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T. Mochizuki
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
E-mail : takuro@kurims.kyoto-u.ac.jp

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In memory of Tora, *the cat of RIMS*

HOLONOMIC $\mathcal{D}$-MODULES WITH BETTI STRUCTURE

Takuro Mochizuki

***Abstract.* —** We define the notion of Betti structure for holonomic $\mathcal{D}$-modules which are not necessarily regular singular. We establish the fundamental functorial properties. We also give auxiliary analysis of holomorphic functions of various types on the real blow up.

***Résumé* ($\mathcal{D}$-modules holonomes munis d'une structure de Betti)**
Nous définissons la notion de structure Betti pour les $\mathcal{D}$-modules holonomes qui ne sont pas nécessairement singuliers réguliers. Nous établissons leurs propriétés fonctorielles principales. Nous donnons également une analyse supplémentaire des fonctions holomorphes de divers types sur l'éclatement réel.

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CHAPTER 1

INTRODUCTION

In this paper, we introduce the notion of Betti structure for holonomic $\mathcal{D}$-modules, motivated by a question in [**13**]. For regular holonomic $\mathcal{D}$-modules, it is clearly defined by the Riemann-Hilbert correspondence, which is a basis of the theory of mixed Hodge modules (see [**55**]–[**58**]). Namely, a Betti structure of a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is defined to be a $\mathbb{Q}$-perverse sheaf $\mathcal{F}$ with an isomorphism $\alpha : \mathcal{F} \otimes \mathbb{C} \simeq \mathrm{DR}_X \mathcal{M}$. It has a nice functorial property for some of standard functors such as pull back, push-forward, dual, etc., in the algebraic situation.

As for the non-regular case, there has been a significant progress toward a generalized Riemann-Hilbert correspondence between holonomic $\mathcal{D}$-modules and some topological objects, a kind of perverse sheaves equipped with "Stokes structure" in some sense. The asymptotic analysis for good meromorphic flat bundles (see [**33**], [**52**] and [**47**]) and the existence of resolution of turning points (see [**26**], [**27**], [**47**]) lead us a rather satisfactory understanding of the structure of meromorphic flat bundles. Moreover, the recent work of A. D'Agnolo and M. Kashiwara [**10**], [**11**] based on the theory of Ind-sheaves [**24**] gives us a description of holonomic $\mathcal{D}$-modules in terms of some topological objects. It should also lead us to a thorough theory of Betti structure of holonomic $\mathcal{D}$-modules.

However, except in the one dimensional case, it turned out that a rather complicated machinery is necessary for the complete description of generalized Riemann-Hilbert correspondence. (See [**11**] and [**24**]; see also [**54**].) In this study, we shall directly define the notion of "Betti structure" for holonomic $\mathcal{D}$-modules with functorial property by using only the classical machinery of holonomic $\mathcal{D}$-modules and perverse sheaves. It still requires non-trivial tasks, and provides us with non-trivial consequences on the compatibility of the

Stokes structure and the $\mathbb{Q}$-structure. We hope that it would be useful for direct understanding of Betti structures and for a further study toward the generalized Riemann-Hilbert correspondence, at least temporarily.

1.1. Pre-Betti structure

To define the notion of Betti structure of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, it is a most naive idea to consider a pair of $\mathbb{Q}$-perverse sheaf $\mathcal{F}$ and an isomorphism

$$\alpha : \mathcal{F} \otimes \mathbb{C} \simeq \mathrm{DR}_X(\mathcal{M})$$

as above, which is called a pre-Betti structure of $\mathcal{M}$ in this paper.A holonomic $\mathcal{D}_X$-module with a pre-Betti structure is called a pre-$\mathbb{Q}$-holonomic $\mathcal{D}_X$-module. We should say that pre-Betti structure is too naive for the following reasons:

- ▷ It is not so intimately related with Stokes structure.
- ▷ Although pre-Betti structures have nice functoriality with respect to dual and proper push-forward, they are not functorial with respect to the push-forward for open immersion, the pull back, the nearby cycle and vanishing cycle functors. Recall that the de Rham functor is not compatible with the latter class of functors, when irregular singularities are present.

It is the main goal in this paper to introduce a condition for a pre-Betti structure to be a "Betti structure". We use an inductive way on the dimension of the support, which was a strategy of M. Saito to define his mixed and pure Hodge modules [**55**] and [**57**].

In the following, a $\mathbb{Q}$-structure of a $\mathbb{C}$-perverse sheaf $\mathcal{F}_{\mathbb{C}}$ is a $\mathbb{Q}$-perverse sheaf $\mathcal{F}_{\mathbb{Q}}$ with an isomorphism $\mathcal{F}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{F}_{\mathbb{C}}$.

1.2. Betti structure in the one dimensional case

We explain our condition for Betti structure in the one dimensional case.

1.2.1. The generalized Riemann-Hilbert correspondence in the one dimensional case. — We know the well established theory on the general structure of holonomic $\mathcal{D}$-modules on curves (the generalized Riemann-Hilbert correspondence). Namely, in the one dimensional case, we have a natural bijective correspondence between meromorphic flat bundles and local systems with Stokes structure, and any holonomic $\mathcal{D}$-modules are described as the gluing of meromorphic flat bundles and skyscraper $\mathcal{D}$-modules. We shall review it very briefly. For simplicity, we consider holonomic $\mathcal{D}$-modules on $X = \Delta = \{|z| < 1\}$ which may have a singularity at the origin $D = \{O\}$.

1.2.1.1. The Stokes structure of meromorphic flat bundles. — Let V be a meromorphic flat bundle on (X, D). Let $\pi : \widetilde{X}(D) \to X$ be the real blow up along D. Let $\mathcal{L}$ be the local system on $\widetilde{X}(D)$ associated to the flat bundle $V_{|X-D}$. Let P be any point of $\pi^{-1}(D)$. According to the classical asymptotic analysis, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$ given by the growth order of flat sections with respect to any meromorphic frame of V. The meromorphic flat bundle V can be reconstructed from the flat bundle $V_{|X-D}$ and the system of filtrations $\{\mathcal{F}^P \bigm| P \in \pi^{-1}(D)\}$, which is the Riemann-Hilbert-Birkhoff correspondence for meromorphic flat bundles on curves.

Let $V^\vee$ be the dual of V as a meromorphic flat bundle, and let $V_! := \boldsymbol{D}_X V^\vee$ be the dual of $V^\vee$ as a $\mathcal{D}_X$-module. Let us recall that the de Rham complexes $\mathrm{DR}_X(V)$ and $\mathrm{DR}_X(V_!)$ can be described in terms of Stokes filtrations. Let $\mathcal{L}^{\leq D}$ and $\mathcal{L}^{<D}$ be the constructible subsheaves of $\mathcal{L}$ such that $\mathcal{L}^{\leq D}_P = \mathcal{F}^P_{\leq 0}(\mathcal{L}_P)$ and $\mathcal{L}^{<D}_P = \mathcal{F}^P_{<0}(\mathcal{L}_P)$. Then, we have natural isomorphisms:

$$\mathrm{DR}(V) \simeq R\pi_* \mathcal{L}^{\leq D}[1], \quad \mathrm{DR}(V_!) \simeq R\pi_* \mathcal{L}^{<D}[1]. \tag{1}$$

1.2.1.2. Gluing of holonomic $\mathcal{D}$-modules. — Let us very briefly recall a key construction due to A. Beilinson [**4**] on the gluing of holonomic $\mathcal{D}$-modules, which we will review in §2.2 in more details. (See also [**32**] and [**59**] for the other formalisms for gluing.) Let $\mathcal{M}$ be any holonomic $\mathcal{D}_X$-module such that $V := \mathcal{M}(*D)$ is a meromorphic flat bundle on (X, D). We have the natural morphisms $V_! \xrightarrow{a_0} \mathcal{M} \xrightarrow{b_0} V$. According to [**4**], we have the $\mathcal{D}$-modules $\Xi_z(V)$ and $\psi_z(V)$ associated to V, with morphisms

$$\psi_z(V) \xrightarrow{a_1} \Xi_z(V) \xrightarrow{b_1} \psi_z(V), \qquad V_! \xrightarrow{a_2} \Xi_z(V) \xrightarrow{b_2} V. \tag{2}$$

It can be shown that $b_0 \circ a_0 = b_2 \circ a_2$. We also have $b_2 \circ a_1 = 0$ and $b_1 \circ a_2 = 0$. We obtain the $\mathcal{D}$-module $\phi_z(\mathcal{M})$ as the cohomology of the naturally associated complex:

$$V_! \longrightarrow \Xi_z(V) \oplus \mathcal{M} \longrightarrow V \tag{3}$$

We have the naturally induced morphisms $\psi_z(V) \xrightarrow{\mathrm{can}} \phi_z(\mathcal{M}) \xrightarrow{\mathrm{var}} \psi_z(V)$. Then, $\mathcal{M}$ is reconstructed as the cohomology of the complex:

$$\psi_z(V) \longrightarrow \Xi_z(V) \oplus \phi_z(\mathcal{M}) \longrightarrow \psi_z(V) \tag{4}$$

Recall that $\Xi_z(V)$, $\psi_z(V)$, and $\phi_z(\mathcal{M})$ are called the maximal extension, the nearby cycle sheaf, and the vanishing cycle sheaf of $\mathcal{M}$.

1.2.2. Betti structure of holonomic $\mathcal{D}$-modules on curves. — We explain when a pre-Betti structure of holonomic $\mathcal{D}$-modules seems eligible to be called a Betti structure in the one dimensional case. Essentially, the condition describes a compatibility with the Stokes structure.

1.2.2.1. Good $\mathbb{Q}$-structure of meromorphic flat bundles. — Let V be a meromorphic flat bundle on (X, D), and let $\mathcal{L}$ denote the associated local system on $\widetilde{X}(D)$ with the Stokes structure. A $\mathbb{Q}$-structure of V is a $\mathbb{Q}$-structure of the associated local system on $X \setminus D$, which is equivalent to a $\mathbb{Q}$-structure of $\mathcal{L}$. It is called a good $\mathbb{Q}$-structure of V if the Stokes filtrations $\mathcal{F}^P$ $(P \in \pi^{-1}(D))$ are defined over $\mathbb{Q}$, with respect to the induced $\mathbb{Q}$-structure of $\mathcal{L}$. By the isomorphisms (1), we obtain the pre-Betti structures of V and $V_!$. Moreover, it is easy to observe that $\psi_z(V)$ and $\Xi_z(V)$ are also naturally equipped with pre-Betti structures such that the morphisms a_i and b_i $(i = 1, 2)$ are compatible with pre-Betti structures.

1.2.2.2. Betti structure of holonomic $\mathcal{D}$-modules on curves. — Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on (X, D) such that $V := \mathcal{M}(*D)$ is a meromorphic flat bundle. Let $(\mathcal{F}, \alpha)$ be a pre-Betti structure of $\mathcal{M}$. We call it a Betti structure if the following holds:

- ▷ The induced $\mathbb{Q}$-structure on $\mathrm{DR}(V_{|X-D})$ induces a good $\mathbb{Q}$-structure of V. As remarked above, we have the induced pre-Betti structures on V and $V_!$.
- ▷ The natural morphisms a_0 and b_0 are compatible with the pre-Betti structures.

Note that we obtain a pre-Betti structure on $\phi_z(\mathcal{M})$ from the expression as the cohomology of the complex (3), and the morphisms var and can are compatible with the pre-Betti structures. The pre-Betti structure of $\mathcal{M}$ can be reconstructed from the pre-Betti structure of $\phi_z(\mathcal{M})$ and the good $\mathbb{Q}$-structure of V.

1.3. Betti structure in the higher dimensional case

We would like to generalize the notion of Betti structure in the higher dimensional case.

1.3.1. Good meromorphic flat bundle and good $\mathbb{Q}$-structure. — Let X be any complex manifold with a simple normal crossing hypersurface D.

It is fundamental to understand the structure of good meromorphic flat bundles on (X, D), which is now well established after the work of H. Majima, C. Sabbah and the author. (See [**33**], [**47**], [**48**], [**52**] and [**54**]; see [**49**] for a survey.) Very briefly, the asymptotic analysis for meromorphic flat bundles on curves can be naturally generalized for good meromorphic flat bundles in the higher dimensional case, and we obtain the Riemann-Hilbert-Birkhoff correspondence, which is a natural correspondence between good meromorphic flat bundles and local systems with Stokes structure.

Let us recall it very briefly. Let (V, ∇) be a good meromorphic flat bundle. Let $\pi : \widetilde{X}(D) \to X$ be the real blow up along D, which means in this paper the fiber product of the real blow up along the irreducible components of D taken over X. Let $\mathcal{L}$ be the local system on $\widetilde{X}(D)$ associated to $V_{|X-D}$. For any point $P \in \pi^{-1}(D)$, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$. It satisfies a compatibility condition with the Stokes filtrations $\mathcal{F}^Q$ for Q which are close to P. We can reconstruct V from $V_{|X-D}$ and the system of filtrations $\{\mathcal{F}^P \bigm| P \in \pi^{-1}(D)\}$. Moreover, if we are given a local system with the family of Stokes filtrations $\{\mathcal{F}^P \mid P \in \pi^{-1}(D)\}$ satisfying the compatibility condition, we have the corresponding good meromorphic flat bundle V. This is the Riemann-Hilbert-Birkhoff correspondence for good meromorphic flat bundles.

As in the one dimensional case, the de Rham complexes of V and $V_!$ are described in terms of the local system $\mathcal{L}$ with the Stokes structure. We obtain the constructible subsheaf $\mathcal{L}^{\leq D}$ of $\mathcal{L}$ which consists of flat sections with the moderate growth. It is described as $\mathcal{L}^{\leq D}_P = \mathcal{F}^P_{\leq 0}(\mathcal{L}_P)$ $(P \in \pi^{-1}(D))$ in terms of the Stokes filtrations. Let $\mathcal{L}^{<D}$ be the constructible subsheaf of $\mathcal{L}$, which consists of flat sections with rapid decay along D. It is also described in terms of the Stokes filtration (see §5.1.2). Then, we have $\mathrm{DR}_X(V) \simeq R\pi_*\mathcal{L}^{\leq D}[\dim X]$ and $\mathrm{DR}_X(V_!) \simeq R\pi_*\mathcal{L}^{<D}[\dim X]$ as in (1).

For any holomorphic function g on X such that $g^{-1}(0) = D$, we obtain $\mathcal{D}_X$-modules $\psi_g(V)$ and $\Xi_g(V)$ with morphisms as in (2) by using the formalism of Beilinson. Their de Rham complexes are also described in terms of the local system $\mathcal{L}$ with the Stokes filtrations.

As in the one dimensional case, a $\mathbb{Q}$-structure of V is a $\mathbb{Q}$-structure of the associated local system on $X \setminus D$, which is equivalent to a $\mathbb{Q}$-structure of $\mathcal{L}$. It is called a good $\mathbb{Q}$-structure of V if the Stokes filtrations are defined over $\mathbb{Q}$. If V is equipped with a good $\mathbb{Q}$-structure, the $\mathcal{D}_X$-modules V, $V_!$, $\Xi_g(V)$ and $\psi_g(V)$ are naturally equipped with pre-Betti structures, and the natural morphisms as in (2) are compatible with the pre-Betti structures.

1.3.2. Good $\mathbb{Q}$-structure of meromorphic flat connections. — In the higher dimensional case, not all meromorphic flat bundles are good, which is one of the main difficulties. Let us recall local resolutions of turning points due to K. Kedlaya [**26**], [**27**]. (See [**52**] for the original conjecture; see also [**44**] and [**47**] for the algebraic case.)

Let X be a complex manifold with a hypersurface D. Let V be a reflexive $\mathcal{O}_X(*D)$-module with a flat connection, which is called a meromorphic flat connection [**38**]. For any $P \in X$, there exist a neighbourhood X_P of P in X and a projective birational morphism $\lambda_P : \check{X}_P \to X_P$ such that

(i) $\check{X}_P$ is smooth and $\check{D}_P := \lambda_P^{-1}(D)$ is normal crossing,

(ii) $\check{X}_P \setminus \check{D}_P \simeq X_P \setminus D$,

(iii) $\check{V}_P := \lambda_P^* V$ is a good meromorphic flat bundle on $(\check{X}_P, \check{D}_P)$. (See Theorem 8.2.2 of [**27**].)

Such (X_P, λ_P) is called a local resolution of V in this paper. If X and V are algebraic, we have a global resolution. (See Theorem 8.1.3 of [**27**] or Theorem 16.2.1 of [**47**].)

Then, the notion of good $\mathbb{Q}$-structure is generalized for meromorphic flat connections which are not necessarily good. Namely, a $\mathbb{Q}$-structure of V is called good if the induced $\mathbb{Q}$-structure of good meromorphic flat bundles $\check{V}_P$ are good for any local resolutions (X_P, λ_P). Even in this case, the de Rham complexes $\mathrm{DR}_X(V)$ and $\mathrm{DR}_X(V_!)$ have naturally induced $\mathbb{Q}$-structures. Moreover, if we are given a holomorphic function g on X such that $g^{-1}(0) = D$, the holonomic $\mathcal{D}_X$-modules $\psi_g(V)$ and $\Xi_g(V)$ are naturally equipped with pre-Betti structures, with which the morphisms in (2) are compatible.

1.3.3. Cells and gluing. — Let us recall that any holonomic $\mathcal{D}$-module $\mathcal{M}$ can be described as the gluing of a "cell" and a holonomic $\mathcal{D}$-module $\mathcal{M}'$ whose support $\operatorname{Supp}\mathcal{M}'$ is strictly smaller than $\operatorname{Supp}\mathcal{M}$. Namely, for any $P \in \operatorname{Supp}\mathcal{M}$, there exists a tuple $\mathcal{C} = (Z, U, \varphi, V)$ as follows:

(Cell 1) $\varphi : Z \to X$ is a morphism of complex manifolds such that $P \in \varphi(Z)$ and that $\dim Z$ is equal to the dimension of $\operatorname{Supp}\mathcal{M}$ at P. We impose that there exists a neighbourhood X_P of P in X such that $\varphi : Z \to X_P$ is projective.

(Cell 2) $U \subset Z$ is the complement of a hypersurface D_Z. We impose that the restriction $\varphi_{|U}$ is an immersion, and that there exists a hypersurface H of X_P such that $\varphi^{-1}(H) = D_Z$.

(Cell 3) V is a good meromorphic flat bundle on (Z, D_Z). We impose $\mathcal{M}(*H) = \varphi_\dagger V$ for a hypersurface H as in (Cell 2). Note that we obtain the natural morphisms $\varphi_\dagger V_! \to \mathcal{M} \to \varphi_\dagger V$.

Such $\mathcal{C}$ is called a cell of $\mathcal{M}$ at P. A holomorphic function g on X is called a cell function for $\mathcal{C}$ if $\varphi(U) = \operatorname{Supp}\mathcal{M} \setminus g^{-1}(0)$. We set $g_Z := g \circ \varphi$. We have natural isomorphisms $\varphi_\dagger \Xi_{g_Z}(V) \simeq \Xi_g \varphi_\dagger(V)$ and $\varphi_\dagger \psi_{g_Z}(V) \simeq \psi_g \varphi_\dagger(V)$. By the formalism of Beilinson, the $\mathcal{D}_X$-module $\phi_g(\mathcal{M})$ is obtained as the cohomology of the complex

$$\varphi_\dagger V_! \longrightarrow \Xi_g \varphi_\dagger(V) \oplus \mathcal{M} \longrightarrow \varphi_\dagger V. \tag{5}$$

We have the description of $\mathcal{M}$ around P as the cohomology of the complex

$$\psi_g(\varphi_\dagger V) \longrightarrow \Xi_g(\varphi_\dagger V) \oplus \phi_g(\mathcal{M}) \longrightarrow \psi_g(\varphi_\dagger V).$$

In other words, $\mathcal{M}$ is described as the gluing of the cell $\mathcal{C}$ and $\phi_g(\mathcal{M})$.

1.3.4. Betti structure

1.3.4.1. Compatibility of cell and pre-Betti structure. — We introduce the compatibility condition of a cell $\mathcal{C}$ and a pre-Betti structure $\mathcal{F}$ of $\mathcal{M}$. We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible if the following holds:

- ▷ Note that the flat bundle $V_{|U}$ has an induced $\mathbb{Q}$-structure. We suppose that it is a good $\mathbb{Q}$-structure in the sense of §1.3.2.
- ▷ By the first condition, $\varphi_\dagger V$, $\varphi_\dagger V_!$, $\Xi_g \varphi_\dagger V$ and $\psi_g \varphi_\dagger V$ are equipped with the induced pre-Betti structures. Then, we impose that the morphisms $\varphi_\dagger V_! \to \mathcal{M} \to \varphi_\dagger V$ are compatible with pre-Betti structures.

Such a cell $\mathcal{C}$ is called a $\mathbb{Q}$-cell of $\mathcal{M}$ at P. Since $\phi_g(\mathcal{M})$ is the cohomology of the complex (5), it is equipped with the induced pre-Betti structure.

1.3.4.2. Inductive definition of Betti structure. — Let us define the notion of Betti structure of $\mathcal{M}$ at P, inductively on the dimension of $\operatorname{Supp}\mathcal{M}$. If $\dim_P \operatorname{Supp}\mathcal{M} = 0$, a Betti structure is defined to be a pre-Betti structure. Let us consider the case $\dim_P \operatorname{Supp}\mathcal{M} \leq n$. We say that a pre-Betti structure of $\mathcal{M}$ is a Betti structure at P if there exists an n-dimensional $\mathbb{Q}$-cell $\mathcal{C} = (Z, \varphi, U, V)$ at P with the following properties:

- ▷ $\dim_P((\operatorname{Supp}\mathcal{M} \cap X_P) \setminus \varphi(Z)) < n$ for some neighbourhood X_P of P in X.
- ▷ For a cell function g for $\mathcal{C}$, the induced pre-Betti structure of $\phi_g(\mathcal{M})$ is a Betti structure at P. Note that $\dim \operatorname{Supp}\phi_g(\mathcal{M}) < n$ by the first condition.

A holonomic $\mathcal{D}$-module with Betti structure is called a $\mathbb{Q}$-holonomic $\mathcal{D}$-module. Morphisms of $\mathbb{Q}$-holonomic $\mathcal{D}_X$-modules are defined to be morphisms of pre-$\mathbb{Q}$-holonomic $\mathcal{D}_X$-modules.

REMARK 1.3.1. — The above is not exactly the same as the definition in §7.2, but they give equivalent objects. □

1.4. Main goal

1.4.1. The category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules. — Besides giving the details on the above arguments, it is our main purpose to show that our notion of Betti structure is nice. The category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules should contain the holonomic $\mathcal{D}$-modules naturally induced from any meromorphic flat connections with a good $\mathbb{Q}$-structure, for which we have the following theorem.

THEOREM 1.4.1. — *Let X be any complex manifold with a hypersurface D. Let V be any meromorphic flat connection on (X, D) with a good $\mathbb{Q}$-structure. Then, the natural pre-Betti structures of V and $V_!$ are Betti structures.*

See Theorem 8.1.3 for a refined result. Some of the functors for holonomic $\mathcal{D}$-modules should be enriched with Betti structures, as in the following theorems.

THEOREM 1.4.2 (Theorem 8.1.1). — *Let $F : X \to Y$ be any projective morphism of complex manifolds. For any $\mathbb{Q}$-holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the push-forward $F_\dagger^i\mathcal{M}$ are also naturally $\mathbb{Q}$-holonomic for any i.*

THEOREM 1.4.3 (Theorem 8.1.4). — *Let X be any complex manifold with a hypersurface D. Let $\mathcal{M}$ be any $\mathbb{Q}$-holonomic $\mathcal{D}_X$-module. Then, $\mathcal{M} \otimes \mathcal{O}_X(*D)$ has a unique Betti structure, for which $\mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_X(*D)$ is compatible with the Betti structures.*

THEOREM 1.4.4 (Proposition 8.3.7). — *Let X be any complex manifold with a hypersurface D. Let $\mathcal{M}$ be any $\mathbb{Q}$-holonomic $\mathcal{D}_X$-module. Let V be any meromorphic connection on (X, D) with a good $\mathbb{Q}$-structure. Then, $\mathcal{M} \otimes V$ is naturally a $\mathbb{Q}$-holonomic $\mathcal{D}_X$-module.*

The following is an easier result.

THEOREM 1.4.5

- ▷ *The category of $\mathbb{Q}$-holonomic $\mathcal{D}_X$-modules is abelian.*
- ▷ *The dual of $\mathbb{Q}$-holonomic $\mathcal{D}_X$-modules are naturally $\mathbb{Q}$-holonomic.*

▷ *Let $\mathcal{M}$ be a $\mathbb{Q}$-holonomic $\mathcal{D}_X$-module. Let $\mathcal{M}' \subset \mathcal{M}$ be a subobject in the category of pre-$\mathbb{Q}$-holonomic $\mathcal{D}_X$-modules. Then, $\mathcal{M}'$ is also $\mathbb{Q}$-holonomic. We have a similar claim for quotients.*

By using the theorems, we obtain that the category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules contains expected objects. For example, it contains the holonomic $\mathcal{D}$-modules obtained from the structure sheaf of any algebraic variety by successive use of the pull back and the push-forward by algebraic morphisms, and the exponential twist by algebraic functions. (This type of holonomic $\mathcal{D}$-modules are closely related with extended exponential-motivic $\mathcal{D}$-modules in [**28**].) It implies the compatibility of the $\mathbb{Q}$-structure and the Stokes structure for some naturally obtained meromorphic flat bundles. Such phenomena are expected in the non-commutative Hodge theory [**25**].

In the algebraic case, the derived category of $\mathbb{Q}$-holonomic $\mathcal{D}$-modules is equipped with standard functoriality.

THEOREM 1.4.6. — *The category of $\mathbb{Q}$-holonomic algebraic $\mathcal{D}$-modules is equipped with the standard functors such as dual, push-forward, pull-back, tensor product, inner homomorphism, the nearby and vanishing cycle functors, compatible with those for the category of holonomic algebraic $\mathcal{D}$-modules with respect to the forgetful functor.*

1.4.2. Analysis on real blow up. — We also give some analysis on the real blow up, which is a complement to [**54**]. Very briefly, we can capture the Stokes structure by considering the de Rham complex on the real blow up, at least in the case of good meromorphic flat bundles. We have several useful classes of functions on the real blow up, the moderate growth, the rapid decay, and the Nilsson type. We study or review the fundamental property of the sheaves of such functions and the corresponding de Rham complexes. We will not restrict ourselves to our main purpose, i.e., the study on Betti structure. For example, we shall prove that the sheaf of holomorphic functions of moderate growth is flat over the sheaf of holomorphic functions on the underlying space (Theorem 4.1.1). Although we will not use it in this paper, it is quite basic, and the author expects that it would be useful for a further study.

