set

$$H_{T_{k+1}}^A(X_{k+1}) = F_S\big(Z_{T_{k+1}-}^A(X_{k+1}), V_{k+1}\big),$$

  put $Z_{T_{k+1}}^A(X_{k+1}) = Z_{T_{k+1}-}^A(X_{k+1})$ and

$$\big(Z_{T_{k+1}}^A(x), H_{T_{k+1}}^A(x)\big) = \big(Z_{T_{k+1}-}^A(x), H_{T_{k+1}-}^A(x)\big)$$

  for all $x \in [-A, A] \setminus \{X_{k+1}\}$.
- ▷ Using the values of $(Z_{T_{k+1}}^A(x), H_{T_{k+1}}^A(x))_{x\in[-A,A]}$, compute $(D_{T_{k+1}}^A(x))_{x\in[-A,A]}$ as in (3.6.1).

We now state a refined version of Theorems 2.3.2 and 2.4.3.

Proposition 3.6.4. — *Let $\beta \in \{\infty, BS\}$. Let π_M be a Poisson measure on $[0, \infty) \times \mathbb{R} \times [0, 1]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$.*

(i) *There exists a unique* LFF(β)*-process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$.*

(ii) *It can be perfectly simulated on $[0, T] \times [-n, n]$ for any $T > 0$, any $n > 0$.*

(iii) *For $A > 0$, let $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ be the unique* $\mathrm{LFF}_A(\beta)$*-process. There holds*

$$\begin{aligned} \Pr\big[&(Z_t(x), D_t(x), H_t(x))_{t\in[0,T], x\in[-A/2,A/2]} \\ &= (Z_t^A(x), D_t^A(x), H_t^A(x))_{t\in[0,T], x\in[-A/2,A/2]}\big] \geq 1 - C_T\, \mathrm{e}^{-\alpha_T A}, \end{aligned} \tag{3.6.2}$$

for some constants $\alpha_T > 0$ and $C_T > 0$ not depending on $A > 0$.

To prove this result, we need a lower-bound of the length of the barriers.

Lemma 3.6.5. — *Let $\beta \in \{\infty, BS\}$. If $\beta = \infty$, consider F_S as in Remark* 3.6.1. *If $\beta = BS$, consider F_S as in Definition* 2.4.1. *There exists $v_0 \in [0, 1)$ such that for all $z \in [\frac{3}{4}, 1)$, all $v \in [v_0, 1]$, $F(z, v) \geq \frac{1}{2}$.*

Proof. — If $\beta = \infty$, the result is obvious with $v_0 = 0$, since $F_S(z,v) = z \geq \frac{1}{2}$ for all $z \in [\frac{1}{2}, 1]$, $v \in [0,1]$. Consider now the case $\beta = BS$. First observe that

$$g_S(t,s) \leq \Pr\left[N^S_{T_S(t+s)} - N^S_{T_S t} > 0\right] = \nu_S\big([0, T_S s]\big).$$

Hence for all $z \in [\frac{3}{4}, 1)$,

$$\theta_z\big([0,\tfrac{1}{2})\big) \leq \nu_S\big([\tfrac{3}{4}T_S, T_S]\big) + \frac{\nu_S([\frac{3}{4}T_S, T_S])^2}{\nu_S([\frac{1}{2}T_S, T_S])^2}\,\nu_S\big([0, \tfrac{1}{2}T_S]\big)$$
$$\leq \nu_S\big([0, \tfrac{1}{2}T_S] \cup [\tfrac{3}{4}T_S, T_S]\big) =: v_0 < 1,$$

since $\operatorname{supp} \nu_S = [0, T_S]$. We deduce that for $z \in [\frac{3}{4}, 1]$,

$$\int_0^1 \mathrm{d}v \mathbb{1}_{\{F_S(z,v) < \frac{1}{2}\}} = \theta_z\big([0, \tfrac{1}{2})\big) \leq v_0.$$

Recalling that $v \mapsto F_S(z,v)$ is nondecreasing, we deduce that $F_S(z,v) \geq \frac{1}{2}$ for $v \in [v_0, 1]$. □

Proof of Proposition 3.6.4. — We split the proof into steps. We work on $[0,T]$.

Step 1. — We observe that for a mark (τ, X, V) of π_M with $X \in [-A, A]$ and $V \geq v_0$ (see Lemma 3.6.5), we have $H^A_t(X) > 0$ or $Z^A_t(X) < 1$ for all $t \in [\tau, \tau + \frac{1}{4}]$ (and the same result applies to the LFF(β)-process if it exists).

Indeed, assume first that $Z^A_{\tau-}(X) \in [0, \frac{3}{4})$. Then for $t \in [\tau, \tau + \frac{1}{4}]$,

$$Z^A_t(X) = Z^A_{\tau-}(X) + t - \tau < 1.$$

Assume next that $Z^A_{\tau-}(X) \in [\frac{3}{4}, 1)$. Then

$$H_\tau(X) = F_S(Z^A_{\tau-}(X), V) \geq \tfrac{1}{2}$$

due to Lemma 3.6.5, so that for $t \in [\tau, \tau + \frac{1}{2}) \supset [\tau, \tau + \frac{1}{4}]$

$$H_t(X) = H_\tau(X) - t + \tau > 0.$$

If finally $Z^A_{\tau-}(X) = 1$, then $Z^A_\tau(X) = 0$, whence, for $t \in [\tau, \tau+1) \supset [\tau, \tau + \frac{1}{4}]$

$$Z^A_t(X) = t - \tau < 1.$$

Step 2. — For $a \in \mathbb{R}$, we consider the event Ω_a defined as follows: for $\{(T_k, X_k, V_k)\}_{k=1,\dots,n}$ the marks of π_M restricted to $[0,T] \times [a, a+1) \times [v_0, 1]$ ordered chronologically, for $T_0 = 0$, $T_{n+1} = T$, we put

$$\Omega_a = \big\{ \max_{i=0,\dots,n} (T_{i+1} - T_i) < \tfrac{1}{4} \big\}.$$

We immediately deduce from step 1 that for any $a \in \mathbb{R}$, any $A > |a| + 1$,

$$\Omega_a \subset \big\{\forall\, t \in [0,T],\ \exists\, x \in (a, a+1), H^A_t(x) > 0 \text{ or } Z^A_t(x) < 1\big\}.$$

Thus on Ω_a, clusters on the left of a cannot be connected to clusters on the right of $a+1$ during $[0,T]$. Hence matches falling at the right of $a+1$ (resp. on the left of a) do not affect the zone $(-\infty, a)$ (resp. $(a+1, \infty)$) during $[0,T]$.

Step 3. — Obviously, $q_T := \Pr(\Omega_a)$ is positive and does not depend on a. Furthermore, Ω_a is independent of Ω_b for all $a, b \in \mathbb{Z}$ with $a \neq b$. Hence there are a.s. infinitely many $a \in \mathbb{Z}$ such that Ω_a is realized.

Then it is routine to deduce the well-posedness of the LFF(β)-process. The perfect simulation algorithm on a finite-box $[0, T] \times [-n, n]$ is also easy: simulate π_M on $[0, T] \times [a_1, a_2]$ in such a way that $\Omega_{a_1} \cap \Omega_{a_2}$ is realized and that $a_1 + 1 < -n < n < a_2$. Then apply the same rules as for the LFF$_A(\beta)$-process. This will give the true LFF(β)-process inside $[a_1 + 1, a_2] \supset [-n, n]$, because matches falling outside $[a_1, a_2 + 1]$ have no effect on the process in the box $[a_1 + 1, a_2]$ during $[0, T]$.

Finally, we can clearly bound from below the left hand side of (3.6.2) by

$$\begin{aligned}\Pr\Big[\Big(\bigcup_{a \in [-A, -\frac{1}{2}A-1] \cap \mathbb{Z}} \Omega_a\Big) \cap \Big(\bigcup_{a \in [\frac{1}{2}A, A-1] \cap \mathbb{Z}} \Omega_a\Big)\Big] &\geq 1 - 2\Pr[\Omega_0^c]^{\lfloor A \rfloor - \lfloor \frac{1}{2}A \rfloor - 2} \\ &\geq 1 - 2(1 - q_T)^{\frac{1}{2}A - 4},\end{aligned}$$

whence (3.6.2) with $C_T = 2/(1 - q_T)^4$ and $\alpha_T = -\frac{1}{2}\log(1 - q_T)$. □

3.7. Localization of the discrete processes when $\beta \in \{\infty, BS\}$

We recall that $\boldsymbol{a}_\lambda$, $\boldsymbol{n}_\lambda$ and $\boldsymbol{m}_\lambda$ are defined in (2.2.1), (2.2.2) and (2.2.4). For $A > 0$ and $i \in \mathbb{Z}$, we set

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda \rfloor, \quad I_A^\lambda = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = \big[i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda\big).$$

For $\eta \in \{0, 1\}^{I_A^\lambda}$ and $i \in I_A^\lambda$, we define the occupied connected component around i as

$$C_A(\eta, i) = \begin{cases} \varnothing & \text{if } \eta(i) = 0, \\ [\![\ell_A(\eta, i), r_A(\eta, i)]\!] & \text{if } \eta(i) = 1, \end{cases}$$

where

$$\begin{aligned}\ell_A(\eta, i) &= (-A_\lambda) \vee \big(\sup\{k < i : \eta(k) = 0\} + 1\big), \\ r_A(\eta, i) &= A_\lambda \wedge \big(\inf\{k > i : \eta(k) = 0\} - 1\big).\end{aligned}$$

Definition 3.7.1. — *Assume* (H_M) *and* $(H_S(\beta))$ *with* $\beta \in \{\infty, BS\}$. *Let* $\lambda \in (0, 1]$ *and* $A > 0$ *be fixed. For each* $i \in I_A^\lambda$, *we consider a* SR(μ_S)*-process* $(N_t^S(i))_{t \geq 0}$ *and a* SR(μ_M^λ)*-process* $(N_t^{M,\lambda}(i))_{t \geq 0}$, *all these processes being independent. Consider a* $\{0, 1\}$*-valued process* $(\eta_t^{\lambda,A}(i))_{i \in I_A^\lambda, t \geq 0}$ *such that a.s., for all* $i \in I_A^\lambda$, $(\eta_t^{\lambda,A}(i))_{t \geq 0}$ *is càdlàg. We say that* $(\eta_t^{\lambda,A}(i))_{i \in I_A^\lambda, t \geq 0}$ *is a* FF$_A(\mu_S, \mu_M^\lambda)$*-process if a.s., for all* $i \in I_A^\lambda$, *all* $t \geq 0$,

$$\eta_t^{\lambda,A}(i) = \int_0^t \mathbb{1}_{\{\eta_{s-}^{\lambda,A}(i) = 0\}} \, dN_s^S(i) - \sum_{j \in I_A^\lambda} \int_0^t \mathbb{1}_{\{j \in C_A(\eta_{s-}^{\lambda,A}, i)\}} \, dN_s^{M,\lambda}(j).$$

For $x \in [-A, A]$ and $t \geq 0$, we introduce

$$(3.7.1)\quad D_t^{\lambda,A}(x) = \frac{1}{\boldsymbol{n}_\lambda} C_A\big(\eta^{\lambda,A}_{\boldsymbol{a}_\lambda t}, \lfloor \boldsymbol{n}_\lambda x \rfloor\big) \subset [-A_\lambda/\boldsymbol{n}_\lambda, A_\lambda/\boldsymbol{n}_\lambda] \simeq [-A, A],$$

$$K_t^{\lambda,A}(x) = \frac{|\{i \in [\![\lfloor \boldsymbol{n}_\lambda x\rfloor - \boldsymbol{m}_\lambda, \lfloor \boldsymbol{n}_\lambda x\rfloor + \boldsymbol{m}_\lambda]\!] \cap I_A^\lambda \;:\; \eta^{\lambda,A}_{\boldsymbol{a}_\lambda t}(x) = 1\}|}{|[\![\lfloor \boldsymbol{n}_\lambda x\rfloor - \boldsymbol{m}_\lambda, \lfloor \boldsymbol{n}_\lambda x\rfloor + \boldsymbol{m}_\lambda]\!] \cap I_A^\lambda|} \in [0,1],$$

$$(3.7.2)\quad Z_t^{\lambda,A}(x) = \frac{\psi_S(K_t^{\lambda,A}(x))}{\boldsymbol{a}_\lambda} \wedge 1 \in [0,1].$$

We generalize [**15**, Proposition 11], with a consequently less intricate proof.

PROPOSITION 3.7.2. — *Assume (H_M) and $(H_S(\beta))$, for some $\beta \in \{\infty, BS\}$. Let $T > 0$ and $\lambda \in (0,1)$. For each $i \in \mathbb{Z}$, we consider a $\mathrm{SR}(\mu_S)$-process $(N_t^S(i))_{t\geq 0}$ and a $\mathrm{SR}(\mu_M^\lambda)$-process $(N_t^{M,\lambda}(i))_{t\geq 0}$, all these processes being independent. Let $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ be the corresponding $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process, and for each $A > 0$, let $(\eta_t^{\lambda,A}(i))_{t\geq 0, i\in I_A^\lambda}$ be the corresponding $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$-process. Recall* (2.2.3)–(2.2.5) *and* (3.7.1)-(3.7.2). *There are some constants $\alpha_T > 0$ and $C_T > 0$ such that for all $A \geq 1$, all $\lambda \in (0,1]$ small enough,*

$$\Pr\Big[(\eta_t^\lambda(i))_{t\in[0,\boldsymbol{a}_\lambda T], i\in I^\lambda_{A/2}} = \big(\eta_t^{\lambda,A}(i)\big)_{t\in[0,\boldsymbol{a}_\lambda T], i\in I^\lambda_{A/2}},$$
$$\big(Z_t^\lambda(x), D_t^\lambda(x)\big)_{t\in[0,T], x\in[-\frac12 A, \frac12 A]} = \big(Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x)\big)_{t\in[0,T], x\in[-\frac12 A, \frac12 A]}\Big]$$
$$\geq 1 - C_T\, \mathrm{e}^{-\alpha_T A}.$$

Proof in the case where $\beta = \infty$. — It of course suffices to prove the result for all A large enough (we will assume that $A > 8T$). We consider the true $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. For $a \in \mathbb{R}$, we introduce

$$J_a^\lambda := [\![\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor (a+1)\boldsymbol{n}_\lambda\rfloor - 1]\!].$$

Step 1. — We show here that for all $a \in \mathbb{R}$, there exists an event $\Omega^\lambda_{a,0}$, depending only on $(N_s^S(i), N_s^{M,\lambda}(i))_{i\in J_a^\lambda, s\in[0,3\boldsymbol{a}_\lambda/4]}$ such that

(i) on $\Omega^\lambda_{a,0}$, a.s., there is $i \in J_a^\lambda$ such that $\eta^\lambda_{\boldsymbol{a}_\lambda s}(i) = 0$ for all $s \in [0, \frac34]$;

(ii) $\lim_{\lambda\to 0} \Pr[\Omega^\lambda_{a,0}] = 1$.

This is very easy: consider simply $\Omega^\lambda_{a,0} = \{\exists\ i \in J_a^\lambda, N^S_{3\boldsymbol{a}_\lambda/4}(i) = 0\}$. Clearly, point (i) is satisfied, since there is a site of J_a^λ on which no seed falls during $[0, \frac34 \boldsymbol{a}_\lambda]$. Since $|J_a^\lambda| = \boldsymbol{n}_\lambda \sim 1/(\lambda \boldsymbol{a}_\lambda) = 1/\nu_S((\boldsymbol{a}_\lambda, \infty))$, we deduce from $(H_S(\infty))$ that

$$\Pr[\Omega^\lambda_{a,0}] = 1 - \nu_S\big((0, \tfrac34 \boldsymbol{a}_\lambda)\big)^{\boldsymbol{n}_\lambda} = 1 - \big(1 - \nu_S((\tfrac34 \boldsymbol{a}_\lambda, \infty))\big)^{\boldsymbol{n}_\lambda}$$
$$\simeq 1 - \mathrm{e}^{-\nu_S((\frac34 \boldsymbol{a}_\lambda,\infty))/\nu_S((\boldsymbol{a}_\lambda,\infty))} \longrightarrow 1$$

as $\lambda \to 0$, whence (ii).

Step 2. — We now check that for all $a \in \mathbb{R}$, all $t \geq \frac12$, there exists an event $\Omega^\lambda_{a,t}$, depending only on $(N_s^S(i), N_s^{M,\lambda}(i))_{i\in J_a^\lambda, s\in[(t-\frac12)\boldsymbol{a}_\lambda, (t+\frac14)\boldsymbol{a}_\lambda]}$ such that

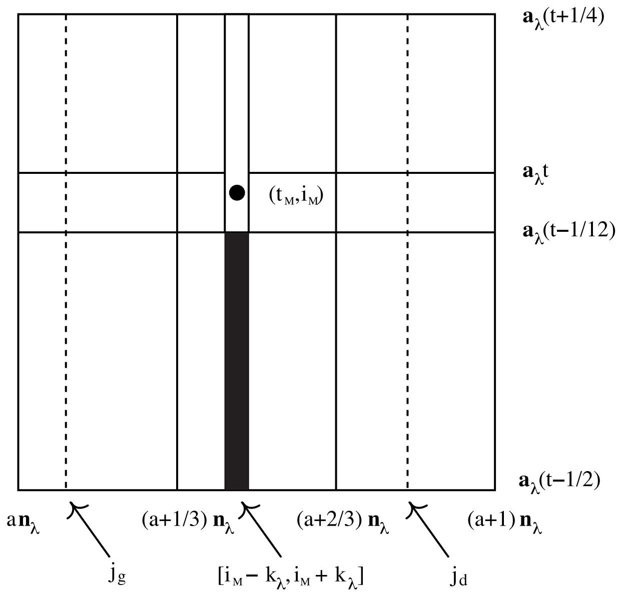

FIGURE 7. The event $\Omega^\lambda_{a,t}$.
A match falls on i_M at time t_M, no seed fall on j_g and j_d during $[\boldsymbol{a}_\lambda(t-\frac{1}{2}), \boldsymbol{a}_\lambda(t+\frac{1}{4})]$. All the sites of $[\![i_M-\boldsymbol{k}_\lambda, i_M+\boldsymbol{k}_\lambda]\!]$ receive at least one seed during $[\boldsymbol{a}_\lambda(t-\frac{1}{2}), \boldsymbol{a}_\lambda(t-\frac{1}{12})]$. Finally, there is at least one site of $[\![i_M-\boldsymbol{k}_\lambda, i_M+\boldsymbol{k}_\lambda]\!]$ on which no seed falls during $[\boldsymbol{a}_\lambda(t-\frac{1}{12}), \boldsymbol{a}_\lambda(t+\frac{1}{4})]$.

(i) on $\Omega^\lambda_{a,t}$, a.s., there is $i \in J^\lambda_a$ such that $\eta^\lambda_{\boldsymbol{a}_\lambda s}(i) = 0$ for all $s \in [t, t+\frac{1}{4}]$;

(ii) $q_\lambda := \Pr[\Omega^\lambda_{a,t}]$ does not depend on t, a and $q := \liminf_{\lambda\to 0} q_\lambda > 0$.

This is much more delicate. We put

$$\boldsymbol{k}_\lambda = \left\lfloor 1/\nu_S((\tfrac{3}{8}\boldsymbol{a}_\lambda, \infty))\right\rfloor.$$

Observe that due to $(H_S(\infty))$, $\boldsymbol{k}_\lambda \ll \boldsymbol{n}_\lambda = \lfloor 1/\nu_S((\boldsymbol{a}_\lambda, \infty))\rfloor$

We introduce the event $\Omega^\lambda_{a,t}$ on which (see Figure 7):

(a) we have $\Delta N^{M,\lambda}_{t_M}(i_M) > 0$ for some $i_M \in [\![\lfloor (a+\frac{1}{3})\boldsymbol{n}_\lambda\rfloor, \lfloor (a+\frac{2}{3})\boldsymbol{n}_\lambda\rfloor]\!]$, some t_M in $[(t-\frac{1}{12})\boldsymbol{a}_\lambda, t\boldsymbol{a}_\lambda]$ and this is the only match falling in J^λ_a during $[(t-\frac{1}{2})\boldsymbol{a}_\lambda, t\boldsymbol{a}_\lambda]$;

(b) there are $j_g \in [\![\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor (a+\frac{1}{4})\boldsymbol{n}_\lambda\rfloor]\!]$ and $j_d \in [\![\lfloor (a+\frac{3}{4})\boldsymbol{n}_\lambda\rfloor, \lfloor (a+1)\boldsymbol{n}_\lambda - 1\rfloor]\!]$ such that $N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(j_g) - N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{2})}(j_g) = N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(j_d) - N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{2})}(j_d) = 0$;

(c) for all $i \in [\![i_M-\boldsymbol{k}_\lambda, i_M+\boldsymbol{k}_\lambda]\!]$, $N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{12})}(i) - N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{2})}(i) > 0$;

(d) there is $j_0 \in [\![i_M-\boldsymbol{k}_\lambda, i_M+\boldsymbol{k}_\lambda]\!]$ such that $N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(j_0) - N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{12})}(j_0) = 0$.

We first prove point (i), considering two cases.

$\triangleright$ If the zone $[\![i_M - \boldsymbol{k}_\lambda, i_M + \boldsymbol{k}_\lambda]\!]$ is completely occupied at time t_M-, then it burns at time t_M and since no seed falls on j_0, which belongs to this zone, during $[t_M, \boldsymbol{a}_\lambda(t+\frac{1}{4})] \supset [\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\frac{1}{4})]$, we deduce that $\eta^\lambda_{\boldsymbol{a}_\lambda s}(j_0) = 0$ for all $s \in [t, t+\frac{1}{4}]$.

$\triangleright$ Assume now that there is $i_0 \in [\![i_M - \boldsymbol{k}_\lambda, i_M + \boldsymbol{k}_\lambda]\!]$ that is vacant at time t_M-. Recall that there is no fire in J^λ_a during $[\boldsymbol{a}_\lambda(t-\frac{1}{2}), t_M)$ and that on each site of $[\![i_M - \boldsymbol{k}_\lambda, i_M + \boldsymbol{k}_\lambda]\!]$, at least one seed falls during $[\boldsymbol{a}_\lambda(t-\frac{1}{2}), \boldsymbol{a}_\lambda(t-\frac{1}{12})] \subset [\boldsymbol{a}_\lambda(t-\frac{1}{2}), t_M)$. Then necessarily, a fire starting at some $i'_M \notin J^\lambda_a$ at some time $t'_M \in [\boldsymbol{a}_\lambda(t-\frac{1}{2}), t_M)$ has made vacant i_0. Assume e.g. that $i'_M < \lfloor a\boldsymbol{n}_\lambda \rfloor$ and observe that $i'_M < j_g < i_0$. The fire (t'_M, i'_M) has then also necessarily made vacant j_g. Since no seed falls on j_g during $[\boldsymbol{a}_\lambda(t-\frac{1}{2}), \boldsymbol{a}_\lambda(t+\frac{1}{4})]$, we deduce that j_g remains vacant during $[t'_M, \boldsymbol{a}_\lambda(t+\frac{1}{4})] \supset [\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\frac{1}{4})]$.

We now prove (ii). The quantity $\Pr[\Omega^\lambda_{a,t}]$ does obviously not depend on $a \in \mathbb{R}$ nor on $t \geq \frac{1}{2}$ by invariance by spatial translation and by time stationarity. We infer from Proposition 3.2.1 that for $\pi_M(\mathrm{d}s, \mathrm{d}x)$ a Poisson measure on $[0,\infty) \times \mathbb{R}$ with intensity measure $\mathrm{d}s\,\mathrm{d}x$, the probability of (a) tends, as $\lambda \to 0$, to

$$\begin{aligned} q := \Pr\Big(&\pi_M\big([t-\tfrac{1}{12}, t] \times [a+\tfrac{1}{3}, a+\tfrac{2}{3}]\big) = 1,\\ &\pi_M\big(([t-\tfrac{1}{2}, t] \times [a, a+1])\backslash([t-\tfrac{1}{12}, t] \times [a+\tfrac{1}{3}, a+\tfrac{2}{3}])\big) = 0\Big), \end{aligned}$$

which is clearly positive. Next, the probability of (b) tends to 1. Indeed, treating e.g. the case of j_g, there holds, recalling that $\boldsymbol{n}_\lambda \simeq 1/\nu_S((\boldsymbol{a}_\lambda, \infty))$,

$$\begin{aligned} &\Pr\Big[\exists\, j \in [\![\lfloor a\boldsymbol{n}_\lambda \rfloor, \lfloor (a+\tfrac{1}{4})\boldsymbol{n}_\lambda \rfloor]\!], N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(j) = N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{2})}(j)\Big]\\ &\qquad \simeq 1 - \nu_S\big((0, \tfrac{3}{4}\boldsymbol{a}_\lambda)\big)^{\frac{1}{4}\boldsymbol{n}_\lambda}\\ &\qquad \simeq 1 - \mathrm{e}^{-\nu_S((\frac{3}{4}\boldsymbol{a}_\lambda/4, \infty))/[4\nu_S((\boldsymbol{a}_\lambda, \infty))]}, \end{aligned}$$

which tends to 1 as $\lambda \to 0$ due to $(H_S(\infty))$. The probability of (c) (conditionally on (a)) also tends to 1. Indeed, its value is nothing but

$$\nu_S\big((0, \tfrac{5}{12}\boldsymbol{a}_\lambda)\big)^{2\boldsymbol{k}_\lambda+1} \simeq \mathrm{e}^{-2\nu_S((\frac{5}{12}\boldsymbol{a}_\lambda, \infty))/\nu_S((\frac{3}{8}\boldsymbol{a}_\lambda, \infty))}$$

which tends to 1 due to $(H_S(\infty))$, since $\frac{5}{12} > \frac{3}{8}$. Finally, the probability of (d) (conditionally on (a)) also tends to 1, since it equals

$$1 - \big(\nu_S\big((0, \tfrac{1}{3}\boldsymbol{a}_\lambda)\big)\big)^{2\boldsymbol{k}_\lambda+1} \simeq 1 - \mathrm{e}^{-2\nu_S((\frac{1}{3}\boldsymbol{a}_\lambda, \infty))/\nu_S((\frac{3}{8}\boldsymbol{a}_\lambda, \infty))},$$

which tends to 1 due to $(H_S(\infty))$, since $\frac{1}{3} < \frac{3}{8}$.

Step 3. — Let now $T > 0$ be fixed. Set $K = \lfloor 4T \rfloor$. For $a \in \mathbb{R}$, we set

$$\widetilde{\Omega}^\lambda_{a,T} = \Omega^\lambda_{a,0} \cap \bigcap_{k=2}^{K} \Omega^\lambda_{a+(k-1), \frac{1}{4}k}.$$

Then it is clear from steps 1 and 2 (observe that $(\frac{1}{4}K + \frac{1}{4} \geq T)$) that

(i) on $\widetilde{\Omega}^{\lambda}_{a,T}$, for all $t \in [0,T]$ there is $i \in [\![\lfloor a\boldsymbol{n}_\lambda\rfloor, [\![\lfloor (a+K)\boldsymbol{n}_\lambda - 1\rfloor]\!]$ such that $\eta^{\lambda}_{\boldsymbol{a}_\lambda t}(i) = 0$;

(ii) $p_\lambda = \Pr[\widetilde{\Omega}^{\lambda}_{a,T}]$ does not depend on a and $p := \liminf_{\lambda\to 0} p_\lambda \geq q^{K-1} > 0$;

(iii) $\widetilde{\Omega}^{\lambda}_{a,T}$ depends only on $(N^S_{\boldsymbol{a}_\lambda t}(i), N^{M,\lambda}_{\boldsymbol{a}_\lambda t}(i))_{t\in[0,T+1], i\in[\![\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor (a+K)\boldsymbol{n}_\lambda\rfloor - 1]\!]}$.

Step 4. — We deduce that for all $a \in \mathbb{Z}$, conditionally on $\Omega^{\lambda}_{a,T}$, clusters on the left of $\lfloor a\boldsymbol{n}_\lambda\rfloor - 1$ are never connected (during $[0, \boldsymbol{a}_\lambda T]$) to clusters on the right of $\lfloor (a+K)\boldsymbol{n}_\lambda\rfloor$. Thus on $\Omega^{\lambda}_{a,T}$, fires starting on the left of $\lfloor a\boldsymbol{n}_\lambda\rfloor - 1$ do not affect the zone $[\lfloor (a+K)\boldsymbol{n}_\lambda\rfloor, \infty) \cap \mathbb{Z}$ and fires starting on the right of $\lfloor (a+K)\boldsymbol{n}_\lambda\rfloor$ do not affect the zone $(-\infty, \lfloor a\boldsymbol{n}_\lambda\rfloor - 1] \cap \mathbb{Z}$.

We deduce that for $A \geq 2K$, the $\mathrm{FF}_A(\mu_S, \mu^{\lambda}_M)$-process and the $\mathrm{FF}(\mu_S, \mu^{\lambda}_M)$-process coincide on $I^{\lambda}_{A/2}$ during $[0, \boldsymbol{a}_\lambda T]$ as soon as there are $a_1 \in [-A, -\frac{1}{2}A - K]$ and $a_2 \in [\frac{1}{2}A, A-K]$ with $\Omega^{\lambda}_{a_1,T} \cap \Omega^{\lambda}_{a_2,T}$ realized. Furthermore, $\Omega^{\lambda}_{a,T}$ is independent of $\Omega^{\lambda}_{b,T}$ for all $a, b \in \mathbb{Z}$ with $|a-b| > K$. Thus we can bound the probabilities of the statement from below, for $A \geq 2K$ and $\lambda > 0$ small enough (so that $\Pr[\Omega^{\lambda}_{a,T}] \geq \frac{1}{2}p$), by

$$
\begin{aligned}
1 - \Pr\Big[\bigcap_{\ell=1}^{\lfloor A/(2K)\rfloor} (\Omega^{\lambda}_{\lfloor -A\rfloor+\ell K, T})^c\Big] - \Pr\Big[\bigcap_{\ell=1}^{\lfloor A/(2K)\rfloor} (\Omega^{\lambda}_{\lfloor A/2\rfloor+\ell K, T})^c\Big] \\
\geq 1 - 2\big(1 - \tfrac{1}{2}p\big)^{\lfloor A/(2K)\rfloor} \geq 1 - 2\big(1 - \tfrac{1}{2}p\big)^{A/(2K)-1}.
\end{aligned}
$$

This concludes the proof: choose $C_T = 2/(1 - \frac{1}{2}p) > 0$ (p depends only on T) and $\alpha_T = -\log(1 - \frac{1}{2}p)/(2K) > 0$. □

When $\beta = BS$, the proof is similar, but consequently simpler.