REMARK 1.4.7. — G. Morando informed the author that the theory of ind-sheaves [**24**] provides us with a powerful method to study analysis on the real blow up. (See also the recent work by A. D'Agnolo and M. Kashiwara [**10**].) While the author hopes that it would make the subject more transparent, he also hopes that his direct way would also be significant for our understanding at this moment. □

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CHAPTER 2

PRELIMINARY

2.1. Notation and words

2.1.1. Dual, push-forward and de Rham functor. — We prepare some notation. See very useful text books [**17**] and [**22**] for more details and precisions on $\mathcal{D}$-modules. Let X be a complex manifold with $\dim X = d_X$. Let $\mathcal{D}_X$ denote the sheaf of holomorphic differential operators on X. In this paper, $\mathcal{D}_X$-module means left $\mathcal{D}_X$-module. Let $\mathrm{Hol}(X)$ be the category of holonomic $\mathcal{D}_X$-modules, and let $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ be the derived category of cohomologically bounded holonomic $\mathcal{D}_X$-complexes.Let Ω^j_X denote the sheaf of holomorphic j-forms. The invertible sheaf $\Omega^{d_X}_X$ is denoted by Ω_X. The sheaves of C^∞-(p,q)-forms are denoted by $\Omega^{p,q}_X$. The dual functor on the derived category of $\mathcal{D}_X$-modules is denoted by $\boldsymbol{D}_X$, i.e.,

$$\boldsymbol{D}_X\mathcal{M}^\bullet := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X \otimes \Omega_X^{\otimes -1})[d_X].$$

Recall that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $\boldsymbol{D}_X\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. For $\mathcal{D}_X$-modules $\mathcal{M}_i$ $(i = 1, 2)$, the tensor product $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$ is naturally a $\mathcal{D}_X$-module. For any tangent vector field v, we have

$$v(m_1 \otimes m_2) = (vm_1) \otimes m_2 + m_1 \otimes (vm_2).$$

The $\mathcal{D}_X$-module is denoted by $\mathcal{M}_1 \otimes^D \mathcal{M}_2$. It is also denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2$ if there is no risk of confusion.

Lemma 2.1.1. — *Let $\mathcal{M}$ be any holonomic $\mathcal{D}_X$-module. Let V be any $\mathcal{D}_X$-module, which is coherent and locally free as an $\mathcal{O}_X$-module. Its dual is denoted by $V^\vee$. Then, we have a natural isomorphism*

$$\boldsymbol{D}_X(\mathcal{M} \otimes^D V) \simeq (\boldsymbol{D}_X\mathcal{M}) \otimes^D V^\vee.$$

Proof. — We recall Remark 3.4 in [**22**]. For any left $\mathcal{D}_X$-module $\mathcal{N}$, we have the left $\mathcal{D}_X$-action on $\mathcal{D}_X \otimes^D \mathcal{N}$. It is also equipped with a right $\mathcal{D}_X$-action given by the multiplication $(f \otimes m) \cdot g = fg \otimes m$ for $g \in \mathcal{D}_X$. The two-sided $(\mathcal{D}_X, \mathcal{D}_X)$-module is denoted by $\mathcal{N}_1$. Similarly, we have a left action of $\mathcal{D}_X$ on $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{N}$ (the tensor product $\otimes_{\mathcal{O}_X}$ is taken for the $\mathcal{O}_X$-module structure of $\mathcal{D}_X$ given by the right multiplication) given by the multiplication $g \cdot (f \otimes m) = gf \otimes m$ for $g \in \mathcal{D}_X$, and a right $\mathcal{D}_X$-action given by $(f \otimes m) \cdot v = fv \otimes m - f \otimes vm$ for a tangent vector v. The two-sided $(\mathcal{D}_X, \mathcal{D}_X)$-module is denoted by $\mathcal{N}_2$. We have a naturally defined $\mathcal{O}_X$-morphism $\mathcal{N} \to \mathcal{N}_1$ given by $m \mapsto 1 \otimes m$. It is naturally extended to a morphism of left $\mathcal{D}_X$-modules $\mathcal{N}_2 \to \mathcal{N}_1$. Actually, it is an isomorphism and compatible with the right $\mathcal{D}_X$-action, as remarked in [**22**].

We have two left $\mathcal{D}_X$-actions on $\mathcal{D}_X \otimes \Omega_X^{\otimes -1}$. The first one is the natural one, and the second one is induced by the right $\mathcal{D}_X$-action. They induce two $\mathcal{O}_X$-actions. Let $(\mathcal{D}_X \otimes \Omega_X^{\otimes -1}) \otimes^i_{\mathcal{O}_X} \mathcal{N}$ denote the tensor product with respect to the i-th one. Each is equipped with two left $\mathcal{D}_X$-actions. From the consideration in the previous paragraph, we obtain a natural isomorphism

$$\iota : \mathcal{N} \otimes^1_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1}) \longrightarrow \mathcal{N} \otimes^2_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1}),$$

compatible with the $\mathcal{D}_X$-actions.

Let us return to Lemma 2.1.1. We have the following natural isomorphisms of $\mathcal{D}_X$-modules:

$$\begin{aligned} (6)\quad \boldsymbol{D}_X(\mathcal{M} \otimes^D V) &= R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes^D V,\, \mathcal{D}_X \otimes \Omega_X^{\otimes -1}) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M},\, V^\vee \otimes^1_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1})\big) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M},\, V^\vee \otimes^2_{\mathcal{O}_X} (\mathcal{D}_X \otimes \Omega_X^{\otimes -1})\big) = (\boldsymbol{D}_X\mathcal{M}) \otimes^D V^\vee. \end{aligned}$$

Here, the first one is obtained by using Godement type injective resolution, and the second one is induced by ι above. □

For any field R, let R_X denote the sheaf on X associated to the constant presheaf valued in R. Let $D^b(R_X)$ (resp. $D^b_c(R_X)$) denote the derived category of cohomologically bounded (resp. bounded constructible) R_X-complexes, and let $\mathrm{Per}(X, R)$ denote the category of R-perverse sheaves. Let $\omega_{X,R}$ denote the dualizing complex of R_X-modules. It will be denoted by ω_X if there is no risk of confusion.

The dual functor on the derived category of R_X-modules is also denoted by $\boldsymbol{D}_X$, i.e., for an R_X-complex $\mathcal{F}^\bullet$, let

$$\boldsymbol{D}_X\mathcal{F}^\bullet := R\mathcal{H}om_{R_X}(\mathcal{F}^\bullet, \omega_{X,R}).$$

The de Rham functor is denoted by DR_X, i.e.,

$$\mathrm{DR}_X \mathcal{M} := \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M} = \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{M}[d_X].$$

According to [**19**], it gives a functor of triangulated categories

$$\mathrm{DR}_X : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_X)$$

compatible with the t-structures, where the t-structure of $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ is the natural one, and the t-structure of $D^b_c(\mathbb{C}_X)$ is given by the middle perversity. In particular, it induces an exact functor $\mathrm{DR}_X : \mathrm{Hol}(X) \to \mathrm{Per}(X, \mathbb{C})$. We can identify

$$\omega_X = \mathrm{DR}_X \mathcal{O}_X[d_X].$$

It is easy to observe that $\mathrm{DR}_X \mathcal{M} = 0$ implies $\mathcal{M} = 0$ for $\mathcal{M} \in \mathrm{Hol}(X)$. The functor $\mathrm{DR}_X : \mathrm{Hol}(X) \to \mathrm{Per}(X, \mathbb{C})$ is faithful, although it is not full in general.

Let $F : X \to Y$ be a morphism of complex manifolds. The push-forward for $\mathbb{C}_X$-complexes in the derived category is denoted by RF_*. (It is also denoted by F_* if there is no risk of confusion.) Its i-th perverse cohomology is denoted by $F^i_\dagger$. Put

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}\mathcal{D}_Y,$$
$$\mathcal{D}_{Y\leftarrow X} := \Omega_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}).$$

The push-forward for $\mathcal{D}_X$-complexes is denoted by $F_\dagger$, i.e.,

$$F_\dagger \mathcal{M} = RF_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}).$$

Its i-th cohomology is denoted by $F^i_\dagger$.

Recall that these functors are compatible on the derived categories. Let $F : X \to Y$ be a proper morphism of complex manifolds. We have natural transformations

$$\mathrm{DR}_Y \circ F_\dagger \simeq RF_* \circ \mathrm{DR}_X, \quad \boldsymbol{D}_X \circ \mathrm{DR}_X \simeq \mathrm{DR}_X \circ \boldsymbol{D}_X, \quad \boldsymbol{D}_Y \circ F_\dagger \simeq F_\dagger \circ \boldsymbol{D}_X.$$

In [**58**], the following diagram is constructed and it is proved to be commutative (see Theorem 3.3 of [**58**]):

$$\begin{array}{ccccc} RF_* \boldsymbol{D}_X \mathrm{DR}_X & \xrightarrow{\simeq} & RF_* \mathrm{DR}_X \boldsymbol{D}_X & \xrightarrow{\simeq} & \mathrm{DR}_Y F_\dagger \boldsymbol{D}_X \\ \simeq\downarrow & & & & \simeq\downarrow \\ \boldsymbol{D}_Y RF_* \mathrm{DR}_X & \xrightarrow{\simeq} & \boldsymbol{D}_Y \mathrm{DR}_Y F_\dagger & \xrightarrow{\simeq} & \mathrm{DR}_Y \boldsymbol{D}_Y F_\dagger\,. \end{array} \tag{7}$$

2.1.2. Hypersurfaces. — For any hypersurface $D \subset X$, let $\mathcal{O}_X(*D)$ denote the sheaf of meromorphic functions whose poles are contained in D. For $\mathcal{M} \in \mathrm{Hol}(X)$, we have $\mathcal{M}(*D), \mathcal{M}(!D) \in \mathrm{Hol}(X)$ given as follows:

$$\mathcal{M}(*D) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D), \quad \mathcal{M}(!D) := \boldsymbol{D}_X\big((\boldsymbol{D}_X\mathcal{M})(*D)\big).$$

We have naturally defined morphism $\mathcal{M} \to \mathcal{M}(*D)$. The morphism $\boldsymbol{D}_X(\mathcal{M}) \to \boldsymbol{D}_X(\mathcal{M})(*D)$ and the natural transformation $\boldsymbol{D}_X \circ \boldsymbol{D}_X \simeq \mathrm{id}_X$ induce $\mathcal{M}(!D) \to \mathcal{M}$. (See §3.3 and §A3.3 of [**22**] for $\boldsymbol{D}_X \circ \boldsymbol{D}_X \simeq \mathrm{id}$.) They are uniquely characterized that the restrictions to $X \setminus D$ are the identities. If D is given as the zero set of a holomorphic function f, they are denoted by $\mathcal{M}(*f)$ and $\mathcal{M}(!f)$, respectively. If we are given two hypersurfaces D_i $(i = 1, 2)$, we set

$$\mathcal{M}(\star_1 D_1)(\star_2 D_2) := \big(\mathcal{M}(\star_1 D_1)\big)(\star_2 D_2),$$

where $\star_i \in \{*, !\}$.

We put $\mathcal{D}_{X(*D)} := \mathcal{D}_X \otimes \mathcal{O}_X(*D)$.

A $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$ is called holonomic, if it is holonomic as a $\mathcal{D}_X$-module. Let $\mathrm{Hol}(X, *D)$ be the category of holonomic $\mathcal{D}_{X(*D)}$-modules, which is naturally a full subcategory of $\mathrm{Hol}(X)$. The dual functor on $\mathrm{Hol}(X, *D)$ is denoted by $\boldsymbol{D}_{X(*D)}$, i.e.,

$$\boldsymbol{D}_{X(*D)}(\mathcal{M}) = \boldsymbol{D}_X(\mathcal{M})(*D).$$

Let $j : X \setminus D \to X$ be the inclusion. We define a functor

$$j^* : \mathrm{Hol}(X) \longrightarrow \mathrm{Hol}(X, *D), \quad j^*(\mathcal{M}) = \mathcal{M}(*D).$$

The natural inclusion $\mathrm{Hol}(X, *D) \to \mathrm{Hol}(X)$ is denoted by j_*. Another functor $j_! : \mathrm{Hol}(X, *D) \to \mathrm{Hol}(X)$ is defined by $j_!(\mathcal{M}) := (j_*\mathcal{M})(!D)$. The functors j^*, j_* and $j_!$ are exact. In this notation, we have $\mathcal{M}(*D) = j_*j^*\mathcal{M}$ and $\mathcal{M}(!D) = j_!j^*\mathcal{M}$ for $\mathcal{M} \in \mathrm{Hol}(X)$.

It is generalized as follows.

Let H be a hypersurface of X and $k : X \setminus H \to X$ denote the inclusion. For $\mathcal{M} \in \mathrm{Hol}(X, *D)$, we define $k^*\mathcal{M} := \mathcal{M}(*H)$. We can naturally regard $\mathrm{Hol}(X, *(D \cup H))$ as a full subcategory of $\mathrm{Hol}(X, *D)$. The natural inclusion is denoted by k_*. We define another functor

$$k_! : \mathrm{Hol}\big(X, *(D \cup H)\big) \longrightarrow \mathrm{Hol}(X, *D), \quad k_!\mathcal{M} = j^*\Big(\big((j \circ k)_*\mathcal{M}\big)\big(!(D \cup H)\big)\Big).$$

Later (§6.4), we shall consider a successive composition of the operations.

2.1.3. Pre-K-holonomic $\mathcal{D}$-modules. — Let $\mathcal{M}$ be any holonomic $\mathcal{D}_X$-module. Let K be any subfield of $\mathbb{C}$. A pre-K-Betti structure of $\mathcal{M}$ is defined to be a K-perverse sheaf $\mathcal{F}$ with an isomorphism $\lambda : \mathcal{F} \otimes_K \mathbb{C} \simeq \mathrm{DR}_X \mathcal{M}$. Such a tuple $(\mathcal{M}, \mathcal{F}, \lambda)$ is called a pre-K-holonomic $\mathcal{D}_X$-module. We will often omit to denote λ. A morphism of K-holonomic $\mathcal{D}_X$-modules $(\mathcal{M}_1, \mathcal{F}_1) \to (\mathcal{M}_2, \mathcal{F}_2)$ is defined to be a pair of a morphism of $\mathcal{D}_X$-modules $\mathcal{M}_1 \to \mathcal{M}_2$ and a morphism of perverse sheaves $\mathcal{F}_1 \to \mathcal{F}_2$ such that the following induced diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}_1 \otimes_K \mathbb{C} & \xrightarrow{\simeq} & \mathrm{DR}_X(\mathcal{M}_1) \\ \downarrow & & \downarrow \\ \mathcal{F}_2 \otimes_K \mathbb{C} & \xrightarrow{\simeq} & \mathrm{DR}_X(\mathcal{M}_2) \end{array} .$$

The category of pre-K-holonomic $\mathcal{D}_X$-modules is denoted by $\mathrm{Hol}^{\mathrm{pre}}(X, K)$.

The following lemma is clear.

LEMMA 2.1.2. — $\mathrm{Hol}^{\mathrm{pre}}(X, K)$ *is abelian.* □

Let $\mathcal{F}$ be a pre-K-Betti structure of $\mathcal{M}$. We have induced pre-K-Betti structures $\boldsymbol{D}\mathcal{F}$ and $F^i_\dagger \mathcal{F}$ of $\boldsymbol{D}\mathcal{M}$ and $F^i_\dagger \mathcal{M}$, where $F : X \to Y$ be a proper morphism. We put

$$\boldsymbol{D}(\mathcal{M}, \mathcal{F}) := (\boldsymbol{D}\mathcal{M}, \boldsymbol{D}\mathcal{F}) \quad \text{and} \quad F^i_\dagger(\mathcal{M}, \mathcal{F}) := (F^i_\dagger \mathcal{M}, F^i_\dagger \mathcal{F}).$$

LEMMA 2.1.3. — *The isomorphism* $\boldsymbol{D}F_\dagger \mathcal{M} \simeq F_\dagger \boldsymbol{D}\mathcal{M}$ *is compatible with the induced pre-K-Betti structures.*

Proof. — Because (7) is commutative, we have the commutativity of the following naturally induced diagram:

$$\begin{array}{ccccc} \mathrm{DR}\, \boldsymbol{D}F_\dagger \mathcal{M} & \xrightarrow{\simeq} & \boldsymbol{D}F_\dagger \,\mathrm{DR}\, \mathcal{M} & \xrightarrow{\simeq} & \boldsymbol{D}F_\dagger \mathcal{F} \otimes \mathbb{C} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ \mathrm{DR}\, F_\dagger \boldsymbol{D}\mathcal{M} & \xrightarrow{\simeq} & F_\dagger \boldsymbol{D}\,\mathrm{DR}\, \mathcal{M} & \xrightarrow{\simeq} & F_\dagger \boldsymbol{D}\mathcal{F} \otimes \mathbb{C} \end{array} .$$

It means the claim of the lemma. □

2.1.4. Formal completion. — Let Y be a real analytic manifold. Let $\mathcal{C}^\infty_Y$ denote the sheaf of C^∞-functions on Y. For any real analytic subset Z, let $\mathcal{C}^{\infty<Z}_Y$ denote the subsheaf of $\mathcal{C}^\infty_Y$ which consists of the sections f such that the Taylor series of f at each point $P \in Z$ is 0. We set $\mathcal{C}^\infty_{\widehat{Z}} := \mathcal{C}^\infty_Y / \mathcal{C}^{\infty<Z}_Y$. We have other descriptions:

(i) It is the sheaf of Whitney functions of class C^∞ on Z, i.e., sections of ∞-jets along Z satisfying the conditions in Theorem I.2.2 of [**34**].

(ii) Let $\mathcal{I}_{Z,\infty}$ be the ideal sheaf of $\mathcal{C}^\infty_Y$ corresponding to Z. Then, $\mathcal{C}^\infty_{\widehat{Z}}$ is also isomorphic to $\varprojlim \mathcal{C}^\infty_Y/\mathcal{I}^m_{Z,\infty}$. (See the proof of Theorem I.4.1 of [**34**].)

For any $\mathcal{C}^\infty_Y$-module $\mathcal{F}$, let $\mathcal{F}_{|\widehat{Z}}$ denote $\mathcal{F}\otimes_{\mathcal{C}^\infty_Y}\mathcal{C}^\infty_{\widehat{Z}}$. Let Z_i $(i=1,2)$ be real analytic subsets in Y. According to Corollary IV.4.4 with Definition I.5.4 of [**34**], the following natural sequence is exact:

$$0\longrightarrow \mathcal{C}^\infty_{\widehat{Z_1\cup Z_2}}\longrightarrow \mathcal{C}^\infty_{\widehat{Z}_1}\oplus\mathcal{C}^\infty_{\widehat{Z}_2}\longrightarrow \mathcal{C}^\infty_{\widehat{Z_1\cap Z_2}}\longrightarrow 0.$$

Let Z_i $(i\in\Lambda)$ be real analytic subsets of Y. For any subset $I\subset\Lambda$, we put

$$Z_I:=\bigcap_{i\in I}Z_i\quad\text{and}\quad Z(I):=\bigcup_{i\in I}Z_i.$$

We fix a total order on Λ. For $J\subset K\subset\Lambda$, we have the restriction $r_{J,K}:\mathcal{C}^\infty_{\widehat{Z}_J}\to\mathcal{C}^\infty_{\widehat{Z}_K}$. If $K=J\sqcup\{i\}$, we put

$$\kappa(J,K):=\big\{k\in J\mid k<i\big\}\quad\text{and}\quad d_{J,K}:=(-1)^{\kappa(J,K)}r_{J,K}.$$

We set

$$\mathcal{K}^m(\mathcal{C}^\infty_{\widehat{Z}(I)}):=\bigoplus_{\substack{|J|=m+1\\ J\subset I}}\mathcal{C}^\infty_{\widehat{Z}_J}.$$

The above morphisms $d_{J,K}$ induce $d_m:\mathcal{K}^m(\mathcal{C}^\infty_{\widehat{Z}(I)})\to\mathcal{K}^{m+1}(\mathcal{C}^\infty_{\widehat{Z}(I)})$. Thus, we obtain a complex $\mathcal{K}^\bullet(\mathcal{C}^\infty_{\widehat{Z}(I)})$. By using the exactness in the previous paragraph, it can be proved that the natural inclusion $\mathcal{C}^\infty_{\widehat{Z}(I)}\to\mathcal{K}^0(\mathcal{C}^\infty_{\widehat{Z}(I)})$ induces a quasi-isomorphism $\mathcal{C}^\infty_{\widehat{Z}(I)}\simeq\mathcal{K}^\bullet(\mathcal{C}^\infty_{\widehat{Z}(I)})$. (See [**52**], for example.)

Let X be a complex manifold. For a complex analytic subset Z, we set

$$\mathcal{O}_{\widehat{Z}}:=\varprojlim\mathcal{O}_X/\mathcal{I}^m_Z,$$

where $\mathcal{I}_Z$ denote the ideal sheaf of Z.We set

$$\Omega^{\bullet,\bullet}_{\widehat{Z}}:=\Omega^{\bullet,\bullet}_{X|\widehat{Z}}$$

which is equipped with the differential operators ∂ and $\overline{\partial}$. If Z is smooth, it is easy to see that the natural inclusion $\mathcal{O}_{\widehat{Z}}\to\Omega^{0,\bullet}_{\widehat{Z}}$ is a quasi-isomorphism.

Let D be a simple normal crossing hypersurface with the irreducible decomposition $D=\bigcup_{i\in\Lambda}D_i$. By the above procedures, we obtain the complexes $\mathcal{K}^\bullet(\mathcal{O}_{\widehat{D}(I)})$. It is known that the natural inclusion $\mathcal{O}_{\widehat{D}(I)}\to\mathcal{K}^0(\mathcal{O}_{\widehat{D}(I)})$ induces a quasi-isomorphism $\mathcal{O}_{\widehat{D}(I)}\simeq\mathcal{K}^\bullet(\mathcal{O}_{\widehat{D}(I)})$. (See [**14**] and [**52**].) We also have $\Omega^{0,\bullet}_{\widehat{D}(I)}\simeq\mathcal{K}^\bullet(\Omega^{0,\bullet}_{\widehat{D}(I)})$. Then, we obtain $\mathcal{O}_{\widehat{D}(I)}\simeq\Omega^{0,\bullet}_{\widehat{D}(I)}$.

We recall a useful isomorphism due to Z. Mebkhout (Lemma 2.2.1.3 of [**43**]).[1]

PROPOSITION 2.1.4 (Z. Mebkhout). — *Let $\mathcal{M}$ be any coherent $\mathcal{D}_X$-module. Let Z be any hypersurface of X. Then,*

$$R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}(*Z), \mathcal{O}_{\widehat{Z}}\big) = 0 \quad \textit{and} \quad \mathcal{M}(!Z) \otimes^L_{\mathcal{D}_X} \mathcal{O}_{\widehat{Z}} = 0. \qquad \square$$

See (3.10) of [**22**] to deduce the second vanishing from the first.

2.2. Beilinson's construction

Let us recall Beilinson's beautiful construction of the nearby cycle functor, the vanishing cycle functor and the maximal functor, which is essential for our purpose. It is particularly convenient for the study of functoriality. See [**4**] for more details and precisions (see also [**32**] and [**59**]).

2.2.1. Preliminary. — Let k be any field of characteristic 0. Let

$$A := k((s)) \quad \text{and} \quad A^i := s^i k[[s]].$$

For $a \le b$, we put $A^{a,b} := A^a/A^b$. The multiplication of s induces a nilpotent endomorphism N_A of $A^{a,b}$. We put

$$G_m := \operatorname{Spec} k[t, t^{-1}].$$

We define

$$\mathfrak{I}^{a,b} := \mathcal{O}_{G_m} \otimes A^{a,b}.$$

It is equipped with the connection given by $\nabla\alpha = N_A(\alpha)(\mathrm{d}t/t)$ for $\alpha \in A^{a,b}$. We have natural morphisms $\mathfrak{I}^{a,b} \to \mathfrak{I}^{c,d}$ for $a \ge c$ and $b \ge d$, which are compatible with the connections. We have a natural isomorphism

$$\mathfrak{I}^{a,a+1} \simeq \mathfrak{I}^{0,1} = \mathcal{O}_{G_m}, \quad s^a \longleftrightarrow 1.$$

This construction makes sense also in the analytic situation. The multi-valued flat sections are formally given by $\alpha \cdot \exp(-s \log t)$ for $\alpha \in A^{a,b}$.

[1] The author thanks the referee who informed this result to him.

2.2.2. Nearby cycle functor and maximal functor. — Let X be any complex manifold with a hypersurface D. Let f be a meromorphic function on (X, D), i.e., the poles of f are contained in D. We set

$$\mathfrak{I}_f^{a,b} := f^*\mathfrak{I}^{a,b}(*D),$$

which are meromorphic flat bundles on $(X, f^{-1}(0) \cup D)$. Let

$$j : X - f^{-1}(0) \longrightarrow X.$$

For a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$, we obtain the holonomic $\mathcal{D}_{X(*D)}$-modules

$$\mathcal{M}_f^{a,b} := \mathcal{M} \otimes \mathfrak{I}_f^{a,b} = j_*j^*(\mathcal{M} \otimes \mathfrak{I}_f^{a,b}).$$

We obtain $\mathcal{D}_{X(*D)}$-modules $\Pi_{f!}^{a,b}\mathcal{M} := j_!j^*\mathcal{M}_f^{a,b}$ and $\Pi_{f*}^{a,b}\mathcal{M} := j_*j^*\mathcal{M}_f^{a,b}$. We define

$$\Pi_{f*!}^{a,b}(\mathcal{M}) := \varprojlim_{N\to\infty} \operatorname{Cok}(\Pi_!^{b,N}\mathcal{M} \to \Pi_*^{a,N}\mathcal{M}).$$

The following lemma is easy to see.

Lemma 2.2.1. — *For any point $P \in X$, there exists a neighbourhood X_P and a large integer N_0 such that the following natural morphisms are isomorphisms on X_P for any $N \geq N_0$:*

$$\operatorname{Cok}(\Pi_!^{b,N+1}\mathcal{M} \to \Pi_*^{a,N+1}\mathcal{M}) \longrightarrow \operatorname{Cok}(\Pi_!^{b,N}\mathcal{M} \to \Pi_*^{a,N}\mathcal{M}).$$

Proof. — See the proof of Lemma 4.1.1 of [**50**], for example. □

Beilinson defined the functors $\psi_f^{(a)} := \Pi_{f*!}^{a,a}$ and $\Xi_f^{(a)} := \Pi_{f*!}^{a,a+1}$. In the case $a = 0$, they are denoted by $\psi_f\mathcal{M}$ and $\Xi_f\mathcal{M}$, respectively. The multiplication of s naturally induces isomorphisms $\psi_f^{(a)}\mathcal{M} \simeq \psi_f^{(a+1)}\mathcal{M}$ and $\Xi_f^{(a)}\mathcal{M} \simeq \Xi_f^{(a+1)}\mathcal{M}$. Note that we have natural isomorphisms $\Pi_{f\star}^{a,a+1}(\mathcal{M}) \simeq j_\star j^*\mathcal{M}$ for $\star = *, !$ induced by the multiplication of a power of s. They will be implicitly identified. We have the exact sequences of holonomic $\mathcal{D}_{X(*D)}$-modules:

$$0 \to \Pi_{f!}^{a,a+1}\mathcal{M} \xrightarrow{c_1^{(a)}} \Xi_f^{(a)}\mathcal{M} \xrightarrow{c_2^{(a)}} \psi_f^{(a)}\mathcal{M} \to 0,$$

$$0 \to \psi_f^{(a+1)}\mathcal{M} \xrightarrow{d_1^{(a)}} \Xi_f^{(a)}\mathcal{M} \xrightarrow{d_2^{(a)}} \Pi_{f*}^{a,a+1}\mathcal{M} \to 0.$$

The multiplication of s and the endomorphism $c_2^{(a)} \circ d_1^{(a)}$ induce an endomorphism $N^{(a+1)}$ of $\psi_f^{(a+1)}\mathcal{M}$.

Recall the important observation due to Beilinson (see [**4**] for $\underset{\longleftrightarrow}{\lim}$):

$$\underset{\longleftrightarrow}{\lim}\Pi_{f!}^{a,b}\mathcal{M} \simeq \underset{\longleftrightarrow}{\lim}\Pi_{f*}^{a,b}\mathcal{M}.$$

In particular, it implies that $N^{(a+1)}$ is locally nilpotent. We also obtain the isomorphism

$$\Pi^{a,b}_{f*!}(\mathcal{M}) \simeq \varinjlim_{N\to\infty} \operatorname{Ker}\bigl(\Pi^{-N,b}_{f!}\mathcal{M} \to \Pi^{-N,a}_{f*}\mathcal{M}\bigr).$$

As in Lemma 2.2.1, $\operatorname{Ker}\bigl(\Pi^{-N,b}_{f!}\mathcal{M} \to \Pi^{-N,a}_{f*}\mathcal{M}\bigr)$ is locally independent of the choice of a large N. See §4.1 of [**50**] for an elementary argument. In particular, we have the identifications

$$\psi^{(a)}_f\mathcal{M} \simeq \varinjlim_{N\to\infty} \operatorname{Ker}\bigl(\Pi^{-N,a}_{f!}\mathcal{M} \to \Pi^{-N,a}_{f*}\mathcal{M}\bigr), \tag{8}$$

$$\Xi^{(a)}_f\mathcal{M} \simeq \varinjlim_{N\to\infty} \operatorname{Ker}\bigl(\Pi^{-N,a+1}_{f!}\mathcal{M} \to \Pi^{-N,a}_{f*}\mathcal{M}\bigr). \tag{9}$$

Remark 2.2.2. — When we distinguish that we work on the category of $\mathcal{D}_{X(*D)}$-modules, we will use the symbols $\psi^{(a)}_f(\mathcal{M},*D)$, $\Xi^{(a)}_f(\mathcal{M},*D)$, etc. □

2.2.3. Vanishing cycle functor and gluing. — Let f be as above. Let $\mathcal{M}_X$ be any holonomic $\mathcal{D}_{X(*D)}$-module. We set $\mathcal{M} := \mathcal{M}_X(*f)$. We have the natural identifications $\Pi^{a,b}_{f\star}\mathcal{M}_X = \Pi^{a,b}_{f\star}\mathcal{M}$ for $\star = *,!$. We also have $\Pi^{a,b}_{f*!}\mathcal{M}_X = \Pi^{a,b}_{f*!}\mathcal{M}$. In particular, $\psi^{(a)}_f\mathcal{M}_X = \psi^{(a)}_f\mathcal{M}$ and $\Xi^{(a)}_f\mathcal{M}_X = \Xi^{(a)}_f\mathcal{M}$. We set

$$\mathcal{M}^{(a)}_X := \mathcal{M}_X \otimes A^{a,a}.$$

We have the naturally defined morphisms:

$$\Pi^{a,a+1}_{f!}\mathcal{M} \xrightarrow{c^{(a)}_{1,X}} \mathcal{M}^{(a)}_X \xrightarrow{d^{(a)}_{2,X}} \Pi^{a,a+1}_{f*}\mathcal{M}.$$

Beilinson defined the vanishing cycle functor $\phi^{(a)}_f\mathcal{M}_X$ as the H^1-cohomology of the following sequence of holonomic $\mathcal{D}_{X(*D)}$-modules:

$$\Pi^{a,a+1}_{f!}\mathcal{M} \xrightarrow{c^{(a)}_1\oplus c^{(a)}_{1,X}} \Xi^{(a)}_f\mathcal{M}\oplus\mathcal{M}^{(a)}_X \xrightarrow{d^{(a)}_2 - d^{(a)}_{2,X}} \Pi^{a,a+1}_{f*}\mathcal{M}.$$

The morphisms $d^{(a)}_1$ and $c^{(a)}_2$ induce can and var:

$$\psi^{(a+1)}_f\mathcal{M} \xrightarrow{\text{can}} \phi^{(a)}_f\mathcal{M} \xrightarrow{\text{var}} \psi^{(a)}_f\mathcal{M}.$$

By construction, we have $\text{var}\circ\text{can} = c^{(a)}_2\circ d^{(a)}_1$.

Conversely, let $\mathcal{M}_Y$ be a holonomic $\mathcal{D}_{X(*D)}$-module whose support is contained in $Y = f^{-1}(0)$, with morphisms

$$\psi^{(1)}_f\mathcal{M} \xrightarrow{u} \mathcal{M}_Y \xrightarrow{v} \psi^{(0)}_f\mathcal{M}, \qquad v\circ u = c^{(0)}_2\circ d^{(0)}_1.$$

Then, we obtain a holonomic $\mathcal{D}_{X(*D)}$-module $\mathrm{Glue}(\mathcal{M}_Y, u, v)$ as the cohomology of the complex:

$$\psi_f^{(1)}\mathcal{M} \xrightarrow{d_1^{(0)}\oplus u} \Xi_f(\mathcal{M}) \oplus \mathcal{M}_Y \xrightarrow{c_2^{(0)}-v} \psi_f^{(0)}\mathcal{M}.$$

Beilinson made an excellent observation that the above two operations are mutually inverse. See [**4**] for more details.

2.2.4. Comparison with ordinary definitions. — Let $\widetilde{\psi}_{f,-1}$ and $\widetilde{\phi}_f$ be the nearby cycle functor and the vanishing cycle functor defined in terms of V-filtrations, i.e., $\widetilde{\psi}_{f,-1}(\mathcal{M}) := \mathrm{Gr}^V_{-1}(\iota_{f\dagger}\mathcal{M})$ and $\widetilde{\phi}_f(\mathcal{M}_X) := \mathrm{Gr}^V_0(\iota_{f\dagger}\mathcal{M}_X)$, where $\iota_f : X \to X \times \mathbb{C}$ denotes the graph, and V denotes a V-filtration of $\iota_{f\dagger}\mathcal{M}_X$ along t.

For simplicity, $\widetilde{\psi}_{f,-1}$ is denoted by $\widetilde{\psi}_f$ in the following.

Lemma 2.2.3. — *We have natural isomorphisms $\psi_f \simeq \widetilde{\psi}_f$, and $\phi_f \simeq \widetilde{\phi}_f$.*

Proof. — Recall that $\widetilde{\phi}_f(\mathcal{M}_X)$ and $\widetilde{\psi}_f(\mathcal{M}_X)$ are naturally equipped with the nilpotent endomorphisms N, which are the nilpotent part of the multiplication of $-\partial_t t$. We have natural identifications

$$\widetilde{\phi}_f(\Pi^{a,b}_{f!}\mathcal{M}) \simeq \widetilde{\phi}_f(\Pi^{a,b}_{f*}\mathcal{M}) \simeq \widetilde{\psi}_f\mathcal{M} \otimes A^{a,b}.$$

The natural nilpotent endomorphisms are given by

$$N \otimes \mathrm{id} - \mathrm{id} \otimes (s\bullet),$$

which is denoted by $N - s$. Here, $s\bullet$ denotes the multiplication of s on $A^{a,b}$. In the following, we argue on any compact subset of X.

Let us look at the natural morphism $G^{a,b} : \Pi^{a,b}_{f!}\mathcal{M} \to \Pi^{a,b}_{f*}\mathcal{M}$. The supports of the kernel and the cokernel are contained in $f^{-1}(0)$. The morphism

$$\widetilde{\phi}_f(G^{a,b}) : \widetilde{\phi}_f(\Pi^{a,b}_{f!}\mathcal{M}) \longrightarrow \widetilde{\phi}_f(\Pi^{a,b}_{f*}\mathcal{M})$$

is naturally identified with

$$N - s : \widetilde{\psi}_f\mathcal{M} \otimes A^{a,b} \longrightarrow \widetilde{\psi}_f\mathcal{M} \otimes A^{a,b}.$$

Hence, if b is sufficiently larger than a, $\mathrm{Cok}(G^{a,b})$ is isomorphic to $\widetilde{\psi}_f\mathcal{M} \otimes A^{a,a+1}$, independently of b. Therefore, we obtain $\psi_f^{(a)}\mathcal{M} \simeq \widetilde{\psi}_f\mathcal{M} \otimes A^{a,a+1}$. In particular, we naturally have $\psi_f^{(0)}\mathcal{M} = \widetilde{\psi}_f\mathcal{M}$.

It follows that $\mathrm{Cok}(\Pi^{a+1,M}_{f!}\mathcal{M} \to \Pi^{a,M}_{f*}\mathcal{M})$ are independent of any sufficiently large M, which should be isomorphic to $\Xi_f^{(a)}\mathcal{M}$. We obtain

$$\widetilde{\phi}_f(\Xi_f^{(a)}\mathcal{M}) \simeq \mathrm{Cok}\left(N - s : \psi_f\mathcal{M} \otimes A^{a+1,M} \to \psi_f\mathcal{M} \otimes A^{a,M}\right)$$

for any sufficiently large M. Because $\phi_f^{(0)}(\mathcal{M}_X)$ is naturally isomorphic to the cohomology of the complex

$$\widetilde{\phi}_f(\Pi_{f!}^{0,1}\mathcal{M}) \longrightarrow \widetilde{\phi}_f(\Xi_f^{(0)}\mathcal{M}) \oplus \widetilde{\phi}_f(\mathcal{M}_X) \longrightarrow \widetilde{\phi}_f(\Pi_{f*}^{0,1}\mathcal{M}),$$

it is easy to obtain $\phi_f^{(0)}(\mathcal{M}) \simeq \widetilde{\phi}_f(\mathcal{M})$ by a direct calculation. □

2.2.5. Compatibility with dual. — In [**4**], the pairing $A \times A \to k = A^{-1}/A^0$ is given by

$$\big\langle f(s), g(s)\big\rangle = \mathrm{Res}_{s=0}\big(f(s)\, g(-s)\mathrm{d}s\big).$$

It induces pairings $A^{a,b} \otimes A^{-b,-a} \to A^{-1}/A^0$. Then, we obtain flat pairings

$$\mathfrak{I}^{a,b} \otimes \mathfrak{I}^{-b,-a} \longrightarrow \mathfrak{I}^{-1,0}.$$

We can identify $\mathfrak{I}^{a,b}$ with the dual of $\mathfrak{I}^{-b,-a}$ by the pairing.

Let $\boldsymbol{D}$ denote the dual functor on the category of holonomic $\mathcal{D}_{X(*D)}$-modules. By using the $\mathcal{D}_{X(*D)}$-version of Lemma 2.1.1, we obtain identifications:

$$\boldsymbol{D}(\Pi_{f*}^{a,b}\mathcal{M}) \simeq \Pi_{f!}^{-b,-a}(\boldsymbol{D}\mathcal{M}), \quad \boldsymbol{D}(\Pi_{f!}^{a,b}\mathcal{M}) \simeq \Pi_{f*}^{-b,-a}(\boldsymbol{D}\mathcal{M}).$$

By (8) and (9), we obtain the identifications

$$\boldsymbol{D}_X\psi_f^{(a)}(\mathcal{M}) \simeq \psi_f^{(-a)}(\boldsymbol{D}_X\mathcal{M}) \quad \text{and} \quad \boldsymbol{D}_X\Xi_f^{(a)}(\mathcal{M}) \simeq \Xi_f^{(-a-1)}(\boldsymbol{D}_X\mathcal{M}).$$

We have $\boldsymbol{D}_X(c_1^{(a)}) = d_2^{(-a-1)}$, $\boldsymbol{D}_X(c_2^{(a)}) = d_1^{(-a-1)}$ and $\boldsymbol{D}_X(c_{1,X}^{(a)}) = d_{2,X}^{(-a-1)}$. Hence, we obtain $\boldsymbol{D}_X\phi_f^{(a)}(\mathcal{M}_X) \simeq \phi_f^{(-a-1)}(\boldsymbol{D}_X\mathcal{M}_X)$. The morphisms

$$\boldsymbol{D}_X\psi_f^{(a)}\mathcal{M} \xrightarrow{\boldsymbol{D}\,\mathrm{var}} \boldsymbol{D}_X\phi_f^{(a)}\mathcal{M}_X \xrightarrow{\boldsymbol{D}\,\mathrm{can}} \boldsymbol{D}_X\psi_f^{(a-1)}\mathcal{M}$$

are identified with

$$\psi_f^{(-a+1)}\mathcal{M} \xrightarrow{\mathrm{can}} \phi_f^{(-a)}\mathcal{M}_X \xrightarrow{\mathrm{var}} \psi_f^{(-a)}\mathcal{M}.$$

The multiplication of s induces an isomorphism $\Phi_s : \psi^{(a)}(\mathcal{M}) \simeq \psi^{(a+1)}(\mathcal{M})$, etc. Under the above identifications, we have $\boldsymbol{D}\Phi_s = -\Phi_s$.

Remark 2.2.4. — In [**50**], we use the pairing $A \times A \to k$ given by $\langle f(s), g(s)\rangle = \mathrm{Res}_{s=0}(f(s)g(-s)\mathrm{d}s/s)$. It makes an inessential shift of the indexes in the formulas. □

2.2.6. Compatibility with push-forward. — Let $F : X \to Y$ be any proper morphism. Assume that $D = F^{-1}(D_Y)$, for simplicity. Let g be any holomorphic function on Y. Let $\mathcal{M}$ be any holonomic $\mathcal{D}_{X(*D)}$-module. We set $\widetilde{g} := F^*g$. Let $j_Y : Y - g^{-1}(0) \to Y$ and $j_X : X - \widetilde{g}^{-1}(0) \to X$. We have natural isomorphisms

$$F_\dagger^i(\mathcal{M} \otimes \mathfrak{I}_{\widetilde{g}}^{a,b}) \simeq F_\dagger^i(\mathcal{M}) \otimes \mathfrak{I}_g^{a,b}$$

of $\mathcal{D}_{Y(*D_Y)}$-modules. We naturally have $(j_{Y\star}j_Y^*)F_\dagger^i \simeq F_\dagger^i \circ (j_{X\star}j_X^*)$ for $\star = *, !$. Hence, it is easy to obtain the identifications

$$F_\dagger^i \psi_{\widetilde{g}}^{(a)} \mathcal{M} = \psi_g^{(a)} F_\dagger^i \mathcal{M}, \quad F_\dagger^i \Xi_g^{(a)} \mathcal{M} = \Xi_g^{(a)} F_\dagger \mathcal{M}, \quad F_\dagger^i \phi_g^{(a)} \mathcal{M} = \phi_g^{(a)} F_\dagger^i \mathcal{M}.$$

2.2.7. Choice of a function. — Let f and h be meromorphic functions on (X, D). We suppose that h is nowhere vanishing on $X \setminus D$. We have natural isomorphisms of $\mathcal{O}_X$-modules

$$\mathfrak{I}_f^{a,b} \simeq \mathfrak{I}_{hf}^{a,b} \simeq A^{a,b} \otimes \mathcal{O}_{X(*D)}(*f).$$

For their flat connections ∇_f and ∇_{hf} and for $\alpha \in A^{a,b}$, we have the formulas:

$$\nabla_f \alpha = \alpha \cdot s \frac{\mathrm{d}f}{f}, \quad \nabla_{hf} \alpha = \alpha \cdot s \Big(\frac{\mathrm{d}f}{f} + \frac{\mathrm{d}h}{h} \Big).$$

If we have $\log h$ on X, we have a flat isomorphism $\Phi : \mathfrak{I}_f^{a,b} \simeq \mathfrak{I}_{hf}^{a,b}$ given by $\Phi(\alpha) = \exp(-s \log h)\, \alpha$. It induces isomorphisms:

$$\Xi_f^{(a)} \simeq \Xi_{hf}^{(a)}, \quad \psi_f^{(a)} \simeq \psi_{hf}^{(a)}, \quad \phi_f^{(a)} \simeq \phi_{hf}^{(a)}. \tag{10}$$

They depend on the choice of a branch of $\log h$.

2.2.8. $\mathbb{Q}$-structure of $\mathfrak{I}^{a,b}$. — In the analytic case, the $\mathbb{Q}$-structure of $A^{a,b}$ is given as follows:

$$\mathbb{C} \cdot s^j \supset \mathbb{Q} \cdot \big(2\pi\sqrt{-1}\big)^j s^j \ .$$

It gives a $\mathbb{Q}$-structure of the fiber of $\mathfrak{I}^{a,b}$ over $1 \in \mathbb{C}^*$. We extend it to a flat $\mathbb{Q}$-structure of the flat bundle $\mathfrak{I}_{|\mathbb{C}^*}$. Let $u := 2\pi\sqrt{-1}\, s$. The connection of $\mathfrak{I}^{a,b}$ is expressed as

$$\nabla(u^a, \ldots, u^{b-1}) = (u^a, \ldots, u^{b-1}) N \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}t}{t}.$$

Here, N denotes the constant matrix such that $N_{i,i+1} = 1$ and $N_{i,j} = 0$ otherwise. Since the monodromy is expressed by $\exp(-N)$, the $\mathbb{Q}$-structure is well defined. More generally, for any subfield $K \subset \mathbb{C}$, we obtain a K-structure of $\mathfrak{I}^{a,b}$ in this way. The pairing $\langle .\, , . \rangle : \mathfrak{I}^{a,b} \otimes \mathfrak{I}^{-b,-a} \to \mathfrak{I}^{-1,0}$ is defined over $\mathbb{Q}$.

Under the identification $\mathfrak{I}^{-1,0} \simeq \mathfrak{I}^{0,1}$ by the multiplication of s, the pairing takes values in $(2\pi\sqrt{-1}\,)^{-1}\mathbb{Q}$.

2.2.9. Comparison with the functors for perverse sheaves. — Let $\mathrm{Loc}(\mathfrak{I}^{a,b})_{\mathbb{Q}}$ denote the $\mathbb{Q}$-local system associated to $\mathfrak{I}^{a,b}$. The fiber over 1 is $u^a\mathbb{Q}[[u]]/u^b\mathbb{Q}[[u]]$, and the monodromy along the loop with the clockwise direction is given by the multiplication of $\exp(u)$. Taking the limit, we have a $\mathbb{Q}$-local system $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$, whose fiber over 1 is $\mathbb{Q}((u))$, and the monodromy is given by the multiplication of $\exp(u)$. We have subsystems $\mathrm{Loc}(\mathfrak{I}^a)_{\mathbb{Q}} \subset \mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$ whose fiber over 1 is $u^a\mathbb{Q}[[u]]$. We have

$$\mathrm{Loc}(\mathfrak{I}^{a,b})_{\mathbb{Q}} \simeq \mathrm{Loc}(\mathfrak{I}^a)_{\mathbb{Q}}/\operatorname{Loc}(\mathfrak{I}^b)_{\mathbb{Q}}.$$

Recall another expression of these local systems as in [**4**].

Let $A_{\mathcal{P}} := \mathbb{Q}((v))$. We set $t := v + 1$. The pairing $A_{\mathcal{P}} \times A_{\mathcal{P}} \to \mathbb{Q}(-1)$ is given as follows:

$$\langle f(t), g(t)\rangle = \operatorname*{Res}_{t=1}\big(f(t)\,g(t^{-1})\frac{\mathrm{d}t}{t}\big)\frac{1}{2\pi\sqrt{-1}}.$$

We have a $\mathbb{Q}$-local system $\mathfrak{I}_{\mathcal{P}}$ on $\mathbb{C}^*$ such that the fiber over 1 is $A_{\mathcal{P}}$, and the monodromy along the loop with the clockwise direction is given by the multiplication of $t = 1 + v$. Let us compare $\mathfrak{I}_{\mathcal{P}}$ and $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$. We take an algebra homomorphism $\Phi : \mathbb{Q}((u)) \to \mathbb{Q}((v))$ determined by $\Phi(\exp(u)) = 1+v$. We identify the fibers of $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}}$ and $\mathfrak{I}_{\mathcal{P}}$ by Φ. Because it is compatible with the monodromy, it induces the identification $\mathrm{Loc}(\mathfrak{I})_{\mathbb{Q}} \simeq \mathfrak{I}_{\mathcal{P}}$. Note that $\Phi(f(-u)) = \Phi(f)(t^{-1})$ and $\Phi(\mathrm{d}u) = \mathrm{d}t/t$. Hence the pairing is preserved.

REMARK 2.2.5. — Recall that the functors ψ, Ξ and ϕ for perverse sheaves are given in terms of $\mathfrak{I}_{\mathcal{P}}$, according to [**4**]. The above comparison gives the compatibility of the de Rham functor DR with ϕ, ψ and Ξ in the regular singular case. □

CHAPTER 3

GOOD HOLONOMIC $\mathcal{D}$-MODULES AND THEIR DE RHAM COMPLEXES

3.1. Good holonomic $\mathcal{D}$-modules

We shall introduce the notion of good holonomic $\mathcal{D}$-modules on any complex manifold X with a normal crossing hypersurface $D = \bigcup_{i\in\Lambda} D_i$. They are $\mathcal{D}$-modules locally described as the gluing of meromorphic flat bundles on $\bigcap_{j\in J} D_j$ $(J \subset \Lambda)$. In §§3.1.1–3.1.3, we study the local case. We explain the global case in §3.1.4. We explain a kind of quiver description of good holonomic $\mathcal{D}$-modules in the local case in §3.1.5.

In the local case, for any good holonomic $\mathcal{D}$-modules, we have various commutativity of functors such as $\phi_i^{(a)}\phi_j^{(b)}(\mathcal{M}) \simeq \phi_j^{(b)}\phi_i^{(a)}(\mathcal{M})$, for which goodness seems truly used.

3.1.1. $\mathcal{I}$-good meromorphic flat bundles. — Let Δ^n denote a multi-disc in $\mathbb{C}^n$, i.e., $\Delta^n := \{(z_1,\ldots,z_n)\in\mathbb{C}^n\,;\,|z_i|<1\}$. We consider the case

$$X := \Delta^n,\quad D_i := \{z_i = 0\}\quad\text{and}\quad D := \bigcup_{i=1}^{\ell} D_i.$$

We set $\underline{\ell} := \{1,\ldots,\ell\}$. For $I \subset \underline{\ell}$, we set

$$D(I) := \bigcup_{i\in I} D_i\quad\text{and}\quad D_I := \bigcap_{i\in I} D_i.$$

We put

$$\partial D_I := D_I \cap D(I^c),\quad\text{where}\quad I^c := \underline{\ell} - I.$$

Let $M(X,D)$ be the set of meromorphic functions on X whose poles are contained in D. Let $H(X)$ be the set of holomorphic functions on X. We give a review on good meromorphic flat bundles. See [**45**], [**48**] and [**49**] for more detailed reviews.

3.1.1.1. Good set of irregular values. — Let $f \in M(X, D)$. Suppose that there exists $\boldsymbol{m} = (m_i) \in \mathbb{Z}^{\ell}_{\geq 0}$ such that

(i) $\boldsymbol{z}^{\boldsymbol{m}} f = \prod z_i^{m_i} f$ is holomorphic,

(ii) if $\boldsymbol{m} \neq (0, \dots, 0)$, we have $(\boldsymbol{z}^{\boldsymbol{m}} f)(O) \neq 0$.

Then, we set $\mathrm{ord}(f) := -\boldsymbol{m}$. In general, such $\boldsymbol{m}$ does not exist. For any holomorphic function f, we have $\mathrm{ord}(f) = (0, \dots, 0)$. If $\mathrm{ord}(g)$ exists for $g \in \mathcal{O}_X(*D)$, then $\mathrm{ord}(g + f) = \mathrm{ord}(g)$ for any holomorphic function f. So, the notion ord is considered for elements in $M(X, D)/H(X)$.

We use the order $\leq$ on $\mathbb{Z}^{\ell}$ given by $\boldsymbol{m} \leq \boldsymbol{n}$ if $m_i \leq n_i$ for any i. A finite subset $\mathcal{I} \subset M(X, D)/H(X)$ is called good if the following holds:

- ▷ For any $f \in \mathcal{I}$, there exists $\mathrm{ord}(f)$.
- ▷ For any $f, g \in \mathcal{I}$, there exists $\mathrm{ord}(f - g)$, and the set $\{\mathrm{ord}(f - g) \mid f, g \in \mathcal{I}\}$ is totally ordered.

For any good set of irregular values $\mathcal{I} \subset M(X, D)/H(X)$ and for any subset $I \subset \underline{\ell}$, let $\mathcal{I}'(I)$ be the set of the elements $\mathfrak{a} \in \mathcal{I}$ which are regular along z_i ($i \in I$), and we put $\mathcal{I}(I) := \{\mathfrak{a}_{|D_I} \mid \mathfrak{a} \in \mathcal{I}'(I)\}$. It is a good set of irregular values on $(D_I, \partial D_I)$.

3.1.1.2. Unramifiedly $\mathcal{I}$-good meromorphic flat bundle. — Let

$$\mathcal{I} \subset M(X, D)/H(X)$$

be a good set of irregular values. Recall that a meromorphic flat bundle $(\mathcal{E}, \nabla)$ on (X, D) is called unramifiedly $\mathcal{I}$-good if the following holds:

- ▷ Let $\mathcal{I}_I$ denote the image of $\mathcal{I}$ to $M(X, D)/M(X, D(I^c))$. For any $P \in D_I \setminus \partial D_I$, the formal completion $(\mathcal{E}, \nabla)_{|\widehat{P}}$ is decomposed into $\bigoplus_{\mathfrak{b} \in \mathcal{I}_I} (\widehat{\mathcal{E}}_{P,\mathfrak{b}}, \widehat{\nabla}_{P,\mathfrak{b}})$ such that $\widehat{\nabla}_{P,\mathfrak{b}} - d\widetilde{\mathfrak{b}}\, \mathrm{id}_{\widehat{\mathcal{E}}_{P,\mathfrak{b}}}$ are regular singular, where $\widetilde{\mathfrak{b}}$ are any lifts of $\mathfrak{b}$ to $M(X, D)$.