Proof when $\beta = BS$. — Recall that $\boldsymbol{a}_\lambda = T_S$ and consider the true $\mathrm{FF}(\mu_S, \mu^{\lambda}_M)$-process $(\eta^{\lambda}_t(i))_{t\geq 0, i\in\mathbb{Z}}$. For $a \in \mathbb{R}$, let

$$J^{\lambda}_a = \big[\!\big[\lfloor a\boldsymbol{n}_\lambda\rfloor, \lfloor (a+1)\boldsymbol{n}_\lambda\rfloor - 1\big]\!\big].$$

Step 1. — We show here that for all $a \in \mathbb{R}$, there exists an event $\Omega^{\lambda}_{a,0}$, depending only on $(N^S_s(i), N^{M,\lambda}_s(i))_{i\in J^{\lambda}_a, s\in[0,\frac{3}{4}\boldsymbol{a}_\lambda]}$ such that

(i) on $\Omega^{\lambda}_{a,0}$, a.s., there is $i \in J^{\lambda}_a$ such that $\eta^{\lambda}_{\boldsymbol{a}_\lambda s}(i) = 0$ for all $s \in [0, \frac{3}{4}]$;

(ii) $\lim_{\lambda\to 0} \Pr[\Omega^{\lambda}_{a,0}] = 1$.

This is done as in the case where $\beta = \infty$. Consider simply

$$\Omega^{\lambda}_{a,0} = \big\{\exists\, i \in J^{\lambda}_a, N^S_{\frac{3}{4}\boldsymbol{a}_\lambda}(i) = 0\big\}.$$

Clearly, (i) is satisfied. To check (ii), recall that $|J^{\lambda}_a| = \boldsymbol{n}_\lambda \to \infty$, whence

$$\Pr[\Omega^{\lambda}_{a,0}] = 1 - \nu_S\big((0, \tfrac{3}{4}T_S)\big)^{\boldsymbol{n}_\lambda} \longrightarrow 1,$$

because $\nu_S((0, \frac{3}{4}T_S)) < 1$ (recall that $\mathrm{supp}\, \nu_S = [0, T_S]$).

Step 2. — We now check that for all $a \in \mathbb{R}$, all $t \geq \frac{1}{2}$, there exists an event $\Omega^\lambda_{a,t}$, depending only on $(N^S_s(i), N^{M,\lambda}_s(i))_{i \in J^\lambda_a, s \in [(t-\frac{1}{4})\boldsymbol{a}_\lambda, (t+\frac{1}{4})\boldsymbol{a}_\lambda]}$ such that

(i) on $\Omega^\lambda_{a,t}$, a.s., there is $i \in J^\lambda_a$ such that $\eta^\lambda_{\boldsymbol{a}_\lambda s}(i) = 0$ for all $s \in [t, t+\frac{1}{4}]$;

(ii) $q_\lambda := \Pr[\Omega^\lambda_{a,t}]$ does not depend on t, a and $q := \liminf_{\lambda \to 0} q_\lambda > 0$.

This is much easier than in the case where $\beta = \infty$: simply set

$$\Omega^\lambda_{a,t} = \big\{\exists\, i_0 \in J^\lambda_a, N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(i_0) = N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{4})}(i_0), N^{M,\lambda}_{\boldsymbol{a}_\lambda t}(i_0) > N^{M,\lambda}_{\boldsymbol{a}_\lambda(t-\frac{1}{4})}(i_0)\big\}.$$

Point (i) is obviously checked since no seed fall on i_0 during $[\boldsymbol{a}_\lambda(t-\frac{1}{4}), \boldsymbol{a}_\lambda(t+\frac{1}{4})]$ and a match falls on i_0 during $[\boldsymbol{a}_\lambda(t-\frac{1}{4}), \boldsymbol{a}_\lambda t]$. Next, $\Pr\big(\Omega^\lambda_{a,t}\big) = 1 - r_\lambda^{|J^\lambda_a|}$, where (for any $i \in \mathbb{Z}$, any $t \geq \frac{1}{4}$)

$$\begin{aligned} r_\lambda :&= \Pr\big[N^S_{\boldsymbol{a}_\lambda(t+\frac{1}{4})}(i_0) > N^S_{\boldsymbol{a}_\lambda(t-\frac{1}{4})}(i_0) \text{ or } N^{M,\lambda}_{\boldsymbol{a}_\lambda t}(i_0) = N^{M,\lambda}_{\boldsymbol{a}_\lambda(t-\frac{1}{4})}(i_0)\big] \\ &= \Pr\big[N^S_{\frac{1}{2}\boldsymbol{a}_\lambda}(i) > 0 \text{ or } N^M_{\frac{1}{4}\boldsymbol{a}_\lambda}(i) = 0\big] \\ &= \nu_S\big([0, \tfrac{1}{2}T_S]\big) + \nu^\lambda_M\big((\tfrac{1}{4}T_S, \infty)\big) - \nu_S\big([0, \tfrac{1}{2}T_S]\big)\nu^\lambda_M\big((\tfrac{1}{4}T_S, \infty)\big). \end{aligned}$$

Due to (H_M),

$$\nu^\lambda_M\big((\tfrac{1}{4}T_S, \infty)\big) = 1 - \lambda \int_0^{\frac{1}{4}T_S} \mu^1_M(\lambda t, \infty)\,\mathrm{d}t = 1 - \lambda \tfrac{1}{4}T_S\big(1 + \varepsilon(\lambda)\big),$$

for some function ε such that $\lim_{\lambda \to 0} \varepsilon(\lambda) = 0$. Setting $\alpha = \nu_S([0, \frac{1}{2}T_S]) \in (0,1)$, we deduce that

$$\begin{aligned} r_\lambda &= \alpha + 1 - \tfrac{1}{4}\lambda T_S\big(1 + \varepsilon(\lambda)\big) - \alpha\big(1 - \tfrac{1}{4}\lambda T_S(1 + \varepsilon(\lambda))\big) \\ &= 1 - \tfrac{1}{4}\lambda(1-\alpha)T_S\big(1 + \varepsilon(\lambda)\big). \end{aligned}$$

Recalling that $|J^\lambda_a| \simeq \boldsymbol{n}_\lambda \simeq 1/(\lambda T_S)$, we finally conclude that

$$\Pr\big(\Omega^\lambda_{a,t}\big) \simeq 1 - r_\lambda^{1/(\lambda T_S)} \to 1 - \mathrm{e}^{-\frac{1}{4}(1-\alpha)} =: q > 0.$$

Steps 3 and 4 are exactly the same as when $\beta = \infty$. □

3.8. Localization of the results when $\beta \in \{\infty, BS\}$

We recall that $\boldsymbol{a}_\lambda$, $\boldsymbol{n}_\lambda$ are defined in (2.2.1) and (2.2.2). For $A > 0$ and $i \in \mathbb{Z}$, we set as usual

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda \rfloor, \quad I^\lambda_A = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = [i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda).$$

In the next sections, we will prove the following localized version of Theorems 2.3.3 and 2.4.4, separating the cases $\beta = \infty$ and $\beta = BS$.

PROPOSITION 3.8.1. — *Let $\beta \in \{\infty, BS\}$. Assume (H_M) and $(H_S(\beta))$. Let $A > 0$ be fixed. Consider, for each $\lambda \in (0,1]$, the process $(Z^{\lambda,A}_t(x), D^{\lambda,A}_t(x))_{t \geq 0, x \in [-A,A]}$ associated with the* $\mathrm{FF}_A(\mu_S, \mu^\lambda_M)$*-process and the* $\mathrm{LFF}_A(\beta)$*-process*

$$\big(Z^A_t(x), D^A_t(x), H^A_t(x)\big)_{t \geq 0, x \in [-A,A]}.$$

(a) *For any* $T > 0$, *any* $\{x_1, \dots, x_p\} \subset [-A, A]$, $(Z_t^{\lambda,A}(x_i), D_t^{\lambda,A}(x_i))_{t\in[0,T],i=1,\dots,p}$ *goes in law to*

$$\big(Z_t^A(x_i), D_t^A(x_i)\big)_{t\in[0,T],i=1,\dots,p},$$

in $\mathbb{D}([0,T], \mathbb{R} \times \mathcal{I} \cup \{\varnothing\})^p$, *as* λ *tends to* 0. *Here* $\mathbb{D}([0,\infty), \mathbb{R} \times \mathcal{I} \cup \{\varnothing\})$ *is endowed with the distance* $\boldsymbol{d}_T$.

(b) *For any* $\{(t_1, x_1), \dots, (t_p, x_p)\} \subset [0,\infty) \times [-A, A]$ (*assume also that* $t_k \neq 1$ *for* $k = 1, \dots, p$ *if* $\beta = \infty$), $(Z_{t_i}^{\lambda,A}(x_i), D_{t_i}^{\lambda,A}(x_i))_{i=1,\dots,p}$ *goes in law to*

$$\big(Z_{t_i}^A(x_i), D_{t_i}^A(x_i)\big)_{i=1,\dots,p}$$

in $(\mathbb{R} \times \mathcal{I} \cup \{\varnothing\})^p$. *Here* $\mathcal{I} \cup \{\varnothing\}$ *is endowed with* $\boldsymbol{\delta}$.

(c)-(i) *Assume first that* $\beta = \infty$. *For all* $t > 0$,

$$\Big(\frac{\psi_S(1 - 1/|C_A(\eta_{\boldsymbol{a}_\lambda t}^{\lambda,A}, 0)|)}{\boldsymbol{a}_\lambda}\mathbb{1}_{\{|C_A(\eta_{\boldsymbol{a}_\lambda t}^{\lambda,A}, 0)|\geq 1\}}\Big) \wedge 1$$

goes in law to $Z_t^A(0)$ *as* $\lambda \to 0$.

(c)-(ii) *Assume next that* $\beta = BS$. *For any* $t \geq 0$, *any* $k \in \mathbb{N}$, *there holds*

$$\lim_{\lambda\to 0} \Pr\big[|C_A(\eta_{T_S t}^{\lambda,A}, 0)| = k\big] = \mathbb{E}\big[q_k(Z_t^A(0))\big],$$

where $q_k(z)$ *was defined, for* $k \geq 0$ *and* $z \in [0,1]$, *in* (2.4.2).

Assuming for a moment that this proposition holds true, we conclude the proofs of Theorems 2.3.3 and 2.4.4.

Proof of Theorem 2.3.3. — Let us first prove (a). Consider a continuous bounded functional $\Psi : \mathbb{D}([0,T], \mathbb{R}\times\mathcal{I}\cup\{\varnothing\})^p \mapsto \mathbb{R}$. We have to prove that $\lim_{\lambda\to 0} G_\lambda(\Psi) = 0$, where

$$G_\lambda(\Psi) = \mathbb{E}\big[\Psi\big((Z_t^\lambda(x_i), D_t^\lambda(x_i))_{t\in[0,T],i=1,\dots,p}\big)\big] - \mathbb{E}\big[\Psi\big((Z_t(x_i), D_t(x_i))_{t\in[0,T],i=1,\dots,p}\big)\big].$$

Using now Propositions 3.6.4 and 3.7.2, we observe that for any $A > 2\max_{i=1,\ldots,p}|x_i|$, for all $\lambda \in (0,1]$ small enough,

$$\begin{aligned}\big|G_\lambda(\Psi)\big| \leq 2\|\Psi\|_\infty \Pr\Big[&\big(Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x)\big)_{t\in[0,T],x\in[-\frac12 A,\frac12 A]}\\ &\qquad\neq \big(Z_t^{\lambda}(x), D_t^{\lambda}(x)\big)_{t\in[0,T],x\in[-\frac12 A,\frac12 A]}\Big]\\ &+ 2\|\Psi\|_\infty \Pr\Big[\big(Z_t^{A}(x), D_t^{A}(x)\big)_{t\in[0,T],x\in[-\frac12 A,\frac12 A]}\\ &\qquad\neq \big(Z_t(x), D_t(x)\big)_{t\in[0,T],x\in[-\frac12 A,\frac12 A]}\Big]\\ &+ \Big|\mathbb{E}\big[\Psi\big((Z_t^{\lambda,A}(x_i), D_t^{\lambda,A}(x_i))_{t\in[0,T],i=1,\ldots,p}\big)\big]\\ &\qquad - \mathbb{E}\big[\Psi\big((Z_t^{A}(x_i), D_t^{A}(x_i))_{t\in[0,T],i=1,\ldots,p}\big)\big]\Big|\\ &\leq 4\|\Psi\|_\infty C_T\,\mathrm{e}^{-\alpha_T A} + \Big|\mathbb{E}\big[\Psi\big((Z_t^{\lambda,A}(x_i), D_t^{\lambda,A}(x_i))_{t\in[0,T],i=1,\ldots,p}\big)\big]\\ &\qquad - \mathbb{E}\big[\Psi\big((Z_t^{A}(x_i), D_t^{A}(x_i))_{t\in[0,T],i=1,\ldots,p}\big)\big]\Big|.\end{aligned}$$

Thus Proposition 3.8.1 (a) implies that $\limsup_{\lambda\to 0}|G_\lambda(\Psi)| \leq 4\|\Psi\|_\infty C_T\,\mathrm{e}^{-\alpha_T A}$. We conclude by making A tend to infinity.

Point (b) is checked similarly. The proof of (c) is also similar, since $D_t^\lambda(0) = D_t^{\lambda,A}(0)$ implies that $C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0) = C_A(\eta^{\lambda,A}_{\boldsymbol{a}_\lambda t}, 0)$. □

Proof of Theorem 2.4.4. — It is deduced from Propositions 3.6.4, 3.7.2 and 3.8.1 exactly as Theorem 2.3.3. □

3.9. Convergence proof when $\beta = BS$

The aim of this section is to prove Proposition 3.8.1 in the case where $\beta = BS$ and this will conclude the proof of Theorem 2.4.4. In the whole section, we thus assume (H_M) and $(H_S(BS))$. The parameters $A > 0$ and $T > 0$ are fixed and we omit the subscript/superscript A in the whole proof.

We recall that $\boldsymbol{a}_\lambda$, $\boldsymbol{n}_\lambda$ and $\boldsymbol{m}_\lambda$ are defined in (2.2.1), (2.2.2) and (2.2.4). For $A > 0$ and $i \in \mathbb{Z}$, we set as usual

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda\rfloor, \quad I_A^\lambda = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = \big[i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda\big).$$

For $[a,b]$ an interval of $[-A,A]$ and $\lambda \in (0,1)$, we introduce, assuming $-A < a < b < A$,

$$\begin{cases}[a,b]_\lambda = [\![\lfloor \boldsymbol{n}_\lambda a + \boldsymbol{m}_\lambda\rfloor + 1, \lfloor \boldsymbol{n}_\lambda b - \boldsymbol{m}_\lambda\rfloor - 1]\!] \subset \mathbb{Z},\\ [-A,b]_\lambda = [\![-A_\lambda, \lfloor \boldsymbol{n}_\lambda b - \boldsymbol{m}_\lambda\rfloor - 1]\!] \subset \mathbb{Z},\\ [a,A]_\lambda = [\![\lfloor \boldsymbol{n}_\lambda a + \boldsymbol{m}_\lambda\rfloor + 1, A_\lambda]\!] \subset \mathbb{Z}.\end{cases} \tag{3.9.1}$$

For $x \in (-A,A)$ and $\lambda \in (0,1)$, we introduce

$$x_\lambda = \big[\big[\lfloor \boldsymbol{n}_\lambda x - \boldsymbol{m}_\lambda\rfloor, \lfloor \boldsymbol{n}_\lambda x + \boldsymbol{m}_\lambda\rfloor\big]\big] \subset \mathbb{Z}. \tag{3.9.2}$$

3.9.1. Height of the barriers. — We need the following lemma. It describes the time needed for a destroyed (microscopic) cluster to be regenerated. Below, we assume that the zone around 0 is completely vacant at time $T_S t_0$. Then we consider the situation where a match falls on the site 0 at some time $T_S t_1 \in (T_S t_0, T_S(t_0+1))$ and we compute the law of Θ_{t_0,t_1}, which is the delay needed for the destroyed cluster to be fully regenerated (divided by T_S).

LEMMA 3.9.1. — *Consider a family of i.i.d.* $\mathrm{SR}(\mu_S)$*-processes* $(N_t^S(i))_{t\ge 0, i\in\mathbb{Z}}$*. Let* $0 \le t_0 < t_1 < t_0+1$ *be fixed. Put*

$$\zeta_{t_0,t}(i) = \min\big(N^S_{T_S(t_0+t)}(i) - N^S_{T_S t_0}(i), 1\big),$$
$$\zeta_{t_1,t}(i) = \min\big(N^S_{T_S(t_1+t)}(i) - N^S_{T_S t_1}(i), 1\big)$$

for all $t > 0$ *and* $i \in \mathbb{Z}$*. Define*

$$\Theta_{t_0,t_1} = \inf\big\{t>0 :\ \forall\, i \in C(\zeta_{t_0,t_1-t_0},0),\ \zeta_{t_1,t}(i)=1\big\} \in [0,1].$$

The the law of Θ_{t_0,t_1} *is* $\theta_{t_1-t_0}$*, recall Definition* 2.4.1.

Proof. — We can assume that $t_0 = 0$ by stationarity. We put $u = t_1 = t_1 - t_0$ and write, for $h \in [0,1]$,

$$\begin{aligned}\Pr[\Theta_{t_0,t_1} \le h] &= \Pr\big[N^S_{T_S u}(0)=0\big]\\ &\quad + \sum_{k\ge1}\sum_{j=0}^{k-1}\Pr\Big[N^S_{T_S u}(j-k) = N^S_{T_S u}(j+1) = 0, \forall i \in [\![j-k+1, j]\!],\\ &\qquad\qquad N^S_{T_S u}(i) > 0,\ N^S_{T_S(u+h)}(i) > N^S_{T_S u}(i)\Big].\end{aligned}$$

This yields, since $g_S(u,h) = \Pr[N^S_{T_S u} > 0, N^S_{T_S(u+h)} > N^S_{T_S u}]$,

$$\begin{aligned}\Pr[\Theta_{t_0,t_1} \le h] &= \nu_S\big([T_S u, T_S]\big) + \sum_{k\ge1} k\big[\nu_S([T_S u, T_S])\big]^2\cdot\big[g_S(u,h)\big]^k\\ &= \nu_S\big([T_S u, T_S]\big) + \frac{[\nu_S([T_S u, T_S])]^2}{[1-g_S(u,h)]^2} g_S(u,h) = \theta_u\big([0,h]\big),\end{aligned}$$

recall Definition 2.4.1. □

3.9.2. Persistent effect of microscopic fires. — Here we study the effect of microscopic fires. First, they produce a barrier, and then, if there are alternatively macroscopic fires on the left and right, they still have an effect. This phenomenon is illustrated on Figure 8 next page in the case of the limit process.

We say that $\mathcal{R} = (\varepsilon; t_0, t_1, \dots, t_K; s)$ satisfies (PP) (like ping-pong) if

(i) $K \ge 2$, $\varepsilon \in \{-1, 1\}$;
(ii) $0 < t_0 < t_1 < \cdots < t_K < s < t_K + 1$;
(iii) for all $k = 0, \dots, K-1$, $t_{k+1} - t_k < 1$;
(iv) $t_2 - t_0 > 1$ and for all $k = 2, \dots, K-2$, $t_{k+2} - t_k > 1$.

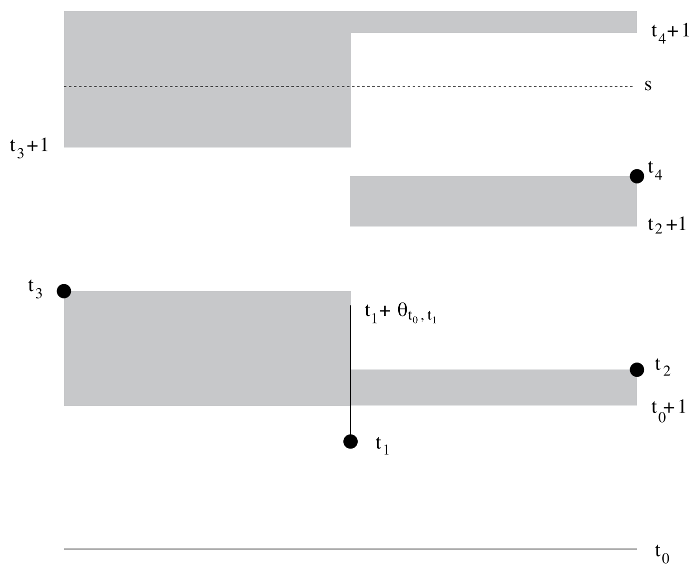

FIGURE 8. Persistent effect of a microscopic fire. Here $\mathcal{R} = (1; t_0, t_1, t_2, t_3, t_4; s)$.

We set $\varepsilon_k = (-1)^k \varepsilon$ for $k \geq 0$.

Consider a family of i.i.d. SR(μ_S)-processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$.

We introduce, for each $\lambda \in (0,1)$, the process $(\zeta_t^{\lambda,\mathcal{R}}(i))_{t\geq t_0, i\in[\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]}$ defined as follows:

▷ for all $t \in [t_0, t_1)$, all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$, $\zeta_t^{\lambda,\mathcal{R}}(i) = \min(N^S_{T_S t}(i) - N^S_{T_S t_0}(i), 1)$;

▷ for all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$, $\zeta_{t_1}^{\lambda,\mathcal{R}}(i) = \zeta_{t_1-}^{\lambda,\mathcal{R}}(i)\mathbb{1}_{\{i \notin C(\zeta_{t_1-}^{\lambda,\mathcal{R}}, 0)\}}$;

▷ for $k = 1, \dots, K-1$,

(*) for all $t \in (t_k, t_{k+1})$, $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_t^{\lambda,\mathcal{R}}(i) = \min\left(\zeta_{t_k}^{\lambda,\mathcal{R}}(i) + N^S_{T_S t}(i) - N^S_{T_S t_k}(i), 1\right),$$

(*) for all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_{t_{k+1}}^{\lambda,\mathcal{R}}(i) = \zeta_{t_{k+1}-}^{\lambda,\mathcal{R}}(i)\mathbb{1}_{i \notin C(\zeta_{t_{k+1}-}^{\lambda,\mathcal{R}}, \epsilon_k \boldsymbol{m}_\lambda)};$$

▷ for all $t \in (t_K, \infty)$, $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_t^{\lambda,\mathcal{R}}(i) = \min\left(\zeta_{t_K}^{\lambda,\mathcal{R}}(i) + N^S_{T_S t}(i) - N^S_{T_S t_K}(i), 1\right).$$

Roughly, we start at time $T_S t_0$ with an empty configuration and seeds fall according to $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$. At time $T_S t_1$, there is a (microscopic) fire at 0. Then alternatively on the left and right, far away from 0 (at $-\boldsymbol{m}_\lambda$ or at $\boldsymbol{m}_\lambda$), there is a (macroscopic) fire at time $T_S t_k$.

Consider the event

$$\Omega^S_{\mathcal{R}}(\lambda) = \big\{\exists -\boldsymbol{m}_\lambda < i_1 < i_2 < i_3 < \boldsymbol{m}_\lambda :\ \zeta^{\lambda,\mathcal{R}}_s(i_1) = \zeta^{\lambda,\mathcal{R}}_s(i_3) = 0, \zeta^{\lambda,\mathcal{R}}_s(i_2) = 1\big\}.$$

Lemma 3.9.2. — *Let $\mathcal{R} = (\varepsilon; t_0, t_1, \dots, t_K; s)$ satisfy* (PP). *Consider Θ_{t_0,t_1} defined in Lemma* 3.9.1 *and $(\zeta^{\lambda,\mathcal{R}}_t(i))_{t\geq t_0, i\in[\![-\boldsymbol{m}_\lambda,\boldsymbol{m}_\lambda]\!]}$ defined above. There holds*

$$\lim_{\lambda\to 0} \Pr\big(\Omega^S_{\mathcal{R}}(\lambda) \mid \Theta_{t_0,t_1} > t_2 - t_1\big) = 1.$$

Proof. — We assume that $\varepsilon = 1$ and that K is even for simplicity. Fix $\alpha = 1/K$.

First fire. — We put $C = C(\zeta^{\lambda,\mathcal{R}}_{t_1-}, 0)$. Since $t_1 - t_0 < 1$ (so that each site is vacant with probability $\nu_S((T_S(t_1 - t_0), T_S)) > 0$ at time t_1), the probability that $C \subset [\![-\lfloor\alpha\boldsymbol{m}_\lambda\rfloor, \lfloor\alpha\boldsymbol{m}_\lambda\rfloor]\!]$ clearly tends to 1. Thus the match falling at time t_1 at 0 destroys nothing outside $[\![-\lfloor\alpha\boldsymbol{m}_\lambda\rfloor, \lfloor\alpha\boldsymbol{m}_\lambda\rfloor]\!]$ (with probability tending to 1).

Second fire. — Since $t_2 - t_0 > 1$ (so that $T_S(t_2 - t_0) > T_S$), at least one seed has fallen, during $[t_0, t_2)$ on each site of $[\![\lfloor\alpha\boldsymbol{m}_\lambda\rfloor + 1, \boldsymbol{m}_\lambda]\!]$. Thus the fire at time t_2 destroys completely this zone, but does not affect $[\![-\boldsymbol{m}_\lambda, -\lfloor\alpha\boldsymbol{m}_\lambda\rfloor - 1]\!]$, because $t_2 < t_1 + \Theta_{t_0,t_1}$ and because by definition of Θ_{t_0,t_1}, there is an empty site in $C \subset [\![-\lfloor\alpha\boldsymbol{m}_\lambda\rfloor, \lfloor\alpha\boldsymbol{m}_\lambda\rfloor]\!]$ during $[t_1, t_1 + \Theta_{t_0,t_1}]$.

Third fire. — Since $t_3 - t_2 < 1$, the probability that there is a vacant site in $[\![\lfloor\alpha\boldsymbol{m}_\lambda\rfloor + 1, \lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor]\!]$ at time t_3 tends to 1 as $\lambda \to 0$.

Next, all the sites of $[\![-\boldsymbol{m}_\lambda, -\lfloor\alpha\boldsymbol{m}_\lambda\rfloor - 1]\!]$ are occupied at time t_3- (because they have not been affected by a fire and because $t_3 - t_0 > t_2 - t_0 > 1$). Thus the fire at time t_3 destroys the zone $[\![-\boldsymbol{m}_\lambda, -\lfloor\alpha\boldsymbol{m}_\lambda\rfloor - 1]\!]$ and does not affect the zone $[\![\lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor, \boldsymbol{m}_\lambda]\!]$.

Fourth fire. — Since $t_4 - t_3 < 1$, the probability that there is (at least) a vacant site in $[\![-\lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor, -\lfloor\alpha\boldsymbol{m}_\lambda\rfloor - 1]\!]$ at time t_4 tends to 1 as $\lambda \to 0$.

Next, all the sites of $[\![\lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor, \boldsymbol{m}_\lambda]\!]$ are occupied at time t_4- (because they have not been affected by a fire during (t_2, t_4) with $t_4 - t_2 > 1$). Thus the fire at time t_4 destroys the zone $[\![\lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor, \boldsymbol{m}_\lambda]\!]$ and does not affect the zone $[\![-\boldsymbol{m}_\lambda, -\lfloor 2\alpha\boldsymbol{m}_\lambda\rfloor]\!]$.

Last fire and conclusion. — Iterating the procedure, we see that with a probability tending to 1 as $\lambda \to 0$, the fire at time t_K destroys the zone

$$\Big[\!\!\Big[\lfloor(\tfrac{1}{2}K\alpha)\boldsymbol{m}_\lambda\rfloor, \boldsymbol{m}_\lambda\Big]\!\!\Big] = \Big[\!\!\Big[\lfloor\tfrac{1}{2}\boldsymbol{m}_\lambda\rfloor, \boldsymbol{m}_\lambda\Big]\!\!\Big].$$

Then one easily concludes: since $0 < s - t_K < 1$, the probability that there is at least one site in $[\![\lfloor\frac{1}{2}\boldsymbol{m}_\lambda\rfloor, \lfloor\frac{2}{3}\boldsymbol{m}_\lambda\rfloor]\!]$ with no seed falling during $[t_K, s]$ tends to 1, the probability that there is at least one site in $[\![\lfloor\frac{2}{3}\boldsymbol{m}_\lambda\rfloor + 1, \lfloor\frac{5}{6}\boldsymbol{m}_\lambda\rfloor]\!]$ with at least one seed falling during $[t_K, s]$ tends to 1, and the probability that there is at least one site in $[\![\lfloor\frac{5}{6}\boldsymbol{m}_\lambda\rfloor + 1, \boldsymbol{m}_\lambda]\!]$ with no seed falling during $[t_K, s]$ tends to 1. □

3.9.3. The coupling. — We are going to construct a coupling between the $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$-process (on the time interval $[0, T_S T]$) and the $\mathrm{LFF}_A(BS)$-process (on $[0, T]$).

First, we couple a family of i.i.d. $\mathrm{SR}(\mu_M^\lambda)$-processes $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$ with a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x)$ on $[0,T]\times[-A,A]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$ as in Proposition 3.2.1.

We call $n := \pi_M([0,T]\times[-A,A])$ and we consider the marks $(T_q, X_q)_{q=1,\dots,n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$.

Next, we introduce some i.i.d. families of i.i.d. $\mathrm{SR}(\mu_S)$-processes $(N_t^{S,q}(i))_{t\geq 0, i\in\mathbb{Z}}$, for $q = 0, 1, \dots$, independent of π_M and $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$.

Then we build a family of i.i.d. $\mathrm{SR}(\mu_S)$-processes, independent of $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$ and of π_M, as follows.

▷ For $q \in \{1, \dots, n\}$, for all $i \in (X_q)_\lambda$ (recall that $(X_q)_\lambda = [\![\lfloor \boldsymbol{n}_\lambda X_q - \boldsymbol{m}_\lambda\rfloor, \lfloor \boldsymbol{n}_\lambda X_q + \boldsymbol{m}_\lambda\rfloor]\!]$) set

$$\big(N_t^{S,\lambda}(i)\big)_{t\geq 0} = \big(N_t^{S,q}(i - \lfloor \boldsymbol{n}_\lambda X_q\rfloor)\big)_{t\geq 0}.$$

(We have a problem if i belongs to $(X_q)_\lambda \cap (X_r)_\lambda$ for some $q < r$. Then set e.g. $(N_t^{S,\lambda}(i))_{t\geq 0} = (N_t^{S,q}(i - \lfloor \boldsymbol{n}_\lambda X_q\rfloor))_{t\geq 0}$. This will occur with a very small probability, so that this choice is not important).