In this paper, we say that a meromorphic flat bundle $(\mathcal{E}, \nabla)$ on $(D_I, \partial D_I)$ is unramifiedly $\mathcal{I}$-good if it is unramifiedly $\mathcal{I}(I)$-good.

3.1.1.3. Ramified case. — For a positive integer m, let

$$X^{(m)} := \Delta^n = \big\{|\zeta_i| < 1\big\}, \quad D_i^{(m)} := \{\zeta_i = 0\} \quad \text{and} \quad D^{(m)} = \bigcup_{i=1}^{\ell} D_i^{(m)}.$$

We have a natural ramified covering $\varphi_m : X^{(m)} \to X$ along D given by

$$\varphi_m(\zeta_1, \dots, \zeta_n) = (\zeta_1^m, \dots, \zeta_\ell^m, \zeta_{\ell+1}, \dots, \zeta_n),$$

and the induced ramified coverings $D_I^{(m)} \to D_I$. Let

$$\mathcal{I} \subset M(X^{(m)}, D^{(m)})/H(X^{(m)})$$

be any good set of irregular values which is preserved by the action of the Galois group of the ramified covering $X^{(m)}/X$. In this paper, a meromorphic flat bundle $\mathcal{E}$ on $(D_I, \partial D_I)$ is called $\mathcal{I}$-good if it is the descent of an unramifiedly $\mathcal{I}$-good meromorphic flat bundle $\mathcal{E}^{(m)}$ on $(D_I^{(m)}, \partial D_I^{(m)})$.

3.1.1.4. Some functors along the divisors. — In this subsection, we use the following notation for simplicity of the description.

NOTATION 3.1.1. — The vanishing cycle functors $\phi_{z_i}^{(a)}$ are denoted by $\phi_i^{(a)}$. For any $I = (i_1, \dots, i_m) \in \{1, \dots, \ell\}^m$ and any $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set

$$\phi_I^{(\boldsymbol{a})} = \phi_{i_1}^{(a_1)} \circ \cdots \circ \phi_{i_m}^{(a_m)}.$$

If $\boldsymbol{a} = (0, \dots, 0)$, it is often denoted just by ϕ_I. We use the symbols $\psi_I^{(\boldsymbol{a})}$, $\Xi_I^{(\boldsymbol{a})}$ and $\Pi_{i\star}^{a,b}$ with a similar meaning. For any holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we set

$$\mathcal{M}(*i) := \mathcal{M}(*D_i) \quad \text{and} \quad \mathcal{M}(!i) := \mathcal{M}(!D_i).$$

If we are given a subset $I \subset \underline{\ell}$, we put

$$\mathcal{M}(!I) := \mathcal{M}\big(!D(I)\big) \quad \text{and} \quad \mathcal{M}(*I) := \mathcal{M}\big(*D(I)\big). \qquad \square$$

LEMMA 3.1.2. — *Let $(\mathcal{E}, \nabla)$ be any $\mathcal{I}$-good meromorphic flat bundle on (X, D). For $1 \leq i, j \leq \ell$ with $i \neq j$, the natural morphism $\phi_i^{(a)}(\mathcal{E}) \to \phi_i^{(a)}(\mathcal{E})(*j)$ is an isomorphism.*

Proof. — Because the support of $\phi_i^{(a)}(\mathcal{E})$ and $\phi_i^{(a)}(\mathcal{E})(*j)$ are contained in D_i, it is enough to prove that the induced morphism for the formal completions

$$\phi_i^{(a)}(\mathcal{E})_{|\widehat{P}} \longrightarrow \phi_i^{(a)}(\mathcal{E})(*j)_{|\widehat{P}}$$

is an isomorphism for each $P \in D_i$. We have only to consider the case $P = (0, \dots, 0)$. We use the notation introduced in §3.1.1.3. Take lifts $\widetilde{\mathfrak{a}}$ of $\mathfrak{a} \in \mathcal{I}$. We have regular singular meromorphic flat bundles $(R_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$ on $(X^{(m)}, D^{(m)})$ for $\mathfrak{a} \in \mathcal{I}$, and an action of the Galois group G of φ_m on $(\mathcal{E}', \nabla') = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (R_{\mathfrak{a}}, \nabla_{\mathfrak{a}} + \mathrm{d}\,\widetilde{\mathfrak{a}})$, such that the formal completions of $(\mathcal{E}', \nabla')$ and $\varphi_m^*(\mathcal{E}, \nabla)$ at $(0, \dots, 0)$ are isomorphic in a G-equivariant way. Let $(\mathcal{E}'', \nabla'')$ be the meromorphic flat bundle on (X, D) obtained as the descent of $(\mathcal{E}', \nabla')$. The formal completions of $(\mathcal{E}'', \nabla'')$ and $(\mathcal{E}, \nabla)$ at P are isomorphic. Then, by using the standard argument to prove the uniqueness of V-filtrations, the isomorphism $\mathcal{E}''_{|\widehat{P}} \simeq \mathcal{E}_{|\widehat{P}}$ is compatible with the V-filtrations along z_i. Therefore, it is enough to prove the claim for $\mathcal{E}''$.

Let (R, ∇) be a regular singular meromorphic flat bundle on (X, D). Let $\mathfrak{b} \in M(X^{(m)}, D^{(m)})$ such that $\mathrm{ord}(\mathfrak{b})$ exists. We set $L(\mathfrak{b}) := \mathcal{O}_{X^{(m)}}(*D^{(m)})\, e$ with the connection $\nabla e = e\, d\mathfrak{b}$. We obtain a meromorphic flat bundle $\varphi_{m*}(L(\mathfrak{b}))$ on (X, D). By the previous consideration, it is enough to prove the claim for any direct summand of the meromorphic flat bundle $\mathcal{E}_1 = R \otimes \varphi_{m*} L(\mathfrak{b})$, which follows from the claim for $\mathcal{E}_1$. We may assume that $\mathfrak{b} = \prod_{j=1}^{\ell} \zeta_j^{b_j}$ for some $b_j \leq 0$.

Let $V(R)$ denote the V-filtration along z_i. For $\boldsymbol{m} \in S := \{0, 1, \ldots, m-1\}^{\ell}$, let $\boldsymbol{\zeta}^{\boldsymbol{m}} := \prod_{k=1}^{\ell} \zeta_k^{m_k}$. We have

$$\varphi_* L(\mathfrak{b}) = \bigoplus_{\boldsymbol{m} \in S} \mathcal{O}_X(*D) \boldsymbol{\zeta}^{\boldsymbol{m}} e.$$

If $b_i < 0$, the V-filtration $V(\mathcal{E}_1)$ of $\mathcal{E}_1$ is given by $V_\alpha(\mathcal{E}_1) = \mathcal{E}_1$ for any $\alpha \in \mathbb{C}$. If $b_i = 0$, we have $V_\alpha(\mathcal{E}_1) = \bigoplus V_{\alpha + m_i/m}(R) \otimes \mathcal{O}_X \boldsymbol{\zeta}^{\boldsymbol{m}} e$. Hence, the natural morphism $\phi_i(\mathcal{E}_1) \to \phi_i(\mathcal{E}_1)(*D_j)$ $(j \neq i)$ is an isomorphism in the both cases. □

Lemma 3.1.3. — *If $i \neq j$, the natural morphism $\mathcal{E}(!i) \to \mathcal{E}(!i)(*j)$ is an isomorphism.*

Proof. — Let N denote the nilpotent part of the action of $-\partial_i z_i$ on $\phi_i(\mathcal{E})$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\, N & \longrightarrow & \mathcal{E}(!i) & \longrightarrow \mathcal{E} \longrightarrow & \mathrm{Cok}\, N & \longrightarrow & 0 \\
 & & \big\downarrow a & & \big\downarrow b & \big\downarrow = & \big\downarrow c & & \\
0 & \longrightarrow & \mathrm{Ker}\, N(*j) & \longrightarrow & \mathcal{E}(!i)(*j) & \longrightarrow \mathcal{E} \longrightarrow & \mathrm{Cok}\, N(*j) & \longrightarrow & 0\,.
\end{array}$$

By Lemma 3.1.2, we obtain that a and c are isomorphisms. Hence, b is also an isomorphism. □

3.1.2. $\mathcal{I}$-good holonomic $\mathcal{D}$-modules. — We continue to use the notation introduced in §3.1.1.

Definition 3.1.4. — A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called $\mathcal{I}$-good on (X, D) if the following holds:

- ▷ $\mathcal{M}(*D)$ is an $\mathcal{I}$-good meromorphic flat bundle on (X, D).
- ▷ For any $I = (i_1, \ldots, i_m) \in \{1, \ldots, \ell\}^m$, $\phi_I(\mathcal{M})(*I^c)$ is the push-forward of an $\mathcal{I}$-good meromorphic flat bundle on $(D_I, \partial D_I)$ by $D_I \to X$. □

The full subcategory of $\mathcal{I}$-good holonomic $\mathcal{D}$-modules is abelian, and it is closed under extensions. If V is a good meromorphic flat bundle, it is a good holonomic $\mathcal{D}_X$-module in the above sense. When we do not have to distinguish $\mathcal{I}$, we will omit to denote it. We will implicitly use the following obvious lemma.

LEMMA 3.1.5. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Suppose that* (i) $\mathcal{M}(*D)$ *is an $\mathcal{I}$-good meromorphic flat bundle,* (ii) $\phi_i(\mathcal{M})$ *are $\mathcal{I}$-good for any $i = 1, \dots, \ell$. Then, $\mathcal{M}$ is $\mathcal{I}$-good.* □

LEMMA 3.1.6. — *Let $\mathcal{M}$ be an $\mathcal{I}$-good holonomic $\mathcal{D}$-module on (X, D). Then $\boldsymbol{D}_X\mathcal{M}$ is $-\mathcal{I}$-good, where $-\mathcal{I} = \{-\mathfrak{a} \mid \mathfrak{a} \in \mathcal{I}\}$.*

Proof. — We use an induction on the dimension of the support of $\mathcal{M}$. It is easy to check that $\boldsymbol{D}_X\mathcal{M}(*D)$ is a good meromorphic flat bundle. By the inductive assumption, $\phi_i^{(a)}(\boldsymbol{D}_X\mathcal{M}) \simeq \boldsymbol{D}_X\phi_i^{(-a-1)}(\mathcal{M})$ are also good. Hence, we obtain that $\mathcal{M}$ is good. □

For any good holonomic $\mathcal{D}$-module $\mathcal{M}$, let $\rho(\mathcal{M}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ denote the pair of $\dim \operatorname{Supp} \mathcal{M}$ and the number of the irreducible components of $\operatorname{Supp} \mathcal{M}$ with the maximal dimension. We use the lexicographic order on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$. For any good holonomic $\mathcal{D}$-module $\mathcal{M}$, there exists $J \subset \underline{\ell}$ with $\dim \operatorname{Supp} \mathcal{M} = n - |J|$ such that $\mathcal{M}(*J^c) \neq 0$. The kernel $\mathcal{N}_1$ and the cokernel $\mathcal{N}_2$ of the natural morphism $\mathcal{M} \to \mathcal{M}(*J^c)$ satisfy $\rho(\mathcal{N}_i) < \rho(\mathcal{M})$ $(i = 1, 2)$.

LEMMA 3.1.7. — *Let $\mathcal{M}$ be $\mathcal{I}$-good on (X, D). Then, $\psi_i^{(a)}(\mathcal{M})$ are also $\mathcal{I}$-good for any $i = 1, \dots, \ell$.*

Proof. — We use an induction on $\rho(\mathcal{M})$. Let J and $\mathcal{N}_j$ $(j = 1, 2)$ be as above. By the assumption of the induction, $\psi_i^{(a)}(\mathcal{N}_j)$ $(j = 1, 2)$ are good. The $\mathcal{D}_X$-module $\mathcal{M}(*J^c)$ is the push-forward of an $\mathcal{I}$-good meromorphic flat bundle $\mathcal{E}_J$ on $(D_J, \partial D_J)$ by the inclusion $\iota_J : D_J \to X$. If $i \in J$, we have $\psi_i^{(a)}(\mathcal{M}(*J^c)) = 0$. If $i \notin J$, $\psi_i^{(a)}(\mathcal{M}(*J^c))$ is isomorphic to $\iota_{J\dagger}\psi_i^{(a)}(\mathcal{E}_J)$. By computing the formal completion $\psi_i^{(a)}(\mathcal{E}_J)_{|\widehat{P}}$ of $P \in \partial D_J$ as in the proof of Lemma 3.1.2, we can prove that $\psi_i^{(a)}(\mathcal{E}_J)_{|\widehat{P}}$ is $\mathcal{I}$-good on $(D_J, \partial D_J)$. Hence, we obtain that $\psi_i^{(a)}(\mathcal{M})$ is also $\mathcal{I}$-good. □

3.1.3. Commutativity of the functors along the coordinate functions. — Let $\mathcal{M}$ be good on (X, D).

LEMMA 3.1.8. — *For any $i \neq j$, we have natural isomorphisms*

$$\phi_i\big(\mathcal{M}(*j)\big) \simeq \phi_i(\mathcal{M})(*j) \quad \text{and} \quad \phi_i\big(\mathcal{M}(!j)\big) \simeq \phi_i(\mathcal{M})(!j).$$

Proof. — The second isomorphism is obtained as the dual of the first one. Let us consider the first isomorphism. We have the following naturally defined morphisms:

$$\phi_i\big(\mathcal{M}(*j)\big) \xrightarrow{a} \phi_i\big(\mathcal{M}(*j)\big)(*j) \xleftarrow{b} \phi_i(\mathcal{M})(*j)\,.$$

Because the restriction of b to $X - D_j$ is an isomorphism, it is easy to see that b is an isomorphism. Let us prove that a is an isomorphism by using an induction on $\rho(\mathcal{M})$. As in the proof of Lemma 3.1.7, the issue can be reduced to the case where $\mathcal{M}$ is a good meromorphic flat bundle, which is given in Lemma 3.1.2. □

LEMMA 3.1.9. — *$\mathcal{M}(*j)$ and $\mathcal{M}(!j)$ are also good.*

Proof. — Because $\phi_j(\mathcal{M}(*j)) \simeq \psi_j(\mathcal{M})$, we obtain that $\mathcal{M}(*j)$ is good from Lemmas 3.1.5, 3.1.7 and 3.1.8. By using Lemma 3.1.6, we obtain that $\mathcal{M}(!j)$ is also good. □

We have the following corollary of Lemma 3.1.9.

COROLLARY 3.1.10. — *Let f be a meromorphic function on (X, D) whose zeros and poles are contained in D. Take $D^{(1)} \subset D$ such that the poles of f are contained in $D^{(1)}$. The holonomic $\mathcal{D}_X$-module $\Pi^{a,b}_{f\star}(\mathcal{M}, *D^{(1)})$ is good on (X, D). Hence, $\psi^{(a)}_f(\mathcal{M}, *D^{(1)})$, $\Xi^{(a)}_f(\mathcal{M}, *D^{(1)})$ and $\phi^{(a)}_f(\mathcal{M}, *D^{(1)})$ are also good on (X, D).* □

We have the following naturally defined morphisms:

$$\mathcal{M}(*i)(!j) \xrightarrow{a} \mathcal{M}(*i)(!j)(*i) \xleftarrow{b} \mathcal{M}(!j)(*i)\,.$$

It is easy to prove that b is an isomorphism for $i \neq j$.

LEMMA 3.1.11. — *The morphism a is also an isomorphism, by which we can identify $\mathcal{M}(*i)(!j)$ and $\mathcal{M}(!j)(*i)$.*

Proof. — By using an induction on $\rho(\mathcal{M})$, we can reduce the issue to the case where $\mathcal{M}$ is a good meromorphic flat bundle, which is given in Lemma 3.1.3. □

In the following, we will not distinguish $\mathcal{M}(*i)(!j)$ and $\mathcal{M}(!j)(*i)$ for $i \neq j$, which will be denoted by $\mathcal{M}(*i!j)$. For $I \sqcup J \subset \underline{\ell}$, we have the natural identification

$$\mathcal{M}(!I*J) \simeq \mathcal{M}(*J!I),$$

which will be used implicitly.

Lemma 3.1.12. — *We have the commutativity*

$$\Xi_i^{(a)} \circ \Xi_j^{(b)} = \Xi_j^{(b)} \circ \Xi_i^{(a)}, \quad \psi_i^{(a)} \circ \psi_j^{(b)} = \psi_j^{(b)} \circ \psi_i^{(a)} \quad \text{and} \quad \phi_i^{(a)} \circ \phi_j^{(b)} = \phi_j^{(b)} \circ \phi_i^{(a)}.$$

Moreover, the functors $\Xi_i^{(a)}$, $\psi_j^{(b)}$ and $\phi_k^{(c)}$ are mutually commutative, where i, j, k are mutually distinct. In the following, we will not care about the order of these functors for good holonomic $\mathcal{D}$-modules on (X,D).

Proof. — We obtain the natural identification $\Pi_{i\star}^{a,b} \circ \Pi_{j\star'}^{c,d} = \Pi_{j\star'}^{c,d} \circ \Pi_{i\star}^{a,b}$ from Lemma 3.1.11. Then, the claim of the lemma is clear. □

3.1.4. Globalization. — Let X be a complex manifold with a normal crossing hypersurface D.

Definition 3.1.13. — A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called good on (X,D) if the following holds:

▷ Let P be any point of D. Let $(U, z_1, \dots, z_n)$ be a coordinate neighbourhood around P such that $D \cap U = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Then, $\mathcal{M}_{|U}$ is good in the sense of Definition 3.1.4. □

We obtain the following from the results in §3.1.2–§3.1.3.

Lemma 3.1.14. — *Let $\mathcal{M}$ be good on (X,D).*

- ▷ *The dual $\boldsymbol{D}_X\mathcal{M}$ is also good on (X,D).*
- ▷ *Let $D^{(1)} \subset D$ be the union of some irreducible components. Then, $\mathcal{M}(*D^{(1)})$ and $\mathcal{M}(!D^{(1)})$ are also good on (X,D).*
- ▷ *Let $D^{(i)} \subset D$ $(i = 1,2)$ be the unions of some irreducible components such that $\dim D^{(1)} \cap D^{(2)} < \dim X - 1$. We have a natural isomorphism $\mathcal{M}(*D^{(1)})(!D^{(2)}) \simeq \mathcal{M}(!D^{(2)})(*D^{(1)})$.*
- ▷ *Let f be a meromorphic function on (X,D) which is invertible on $X \setminus D$. Take $D^{(1)} \subset D$ such that the poles of f are contained in $D^{(1)}$. Then, $\psi_f^{(a)}(\mathcal{M}, *D^{(1)})$, $\Xi_f^{(a)}(\mathcal{M}, *D^{(1)})$ and $\phi_f^{(a)}(\mathcal{M}, *D^{(1)})$ are also good on (X,D).* □

3.1.5. A quiver description in the local case. — We set

$$X := \Delta^n, \quad D_i = \{z_i = 0\} \quad \text{and} \quad D = \bigcup_{i=1}^{\ell} D_i.$$

We use the notation introduced in §3.1.1. Let $\mathcal{I} \subset M(X^{(m)}, D^{(m)})/H(X^{(m)})$ be a good set of irregular values which is preserved by the action of the Galois group of the ramified covering $X^{(m)} \to X$.

We consider tuples of $\mathcal{I}$-good meromorphic flat bundles V_I on $(D_I, \partial D_I)$ $(I \subset \underline{\ell})$, with a tuple of morphisms

$$\psi_i^{(1)}(V_I) \xrightarrow{g_{I,i}} V_{Ii} \xrightarrow{f_{I,i}} \psi_i^{(0)}(V_I)$$

for $I \subset \underline{\ell}$ and $i \in \underline{\ell} \setminus I$. Here $Ii := I \cup \{i\}$. We impose the following conditions:

- ▷ $f_{I,i} \circ g_{I,i}$ is equal to $\mathrm{var} \circ \mathrm{can} : \psi_i^{(1)}(V_I) \to \psi_i^{(0)}(V_I)$;
- ▷ for any $I \sqcup \{i\} \sqcup \{j\} \subset \underline{\ell}$, we have the commutativity

$$\psi_j^{(0)}(f_{I,i}) \circ f_{Ii,j} = \psi_i^{(0)}(f_{I,j}) \circ f_{Ij,i},$$
$$g_{Ii,j} \circ \psi_j^{(1)}(g_{I,i}) = g_{Ij,i} \circ \psi_i^{(1)}(g_{I,j}),$$
$$f_{Ij,i} \circ g_{Ii,j} = \psi_i^{(0)}(g_{I,j}) \circ \psi_j^{(1)}(f_{I,i}).$$

For such $\mathcal{C}^{(a)} = ((V_I^{(a)}), (f_{I,i}^{(a)}, g_{I,i}^{(a)}))$ $(a = 1, 2)$, morphisms $\mathcal{C}^{(1)} \to \mathcal{C}^{(2)}$ are defined to be a tuple of morphisms $\varphi_I : V_I^{(1)} \to V_I^{(2)}$ of meromorphic flat bundles such that the following diagram is commutative:

$$\begin{array}{ccccc}
\psi_i^{(1)}(V_I^{(1)}) & \xrightarrow{g_{I,i}^{(1)}} & V_{Ii}^{(1)} & \xrightarrow{f_{I,i}^{(1)}} & \psi_i^{(0)}(V_I^{(1)}) \\
{\scriptstyle \psi_i^{(1)}(\varphi_I)}\downarrow & & {\scriptstyle \varphi_{Ii}}\downarrow & & {\scriptstyle \psi_i^{(0)}(\varphi_I)}\downarrow \\
\psi_i^{(1)}(V_I^{(2)}) & \xrightarrow{g_{I,i}^{(2)}} & V_{Ii}^{(2)} & \xrightarrow{f_{I,i}^{(2)}} & \psi_i^{(0)}(V_I^{(2)}).
\end{array}$$

Let $C(X, D)$ denote the category of such objects and morphisms (we do not fix $\mathcal{I}$).

Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X, D). Set $V_I(\mathcal{M}) := \phi_I^{(\mathbf{0})}(\mathcal{M})(*\partial D_I)$ and $V_\varnothing(\mathcal{M}) := \mathcal{M}(*D)$, which are naturally equipped with morphisms

$$\psi_i^{(1)}\big(V_I(\mathcal{M})\big) \xrightarrow{g_{I,i}(\mathcal{M})} V_{Ii}(\mathcal{M}) \xrightarrow{f_{I,i}(\mathcal{M})} \psi_i^{(0)}\big(V_I(\mathcal{M})\big).$$

Thus, we obtain an object in $C(X, D)$ denoted by $\Phi(\mathcal{M})$. The construction gives a functor $\mathcal{M} : \mathrm{Hol}^{\mathrm{good}}(X, D) \to C(X, D)$.

Proposition 3.1.15. — *Φ is an equivalence of categories.*

Proof. — Let us construct a quasi-inverse functor $\Upsilon : C(X,D) \to \mathrm{Hol}^{\mathrm{good}}(X,D)$. Let $\iota_I : D_I \to X$ denote the inclusion. For any $I \subset \underline{\ell}$, we set $\mathcal{M}_I^{(0)} := \iota_{I\dagger} V_I$. For $I \subset \underline{\ell}$ with $1 \notin I$, we define $\mathcal{M}_I^{(1)}$ as the gluing of V_I and V_{I1} by $f_{I,1}$ and $g_{I,1}$, i.e., $\mathcal{M}_I^{(1)}$ is the cohomology of the complex

$$\iota_{I1\dagger}\psi_1^{(1)}(V_I) \xrightarrow{d_1^{(1)}+g_{I,1}} \iota_{I\dagger}\Xi_1^{(1)}(V_I) \oplus \iota_{I1\dagger}V_{I1} \xrightarrow{c_2^{(0)}-f_{I,1}} \iota_{I1\dagger}\psi_1^{(0)}(V_I).$$

For $I \sqcup \{i\} \subset \underline{\ell} \setminus \{1\}$, we have naturally induced morphisms

$$\psi_i^{(1)}(\mathcal{M}_I^{(1)}) \xrightarrow{g_{I,i}^{(1)}} \mathcal{M}_{I1}^{(1)} \xrightarrow{f_{I,i}^{(1)}} \psi_i^{(0)}(\mathcal{M}_I^{(1)}).$$

Then,

(i) $f_{I,i}^{(1)} \circ g_{I,i}^{(1)}$ is equal to the canonical morphism;

(ii) for any $I \sqcup \{i\} \sqcup \{j\} \subset \underline{\ell} \setminus \{1\}$, we have the commutativity

$$\psi_j^{(0)}(f_{I,i}^{(1)}) \circ f_{Ii,j}^{(1)} = \psi_i^{(0)}(f_{I,j}^{(1)}) \circ f_{Ij,i}^{(1)},$$
$$g_{Ii,j}^{(1)} \circ \psi_j^{(0)}(g_{I,i}^{(1)}) = g_{Ij,i}^{(1)} \circ \psi_i^{(0)}(g_{I,j}^{(1)}),$$
$$f_{Ij,i}^{(1)} \circ g_{Ii,j}^{(1)} = \psi_i^{(0)}(g_{I,j}^{(1)}) \circ \psi_j^{(1)}(f_{I,i}^{(1)}).$$

Inductively on m, we can introduce good holonomic $\mathcal{D}$-modules $\mathcal{M}_I^{(m)}$ on (X,D) for $I \subset \underline{\ell} \setminus \underline{m}$, and morphisms for $I \sqcup \{i\} \subset \underline{\ell} \setminus \underline{m}$

$$\psi_i^{(1)}(\mathcal{M}_I^{(m)}) \xrightarrow{g_{I,i}^{(m)}} \mathcal{M}_{I1}^{(m)} \xrightarrow{f_{I,i}^{(m)}} \psi_i^{(0)}(\mathcal{M}_I^{(m)}) \tag{11}$$

such that

$$\psi_j^{(0)}(f_{I,i}^{(m)}) \circ f_{Ii,j}^{(m)} = \psi_i^{(0)}(f_{I,j}^{(m)}) \circ f_{Ij,i}^{(m)},$$
$$g_{Ii,j}^{(m)} \circ \psi_j^{(0)}(g_{I,i}^{(m)}) = g_{Ij,i}^{(m)} \circ \psi_i^{(0)}(g_{I,j}^{(m)}),$$
$$f_{Ij,i}^{(m)} \circ g_{Ii,j}^{(m)} = \psi_i^{(0)}(g_{I,j}^{(m)}) \circ \psi_j^{(1)}(f_{I,i}^{(m)}).$$

Indeed, suppose we are given such holonomic $\mathcal{D}$-modules for $m-1$, we define $\mathcal{M}_I^{(m)}$ for $I \subset \underline{\ell} \setminus m$ as the gluing of $\mathcal{M}_I^{(m-1)}$ and $\mathcal{M}_{Im}^{(m-1)}$ by $g_{I,m}^{(m-1)}$ and $f_{I,m}^{(m-1)}$. By the construction, we have the induced morphisms as in (11) with the desired property. After the procedure, we obtain a good holonomic $\mathcal{D}$-module

$$\Upsilon\big((V_I \mid I \subset \underline{\ell}), (f_{I,i}, g_{I,i} \mid I \sqcup \{i\} \subset \underline{\ell})\big) := \mathcal{M}^{(\ell)}.$$

Clearly, Υ and Φ are mutually quasi-inverse. □

We can describe some functors on $\mathrm{Hol}^{\mathrm{good}}(X,D)$ in terms of $C(X,D)$. Let

$$\mathcal{C} = \big((V_I), (g_{I,i}, f_{I,i})\big).$$

▷ For i, we define $\mathcal{C}(*D_i) = ((V'_I), (g'_{I,i}, f'_{I,i}))$ as follows.

We set $V'_I := V_I$ $(i \notin I)$ or $V'_I := \psi_i^{(0)}(V_{I\setminus\{i\}})$ $(i \notin I)$. If $j \neq i$, $g'_{I,j}$ and $f'_{I,j}$ are the naturally induced morphisms, and $g'_{I,i}$ and $f'_{I,i}$ are given by the canonical morphisms

$$\psi_i^{(1)}(V_I) \xrightarrow{\text{can}} \psi_i^{(0)}(V_I) \xrightarrow{\text{id}} \psi_i^{(0)}(V_I).$$

▷ We define $\mathcal{C}(!D_i)$ as follows.

We set $V'_I := V_I$ $(i \notin I)$ or $V'_I := \psi_i^{(1)}(V_{I\setminus\{i\}})$ $(i \notin I)$. If $j \neq i$, $g'_{I,j}$ and $f'_{I,j}$ are the naturally induced morphisms, and $g'_{I,i}$ and $f'_{I,i}$ are given by the canonical morphisms $\psi_i^{(1)}(V_I) \xrightarrow{\text{id}} \psi_i^{(1)}(V_I) \xrightarrow{\text{var}} \psi_i^{(0)}(V_I)$. We have naturally defined morphisms $\mathcal{C}(!D_i) \to \mathcal{C} \to \mathcal{C}(*D_i)$. It is easy to observe

$$\Phi\big(\mathcal{M}(\star D_i)\big) \simeq \Phi(\mathcal{M})(\star D_i).$$

▷ We define $\psi_i^{(a)}(\mathcal{C}) = ((V'_I), (g'_{I,i}, f'_{I,i}))$ as follows.

If $i \notin I$, we set $V'_I = 0$. If $i \in I$, we set $V'_I := \psi_i^{(a)}(V_{I\setminus i})$. The morphisms $g'_{I,i}$ and $f'_{I,i}$ are the naturally induced ones. Then, we have a natural isomorphism

$$\Phi\psi_i^{(a)}(\mathcal{M}) \simeq \psi_i^{(a)}\Phi(\mathcal{M}).$$

▷ We define $\phi_i^{(a)}(\mathcal{C}) = ((V'_I), (g'_{I,i}, f'_{I,i}))$ as follows.

If $i \notin I$, we set $V'_I = 0$. If $i \in I$, we set $V'_I := V_I^{(a)}$. The morphisms $g'_{I,i}$ and $f'_{I,i}$ are the naturally induced ones. Then, we have a natural isomorphism

$$\Phi\phi_i^{(a)}(\mathcal{M}) \simeq \phi_i^{(a)}\Phi(\mathcal{M}).$$

▷ We define $\boldsymbol{D}(\mathcal{C}) = ((V'_I), (g'_{I,i}, f'_{I,i}))$ as follows.

We set $V'_I := \boldsymbol{D}(V_I^{(-1)})(*\partial D_I)$. The morphisms $g'_{I,i}$ and $f'_{I,i}$ are the naturally induced ones. Then, we have a natural isomorphism

$$\Phi\boldsymbol{D}(\mathcal{M}) \simeq \boldsymbol{D}\Phi(\mathcal{M}).$$

3.1.6. Appendix. — The category $\mathrm{Hol}^{\mathrm{good}}(X, D)$ of good holonomic $\mathcal{D}$-modules on (X, D) is not abelian. Indeed, a direct sum of good holonomic $\mathcal{D}$-modules is not necessarily good. If we would like to work on an abelian category, it would be convenient to restrict ourselves to a smaller category.

We generalize the notion of good system of irregular values in §2.4.1 of [**47**]. For any point $P \in D$, we introduce some rings. To define them, we introduce a category C_P.

▷ Objects in C_P are holomorphic maps $\varphi : (Z,Q) \to (X,P)$ of smooth complex manifolds which are coverings with ramification along D on a neighbourhood of P. We set $D_Z := \varphi^{-1}(D)$.

▷ Morphisms $F : ((Z,Q),\varphi) \to ((Z',Q'),\varphi')$ are holomorphic maps

$$F : (Z,Q) \longrightarrow (Z',Q') \quad \text{such that } \varphi' \circ F = \varphi.$$

Such morphisms induce the morphisms $\mathcal{O}_{Z'}(*D_{Z'})_{Q'} \to \mathcal{O}_Z(*D_Z)_Q$ over $\mathcal{O}_X(*D)_Q$. Let $\widetilde{\mathcal{O}}_X(*D)_P$ denote a colimit of $\mathcal{O}_Z(*D_Z)_Q$. Similarly, let $\widetilde{\mathcal{O}}_{X,P}$ denote the colimit of $\mathcal{O}_{Z,Q}$.

We have another more direct description. Let $\mathbb{C}\{z_1,\dots,z_n\}$ denote the ring of convergent power series. Let $\mathbb{C}\{z_1,\dots,z_n\}_{z_1\cdots z_\ell}$ denote its localization with respect to $z_1\cdots z_\ell$. For a coordinate system $(z_1,\dots,z_n)$ such that $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$, we have natural isomorphisms

$$\widetilde{\mathcal{O}}_{X,P} \simeq \varinjlim_e \mathbb{C}\{z_1^{1/e},\dots,z_\ell^{1/e},z_{\ell+1},\dots,z_n\},$$
$$\widetilde{\mathcal{O}}_X(*D)_P \simeq \varinjlim_e \mathbb{C}\{z_1^{1/e},\dots,z_\ell^{1/e},z_{\ell+1},\dots,z_n\}_{z_1^{1/e}\cdots z_\ell^{1/e}}.$$

A finite subset $\mathcal{I} \subset \widetilde{\mathcal{O}}_X(*D)_P/\widetilde{\mathcal{O}}_{X,P}$ can be regarded as $\mathcal{I} \subset \mathcal{O}_Z(*D_Z)_Q/\mathcal{O}_{Z,Q}$ for some $((Z,Q),\varphi) \in C_P$. It is called a good set of ramified irregular values if:

(i) it is a good set of irregular values on (Z, D_Z),

(ii) it is stable under the action of the Galois group of φ.

Note that if P_1 is close to P, we choose $Q_1 \in \varphi^{-1}(P_1)$, and we obtain a natural map $\mathcal{I}_P \to \mathcal{O}_Z(*D_Z)_{Q_1}/\mathcal{O}_{Z,Q_1} \to \widetilde{\mathcal{O}}_X(*D)_{P_1}/\widetilde{\mathcal{O}}_{X,P_1}$. The image is well defined.

Definition 3.1.16. — A good system of ramified irregular values on (X,D) is a family of good sets of ramified irregular values $\boldsymbol{\mathcal{I}} = \{\mathcal{I}_P \mid P \in D\}$ satisfying the following condition.

▷ If P_1 is sufficiently close to P, we impose that the image of $\mathcal{I}_P$ in the image of $\mathcal{I}_P$ in $\widetilde{\mathcal{O}}_X(*D)_{P_1}/\widetilde{\mathcal{O}}_{X,P_1}$ is equal to $\mathcal{I}_{P_1}$. □

Let $\boldsymbol{\mathcal{I}} = (\mathcal{I}_P \mid P \in D)$ be a good system of ramified irregular values on (X,D). A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called $\mathcal{I}$-good if for any $P \in D$ there exists a neighborhood X_P such that $\mathcal{M}_{|X_P}$ is $\mathcal{I}_P$-good. Then, the category of $\boldsymbol{\mathcal{I}}$-good holonomic $\mathcal{D}$-modules on (X,D) is an abelian full subcategory of $\mathrm{Hol}(X)$.

3.2. De Rham complexes

3.2.1. De Rham complex with infinite decay. — For any complex manifold X, let $\Omega_X^{p,q}$ denote the sheaf of C^∞-(p,q)-forms on X.We set

$$d_X := \dim X.$$

For any analytic subset $Z \subset X$, we set

$$\Omega_{\widehat{Z}}^{p,q} := \Omega_X^{p,q} \otimes_{\mathcal{C}_X^\infty} \mathcal{C}_{\widehat{Z}}^\infty.$$

For any hypersurface $D \subset X$, we set

$$\Omega_{\widehat{Z}}^{p,q}(*D) := \Omega_{\widehat{Z}}^{p,q} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D).$$

We say that $D_1 \cup D_2 = D$ is a decomposition of D if $D_i \subset X$ $(i = 1, 2)$ are hypersurfaces such that $\operatorname{codim}_X(D_1 \cap D_2) > 1$.

In that situation, we say that D_2 is the complement of D_1 in D. In other words, the complement of D_1 in D is the union of the irreducible components of D which are not contained in D_1.

When we are given a hypersurface $D \subset X$ with a decomposition $D = D_1 \cup D_2$, we denote the kernel of $\Omega_X^{p,q}(*D_2) \to \Omega_{\widehat{D}_1}^{p,q}(*D_2)$ by

$$\Omega_X^{p,q}(*D_2)^{<D_1}.$$

Let D_0 be a normal crossing hypersurface of X with a decomposition $D_0 = D_1 \cup D_2$. For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, we define $\mathrm{DR}_X^{<D_1 \leq D_2} \mathcal{M}$ as

$$\operatorname{Cone}\big(\mathrm{DR}_X(\mathcal{M}(*D_2)) \to \mathrm{DR}_{\widehat{D}_1}\big(\mathcal{M}(*D_2)\big)\big)[-1]$$

in the derived category $D^b(\mathbb{C}_X)$. We have the following natural quasi-isomorphisms:

$$\mathrm{DR}_X^{<D_1 \leq D_2} \mathcal{M} \simeq \Omega_X^{d_X, \bullet\, <D_1}(*D_2) \otimes_{\mathcal{D}_X}^L \mathcal{M} \simeq \operatorname{Tot} \Omega_X^{\bullet,\bullet\, <D_1}(*D_2) \otimes_{\mathcal{O}_X} \mathcal{M}[d_X].$$

Here, Tot means the total complex associated to the double complex. In the following, we shall often omit to denote Tot. It is easy to observe that the natural morphism $\mathrm{DR}_X^{<D_1 \leq D_2} \mathcal{M} \to \mathrm{DR}_X^{<D_1 \leq D_2}(\mathcal{M}(*D_0))$ is an isomorphism. We also have the following natural isomorphisms in $D^b(\mathbb{C}_X)$:

$$\begin{aligned}\mathrm{DR}_X^{<D_1}\big(\boldsymbol{D}_X \mathcal{M}(*D_0)\big) &\simeq \Omega_X^{d_X,\bullet}(*D_2)^{<D_1} \otimes_{\mathcal{D}_X}^L \boldsymbol{D}_X \mathcal{M}(*D_0) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \Omega_X^{0,\bullet}(*D_2)^{<D_1}\big)[d_X].\end{aligned}$$

The following proposition is an immediate consequence of the isomorphism of Mebkhout recalled in Proposition 2.1.4.

Proposition 3.2.1. — *If* $(\mathcal{M}(*D_2))(!D_1) \simeq \mathcal{M}(*D_2)$, *the natural morphism*

$$\mathrm{DR}_X^{<D_1 \leq D_2}(\mathcal{M}) \longrightarrow \mathrm{DR}_X^{\leq D_2}(\mathcal{M})$$

is an isomorphism in $D_c^b(\mathbb{C}_X)$. □

3.2.2. The identification in the case of good holonomic $\mathcal{D}$-modules

Let X be a complex manifold with a normal crossing hypersurface D. Let $D_0 \subset D$ be the union of some irreducible components with a decomposition $D_0 = D_1 \cup D_2$. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X, D).

The following proposition is a special case of Proposition 3.2.1.[1]

Proposition 3.2.2. — *If* $\mathcal{M}(!D_1) = \mathcal{M}$, *the natural morphism*

$$\mathrm{DR}_X^{<D_1 \leq D_2}\,\mathcal{M} \longrightarrow \mathrm{DR}_X^{\leq D_2}\,\mathcal{M}$$

is a quasi-isomorphism. □

We obtain the following isomorphisms in $D_c^b(\mathbb{C}_X)$:

$$\mathrm{DR}_X^{<D_1 \leq D_2}(\mathcal{M}) \xleftarrow{\simeq} \mathrm{DR}_X^{<D_1 \leq D_2}\big(\mathcal{M}(!D_1)\big) \xrightarrow{\simeq} \mathrm{DR}_X^{\leq D_2}\big(\mathcal{M}(!D_1)\big). \tag{12}$$

We have already seen the right isomorphism. For the left isomorphism, we may use

$$\Omega_X^{p,q\,<D_1} \simeq \Omega_X^{p,q\,<D_1}(*D_1).$$

We will identify $\mathrm{DR}_X^{<D_1 \leq D_2}(\mathcal{M})$ and $\mathrm{DR}_X^{\leq D_2}(\mathcal{M}(!D_1))$ by (12).

Lemma 3.2.3. — *If* $D_1 \subset D_1' \subset D$, *then the following diagram of the natural morphisms is commutative:*

$$\begin{array}{ccc}
\mathrm{DR}_X^{<D_1'}\,\mathcal{M} & \xrightarrow{\simeq} & \mathrm{DR}_X\,\mathcal{M}(!D_1') \\
\downarrow & & \downarrow \\
\mathrm{DR}_X^{<D_1}\,\mathcal{M} & \xrightarrow{\simeq} & \mathrm{DR}_X\,\mathcal{M}(!D_1).
\end{array}$$

It is also factorized as follows:

$$\begin{array}{ccccc}
\mathrm{DR}_X^{<D_1'}\,\mathcal{M} & \xleftarrow{\simeq} & \mathrm{DR}^{<D_1}\,\mathcal{M}(!D_1') & \xrightarrow{\simeq} & \mathrm{DR}_X\,\mathcal{M}(!D_1') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{DR}_X^{<D_1}\,\mathcal{M} & \xleftarrow{\simeq} & \mathrm{DR}^{<D_1}\,\mathcal{M} & \xrightarrow{\simeq} & \mathrm{DR}_X\,\mathcal{M}(!D_1).
\end{array}$$

[1] The author thanks the referee for the simplified proof of the proposition.

Proof. — We have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{DR}_X^{<D_1'}\mathcal{M} & \xleftarrow{\simeq} & \mathrm{DR}^{<D_1'}\mathcal{M}(!D_1') & \xrightarrow{\simeq} & \mathrm{DR}_X\mathcal{M}(!D_1') \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{DR}_X^{<D_1}\mathcal{M} & \xleftarrow{\simeq} & \mathrm{DR}^{<D_1}\mathcal{M}(!D_1) & \xrightarrow{\simeq} & \mathrm{DR}_X\mathcal{M}(!D_1). \end{array}$$

Then, the claim of the lemma is clear. □

3.2.3. Duality. — We continue to use the notation in §3.2.2. For simplicity, we assume $D = D_0$. We have a morphism of complexes

(13) $\mathrm{Tot}\big(\mathrm{Tot}\,\Omega^{\bullet,\bullet\,<D_2}(*D_1)[d_X]\otimes\mathrm{Tot}\,\Omega^{0,\bullet\,<D_1}(*D_2)[d_X]\big)\longrightarrow\mathrm{Tot}\,\Omega^{\bullet,\bullet}[2d_X]$

by $\xi\otimes\eta\longmapsto(-1)^{pd_X}\xi\wedge\eta$, where ξ and η are local sections of

$$\big(\mathrm{Tot}\,\Omega^{\bullet,\bullet\,<D_2}(*D_1)\big)^{p+d_X}\quad\text{and}\quad\big(\mathrm{Tot}\,\Omega^{0,\bullet\,<D_1}(*D_2)\big)^{q+d_X}$$

respectively. Let $\mathcal{I}_1^\bullet$ be a $\mathcal{D}_X$-injective resolution of $\mathrm{Tot}\,\Omega^{0,\bullet\,<D_1}(*D_2)[d_X]$, and let $\mathcal{I}_2^\bullet$ be a $\mathbb{C}_X$-injective resolution of $\mathrm{Tot}\,\Omega^{\bullet,\bullet}[2d_X]$. Then, the morphism is extended to a $\mathbb{C}_X$-homomorphism $\mathrm{DR}_X^{\leq D_1<D_2}(\mathcal{I}_1^\bullet)\to\mathcal{I}_2^\bullet$.

For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have the following natural morphism:

$$\mathrm{DR}_X^{<D_1\leq D_2}(\boldsymbol{D}_X\mathcal{M})\longrightarrow\boldsymbol{D}_X\,\mathrm{DR}_X^{<D_2\leq D_1}(\mathcal{M}).\tag{14}$$

Indeed, $\mathrm{DR}_X^{<D_1\leq D_2}\boldsymbol{D}_X\mathcal{M}$ is represented by $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{I}_1^\bullet)$. Hence, we have the desired morphism given as follows:

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{I}_1^\bullet)&\longrightarrow\mathcal{H}om_{\mathbb{C}_X}(\mathrm{DR}_X^{<D_2\leq D_1}\mathcal{M},\mathrm{DR}_X^{<D_2\leq D_1}\mathcal{I}_1^\bullet)\\&\longrightarrow\mathcal{H}om_{\mathbb{C}_X}(\mathrm{DR}_X^{<D_2\leq D_1}\mathcal{M},\mathcal{I}_2^\bullet).\end{aligned}$$

Theorem 3.2.4. — *Let V be a good meromorphic flat bundle on (X,D). The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{DR}^{<D_1\leq D_2}(V^\vee) & \xrightarrow{G_1} & \boldsymbol{D}_X\,\mathrm{DR}^{<D_2\leq D_1}(V) \\ \simeq\big\downarrow & & \simeq\big\uparrow \\ \mathrm{DR}\,V^\vee(!D_1) & \xrightarrow[\simeq]{G_2} & \boldsymbol{D}_X\,\mathrm{DR}_X\big(V(!D_2)\big). \end{array}\tag{15}$$

Here, G_1 is induced by (14) *and* $\mathrm{DR}_X^{<D_1\leq D_2}(\boldsymbol{D}_XV)\simeq\mathrm{DR}_X^{<D_1\leq D_2}(V^\vee)$. *The vertical isomorphisms are given by* (12), *and G_2 is induced by the natural isomorphism of $\mathcal{D}$-modules $V^\vee(!D_1)\simeq\boldsymbol{D}_X(V(!D_2))$.* (*See* §3.1.3.) *In particular, G_1 is also an isomorphism.*

Proof. — We have the commutativity of the natural morphisms

$$\begin{array}{ccccc}
\mathrm{DR}_X^{<D_1\le D_2}(V^\vee) & \xrightarrow{\simeq} & \mathrm{DR}_X^{<D_1\le D_2}(\boldsymbol{D}_X V) & \longrightarrow & \boldsymbol{D}_X\,\mathrm{DR}_X^{<D_2\le D_1}(V) \\
\simeq\uparrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathrm{DR}_X^{<D_1\le D_2}\big(V^\vee(!D_1)\big) & \xrightarrow{\simeq} & \mathrm{DR}_X^{<D_1\le D_2}\big(\boldsymbol{D}_X(V(!D_2))\big) & \longrightarrow & \boldsymbol{D}_X\,\mathrm{DR}_X^{<D_2\le D_1}\big(V(!D_2)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{DR}_X\big(V^\vee(!D_1)\big) & \xrightarrow{\simeq} & \mathrm{DR}_X\big(\boldsymbol{D}_X(V(!D_2))\big) & \longrightarrow & \boldsymbol{D}_X\,\mathrm{DR}_X\big(V(!D_2)\big).
\end{array}$$

Then, the claim of the theorem is clear. □

3.2.4. Functoriality for birational morphisms. — Let X be a complex manifold, and let D be a normal crossing hypersurface with a decomposition $D = D_1 \cup D_2$. Let D_3 be a hypersurface of X. Let $\varphi : X' \to X$ be a proper birational morphism such that

(i) $D' = \varphi^{-1}(D \cup D_3)$ is normal crossing,

(ii) $X' \setminus D' \simeq X \setminus (D \cup D_3)$.

We put $D_1' := \varphi^{-1}(D_1)$. Let D_2' be the complement of D_1' in D'.

Let $\mathcal{M}'$ be any coherent $\mathcal{D}_{X'}$-module having a good filtration in the neighbourhood of fibers of φ. We have the natural morphism

$$\mathrm{DR}_X^{<D_1\le D_2}\,\varphi_\dagger\mathcal{M}' \longrightarrow R\varphi_*\,\mathrm{DR}_{X'}^{<D_1'\le D_2'}\,\mathcal{M}'. \tag{16}$$

Indeed, we have

$$\begin{aligned}
\mathrm{DR}_X^{<D_1\le D_2}\,\varphi_\dagger\mathcal{M}' &\simeq R\varphi_*\big(\Omega_{X'}\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}(\Omega_X^{0,\bullet<D_1}(*D_2))\otimes^L_{\mathcal{D}_{X'}}\mathcal{M}'\big) \\
&\longrightarrow R\varphi_*\big((\Omega_{X'}\otimes\Omega_{X'}^{0,\bullet<D_1'}(*D_2'))\otimes^L_{\mathcal{D}_{X'}}\mathcal{M}'\big) \\
&\simeq R\varphi_*\big(\mathrm{DR}_{X'}^{<D_1'\le D_2'}(\mathcal{M}')\big).
\end{aligned} \tag{17}$$

Let V be a good meromorphic flat bundle on (X,D), and we set

$$V' := \varphi^*V\otimes\mathcal{O}_{X'}(*D').$$

We have a natural isomorphism

$$\big(V(*D_3)\big)(!D_1)\simeq\varphi_\dagger\big(V'(!D_1')\big).$$

Hence, we have a morphism of $\mathcal{D}_X$-modules $V(!D_1)\to\varphi_\dagger(V'(!D_1'))$. We obtain the following morphism from (16) and $V\to\varphi_\dagger V'$:

$$\mathrm{DR}_X^{<D_1\le D_2}(V)\longrightarrow R\varphi_*\,\mathrm{DR}_{X'}^{<D_1'\le D_2'}(V'). \tag{18}$$

It is equal to the one induced by $\varphi^{-1}(\Omega_X^{\bullet,\bullet<D_1}(*D_2)\otimes V)\to\Omega_{X'}^{\bullet,\bullet<D'_1}(*D'_2)\otimes V'$. Note that we have natural isomorphisms

$$(\Omega_{X'}\otimes V')\otimes^L_{\mathcal{D}_{X'}}(\mathcal{O}_{X'}\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}(\mathcal{D}_X\otimes\Omega_X^{-1}))\simeq(\Omega_{X'}\otimes V')\otimes^L_{\mathcal{D}_{X'}(*D')}\big(\mathcal{O}_{X'}(*D')\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}(\mathcal{D}_X\otimes\Omega_X^{-1})\big)\simeq(\Omega_{X'}\otimes V')\otimes^L_{\mathcal{D}_{X'}(*D')}(\mathcal{D}_{X'}(*D')\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}\Omega_X)\simeq V'. \tag{19}$$

By considering the dual with $V^\vee$ (see Theorem 3.2.4), we also obtain the following morphism:

$$R\varphi_*\,\mathrm{DR}_{X'}^{<D'_2\le D'_1}(V')\longrightarrow\mathrm{DR}_X^{<D_2\le D_1}(V). \tag{20}$$

THEOREM 3.2.5. — *We have the commutative diagram*

$$\begin{array}{ccc}\mathrm{DR}_X^{<D_1\le D_2}V & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}^{<D'_1\le D'_2}V' \\ \simeq\downarrow & & \simeq\downarrow \\ \mathrm{DR}_X V(!D_1) & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}V'(!D'_1).\end{array} \tag{21}$$

Here, the vertical isomorphisms are given in (12)*, the upper horizontal arrow is* (18)*, and the lower horizontal arrow is induced by the morphism of $\mathcal{D}_X$-modules $V(!D_1)\to\varphi_\dagger(V'(!D'_1))$. Similarly, we have the commutative diagram*

$$\begin{array}{ccc}R\varphi_*\,\mathrm{DR}_{X'}^{<D'_2\le D'_1}V' & \longrightarrow & \mathrm{DR}_X^{<D_2\le D_1}V \\ \simeq\downarrow & & \simeq\downarrow \\ R\varphi_*\,\mathrm{DR}_{X'}V'(!D'_2) & \longrightarrow & \mathrm{DR}_X V(!D_2).\end{array} \tag{22}$$

Here, the vertical isomorphisms are given in (12)*, the upper horizontal arrow is* (20)*, and the lower horizontal arrow is induced by the natural morphism of $\mathcal{D}_X$-modules $\varphi_\dagger(V'(!D'_2))\to V(!D_2)$.*

Proof. — We have the commutative diagram

$$\begin{array}{ccccc}\mathrm{DR}_X^{<D_1\le D_2}(V) & \longrightarrow & \mathrm{DR}_X^{<D_1\le D_2}(\varphi_\dagger V') & \longrightarrow & R\varphi_*\,\mathrm{DR}_X^{<D'_1\le D'_2}V' \\ \simeq\uparrow & & \uparrow & & \simeq\uparrow \\ \mathrm{DR}_X^{<D_1\le D_2}\big(V(!D_1)\big) & \longrightarrow & \mathrm{DR}_X^{<D_1\le D_2}\big(\varphi_\dagger V'(!D'_1)\big) & \longrightarrow & R\varphi_*\,\mathrm{DR}_X^{<D'_1\le D'_2}V'(!D'_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{DR}_X V(!D_1) & \longrightarrow & \mathrm{DR}_X\,\varphi_\dagger V'(!D'_1) & \longrightarrow & R\varphi_*\,\mathrm{DR}_X V'(!D'_1).\end{array}$$

Then, we obtain the commutativity of (21).