▷ For all other $i \in \mathbb{Z}$ set

$$\big(N_t^{S,\lambda}(i)\big)_{t\geq 0} = \big(N_t^{S,0}(i)\big)_{t\geq 0}.$$

The $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in I_A^\lambda}$ is built upon the seed processes $(N_t^{S,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$ and match processes $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$.

The advantage of the previous construction is the following. When a match falls at some X_q for the $\mathrm{LFF}_A(BS)$-process, it will fall at $\lfloor \boldsymbol{n}_\lambda X_q\rfloor$ in the discrete process, and thus if it is *microscopic*, it will involve the same seed processes for all values of λ.

It also considerably simplifies the dependence/independence considerations.

Finally, we build the $\mathrm{LFF}_A(BS)$-process. We consider the Poisson measure π_M previously introduced, and for all $0 < t_0 < t_1 < t_0 + 1$, for all $q = 1, \dots, n$, we consider Θ_{t_0,t_1}^q defined from $(N_t^{S,q}(i))_{t\geq 0, i\in\mathbb{Z}}$ as in Lemma 3.9.1. We define $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T], x\in[-A,A]}$ as follows:

Algorithm 3.9.3. — Consider the marks $(T_k, X_k)_{k=1,\dots,n}$ of π_M in $[0,T]\times[-A,A]$, ordered chronologically and set $T_0 = 0$.

Step 0. — Put $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in [-A, A]$.

Assume that for some $k \in \{0, \dots, n-1\}$, $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T_k], x\in[-A,A]}$ has been built.

Step $k+1$. — For $t \in (T_k, T_{k+1})$ and $x \in [-A, A]$, put

$$Z_t(x) = \min(1, Z_{T_k}(x) + t - T_k), \quad H_t(x) = \max(0, H_{T_k}(x) - t + T_k),$$

and then define $D_t(x)$ as in (3.6.1). Finally, build $(Z_{T_{k+1}}(x), D_{T_{k+1}}(x), H_{T_{k+1}}(x))$ as follows.

(i) If $Z_{T_{k+1}-}(X_{k+1}) = 1$, set $H_{T_{k+1}}(x) = H_{T_{k+1}-}(x)$ for all $x \in [-A, A]$ and consider $[a, b] := D_{T_{k+1}-}(X_{k+1})$. Set

$$Z_{T_{k+1}}(x) = \begin{cases} 0 & \text{for all } x \in (a, b), \\ Z_{T_{k+1}-}(x) & \text{for all } x \in [-A, A] \setminus [a, b]. \end{cases}$$

$$Z_{T_{k+1}}(a) = \begin{cases} 0 & \text{if } Z_{T_{k+1}-}(a) = 1, \\ Z_{T_{k+1}-}(a) & \text{if } Z_{T_{k+1}-}(a) < 1, \end{cases}$$

$$Z_{T_{k+1}}(b) = \begin{cases} 0 & \text{if } Z_{T_{k+1}-}(b) = 1, \\ Z_{T_{k+1}-}(b) & \text{if } Z_{T_{k+1}-}(b) < 1. \end{cases}$$

(ii) If $Z_{T_{k+1}-}(X_{k+1}) < 1$, set

$$H_{T_{k+1}}(X_{k+1}) = \Theta^{k+1}_{T_{k+1}-Z_{T_{k+1}-}(X_{k+1}),T_{k+1}},$$
$$Z_{T_{k+1}}(X_{k+1}) = Z_{T_{k+1}-}(X_{k+1}),$$

and for all $x \in [-A, A] \setminus \{X_{k+1}\}$,

$$\big(Z_{T_{k+1}}(x), H_{T_{k+1}}(x)\big) = \big(Z_{T_{k+1}-}(x), H_{T_{k+1}-}(x)\big).$$

(iii) Using the values of $(Z_{T_{k+1}}(x), H_{T_{k+1}}(x))_{x\in[-A,A]}$, compute $(D_{T_{k+1}}(x))_{x\in[-A,A]}$ as in (3.6.1).

LEMMA 3.9.4. — *The process* $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T],x\in[-A,A]}$ *built in Algorithm* 3.9.3 *is a* $\mathrm{LFF}_A(BS)$*-process.*

Proof. — The only difference between Algorithms 3.6.3 and 3.9.3 is that in step $k+1$, point (ii), we use $\Theta^{k+1}_{T_{k+1}-Z_{T_{k+1}-}(X_{k+1}),T_{k+1}}$ instead of $F_S(Z_{T_{k+1}-}(X_{k+1}), V_{k+1})$.

But due to Lemma 3.9.1 and Definition 2.4.1, these two variables have the same law $\theta_{Z_{T_{k+1}-}(X_{k+1})}$ (conditionally on T_{k+1}, X_{k+1} and $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T_{k+1}),x\in[-A,A]}$). Indeed, it suffices to use that in Algorithm 3.6.3, V_{k+1} is independent of $Z_{T_{k+1}-}(X_{k+1})$, while in Algorithm 3.9.3, the family $(N^{S,k+1}_t(i))_{t\geq 0,i\in\mathbb{Z}}$ is independent of $(T_{k+1}, Z_{T_{k+1}-}(X_{k+1}))$. □

Finally, we observe that $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T],x\in[-A,A]}$ depends only on π_M and on $((N^{S,q}_t(i))_{t\in[0,T],i\in\mathbb{Z}})_{q\geq 1}$. It is independent of $(N^{S,0}_t(i))_{t\in[0,T],i\in\mathbb{Z}}$.

3.9.4. A favorable event. — First, we know from Proposition 3.2.1 that

$$\Omega^M_{A,T}(\lambda) := \big\{\forall t \in [0, T],\ \forall i \in I^\lambda_A,\ \Delta N^{M,\lambda}_{T_S t}(i) \neq 0 \text{ iff } \pi_M(\{t\} \times i_\lambda) \neq 0\big\}$$

satisfies

$$\lim_{\lambda\to 0} \Pr[\Omega^M_{A,T}(\lambda)] = 1.$$

Next, we recall that the marks of π_M are called $(T_1, X_1), \dots, (T_n, X_n)$ and are ordered chronologically. We introduce $\mathcal{T}_M = \{0, T_1, \dots, T_n\}$, $\mathcal{B}_M = \{X_1, \dots, X_n\}$, as well as the set $\mathcal{C}_M$ of connected components of $[-A, A] \setminus \mathcal{B}_M$ (sometimes referred to as *cells*).

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \Big\{ \min_{\substack{s,t \in \mathcal{T}_M \\ s \neq t}} |t-s| \geq \alpha, \min_{\substack{x,y \in \mathcal{B}_M \cup \{-A,A\} \\ x \neq y}} |x-y| \geq \alpha, \Big\},$$

which clearly satisfies $\lim_{\alpha \to 0} \Pr[\Omega_M(\alpha)] = 1$. Observe that for any given $\alpha > 0$, there is $\lambda_\alpha > 0$ such that for all $\lambda \in (0, \lambda_\alpha)$, on $\Omega_M(\alpha)$,

▷ for all $x, y \in \mathcal{B}_M \cup \{-A, A\}$ with $x \neq y$, $x_\lambda \cap y_\lambda = \varnothing$;

▷ the family $\{c_\lambda, c \in \mathcal{C}_M\} \cup \{x_\lambda, x \in \mathcal{B}_M\}$ is a partition of I_A^λ (recall (3.9.1) and (3.9.2)).

Indeed, it suffices that $\sup_{(0,\lambda_\alpha)}[\boldsymbol{m}_\lambda / \boldsymbol{n}_\lambda] < \frac{1}{4}\alpha$.

Let $q \in \{1, \dots, n\}$. We call $\mathcal{U}_q$ the set of all possible $\mathcal{R} = (\varepsilon, t_0, \dots, t_K; s)$ satisfying (PP) with $\varepsilon \in \{-1, 1\}$, with $\{t_0, \dots, t_K, s\} \subset \mathcal{T}_M$ and with $\Theta^q_{t_0,t_1} > t_2 - t_1$. We introduce, for $q = 1, \dots, n$ and $\mathcal{R} \in \mathcal{U}_q$, the event $\Omega^{S,q}_{\mathcal{R}}(\lambda)$ defined as in subsection 3.9.2 with the SR(μ_S)-processes $(N^{S,q}_t(i))_{t \geq 0, i \in \mathbb{Z}}$. Then we put

$$\Omega^S_1(\lambda) = \bigcap_{q=1}^{n} \bigcap_{\mathcal{R} \in \mathcal{U}_q} \Omega^{S,q}_{\mathcal{R}}(\lambda),$$

which satisfies $\lim_{\lambda \to 0} \Pr\left(\Omega^S_1(\lambda)\right) = 1$ thanks to Lemma 3.9.2 (since for each q, $(N^{S,q}_t(i))_{t \geq 0, i \in \mathbb{Z}}$ is independent of π_M and since conditionally on π_M, the set $\mathcal{U}_q$ is finite).

We also consider the event $\Omega^S_2(\lambda)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\boldsymbol{m}_\lambda < i_1 < i_2 < -\tfrac{1}{2}\boldsymbol{m}_\lambda < i_3 < 0 < i_4 < \tfrac{1}{2}\boldsymbol{m}_\lambda < i_5 < i_6 < \boldsymbol{m}_\lambda$$

such that

▷ for $j = 1, 3, 4, 6$, $N^{S,q}_{T_S t_2}(i_j) - N^{S,q}_{T_S t_1}(i_j) = 0$;

▷ for $j = 2, 5$, $N^{S,q}_{T_S t_2}(i_j) - N^{S,q}_{T_S t_1}(i_j) > 0$.

There holds $\lim_{\lambda \to 0} \Pr\left(\Omega^S_2(\lambda)\right) = 1$. Indeed, it suffices to prove that almost surely, $\lim_{\lambda \to 0} \Pr\left(\Omega^S_2(\lambda) | \pi_M\right) = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N^{S,q}_t(i))_{t \geq 0, i \in \mathbb{Z}}$, it suffices to work with a fixed $q \in \{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$.

Observe that for each i, $\Pr(N^{S,q}_{T_S t_2}(i) - N^{S,q}_{T_S t_1}(i) = 0) = \nu_S((T_S(t_2 - t_1), T_S)) < 1$ and $\Pr(N^{S,q}_{T_S t_2}(i) - N^{S,q}_{T_S t_1}(i) > 0) = \nu_S((0, T_S(t_2 - t_1))) < 1$ by definition of T_S and since $t_2 - t_1 < 1$. Recall also that $\boldsymbol{m}_\lambda$ tends to infinity. Thus during $[T_S t_1, T_S t_2]$, the probability that a seed falls on each site of $[\![-\boldsymbol{m}_\lambda + 1, -\lfloor \frac{1}{4}\boldsymbol{m}_\lambda \rfloor]\!]$ tends to 0, the probability that no seed at all falls on $[\![-\lfloor\frac{1}{4}\boldsymbol{m}_\lambda\rfloor+1,-\lfloor\frac{1}{2}\boldsymbol{m}_\lambda\rfloor-1]\!]$ tends to 0, the probability a seed falls on each site of $[\![-\lfloor\frac{1}{2}\boldsymbol{m}_\lambda\rfloor,-1]\!]$ tends to 0, etc.

We finally introduce the event

$$\Omega(\alpha,\lambda)=\Omega^M_{A,T}(\lambda)\cap\Omega_M(\alpha)\cap\Omega^S_1(\lambda)\cap\Omega^S_2(\lambda).$$

We observe that $\Omega(\alpha,\lambda)$ is independent of $(N_t^{S,0}(i))_{t\geq 0,i\in\mathbb{Z}}$ and that for any $\varepsilon>0$, choosing $\alpha>0$ small enough, $\Pr[\Omega(\alpha,\lambda)]>1-\varepsilon$ for all $\lambda>0$ small enough.

3.9.5. Heart of the proof. — We now handle the main part of the proof.

Consider the $\text{LFF}_A(BS)$-process. Observe that by construction, we have, for $c\in\mathcal{C}_M$ and $x,y\in c$, $Z_t(x)=Z_t(y)$ for all $t\in[0,T]$, thus we can introduce $Z_t(c)$.

If $x\in\mathcal{B}_M$, it is at the boundary of two cells $c_-,c_+\in\mathcal{C}_M$ and then for all $t\in[0,T]$ we set

$$Z_t(x_-)=Z_t(c_-)\quad\text{and}\quad Z_t(x_+)=Z_t(c_+).$$

If $x\in(-A,A)\setminus\mathcal{B}_M$, we put $Z_t(x_-)=Z_t(x_+)=Z_t(x)$ for all $t\in[0,T]$.

For $x\in\mathcal{B}_M$ and $t\geq 0$ we set

$$\widetilde{H}_t(x)=\max\big(H_t(x),1-Z_t(x),1-Z_t(x_-),1-Z_t(x_+)\big).$$

Observe that x is *microscopic* or *acts like a barrier* at time t if and only if $\widetilde{H}_t(x)>0$.

Actually $Z_t(x)$ always equals either $Z_t(x_-)$ or $Z_t(x_+)$ and these can be distinct only at a point where has occurred a microscopic fire (that is if $x=X_q$ for some $q\in\{1,\dots,n\}$, if $Z_{T_q-}(X_q)<1$ and if $t>T_q$).

For all $x\in(-A,A)$ and $t\in[0,T]$, we put

$$\tau_t(x)=\sup\big\{s\leq t:\ Z_s(x_+)=Z_s(x_-)=Z_s(x)=0\big\}\in[0,t]\cap\mathcal{T}_M.$$

For $c\in\mathcal{C}_M$ and $t\in[0,T]$, we clearly have $\tau_t(x)=\tau_t(y)$ for all $x,y\in c$, so that we can also define $\tau_t(c)$.

Observe, using Algorithm 3.9.3, that

$$\text{for } x\notin\mathcal{B}_M,\quad Z_t(x)=\min\left(t-\tau_t(x),1\right)\text{ for all } t\in[0,T];\tag{3.9.3}$$

$$\text{for } q=1,\dots,n,\quad Z_t(X_q)=\min\left(t-\tau_t(X_q),1\right)\text{ for all } t\in[0,T_q).\tag{3.9.4}$$

Indeed, $\tau_t(x)$ stands for the last time before t where x was involved in a macroscopic fire (with the convention that a macroscopic fire occurs at time 0). Thus for $x\notin\mathcal{B}_M$, if $t-\tau_t(x)\geq 1$, $Z_t(x)=1$, and if $t-\tau_t(x)<1$, $Z_t(x)=t-\tau_t(x)$. For $x=X_q$, the same reasoning holds during $[0,T_q)$.

We also define for all $t \in [0,T]$, all $c \in \mathcal{C}_M$ and all $x \in (-A,A)$ here (c_λ is defined by (3.9.1) and x_λ by (3.9.2))

$$\tau_t^\lambda(c) = \sup\big\{s \le t:\ \forall i \in c_\lambda, \eta^\lambda_{T_S t-}(i) = 1 \text{ and } \eta^\lambda_{T_S t}(i) = 0\big\} \in [0,t],$$
$$\rho_t^\lambda(c) = \sup\big\{s \le t:\ \exists i \in c_\lambda, \eta^\lambda_{T_S t-}(i) = 1 \text{ and } \eta^\lambda_{T_S t}(i) = 0\big\} \in [0,t],$$
$$\tau_t^\lambda(x) = \sup\big\{s \le t:\ \forall i \in x_\lambda, \eta^\lambda_{T_S t-}(i) = 1 \text{ and } \eta^\lambda_{T_S t}(i) = 0\big\} \in [0,t]$$

with the convention that $\eta^\lambda_{0-}(i) = 1$ for all $i \in I_A^\lambda$. Observe that on $\Omega^M_{A,T}(\lambda)$, we have $\tau_t^\lambda(c), \rho_t^\lambda(c), \tau_t^\lambda(x) \in [0,t] \cap \mathcal{T}_M$ for all $t \in [0,T]$, all $c \in \mathcal{C}_M$ and all $x \in (-A,A)$.

For $t \in [0,T]$, consider the event

$$\Omega_t^\lambda = \big\{\forall s \in [0,t], \forall c \in \mathcal{C}_M, \tau_s^\lambda(c) = \rho_s^\lambda(c) = \tau_s(c) \text{ and } \forall x \in \mathcal{B}_M, \tau_s^\lambda(x) = \tau_s(x)\big\}.$$

We define Ω^λ_{t-} similarly, replacing $[0,t]$ by $[0,t)$. The aim of the subsection is to prove the following result.

Lemma 3.9.5. — *For any $\alpha > 0$, any $\lambda \in (0,\lambda_\alpha)$, Ω^λ_T a.s. holds on $\Omega(\alpha,\lambda)$.*

Proof. — We work on $\Omega(\alpha,\lambda)$ and assume that $\lambda \in (0,\lambda_\alpha)$. Clearly, $\tau_0(x) = \tau_0^\lambda(x) = 0$ and $\tau_0(c) = \tau_0^\lambda(c) = \rho_0^\lambda(c) = 0$ for all $x \in \mathcal{B}_M$, all $c \in \mathcal{C}_M$, so that Ω_0^λ a.s. holds. We will show that for $q = 0,\dots,n-1$, $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}}$. This will prove that $\Omega^\lambda_{T_n}$ holds. The extension to Ω^λ_T will be straightforward (see step 1 below).

We thus fix $q \in \{0,\dots,n-1\}$ and assume $\Omega^\lambda_{T_q}$. We repeatedly use below that on the time interval (T_q, T_{q+1}), there are no fires at all (in $[-A,A]$) for the $\mathrm{LFF}_A(BS)$-process and no fires at all (in I_A^λ) during $(T_S T_q, T_S T_{q+1})$ for the $\mathrm{FF}_A(\mu_S,\mu_M^\lambda)$-process (use $\Omega^M_{A,T}(\lambda)$).

Step 1. — To start with, we observe that since there are no fires between $T_S T_q$ and $T_S T_{q+1}$, we have

$$\tau_t^\lambda(x) = \tau^\lambda_{T_q}(x), \quad \tau_t^\lambda(c) = \tau^\lambda_{T_q}(c) \quad \text{and} \quad \rho_t^\lambda(c) = \rho^\lambda_{T_q}(c)$$

for all $x \in \mathcal{B}_M$, all $c \in \mathcal{C}_M$, all $t \in [T_q, T_{q+1})$ (because $\eta^\lambda_{T_S t}(i)$ is nondecreasing on $[T_q, T_{q+1})$ for all $i \in I_A^\lambda$). By the same way,

$$\tau_t(x) = \tau_{T_q}(x) \quad \text{and} \quad \tau_t(c) = \tau_{T_q}(c)$$

for all $x \in \mathcal{B}_M$, all $c \in \mathcal{C}_M$, all $t \in [T_q, T_{q+1})$ (because $Z_t(x), Z_t(x_+), Z_t(x_-)$ are nondecreasing on $[T_q, T_{q+1})$ for all $x \in [-A,A]$). Hence for $t \in [T_q, T_{q+1})$, $\Omega_t^\lambda = \Omega^\lambda_{T_q}$. Thus $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}-}$.

Step 2. — Let $c \in \mathcal{C}_M$. Observe that on $\Omega^\lambda_{T_{q+1}-}$, there holds, for all $i \in c_\lambda$,

$$\eta^\lambda_{T_S T_{q+1}-}(i) = \min\big(N^{S,0}_{T_S T_{q+1}-}(i) - N^{S,0}_{T_S \tau_{T_q}(c)}(i), 1\big). \tag{3.9.5}$$

Indeed, seeds are falling on i according to $(N_t^{S,0}(i))_{t\ge 0}$. Furthermore, we know from step 1 that

$$\rho^\lambda_{T_{q+1}-}(c) = \tau^\lambda_{T_{q+1}-}(c) = \tau_{T_{q+1}-}(c) = \tau_{T_q}(c).$$

By definition of $\tau^{\lambda}_{T_{q+1}-}(c)$, $\eta^{\lambda}_{T_S\tau_{T_q}}(i)=0$ for all $i\in c_\lambda$. And by definition of $\rho^{\lambda}_{T_{q+1}-}(c)$, no fire affects c_λ during $(T_S\rho^{\lambda}_{T_{q+1}-}(c), T_ST_{q+1})$.

Step 3. — We show here that if $Z_{T_{q+1}-}(X_{q+1})<1$, there exist j_1,j_2,j_3,j_4 in $(X_{q+1})_\lambda$ such that

$$j_1<j_2<\lfloor \boldsymbol{n}_\lambda X_{q+1}\rfloor<j_3<j_4,$$
$$\eta^{\lambda}_{T_ST_{q+1}-}(j_2)=\eta^{\lambda}_{T_ST_{q+1}-}(j_3)=0,$$
$$\eta^{\lambda}_{T_ST_{q+1}-}(j_1)=\eta^{\lambda}_{T_ST_{q+1}-}(j_4)=1.$$

Recall that for $i\in(X_{q+1})_\lambda$, the seeds fall according to $(N^{S,q+1}_t(i-\lfloor \boldsymbol{n}_\lambda X_{q+1}\rfloor))_{t\geq 0}$. Recall also that $\tau^{\lambda}_{T_{q+1}-}(X_{q+1})=\tau_{T_{q+1}-}(X_{q+1})$ (by Step 1), so that by definition, $(X_{q+1})_\lambda$ is completely vacant at time $T_S\tau_{T_{q+1}-}(X_{q+1})$. Recall finally that $\tau_{T_{q+1}-}(X_{q+1})\in\mathcal{T}_M$ (and so does T_{q+1}).

Observe that by (3.9.4), $Z_{T_{q+1}-}(X_{q+1})<1$ implies that $T_{q+1}-\tau_{T_{q+1}-}(X_{q+1})<1$. Since we work on $\Omega^S_2(\lambda)$, we know that there are some sites

$$i_1<i_2<i_3<\lfloor \boldsymbol{n}_\lambda X_{q+1}\rfloor<i_4<i_5<i_6$$

in $(X_{q+1})_\lambda$ such that at least one seed has fallen on i_2 and i_5 and no seed has fallen on i_1,i_3,i_4,i_6 during $[T_S\tau_{T_{q+1}-}(X_{q+1}),T_ST_{q+1})$. All this implies that

$$\eta^{\lambda}_{T_ST_{q+1}-}(i_2)=\eta^{\lambda}_{T_ST_{q+1}-}(i_5)=1 \quad\text{and}\quad \eta^{\lambda}_{T_ST_{q+1}-}(i_3)=\eta^{\lambda}_{T_ST_{q+1}-}(i_4)=0$$

(because the vacant sites i_1,i_6 protect the occupied sites i_2,i_4 from fires falling outside $(X_{q+1})_\lambda$ and because no fire falls on $(X_{q+1})_\lambda$ during $[0,T_ST_{q+1})$).

Step 4. — Next we check that if $Z_{T_{q+1}-}(c)=1$ for some $c\in\mathcal{C}_M$, then

$$\eta^{\lambda}_{T_ST_{q+1}-}(i)=1 \quad \text{for all } i\in c_\lambda.$$

Recalling (3.9.3), we see that $Z_{T_{q+1}-}(c)=1$ implies that $T_{q+1}-\tau_{T_{q+1}-}(c)\geq 1$ and thus $T_{q+1}-\tau_{T_q}(c)\geq 1$ by step 1. Using (3.9.5), we conclude that for all $i\in c_\lambda$,

$$\eta^{\lambda}_{T_ST_{q+1}-}(i)=\min\left(N^{S,0}_{T_ST_{q+1}-}(i)-N^{S,0}_{T_S\tau_{T_q}(c)(i)},1\right)=1$$

(at least one seed falls on each site during a time interval of length greater than T_S).

Step 5. — We now prove that if $\widetilde{H}_{T_{q+1}-}(x)=0$ for some $x\in\mathcal{B}_M$,

$$\eta^{\lambda}_{T_ST_{q+1}-}(i)=1 \quad \text{then for all } i\in x_\lambda.$$

Preliminary considerations. — Let $k\in\{1,\dots,n\}$ such that $x=X_k$, which is at the boundary of two cells $c_-,c_+\in\mathcal{C}_M$. We know that $\widetilde{H}_{T_{q+1}-}(x)=0$, whence $H_{T_{q+1}-}(x)=0$ and $Z_{T_{q+1}-}(x)=Z_{T_{q+1}-}(c_+)=Z_{T_{q+1}-}(c_-)=1$. This implies that $T_{q+1}\geq 1$, because $Z_t(x)=t$ for all $t<1$, all $x\in[-A,A]$.

No fire has concerned $(c_-)_\lambda$ during $(T_S\rho^{\lambda}_{T_{q+1}-}(c_-),T_ST_{q+1})$ (by definition of $\rho^{\lambda}_{T_{q+1}-}(c_-)$). But step 1 implies that $\rho^{\lambda}_{T_{q+1}-}(c_-)=\tau_{T_{q+1}-}(c_-)\leq T_{q+1}-1$, because $Z_{T_{q+1}-}(c_-)=1$, see (3.9.3). Using a similar argument for c_+, we conclude that no match falling outside $(X_k)_\lambda$ can affect $(X_k)_\lambda$ during $(T_S(T_{q+1}-1), T_S T_{q+1})$ (because to affect $(X_k)_\lambda$, a match falling outside $(X_k)_\lambda$ needs to cross c_- or c_+).

Case 1. – First assume that $k \geq q+1$. Then we know that no fire has fallen on $(X_k)_\lambda$ during $[0, T_S T_{q+1})$. Due to the preliminary considerations, we deduce that no fire at all has concerned $(X_k)_\lambda$ during $(T_S(T_{q+1}-1), T_S T_{q+1})$. This time interval is of length greater than T_S. Thus $(X_k)_\lambda$ is completely occupied at time $T_S T_{q+1}-$.

Case 2. – Assume that $k \leq q$ and $Z_{T_k-}(X_k) = 1$, so that there already has been a macroscopic fire in $(X_k)_\lambda$ (at time $\boldsymbol{a}_\lambda T_k$). Since then $Z_{T_k}(X_k) = 0$ and $Z_{T_{q+1}-}(X_k) = 1$, we deduce that $T_{q+1} - T_k \geq 1$. We conclude as in case 1 that no fire at all has concerned $(X_k)_\lambda$ during $(T_S(T_{q+1}-1), T_S T_{q+1})$, which implies the claim.

Case 3. – Assume that $k \leq q$ and $Z_{T_k-}(X_k) < 1$ and $T_{q+1} - T_k \geq 1$. Then there already has been a microscopic fire in $(X_k)_\lambda$ (at time $T_S T_k$). But there are no fire in $(X_k)_\lambda$ during $(T_S T_k, T_S T_{q+1})$ and we conclude as in case 2.

Case 4. – Assume finally that $k \leq q$ and $Z_{T_k-}(X_k) < 1$ and $T_{q+1} - T_k < 1$. There has been a microscopic fire in $(X_k)_\lambda$ (at time $T_S T_k$). Since $H_{T_{q+1}-}(X_k) = 0$, we deduce (see Algorithm 3.9.3) that $T_k + \Theta^k_{T_k - Z_{T_k-}(X_k), T_k} \leq T_{q+1}$.

Consider the zone $C(\eta^\lambda_{T_S T_k -}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$ destroyed by the match falling at time $T_S T_k$. This zone is completely occupied at time $T_S(T_k + \Theta_{T_k - Z_{T_k-}(X_k), T_k}) \leq T_S T_{q+1}$ by definition of $\Theta_{T_k - Z_{T_k-}(X_k), T_k}$, see Lemma 3.9.1, using here again the preliminary considerations.

We deduce that $C(\eta^\lambda_{T_S T_k -}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$ is completely occupied at time $T_S T_{q+1}-$.

Consider now $i \in (X_k)_\lambda \setminus C(\eta^\lambda_{T_S T_k -}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$. Then i has not been killed by the fire falling on $\lfloor \boldsymbol{n}_\lambda X_k \rfloor$. Thus i cannot have been killed during $(T_S(T_{q+1}-1), T_S T_{q+1})$ (due to the preliminary considerations) and is thus occupied at time $T_S T_{q+1}-$. This implies the claim.

Step 6. — Let us now prove that if $\widetilde{H}_{T_{q+1}-}(x) > 0$ and $Z_{T_{q+1}-}(x_+) = 1$ for some $x \in \mathcal{B}_M$, there are $i_1, i_2 \in x_\lambda$ such that

$$i_1 < i_2 \quad \text{and} \quad \eta^\lambda_{T_S T_{q+1}-}(i_1) = 1, \quad \eta^\lambda_{T_S T_{q+1}-}(i_2) = 0.$$

Recall that x is at the boundary of two cells c_-, c_+. We have either $H_{T_{q+1}-}(x) > 0$ or $Z_{T_{q+1}-}(c_-) < 1$ (because $Z_{T_{q+1}-}(c_+) = 1$ by assumption). Clearly, $x = X_k$ for some $k \leq q$, with $Z_{T_k-}(X_k) < 1$ (else, we would have $H_t(x) = 0$ and $Z_t(c_-) = Z_t(c_+)$ for all $t \in [0, T_{q+1})$). Thus, recalling (3.9.4), $T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = \tau^\lambda_{T_k-}(X_k)$, so that $(X_k)_\lambda$ is completely empty at time $T_S(T_k - Z_{T_k-}(X_k))$.