Let us consider the commutativity of (22). Recall the commutativity of (7). We have the commutative diagram for $\mathcal{N} \to \varphi_\dagger \mathcal{N}'$, where $\mathcal{N}$ (resp. $\mathcal{N}'$) is a coherent $\mathcal{D}_X$-module (resp. $\mathcal{D}_{X'}$-module):

$$\begin{array}{ccccccc} R\varphi_* \operatorname{DR}_X \boldsymbol{D}\mathcal{N}' & \simeq & \operatorname{DR} \varphi_\dagger \boldsymbol{D}\mathcal{N}' & \simeq & \operatorname{DR} \boldsymbol{D}\varphi_\dagger \mathcal{N}' & \to & \operatorname{DR} \boldsymbol{D}\mathcal{N} \\ \downarrow & & & & \downarrow & & \downarrow \\ R\varphi_* \boldsymbol{D} \operatorname{DR}_X \mathcal{N}' & \simeq & \boldsymbol{D} R\varphi_* \operatorname{DR} \mathcal{N}' & \simeq & \boldsymbol{D} \operatorname{DR} \varphi_\dagger \mathcal{N}' & \to & \boldsymbol{D} \operatorname{DR} \mathcal{N}. \end{array}$$

The vertical arrows are also isomorphisms. Hence, the lower horizontal arrow in (22) is obtained as the dual of $\operatorname{DR}_X V^\vee(D_1) \to R\varphi_* \operatorname{DR}_{X'} V'^\vee(!D_1')$ in $D_c^b(\mathbb{C}_X)$. Then, the commutativity of (22) follows from the commutativity of (21). Thus, the proof of Theorem 3.2.5 is finished. □

CHAPTER 4

SOME SHEAVES ON THE REAL BLOW UP

4.1. Holomorphic functions

We shall introduce the sheaves of holomorphic functions of various types. We give some statements mainly on flatness. The proof will be given later.

4.1.1. Preliminary. — Let X be an n-dimensional complex manifold with a simply normal crossing hypersurface D with the irreducible decomposition $\bigcup_{i\in\Lambda} D_i$. In this paper, the real blow up $\pi : \widetilde{X}(D) \to X$ means the fiber product of $\widetilde{X}(D_i)$ over X.For any subset $I \subset \Lambda$, we set

$$D_I := \bigcap_{i\in I} D_i \quad \text{and} \quad D(I) := \bigcup_{i\in I} D_i.$$

Formally, $D_\varnothing := X$. For $J \subset I^c := \Lambda \setminus I$, we put

$$D_I(J) := D_I \cap D(J).$$

In particular, $\partial D_I := D_I(I^c)$.

4.1.2. Holomorphic functions with moderate growth or rapid decay

Recall that holomorphic functions on an open subset $U \subset \widetilde{X}(D)$ are defined to be C^∞-functions on U whose restriction to $U \setminus \pi^{-1}(D)$ are holomorphic. A holomorphic function f on U is called of rapid decay if the following holds:

- ▷ Let P be any point of $\pi^{-1}(D) \cap U$. We take a holomorphic coordinate system $(z_1, \dots, z_n)$ around $\pi(P)$ such that $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Then, we have $f = O(\prod_{i=1}^{\ell} |z_i|^N)$ for any N around P.

The sheaf of holomorphic functions on $\widetilde{X}(D)$ is denoted $\mathcal{O}_{\widetilde{X}(D)}$ and the sheaf of holomorphic functions with rapid decay is denoted $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D)}$.

Let U be any open subset in $\widetilde{X}(D)$. A holomorphic function f on $U\setminus\pi^{-1}(D)$ is called of moderate growth if the following holds:

- ▷ Let P be any point of $\pi^{-1}(P)\cap U\neq\varnothing$. We take a holomorphic coordinate system $(z_1,\dots,z_n)$ around $\pi(P)$ such that $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. Then, we have $f=O(\prod_{i=1}^{\ell}|z_i|^{-N})$ for some N around P.

In this paper, the sheaf of holomorphic functions with moderate growth is denoted $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)}$.

We shall prove the following (Proposition 4.2.4, Theorem 4.6.1).

Theorem 4.1.1. — *The sheaves $\mathcal{O}_{\widetilde{X}(D)}$, $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D)}$ and $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)}$ are flat over $\pi^{-1}(\mathcal{O}_X)$.*

4.1.3. Partially rapid decay functions on completions. — Suppose that Z is $\pi^{-1}(D_I(J))$ for some $I\sqcup J\subset\Lambda$. Let $\mathcal{I}_Z\subset\mathcal{O}_{\widetilde{X}(D)}$ be the ideal sheaf of Z, and put

$$\mathcal{O}_{\widehat{Z}}:=\varprojlim\mathcal{O}_X/\mathcal{I}_Z^m.$$

For a given $\mathcal{O}_{\widetilde{X}(D)}$-module $\mathcal{F}$, we set $\mathcal{F}_{|\widehat{Z}}:=\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}(D)}}\mathcal{O}_{\widehat{Z}}$. According to a generalized Borel-Ritt theorem due to Majima and Sabbah (see [**33**], Proposition II.1.1.16 of [**52**]), the natural morphism $\mathcal{O}_{\widehat{\pi^{-1}(D_I)}}\to\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ is surjective. The kernel is denoted by $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$. If $D_I=X$ and $D(J)=D$, it is equal to $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D)}$.

We shall prove the following theorem. (See Proposition 4.2.4 for a refined claim.)

Proposition 4.1.2. — *The sheaves $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{O}_{\widehat{\pi^{-1}D_I(J)}}$ are flat over $\pi^{-1}(\mathcal{O}_X)$.*

4.1.4. Holomorphic functions of Nilsson type

4.1.4.1. Preliminary. — We set

$$\mathrm{Nil}(z):=\bigoplus_{\alpha\in\mathbb{C}}z^{\alpha}\mathbb{C}[\log z].$$

For $(\alpha,k)\in\mathbb{C}\times\mathbb{Z}_{\geq 0}$, we put $\varphi_{\alpha,k}(z):=z^{\alpha}(\log z)^k\in\mathrm{Nil}(z)$. Let T be any finite subset contained in $\{\alpha\in\mathbb{C}\,|\,0\leq\mathrm{Re}(\alpha)<1\}$. For simplicity, we assume $0\in T$. Let N be a non-negative integer. We set

$$\mathrm{Nil}_{T,N}(z):=\Big\{\sum a_{\alpha,j,k}\,\varphi_{\alpha+j,k}(z)\in\mathrm{Nil}(z)\,\Big|\,a_{\alpha,j,k}\in\mathbb{C},\ j\geq -N,\ k\leq N,\ \alpha\in T\Big\}.$$

Note that $\mathrm{Nil}_{T,N}(z)$ is a finitely generated free $\mathbb{C}[z]$-module. For $T \subset T'$ and $N \leq N'$, we have a natural inclusion $\mathrm{Nil}_{T,N}(z) \subset \mathrm{Nil}_{T',N'}(z)$. We have

$$\mathrm{Nil}(z) = \varinjlim \mathrm{Nil}_{T,N}(z).$$

Let $\widetilde{\mathbb{C}}_z$ be the real blow up of $\mathbb{C}_z$ along 0. Let ι be the inclusion $\iota : \mathbb{C}_z^* \to \widetilde{\mathbb{C}}_z$. We have the subsheaves of $\iota_* \mathcal{O}_{\mathbb{C}^*}$ on $\widetilde{\mathbb{C}}$ corresponding to $\mathrm{Nil}(z)$ and $\mathrm{Nil}_{T,N}(z)$. The sheaves are also denoted by $\mathrm{Nil}(z)$ and $\mathrm{Nil}_{T,N}(z)$.

For $\ell \geq 1$, put

$$\mathrm{Nil}(z_1, \ldots, z_\ell) := \mathrm{Nil}(z_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathrm{Nil}(z_\ell),$$
$$\mathrm{Nil}_{T,N}(z_1, \ldots, z_\ell) := \mathrm{Nil}_{T,N}(z_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathrm{Nil}_{T,N}(z_\ell).$$

We naturally regard $\mathrm{Nil}(z_1, \ldots, z_\ell)$ as a subsheaf of $\iota_* \mathcal{O}_{\mathbb{C}^n - D}$ on the real blow up $\widetilde{\mathbb{C}}(D)$, where $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$ and $\iota : \mathbb{C}^n - D \to \widetilde{\mathbb{C}}^n(D)$. For $(\boldsymbol{\alpha}, \boldsymbol{k}) \in \mathbb{C}^\ell \times \mathbb{Z}_{\geq 0}^\ell$, we put

$$\varphi_{\boldsymbol{\alpha},\boldsymbol{k}}(z_1, \ldots, z_n) := \prod_{i=1}^{\ell} \varphi_{\alpha_i, k_i}(z_i),$$

which are regarded as multi-valued flat sections of $\mathrm{Nil}(z_1, \ldots, z_\ell)$.

4.1.4.2. Holomorphic functions of Nilsson type. — Let X be an n-dimensional complex manifold with a simply normal crossing hypersurface D. Let $D = D^{(1)} \cup D^{(2)}$ be a decomposition.

We shall introduce a sheaf $\mathcal{A}_{\widetilde{X}(D)}^{<D^{(1)} \leq D^{(2)}}$ on $\widetilde{X}(D)$.

First, let us consider the case $X = \Delta^n$, $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let $\underline{\ell} = I_1 \sqcup I_2$ be determined by $D^{(j)} = \bigcup_{i \in I_j} \{z_i = 0\}$ for $j = 1, 2$. Let $\widetilde{\jmath}$ denote the inclusion $X - D \to \widetilde{X}(D)$. Let $\mathcal{A}_{\widetilde{X}(D)}^{<D^{(1)} \leq D^{(2)}}$ be the image of the naturally defined morphisms

$$\mathcal{O}_{\widetilde{X}(D)}^{<D^{(1)}} \otimes \mathrm{Nil}(z_i \mid i \in I_2) \longrightarrow \widetilde{\jmath}_* \mathcal{O}_{X-D}.$$

We can observe that they are independent of the choice of a coordinate system $(z_1, \ldots, z_n)$. Hence, we obtain globally defined sheaf $\mathcal{A}_{\widetilde{X}(D)}^{<D^{(1)} \leq D^{(2)}}$ on $\widetilde{X}(D)$. It is also denoted by $\mathcal{A}_{\widetilde{X}(D)}^{\mathrm{nil}\,<D^{(1)}}$.

We shall prove the following. (See Theorem 4.3.1 and Corollary 4.3.3 for refined claims.)

Theorem 4.1.3. — *$\mathcal{A}^{<D^{(1)} \leq D^{(2)}}$ is flat over $\pi^{-1}\mathcal{O}_X$. We also have*

$$R\pi_* \mathcal{A}_{\widetilde{X}(D)}^{\mathrm{nil}} \simeq \mathcal{O}_X(*D).$$

REMARK 4.1.4. — This type of sheaves are useful when we study the de Rham complex of $V(!D^{(1)}{*}D^{(2)})$ for a good meromorphic flat bundle on (X,D). Compared with functions with moderate growth, we may consider functions with rapid decay along some direction and of Nilsson type along other direction. □

4.1.5. Real blow up along holomorphic functions

4.1.5.1. Category of complex manifolds over $\mathbb{C}^\ell$. — It is convenient to consider the category Cat_ℓ of complex manifolds over $\mathbb{C}^\ell$ given as follows.

▷ An object of Cat_ℓ is a morphism $f: X \to \mathbb{C}^\ell$ of complex manifolds.

▷ Morphisms $\varphi: (X_1, f_1) \to (X_2, f_2)$ in Cat_ℓ are morphisms of complex manifolds $\varphi: X_1 \to X_2$ such that $f_1 = f_2 \circ \varphi$.

We say that φ has *some property* when the underlying φ has the property. For example, we say that $\varphi: (X_1, f_1) \to (X_2, f_2)$ is a closed immersion when $\varphi: X_1 \to X_2$ is a closed immersion. For a given object (X,f) in Cat_ℓ, we set

$$D_X := f^{-1}(D_0),$$

where $D_0 := \bigcup_{i=1}^{\ell}\{z_i = 0\}$.

Let $\widetilde{\mathbb{C}}$ denote the real blow up of $\mathbb{C}$ along $z=0$. We have $\widetilde{\mathbb{C}}^\ell(D_0) = \widetilde{\mathbb{C}}^\ell$.

For any object (X,f) in Cat_ℓ, we have the naturally defined map

$$\Gamma_f : X \longrightarrow X \times \mathbb{C}^\ell, \quad \Gamma_f(x) = \big(x, f(x)\big).$$

A morphism $\varphi: (X_1, f_1) \to (X_2, f_2)$ induces maps $X_1 \times \mathbb{C}^\ell \to X_2 \times \mathbb{C}^\ell$ and $X_1 \times \widetilde{\mathbb{C}}^\ell \to X_2 \times \widetilde{\mathbb{C}}^\ell$, which are denoted by φ_1 and $\widetilde{\varphi}_1$, respectively.

4.1.5.2. Real blow up along functions. — Let (X,f) be an object in Cat_ℓ. Let $j: X \times (\mathbb{C}^*)^\ell \to X \times \widetilde{\mathbb{C}}^\ell$ denote the inclusion. Let $\widetilde{X}(f)$ denote the topological space obtained as the closure of $j(\Gamma_f(X \setminus D_X))$ in $X \times \widetilde{\mathbb{C}}^\ell$, which is called the real blow up of X along f [**54**]. The projection $\widetilde{X}(f) \to X$ is denoted by π_f. The inclusion $\widetilde{X}(f) \to X \times \widetilde{\mathbb{C}}^\ell$ is denoted by $\widetilde{\Gamma}_f$. If there is no risk of confusion, we shall omit to denote the subscript f to simplify the notation. If f is submersive, $\widetilde{X}(f)$ is naturally diffeomorphic to $\widetilde{X}(D_X)$. A morphism $\varphi: (X_1, f_1) \to (X_2, f_2)$ in Cat_ℓ naturally induces a continuous map $\widetilde{\varphi}: \widetilde{X}_1(f_1) \to \widetilde{X}_2(f_2)$.

4.1.5.3. Moderate growth and rapid decay. — Let $(X,f) \in \mathrm{Cat}_\ell$. Let U be any open subset of $\widetilde{X}(f)$. A holomorphic function s on $U \setminus \pi_f^{-1}(D_X)$ is called of moderate growth if we have $|s| = O(\prod |f_i|^{-N})$ for some N locally around any point of $U \cap \pi^{-1}(D_X)$. A holomorphic function s on $U \setminus \pi_f^{-1}(D_X)$ is called of rapid decay if we have $|s| = O(\prod |f_i|^N)$ for any N locally around any point of $U \cap \pi^{-1}(D_X)$. The sheaf of holomorphic functions with moderate growth (resp. rapid decay) is denoted by $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$ (resp. $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$).

We shall prove the following theorem. (See Theorems 4.5.1, 4.5.3, and Theorems 4.4.3, 4.5.4 for refined claims.)

THEOREM 4.1.5

▷ *The sheaves* $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$ *and* $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$ *are flat over* $\pi_f^{-1}(\mathcal{O}_X)$.

▷ *Let* $\widetilde{\Gamma}_f : \widetilde{X}(f) \to X \times \widetilde{\mathbb{C}}^\ell$ *denote the inclusion. Then, we naturally have*

$$\widetilde{\Gamma}_{f*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)} \simeq \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell},$$
$$\widetilde{\Gamma}_{f*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \simeq \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}.$$

▷ *Let* $\rho_0 : \widetilde{X}(D_X) \to \widetilde{X}(f)$ *denote the naturally induced map. Then, we naturally have*

$$R\rho_{0*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)} \simeq \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}, \qquad R\rho_{0*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}.$$

▷ *Let* $\varphi : (Y, g) \to (X, f)$ *be a projective morphism in* Cat_ℓ. *Let* M *be a coherent* $\mathcal{O}_Y$*-module. Then, we have the following natural isomorphism:*

$$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes_{\pi_f^{-1}\mathcal{O}_X} \pi_f^{-1}R\varphi_*M \longrightarrow R\widetilde{\varphi}_*(\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)} \otimes_{\pi_g^{-1}\mathcal{O}_Y} \pi_g^{-1}M).$$

4.2. C^∞-functions

4.2.1. Preliminary. — Let X be any n-dimensional complex manifold with a simply normal crossing hypersurface D with the irreducible decomposition $\bigcup_{i\in\Lambda} D_i$. We use the notation in §4.1.1. Let D° be a (possibly empty) hypersurface of X such that

(i) $D \cup D^\circ$ is simply normal crossing,

(ii) $\dim D \cap D^\circ < n-1$.

For $J \subset \Lambda$, we set

$$D(\overline{J}) := D(J) \cup D^\circ.$$

For $I \sqcup J \subset \Lambda$, we put

$$D_I(\overline{J}) := D_I \cap D(\overline{J}).$$

Let $\Omega^{0,q}_{\widetilde{X}(D)}$ denote the sheaf of C^∞-logarithmic $(0,q)$-forms on $\widetilde{X}(D)$, i.e., a section of $\Omega^{0,q}_{\widetilde{X}(D)}$ is locally described as a linear combination of

$$f \cdot \mathrm{d}\overline{z}_{i_1}/\overline{z}_{i_1} \cdots \mathrm{d}\overline{z}_{i_m}/\overline{z}_{i_m} \cdot \mathrm{d}\overline{z}_{j_1} \cdots \mathrm{d}\overline{z}_{j_k} \quad (1 \le i_p \le \ell,\ \ell+1 \le j_q \le n,\ f \in \mathcal{C}^\infty_{\widetilde{X}(D)})$$

in terms of a local holomorphic coordinate system $(z_1,\dots,z_n)$ such that D is locally described as $\bigcup_{i=1}^{\ell}\{z_i=0\}$. We have the naturally defined operator

$$\overline{\partial}:\Omega^{0,q}_{\widetilde{X}(D)}\longrightarrow\Omega^{0,q+1}_{\widetilde{X}(D)}.$$

The complex $\Omega^{0,\bullet}_{\widetilde{X}(D)}$ is called the Dolbeault complex of $\widetilde{X}(D)$. We put

$$\Omega^{0,\bullet}_{\widehat{Z}}:=\Omega^{0,\bullet}_{\widetilde{X}(D)|\widehat{Z}}$$

for any real analytic subset $Z\subset\widetilde{X}(D)$.

For a given C^{∞}-manifold Y and a real analytic subset $W\subset X$, let

$$\mathcal{C}^{\infty\,<W}_{\widehat{\pi^{-1}(D_I)}\times Y}$$

denote the sheaf $\mathcal{C}^{\infty\,<\pi^{-1}(W)\times Y}_{\widehat{\pi^{-1}(D_I)}\times Y}$ on $\widetilde{X}(D)\times Y$, for simplicity of the description. We also put on $\widetilde{X}(D)\times Y$

$$\Omega^{0,\bullet<W}_{\widehat{\pi^{-1}(D_I)}\times Y}:=\Omega^{0,\bullet}_{\widetilde{X}(D)}\otimes_{\mathcal{C}^{\infty}_{\widetilde{X}(D)}}\mathcal{C}^{\infty<W}_{\widehat{\pi^{-1}(D_I)}\times Y}.$$

Let q_I denote the projection $\pi^{-1}(D_I)\to\widetilde{D}_I(\partial D_I)$. If we are given a holomorphic coordinate system $(z_1,\dots,z_n)$ as above, then

$$\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}=q_I^{-1}\mathcal{O}^{<D_I(J)}_{\widetilde{D}_I(\partial D_I)}[\![z_i\mid i\in I]\!].$$

By a natural diffeomorphism $\pi^{-1}(D_I)\simeq\widetilde{D}_I(\partial D_I)\times(S^1)^{|I|}$, we can locally identify

$$\mathcal{C}^{\infty\,<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}}=\mathcal{C}^{\infty\,<D_I(\bar{J})}_{\widetilde{D}_I(\partial D_I)\times(S^1)^{|I|}}[\![z_i\mid i\in I]\!].$$

For $I\subset J$ and $m\geq 0$, put

$$\mathcal{T}(m,I,J):=\{K\subset J\,\big|\,I\subset K,\,|K|=|I|+m+1\}.$$

We set

$$\mathcal{K}^m(\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}):=\bigoplus_{K\in\mathcal{T}(m,I,J)}\mathcal{O}_{\widehat{\pi^{-1}(D_K)}}.$$

We obtain a complex $\mathcal{K}^{\bullet}(\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}})$ as in §2.1.4. Similarly, we obtain a complex $\mathcal{K}^{\bullet}(\Omega^{0,\bullet\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}\times Y})$. See §I.5 of [**34**] and §II.1.1 of [**52**] for the following.

LEMMA 4.2.1. — *Let $\mathcal{B}$ be $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ or $\Omega^{0,\bullet\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}\times Y}$. The natural inclusion $\mathcal{B}\to\mathcal{K}^0(\mathcal{B})$ induces a quasi-isomorphism $\mathcal{B}\to\mathcal{K}^{\bullet}(\mathcal{B})$.* □

4.2.2. Dolbeault resolution. — In this subsection, we suppose $D^\circ = \varnothing$.

PROPOSITION 4.2.2 (See [**33**], [**52**]). — *$\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\Omega^{0,\bullet\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ are c-soft resolutions of $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ respectively, where $J \subset I^c$.*

Proof. — We give only an outline. In each case, it is easy to compute the 0-th cohomology of the Dolbeault complexes. It is enough to prove the vanishing of the higher cohomology. We may assume $X = \Delta^n$, $D_i = \{z_i = 0\}$ and $D = \bigcup_{i=1}^{\ell} D_i$.

First, let us look at $\Omega^{0,\bullet}_{\widetilde{X}(D)}$. For $1 \leq j \leq n$, let $\mathcal{P}^0_{\leq j}$ be the sheaf of C^∞-functions on $\widetilde{X}(D)$ which are $\overline{\partial}_i$-holomorphic for $i > j$. We set

$$X_j := \Delta^j = \big\{(z_1, \dots, z_j)\big\} \quad \text{and} \quad D_{j,\ell} := \bigcup_{i \leq \min\{j,\ell\}} \{z_i = 0\}.$$

Let $q_{\leq j}$ be the projection $\widetilde{X}(D) \to \widetilde{X}_j(D_{j,\ell})$. Let $\mathcal{P}^1_{\leq j}$ be the sheaf of C^∞-sections of $q^{-1}_{\leq j}\Omega^{0,1}_{\widetilde{X}_j(D_{j,\ell})}$, which are $\overline{\partial}_i$-holomorphic for $i > j$. We set

$$\mathcal{P}^\bullet_{\leq j} := \bigwedge^{\bullet} \mathcal{P}^1_{\leq j}$$

over $\mathcal{P}^0_{\leq j}$. We have the naturally defined operator

$$\overline{\partial} : \mathcal{P}^\bullet_{\leq j} \longrightarrow \mathcal{P}^{\bullet+1}_{\leq j}.$$

Because $\mathcal{P}^\bullet_{\leq 0} = \mathcal{O}_{\widetilde{X}(D)}$ and $\mathcal{P}^\bullet_{\leq n} = \Omega^{0,\bullet}_{\widetilde{X}(D)}$, it is enough to prove that the natural inclusions $\mathcal{P}^\bullet_{\leq j} \to \mathcal{P}^\bullet_{\leq j+1}$ are quasi-isomorphisms for the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widetilde{X}(D)}$. Let $\mathcal{Q}^0_{\leq j} = \mathcal{P}^0_{\leq j+1}$. Let $\mathcal{Q}^1_{\leq j}$ be the sheaf of $q^{-1}_{\leq j}\Omega^{0,1}_{\widetilde{X}_j(D_{j,\ell})}$ which are $\overline{\partial}_i$-holomorphic for $i > j+1$. We take the exterior product $\mathcal{Q}^\bullet_{\leq j} = \bigwedge^\bullet \mathcal{Q}^1_{\leq j}$ over $\mathcal{Q}^0_{\leq j}$. We have the naturally defined operator

$$\begin{aligned} \overline{\partial}_{j+1} : \mathcal{Q}^\bullet_{\leq j} &\to \mathcal{Q}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1}/\overline{z}_{j+1} && (j-1 \leq \ell), \\ \overline{\partial}_{j+1} : \mathcal{Q}^\bullet_{\leq j} &\to \mathcal{Q}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1} && (j \geq \ell). \end{aligned}$$

We clearly have $\operatorname{Ker} \overline{\partial}_{j+1} = \mathcal{P}^\bullet_{\leq j}$. Let us prove $\operatorname{Cok} \overline{\partial}_{j+1} = 0$. In the case $j \geq \ell$, it can be proved by the argument for the standard Dolbeault's lemma. Let us consider the case $j < \ell$.

LEMMA 4.2.3. — *$\overline{\partial}_{j+1} : \mathcal{Q}^\bullet_{\leq j|\widehat{\pi^{-1}(D_{j+1})}} \to \mathcal{Q}^\bullet_{\leq j|\widehat{\pi^{-1}(D_{j+1})}} \wedge \mathrm{d}\overline{z}_{j+1}/\overline{z}_{j+1}$ is an epimorphism.*

Proof. — We use the polar coordinate system $z_{j+1} = r_{j+1}\mathrm{e}^{\sqrt{-1}\,\theta_{j+1}}$. The action of $\overline{\partial}_{j+1}$ is expressed as follows:

$$\overline{\partial}_{j+1}\Big(\sum_n f_n(\theta_{j+1})\,z_{j+1}^n\Big) = \sum_n \big(\tfrac{1}{2}\sqrt{-1}\,\partial_{\theta_{j+1}}\big) f_n(\theta_{j+1})\,z_{j+1}^n \cdot \mathrm{d}\overline{z}_{j+1}/\overline{z}_{j+1}$$

Then, it is easy to prove the claim of Lemma 4.2.3. □

Put $D' := \bigcup_{i=1, i\neq j+1}^{\ell}\{z_i = 0\}$, and consider the real blow up

$$\pi' : \widetilde{X}(D') \longrightarrow X.$$

We have a naturally induced morphism

$$q'_{\leq j} : \widetilde{X}(D') \longrightarrow \widetilde{X}_j(D_{j,\ell}).$$

Let $\mathcal{S}^1_{\leq j,X}$ be the sheaf of sections of $(q'_{\leq j})^{-1}\Omega^{0,1}_{\widetilde{X}_j(D_{j,\ell})}$ on $\widetilde{X}(D')$, which are $\overline{\partial}_i$-holomorphic for $i > j+1$. Let $\mathcal{S}^0_{\leq j,X}$ be the sheaf of C^∞-functions on $\widetilde{X}(D')$, which are $\overline{\partial}_i$-holomorphic for $i > j+1$. We set

$$\mathcal{S}^\bullet_{\leq j} := \bigwedge^\bullet \mathcal{S}^1_{\leq j}.$$

It is easy to prove the vanishing of the cokernel of $\overline{\partial}_{j+1} : \mathcal{S}^\bullet_{\leq j} \to \mathcal{S}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1}$ by using the argument for standard Dolbeault's lemma.