Case 1. – Assume first that $H_{T_{q+1}-}(x) > 0$. Then by construction, see Algorithm 3.9.3,

$$T_k + \Theta^k_{T_k - Z_{T_k-}(X_k), T_k} > T_{q+1} > T_k.$$

Consider $C = C(\eta^\lambda_{T_S T_k -}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$. By $\Omega_2^S(\lambda)$, we have

$$C \subset \big[\!\big[\lfloor \boldsymbol{n}_\lambda X_k - \tfrac{1}{2}\boldsymbol{m}_\lambda \rfloor, \lfloor \boldsymbol{n}_\lambda X_k + \tfrac{1}{2}\boldsymbol{m}_\lambda \rfloor \big]\!\big],$$

because $T_k - Z_{T_k-}(X_k)$ and T_k belong to $\mathcal{T}_M$ and $0 < Z_{T_k-}(X_k) < 1$. The component C is destroyed at time $T_S T_k$. By Definition of $\Theta^k_{T_k - Z_{T_k-}(X_k), T_k}$, see Lemma 3.9.1, we deduce that C is not completely occupied at time

$$T_S T_{q+1} < T_S(T_k + \Theta^k_{T_k - Z_{T_k-}(X_k), T_k}).$$

Consequenty, there is $i_2 \in [\![\lfloor \boldsymbol{n}_\lambda X_k - \frac{1}{2}\boldsymbol{m}_\lambda \rfloor, \lfloor \boldsymbol{n}_\lambda X_k + \frac{1}{2}\boldsymbol{m}_\lambda \rfloor]\!]$ such that

$$\eta^\lambda_{T_S T_{q+1} -}(i_2) = 0.$$

Finally, using again $\Omega_2^S(\lambda)$ there is necessarily (at least) one seed falling on a site in $[\![\lfloor \boldsymbol{n}_\lambda X_k - \boldsymbol{m}_\lambda + 1 \rfloor, \lfloor \boldsymbol{n}_\lambda X_k - \frac{1}{2}\boldsymbol{m}_\lambda - 1 \rfloor]\!] \subset (X_k)_\lambda$ during $(T_S T_q, T_S T_{q+1})$. This shows the result.

Case 2. – Assume next that $H_{T_{q+1}-}(x) = 0$ and that

$$T_{q+1} - \big[T_k - Z_{T_k-}(X_k)\big] < 1.$$

Recall that $(X_k)_\lambda$ is completely empty at time $T_S(T_k - Z_{T_k-}(X_k))$. Since $T_k - Z_{T_k-}(X_k)$ and T_{q+1} belong to $\mathcal{T}_M$ and since their difference is smaller than 1 by assumption, $\Omega_2^S(\lambda)$ guarantees us the existence of $i_1 < i_2 < i_3$, all in $(X_k)_\lambda$, such that (at least) one seed falls on i_2 and no seed fall on i_1 nor on i_3 during $(T_S(T_k - Z_{T_k-}(X_k)), T_S T_{q+1})$. One easily concludes that i_2 is occupied and i_3 is vacant at time $T_S T_{q+1}-$, as desired.

Case 3. – Assume finally that $H_{T_{q+1}-}(x) = 0$ and that

$$T_{q+1} - \big[T_k - Z_{T_k-}(X_k)\big] \geq 1.$$

Since $H_{T_{q+1}-}(x) = 0$, there holds $Z_{T_{q+1}-}(c_-) < 1 = Z_{T_{q+1}-}(c_+)$ and $T_k + \Theta^k_{T_k - Z_{T_k-}(X_k), T_k} \leq T_{q+1}$. We aim to use the event $\Omega_1^S(\lambda)$. We introduce

$$t_0 = T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = \tau^\lambda_{T_k-}(X_k).$$

Observe that $\tau_{T_k-}(c_-) = \tau_{T_k-}(c_+) = \tau_{T_k-}(x)$ because there is no match falling (exactly) on x during $[0, T_k)$. Thus $Z_{t_0}(x) = Z_{t_0}(c_-) = Z_{t_0}(c_+) = 0$.

Set now $t_1 = T_k$ and $s = T_{q+1}$. Observe that $0 < t_1 - t_0 < 1$ (because $Z_{T_k-}(X_k) < 1$). Necessarily, $Z_t(c_-)$ has jumped to 0 at least one time between t_0 and $T_{q+1}-$ (else, one would have $Z_{T_{q+1}-}(c_-) = 1$, since $T_{q+1} - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$, and since for all $t \in [t_0, t_0 + 1)$, $Z_t(c_-) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \cdots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_t(c_-), Z_t(c_+))$ during $(t_0 + 1, s)$. We also put $\varepsilon = 1$ if t_2 is a jump of $Z_t(c_+)$ and $\varepsilon = -1$ else. Then we observe that $Z_t(c_-)$ and $Z_t(c_+)$ do never jump to 0 at the same time during $(t_0, s]$ (else, it would mean that they are killed by the same fire at some time u, whence necessarily, $H_r(u) = 0$ and $Z_r(c_-) = Z_r(c_+)$ for all $r \in (u, s]$).

Furthermore, there is always at least one jump of $(Z_t(c_-), Z_t(c_+))$ in any time interval of length 1 (during $[t_0 + 1, s)$), because else, $Z_t(c_+)$ and $Z_t(c_-)$ would both become equal to 1 and thus would remain equal forever.

Finally, observe that two jumps of $Z_t(c_-)$ cannot occur in a time interval of length 1 (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$) and the same thing holds for c_+.

Consequently, the family $\mathcal{R} = \{\varepsilon, t_0, \dots, t_K; s\}$ necessarily satisfies condition (PP).

Next, $t_2 - t_1 < \Theta^k_{T_k - Z_{T_k-}(X_k), T_k} = \Theta^k_{t_0,t_1}$, because else, we would have $H_{t_2-}(X_k) = 0$ and thus the fire destroying c_+ (or c_-) at time t_2 would also destroy c_- (or c_+), we thus would have $Z_{t_2}(c_+) = Z_{t_2}(c_-) = 0$, so that $Z_t(c_+)$ and $Z_t(c_-)$ would remain equal forever.

Finally, we check that $(\eta^\lambda_{T_S t}(i))_{t \geq t_0, i \in x_\lambda} = (\zeta^{\lambda,\mathcal{R},k}_t(i + \lfloor \boldsymbol{n}_\lambda x \rfloor))_{t \geq t_0, i \in x_\lambda}$, this last process being built with the family of seed processes $(N^{S,k}_{T_S t}(i))_{t \geq t_0, i \in x_\lambda}$ as in subsection 3.9.2. Both are empty at time t_0. Seeds fall according to the same processes. In both cases, a first match falls on $\lfloor \boldsymbol{n}_\lambda x \rfloor$ at time t_1. In both cases (say that $\varepsilon = 1$) a fire destroys the occupied connected component containing $\lfloor \boldsymbol{n}_\lambda x \rfloor + \boldsymbol{m}_\lambda$ at time t_2 (by definition for $\zeta^{\lambda,\mathcal{R}}$ and since $Z_{t_2-}(c_+) = 1$ implies, exactly as in step 4, that $\eta^\lambda_{T_S t_2-}(i) = 1$ for all i in $(c_+)_\lambda$, so that the fire destroying c_+ at time t_2 also destroys the occupied connected component around $\lfloor \boldsymbol{n}_\lambda x \rfloor + \boldsymbol{m}_\lambda$, which is at the boundary of c_+). And so on.

We thus can use $\Omega^S_1(\lambda)$ and conclude that there are some sites $i_1 < i_2$ in x_λ with $\eta^\lambda_{T_S T_{q+1}-}(i_1) = 1$ and $\eta^\lambda_{T_S T_{q+1}-}(i_2) = 0$ as desired.

Step 7. — We finally conclude the proof. We put $z := Z_{T_{q+1}-}(X_{q+1})$ and consider separately the cases where $z \in (0, 1)$ and $z = 1$. Observe that $z = 0$ never happens, since by construction, $Z_{T_{q+1}-}(X_{q+1}) = \min(Z_{T_q}(X_{q+1}) + (T_{q+1} - T_q), 1) > 0$ and since $T_{q+1} > T_q$.

Case $z \in (0, 1)$. – Then in the $\mathrm{LFF}_A(BS)$-process, see Algorithm 3.9.3,

$$Z_{T_{q+1}}(x) = Z_{T_{q+1}-}(x) > 0 \quad \text{for all } x \in [-A, A],$$

whence $\tau_{T_{q+1}}(x) = \tau_{T_{q+1}-}(x)$ and $\tau_{T_{q+1}}(c) = \tau_{T_{q+1}-}(c)$ for all $x \in \mathcal{B}_M$, all $c \in \mathcal{C}_M$.

Using step 3, we see that the match falling on $\lfloor \boldsymbol{n}_\lambda X_{q+1} \rfloor$ at time $T_S T_{q+1}$ destroys nothing outside $[\![j_2 + 1, j_3 - 1]\!]$. As a conclusion, we obviously have

$$\tau^\lambda_{T_{q+1}}(x) = \tau^\lambda_{T_{q+1}-}(x) \quad \text{and} \quad \rho^\lambda_{T_{q+1}}(c) = \tau^\lambda_{T_{q+1}}(c) = \tau^\lambda_{T_{q+1}-}(c)$$

for all $x \in \mathcal{B}_M \setminus \{X_{q+1}\}$ and all $c \in \mathcal{C}_M$. There also holds $\tau^\lambda_{T_{q+1}}(X_{q+1}) = \tau^\lambda_{T_{q+1}-}(x)$ because j_1 (see step 3), which is occupied at time $T_S T_{q+1}-$ and not killed at time $T_S T_{q+1}$ (thanks to j_2), does belong to $(X_{q+1})_\lambda$.

We conclude that when $z \in (0,1)$, $\Omega^\lambda_{T_{q+1}-}$ implies $\Omega^\lambda_{T_{q+1}}$. Using Step 1, we deduce that $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}}$ when $z \in (0,1)$.

Case 1. – Then there are $a, b \in \mathcal{B}_M \cup \{-A, A\}$ such that $D_{T_{q+1}-}(X_{q+1}) = [a,b]$. We assume that $a, b \in \mathcal{B}_M$, the other cases being treated similarly. Recalling Algorithm 3.9.3, we know that for all $c \in \mathcal{C}_M$ with $c \subset (a,b)$, $Z_{T_{q+1}-}(c) = 1$, for all $x \in \mathcal{B}_M \cap (a,b)$, $\widetilde{H}_{T_{q+1}-}(x) = 0$, while finally $\widetilde{H}_{T_{q+1}-}(a) > 0$ and $\widetilde{H}_{T_{q+1}-}(b) > 0$. For the $\mathrm{LFF}_A(BS)$-process, we have

(i) $\tau_{T_{q+1}}(c) = T_{q+1}$ for all $c \in \mathcal{C}_M$ with $c \subset (a,b)$;
(ii) $\tau_{T_{q+1}}(x) = T_{q+1}$ for all $x \in \mathcal{B}_M \cap (a,b)$;
(iii) $\tau_{T_{q+1}}(c) = \tau_{T_{q+1}-}(c)$ for all $c \in \mathcal{C}_M$ with $c \cap (a,b) = \varnothing$;
(iv) $\tau_{T_{q+1}}(x) = \tau_{T_{q+1}-}(x)$ for all $x \in \mathcal{B}_M \setminus (a,b)$.

Next, using steps 4, 5, using step 6 for a (and a very similar result for b), we immediately check that the fire occurring at $\lfloor \boldsymbol{n}_\lambda X_{q+1} \rfloor$ at time $T_S T_{q+1}$

- ▷ destroys completely all the cells $c \in \mathcal{C}_M$ with $c \subset (a,b)$;
- ▷ destroys completely all the zones x_λ with $x \in \mathcal{B}_M \cap (a,b)$;
- ▷ does not destroy at all the cells $c \in \mathcal{C}_M$ with $c \cap (a,b) = \varnothing$ and the zones x_λ with $x \in \mathcal{B}_M \setminus [a,b]$;
- ▷ does not destroy completely a_λ nor b_λ.

Consequently, we have

(i) $\rho^\lambda_{T_{q+1}}(c) = \tau^\lambda_{T_{q+1}}(c) = T_{q+1}$ for all $c \in \mathcal{C}_M$ with $c \subset (a,b)$;
(ii) $\tau^\lambda_{T_{q+1}}(x) = T_{q+1}$ for all $x \in \mathcal{B}_M \cap (a,b)$;
(iii) $\rho^\lambda_{T_{q+1}}(c) = \rho^\lambda_{T_{q+1}-}(c)$ and $\tau^\lambda_{T_{q+1}}(c) = \tau^\lambda_{T_{q+1}-}(c)$ for all $c \in \mathcal{C}_M$ with $c \cap (a,b) = \varnothing$;
(iv) $\tau^\lambda_{T_{q+1}}(x) = \tau^\lambda_{T_{q+1}-}(x)$ for all $x \in \mathcal{B}_M \setminus (a,b)$.

We conclude that when $z = 1$, $\Omega^\lambda_{T_{q+1}-}$ implies $\Omega^\lambda_{T_{q+1}}$. Using step 1, we deduce that $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}}$ when $z = 1$. □

3.9.6. Conclusion. — To achieve the proof, we will need the following result.

LEMMA 3.9.6. — *Let $(N^S_t(i))_{t \geq 0, i \in \mathbb{Z}}$ be a family of i.i.d.* $\mathrm{SR}(\mu_S)$*-processes.*

(i) *Put* $K^\lambda_t = (2\boldsymbol{m}_\lambda + 1)^{-1} |\{i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!] : N^S_{T_S t}(i) > 0\}|$ *and*

$$U^\lambda_t = \Big(\frac{\psi_S(K^\lambda_t)}{T_S}\Big) \wedge 1,$$

recall notation 2.1.5. *Then for any* $T > 0$, $\sup_{[0,T]} |U^\lambda_t - t \wedge 1|$ *tends a.s. to* 0 *as* λ *tends to* 0.

(ii) *For any* $k \geq 0$, $\Pr[|C(\min(N^S_{T_S t}, 1), 0)| = k] = q_k(t \wedge 1)$, *where* $q_k(z)$ *was defined* (2.4.2).

Proof. — We start with (i). First observe that $t \mapsto U_t^\lambda$ and $t \mapsto t \wedge 1$ are nondecreasing and $t \mapsto t \wedge 1$ is continuous. By the Dini Theorem, it suffices to prove that for all $t \in [0,T]$, a.s., $\lim U_t^\lambda = t \wedge 1$. To do so, observe that $(2\boldsymbol{m}_\lambda + 1)K_t^\lambda$ has a binomial distribution with parameters $2\boldsymbol{m}_\lambda + 1$ and $\nu_S((0,T_S t))$. Thus K_t^λ tends a.s. to $\nu_S((0,T_S t))$. Hence U_t^λ tends a.s. to $(\psi_S(\nu_S((0,T_S t)))/T_S) \wedge 1 = t \wedge 1$ by definition of ψ_S.

We now check (ii). If $t \geq 1$, then obviously, $\min(N_{T_S t}^S(i), 1) = 1$ for all $i \in \mathbb{Z}$, whence $|C(\min(N_{T_S t}^S, 1), 0)| = \infty$ a.s. Consequently,

$$\Pr\left[|C(\min(N_{T_S t}^S, 1), 0)| = k\right] = 0 = q_k(1).$$

For $t < 1$, the result relies on a simple computation involving the i.i.d. random variables $\min(N_{T_S t}^S(i), 1)$, which have a Bernoulli distribution with parameter $\nu_S((0, T_S t))$: if $k = 0$, there holds

$$\Pr\left[|C(\min(N_{T_S t}^S, 1), 0)| = 0\right] = \Pr\left[N_{T_S t}(i) = 0\right] = \nu_S\big((T_S t, T_S)\big) = q_0(t).$$

For $k \geq 1$,

$$\begin{aligned} &\Pr\left[|C(\min(N_{T_S t}^S, 1), 0)| = k\right] \\ &\quad = \sum_{j=0}^{k-1} \Pr\left[N_{T_S t}^S(j-k) = N_{T_S t}^S(j+1) = 0, \forall i \in [\![j-k+1, j]\!], N_{T_S t}^S(i) = 1\right] \\ &\quad = k\big[\nu_S((T_S t, T_S))\big]^2 \cdot \big[\nu_S((0, T_S t))\big]^k = q_k(t), \end{aligned}$$

which ends the proof. □

We finally give the

Proof of Proposition 3.8.1 when $\beta = BS$. — Let us fix $x_0 \in (-A, A)$, $t_0 \in (0, T]$ and $\varepsilon > 0$. We will prove that with our coupling (see subsection 3.9.3), there holds

(a) $\lim_{\lambda\to 0} \Pr\left[\boldsymbol{\delta}(D_{t_0}^\lambda(x_0), D_{t_0}(x_0)) > \varepsilon\right] = 0$;

(b) $\lim_{\lambda\to 0} \Pr\left[\boldsymbol{\delta}_T(D^\lambda(x_0), D(x_0)) > \varepsilon\right] = 0$;

(c) $\lim_{\lambda\to 0} \Pr\left[\sup_{[0,T]} |Z_t^\lambda(x_0) - Z_t^\lambda(x_0)| \geq \varepsilon\right] = 0$;

(d) $\lim_{\lambda\to 0} \Pr\left[|C(\eta_{T_S t_0}^\lambda, \lfloor \boldsymbol{n}_\lambda x_0 \rfloor)| = k\right] = \mathbb{E}\big[q_k(Z_t(x_0))\big]$.

Recall that $q_k(z)$ was defined, for $k \geq 0$ and $z \in [0,1]$ in (2.4.2). These points will clearly imply the result.

First, we introduce, for $\zeta > 0$, the event $\Omega_{A,T}^{x_0}(\zeta)$ on which $x_0 \notin \bigcup_{q=1}^n [X_q - \zeta, X_q + \zeta]$. The probability of this event obviously tends to 1 as $\zeta \to 0$.

On $\Omega_{A,T}^{x_0}(\zeta)$, for $\lambda > 0$ small enough (say, small enough such that $4\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda < \zeta$), $\lfloor \boldsymbol{n}_\lambda x_0 \rfloor \notin \bigcup_{q=1}^n (X_q)_\lambda$. We then call $c_0 \in \mathcal{C}_M$ the cell containing x_0.

Step 1. — We first show that (a) (which holds for an arbitrary value of $t_0 \in (0, T]$) implies (b).

Indeed, we have by construction, for any $t \in [0,T]$, $\boldsymbol{\delta}(D_t^\lambda(x_0), D_t(x_0)) < 4A$. Hence by dominated convergence, (a) implies that $\lim_{\lambda\to 0} \mathbb{E}\left[\boldsymbol{\delta}(D_t^\lambda(x_0), D_t(x_0))\right] = 0$, whence again by dominated convergence, $\lim_{\lambda\to 0} \mathbb{E}\left[\boldsymbol{\delta}_T(D^\lambda(x_0), D(x_0))\right] = 0$.

Step 2. — Due to Lemma 3.9.5, we know that on $\Omega(\alpha,\lambda) \cap \Omega_{A,T}^{x_0}(\zeta)$, we have

$$\tau_t^\lambda(c_0) = \rho_t^\lambda(c_0) = \tau_t(x_0) \quad \text{for all } t \in [0,T].$$

This implies that for all $i \in (c_0)_\lambda$, for all $t \in [0,T]$,

$$\eta_{T_S t}^\lambda(i) = \min\left(N_{T_S t}^{S,0}(i) - N_{T_S \tau_t(x_0)}^{S,0}(i), 1\right).$$

We also recall that by construction, $(\tau_t(x_0))_{t\geq 0}$ is independent of $(N_t^{S,0}(i))_{t\geq 0, i\in\mathbb{Z}}$.

Step 3. — Here we prove (d), for some fixed $k \geq 0$. Let $\delta > 0$ be fixed. We first consider $\alpha_0 > 0$, $\zeta_0 > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$,

$$\Pr\left[\Omega(\alpha_0,\lambda) \cap \Omega_{A,T}^{x_0}(\zeta_0)\right] > 1 - \delta.$$

Then we consider $\lambda_k \leq \lambda_0$ in such a way that for $\lambda \in (0,\lambda_k)$,

$$\left[\!\left[\lfloor \boldsymbol{n}_\lambda x_0 \rfloor - k - 1, \lfloor \boldsymbol{n}_\lambda x_0 \rfloor + k + 1\right]\!\right] \subset (c_0)_\lambda$$

on $\Omega_{A,T}^{x_0}(\zeta_0)$ (it suffices that $2k < \zeta_0 \boldsymbol{n}_\lambda$ for all $\lambda \in (0,\lambda_k)$).

We easily conclude: for $\lambda \in (0,\lambda_k)$, recalling (3.9.3), using Lemma 3.9.6 (ii) together with a (spatial and temporal) stationarity argument, using step 2 and that $(N_t^{S,0}(i))_{t\geq 0, i\in\mathbb{Z}}$ is independent of $\Omega_{A,T}^{x_0}(\zeta) \cap \Omega(\alpha,\lambda)$ and $\tau_t(x_0)$, we obtain

$$\begin{aligned}
&\left|\Pr\left[|C(\eta_{T_S t}^\lambda, \lfloor \boldsymbol{n}_\lambda x_0 \rfloor)| = k\right] - \mathbb{E}\left[q_k(Z_t(x_0))\right]\right| \\
&\quad = \left|\Pr\left[|C(\eta_{T_S t}^\lambda, \lfloor \boldsymbol{n}_\lambda x_0 \rfloor)| = k\right] - \mathbb{E}\left[q_k(\min(t - \tau_t(x_0), 1))\right]\right| \\
&\quad \leq \Pr\left[\left(\Omega(\alpha,\lambda) \cap \Omega_{A,T}^{x_0}(\zeta)\right)^c\right] < \delta.
\end{aligned}$$

This concludes the proof of (d).

Step 4. — We next prove (c). For $\delta > 0$ fixed, we consider $\alpha_0 > 0$, $\zeta_0 > 0$ and $\lambda_0 > 0$ be as in step 3. Consider the successive values $0 = s_0 < s_1 < \cdots < s_\ell < T$ of $(\tau_t(x_0))_{t\in[0,T]}$. Set also $s_{\ell+1} = T$. Recall the definition of $Z_t^\lambda(x)$, see (2.2.5), and compare to Lemma 3.9.6 (i). Let $k \in \{0,\ldots,\ell\}$ be fixed. Denote by $(U_t^{k,\lambda})_{t\geq 0}$ the process defined as in Lemma 3.9.6 (i) with the seed process

$$\left(N_{s_k/T_S + t}^{S,0}(i - \lfloor \boldsymbol{n}_\lambda x_0 \rfloor) - N_{s_k/T_S}^{S,0}(i - \lfloor \boldsymbol{n}_\lambda x_0 \rfloor)\right)_{t\geq 0, i\in\mathbb{Z}}$$

(this is indeed a family of $\mathrm{SR}(\mu_S)$-processes by stationarity and since $s_1,\ldots,s_\ell$ are independent of $(N_t^{S,0}(i))_{t\geq 0, i\in\mathbb{Z}}$). Then due to Lemma 3.9.6 (i), for all $\lambda > 0$ small enough, say $\lambda \in (0,\lambda_1)$,

$$\Pr\Big(\sup_{[s_k,s_{k+1})} |U_{t-s_k}^{k,\lambda} - (t-s_k)\wedge 1| \geq \varepsilon\Big) \leq \delta.$$

But on $\Omega(\alpha,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta)$, we have $Z^\lambda_t(x_0) = U^{k,\lambda}_{t-s_k}$ for all $t\in[s_k,s_{k+1})$, see step 2. It also holds, recall (3.9.3), that $Z_t(x_0) = (t-s_k)\wedge 1$ for $t\in[s_k,s_{k+1})$. As a conclusion, for all $\lambda>0$ small enough,

$$\Pr\big(\sup_{[s_k,s_{k+1})}|Z^\lambda_t(x_0)-Z_t(x_0)|\geq\varepsilon\big)\leq \Pr\big((\Omega(\alpha,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta))^c\big) + \Pr\big(\sup_{[s_k,s_{k+1})}|U^{k,\lambda}_{t-s_k}-(t-s_k)\wedge 1|\geq\varepsilon\big)\leq 2\delta.$$

Observing finally that $\ell\leq\pi_M([0,T]\times(-A,A))$ and that

$$\mathbb{E}\big[\pi_M([0,T]\times(-A,A))\big] = 2TA,$$

we easily deduce that for all $\lambda>0$ small enough,

$$\Pr\big(\sup_{[0,T]}|Z^\lambda_t(x_0)-Z_t(x_0)|\geq\varepsilon\big)\leq 2TA\delta.$$

Point (c) immediately follows.

Step 5. — It remains to prove (a). Let $\delta>0$. Put $\mathcal{T}^*_M = \mathcal{T}_M\cup\{t_0\}$. Define the events $\Omega^*_M(\alpha)$, $\Omega^{S,*}_1(\lambda)$ and $\Omega^{S,*}_2(\lambda)$ as $\Omega_M(\alpha)$, $\Omega^S_1(\lambda)$ and $\Omega^S_2(\lambda)$, replacing $\mathcal{T}_M$ by $\mathcal{T}^*_M$. Define also $\Omega^*(\lambda,\alpha) = \Omega^M_{A,T}(\lambda)\cap\Omega^*_M(\alpha)\cap\Omega^{S,*}_1(\lambda)\cap\Omega^{S,*}_2(\lambda)$. Clearly, choosing $\alpha_1>0$ and $\zeta_1>0$ small enough, we have $\Pr[\Omega^*(\lambda,\alpha_1)\cap\Omega^{x_0}_{A,T}(\zeta_1)]\geq 1-\delta$ for all $\lambda>0$ small enough, say $\lambda\in(0,\lambda_2)$. On $\Omega^*(\lambda,\alpha_1)\cap\Omega^{x_0}_{A,T}(\zeta_1)$, we can argue exactly as in the proof of Lemma 3.9.5 to check that

(i) if $Z_{t_0}(x_0)<1$, then $D_{t_0}(x_0)=\{x_0\}$ and $C(\eta^\lambda_{T_S t},\lfloor \boldsymbol{n}_\lambda x_0\rfloor)\subset(x_0)_\lambda$ (see Step 3 of the proof of Lemma 3.9.5), whence $D^\lambda_{t_0}(x_0)\subset[x_0-\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda, x_0+\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda]$. We deduce that $\boldsymbol{\delta}(D_{t_0}(x_0),D^\lambda_{t_0}(x_0))\leq 2\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda$;

(ii) if $Z_{t_0}(x_0)=1$ and $D_{t_0}(x_0)=[a,b]$ for some $a,b\in\mathcal{B}_M\cup\{-A,A\}$, then
   - ▷ for all $c\in\mathcal{C}_M$ with $c\subset[a,b]$, $\eta^\lambda_{T_S t}(i)=1$ for all $i\in c_\lambda$ (see Step 4 of the preceding proof);
   - ▷ for all $x\in\mathcal{B}_M\cap(a,b)$, $\eta^\lambda_{T_S t}(i)=1$ for all $i\in x_\lambda$ (see step 5 of the preceding proof);
   - ▷ there are $i\in a_\lambda$ and $j\in b_\lambda$ such that $\eta^\lambda_{T_S t}(i)=\eta^\lambda_{T_S t}(j)=0$ (see step 6 of the preceding proof);

so that

$$[\![\lfloor \boldsymbol{n}_\lambda a\rfloor+\boldsymbol{m}_\lambda+1,\lfloor \boldsymbol{n}_\lambda b\rfloor-\boldsymbol{m}_\lambda-1]\!]\subset C(\eta^\lambda_{T_S t},\lfloor \boldsymbol{n}_\lambda x_0\rfloor)\subset[\![\lfloor \boldsymbol{n}_\lambda a\rfloor-\boldsymbol{m}_\lambda,\lfloor \boldsymbol{n}_\lambda b\rfloor+\boldsymbol{m}_\lambda]\!],$$

and thus

$$[a+\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda, b-\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda]\subset D^\lambda_{t_0}(x_0)\subset[a-\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda, b+\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda],$$

whence as previously, $\boldsymbol{\delta}(D_{t_0}(x_0),D^\lambda_{t_0}(x_0))\leq 2\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda$. Thus for all $\lambda\in(0,\lambda_2)$, on $\Omega^*(\lambda,\alpha_1)\cap\Omega^{x_0}_{A,T}(\zeta_1)$, we always have

$$\boldsymbol{\delta}\big(D_{t_0}(x_0),D^\lambda_{t_0}(x_0)\big)\leq 2\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda.$$

We conclude that for $\delta > 0$, for all $\lambda \in (0, \lambda_2)$ small enough (so that $2\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda < \varepsilon$), there holds

$$\Pr\left[\boldsymbol{\delta}(D_{t_0}(x_0), D^\lambda_{t_0}(x_0)) > \varepsilon\right] \leq \Pr\left[(\Omega^*(\lambda,\alpha) \cap \Omega^{x_0}_{A,T}(\zeta))^c\right] < \delta.$$

This concludes the proof. □

3.10. Convergence proof when $\beta = \infty$

The aim of this section is to prove Proposition 3.8.1 in the case where $\beta = \infty$ and this will conclude the proof of Theorem 2.3.3. This section generalizes consequently [**15**, section 4] and the proof we present here is quite different and slightly less intricate. We follow essentially the ideas of the previous section. Some points are easier (because the height of the barriers are deterministic in the limit process), but some other points are more complicated (in particular, the height of the barriers are not constant as a function of λ).

In the whole section, we assume (H_M) and $(H_S(\infty))$. The parameters $A > 0$ and $T > 0$ are fixed and we omit the subscript/superscript A in the whole proof.

We recall that $\boldsymbol{a}_\lambda$, $\boldsymbol{n}_\lambda$ and $\boldsymbol{m}_\lambda$ are defined in (2.2.1), (2.2.2) and (2.2.4). For $A > 0$ and $i \in \mathbb{Z}$, we set as usual

$$A_\lambda = \lfloor A\boldsymbol{n}_\lambda \rfloor, \quad I^\lambda_A = [\![-A_\lambda, A_\lambda]\!], \quad i_\lambda = \left[i/\boldsymbol{n}_\lambda, (i+1)/\boldsymbol{n}_\lambda\right).$$

For $[a,b]$ an interval of $[-A,A]$ and $\lambda \in (0,1)$, we introduce, assuming $-A < a < b < A$,

$$(3.10.1) \qquad \begin{cases} [a,b]_\lambda &= [\![\lfloor \boldsymbol{n}_\lambda a + \boldsymbol{m}_\lambda \rfloor + 1, \lfloor \boldsymbol{n}_\lambda b - \boldsymbol{m}_\lambda \rfloor - 1]\!] \subset \mathbb{Z}, \\ [-A,b]_\lambda &= [\![-A_\lambda, \lfloor \boldsymbol{n}_\lambda b - \boldsymbol{m}_\lambda \rfloor - 1]\!] \subset \mathbb{Z}, \\ [a,A]_\lambda &= [\![\lfloor \boldsymbol{n}_\lambda a + \boldsymbol{m}_\lambda \rfloor + 1, A_\lambda]\!] \subset \mathbb{Z}. \end{cases}$$

For $x \in (-A,A)$ and $\lambda \in (0,1)$, we introduce as usual

$$(3.10.2) \qquad x_\lambda = \left[\!\left[\lfloor \boldsymbol{n}_\lambda x - \boldsymbol{m}_\lambda \rfloor, \lfloor \boldsymbol{n}_\lambda x + \boldsymbol{m}_\lambda \rfloor\right]\!\right] \subset \mathbb{Z}.$$

3.10.1. Speed of occupation. — We start with some easy estimates.

LEMMA 3.10.1. — *Consider a family of i.i.d.* $\mathrm{SR}(\mu_S)$*-processes* $(N^S_t(i))_{t\geq 0, i\in\mathbb{Z}}$. *Let* $a < b$.

(i) *For* $t < 1$, $\lim_{\lambda\to 0} \Pr[\forall i \in [\![\lfloor a\boldsymbol{m}_\lambda \rfloor, \lfloor b\boldsymbol{m}_\lambda \rfloor]\!], N^S_{\boldsymbol{a}_\lambda t}(i) > 0] = 0$.

(ii) *For* $t \geq 1$, $\lim_{\lambda\to 0} \Pr[\forall i \in [\![\lfloor a\boldsymbol{m}_\lambda \rfloor, \lfloor b\boldsymbol{m}_\lambda \rfloor]\!], N^S_{\boldsymbol{a}_\lambda t}(i) > 0] = 1$.

(iii) *For* $t < 1$, $\lim_{\lambda\to 0} \Pr[\forall i \in [\![\lfloor a\boldsymbol{n}_\lambda \rfloor, \lfloor b\boldsymbol{n}_\lambda \rfloor]\!], N^S_{\boldsymbol{a}_\lambda t}(i) > 0] = 0$.

(iv) *For* $t > 1$, $\lim_{\lambda\to 0} \Pr[\forall i \in [\![\lfloor a\boldsymbol{n}_\lambda \rfloor, \lfloor b\boldsymbol{n}_\lambda \rfloor]\!], N^S_{\boldsymbol{a}_\lambda t}(i) > 0] = 1$.

(v) *For* $t > 0$ *and* $i \in \mathbb{Z}$, $\lim_{\lambda\to 0} \Pr[N^S_{\boldsymbol{a}_\lambda t}(i) > 0] = 1$.

Proof. — To check points (i) and (ii), it suffices to note that

$$\nu_S\big((0, \boldsymbol{a}_\lambda t)\big)^{(b-a)\boldsymbol{m}_\lambda} \sim \mathrm{e}^{-(b-a)\boldsymbol{m}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty))},$$

which tends to 0 if $t < 1$ (see (2.2.4)) and to 1 if $t \geq 1$ (because then $\boldsymbol{m}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty)) \leq \boldsymbol{m}_\lambda \nu_S((\boldsymbol{a}_\lambda, \infty)) \simeq \boldsymbol{m}_\lambda / \boldsymbol{n}_\lambda \to 0$). To check points (iii) and (iv), observe that

$$\nu_S\big((0, \boldsymbol{a}_\lambda t)\big)^{(b-a)\boldsymbol{n}_\lambda} \sim \mathrm{e}^{-(b-a)\boldsymbol{n}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty))} \sim \mathrm{e}^{-(b-a)\nu_S((\boldsymbol{a}_\lambda t, \infty))/\nu_S((\boldsymbol{a}_\lambda, \infty))}$$

tends to 0 if $t < 1$ and to 1 if $t > 1$ due to $(H_S(\infty))$. Finally, (v) follows from the fact that $1 - \nu_S((\boldsymbol{a}_\lambda t, \infty))$ obviously tends to 1. □

3.10.2. Height of the barriers. — We describe here the time needed for a destroyed (microscopic) cluster to be regenerated. Roughly, we assume that the zone around 0 is completely vacant at time $\boldsymbol{a}_\lambda t_0$. Then we consider the situation where a match falls on the site 0 at some time $\boldsymbol{a}_\lambda t_1 \in (\boldsymbol{a}_\lambda t_0, \boldsymbol{a}_\lambda (t_0 + 1))$ and we denote by $\Theta^\lambda_{t_0,t_1}$ the delay needed for the destroyed cluster to be fully regenerated (divided by $\boldsymbol{a}_\lambda$). We show that

$$\Theta^\lambda_{t_0,t_1} \simeq t_1 - t_0 \quad \text{when } \lambda \text{ is small.}$$

LEMMA 3.10.2. — *Consider a family of i.i.d.* $\mathrm{SR}(\mu_S)$*-processes* $(N^S_t(i))_{t \geq 0, i \in \mathbb{Z}}$. *Let* $0 \leq t_0 < t_1 < t_0 + 1$ *be fixed. Put*

$$\begin{aligned} \zeta^\lambda_{t_0,t}(i) &= \min\big(N^S_{\boldsymbol{a}_\lambda(t_0+t)}(i) - N^S_{\boldsymbol{a}_\lambda t_0}(i), 1\big), \\ \zeta^\lambda_{t_1,t}(i) &= \min\big(N^S_{\boldsymbol{a}_\lambda(t_1+t)}(i) - N^S_{\boldsymbol{a}_\lambda t_1}(i), 1\big) \end{aligned}$$

for all $t > 0$ *and* $i \in \mathbb{Z}$. *Define*

$$\Theta^\lambda_{t_0,t_1} = \inf\big\{t > 0 : \ \forall\, i \in C(\zeta^\lambda_{t_0,t_1-t_0}, 0),\ \zeta^\lambda_{t_1,t}(i) = 1\big\} \in [0, 1].$$

Then for all $\delta > 0$,

$$\lim_{\lambda \to 0} \Pr\big[|\Theta^\lambda_{t_0,t_1} - (t_1 - t_0)| \geq \delta\big] = 0.$$

Proof. — We can assume that $t_0 = 0$ by stationarity. We put $u = t_1 = t_1 - t_0$. Exactly as in the case where $\beta = BS$ (see subsection 3.9.1), we obtain, for $h > 0$,

$$\Pr[\Theta^\lambda_{t_0,t_1} \leq h] = \nu_S\big((\boldsymbol{a}_\lambda u, \infty)\big) + \frac{[\nu_S((\boldsymbol{a}_\lambda u, \infty))]^2}{[1 - g^\lambda_S(u,h)]^2} g^\lambda_S(u,h),$$

where

$$g^\lambda_S(u,h) = \Pr\big[N^S_{\boldsymbol{a}_\lambda u}(0) > 0, N^S_{\boldsymbol{a}_\lambda(u+h)}(0) > N^S_{\boldsymbol{a}_\lambda u}(0)\big].$$

For $h > u$, we observe that $g^\lambda_S(u,h) \geq 1 - \nu_S((\boldsymbol{a}_\lambda h, \infty)) - \nu_S((\boldsymbol{a}_\lambda u, \infty))$, whence

$$\Pr[\Theta^\lambda_{t_0,t_1} \leq h] \geq \Big(\frac{\nu_S((\boldsymbol{a}_\lambda u, \infty))}{\nu_S((\boldsymbol{a}_\lambda h, \infty)) + \nu_S((\boldsymbol{a}_\lambda u, \infty))}\Big)^2 \big[1 - \nu_S((\boldsymbol{a}_\lambda h, \infty)) - \nu_S((\boldsymbol{a}_\lambda u, \infty))\big],$$

which tends to 1 as $\lambda \to 0$ due to $(H_S(\infty))$, since $\boldsymbol{a}_\lambda$ increases to infinity and since $h > u$.

For $h < u$, there holds $g_S^\lambda(u,h) \le 1 - \nu_S((\boldsymbol{a}_\lambda h, \infty))$, so that

$$\Pr\left[\Theta_{t_0,t_1}^\lambda \le h\right] \le \nu_S((\boldsymbol{a}_\lambda u, \infty)) + \left(\frac{\nu_S((\boldsymbol{a}_\lambda u, \infty))}{\nu_S((\boldsymbol{a}_\lambda h, \infty))}\right)^2 [1 - \nu_S((\boldsymbol{a}_\lambda h, \infty))],$$

which tends to 0 due to $(H_S(\infty))$ and since $h < u$. This concludes the proof. □

3.10.3. Persistent effect of microscopic fires. — We handle a study similar to subsection 3.9.2. Recall that $\mathcal{R} = (\varepsilon; t_0, t_1, \dots, t_K; s)$ satisfies (PP) if

(i) $K \ge 2$, $\varepsilon \in \{-1, 1\}$;

(ii) $0 < t_0 < t_1 < \dots < t_K < s < t_K + 1$;

(iii) for all $k = 0, \dots, K-1$, $t_{k+1} - t_k < 1$;

(iv) $t_2 - t_0 > 1$ and for all $k = 2, \dots, K-2$, $t_{k+2} - t_k > 1$,

and that we set $\varepsilon_k = (-1)^k \varepsilon$ for $k \ge 0$.

For a family of i.i.d. SR(μ_S)-processes $(N_t^S(i))_{t\ge 0, i\in\mathbb{Z}}$, we introduce, for each λ in $(0,1)$, the process $(\zeta_t^{\lambda,\mathcal{R}}(i))_{t\ge t_0, i\in[\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]}$ defined as follows:

▷ for all $t \in [t_0, t_1)$, all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$, $\zeta_t^{\lambda,\mathcal{R}}(i) = \min(N_{\boldsymbol{a}_\lambda t}^S(i) - N_{\boldsymbol{a}_\lambda t_0}^S(i), 1)$,

▷ for all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$, $\zeta_{t_1}^{\lambda,\mathcal{R}}(i) = \zeta_{t_1-}^{\lambda,\mathcal{R}}(i) \mathbb{1}_{\{i \notin C(\zeta_{t_1-}^{\lambda,\mathcal{R}}, 0)\}}$,

▷ for $k = 1, \dots, K-1$,

(*) for all $t \in (t_k, t_{k+1})$, $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_t^{\lambda,\mathcal{R}}(i) = \min\left(\zeta_{t_k}^{\lambda,\mathcal{R}}(i) + N_{\boldsymbol{a}_\lambda t}^S(i) - N_{\boldsymbol{a}_\lambda t_k}^S(i), 1\right),$$

(*) for all $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_{t_{k+1}}^{\lambda,\mathcal{R}}(i) = \zeta_{t_{k+1}-}^{\lambda,\mathcal{R}}(i) \mathbb{1}_{i \notin C(\zeta_{t_{k+1}-}^{\lambda,\mathcal{R}}, \epsilon_k \boldsymbol{m}_\lambda)};$$

▷ for all $t \in (t_K, \infty)$, $i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$,

$$\zeta_t^{\lambda,\mathcal{R}}(i) = \min\left(\zeta_{t_K}^{\lambda,\mathcal{R}}(i) + N_{\boldsymbol{a}_\lambda t}^S(i) - N_{\boldsymbol{a}_\lambda t_K}^S(i), 1\right).$$

Consider the event

$$\Omega_{\mathcal{R}}^S(\lambda) = \left\{\exists -\boldsymbol{m}_\lambda < i_1 < i_2 < i_3 < \boldsymbol{m}_\lambda : \ \zeta_s^{\lambda,\mathcal{R}}(i_1) = \zeta_s^{\lambda,\mathcal{R}}(i_3) = 0, \zeta_s^{\lambda,\mathcal{R}}(i_2) = 1\right\}.$$

LEMMA 3.10.3. — *Let $\mathcal{R} = (\varepsilon; t_0, t_1, \dots, t_K; s)$ satisfy* (PP). *For each $\lambda \in (0,1]$, consider the process $(\zeta_t^{\lambda,\mathcal{R}}(i))_{t\ge t_0, i\in[\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]}$ defined above. If $t_2 - t_1 < t_1 - t_0$, there holds*

$$\lim_{\lambda\to 0} \Pr\left(\Omega_{\mathcal{R}}^S(\lambda)\right) = 1.$$

Compare to Lemma 3.9.2: the condition $\Theta_{t_0,t_1} > t_2 - t_1$ is replaced by the condition $t_1 - t_0 > t_2 - t_1$. This is very natural, in view of Lemma 3.10.2.

Proof. — In view of Lemma 3.10.1, the proof is very similar to that of Lemma 3.9.2. We assume that $\varepsilon = 1$ and that K is even for simplicity. Fix $\alpha = 1/K$.

First fire. — We put $C = C(\zeta_{t_1-}^{\lambda,\mathcal{R}}, 0)$. Since $t_1 - t_0 < 1$, $C \subset [\![-\lfloor \alpha \boldsymbol{m}_\lambda \rfloor, \lfloor \alpha \boldsymbol{m}_\lambda \rfloor]\!]$ with probability tending to 1 (use Lemma 3.10.1 (i) and space/time stationarity). Thus the match falling at time t_1 destroys nothing outside $[\![-\lfloor \alpha \boldsymbol{m}_\lambda \rfloor, \lfloor \alpha \boldsymbol{m}_\lambda \rfloor]\!]$.

Second fire. — Since $t_2 - t_0 > 1$, at least one seed has fallen, during $[t_0, t_2)$ on each site of $[\![\lfloor \alpha \boldsymbol{m}_\lambda \rfloor + 1, \boldsymbol{m}_\lambda]\!]$ with probability tending to 1 (use Lemma 3.10.1 (ii) and space/time stationarity). Thus the fire at time t_2 destroys completely the zone $[\![\lfloor \alpha \boldsymbol{m}_\lambda \rfloor + 1, \boldsymbol{m}_\lambda]\!]$. Furthermore, it does not affect $[\![-\boldsymbol{m}_\lambda, -\lfloor \alpha \boldsymbol{m}_\lambda \rfloor - 1]\!]$ with probability tending to 1, because $t_2 < t_1 + \Theta_{t_0,t_1}^\lambda$ with probability tending to 1 ($\Theta_{t_0,t_1}^\lambda \simeq t_1 - t_0$ by Lemma 3.10.2 and $t_2 - t_1 < t_1 - t_0$ by assumption) and because there is an empty site in $C \subset [\![-\lfloor \alpha \boldsymbol{m}_\lambda \rfloor, \lfloor \alpha \boldsymbol{m}_\lambda \rfloor]\!]$ during $[t_1, t_1 + \Theta_{t_0,t_1}^\lambda]$ (by definition of Θ_{t_0,t_1}^λ).

Third fire. — Since $t_3 - t_2 < 1$, the probability that there is a vacant site in $[\![\lfloor \alpha \boldsymbol{m}_\lambda \rfloor + 1, \lfloor 2\alpha \boldsymbol{m}_\lambda \rfloor]\!]$ at time t_3 tends to 1 as $\lambda \to 0$ (use Lemma 3.10.1 (i) and space/time stationarity).

Next, all the sites of $[\![-\boldsymbol{m}_\lambda, -\lfloor \alpha \boldsymbol{m}_\lambda \rfloor - 1]\!]$ are occupied at time t_3- with probability tending to 1 (because they have not been affected by a fire during $[t_0, t_3)$ and because $t_3 - t_0 > t_2 - t_0 > 1$, see Lemma 3.10.1 (ii)). Thus the fire at time t_3 destroys the zone $[\![-\boldsymbol{m}_\lambda, -\lfloor \alpha \boldsymbol{m}_\lambda \rfloor - 1]\!]$ and does not affect the zone $[\![\lfloor 2\alpha \boldsymbol{m}_\lambda \rfloor, \boldsymbol{m}_\lambda]\!]$.

Last fire and conclusion. — Iterating the procedure, we see that with a probability tending to 1 as $\lambda \to 0$, the fire at time t_K destroys the zone

$$\Big[\!\Big[\lfloor (\tfrac{1}{2} K\alpha) \boldsymbol{m}_\lambda \rfloor, \boldsymbol{m}_\lambda \Big]\!\Big] = \Big[\!\Big[\lfloor \tfrac{1}{2} \boldsymbol{m}_\lambda \rfloor, \boldsymbol{m}_\lambda \Big]\!\Big].$$

Then one easily concludes: since $0 < s - t_K < 1$, the probability that there is at least one site in $[\![\lfloor \frac{1}{2} \boldsymbol{m}_\lambda \rfloor, \lfloor \frac{2}{3} \boldsymbol{m}_\lambda \rfloor]\!]$ with no seed falling during $[t_K, s]$ tends to 1 (by Lemma 3.10.1 (i)), the probability that there is at least one site in $[\![\lfloor \frac{2}{3} \boldsymbol{m}_\lambda \rfloor + 1, \lfloor \frac{5}{6} \boldsymbol{m}_\lambda \rfloor]\!]$ with at least one seed falling during $[t_K, s]$ tends to 1 (by Lemma 3.10.1 (v)), and the probability that there is at least one site in $[\![\lfloor \frac{5}{6} \boldsymbol{m}_\lambda \rfloor + 1, \boldsymbol{m}_\lambda]\!]$ with no seed falling during $[t_K, s]$ tends to 1 (by Lemma 3.10.1 (i)). □

3.10.4. The coupling. — We are going to construct a coupling between the $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$-process (on the time interval $[0, \boldsymbol{a}_\lambda T]$) and the $\mathrm{LFF}_A(\infty)$-process (on $[0, T]$).

First, we couple a family of i.i.d. $\mathrm{SR}(\mu_M^\lambda)$-processes $(N_t^{M,\lambda}(i))_{t \geq 0, i \in \mathbb{Z}}$ with a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x)$ on $[0, T] \times [-A, A]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x$ as in Proposition 3.2.1.

We call $n := \pi_M([0, T] \times [-A, A])$ and we consider the marks $(T_q, X_q)_{q=1,\dots,n}$ of π_M ordered in such a way that $0 < T_1 < \cdots < T_n < T$.

Next, we introduce some i.i.d. families of i.i.d. $\mathrm{SR}(\mu_S)$-processes $(N_t^{S,q}(i))_{t \geq 0, i \in \mathbb{Z}}$, for $q = 0, 1, \dots$, independent of π_M and $(N_t^{M,\lambda}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Then we build a family of i.i.d. $\mathrm{SR}(\mu_S)$-processes (independent of $(N_t^{M,\lambda}(i))_{t \geq 0, i \in \mathbb{Z}}$ and π_M) as follows.

▷ For $q \in \{1, \dots, n\}$, recall that $(X_q)_\lambda = [\![\lfloor \boldsymbol{n}_\lambda X_q - \boldsymbol{m}_\lambda \rfloor, \lfloor \boldsymbol{n}_\lambda X_q + \boldsymbol{m}_\lambda \rfloor]\!])$, and for all $i \in (X_q)_\lambda$, set

$$\left(N_t^{S,\lambda}(i)\right)_{t\geq 0} = \left(N_t^{S,q}(i - \lfloor \boldsymbol{n}_\lambda X_q \rfloor)\right)_{t\geq 0}.$$

(In the rare case where i belongs to $(X_q)_\lambda \cap (X_r)_\lambda$ for some $q < r$, set e.g. $(N_t^{S,\lambda}(i))_{t\geq 0} = (N_t^{S,q}(i - \lfloor \boldsymbol{n}_\lambda X_q \rfloor))_{t\geq 0}$. This will occur with a very small probability, so that this choice is not important).

▷ For all other $i \in \mathbb{Z}$ set

$$\left(N_t^{S,\lambda}(i)\right)_{t\geq 0} = \left(N_t^{S,0}(i)\right)_{t\geq 0}.$$

The $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in I_A^\lambda}$ is built from the seed processes $(N_t^{S,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$ and from the match processes $(N_t^{M,\lambda}(i))_{t\geq 0, i\in\mathbb{Z}}$.

Finally, we build the $\mathrm{LFF}_A(\infty)$-process $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T], x\in[-A,A]}$ from π_M (use Algorithm 3.9.3 replacing $\Theta^{k+1}_{T_{k+1} - Z_{T_{k+1}-}(X_{k+1}), T_{k+1}}$ by $Z_{T_{k+1}-}(X_{k+1})$) and observe that it is independent of $(N_t^{S,q}(i))_{t\in[0,T], i\in\mathbb{Z}, q\geq 0}$.

3.10.5. A favorable event. — First, we know from Proposition 3.2.1 that

$$\Omega_{A,T}^M(\lambda) := \left\{\forall t \in [0,T],\ \forall i \in I_A^\lambda,\ \Delta N^{M,\lambda}_{\boldsymbol{a}_\lambda t}(i) \neq 0 \text{ iff } \pi_M(\{t\} \times i_\lambda) \neq 0\right\}$$

satisfies $\lim_{\lambda\to 0} \Pr[\Omega_{A,T}^M(\lambda)] = 1$. Next, we recall that the marks of π_M are called (T_1, X_1), ..., (T_n, X_n) and are ordered chronologically. We introduce

$$\mathcal{T}_M = \{0, T_1, \dots, T_n\}, \quad \mathcal{B}_M = \{X_1, \dots, X_n\},$$

as well as the set $\mathcal{C}_M$ of connected components of $[-A, A] \setminus \mathcal{B}_M$ (sometimes referred to as *cells*). We also introduce

$$\mathcal{S}_M = \{2t - s : s, t \in \mathcal{T}_M, s < t\},$$

which has to be seen as the possible limit values of $t + \Theta_{s,t}^\lambda \simeq t + t - s$, recall Lemma 3.10.2.

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \Big\{ \min_{\substack{s,t\in\mathcal{T}_M\cup\mathcal{S}_M\\ s\neq t}} |t-s| \geq \alpha,\ \min_{\substack{s,t\in\mathcal{T}_M\cup\mathcal{S}_M\\ s\neq t}} |t-(s+1)| \geq \alpha,\ \min_{\substack{x,y\in\mathcal{B}_M\cup\{-A,A\}\\ x\neq y}} |x-y| \geq \alpha \Big\},$$

which clearly satisfies $\lim_{\alpha\to 0} \Pr[\Omega_M(\alpha)] = 1$. As in the case $\beta = BS$, for any given $\alpha > 0$, there is $\lambda_\alpha > 0$ such that for all $\lambda \in (0, \lambda_\alpha)$, on $\Omega_M(\alpha)$,

▷ for all $x, y \in \mathcal{B}_M \cup \{-A, A\}$, with $x \neq y$, $x_\lambda \cap y_\lambda = \varnothing$;

▷ the family $\{c_\lambda, c \in \mathcal{C}_M\} \cup \{x_\lambda, x \in \mathcal{B}_M\}$ is a partition of I_A^λ (recall (3.10.1) and (3.10.2)).

Let $q \in \{1, \dots, n\}$. We call $\mathcal{U}_q$ the set of all possible $\mathcal{R} = (\varepsilon, t_0, \dots, t_K; s)$ satisfying (PP) with $\varepsilon \in \{-1, 1\}$, with $\{t_0, \dots, t_K, s\} \subset \mathcal{T}_M$ and with $t_1 - t_0 > t_2 - t_1$. We introduce, for $q = 1, \dots, n$ and $\mathcal{R} \in \mathcal{U}_q$, the event $\Omega^{S,q}_{\mathcal{R}}(\lambda)$ defined as in subsection 3.10.3 with the SR(μ_S)-processes $(N^{S,q}_t(i))_{t\geq 0, i\in\mathbb{Z}}$. Then we put

$$\Omega^S_1(\lambda) = \bigcap_{q=1}^{n} \bigcap_{\mathcal{R}\in\mathcal{U}_q} \Omega^{S,q}_{\mathcal{R}}(\lambda),$$

which satisfies $\lim_{\lambda\to 0} \Pr\left(\Omega^S_1(\lambda)\right) = 1$ thanks to Lemma 3.10.3.

We also consider the event $\Omega^S_2(\lambda)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\boldsymbol{m}_\lambda < i_1 < i_2 < -\tfrac{1}{2}\boldsymbol{m}_\lambda < i_3 < 0 < i_4 < \tfrac{1}{2}\boldsymbol{m}_\lambda < i_5 < i_6 < \boldsymbol{m}_\lambda$$

such that

▷ for $j = 1, 3, 4, 6$, $N^{S,q}_{\boldsymbol{a}_\lambda t_2}(i_j) - N^{S,q}_{\boldsymbol{a}_\lambda t_1}(i_j) = 0$,

▷ for $j = 2, 5$, $N^{S,q}_{\boldsymbol{a}_\lambda t_2}(i_j) - N^{S,q}_{\boldsymbol{a}_\lambda t_1}(i_j) > 0$.

There holds $\lim_{\lambda\to 0} \Pr\left(\Omega^S_2(\lambda)\right) = 1$. Indeed, it suffices to prove that almost surely, $\lim_{\lambda\to 0} \Pr\left(\Omega^S_2(\lambda)|\pi_M\right) = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N^{S,q}_t(i))_{t\geq 0, i\in\mathbb{Z}}$, it suffices to work with a fixed q in $\{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$. The result then follows from Lemma 3.10.1 (i)-(v) together with space/time stationarity.

Next we introduce the event $\Omega^S_3(\lambda)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M$,

▷ for all $c \in \mathcal{C}_M$, if $0 < t_2 - t_1 < 1$, there is $i \in c_\lambda$ with $N^{S,\lambda}_{\boldsymbol{a}_\lambda t_2}(i) - N^{S,\lambda}_{\boldsymbol{a}_\lambda t_1}(i) = 0$;

▷ for all $x \in \mathcal{B}_M$, if $0 < t_2 - t_1 < 1$, there is $i \in x_\lambda$ with $N^{S,\lambda}_{\boldsymbol{a}_\lambda t_2}(i) - N^{S,\lambda}_{\boldsymbol{a}_\lambda t_1}(i) = 0$;

▷ if $t_2 - t_1 > 1$, for all $c \in \mathcal{C}_M$, for all $i \in c_\lambda$, $N^{S,\lambda}_{\boldsymbol{a}_\lambda t_2}(i) - N^{S,\lambda}_{\boldsymbol{a}_\lambda t_1}(i) > 0$;

▷ if $t_2 - t_1 > 1$, for all $x \in \mathcal{B}_M$, for all $i \in x_\lambda$, $N^{S,\lambda}_{\boldsymbol{a}_\lambda t_2}(i) - N^{S,\lambda}_{\boldsymbol{a}_\lambda t_1}(i) > 0$.

There holds $\lim_{\lambda\to 0} \Pr\left(\Omega^S_3(\lambda)\right) = 1$. As previously, it suffices to work with some fixed t_1, t_2, $x \in (-A, A)$ and $c = (a, b) \subset (-A, A)$. Observing that $|x_\lambda| \sim 2\boldsymbol{m}_\lambda$ and that $|c_\lambda| \sim (b - a)\boldsymbol{n}_\lambda$, Lemma 3.10.1 and space/time stationarity shows the result.

We also need $\Omega^S_4(\gamma, \lambda)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$, for all $t_0, t_1 \in \mathcal{T}_M$ with $t_0 < t_1 < t_0 + 1$, there holds $|\Theta^{q,\lambda}_{t_0,t_1} - (t_1 - t_0)| < \gamma$. Here $\Theta^{q,\lambda}_{t_0,t_1}$ is defined as in Lemma 3.10.2 with the seed processes family $(N^{S,q}_t(i))_{t\geq 0, i\in\mathbb{Z}}$. Lemma 3.10.2 directly implies that for any $\gamma > 0$, $\lim_{\lambda\to 0} \Pr[\Omega^S_4(\gamma, \lambda)] = 1$.

We finally introduce the event

$$\Omega(\alpha, \gamma, \lambda) = \Omega^M_{A,T}(\lambda) \cap \Omega_M(\alpha) \cap \Omega^S_1(\lambda) \cap \Omega^S_2(\lambda) \cap \Omega^S_3(\lambda) \cap \Omega^S_4(\gamma, \lambda).$$

We have shown that for any $\varepsilon > 0$, there exists $\alpha > 0$ such that for any $\gamma > 0$, there holds $\liminf_{\lambda\to 0} \Pr[\Omega(\alpha, \gamma, \lambda)] > 1 - \varepsilon$.

3.10.6. Heart of the proof. — We now handle the main part of the proof, following closely subsection 3.9.5.

Consider the $\mathrm{LFF}_A(\infty)$-process. Observe that by construction, we have, for $c \in \mathcal{C}_M$ and $x, y \in c$, $Z_t(x) = Z_t(y)$ for all $t \in [0,T]$, thus we can introduce $Z_t(c)$.

If $x \in \mathcal{B}_M$, it is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_M$ and then we set $Z_t(x_-) = Z_t(c_-)$ and $Z_t(x_+) = Z_t(c_+)$ for all $t \in [0,T]$.

If $x \in (-A,A) \setminus \mathcal{B}_M$, we put $Z_t(x_-) = Z_t(x_+) = Z_t(x)$ for all $t \in [0,T]$.

For $x \in \mathcal{B}_M$ and $t \geq 0$ we set $\widetilde{H}_t(x) = \max(H_t(x), 1 - Z_t(x), 1 - Z_t(x_-), 1 - Z_t(x_+))$.

Actually $Z_t(x)$ always equals either $Z_t(x_-)$ or $Z_t(x_+)$ and these can be distinct only at a point where has occurred a microscopic fire (that is if $x = X_q$ for some $q \in \{1, \dots, n\}$ with $T_q < t$ and $Z_{T_q-}(X_q) < 1$).

For all $x \in (-A,A)$ and $t \in [0,T]$, we put

$$\tau_t(x) = \sup\left\{s \leq t : \; Z_s(x_+) = Z_s(x_-) = Z_s(x) = 0\right\} \in [0,t] \cap \mathcal{T}_M.$$

For $c \in \mathcal{C}_M$ and $t \in [0,T]$, we can define $\tau_t(c)$ as usual.

Observe, using Algorithm 3.9.3, that as when $\beta = BS$,

$$\text{for } x \notin \mathcal{B}_M, \;\; Z_t(x) = \min\left(t - \tau_t(x), 1\right) \text{ for all } t \in [0,T], \tag{3.10.3}$$

$$\text{for } q = 1, \dots, n, \;\; Z_t(X_q) = \min\left(t - \tau_t(X_q), 1\right) \text{ for all } t \in [0, T_q). \tag{3.10.4}$$

We also define for all $t \in [0,T]$, all $c \in \mathcal{C}_M$ and all $x \in (-A,A)$

$$\begin{aligned}
\tau_t^\lambda(c) &= \sup\left\{s \leq t : \; \forall i \in c_\lambda, \eta^\lambda_{\mathbf{a}_\lambda t-}(i) = 1 \text{ and } \eta^\lambda_{\mathbf{a}_\lambda t}(i) = 0\right\} \in [0,t],\\
\rho_t^\lambda(c) &= \sup\left\{s \leq t : \; \exists i \in c_\lambda, \eta^\lambda_{\mathbf{a}_\lambda t-}(i) = 1 \text{ and } \eta^\lambda_{\mathbf{a}_\lambda t}(i) = 0\right\} \in [0,t],\\
\tau_t^\lambda(x) &= \sup\left\{s \leq t : \; \forall i \in x_\lambda, \eta^\lambda_{\mathbf{a}_\lambda t-}(i) = 1 \text{ and } \eta^\lambda_{\mathbf{a}_\lambda t}(i) = 0\right\} \in [0,t]
\end{aligned}$$

with the convention that $\eta^\lambda_{0-}(i) = 1$ for all $i \in I_A^\lambda$. Observe that on $\Omega^M_{A,T}(\lambda)$, we have $\tau_t^\lambda(c), \rho_t^\lambda(c), \tau_t^\lambda(x) \in [0,t] \cap \mathcal{T}_M$ for all $t \in [0,T]$, all $c \in \mathcal{C}_M$ and all $x \in (-A,A)$.