Let $P \in \pi^{-1}(D)$. Let U be a small neighbourhood around P. We will shrink it in the following argument. According to Lemma 4.2.3, for any section φ of $\mathcal{Q}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1}/\overline{z}_{j+1}$ on U, we can take a local section ψ of $\mathcal{Q}^\bullet_{\leq j}$ such that

$$(\varphi - \overline{\partial}_j\psi)_{|\widehat{\pi^{-1}(D_j)}\cap U} = 0.$$

We put $\lambda := \varphi - \overline{\partial}_j\psi$. We take a cut function ρ around P, i.e., ρ is constantly 1 around P and constantly 0 near the boundary of U. We can regard $\rho\,\lambda$ as a section of $\mathcal{S}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1}$. Then, we can find a section κ of $\mathcal{S}^\bullet_{\leq j}$ around $\pi_j(P)$ such that $\overline{\partial}_{j+1}\kappa = \rho\lambda$, where π_j denotes the natural projection $\widetilde{X}(D) \to \widetilde{X}(D')$. We obtain $\varphi = \overline{\partial}_j(\psi + \kappa)$ around P. Thus, we obtain the vanishing of the cokernel of $\overline{\partial}_{j+1} : \mathcal{Q}^\bullet_{\leq j} \to \mathcal{Q}^\bullet_{\leq j} \wedge \mathrm{d}\overline{z}_{j+1}/\overline{z}_{j+1}$, and hence the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widetilde{X}(D)}$.

Because $\pi^{-1}(D_I) = \widetilde{D}_I(\partial D_I) \times (S^1)^{|I|}$, we can reduce the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I)}}$ to the vanishing of $\Omega^{0,\bullet}_{\widetilde{D}_I(\partial D_I)}$ by a formal calculation as in Lemma 4.2.3. By using the resolution in Lemma 4.2.1, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D(I))}}$. We have the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^{<D(I)}_{\widetilde{X}(D)} & \longrightarrow & \mathcal{O}_{\widetilde{X}(D)} & \longrightarrow & \mathcal{O}_{\widehat{\pi^{-1}(D(I))}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^{0,\bullet\,<D(I)}_{\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet}_{\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet}_{\widehat{\pi^{-1}(D(I))}} & \longrightarrow & 0. \end{array}$$

Then, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet\,<D(I)}_{\widetilde{X}(D)}$. By a formal calculation as in Lemma 4.2.3, we obtain the vanishing of the higher cohomology of $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ and $\Omega^{0,\bullet\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$. □

4.2.3. Flatness. — In this subsection, D° is not necessarily empty.

PROPOSITION 4.2.4. — *Let $I \sqcup J \subset \Lambda$. The sheaves*

$$\mathcal{C}^{\infty\,<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}},\quad \mathcal{C}^{\infty\,<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}},\quad \mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}},\quad \mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$$

are flat over $\pi^{-1}\mathcal{O}_X$. In particular, $\mathcal{O}_{\widetilde{X}(D)}$ and $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D)}$ are flat over $\pi^{-1}\mathcal{O}_X$.

Proof. — Let us recall a general result. For a real analytic manifold Y, let $\mathcal{O}^{\mathbb{R}}_Y$ denote the sheaf of real analytic functions on Y. If Y is the product of a complex manifold Y_1 and a real analytic manifold Y_2, let $\mathcal{O}^{Y_1-\mathrm{hol}}_Y$ denote the sheaf of real analytic functions which are holomorphic in the Y_1-direction. The extension $\mathcal{O}^{Y_1-\mathrm{hol}}_Y \subset \mathcal{O}^{\mathbb{R}}_Y$ is faithfully flat.

LEMMA 4.2.5. — *Let $W_1 \subset W_2 \subset Y$ be real analytic subsets. Then, $\mathcal{C}^{\infty\,<W_i}_Y$ and $\mathcal{C}^{\infty\,<W_1}_Y/\mathcal{C}^{\infty\,<W_2}_Y$ are flat over $\mathcal{O}^{\mathbb{R}}_Y$.*

Proof. — The sheaf $\mathcal{C}^\infty_Y$ is faithfully flat over $\mathcal{O}^{\mathbb{R}}_Y$ (Corollary 1.12 of [**34**]). Theorem VI.1.2 of [**34**] implies $\mathfrak{a}\,\mathcal{C}^{\infty\,<W_1}_Y \cap \mathcal{C}^{\infty\,<W_2}_Y = \mathfrak{a}\,\mathcal{C}^{\infty\,<W_2}_Y$ for any real analytic subsets $W_1 \subset W_2 \subset Y$ and for any ideal sheaf $\mathfrak{a}$ of $\mathcal{O}^{\mathbb{R}}_Y$. By using the argument in the proof of Proposition III.4.7 in [**34**], we can prove the following:

- ▷ Let A be a ring. Let M be an A-flat module. Let N be an A-submodule of M. If $\mathfrak{a}M \cap N = \mathfrak{a}N$ for any ideal $\mathfrak{a}$ of A, then N and M/N are also A-flat.

We immediately obtain the claim of Lemma 4.2.5 from these results. □

Let Z_0 be a complex manifold with a normal crossing hypersurface D_0. Let Z_1 be a real analytic manifold. We put $Z := Z_0 \times Z_1$ and $D := D_0 \times Z_1$. Let G denote the composite of the maps $Z \to Z_0 \to Z_0 \times \mathbb{C}^n$, where the latter is induced by the inclusion $\{(0,\dots,0)\}\subset\mathbb{C}^n$. Let $(t_1,\dots,t_n)$ be the standard holomorphic coordinate system of $\mathbb{C}^n$.

Lemma 4.2.6. — *$\mathcal{C}_Z^{\infty\,<D}[[t_1,\dots,t_n]]$ is flat over $G^{-1}\mathcal{O}_{Z_0\times\mathbb{C}^n}$.*

Proof. — Let $\iota_1: Z\to Z_2:=Z\times\mathbb{R}^n$ the inclusion induced by
$$\{(0,\dots,0)\}\longrightarrow\mathbb{R}^n.$$
We put $D_2:=D\times\mathbb{R}^n$. We regard that $(t_1,\dots,t_n)$ is a real coordinate system of $\mathbb{R}^n\subset\mathbb{C}^n$. We have the natural identification
$$\mathcal{C}_Z^{\infty\,<D}[[t_1,\dots,t_n]]=\mathcal{C}_{Z_2}^{\infty\,<D_2}/\mathcal{C}_{Z_2}^{\infty\,<D_2\cup Z}.$$
According to Lemma 4.2.5, it is flat over $\iota_1^{-1}\mathcal{O}_{Z_2}^{\mathbb{R}}$. Let G_1 be the composite of $Z\to Z_0\to Z_0\times\mathbb{R}^n$. We have a natural isomorphism
$$G_1^{-1}\mathcal{O}_{Z_0\times\mathbb{R}^n}^{Z_0-\mathrm{hol}}\simeq G^{-1}\mathcal{O}_{Z_0\times\mathbb{C}^n}.$$
Since the extension $G_1^{-1}\mathcal{O}_{Z_0\times\mathbb{R}^n}^{Z_0-\mathrm{hol}}\subset\mathcal{O}_{Z_2}^{\mathbb{R}}$ is faithfully flat, we obtain the claim of Lemma 4.2.6. □

Let us return to the proof of Proposition 4.2.4. We may assume that $X=\Delta^n$, $D_i=\{z_i=0\}$, $D=\bigcup_{i=1}^{\ell}D_i$ and $D^{\circ}=\bigcup_{i=\ell+1}^{m}D_i$. For $I\subset\underline{\ell}$, let $\pi_I:\widetilde{X}(D(I))\to X$ be the real blow up. We have the natural identification
$$\pi_I^{-1}(D_I)=D_I\times(S^1)^{|I|}\quad\text{and}\quad\pi_I^{-1}\big(D_I(\overline{I}^c)\big)=D_I(\overline{I}^c)\times(S^1)^{|I|}.$$
From Lemma 4.2.6, we obtain that
$$\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}=\mathcal{C}^{\infty\,<D_I(\overline{I}^c)}_{\pi_I^{-1}(D_I)}[[z_i\mid i\in I]]$$
is flat over $\pi_I^{-1}\mathcal{O}_X$.

Lemma 4.2.7. — *$\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$. (Note that $\pi:\widetilde{X}(D)\to X$.)*

Proof. — The claim is clear outside of $\pi^{-1}(\partial D_I)$. Let P be any point of ∂D_I. Let $\mathfrak{a}$ be any finitely generated ideal of $\mathcal{O}_{X,P}$. We take a free resolution $\mathcal{Q}_\bullet$ of $\mathfrak{a}$, i.e., $\cdots\to\mathcal{Q}_1\to\mathcal{Q}_0\to\mathfrak{a}$. We obtain a $\pi^{-1}\mathcal{O}_X$-free resolution $\pi^{-1}\mathcal{Q}_\bullet$ of $\pi^{-1}\mathfrak{a}$. We set $\widetilde{\mathcal{Q}}_j=\mathcal{Q}_j$ for $j\geq 0$ and $\widetilde{\mathcal{Q}}_{-1}:=\mathfrak{a}$ for simplicity of the description.

It is enough to prove that $\pi^{-1}\widetilde{\mathcal{Q}}_\bullet\otimes\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ is exact.

Let $\rho:\widetilde{X}(D)\to\widetilde{X}(D(I))$ be the naturally induced map. Note
$$\rho_*\big(\pi^{-1}\widetilde{\mathcal{Q}}_\bullet\otimes\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}\big)=\pi_I^{-1}(\widetilde{\mathcal{Q}}_\bullet)\otimes\rho_*\big(\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}\big)=\pi_I^{-1}(\widetilde{\mathcal{Q}}_\bullet)\otimes\mathcal{C}^{\infty\,<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}.$$

The first equality is the projection formula. As for the second one, it is enough to observe that the natural morphism $\mathcal{C}^{\infty<D(\overline{I}^c)}_{\pi_I^{-1}(D_I)} \to \rho_*\mathcal{C}^{\infty<D(\overline{I}^c)}_{\pi^{-1}(D_I)}$ is an isomorphism. It is clearly injective. Let f be a section of $\rho_*\mathcal{C}^{<\infty D(\overline{I}^c)}_{\pi^{-1}(D_I)}$. The restriction $g := f_{|\pi_I^{-1}(D_I\setminus D(\overline{I}^c))}$ gives a C^∞-function on $\pi_I^{-1}(D_I \setminus \partial D_I)$. For any differential operator R on $\pi_I^{-1}(D_I)$, $R(g)(P)$ goes to 0 when P goes to a point in $\pi_I^{-1}(\partial D_I)$. Hence, g gives a section of $\mathcal{C}^{\infty<D(\overline{I}^c)}_{\pi_I^{-1}(D_I)}$ which is mapped to f.

Let $Q \in \pi^{-1}(P)$. Take any cycle φ of $\pi^{-1}\widetilde{\mathcal{Q}}_i \otimes \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ at Q. By using a cut function around Q, we can regard it as a global cycle of $\pi^{-1}\widetilde{\mathcal{Q}}_i \otimes \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$ whose support is a small neighbourhood of Q. Then, it can be regarded as a cycle of $\pi_I^{-1}(\widetilde{\mathcal{Q}}_i) \otimes \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$ around $\rho(Q)$. Because $\mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$ is flat over $\pi_I^{-1}\mathcal{O}_X$, we obtain that φ is a boundary in the complex $\pi_I^{-1}(\widetilde{\mathcal{Q}}_\bullet) \otimes \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi_I^{-1}(D_I)}}$. Then, it is easy to deduce that φ is a boundary in $\pi^{-1}(\widetilde{\mathcal{Q}}_\bullet) \otimes \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$. Thus, the proof of Lemma 4.2.7 is finished. □

Let us prove that $\mathcal{C}^{\infty<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$, where $I \sqcup J \subset \underline{\ell}$. We put

$$\mathcal{S}(I,J,m) := \big\{K \subset \underline{\ell} - J \mid I \subset K,\, |K| = m\big\}.$$

Put $\mathcal{G}_{I,\ell+1} := \mathcal{C}^{\infty<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}}$, and descending inductively we set

$$\mathcal{G}_{I,m} := \operatorname{Ker}\Big(\mathcal{G}_{I,m+1} \to \bigoplus_{K\in\mathcal{S}(I,J,m)} \mathcal{C}^{\infty<D(\overline{K}^c)}_{\widehat{\pi^{-1}(D_K)}}\Big).$$

We have $\mathcal{G}_{I,|I|+1} = \mathcal{C}^{\infty<D(\overline{I}^c)}_{\widehat{\pi^{-1}(D_I)}}$, which is flat over $\pi^{-1}\mathcal{O}_X$. By an induction, we obtain that $\mathcal{G}_{I,m}$ are flat over $\pi^{-1}\mathcal{O}_X$.

Hence, we obtain that $\mathcal{C}^{\infty<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}}$ is flat over $\pi^{-1}\mathcal{O}_X$. By using the resolution of $\mathcal{C}^{\infty<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ in Lemma 4.2.1, we obtain that $\mathcal{C}^{\infty<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ is flat over $\pi^{-1}\mathcal{O}_X$. As a result, we obtain that $\Omega^{0,\bullet<D(\bar{J})}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet<D^\circ}_{\widehat{\pi^{-1}(D_I(J))}}$ are flat over $\pi^{-1}\mathcal{O}_X$, where $J \subset I^c$. In particular, $\Omega^{0,\bullet<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet}_{\widehat{\pi^{-1}(D_I(J))}}$ are flat over $\pi^{-1}\mathcal{O}_X$. Then, we obtain the $\pi^{-1}\mathcal{O}_X$-flatness of $\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{O}_{\widehat{\pi^{-1}(D_I(J))}}$ by using Proposition 4.2.2. Thus, the proof of Proposition 4.2.4 is finished. □

4.3. Nilsson type functions

4.3.1. C^∞-functions of Nilsson type. — Let X, D and D° be as in §4.2.1. We put

$$D^{(3)} := D^{(1)} \cup D^\circ.$$

We shall introduce a sheaf $\mathcal{C}^{\infty < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}$ on $\widetilde{X}(D)$. First, let us consider the case

$$X = \Delta^n, \quad D = \bigcup_{i=1}^{\ell} \{z_i = 0\} \quad \text{and} \quad D^\circ = \bigcup_{i=\ell+1}^{m} \{z_i = 0\}.$$

Let $\underline{\ell} = I_1 \sqcup I_2$ be determined by $D^{(j)} = \bigcup_{i \in I_j} \{z_i = 0\}$ for $j = 1, 2$. Let $\widetilde{\jmath}$ denote the inclusion $X - D \to \widetilde{X}(D)$. Let $\mathcal{C}^{\infty\, < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}$ be the image of the naturally defined morphisms:

$$\mathcal{C}^{\infty\, < D^{(3)}}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_i \mid i \in I_2) \longrightarrow \widetilde{\jmath}_* \mathcal{C}^{\infty\, < D^\circ}_{X-D}.$$

We can observe that they are independent of the choice of a coordinate system $(z_1, \ldots, z_n)$. Hence, we obtain a globally defined sheaf $\mathcal{C}^{\infty < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}$ on $\widetilde{X}(D)$. It is also denoted by $\mathcal{C}^{\infty\ \mathrm{nil} < D^{(3)}}_{\widetilde{X}(D)}$. Put

$$\Omega^{0,\bullet\, < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)} := \Omega^{0,\bullet}_{\widetilde{X}(D)} \otimes_{\mathcal{C}^\infty_{\widetilde{X}(D)}} \mathcal{C}^{\infty < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}.$$

We will prove the following theorem in §4.3.6. (More refined claims will be proved.)

Theorem 4.3.1

- ▷ $\Omega^{0,\bullet\, < D^{(1)} \leq D^{(2)}}_{\widetilde{X}(D)}$ *is naturally a c-soft resolution of* $\mathcal{A}^{< D^{(1)} \leq D^{(2)}}_{\widetilde{X}(D)}$ *if* $D^\circ = \varnothing$.
- ▷ *The sheaves* $\mathcal{A}^{< D^{(1)} \leq D^{(2)}}_{\widetilde{X}(D)}$ *and* $\Omega^{0,\bullet\, < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}$ *are flat over* $\pi^{-1}\mathcal{O}_X$.

Let $D^{(i)} = \bigcup_{j \in \Lambda_i} D^{(i)}_j$ $(i = 1, 2)$ be the irreducible decomposition. Fix $k \in \Lambda_1 \sqcup \Lambda_2$. We put

$$E^{(i)} := \bigcup_{j \in \Lambda_i \setminus \{k\}} D^{(i)}_j \qquad (i = 1, 2).$$

We put $E := E^{(1)} \cup E^{(2)}$ and $E^{(3)} := D^{(3)}$. We have the naturally defined projection $\rho : \widetilde{X}(D) \to \widetilde{X}(E)$. We will prove the following theorem in §4.3.7.

Theorem 4.3.2. — *If $k \in \Lambda_1$, the following naturally defined morphism is an isomorphism:*

$$\Omega^{0,\bullet\, < E^{(3)} \leq E^{(2)}}_{\widetilde{X}(E)} \longrightarrow \rho_* \Omega^{0,\bullet\, < D^{(3)} \leq D^{(2)}}_{\widetilde{X}(D)}.$$

If $k \in \Lambda_2$, the following naturally defined morphism is a quasi-isomorphism:

$$\Omega^{0,\bullet\,<E^{(3)}\leq E^{(2)}}_{\widetilde{X}(E)}(*D_k^{(2)}) \longrightarrow \rho_*\Omega^{0,\bullet\,<D^{(3)}\leq D^{(2)}}_{\widetilde{X}(D)}.$$

Corollary 4.3.3. — *The natural morphism*

$$\Omega_X^{0,\bullet\,<D^{(1)}}(*D^{(2)}) \longrightarrow \pi_*\Omega^{0,\bullet\,<D^{(1)}\leq D^{(2)}}_{\widetilde{X}(D)}$$

is a quasi-isomorphism. In particular, $R\pi_*\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \simeq \mathcal{O}_X(*D)$.

For the proof of the theorems, we may assume

$$X = \Delta^n, \quad D = \bigcup_{i=1}^{\ell}\{z_i = 0\} \quad \text{and} \quad D^\circ = \bigcup_{i=\ell+1}^{m}\{z_i = 0\},$$

where $1 \leq \ell \leq m \leq n$. We set $D_i := \{z_i = 0\}$ for $i = 1, \dots, m$. We use the notation in §4.1.1. For a subset $J \subset \underline{\ell}$, we set $\overline{J} := J \sqcup (\underline{m} \setminus \underline{\ell})$.

4.3.2. Refinements. — For any locally closed real analytic subset $Z \subset \widetilde{X}(D)$, we implicitly regard $\mathcal{O}_{\widehat{Z}}$ as a sheaf on $\widetilde{X}(D)$ in a natural way.

For any $I \sqcup J \subset \underline{\ell}$, let $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ denote the image of the naturally defined morphism

$$\mathcal{O}^{<D(J)}_{\widehat{\pi^{-1}(D_I)}} \otimes_{\mathbb{C}[z_1,\dots,z_\ell]} \mathrm{Nil}(z_1,\dots,z_\ell) \longrightarrow \mathcal{O}_{\pi^{-1}\widehat{(D_I\setminus\partial D_I)}} \otimes_{\mathbb{C}[z_i\mid i\in I]} \mathrm{Nil}(z_i \mid i \in I).$$

In the case $I = \varnothing$, it is $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widetilde{X}(D)}$.

For $I \sqcup J \subset \underline{\ell}$, let $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))}}$ denote the image of the naturally defined morphism

$$\begin{aligned}&\mathcal{O}_{\pi^{-1}\widehat{(D_I(J))}} \otimes_{\mathbb{C}[z_1,\dots,z_\ell]} \mathrm{Nil}(z_1,\dots,z_\ell)\\ &\qquad\longrightarrow \bigoplus_{j\in J}\mathcal{O}_{\pi^{-1}\widehat{(D_{Ij}\setminus\partial D_{Ij})}} \otimes_{\mathbb{C}[z_i\mid i\in Ij]} \mathrm{Nil}(z_i \mid i \in Ij).\end{aligned}$$

Here, $Ij := I \sqcup \{j\}$. In particular, $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D(J))}}$ is the image of the morphism

$$\mathcal{O}_{\pi^{-1}\widehat{(D(J))}} \otimes_{\mathbb{C}[z_1,\dots,z_\ell]} \mathrm{Nil}(z_1,\dots,z_\ell) \longrightarrow \bigoplus_{j\in J}\mathcal{O}_{\pi^{-1}\widehat{(D_j\setminus\partial D_j)}} \otimes_{\mathbb{C}[z_j]} \mathrm{Nil}(z_j).$$

Let $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\mathcal{A}^{\mathrm{nil}}_{\pi^{-1}\widehat{(D_I(J))},T,N}$ be the sheaves obtained from

$$\mathrm{Nil}_{T,N}(z_1,\dots,z_\ell) \ \text{instead of} \ \mathrm{Nil}(z_1,\dots,z_\ell).$$

For $T \subset T'$ and $N \leq N'$, we have natural inclusions

$$\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N} \subset \mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T',N'} \quad \text{and} \quad \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N} \subset \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T',N'}.$$

We have the natural isomorphisms

$$\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}} \simeq \varinjlim \mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N} \quad \text{and} \quad \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}} \simeq \varinjlim \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}. \tag{23}$$

Let $q_I : \pi^{-1}(D_I) \to \widetilde{D}_I(\partial D_I)$ denote the projection. Let

$$\pi_I : \widetilde{D}_I(\partial D_I) \longrightarrow D_I$$

be the real blow up. Then, we have

$$\begin{aligned} &\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N} \\ &\qquad = q_I^{-1}\mathcal{A}^{\mathrm{nil}\,<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}[[z_i \mid i \in I]] \otimes_{\mathbb{C}[z_i|i\in I]} \mathrm{Nil}_{T,N}(z_i \mid i \in I), \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N} \\ &\qquad = q_I^{-1}\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi_I^{-1}(D_I(J))},T,N}[[z_i \mid i \in I]] \otimes_{\mathbb{C}[z_i|i\in I]} \mathrm{Nil}_{T,N}(z_i \mid i \in I). \end{aligned} \tag{25}$$

4.3.3. Specialization. — Let us construct for any $I \sqcup J \subset \underline{\ell}$ a morphism

$$\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)}} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}.$$

First, let us construct $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}}$ in the case $D = D_1$. Let Φ denote the natural morphism

$$\Phi : \mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1) \longrightarrow \widetilde{\jmath}_*\mathcal{O}_{X-D},$$

where $\widetilde{\jmath} : X - D \to \widetilde{X}(D)$.

Lemma 4.3.4. — *Assume that $D = D_1$. Let $\mathcal{S} \subset \mathbb{C}$ be a finite subset such that the induced map $\mathcal{S} \to \mathbb{C}/\mathbb{Z}$ is injective. Assume that we are given*

$$f = \sum_{\alpha\in\mathcal{S}} \sum_{j=0}^{M} f_{\alpha,j} \otimes \varphi_{\alpha,j}(z_1) \in \mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1)$$

such that $\Phi(f) \in \mathcal{O}^{<D}_{\widetilde{X}(D)}$. Then, we have $f_{\alpha,j} \in \mathcal{O}^{<D}_{\widetilde{X}(D)}$. In particular, we have the well defined map $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}}$ in the case $D = \{z_1 = 0\}$.

Proof. — Let us consider the growth order of $f_{\alpha,j}\, z_1^{\alpha}(\log z_1)^j$. For the polar coordinate system $z_1 = r\,\mathrm{e}^{\sqrt{-1}\,\theta}$, we have

$$z_1^{\alpha} = \exp\big(\beta \log r - \gamma\theta + \sqrt{-1}\,(\gamma \log r + \beta\theta)\big),$$

where $\beta = \mathrm{Re}\,\alpha$ and $\gamma = \mathrm{Im}\,\alpha$.

Let V be the set of $(\alpha, j) \in \mathcal{S} \times \mathbb{Z}_{\geq 0}$ such that $f_{\alpha,j}$ is not contained in $\mathcal{O}^{<D}_{\widetilde{X}(D)}$. We will derive a contradiction by assuming $V \neq \varnothing$.

For each $(\alpha, j) \in V$, there exists a unique integer $m(\alpha, j)$ such that

(i) $h_{\alpha,j} := z_1^{-m(\alpha,j)} f_{\alpha,j} \in \mathcal{O}_{\widetilde{X}(D)}$,

(ii) $h_{\alpha,j|\pi^{-1}(D)}$ is not constantly 0.