For $t \in [0,T]$, consider the event

$$\Omega_t^\lambda = \left\{\forall s \in [0,t], \forall c \in \mathcal{C}_M, \tau_s^\lambda(c) = \rho_s^\lambda(c) = \tau_s(c) \text{ and } \forall x \in \mathcal{B}_M, \tau_s^\lambda(x) = \tau_s(x)\right\}.$$

Lemma 3.10.4. — *Let $\alpha > \gamma > 0$. For any $\lambda \in (0, \lambda_\alpha)$, Ω_T^λ a.s. holds on $\Omega(\alpha, \gamma, \lambda)$.*

Proof. — We work on $\Omega(\alpha,\gamma,\lambda)$ and assume that $\lambda \in (0, \lambda_\alpha)$. Clearly,

$$\tau_0(x) = \tau_0^\lambda(x) = 0 \quad \text{and} \quad \tau_0(c) = \tau_0^\lambda(c) = \rho_0^\lambda(c) = 0$$

for all $x \in \mathcal{B}_M$, all $c \in \mathcal{C}_M$, so that Ω_0^λ a.s. holds. We will show that for $q = 0, \dots, n-1$, $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}}$. This will prove that $\Omega^\lambda_{T_n}$ holds. The extension to Ω^λ_T will be straightforward (see step 1 below).

We thus fix $q \in \{0, \dots, n-1\}$ and assume $\Omega^\lambda_{T_q}$. We repeatedly use below that on the time interval (T_q, T_{q+1}), there are no fires at all (in $[-A, A]$) for the LFF$_A(BS)$-process and no fires at all (in I^λ_A) during $(\boldsymbol{a}_\lambda T_q, \boldsymbol{a}_\lambda T_{q+1})$ for the FF$_A(\mu_S, \mu^\lambda_M)$-process (use $\Omega^M_{A,T}(\lambda)$).

Step 1. — Exactly as in the proof of Lemma 3.9.5, step 1, $\Omega^\lambda_{T_q}$ implies $\Omega^\lambda_{T_{q+1}-}$.

Step 2. — Exactly as in the proof of Lemma 3.9.5, step 2, we observe that for $c \in \mathcal{C}_M$, on $\Omega^\lambda_{T_{q+1}-}$, there holds, for all $i \in c_\lambda$,

$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i) = \min\left(N^{S,0}_{\boldsymbol{a}_\lambda T_{q+1}-}(i) - N^{S,0}_{\boldsymbol{a}_\lambda \tau_{T_q}(c)}(i), 1\right). \tag{3.10.5}$$

Step 3. — If $Z_{T_{q+1}-}(X_{q+1}) < 1$, there exist $j_1, j_2, j_3, j_4 \in (X_{q+1})_\lambda$ such that

$$j_1 < j_2 < \lfloor \boldsymbol{n}_\lambda X_{q+1} \rfloor < j_3 < j_4,$$
$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(j_2) = \eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(j_3) = 0,$$
$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(j_1) = \eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(j_4) = 1.$$

The proof is the same as Lemma 3.9.5, step 3.

Step 4. — Next we check that if $Z_{T_{q+1}-}(c) = 1$ for some $c \in \mathcal{C}_M$, then

$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i) = 1 \quad \text{for all } i \in c_\lambda.$$

Recalling (3.10.3), we see that $Z_{T_{q+1}-}(c) = 1$ implies that $T_{q+1} - \tau_{T_{q+1}-}(c) \geq 1$ and thus $T_{q+1} - \tau_{T_q}(c) \geq 1$ by step 1. Using $\Omega_M(\alpha)$ and that $T_{q+1}, \tau_{T_q}(c) \in \mathcal{T}_M$, we deduce that $T_{q+1} - \tau_{T_q}(c) > 1$. Using (3.10.5), we conclude that for all $i \in c_\lambda$,

$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i) = \min\left(N^{S,0}_{\boldsymbol{a}_\lambda T_{q+1}-}(i) - N^{S,0}_{\boldsymbol{a}_\lambda \tau_{T_q}(c)(i)}, 1\right) = 1$$

by $\Omega^S_3(\lambda)$.

Step 5. — We now prove that if $\widetilde{H}_{T_{q+1}-}(x) = 0$ for some $x \in \mathcal{B}_M$, then

$$\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i) = 1 \quad \text{for all } i \in x_\lambda.$$

Preliminary considerations. — Let $k \in \{1, \dots, n\}$ such that $x = X_k$, which is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_M$. We know that $\widetilde{H}_{T_{q+1}-}(x) = 0$, whence $H_{T_{q+1}-}(x) = 0$ and $Z_{T_{q+1}-}(x) = Z_{T_{q+1}-}(c_+) = Z_{T_{q+1}-}(c_-) = 1$. This implies that $T_{q+1} \geq 1$ (because $Z_t(x) = t$ for all $t < 1$ and all $x \in [-A, A]$) and thus $T_{q+1} \geq 1 + \alpha$ due to $\Omega_M(\alpha)$.

No fire has concerned $(c_-)_\lambda$ during $(\boldsymbol{a}_\lambda \rho^\lambda_{T_{q+1}-}(c_-), \boldsymbol{a}_\lambda T_{q+1})$ (by definition of $\rho^\lambda_{T_{q+1}-}(c_-)$). But step 1 implies that $\rho^\lambda_{T_{q+1}-}(c_-) = \tau_{T_{q+1}-}(c_-) \leq T_{q+1} - 1$ because $Z_{T_{q+1}-}(c_-) = 1$, see (3.10.3). Recalling $\Omega_M(\alpha)$, we deduce that

$$\rho^\lambda_{T_{q+1}-}(c_-) < T_{q+1} - 1 - \alpha.$$

Using a similar argument for c_+, we conclude that no match falling outside $(X_k)_\lambda$ can affect $(X_k)_\lambda$ during $(\boldsymbol{a}_\lambda(T_{q+1} - 1 - \alpha), \boldsymbol{a}_\lambda T_{q+1})$ (because to affect $(X_k)_\lambda$, a match falling outside $(X_k)_\lambda$ needs to cross c_- or c_+).

Case 1. – First assume that $k \geq q+1$. Then we know that no fire has fallen on $(X_k)_\lambda$ during $[0, \boldsymbol{a}_\lambda T_{q+1})$. Due to the preliminary considerations, we deduce that no fire at all has concerned $(X_k)_\lambda$ during $(\boldsymbol{a}_\lambda(T_{q+1}-1-\alpha), \boldsymbol{a}_\lambda T_{q+1})$. Using $\Omega_3^S(\lambda)$, we conclude that $(X_k)_\lambda$ is completely occupied at time $\boldsymbol{a}_\lambda T_{q+1}-$.

Case 2. – Assume that $k \leq q$ and $Z_{T_k-}(X_k)=1$, so that there already has been a macroscopic fire in $(X_k)_\lambda$ (at time $\boldsymbol{a}_\lambda T_k$). Since $Z_{T_k}(X_k)=0$ and $Z_{T_{q+1}-}(X_k)=1$, we deduce that $T_{q+1}-T_k \geq 1$, whence $T_{q+1}-T_k \geq 1+\alpha$ as usual. We conclude as in Case 1 that no fire at all has concerned $(X_k)_\lambda$ during $(T_S(T_{q+1}-1-\alpha), T_S T_{q+1})$, which implies the claim by $\Omega_3^S(\lambda)$.

Case 3. – Assume that $k \leq q$ and $Z_{T_k-}(X_k) < 1$ and $T_{q+1}-T_k \geq 1$, whence $T_{q+1}-T_k \geq 1+\alpha$ due to $\Omega_M(\alpha)$. Then there already has been a microscopic fire in $(X_k)_\lambda$ (at time $\boldsymbol{a}_\lambda T_k$). But there are no fire in $(X_k)_\lambda$ during $(\boldsymbol{a}_\lambda T_k, \boldsymbol{a}_\lambda T_{q+1}) \supset (T_S(T_{q+1}-1-\alpha), T_S T_{q+1})$ and we conclude as in case 2.

Case 4. – Assume finally that $k \leq q$ and $Z_{T_k-}(X_k) < 1$ and $T_{q+1}-T_k < 1$, whence $T_{q+1}-T_k \leq 1-\alpha$ due to $\Omega_M(\alpha)$. There has been a microscopic fire in $(X_k)_\lambda$ (at time $\boldsymbol{a}_\lambda T_k$). Since $H_{T_{q+1}-}(X_k)=0$, we deduce (see Algorithm 3.9.3 and recall that $\Theta^k_{T_k-Z_{T_k-}(X_k),T_k}$ is replaced by $Z_{T_k-}(X_k)$) that $T_k+Z_{T_k-}(X_k) \leq T_{q+1}$, whence $T_k+Z_{T_k-}(X_k) \leq T_{q+1}-\alpha$ by $\Omega_M(\alpha)$ ($\mathcal{S}_M$ was designed for that purpose).

Consider the zone $C = C(\eta^\lambda_{\boldsymbol{a}_\lambda T_k-}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$ destroyed by the match falling at time $\boldsymbol{a}_\lambda T_k$. This zone is completely occupied at time $\boldsymbol{a}_\lambda(T_k+\Theta^{k,\lambda}_{T_k-Z_{T_k-}(X_k),T_k})$: this follows from the definition of $\Theta^{k,\lambda}_{T_k-Z_{T_k-}(X_k),T_k}$, see Lemma 3.9.1 and from the preliminary considerations. Using $\Omega_4^S(\gamma,\lambda)$, we deduce that

$$T_k+\Theta^{k,\lambda}_{T_k-Z_{T_k-}(X_k),T_k} \leq T_k+Z_{T_k-}(X_k)+\gamma < T_{q+1},$$

since $\gamma < \alpha$. Hence C is completely occupied at time $\boldsymbol{a}_\lambda T_{q+1}-$.

Consider now $i \in (X_k)_\lambda \setminus C$. Then i has not been killed by the fire starting at $\lfloor \boldsymbol{n}_\lambda X_k \rfloor$. Thus i cannot have been killed during $(\boldsymbol{a}_\lambda(T_{q+1}-1-\alpha), \boldsymbol{a}_\lambda T_{q+1})$ (due to the preliminary considerations) and we conclude, using $\Omega_3^S(\lambda)$, that i is occupied at time $\boldsymbol{a}_\lambda T_{q+1}-$. This implies the claim.

Step 6. — Let us now prove that if $\widetilde{H}_{T_{q+1}-}(x) > 0$ and $Z_{T_{q+1}-}(x_+)=1$ for some $x \in \mathcal{B}_M$, there are $i_1, i_2 \in x_\lambda$ such that $i_1 < i_2$ and $\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i_1)=1$, $\eta^\lambda_{\boldsymbol{a}_\lambda T_{q+1}-}(i_2)=0$. Recall that x is at the boundary of two cells c_-, c_+.

We have either $H_{T_{q+1}-}(x) > 0$ or $Z_{T_{q+1}-}(c_-) < 1$ (because $Z_{T_{q+1}-}(c_+)=1$ by assumption). Clearly, $x = X_k$ for some $k \leq q$, with $Z_{T_k-}(X_k) < 1$ (else, we would have $H_t(x)=0$ and $Z_t(c_-)=Z_t(c_+)$ for all $t \in [0, T_{q+1})$). Thus, recalling (3.10.4),

$$T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = \tau^\lambda_{T_k-}(X_k),$$

so that $(X_k)_\lambda$ is completely empty at time $\boldsymbol{a}_\lambda(T_k - Z_{T_k-}(X_k))$.

Case 1. – Assume first that $H_{T_{q+1}-}(x) > 0$. Then by construction, see Algorithm 3.9.3 (with $\Theta_{T_k - Z_{T_k-}(X_k), T_k}$ replaced by $Z_{T_k-}(X_k)$), there holds

$$T_k + Z_{T_k-}(X_k) > T_{q+1} > T_k,$$

whence by $\Omega_M(\alpha)$, $T_k + Z_{T_k-}(X_k) > T_{q+1} + \alpha > T_k + 2\alpha$.

Consider $C = C(\eta^\lambda_{\boldsymbol{a}_\lambda T_k-}, \lfloor \boldsymbol{n}_\lambda X_k \rfloor)$. By $\Omega_2^S(\lambda)$, we have

$$C \subset [\![\lfloor \boldsymbol{n}_\lambda X_k - \tfrac{1}{2}\boldsymbol{m}_\lambda \rfloor, \lfloor \boldsymbol{n}_\lambda X_k + \tfrac{1}{2}\boldsymbol{m}_\lambda \rfloor]\!]$$

(because $(X_k)_\lambda$ is completely empty at time $\boldsymbol{a}_\lambda(T_k - Z_{T_k-}(X_k))$, because $T_k - Z_{T_k-}(X_k)$ and T_k belong to $\mathcal{T}_M$ and because $0 < Z_{T_k-}(X_k) < 1$).

The component C is destroyed at time $T_S T_k$. By Definition of $\Theta^{k,\lambda}_{T_k - Z_{T_k-}(X_k), T_k}$, see Lemma 3.10.2, we deduce that C is not completely occupied at time

$$\boldsymbol{a}_\lambda (T_k + \Theta^{k,\lambda}_{T_k - Z_{T_k-}(X_k), T_k}).$$

But by $\Omega_4^S(\gamma, \lambda)$ we see that $\Theta^{k,\lambda}_{T_k - Z_{T_k-}(X_k), T_k} \geq Z_{T_k-}(X_k) - \gamma$, whence

$$T_k + \Theta^{k,\lambda}_{T_k - Z_{T_k-}(X_k), T_k} \geq T_k + Z_{T_k-}(X_k) - \gamma > T_{q+1}$$

since $\gamma < \alpha$. All this implies that C is not completely occupied at time $\boldsymbol{a}_\lambda T_{q+1}-$.

Finally, using again $\Omega_2^S(\lambda)$ there is necessarily (at least) one seed falling on a site in $[\![\lfloor \boldsymbol{n}_\lambda X_k - \boldsymbol{m}_\lambda + 1 \rfloor, \lfloor \boldsymbol{n}_\lambda X_k - \tfrac{1}{2}\boldsymbol{m}_\lambda - 1 \rfloor]\!] \subset (X_k)_\lambda$ during $(\boldsymbol{a}_\lambda T_q, \boldsymbol{a}_\lambda T_{q+1})$. This shows the result.

Case 2. – Assume next that $H_{T_{q+1}-}(x) = 0$ and $T_{q+1} - [T_k - Z_{T_k-}(X_k)] < 1$. Recall that $(X_k)_\lambda$ is completely empty at time $\boldsymbol{a}_\lambda(T_k - Z_{T_k-}(X_k))$. Since $T_k - Z_{T_k-}(X_k)$ and T_{q+1} belong to $\mathcal{T}_M$ and since their difference is smaller than 1 by assumption, $\Omega_2^S(\lambda)$ guarantees us the existence of $i_1 < i_2 < i_3$, all in $(X_k)_\lambda$, such that (at least) one seed falls on i_2 and no seed fall on i_1 nor on i_3 during $(\boldsymbol{a}_\lambda(T_k - Z_{T_k-}(X_k)), \boldsymbol{a}_\lambda T_{q+1})$. One easily concludes that i_2 is occupied and i_3 is vacant at time $\boldsymbol{a}_\lambda T_{q+1}-$, as desired.

Case 3. – Assume finally that $H_{T_{q+1}-}(x) = 0$ and $T_{q+1} - [T_k - Z_{T_k-}(X_k)] \geq 1$, whence $T_{q+1} - [T_k - Z_{T_k-}(X_k)] \geq 1 + \alpha$ by $\Omega_M(\alpha)$. Since $H_{T_{q+1}-}(x) = 0$, there holds $Z_{T_{q+1}-}(c_-) < 1 = Z_{T_{q+1}-}(c_+)$ and $T_k + Z_{T_k-}(X_k) \leq T_{q+1}$, so that $T_k + Z_{T_k-}(X_k) \leq T_{q+1} - \alpha$.

We aim to use the event $\Omega_1^S(\lambda)$. We introduce

$$t_0 = T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = \tau^\lambda_{T_k-}(X_k).$$

Observe that $\tau_{T_k-}(c_-) = \tau_{T_k-}(c_+) = \tau_{T_k-}(x)$ because there has been no fire (exactly) at x during $[0, T_k)$. Thus

$$Z_{t_0-}(x) = Z_{t_0-}(x_-) = Z_{t_0-}(x_+) = 1 \quad \text{and} \quad Z_{t_0}(x) = Z_{t_0}(c_-) = Z_{t_0}(c_+) = 0.$$

Set now $t_1 = T_k$ and $s = T_{q+1}$. Observe that $0 < t_1 - t_0 < 1$. Necessarily, $Z_t(c_-)$ has jumped to 0 at least one time between t_0 and $T_{q+1}-$ (else, one would have $Z_{T_{q+1}-}(c_-) = 1$, since $T_{q+1} - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$, and since for all $t \in [t_0, t_0 + 1)$, $Z_t(c_-) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \cdots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_t(c_-), Z_t(c_+))$ during $(t_0 + 1, s)$. We also put $\varepsilon = 1$ if t_2 is a jump of $Z_t(c_+)$ and $\varepsilon = -1$ else. Then we prove exactly as in Lemma 3.9.5, step 6, case 3 that $\mathcal{R} = \{\varepsilon, t_0, \ldots, t_K; s\}$ necessarily satisfies the condition (PP).

Next, $t_2 - t_1 < Z_{T_k-}(X_k) = t_1 - t_0$, because else, we would have $H_{t_2-}(X_k) = 0$ and thus the fire destroying c_+ (or c_-) at time t_2 would also destroy c_- (or c_+), we thus would have $Z_{t_2}(c_+) = Z_{t_2}(c_-) = 0$, so that $Z_t(c_+)$ and $Z_t(c_-)$ would remain equal forever.

Finally, we check as in Lemma 3.9.5, step 6, case 3 that

$$\big(\eta^\lambda_{\boldsymbol{a}_\lambda t}(i)\big)_{t \geq t_0, i \in x_\lambda} = \big(\zeta^{\lambda,\mathcal{R},k}_t(i + \lfloor \boldsymbol{n}_\lambda x \rfloor)\big)_{t \geq t_0, i \in x_\lambda},$$

this last process being built upon the family $(N^{S,k}_t(i))_{t \geq t_0, i \in x_\lambda}$ as in subsection 3.10.3.

We thus can use $\Omega^S_1(\lambda)$ and conclude that there are some sites $i_1 < i_2$ in x_λ with $\eta^\lambda_{T_S T_{q+1}-}(i_1) = 1$ and $\eta^\lambda_{T_S T_{q+1}-}(i_2) = 0$ as desired.

Step 7. — The conclusion follows from the previous steps exactly as in the proof of Lemma 3.9.5, step 7: it suffices to replace everywhere T_S by $\boldsymbol{a}_\lambda$. □

3.10.7. Conclusion. — To achieve the proof, we will need the following result.

Lemma 3.10.5. — *Let $(N^S_t(i))_{t \geq 0, i \in \mathbb{Z}}$ be a family of i.i.d.* SR(μ_S)*-processes, and define*

$$\zeta^\lambda_t(i) = \min\big(N^S_{\boldsymbol{a}_\lambda t}(i), 1\big).$$

(i) *Put $K^\lambda_t = (2\boldsymbol{m}_\lambda + 1)^{-1} |\{i \in [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!] : \zeta^\lambda_t(i) > 0\}|$ and*

$$U^\lambda_t = \Big(\frac{\psi_S(K^\lambda_t)}{\boldsymbol{a}_\lambda}\Big) \wedge 1,$$

recall notation 2.1.5*. Then for any $\varepsilon > 0$, any $T > 0$,*

$$\lim_{\lambda \to 0} \Pr\Big[\sup_{[0,T]} |U^\lambda_t - t \wedge 1| > \varepsilon\Big] = 0.$$

(ii) *Put also $C^\lambda_t = C(\zeta^\lambda_t, 0)$ and define*

$$V^\lambda_t = \big(\boldsymbol{a}_\lambda^{-1} \psi_S(1 - 1/|C^\lambda_t|) \mathbb{1}_{\{|C^\lambda_t| > 0\}}\big) \wedge 1.$$

Then for any $\varepsilon > 0$, for all $t \in [0, 1)$,

$$\lim_{\lambda \to 0} \Pr\big[C^\lambda_t \subset [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!], |V^\lambda_t - t| < \varepsilon\big] = 1.$$

Proof. — We split the proof into three steps.

Step 1. — Here we show that for $t \geq 0$ fixed,

$$\lim_{\lambda \to 0} \Pr\big[|U^\lambda_t - t \wedge 1| > \varepsilon\big] = 0.$$

Case 1. – Assume first that $t \geq 1$. Then Lemma 3.10.1 (ii) implies that $\lim_{\lambda\to 0} \Pr[K_t^\lambda = 1] = 1$. But $K_t^\lambda = 1$ implies that $U_t^\lambda = [\psi_S(1)/\boldsymbol{a}_\lambda] \wedge 1 = 1$ (because $\psi_S(1) = \infty$).

Case 2. – Assume next that $t < 1$. Then the random variable $X_t^\lambda = (2\boldsymbol{m}_\lambda + 1)K_t^\lambda$ has a binomial distribution with parameters $2\boldsymbol{m}_\lambda + 1$ and $\nu_S((0, \boldsymbol{a}_\lambda t))$. Let $\varepsilon \in (0, t)$ be fixed. Then, using Bienaymé-Chebyshev's inequality,

$$\begin{aligned}
&\Pr\left[K_t^\lambda \leq \nu_S((0, \boldsymbol{a}_\lambda(t-\varepsilon)))\right]\\
&\quad = \Pr\left[X_t^\lambda \leq (2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda(t-\varepsilon)))\right]\\
&\quad \leq \Pr\left[|X_t^\lambda - (2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda t))| \geq (2\boldsymbol{m}_\lambda + 1)\nu_S((\boldsymbol{a}_\lambda(t-\varepsilon), \boldsymbol{a}_\lambda t))\right]\\
&\quad \leq \frac{(2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda t))\nu_S((\boldsymbol{a}_\lambda t, \infty))}{(2\boldsymbol{m}_\lambda + 1)^2\nu_S^2((\boldsymbol{a}_\lambda(t-\varepsilon), \boldsymbol{a}_\lambda t))} \leq \frac{\nu_S((\boldsymbol{a}_\lambda t, \infty))}{(2\boldsymbol{m}_\lambda + 1)\nu_S^2((\boldsymbol{a}_\lambda(t-\varepsilon), \boldsymbol{a}_\lambda t))}.
\end{aligned}$$

This last quantity tends to 0. Indeed, $(H_S(\infty))$ implies that

$$\nu_S\big((\boldsymbol{a}_\lambda(t-\varepsilon), \boldsymbol{a}_\lambda t)\big) \sim \nu_S\big((\boldsymbol{a}_\lambda(t-\varepsilon), \infty)\big) \geq \nu_S\big((\boldsymbol{a}_\lambda t, \infty)\big)$$

and it suffices to use that $\boldsymbol{m}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty)) \to \infty$ by (2.2.4), since $t < 1$.

By the same way, for $\varepsilon > 0$,

$$\begin{aligned}
&\Pr\left[K_t^\lambda \geq \nu_S((0, \boldsymbol{a}_\lambda(t+\varepsilon)))\right]\\
&\quad = \Pr\left[X_t^\lambda \geq (2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda(t+\varepsilon)))\right]\\
&\quad \leq \Pr\left[|X_t^\lambda - (2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda t))| \geq (2\boldsymbol{m}_\lambda + 1)\nu_S((\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\varepsilon)))\right]\\
&\quad \leq \frac{(2\boldsymbol{m}_\lambda + 1)\nu_S((0, \boldsymbol{a}_\lambda t))\nu_S((\boldsymbol{a}_\lambda t, \infty))}{(2\boldsymbol{m}_\lambda + 1)^2\nu_S^2((\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\varepsilon)))} \leq \frac{\nu_S((\boldsymbol{a}_\lambda t, \infty))}{(2\boldsymbol{m}_\lambda + 1)\nu_S^2((\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\varepsilon)))},
\end{aligned}$$

which also tends to 0, because $(H_S(\infty))$ implies that

$$\nu_S\big((\boldsymbol{a}_\lambda t, \boldsymbol{a}_\lambda(t+\varepsilon))\big) \sim \nu_S\big((\boldsymbol{a}_\lambda t, \infty)\big),$$

and because $\boldsymbol{m}_\lambda \nu_S((\boldsymbol{a}_\lambda t, \infty)) \to \infty$, since $t < 1$.

To conclude the step it suffices to note that for $0 < t - \varepsilon < t < t + \varepsilon < 1$, $K_t^\lambda \in (\nu_S((0, \boldsymbol{a}_\lambda(t-\varepsilon))), \nu_S((0, \boldsymbol{a}_\lambda(t+\varepsilon))))$ implies that $U_t^\lambda \in (t-\varepsilon, t+\varepsilon)$ by definition of ψ_S.

Step 2. — Using a well suited version of the Dini theorem, we conclude the proof of (i). Indeed, let $\varepsilon > 0$ and consider a subdivision $0 = t_0 < t_1 < \cdots < t_\ell = T$, with $t_{i+1} - t_i < \frac{1}{2}\varepsilon$. Using step 1, we see that

$$\lim_{\lambda\to 0} \Pr\big[\max_{i=0,\dots,\ell} |U_{t_i}^\lambda - t_i \wedge 1| > \tfrac{1}{2}\varepsilon\big] = 0.$$

Observe that $t \mapsto U_t^\lambda$ is a.s. nondecreasing and that $t \mapsto t \wedge 1$ is nondecreasing and Lipschitz continuous. We deduce that

$$\sup_{[0,T]} |U_t^\lambda - t \wedge 1| \leq \frac{1}{2}\varepsilon + \max_{i=0,\dots,\ell} |U_{t_i}^\lambda - t_i \wedge 1|.$$

One immediately concludes.

Step 3. — It remains to prove (ii). Let thus $t < 1$ and $\varepsilon > 0$ be fixed. We can of course assume that $0 < t - \varepsilon < t < t + \varepsilon < 1$.

First, $\lim_{\lambda\to 0} \Pr[C_t^\lambda \subset [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]] = 1$ due to Lemma 3.10.1 (i).

Next, each site is vacant with probability $\nu_S((\boldsymbol{a}_\lambda t, \infty))$. It is thus classical that as $\lambda \to 0$, $\nu_S((\boldsymbol{a}_\lambda t, \infty))|C_t^\lambda|$ goes in law to a random variable X with density $x\,\mathrm{e}^{-x}\mathbb{1}_{x>0}$. Indeed,

- ▷ for Y_δ a geometric random variable with parameter δ, the random variable δY_δ goes in law, as $\delta \to 0$, to an exponentially distributed random variable with parameter 1;
- ▷ $|C_t^\lambda|$ is the sum of two independent geometric random variables, both with parameter $\nu_S((\boldsymbol{a}_\lambda t, \infty))$;
- ▷ $x\,\mathrm{e}^{-x}\mathbb{1}_{x>0}$ is the density of the sum of two independent exponentially distributed random variables with parameter 1.

For $\delta > 0$, consider $0 < a < 1 < b$ such that $\Pr[X \in (a, b))] \geq 1 - \delta$. Then

$$\lim_{\lambda\to 0} \Pr\left[|C_t^\lambda| \in (a/\nu_S((\boldsymbol{a}_\lambda t, \infty)), b/\nu_S((\boldsymbol{a}_\lambda t, \infty))\right] \geq 1 - \delta.$$

But due to $(H_S(\infty))$, $|C_t^\lambda| \in (a/\nu_S((\boldsymbol{a}_\lambda t, \infty)), b/\nu_S((\boldsymbol{a}_\lambda t, \infty))$ implies, if λ is small enough, that $|C_t^\lambda| \in (1/\nu_S((\boldsymbol{a}_\lambda (t-\varepsilon), \infty)), 1/\nu_S((\boldsymbol{a}_\lambda (t+\varepsilon), \infty))$, whence finally

$$V_t^\lambda \in \left(\boldsymbol{a}_\lambda^{-1}\psi_S(\nu_S((0, \boldsymbol{a}_\lambda(t-\varepsilon)))), \boldsymbol{a}_\lambda^{-1}\psi_S(\nu_S((0, \boldsymbol{a}_\lambda(t+\varepsilon))))\right) = (t-\varepsilon, t+\varepsilon).$$

We have proved that for all $\delta > 0$, $\liminf_{\lambda\to 0} \Pr[|V_t^\lambda - t| < \varepsilon] \geq 1 - \delta$, which concludes the proof. □

We finally give the

Proof of Proposition 3.8.1 when $\beta = \infty$. — Let us fix $x_0 \in (-A, A)$, $t_0 \in (0, T] \setminus \{1\}$ and $\varepsilon > 0$. We will prove that with our coupling (see subsection 3.9.3), there holds

(a) $\lim_{\lambda\to 0} \Pr\left[\boldsymbol{\delta}(D_{t_0}^\lambda(x_0), D_{t_0}(x_0)) > \varepsilon\right] = 0$;
(b) $\lim_{\lambda\to 0} \Pr\left[\boldsymbol{\delta}_T(D^\lambda(x_0), D(x_0)) > \varepsilon\right] = 0$;
(c) $\lim_{\lambda\to 0} \Pr\left[\sup_{[0,T]} |Z_t^\lambda(x_0) - Z_t(x_0)| \geq \varepsilon\right] = 0$;
(d) $\lim_{\lambda\to 0} \Pr\left[|W_{t_0}^\lambda(x_0) - Z_{t_0}(x_0)| > \varepsilon\right] = 0$, where

$$W_{t_0}^\lambda(x_0) = \Big(\frac{\psi_S(1 - 1/|C_A(\eta_{\boldsymbol{a}_\lambda t_0}^\lambda, \lfloor \boldsymbol{n}_\lambda x_0\rfloor)|)\mathbb{1}_{\{|C_A(\eta_{\boldsymbol{a}_\lambda t_0}^\lambda, \lfloor \boldsymbol{n}_\lambda x_0\rfloor)|>0\}}}{\boldsymbol{a}_\lambda}\Big) \wedge 1.$$

These points will clearly imply the result.

First, we introduce, for $\zeta > 0$, the event $\Omega_{A,T}^{x_0}(\zeta)$ on which $x_0 \notin \bigcup_{q=1}^n [X_q - \zeta, X_q + \zeta]$. The probability of this event obviously tends to 1 as $\zeta \to 0$.

On $\Omega_{A,T}^{x_0}(\zeta)$, we have, for $\lambda > 0$ small enough (say, such that $4\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda < \zeta$), $\lfloor \boldsymbol{n}_\lambda x_0 \rfloor \notin \bigcup_{q=1}^n (X_q)_\lambda$. We then call $c_0 \in \mathcal{C}_M$ the cell containing x_0.

Step 1. — As in the case where $\beta = BS$, (a) implies (b) (the fact that $t_0 = 1$ is excluded in (a) is of course not a problem, because $\{1\}$ is Lebesgue-negligible).

Step 2. — Due to Lemma 3.10.4, we know that if $0 < \gamma < \alpha$, on $\Omega(\alpha,\gamma,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta)$, $\tau^\lambda_t(c_0) = \rho^\lambda_t(c_0) = \tau_t(x_0)$ for all $t \in [0,T]$. This implies that for all $i \in (c_0)_\lambda$, for all $t \in [0,T]$,

$$\eta^\lambda_{T_S t}(i) = \min\big(N^{S,0}_{\boldsymbol{a}_\lambda t}(i) - N^{S,0}_{\boldsymbol{a}_\lambda \tau_t(x_0)}(i), 1\big).$$

We also recall that by construction, $(\tau_t(x_0))_{t\geq 0}$ is independent of $(N^{S,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.

Step 3. — Here we prove (d). Let $\delta > 0$ be fixed. We first consider $\alpha_0 > 0$, $\gamma_0 \in (0,\alpha_0)$, $\zeta_0 > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$,

$$\Pr\big[\Omega(\alpha_0,\gamma_0,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta_0)\big] > 1-\delta.$$

Then we consider $\lambda_1 \leq \lambda_0$ in such a way that for $\lambda \in (0,\lambda_1)$,

$$[\![\lfloor \boldsymbol{n}_\lambda x_0\rfloor - \boldsymbol{m}_\lambda, \lfloor \boldsymbol{n}_\lambda x_0\rfloor + \boldsymbol{m}_\lambda]\!] \subset (c_0)_\lambda$$

(this can be done properly by using $\Omega^{x_0}_{A,T}(\zeta)$ and the fact that $\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda \to 0$).

Introduce C^λ_t and V^λ_t as in Lemma 3.10.5 (ii), using the seed processes

$$(N^{S,\lambda}_{t+\tau_t(x_0)/\boldsymbol{a}_\lambda}(i + \lfloor \boldsymbol{n}_\lambda x_0\rfloor) - N^S_{\tau_t(x_0)/\boldsymbol{a}_\lambda}(i+\lfloor \boldsymbol{n}_\lambda x_0\rfloor))_{t\geq 0, i\in\mathbb{Z}}.$$

Then by step 2, we observe that $C^\lambda_{t_0-\tau_t(x_0)} \subset [\![-\boldsymbol{m}_\lambda, \boldsymbol{m}_\lambda]\!]$ implies that, on $\Omega(\alpha,\gamma,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta)$ and for $\lambda < \lambda_1$,

$$C_A\big(\eta^\lambda_{\boldsymbol{a}_\lambda t_0}, \lfloor \boldsymbol{n}_\lambda x_0\rfloor\big) = \big\{i + \lfloor \boldsymbol{n}_\lambda x_0\rfloor : \ i \in C^\lambda_{t_0-\tau_t(x_0)}\big\},$$

whence $W^\lambda_{t_0}(x_0) = V^\lambda_{t_0-\tau_{t_0}(x_0)}$. All this implies, using Lemma 3.10.5 (ii), that

$$\liminf_{\lambda\to 0} \Pr\big[|W^\lambda_{t_0}(x_0) - (t_0 - \tau_{t_0}(x_0))| < \varepsilon \mid t_0 - \tau_{t_0}(x_0) < 1\big] \geq 1-\delta.$$

Recalling finally (3.10.3), we deduce that

$$\liminf_{\lambda\to 0} \Pr\big[|W^\lambda_{t_0}(x_0) - Z_{t_0}(x_0)| < \varepsilon \mid t_0 - \tau_{t_0}(x_0) < 1\big] \geq 1-\delta.$$

If now $t_0 - \tau_{t_0}(x_0) > 1$, then step 2 and $\Omega^S_3(\lambda)$ imply that $(c_0)_\lambda$ is completely occupied at time $\boldsymbol{a}_\lambda t_0$. Hence

$$\big|C(\eta^\lambda_{\boldsymbol{a}_\lambda t_0}, \lfloor \boldsymbol{n}_\lambda x_0\rfloor)\big| \geq \big|(c_0)_\lambda\big| \simeq |c|\boldsymbol{n}_\lambda \geq \alpha \boldsymbol{n}_\lambda$$

by $\Omega_M(\alpha)$. Consequently,

$$W^\lambda_{t_0}(x_0) \geq \big[\boldsymbol{a}_\lambda^{-1}\psi_S(1 - 1/(\alpha\boldsymbol{n}_\lambda))\big]\wedge 1 \simeq \big[\boldsymbol{a}_\lambda^{-1}\psi_S(1-\nu_S((\boldsymbol{a}_\lambda,\infty))/\alpha)\big]\wedge 1.$$

For $\varepsilon > 0$, $\nu_S((\boldsymbol{a}_\lambda,\infty))/\alpha \leq \nu_S(((1-\varepsilon)\boldsymbol{a}_\lambda,\infty))$ for all λ small enough: use $(H_S(\infty))$.

Thus for all λ small enough, on $\Omega(\alpha,\gamma,\lambda)\cap\Omega^{x_0}_{A,T}(\zeta)$, we have

$$W^\lambda_{t_0}(x_0) \geq \big[\boldsymbol{a}_\lambda^{-1}\psi_S(1-\nu_S(((1-\varepsilon)\boldsymbol{a}_\lambda,\infty)))\big]\wedge 1 = 1-\varepsilon$$

by definition of ψ_S. Thus

$$\liminf_{\lambda\to 0}\Pr\big[W^\lambda_{t_0}(x_0) \in (1-\varepsilon,1] \mid t_0 - \tau_{t_0}(x_0) > 1\big] \geq 1-\delta.$$

Recalling (3.10.3), we deduce that

$$\liminf_{\lambda\to 0}\Pr\big[|W^\lambda_{t_0}(x_0) - Z_{t_0}(x_0)| < \varepsilon \mid t_0 - \tau_{t_0}(x_0) > 1\big] \geq 1-\delta.$$

Finally, we observe that a.s., $t_0 - \tau_{t_0}(x_0) \neq 1$. Indeed, we have excluded $t_0 = 1$ and the only value charged with positive probability by $\tau_{t_0}(x_0)$ is 0. Thus

$$\liminf_{\lambda \to 0} \Pr\left[|W^\lambda_{t_0}(x_0) - Z_{t_0}(x_0)| < \varepsilon\right] \geq 1 - \delta.$$

Since this holds for any $\delta > 0$, this concludes the proof of (d).

Step 4. — Next, (c) is proved exactly as when $\beta = BS$ (change the beginning: let first $\delta > 0$, $\alpha_0 > 0$, $\zeta_0 \in (0, \alpha_0)$ and $\lambda_0 > 0$ be as in step 3; replace everywhere T_S by $\boldsymbol{a}_\lambda$; and make use of Lemma 3.10.5 instead of Lemma 3.9.6).

Step 5. — Finally, (a) is also proved as when $\beta = BS$. The only difference is that when put $\mathcal{T}^*_M = \mathcal{T}_M \cup \{t_0\}$, we need that $t_0 \neq 1$ (because $0 \in \mathcal{T}_M$ and $\Omega^*_M(\alpha)$ will thus require that for $|t_0 - 1| > \alpha$). □

3.11. Cluster-size distribution when $\beta \in \{\infty, BS\}$

The aim of this section is to prove Corollaries 2.3.4 and 2.4.5.

3.11.1. Study of the LFF(∞) **and** LFF(BS)**-processes.** — We first extend [**15**, Lemma 17].

Lemma 3.11.1. — *Let $\beta \in \{\infty, BS\}$. Let $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a* LFF(β)*-process. There are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ such that the following estimates hold.*

(i) *For any $t \in (1, \infty)$, any $x \in \mathbb{R}$, any $z \in [0, 1)$, $\Pr[Z_t(x) = z] = 0$.*

(ii) *For any $t \in [0, \infty)$, any $B > 0$, any $x \in \mathbb{R}$, $P[|D_t(x)| = B] = 0$.*

(iii) *For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\Pr[|D_t(x)| \geq B] \leq c_2\,\mathrm{e}^{-\kappa_1 B}$.*

(iv) *For all $t \in [\frac{3}{2}, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\Pr[|D_t(x)| \geq B] \geq c_1\,\mathrm{e}^{-\kappa_2 B}$.*

(v) *For all $t \in [\frac{5}{2}, \infty)$, all $0 \leq a < b < 1$, all $x \in \mathbb{R}$,*

$$c_1(b-a) \leq \Pr\big(Z_t(x) \in [a, b]\big) \leq c_2(b-a).$$

Proof. — By invariance by translation, it suffices to treat the case $x = 0$. When $\beta = BS$, the function F_S was defined in Definition 2.4.1. Recall that the LFF(∞)-process can be viewed as a LFF(BS)-process with the function $F_S(z, v) = z$, see Remark 3.6.1.

We consider a Poisson measure $\pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$ on $[0, \infty) \times \mathbb{R} \times [0, 1]$ with intensity measure $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$. We also denote by $\pi_M(\mathrm{d}t, \mathrm{d}x) = \int_{v\in[0,1]} \pi_M(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v)$.

Point (i). — For $t \in [0, 1]$, we have a.s. $Z_t(0) = t$. But for $t > 1$ and $z \in [0, 1)$, $Z_t(0) = z$ implies that the cluster containing 0 has been killed at time $t - z$, so that necessarily $\pi_M(\{t - z\} \times \mathbb{R}) > 0$. This happens with probability 0.

Point (ii). — For any $t > 0$, $|D_t(0)|$ is either 0 or of the form $|X_i - X_j|$ (with $i \neq j$), where $(T_i, X_i)_{i\geq 1}$ are the marks of the Poisson measure $\pi_M(\mathrm{d}s, \mathrm{d}x)$ restricted to $[0, t] \times \mathbb{R}$. We easily conclude as previously that for $B > 0$, $\Pr(|D_t(0)| = B) = 0$.

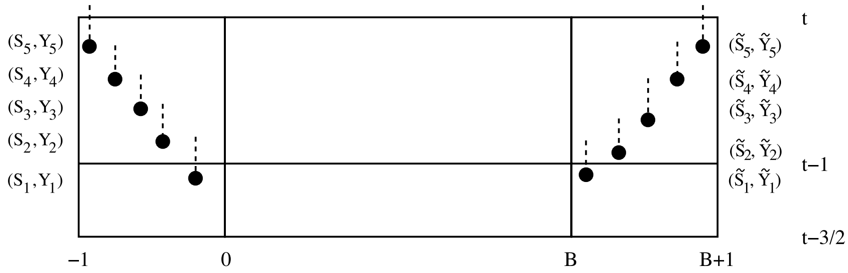

FIGURE 9. The event $\Omega_{t,B}$.

Point (iii). — First if $t \in [0,1)$, we have a.s. $|D_t(0)| = 0$ and the result is obvious. Recall now that $v_0 \in [0,1)$ was defined in Lemma 3.6.5 and that for (τ, X, V) a mark of π_M such that $V \geq v_0$, we have $H_t(X) > 0$ or $Z_t(X) < 1$ for all $t \in [\tau, \tau + \frac{1}{4}]$ (see the proof of Proposition 3.6.4, step 1). This implies that for $t \geq 1$,

$$\begin{aligned}\big\{D_t(0) \geq B\big\} \subset &\big\{[0, \tfrac{1}{2}B] \text{ is connected at time } t \text{ or} \\ &\qquad [-\tfrac{1}{2}B, 0] \text{ is connected at time } t\big\} \\ \subset &\big\{\pi_M([t-\tfrac{1}{4}, t] \times [0, \tfrac{1}{2}B] \times [v_0, 1]) = 0\big\} \\ &\qquad \cup \big\{\pi_M([t-\tfrac{1}{4}, t] \times [-\tfrac{1}{2}B, 0] \times [v_0, 1]) = 0\big\}.\end{aligned}$$

Consequently, $\Pr[|D_t(0)| \geq B] \leq 2\,\mathrm{e}^{-\frac{1}{8}(1-v_0)B}$ as desired.

Point (iv). — Fix $t \geq \frac{3}{2}$ and $B > 0$. Consider the event $\Omega_{t,B} = \Omega^1_{t,B} \cap \Omega^2_t \cap \Omega^3_{t,B}$, illustrated by Figure 9, where

- ▷ $\Omega^1_{t,B} = \{\pi_M([t-\frac{3}{2}, t] \times [0, B] \times [0, 1)) = 0\}$;
- ▷ Ω^2_t is the event that in the box $[t-\frac{3}{2}, t] \times [-1, 0] \times [0, 1]$, π_M has exactly five marks $(S_i, Y_i, V_i)_{i=1,\ldots,5}$ with $Y_5 < Y_4 < Y_3 < Y_2 < Y_1$, $\min_{i=1,\ldots,5} V_i > v_0$ and $t-\frac{3}{2} < S_1 < t-1$, $S_1 < S_2 < S_1+\frac{1}{4}$, $S_2 < S_3 < S_2+\frac{1}{4}$, $S_3 < S_4 < S_3+\frac{1}{4}$, $S_4 < S_5 < S_4 + \frac{1}{4}$ and $S_5 + \frac{1}{4} > t$.
- ▷ $\Omega^3_{t,B}$ is the event that in the box $[t-\frac{3}{2}, t] \times [B, B+1] \times [0, 1]$, π_M has exactly five marks $(\widetilde{S}_i, \widetilde{Y}_i, \widetilde{V}_i)_{i=1,\ldots,5}$ with $\widetilde{Y}_1 < \widetilde{Y}_2 < \widetilde{Y}_3 < \widetilde{Y}_4 < \widetilde{Y}_5$, $\min_{i=1,\ldots,5} \widetilde{V}_i > v_0$ and $t-\frac{3}{2} < \widetilde{S}_1 < t-1$, $\widetilde{S}_1 < \widetilde{S}_2 < \widetilde{S}_1 + \frac{1}{4}$, $\widetilde{S}_2 < \widetilde{S}_3 < \widetilde{S}_2 + \frac{1}{4}$, $\widetilde{S}_3 < \widetilde{S}_4 < \widetilde{S}_3 + \frac{1}{4}$, $\widetilde{S}_4 < \widetilde{S}_5 < \widetilde{S}_4 + \frac{1}{4}$ and $\widetilde{S}_5 + \frac{1}{4} > t$.

We of course have $p := \Pr(\Omega^2_t) = \Pr(\Omega^3_{t,B}) > 0$, and this probability does not depend on $t \geq \frac{3}{2}$ nor on $B > 0$. Furthermore, $\Pr(\Omega^1_{t,B}) = \mathrm{e}^{-\frac{3}{2}B}$. These three events being independent, we conclude that $\Pr(\Omega_{t,B}) \geq p^2\,\mathrm{e}^{-3B/2}$. To conclude the proof of (iv), it thus suffices to check that $\Omega_{t,B} \subset \{[0,B] \subset D_t(0)\}$. But on $\Omega_{t,B}$, using the same arguments as in Point (iii), we observe that:

- ▷ the fire starting at (S_2, Y_2) can not affect $[0, B]$, because since $S_2 \in [S_1, S_1 + \frac{1}{4})$, $H_{S_2-}(Y_1) > 0$ or $Z_{S_2-}(Y_1) > 0$, with $Y_2 < Y_1 < 0$;
- ▷ then the fire starting at (S_3, Y_3) can not affect $[0, B]$, because since $S_3 \in [S_2, S_2 + \frac{1}{4})$, $H_{S_3-}(Y_2) > 0$ or $Z_{S_3-}(Y_2) > 0$, with $Y_3 < Y_2 < 0$;
- ▷ then the fire starting at (S_4, Y_4) can not affect $[0, B]$, because since $S_4 \in [S_3, S_3 + \frac{1}{4})$, $H_{S_4-}(Y_3) > 0$ or $Z_{S_4-}(Y_3) > 0$, with $Y_4 < Y_3 < 0$;
- ▷ then the fire starting at (S_5, Y_5) can not affect $[0, B]$, because since $S_5 \in [S_4, S_4 + \frac{1}{4})$, $H_{S_5-}(Y_4) > 0$ or $Z_{S_5-}(Y_4) > 0$, with $Y_5 < Y_4 < 0$;
- ▷ furthermore, the fires starting on the left at -1 during $(S_1, t]$ cannot affect $[0, B]$, because for all $t \in (S_1, t]$, there is always a site $x_t \in \{Y_1, Y_2, Y_3, Y_4\} \subset [-1, 0]$ with $H_t(x_t) > 0$ or $Z_t(x_t) < 1$;
- ▷ the same arguments apply on the right of B.

As a conclusion, the zone $[0, B]$ is not affected by any fire during $(S_1 \vee \widetilde{S}_1, t]$. Since the length of this time interval is greater than 1, we deduce that for all $x \in [0, B]$,

$$Z_t(x) = \min\big(Z_{S_1\vee\widetilde{S}_1}(x) + t - S_1 \vee \widetilde{S}_1, 1\big) \geq \min(t - S_1 \vee \widetilde{S}_1, 1) = 1,$$
$$H_t(x) = \max\big(H_{S_1\vee\widetilde{S}_1}(x) - (t - S_1 \vee \widetilde{S}_1), 0\big) \leq \max\big(1 - (t - S_1 \vee \widetilde{S}_1), 0\big) = 0,$$

whence $[0, B] \subset D_t(0)$.

Point (v). — For $0 \leq a < b < 1$ and $t \geq 1$, we have $Z_t(0) \in [a, b]$ if and only there is $\tau \in [t - b, t - a]$ such that $Z_\tau(0) = 0$. And this happens if and only if

$$X_{t,a,b} := \int_{t-b}^{t-a} \int_{\mathbb{R}} \mathbb{1}_{\{y \in D_{s-}(0)\}} \pi_M(\mathrm{d}s, \mathrm{d}y) \geq 1.$$

We deduce that

$$\Pr\big(Z_t(0) \in [a, b]\big) = \Pr\big(X_{t,a,b} \geq 1\big) \leq \mathbb{E}\big[X_{t,a,b}\big] = \int_{t-b}^{t-a} \mathbb{E}\big[|D_s(0)|\big]\,\mathrm{d}s \leq C(b - a),$$

where we used point (iii) for the last inequality.

Next, we have $\{\pi_M([t - b, t - a] \times D_{t-b}(0)) \geq 1\} \subset \{X_{t,a,b} \geq 1\}$: it suffices to note that a.s.,

$$\begin{aligned}\{X_{t,a,b} = 0\} &\subset \big\{X_{t,a,b} = 0, D_{t-b}(0) \subset D_s(0) \text{ for all } s \in [t - b, t - a]\big\}\\ &\subset \big\{\pi_M([t - b, t - a] \times D_{t-b}(0)) = 0\big\}.\end{aligned}$$

Now since $D_{t-b}(0)$ is independent of $\pi_M(\mathrm{d}s, \mathrm{d}x)$ restricted to $(t-b, \infty)\times\mathbb{R}$, we deduce that for $t \geq \frac{5}{2}$

$$\begin{aligned}\Pr\big(Z_t(0) \in [a, b]\big) &\geq \Pr\big[\pi_M((t - b, t - a] \times D_{t-b}(0)) \geq 1\big]\\ &\geq \Pr\big[|D_{t-b}(0)| \geq 1\big](1 - \mathrm{e}^{-(b-a)}) \geq c(1 - \mathrm{e}^{-(b-a)}),\end{aligned}$$

where we used point (iv) (here $t - b \geq \frac{3}{2}$) to get the last inequality. This concludes the proof, since $1 - \mathrm{e}^{-x} \geq \frac{1}{2}x$ for all $x \in [0, 1]$. □

3.11.2. The case $\beta = \infty$. — We can now handle the

Proof of Corollary 2.3.4. — We thus assume (H_M) and $(H_S(\infty))$ and consider, for each $\lambda > 0$, a $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. Let also $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a $\mathrm{LFF}(\infty)$-process.

Point (ii). — Using Lemma 3.11.1 (iii)-(iv) and recalling that

$$\big|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)\big| / \boldsymbol{n}_\lambda = \big|D_t^\lambda(0)\big|$$

by (2.2.3), it suffices to check that for all $t \geq \frac{3}{2}$, all $B > 0$,

$$\lim_{\lambda\to 0} \Pr\big[|D_t^\lambda(0)| \geq B\big] = \Pr\big[|D_t(0)| \geq B\big].$$

This follows from Theorem 2.3.3 (b), which implies that $|D_t^\lambda(0)|$ goes in law to $|D_t(0)|$ and from Lemma 3.11.1 (ii).

Point (i). — Due to Lemma 3.11.1 (v) we only need that for all $0 < a < b < 1$, all $t \geq \frac{5}{2}$,

$$\lim_{\lambda\to 0} \Pr\big(|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)| \in [1/\nu_S((\boldsymbol{a}_\lambda a, \infty)), 1/\nu_S((\boldsymbol{a}_\lambda b, \infty))]\big) = \Pr\big(Z_t(0) \in [a, b]\big).$$

But using Theorem 2.3.3 (c) and Lemma 3.11.1 (i), we know that

$$\lim_{\lambda\to 0} \Pr\big[\psi_S\left(1 - 1/|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)|\right) \mathbb{1}_{\{|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)|\geq 1\}} \in [\boldsymbol{a}_\lambda a, \boldsymbol{a}_\lambda b]\big] = \Pr\big(Z_t(0) \in [a, b]\big).$$

Using finally the definition of ψ_S (see notation 2.1.5 (ii)), we see that for all $c \in \mathbb{N}$, all $0 < \alpha < \beta$,

$$\psi_S\left(1 - 1/c\right) \mathbb{1}_{\{c\geq 1\}} \in [\alpha, \beta] \text{ if and only if } c \in \big[1/\nu_S((\alpha, \infty)), 1/\nu_S((\beta, \infty))\big].$$

One immediately concludes. □

3.11.3. The case $\beta = BS$. — We finally give the

Proof of Corollary 2.4.5. — We thus assume (H_M) and $(H_S(BS))$ and consider, for each $\lambda > 0$, a $\mathrm{FF}(\mu_S, \mu_M^\lambda)$-process $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. Let also $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a $\mathrm{LFF}(BS)$-process.

Point (ii). — Using Lemma 3.11.1 (iii)–(iv) and recalling that

$$\big|C(\eta^\lambda_{\boldsymbol{a}_\lambda t}, 0)\big| / \boldsymbol{n}_\lambda = \big|D_t^\lambda(0)\big|$$

by (2.2.3), it suffices to check that for all $t \geq \frac{3}{2}$, all $B > 0$,

$$\lim_{\lambda\to 0} \Pr\big[|D_t^\lambda(0)| \geq B\big] = \Pr\big[|D_t(0)| \geq B\big].$$

This follows from Theorem 2.4.4 (b), which implies that $|D_t^\lambda(0)|$ goes in law to $|D_t(0)|$ and from Lemma 3.11.1 (ii).

Point (i). — Theorem 2.4.4 (c) asserts that for all $t \geq 0$, all $k \geq 0$,

$$\lim_{\lambda\to 0} \Pr\big[|C(\eta^\lambda_{T_S t}, 0)| = k\big] = \mathbb{E}\big[q_k(Z_t(0))\big],$$

where $q_k(z)$ was defined in (2.4.2). Using next Lemma 3.11.1 (v) and recalling that $Z_t(0) \in [0,1]$ a.s., we see that for $t \geq \frac{5}{2}$, the law of $Z_t(0)$ is of the form

$$g_t(z)\mathbb{1}_{\{0\leq z\leq 1\}}\,\mathrm{d}z + \alpha_t\delta_1(\mathrm{d}z),$$

for some function $g_t : [0,1] \mapsto \mathbb{R}_+$ satisfying $c \leq g_t \leq C$, where the constants $0 < c < C$ do not depend on $t \geq \frac{5}{2}$. One immediately deduces that for any $k \geq 0$, $\mathbb{E}[q_k(Z_t(0))] \in [cq_k, Cq_k]$. Indeed, there holds $q_k = \int_0^1 q_k(z)\,\mathrm{d}z$ and $q_k(1) = 0$. This concludes the proof. □

CHAPTER 4

NUMERICAL SIMULATIONS

4.1. Simulations

We would like to present some simulations of the discrete forest fire process. In all the simulations below, we choose $\mu_M^\lambda(\mathrm{d}t) = \lambda \mathrm{e}^{-\lambda t} \mathbb{1}_{t \geq 0} \, \mathrm{d}t$ and we consider different laws μ_S. We simulate the $\mathrm{FF}_A(\mu_S, \mu_M^\lambda)$ process with $A = 2.5$, for some given value of λ. Since there are too much concerned sites, it is not possible to draw the whole picture. We thus extract a zone in which some interesting events occur.

In all the pictures below, time evolves vertically, with $t = 0$ at the bottom. On each site, we plot white (resp. black) segments when the site is vacant (resp. occupied). Matches are represented by bullets.

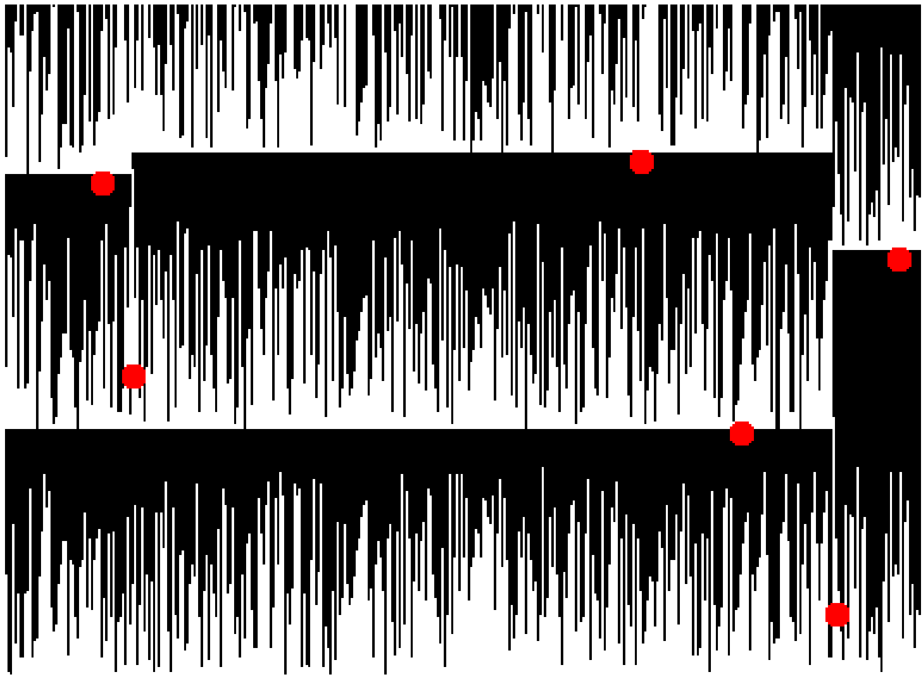

FIGURE 10. Simulation with $\beta = BS$.

We used $\mu_S = \delta_1$, $\nu_S(\mathrm{d}t) = \mathbf{1}_{\{t\in[0,1]\}}\,\mathrm{d}t$, $\boldsymbol{a}_\lambda = T_S = 1$ and $\lambda = 10^{-3}$. Here everything happens, roughly, as described by the limit process (compare with Figure 3).

At the begining, all the sites are vacant. Many sites remain vacant for a while, but we observe that all become occupied after some time, except one, which has burnt due to the first match.

This first match produces a microscopic fire, involving very few sites (we cannot see it on the picture because the bullet is slightly too large, but these sites were occupied just before the match).

The second fire is macroscopic: it concerns many sites. It it is limited on the right by a vacant site, which is due to the effect of the first (microscopic) fire.

The third fire concerns few sites and is microscopic.

The fourth fire is macroscopic and is limited on the left by the a vacant site, produced by the second fire (which was macroscopic and destroyed a large zone which is not filled again).

The fifth fire is macroscopic and is limited by a vacant site produced by the third fire, which was microscopic.

Finally, the last fire is macroscopic, and is limited on both sides by vacant sites let by the two previous (macroscopic) fires.

Observe that the time needed to completely fill again a macroscopic zone is roughly always the same (look at the time needed after time 0, after the second fire, after the fourth fire).

Note also that the effect of the first (microscopic) fire persists for quite a long time: it limits the second fire, which limits the fourth fire, which itself limits the sixth fire.

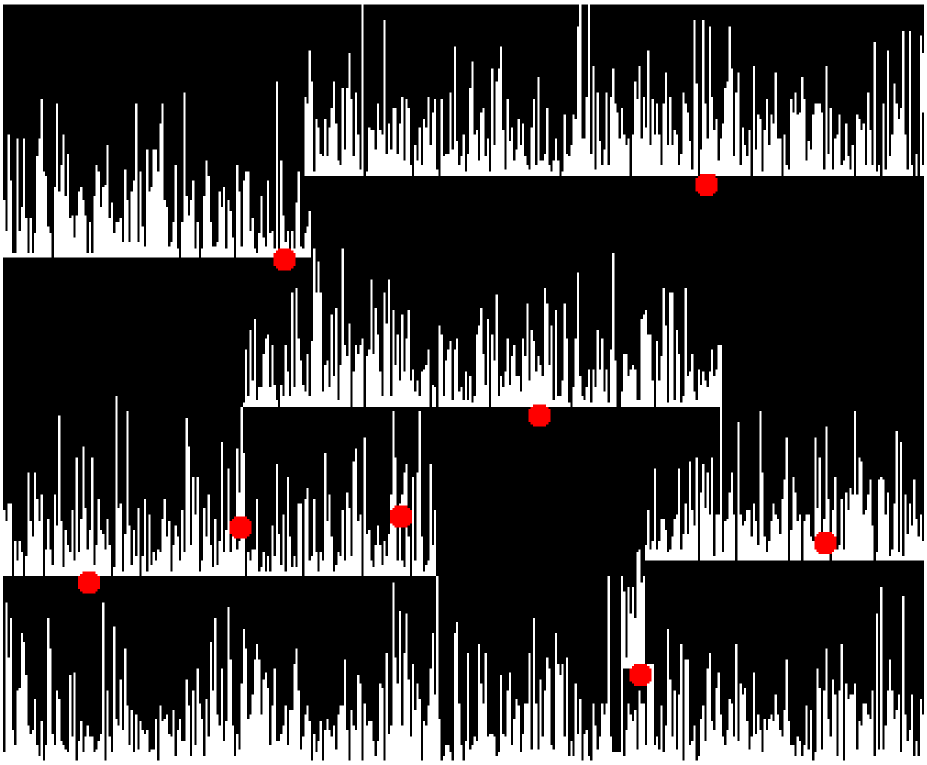

FIGURE 11. Simulation with $\beta = \infty$.

Here $\mu_S((t,\infty)) = \mathrm{e}^{-t^2/2}$, $\nu_S(\mathrm{d}t) = (\sqrt{2/\pi})\mathrm{e}^{-t^2/2}\mathbf{1}_{\{t\geq 0\}}\,\mathrm{d}t$ and $\lambda = 10^{-3}$. We used the approximate value $\boldsymbol{a}_\lambda \simeq \sqrt{2\log(1/\lambda)}$. The picture is not so far from the limit process (compare with Figure 2), but there are some defaults.

The first fire is rather microscopic, but has however quite a large length.

The second fire, which is clearly macroscopic, is limited not by a previous microscopic fire, but by a site where the first seed has needed an unusual large time to fall.

Also, the limit process predicts that the length of the barrier produced by a microscopic fire equals the delay between the time at which the match falls and the last time where the zone was involved in a maroscopic fire. We see here that this is roughly the case for the first fire, but the effect of the third and fourth (microscopic) fires are too long.

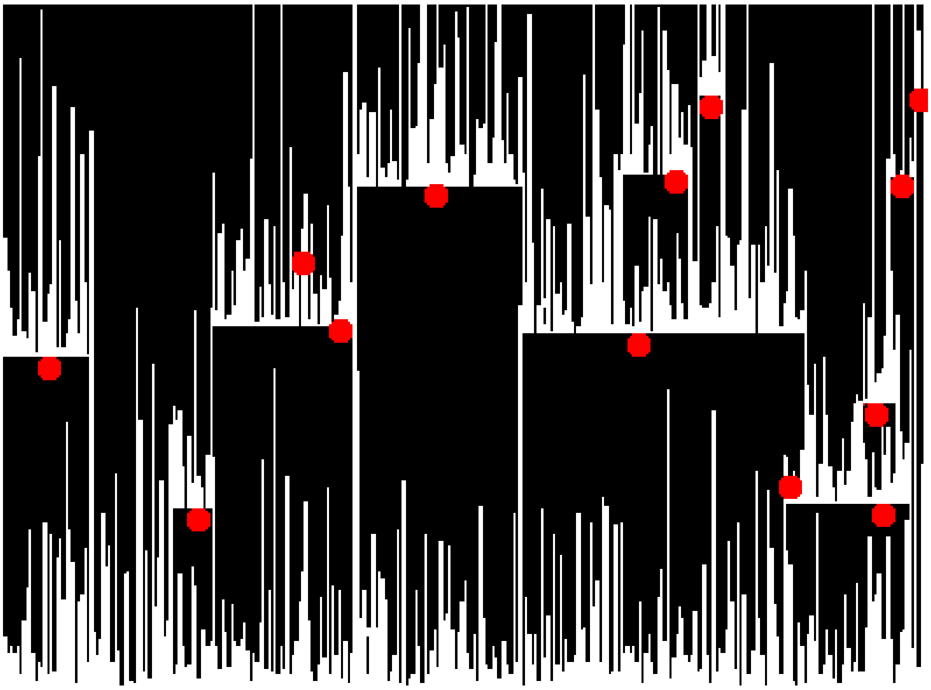

FIGURE 12. Simulation with $\beta = 5$.

We considered $\mu_S((t,\infty)) = (1+t/\beta)^{-\beta-1}$ and $\nu_S((t,\infty)) = (1+t/\beta)^{-\beta}$ with $\beta = 5$ and $\lambda = 5.10^{-3}$. We used the approximate value $\boldsymbol{a}_\lambda \simeq (1/\lambda)^{1/(\beta+1)}$. This picture resembles much the limit process (see Figure 4): all the fires involve a macroscopic number of sites and we observe that sites where no seed fall during a large time interval are rather rare.

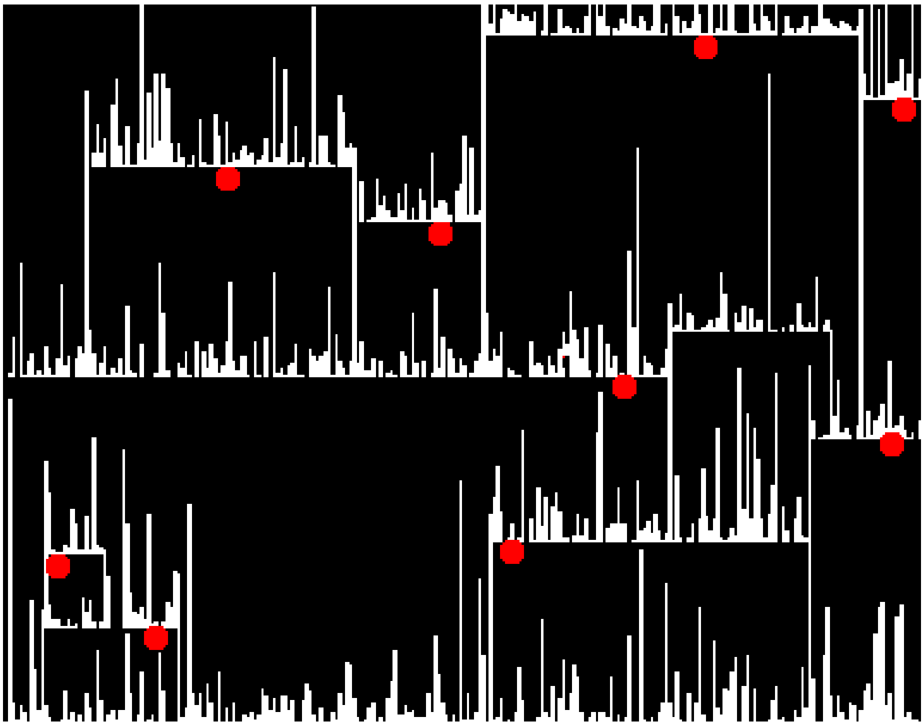

FIGURE 13. Simulation with $\beta = 2$.

Same thing as Figure 12 with $\beta = 2$ and $\lambda = 10^{-3}$. This picture is in perfect adequacy with the limit process (see Figure 4), at least from a qualitative point of view: when a fire starts, it burns a macroscopic zone, which is rather quickly filled again, except for some quite rare sites.

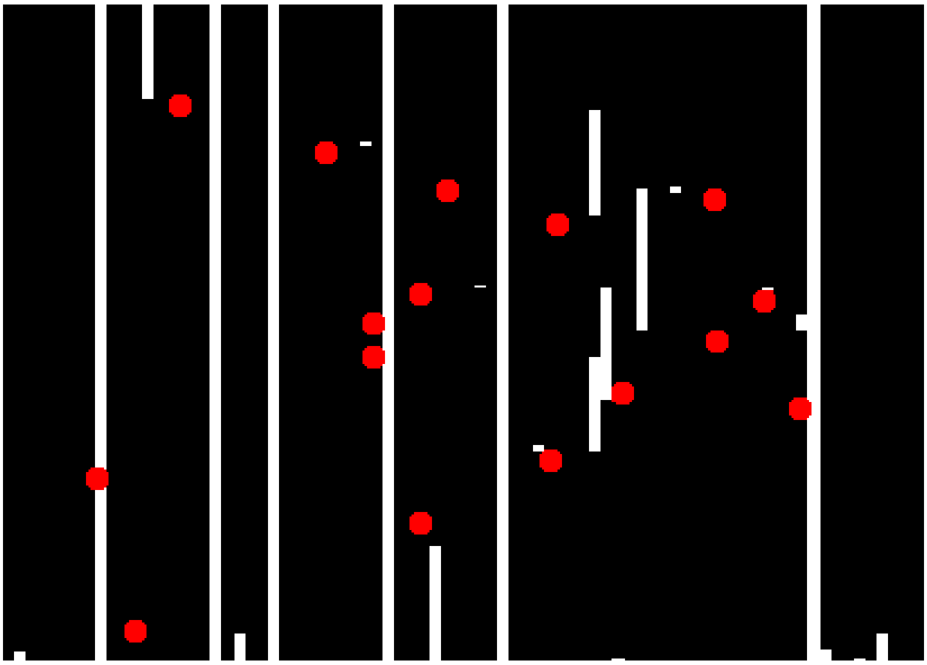

FIGURE 14. Simulation with $\beta = 0$.

We used $\mu_S((t,\infty)) = e(e+t)^{-1}[\log(e+t)]^{-2}$, $\nu_S((t,\infty)) = [\log(e+t)]^{-1}$ and $\lambda = 10^{-7}$. We used the approximate value $\boldsymbol{a}_\lambda \simeq 1/[\lambda \log(1/\lambda)]$. This picture is quite satisfactory when compared to the limit process (see Figure 5): there are six sites where the first seed never falls and the fires have quite a low effect, in the sense that most of the burnt sites become occupied again almost immediately.

CHAPTER 5

APPENDIX

5.1. Appendix

5.1.1. Regularly varying functions. — The proof below is closely related to the theory of regularly varying functions and is probably completely standard.

LEMMA 5.1.1. — *Assume* (H_S). *Then either* $(H_S(BS))$ *holds or there exists* β *in* $[0,\infty)\cup\{\infty\}$ *such that* $(H_S(\beta))$ *holds.*

Proof. — We thus assume (H_S) and that the support of μ_S is unbounded. Hence, for all $t>0$,

$$\varphi(t) := \lim_{x\to\infty} \frac{\nu_S((x,\infty))}{\nu_S((xt,\infty))} \in [0,\infty)\cup\{\infty\}$$

exists. The function φ is clearly nondecreasing and satisfies $\varphi(1)=1$.

Step 1. — We first show that for all $t>0$, $\varphi(1/t)=1/\varphi(t)$, with the convention that $1/0=\infty$ and $1/\infty=0$. This is not hard:

$$\varphi(1/t) = \lim_{x\to\infty} \frac{\nu_S((x,\infty))}{\nu_S((x/t,\infty))} = \lim_{y\to\infty} \frac{\nu_S((yt,\infty))}{\nu_S((y,\infty))} = 1/\varphi(t).$$

Step 2. — By the same way, one easily checks that for $0<s\le t$, one has $\varphi(st)=\varphi(s)\varphi(t)$ as soon as $\varphi(s)>0$ or $\varphi(t)<\infty$. It suffices to write

$$\varphi(st) = \lim_{x\to\infty} \frac{\nu_S((x,\infty))}{\nu_S((xst,\infty))} = \lim_{x\to\infty} \frac{\nu_S((x,\infty))}{\nu_S((xs,\infty))}\frac{\nu_S((xs,\infty))}{\nu_S((xst,\infty))} = \varphi(s)\varphi(t).$$

Step 3. — We assume first that $\varphi(s)>0$ for all $s\in(0,1)$. By Step 1, one easily deduces that $\varphi(s)\in(0,\infty)$ for all $s>0$. We thus have a nondecreasing function $\varphi:(0,\infty)\mapsto(0,\infty)$ such that $\varphi(st)=\varphi(s)\varphi(t)$ for all $0<s\le t$ and such that $\varphi(1)=1$. One classically concludes that there exists $\beta\in[0,\infty)$ such that $\varphi(t)=t^\beta$.

Step 4. — We now assume that $\varphi(\alpha)=0$ for some $\alpha\in(0,1)$. We want to show that if so, then $\varphi(t)=0$ for all $t\in(0,1)$. This will imply that $\varphi(t)=\infty$ for $t>1$ by step 1, whence $\varphi(t)=t^\infty$.

Let thus $\alpha_* = \sup\{\alpha > 0 : \varphi(\alpha) = 0\}$. Suppose by contradiction that $\alpha_* \in (0,1)$. By monotonicity, we have $\varphi(\alpha) = 0$ for all $\alpha \in (0, \alpha_*)$. By step 1, we know that $\varphi(s) \in (0,\infty)$ for all $s \in (\alpha_*, 1/\alpha_*)$. Due to step 2, we deduce that for all small $\varepsilon > 0$, $\varphi((\alpha_*-\varepsilon)(1/\alpha_*-\varepsilon)) = 0$. But for $\varepsilon > 0$ small enough, we have $(\alpha_*-\varepsilon)(1/\alpha_*-\varepsilon) > \alpha_*$ (because $\alpha_* < 1$). This contradicts the definition of α_*. □

Next, we prove the existence of the scale $\boldsymbol{m}_\lambda$ satisfying (2.2.4).

LEMMA 5.1.2. — *Assume* $(H_S(\infty))$. *Recall* (2.2.1), (2.2.2). *There exists a function* $\boldsymbol{m}_\lambda : (0,1] \mapsto \mathbb{N}$ *satisfying* (2.2.4).

Proof. — Recalling that $\lim_{\lambda\to 0} \boldsymbol{a}_\lambda = \infty$ and using $(H_S(\infty))$, we observe that for any $n \geq 1$,

$$\lim_{\lambda\to 0} \nu_S\big(((1-1/n)\boldsymbol{a}_\lambda, \infty)\big)/\nu_S\big((\boldsymbol{a}_\lambda,\infty)\big) = \infty.$$

Thus there exists $\lambda_n \in (0,1]$ such that for all $\lambda \in (0, \lambda_n]$,

$$\nu_S\big(((1-1/n)\boldsymbol{a}_\lambda, \infty)\big)/\nu_S\big((\boldsymbol{a}_\lambda,\infty)\big) \geq n.$$

We of course may choose $\lambda_1 = 1$ and choose the sequence $(\lambda_n)_{n\geq 1}$ decreasing to 0. Then we define $\varepsilon_\lambda : (0,1] \mapsto (0,1]$ by setting, for all $n \geq 1$, $\varepsilon_\lambda = 1/n$ for $\lambda \in (\lambda_{n+1}, \lambda_n]$. There holds $\lim_{\lambda\to 0} \varepsilon_\lambda = 0$. Finally, we put

$$\boldsymbol{m}_\lambda = \big\lfloor 1/\nu_S((\boldsymbol{a}_\lambda(1-\varepsilon_\lambda), \infty))\big\rfloor.$$

This function is obviously non-increasing. Next, recalling that $\boldsymbol{n}_\lambda = \lfloor 1/\nu_S((\boldsymbol{a}_\lambda, \infty))\rfloor$, we see that for all $n \geq 1$, all $\lambda \in (\lambda_{n+1}, \lambda_n)$,

$$\frac{\boldsymbol{m}_\lambda}{\boldsymbol{n}_\lambda} \simeq \frac{\nu_S((\boldsymbol{a}_\lambda,\infty))}{\nu_S((\boldsymbol{a}_\lambda(1-\varepsilon_\lambda),\infty))} = \frac{\nu_S((\boldsymbol{a}_\lambda,\infty))}{\nu_S((\boldsymbol{a}_\lambda(1-1/n),\infty))} \leq \frac{1}{n},$$

whence $\lim_{\lambda\to 0}(\boldsymbol{m}_\lambda/\boldsymbol{n}_\lambda) = 0$. Finally, fix $z \in (0,1)$ and consider n large enough, so that $1 - 1/n > z$. Then for $\lambda \in (0, \lambda_n)$, there holds $\varepsilon_\lambda \leq 1/n$, whence

$$\nu_S\big((\boldsymbol{a}_\lambda z,\infty)\big)\boldsymbol{m}_\lambda \simeq \frac{\nu_S((\boldsymbol{a}_\lambda z,\infty))}{\nu_S((\boldsymbol{a}_\lambda(1-\varepsilon_\lambda),\infty))} \geq \frac{\nu_S((\boldsymbol{a}_\lambda z,\infty))}{\nu_S((\boldsymbol{a}_\lambda(1-1/n),\infty))} \to \infty$$

as $\lambda \to 0$ due to $(H_S(\infty))$, since $z < 1 - 1/n$. □

5.1.2. Coupling. — Finally, we recall some well-known facts about coupling.

LEMMA 5.1.3. — (i) *Let* $(p_k)_{k\geq 0}$ *and* $(q_k)_{k\geq 0}$ *be two probability laws on* $\{0,1,\dots\}$. *One can couple*

$$X \sim (p_k)_{k\geq 0} \quad \textit{and} \quad Y \sim (q_k)_{k\geq 0}$$

such that for all $k \geq 0$,

$$\Pr[X = Y = k] \geq p_k \wedge q_k.$$

(ii) *For f, g two probability densities on $\mathbb{R}$, one can couple $X \sim f(x)\,\mathrm{d}x$ and $Y \sim g(x)\,\mathrm{d}x$ in such a way that*

$$\Pr[X = Y] \geq \int_{\mathbb{R}} \min(f(x), g(x))\,\mathrm{d}x.$$

(iii) *If we have a sequence of laws μ_n on some Polish space, converging weakly to some law μ, then it is possible to find some random variables $X_n \sim \mu_n$ and $X \sim \mu$ such that a.s., $\lim_{n\to\infty} X_n = X$.*

Proof. — First observe that (iii) is nothing but the Skorokhod representation theorem. To prove (i), set $r_k = p_k \wedge q_k$ and $r = \sum_0^\infty r_k$. Consider a Bernoulli r.v. C with parameter r, a $(r_k/r)_{k\geq 0}$-distributed r.v. Z, a $((p_k - r_k)/(1-r))_{k\geq 0}$-distributed r.v. U and a $((q_k - r_k)/(1-r))_{k\geq 0}$-distributed r.v. V. Assume that all these objects are independent and put $(X, Y) = C(Z, Z) + (1-C)(U, V)$. Some immediate computations show that $X \sim (p_k)_{k\geq 0}$ and $Y \sim (q_k)_{k\geq 0}$ and for $k \geq 0$, $\Pr[X = Y = k] \geq r_k$.

The proof of (ii) is similar: put $h = \min(f, g)$ and $r = \int_{\mathbb{R}} h(x)\,\mathrm{d}x$. Consider a Bernoulli r.v. C with parameter r, a r.v. Z with density h/r, a r.v. U with density $(f-h)/(1-r)$ and a r.v. V with density $(g-h)/(1-r)$. Assume that all these objects are independent and put $(X, Y) = C(Z, Z) + (1-C)(U, V)$. Some immediate computations show that $X \sim f(x)\,\mathrm{d}x$, $Y \sim g(y)\,\mathrm{d}y$ and $\Pr[X = Y] \geq r$. □

BIBLIOGRAPHY

[1] Aldous (D. J.) – *Emergence of the giant component in special Marcus-Lushnikov processes*, Random Structures Algorithms, t. **12** (1998), pp. 179–196.

[2] ———, *Deterministic and stochastic models for coalescence (aggregation, coagulation): A review of the mean-field theory for probabilists*, Bernoulli, t. **5** (1999), pp. 3–48.

[3] ———, *The percolation process on a tree where infinite clusters are frozen*, Math. Proc. Cambridge Philos. Soc., t. **128** (2000), pp. 465–477.

[4] Amaral (L .A. N.) & Lauritsen (K. B.) – *Self-organized criticality in a rice-pile model*, Phys Rev. E., t. **54** (1996), pp. 4512–4515.

[5] Bak (P.), Tang (C.) & Wiesenfeld (K.) – *Self-organized criticality: an explanation of* $1/f$ *noise*, Phys. Rev. Letters, t. **59** (1987), pp. 381–384.

[6] ———, *Self-organized criticality*, Phys. Rev. A, t. **38** (1988), pp. 364–374.

[7] van den Berg (J.) & Brouwer (R.) – *Self-organized forest-fires near the critical time*, Comm. Math. Phys., t. **267** (2006), pp. 265–277.

[8] van den Berg (J.), Brouwer (R.) & Vágvölgyi (B.) – Box-crossings and continuity results for self-destructive percolation in the plane, pp. 117–136 in *In and Out of Equilibrium 2*, Progress in Probability, vol. 60, Birkhäuser, 2008.

[9] van den Berg (J.) & Járai (A. A.) – *On the asymptotic density in a one-dimensional self-organized critical forest-fire model*, Comm. Math. Phys., t. **253** (2005), pp. 633–644.

[10] van den Berg (J.) & Tóth (B.) – *A signal-recovery system: asymptotic properties, and construction of an infinite-volume process*, Stochastic Process. Appl., t. **96** (2001), pp. 177–190.

[11] Bertoin (J.) – *Random fragmentation and coagulation processes*, Cambridge Studies in Advanced Math., t. **102** (2006).

[12] ______, *Burning cars in parkings*, preprint, `http://hal.archives-ouvertes.fr/ccsd-00505206/en/`, 2010.

[13] Borgne (Y. Le) & Rossin (D.) – *On the identity of the sand pile group*, Discrete Math., t. **256** (2002), pp. 775–790.

[14] Bressaud (X.) & Fournier (N.) – *On the invariant distribution of an avalanche process*, Ann. Probab., t. **37** (2009), pp. 48–77.

[15] ______, *Asymptotics of one-dimensional forest fire processes.*, Ann. Probab., t. **38** (2010), pp. 1783–1816.

[16] Brouwer (R.) – *A modified version of frozen percolation on the binary tree*, preprint, 2005, arXiv: math/0511021v1.

[17] Brouwer (R.) & Pennanen (J.) – *The cluster size distribution for a forest-fire process on $\mathbb{Z}$*, Electron. J. Prob., t. **11** (2006), pp. 1133–1143.

[18] Brylawski (T.) – *The lattice of integer partitions*, Discrete Math., t. **6** (1973), pp. 201–219.

[19] Cafiero (R.), Pietronero (L.) & Vespignani (A.) – *Persistence of screening and self-criticality in the scale invariant dynamics of diffusion limited aggregation*, Phys. Rev. Lett., t. **70** (1993), pp. 3939–3942.

[20] Christensen (K.), Corral (A.), Frette (V.), Feder (J.) & Jossang (T.) – *Tracer dispersion in a self-organized critical system*, Phys. Rev. Lett., t. **77** (1996), pp. 107–110.

[21] Cocozza-Thivent (C.) – *Processus stochastiques et fiabilité des systèmes*, Springer-Smai, 1997.

[22] Corral (A.), Telesca (L.) & Lasaponara (R.) – *Scaling and correlations in the dynamics of forest-fire occurrence*, Phys. Rev. E, t. **77** (2008), 016101, 7 pp.

[23] Cui (W.) & Perera (A. H.) – *What do we know about forest fire size distribution, and why is this knowledge useful for forest management?*, Internat. J. Wildland Fire, t. **17** (2008), pp. 234–244.

[24] Damron (M.), Sapozhnikov (A.) & Vágvölgyi (B.) – *Relations between invasion percolation and critical percolation in two dimensions*, Ann. Probab., t. **37** (2009), pp. 2297–2331.

[25] Dhar (D.) – *Self-organized critical state of sand pile automaton models*, Phys. Rev. Lett., t. **64** (1990), pp. 1613–1616.

[26] ———, *Theoretical studies of self-organized criticality*, Phys. A, t. **369** (2006), pp. 29–70.

[27] Drossel (B.) & Schwabl (F.) – *Self-organized critical forest-fire model*, Phys. Rev. Lett., t. **69** (1992), pp. 1629–1632.

[28] Drossela (B.), Clar (S.) & Schwabl (F.) – *Exact results for the one-dimensional self-organized critical forest-fire model*, Phys. Rev. Lett., t. **71** (1993), pp. 3739–3742.

[29] Dürre (M.) – *Existence of multi-dimensional volume self-organized critical forest-fire models*, Electron. J. Prob., t. **11** (2006), pp. 513–539.

[30] ———, *Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models*, Electronic Comm. Prob., t. **11** (2006), pp. 304–315.

[31] ———, *Self-organized critical phenomena: Forest fire and sand pile models*, Ph.D. Thesis, LMU München, 2009.

[32] Erdős (P.) & Rényi (A.) – *On the evolution of random graphs*, Bull. Inst. Internat. Statist., t. **38** (1961), pp. 343–347.

[33] Fournier (N.) & Laurençot (P.) – *Marcus Lushnikov processes, Smoluchowski's and Flory's models*, Stochastic Process. Appl., t. **119** (2009), pp. 167–189.

[34] Goles (E.), Latapy (M.), Magnien (C.), Morvan (M.) & Phan (H. D.) – *Sand pile models and lattices: a comprehensive survey*, Theoret. Comput. Sci., t. **322** (2004), pp. 383–407.

[35] Grassberger (P.) – *Critical behaviour of the Drossel-Schwabl forest fire model*, New J. Physics, t. **4** (2002), pp. 17.

[36] Grimmett (G.) – *Percolation*, Springer-Verlag, New York, 1989.

[37] Henley (C. L.) – *Self-organized percolation: a simpler model*, Bull. Am. Phys. Soc., t. **34** (1989), pp. 838.

[38] Holmes (T. P.), Huggett (R. J.) & Westerling (A. L.) – Statistical analysis of large wildfires, pp. 59–77 in *The Economics of Forest Disturbances: Wildfires, Storms, and Invasive Species*, Springer, Dordrecht, 2008.

[39] Holroyd (A. E.), Levine (L.), Meszaros (K.), Y. Peres (J. Propp) & Wilson (D.) – Chip-Firing and Rotor-Routing on Directed Graphs, pp. 331–364 in *In and Out of Equilibrium 2*, Progr. Probab., vol. 60, 2008.

[40] Honecker (A.) & Peschel (I.) – *Critical properties of the one-dimensional forest-fire model*, Physica A, t. **229** (1996), pp. 478–500.

[41] Jacod (J.) & Shiryaev (A. N.) – *Limit theorems for stochastic processes*, Springer, 1982.

[42] Járai (A. A.) – *Thermodynamic limit of the abelian sand pile model on $\mathbb{Z}^d$*, Markov Process. Related Fields, t. **11** (2005), pp. 313–336.

[43] Jensen (H. J.) – *Self-organized criticality*, Cambridge Lecture Notes in Physics, vol. 10, Cambridge University Press, 1998.

[44] Kingman (J.) – *The coalescent*, Stochastic Process. Appl., t. **13** (1982), pp. 235–248.

[45] Liggett (T. M.) – *Interacting particle systems*, Springer, 1985.

[46] Lübeck (S.) & Usadel (K. D.) – *The Bak-Tang-Wiesenfeld sand pile model around the upper critical dimension*, Phys. Rev. E, t. **56** (1997), pp. 5138–5143.

[47] Maes (C.), Redig (F.) & Saada (E.) – *The infinite volume limit of dissipative abelian sand piles*, Commun. Math. Phys., t. **244** (2004), pp. 395–417.

[48] Majumdar (S. N.) – *Exact fractal dimension of the loop-erased random walk in two dimensions*, Phys. Rev. Lett., t. **68** (1992), pp. 2329–2331.

[49] Mangiavillano (A.) – *Multi-scalarité du phénomène feu de forêt en régions méditerranéennes françaises de 1973 à 2006*, Ph.D. Thesis, Université d'Avignon, 2008.

[50] Olami (Z.), Feder (H. J. S.) & Christensen (K.) – *Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes*, Physical Review Letters, t. **68** (1992), pp. 1244–1247.

[51] Pietronero (L.), Tartaglia (P.) & Zhang (Y. C.) – *Theoritical studies of self-organized criticality*, Physica A, t. **173** (1991), pp. 22–44.

[52] Priezzhev (V. B.), Dhar (D.), Dhar (A.) & Krishnamurthy (S.) – *Eulerian walkers as a model of self-organised criticality*, Phys. Rev. Lett., t. **77** (1996), pp. 5079–5082.

[53] Pruessner (G.) & Jensen (H. J.) – *A new, efficient algorithm for the forest fire model*, 2003.

[54] Ráth (B.) – *Mean field frozen percolation*, J. Statist. Physics, t. **137** (2009), pp. 459–499.

[55] Ráth (B.) & Tóth (B.) – *Erdős-Rényi random graphs + forest fires = self-organized criticality*, Electronic J. Probab., t. **14** (2009), pp. 1290–1327.

[56] Redig (F.) – Mathematical aspects of the abelian sand pile model, pp. 657–730 in *Mathematical statistical physics*, Elsevier, Amsterdam, 2006.

[57] Scheidegger (A. E.) – *A stochastic model for drainage patterns into a intramontane trench*, Bull. Assoc. Sci. Hydrol., t. **12** (1967), pp. 15–20.

[58] Smoluchowski (M.) – *Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen*, Physik. Zeitschr., t. **17** (1916), pp. 557–599.

[59] Sornette (D.) – *Critical market crashes*, Physics Reports, t. **378** (2003), pp. 1–98.

[60] Stahl (A.) – *Personnal communication*, 2010.

[61] Stauffer (D.) & Sornette (D.) – *Self-organized percolation model for stock market fluctuations*, Physica A, t. **271** (1999), pp. 496–506.

[62] Tardos (G.) – *Polynomial bound for a chip firing game on graphs*, SIAM J. Discrete Math., t. **1** (1988), pp. 397–398.

[63] Velenik (Y.) – *Le modèle d'Ising*, (2009), `http://cel.archives-ouvertes.fr/cel-00392289`.

[64] Volkov (S.) – *Forest fires on $\mathbb{Z}_+$ with ignition only at* 0, preprint, `http://arxiv.org/abs/0907.1821`, 2009.

[65] Zhang (Y. C.) – *Scaling theory of self-organized criticality*, Phys. Rev. Lett., t. **63** (1989), pp. 470–473.

[66] Zinck (R.) & Grimm (V.) – *More realistic than anticipated: a classical forest-fire model from statistical physics captures real fire shapes*, The Open Ecology J., t. **1** (2008), pp. 8–13.

[67] Zinck (R.), Johst (K.) & Grimm (V.) – *Wildfire, landscape diversity and the Drossel-Schwabl model*, Ecological Modelling, t. **221** (2010), pp. 98–105.