We set

$$\kappa := \max_{(\alpha,j)\in V} \big\{ \operatorname{Re}\alpha + m(\alpha, j) \big\}, \quad S := \big\{ (\alpha, j) \in V \mid \operatorname{Re}\alpha + m(\alpha, j) = \kappa \big\}.$$

For $(\alpha_1, j_1), (\alpha_2, j_2) \in S$, we have $\operatorname{Re}\alpha_1 = \operatorname{Re}\alpha_2$ and $m(\alpha_1, j_1) = m(\alpha_2, j_2)$. We also have $\operatorname{Im}\alpha_1 \neq \operatorname{Im}\alpha_2$ if $\alpha_1 \neq \alpha_2$. We obtain the following estimate for some $\epsilon > 0$:

$$\begin{aligned} (26) \qquad & \sum_{(\alpha,j)\in V} h_{\alpha,j|\pi^{-1}(D)}\, z_1^{\alpha+m(\alpha,j)} (\log z_1)^j \\ &= r^{\kappa} \Big(\sum_{(\alpha,j)\in V} h_{\alpha,j|\pi^{-1}(D)}\, \mathrm{e}^{-\operatorname{Im}\alpha\theta + \sqrt{-1}\,(\operatorname{Im}\alpha \log r + \operatorname{Re}\alpha\theta)} (\log z_1)^j \Big) = O(r^{\kappa+\epsilon}). \end{aligned}$$

Let us deduce that $h_{\alpha,j|\pi^{-1}(D)}$ are constantly 0 from (26). Assume the contrary. Let $Q \in \pi^{-1}(D)$ at which $h_{\alpha,j}(Q) \neq 0$ for one of $(\alpha, j) \in V$. We may assume $\theta(Q) = 0$. We obtain the following from (26):

$$(27) \qquad \sum_{(\alpha,j)\in V} h_{\alpha,j}(Q)\, \mathrm{e}^{\sqrt{-1}\,\operatorname{Im}\alpha \log r} (\log r)^j = O(r^{\epsilon}).$$

But, for any $\delta > 0$, we can take $0 < r < \delta$ such that the amplitudes of the complex numbers

$$(-1)^j h_{\alpha,j}(Q)\, \mathrm{e}^{\sqrt{-1}\,\operatorname{Im}\alpha \log r}, \quad (\alpha, j) \in V,$$

are sufficiently close, which contradicts with (27). Hence, $h_{\alpha,j}$ $(\alpha, j) \in V$ are constantly 0. Thus, we obtain Lemma 4.3.4. □

Let us return to the general case. We take $\mathcal{S} \subset \mathbb{C}$ such that the induced map $\mathcal{S} \to \mathbb{C}/\mathbb{Z}$ is bijective. Let $q_i : (\mathcal{S} \times \mathbb{Z})^{\ell} \to \mathcal{S} \times \mathbb{Z}$ be the projection onto the i-th component, and $\pi_i : (\mathcal{S} \times \mathbb{Z})^{\ell} \to (\mathcal{S} \times \mathbb{Z})^{\ell-1}$ be the projection forgetting the i-th component. For a given

$$\sum_{(\boldsymbol{\alpha},\boldsymbol{k})\in\mathcal{S}^{\ell}\times\mathbb{Z}^{\ell}_{\geq 0}} A_{\boldsymbol{\alpha},\boldsymbol{k}} \otimes \varphi_{\boldsymbol{\alpha},\boldsymbol{k}} \in \mathcal{O}_{\widetilde{X}(D)} \otimes \operatorname{Nil}(z_1, \ldots, z_{\ell}),$$

we set

$${}^{i}F_{\beta,j} := \sum_{q_i(\boldsymbol{\alpha},\boldsymbol{k})=(\beta,j)} A_{\boldsymbol{\alpha},\boldsymbol{k}} \cdot \varphi_{\pi_i(\boldsymbol{\alpha},\boldsymbol{k})}(z_j \mid j \neq i).$$

Put $i^c := \underline{\ell} - \{i\}$. If $\sum A_{\boldsymbol{\alpha},\boldsymbol{k}} \cdot \varphi_{\boldsymbol{\alpha},\boldsymbol{k}}$ belongs to $\mathcal{O}^{<D_i}_{\widetilde{X}(D)\setminus\pi^{-1}(D(i^c))}$, we obtain ${}^iF_{\beta,j|\pi^{-1}\widehat{(D_i\setminus\partial D_i)}} = 0$ by applying Lemma 4.3.4 to $\sum {}^iF_{\beta,j} \cdot \varphi_{\beta,j}(z_i)$. It implies that the morphism

$$\mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1,\dots,z_\ell) \longrightarrow \mathcal{O}_{\widehat{\pi^{-1}(D_i)}} \otimes \mathrm{Nil}(z_1,\dots,z_\ell) \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}$$

factors through $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$. Hence, we have a well defined morphism

$$\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}.$$

By construction, it is an epimorphism. We also obtain that the following morphism factors through $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$:

$$\mathcal{O}_{\widetilde{X}(D)} \otimes \mathrm{Nil}(z_1,\dots,z_\ell) \\ \longrightarrow \mathcal{O}_{\widehat{\pi^{-1}(D(I))}} \otimes \mathrm{Nil}(z_1,\dots,z_\ell) \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(I))}} \subset \bigoplus_{i\in I} \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_i)}}.$$

Hence, we obtain the well defined morphism $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(I))}}$. We also obtain $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D),T,N} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(I))},T,N}$. They are surjective by construction. By using (23), (24) and (25), we also obtain epimorphisms

$$\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)}} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}} \quad \text{and} \quad \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)},T,N} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}.$$

LEMMA 4.3.5. — *We have :*

$$\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}} = \mathrm{Ker}\,\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)}} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\big),$$
$$\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N} = \mathrm{Ker}\,\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I)},T,N} \to \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}\big).$$

Proof. — The implication $\subset$ is clear. Let us prove the converse. First, we consider the case $I = \varnothing$. Let $f = \sum A_{\boldsymbol{\alpha},\boldsymbol{k}}\, \varphi_{\boldsymbol{\alpha},\boldsymbol{k}}$ be any section of

$$\mathrm{Ker}\,\big(\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))}}\big).$$

Let us prove the following equality on $\pi^{-1}(\widehat{D_K - \partial D_K})$ for any subset $K \subset \underline{\ell}$ such that $K \cap J \neq \varnothing$:

$$\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} \prod_{i\notin K} \varphi_{\alpha_i,k_i}(z_i) = 0. \tag{28}$$

We use an induction on $|K|$. In the case $|K| = 1$, it follows from the assumption. Let $K = K' \sqcup \{j\}$. Assume that we have already known (28) for K'. Using Lemma 4.3.4, we obtain the claim for K. As a special case of (28), we have $A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_{\underline{\ell}})}} = 0$.

Note that the expression of f is not unique. We would like to replace $A_{\boldsymbol{\alpha},\boldsymbol{k}}$ such that the following holds:

$P(m)$ **:** $A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} = 0$ if $|K| \geq m$ and $K \cap J \neq \varnothing$.

We use a descending induction on m. In the case $m = \ell$, it holds as was already proved. Assume that $P(m+1)$ holds. Take $K \subset \underline{\ell}$ such that $|K| = m$ and $K \cap J \neq \varnothing$. We have

$$A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} \prod_{i \notin K} \varphi_{\alpha_i,k_i}(z_i) \in \mathcal{O}^{<D(K^c)}_{\widehat{\pi^{-1}(D_K)}}.$$

By a generalized Borel-Ritt theorem due to Majima and Sabbah, we can take $G_{\boldsymbol{\alpha},\boldsymbol{k}} \in \mathcal{O}^{<D(K^c)}_{\widetilde{X}(D)}$ satisfying

$$G_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} = A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} \prod_{i \notin K} \varphi_{\alpha_i,k_i}(z_i).$$

By (28), the following holds:

$$\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} G_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_K)}} = 0.$$

We have the equality

$$f = \sum_{\boldsymbol{\alpha},\boldsymbol{k}} \Big(A_{\boldsymbol{\alpha},\boldsymbol{k}} - \frac{G_{\boldsymbol{\alpha},\boldsymbol{k}}}{\prod_{i \notin K} \varphi_{\alpha_i,k_i}(z_i)} \Big) \varphi_{\boldsymbol{\alpha},\boldsymbol{k}}(z_1, \ldots, z_\ell) \\ + \sum_{\boldsymbol{\beta},\boldsymbol{j}} \Big(\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} G_{\boldsymbol{\alpha},\boldsymbol{k}} \Big) \varphi_{\boldsymbol{\beta},\boldsymbol{j}}(z_i \mid i \in K).$$

Note that $\sum_{q_K(\boldsymbol{\alpha},\boldsymbol{k})=(\boldsymbol{\beta},\boldsymbol{j})} G_{\boldsymbol{\alpha},\boldsymbol{k}}$ is 0 on $\widehat{\pi^{-1}(D_K)} \cup \pi^{-1}(\widehat{D(K^c)})$. In particular, it is 0 on $\bigcup_{|K_1|=m} \widehat{\pi^{-1}(D_{K_1})}$. By construction, $A_{\boldsymbol{\alpha},\boldsymbol{k}} - G_{\boldsymbol{\alpha},\boldsymbol{k}} \prod_{i \notin K} \varphi_{\alpha_i,k_i}(z_i)^{-1}$ vanishes on $\widehat{\pi^{-1}(D_K)}$. Moreover, if $A_{\boldsymbol{\alpha},\boldsymbol{k}|\widehat{\pi^{-1}(D_L)}} = 0$ for some $|L| = m$ with $L \cap J \neq \varnothing$, $A_{\boldsymbol{\alpha},\boldsymbol{k}} - G_{\boldsymbol{\alpha},\boldsymbol{k}} \prod_{i \notin K} \varphi_{\alpha_i,k_i}(z_i)^{-1}$ also vanishes on $\widehat{\pi^{-1}(D_L)}$. Hence, by applying the above procedure to each K satisfying $|K| = m$ and $K \cap J \neq \varnothing$, we can arrive at $P(m)$. The status $P(0)$ means

$$f = \sum A_{\boldsymbol{\alpha},\boldsymbol{k}} \, \varphi_{\boldsymbol{\alpha},\boldsymbol{k}}$$

with $A_{\boldsymbol{\alpha},\boldsymbol{k}} \in \mathcal{O}^{<D(J)}_{\widetilde{X}(D)}$, which implies that $f \in \mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)}$.

Thus, we are done in the case $I = \varnothing$.

We can reduce the general case to the case $I = \varnothing$ by using (23), (24) and (25). Thus, the proof of Lemma 4.3.5 is finished. □

4.3.4. A resolution. — For $I \subset J$ and $m \geq 0$, put

$$\mathcal{T}(m,I,J) := \big\{K \subset J \mid I \subset K,\ |K| = |I| + m + 1\big\}.$$

We set

$$\mathcal{K}^m\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}\big) := \bigoplus_{K\in\mathcal{T}(m,I,J)} \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)}}.$$

We obtain a complex $\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}})$ as in §2.1.4.

Lemma 4.3.6. — *The* 0*-th cohomology of* $\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}})$ *is* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$*, and the higher cohomology sheaves are* 0*. A similar claim holds for* $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$*.*

Proof. — It is enough to consider the issue for $\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N})$. First, let us consider the case $I = \varnothing$. We use an induction on $|J|$ and the dimension of X. The cases $|J| = 1$ or $\dim X = 1$ are clear. Let $J = J_0 \sqcup \{j\}$. Assume that the claim holds for J_0. We set

$$\mathcal{L}^m_{T,N} := \bigoplus_{\substack{|K|=m+1\\ j\in K\subset J}} \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)},T,N}.$$

We have the exact sequence

$$0 \longrightarrow \mathcal{L}^\bullet_{T,N} \longrightarrow \mathcal{K}^\bullet\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))},T,N}\big) \longrightarrow \mathcal{K}^\bullet\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J_0))},T,N}\big) \longrightarrow 0.$$

Let $q_j : \pi^{-1}(D_j) \to \widetilde{D}_j(\partial D_j)$ and $\pi_j : \widetilde{D}_j(\partial D_j) \to D_j$ be the projections. We have a natural isomorphism

$$\mathcal{L}^\bullet_{T,N} \simeq \mathrm{Cone}\big(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_j)},T,N} \to q_j^{-1}\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi_j^{-1}(D_j\cap D(J_0))},T,N})[\![z_j]\!] \otimes_{\mathbb{C}[z_j]} \mathrm{Nil}_{T,N}(z_j)\big)[-1].$$

By the inductive assumption, we obtain the vanishing of the higher cohomology sheaves of $\mathcal{L}^\bullet_{T,N}$ and $\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J_0))},T,N})$. Hence, we obtain the vanishing of the higher cohomology of $\mathcal{K}^\bullet(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))},T,N})$. The calculation of the 0-th cohomology is easy.

The general case can be easily reduced to the case $I = \varnothing$ by (23), (24) and (25). ∎

4.3.5. The C^{∞}-version. — Let Y be a C^{∞}-manifold.

For $I\sqcup J\subset\underline{\ell}$, let $\mathcal{C}^{\infty\,\mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}$ denote the image of the morphism

$$\mathcal{C}^{\infty\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}\otimes_{\mathbb{C}[z_i|i\in J^c]}\mathrm{Nil}(z_i\mid i\in J^c)\longrightarrow \mathcal{C}^{\infty\,<D(\overline{J})}_{\pi^{-1}\widehat{(D_I\setminus\partial D_I)}\times Y}\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}(z_i\mid i\in I).$$

Let $\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y}$ be the image of the morphism

$$\mathcal{C}^{\infty\,<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y}\otimes_{\mathbb{C}[z_1,\dots,z_\ell]}\mathrm{Nil}(z_1,\dots,z_\ell)\longrightarrow\bigoplus_{j\in J}\mathcal{C}^{\infty\,<D^{\circ}}_{\pi^{-1}\widehat{(D_{Ij}-\partial D_{Ij})}\times Y}\otimes_{\mathbb{C}[z_i|i\in Ij]}\mathrm{Nil}(z_i\mid i\in Ij).$$

In particular, $\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D(J))}\times Y}$ is the image of the morphism

$$\mathcal{C}^{\infty\,<D^{\circ}}_{\pi^{-1}\widehat{(D(J))}\times Y}\otimes_{\mathbb{C}[z_1,\dots,z_\ell]}\mathrm{Nil}(z_1,\dots,z_\ell)\longrightarrow\bigoplus_{j\in J}\mathcal{C}^{\infty\,<D^{\circ}}_{\pi^{-1}\widehat{(D_j-\partial D_j)}\times Y}\otimes_{\mathbb{C}[z_j]}\mathrm{Nil}(z_j).$$

Similarly, $\mathcal{C}^{\infty\,\mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}$ and $\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y,T,N}$ denote the sheaves obtained from $\mathrm{Nil}_{T,N}(z_1,\dots,z_\ell)$ instead of $\mathrm{Nil}(z_1,\dots,z_\ell)$. We have

$$(29)\quad \mathcal{C}^{\infty\,\mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}=\mathcal{C}^{\infty\,\mathrm{nil}<D_I(\overline{J})}_{\widetilde{D}_I(\partial D_I)\times(S^1)^{|I|}\times Y,T,N}[\![z_i\mid i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i\mid i\in I),$$

$$(30)\quad \mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y,T,N}=\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}\cap D_I}_{\pi_I^{-1}\widehat{(D_I(J))}\times(S^1)^{|I|}\times Y,T,N}[\![z_i\mid i\in I]\!]\otimes_{\mathbb{C}[z_i|i\in I]}\mathrm{Nil}_{T,N}(z_i\mid i\in I).$$

By the argument in §4.3.3, we obtain the well defined surjective morphisms:

$$(31)\quad \mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_I)}\times Y}\longrightarrow\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y},\quad \mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}\longrightarrow\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y,T,N}.$$

By the argument in the proof of Lemma 4.3.5, we can prove that the kernels of the morphisms in (31) are $\mathcal{C}^{\infty\,\mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y}$ and $\mathcal{C}^{\infty\,\mathrm{nil}<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}\times Y,T,N}$, respectively.

We set

$$\mathcal{K}^m\big(\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y}\big):=\bigoplus_{K\in\mathcal{T}(m,I,J)}\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_K)}\times Y}.$$

We obtain a complex $\mathcal{K}^{\bullet}(\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y})$. It is easy to see that the 0-th cohomology is $\mathcal{C}^{\infty\,\mathrm{nil}<D^{\circ}}_{\pi^{-1}\widehat{(D_I(J))}\times Y}$. By using an argument in the proof of Lemma 4.3.6, we can prove the vanishing of the higher cohomology.

Similar claims hold for $\mathcal{K}^{\bullet}(\mathcal{C}^{\infty\ \mathrm{nil}<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))\times Y},T,N})$.

4.3.6. Proof of Theorem 4.3.1. — We do not consider D° in this subsection. We put

$$\Omega^{0,\bullet\ \mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}} := \Omega^{0,\bullet}_{\widetilde{X}(D)} \otimes_{\mathcal{C}^{\infty}_{\widetilde{X}(D)}} \mathcal{C}^{\infty\,\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}},$$
$$\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}} := \Omega^{0,\bullet}_{\widetilde{X}(D)} \otimes_{\mathcal{C}^{\infty}_{\widetilde{X}(D)}} \mathcal{C}^{\infty\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}.$$

We use the symbols $\Omega^{0,\bullet\ \mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ with a similar meaning. The following proposition implies the first claim of Theorem 4.3.1.

Proposition 4.3.7. — *The complexes $\Omega^{0,\bullet\ \mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ are c-soft resolutions of the sheaves $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ respectively. Similar claims hold for $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$.*

Proof. — We use an induction on $\dim X$. In the case $\dim X = 0$, the claim is trivial. Let us prove the claim for $\widehat{\pi^{-1}(D_I)}$. For $I \neq \varnothing$, let $q_I : \pi^{-1}(D_I) \to \widetilde{D}_I(\partial D_I)$ denote the naturally induced morphism. We put $\mathrm{Nil}_{T,N}(I) := \mathrm{Nil}_{T,N}(z_i \mid i \in I)$. By using the inductive assumption and a formal calculation as in Lemma 4.2.3, we can prove that the following morphisms are quasi-isomorphisms:

$$(32)\quad q_I^{-1}\mathcal{A}^{\mathrm{nil}<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i \in I]\!] \otimes \mathrm{Nil}_{T,N}(I) \\ \longrightarrow q_I^{-1}\Omega^{0,\bullet\ \mathrm{nil}<D_I(J)}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i \in I]\!] \otimes \mathrm{Nil}_{T,N}(I) \longrightarrow \Omega^{0,\bullet\ \mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}.$$

It implies the claim for $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$. We obtain the claim for $\mathcal{A}^{\mathrm{nil}<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ from (23). For any subset $I \subset \underline{\ell}$ (I can be $\varnothing$), by using the resolutions $\mathcal{K}^{\bullet}(\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}})$ and $\mathcal{K}^{\bullet}(\Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}})$, we can reduce the claim for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ to the claims for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_K)}}$ ($I \subsetneq K$). The claim for $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ can be obtained in a similar way. By using the exact sequences

$$0 \longrightarrow \Omega^{0,\bullet<D}_{\widetilde{X}(D)} \longrightarrow \Omega^{0,\bullet\ \mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D)}} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}^{<D}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D)}} \longrightarrow 0,$$

we obtain the claim for $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)}$. By using the exact sequences

$$0 \longrightarrow \Omega^{0,\bullet\,<D(J)}_{\widetilde{X}(D)} \longrightarrow \Omega^{0,\bullet\ \mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \Omega^{0,\bullet\ \mathrm{nil}}_{\widehat{\pi^{-1}(D(J))}} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D)} \longrightarrow \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D(J))}} \longrightarrow 0,$$

we obtain the claim for $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D)}$. The claims for $\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D),T,N}$ and $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widetilde{X}(D),T,N}$ can be obtained similarly. □

The following proposition implies the second claim of Theorem 4.3.1.

PROPOSITION 4.3.8. — *$\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)}}$, $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))}}$, $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)}}$ and $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))}}$ are flat over $\pi^{-1}\mathcal{O}_X$. Similar claims hold for $\mathcal{C}^{\infty\ \mathrm{nil}\,<D(\overline{J})}_{\widehat{\pi^{-1}(D_I)},T,N}$, $\mathcal{C}^{\infty\ \mathrm{nil}\,<D^{\circ}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$, $\mathcal{A}^{\mathrm{nil}\,<D(J)}_{\widehat{\pi^{-1}(D_I)},T,N}$ and $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(D_I(J))},T,N}$ are also flat over $\pi^{-1}\mathcal{O}_X$.*

Proof. — We have $\mathcal{C}^{\infty\,\mathrm{nil}\,<D(I^c)}_{\widehat{\pi^{-1}(D_I)}} = \mathcal{C}^{\infty<D(I^c)}_{\widehat{\pi^{-1}(D_I)}} \otimes_{\mathbb{C}[z_i|i\in I]} \mathrm{Nil}(z_i|\,i \in I)$, which is flat over $\pi^{-1}\mathcal{O}_X$, according to Lemma 4.2.7. Then, we can prove Proposition 4.3.8 by the arguments in the last part of the proof of Proposition 4.2.4. □

4.3.7. Proof of Theorem 4.3.2. — The first claim of Theorem 4.3.2 is obvious. We give a preliminary for the second claim. Put

$$X' := \mathbb{C} \times X, \quad X'_0 := \{0\} \times X, \quad D' := (\mathbb{C} \times D) \cup (\{0\} \times X).$$

Let $J \subset \underline{\ell}$. Put $D'(\overline{J}) := \mathbb{C} \times D(\overline{J})$. Let $\pi_0 : \widetilde{X}'(D') \to X'$ and $\pi_1 : \mathbb{C} \times \widetilde{X}(D) \to \mathbb{C} \times X$ be the real blow up. We have a natural diffeomorphism $\pi_0^{-1}(X'_0) \simeq S^1 \times \widetilde{X}(D)$. Let $\rho_0 : \widetilde{X}'(D') \to \mathbb{C} \times \widetilde{X}(D)$ be the naturally induced map. We use the coordinate system $z = r\,\mathrm{e}^{\sqrt{-1}\,\theta}$ of $\mathbb{C}$. We have a natural inclusion:

$$\mathcal{C}^{\infty\ \mathrm{nil}\,<D'(\overline{J})}_{\widehat{\pi_1^{-1}(X'_0)}}(*X'_0) \longrightarrow \rho_{0*}\big(\mathcal{C}^{\infty\ \mathrm{nil}\,<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}}\big). \tag{33}$$

The operator $\overline{z}\,\overline{\partial}_z$ induces endomorphisms of $\mathcal{C}^{\infty\,\mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_1^{-1}(X'_0)}}(*X'_0)$ and $\rho_{0*}(\mathcal{C}^{\infty\,\mathrm{nil}<D'(\overline{J})}_{\widehat{\pi_0^{-1}(X'_0)}})$, which are denoted by F_1 and F_2, respectively.

LEMMA 4.3.9. — *The cokernel of F_i $(i = 1, 2)$ are 0, and (33) induces an isomorphism* $\mathrm{Ker}\,F_1 \simeq \mathrm{Ker}\,F_2$.

Proof. — It is easy to obtain the vanishing of Cok F_1 by a formal calculation. Let us prove the other claims. We take $\mathcal{S} \subset \mathbb{C}$ such that

(i) the induced map $\mathcal{S} \to \mathbb{C}/\mathbb{Z}$ is bijective,

(ii) $0 \in \mathcal{S}$.

According to the decomposition $\mathrm{Nil}(z) = \bigoplus_{\alpha \in \mathcal{S}} z^{\alpha}\mathbb{C}[z, z^{-1}]\,[\log z]$, we have the decomposition

$$\mathcal{C}^{\infty\ \mathrm{nil} < D'(\bar{J})}_{\widehat{\pi_0^{-1}(X_0')}} = \bigoplus_{\alpha \in \mathcal{S}} \mathcal{C}^{\infty\ \mathrm{nil} < D'(\bar{J})}_{\widehat{\pi_0^{-1}(X_0')},\alpha}.$$

Let $U \subset \widetilde{X}(D)$ be an open subset. Let f be a section of $\mathcal{C}^{\infty\ \mathrm{nil} < D'(\bar{J})}_{\widehat{\pi_0^{-1}(X_0')},\alpha}$ on $S^1 \times U \subset \pi_0^{-1}(X_0')$ expressed as follows:

$$f = \sum_{\boldsymbol{\beta},\boldsymbol{k}} \sum_{n,j} f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\, \varphi_{\boldsymbol{\beta},\boldsymbol{k}}\, \mathrm{e}^{-\sqrt{-1}\,\theta\alpha} z^{\alpha+n} \big(\log |z|^2\big)^j \qquad (f_{\boldsymbol{\beta},\boldsymbol{k},n,j} \in \mathcal{C}^{\infty\,<D(\bar{J})}_{S^1 \times \widetilde{X}(D)}).$$

We have the equality

$$\begin{aligned}(34)\quad \bar{z}\,\bar{\partial}_z f = &\sum_{\boldsymbol{\beta},\boldsymbol{k}} \sum_{n,j} \big(\tfrac{1}{2}\sqrt{-1}\ \partial_\theta + \tfrac{1}{2}\alpha\big) f_{\boldsymbol{\beta},\boldsymbol{k},n,j}\, \varphi_{\boldsymbol{\beta},\boldsymbol{k}}\, \mathrm{e}^{-\sqrt{-1}\,\theta\alpha} z^{\alpha+n} \big(\log |z|^2\big)^j \\ &+ \sum_{\boldsymbol{\beta},\boldsymbol{k}} \sum_{n,j} f_{\boldsymbol{\beta},\boldsymbol{k},n,j} \varphi_{\boldsymbol{\beta},\boldsymbol{k}}\, \mathrm{e}^{-\sqrt{-1}\,\theta\alpha} z^{\alpha+n} j \big(\log |z|^2\big)^{j-1}.\end{aligned}$$

For any section g of $\mathcal{C}^{\infty\,<D(\bar{J})}_{S^1 \times \widetilde{X}(D)}$ on $S^1 \times U$, we can solve the equation

$$\partial_\theta G - \sqrt{-1}\,\alpha\, G = g \qquad (\alpha \neq 0)$$

in $\mathcal{C}^{\infty\ \mathrm{nil} < D(\bar{J})}_{S^1 \times \widetilde{X}(D)}$. We remark $\int_0^{2\pi} \mathrm{e}^{-\sqrt{-1}\,\alpha\theta} g(\theta) \mathrm{d}\theta = 0$. It is easy to obtain $\mathrm{Cok}(\bar{z}\,\partial_{\bar{z}}) = 0$ and $\mathrm{Ker}(\bar{z}\,\partial_{\bar{z}}) = 0$ in the part $\alpha \neq 0$ by using (34). Let us consider the part $\alpha = 0$. We use the filtration with respect to the order of $\log |z|^2$. If we take Gr with respect to this filtration, the second term in (34) with $\alpha = 0$ disappears. We obtain $\mathcal{H}^0 \,\mathrm{Gr}_j = \mathcal{H}^1 \,\mathrm{Gr}_j$ for each j, and they are represented by constants with respect to θ. Then, the second term in (34) induces $\mathcal{H}^0 \,\mathrm{Gr}_j \simeq \mathcal{H}^1 \,\mathrm{Gr}_{j-1}$ for $j \geq 1$. Hence, we obtain the vanishing of the cokernel of $\bar{z}\,\partial_{\bar{z}}$, and the kernel is $\mathcal{H}^0 \,\mathrm{Gr}_0$. Then, the remaining claims of Lemma 4.3.9 are clear. ∎

We have the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{0,\bullet\,<D'(\bar{J})\cup X_0'}_{\mathbb{C}\times\widetilde{X}(D)} & \longrightarrow & \Omega^{0,\bullet<D'(\bar{J})}_{\mathbb{C}\times\widetilde{X}(D)}(*X_0') & \longrightarrow & \Omega^{0,\bullet\,<D'(\bar{J})}_{\widehat{\pi_1^{-1}(X_0')}} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{\simeq} & & \\ 0 & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\bar{J})\cup X_0'}_{\widetilde{X}'(D')} & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\bar{J})}_{\widetilde{X}(D')} & \longrightarrow & \rho_{0*}\Omega^{0,\bullet\,<D'(\bar{J})}_{\widehat{\pi_0^{-1}(X_0')}} & \longrightarrow & 0. \end{array}$$

The left vertical arrow is an isomorphism. According to Lemma 4.3.9, the right vertical arrow is a quasi-isomorphism. Thus, the central vertical arrow is also a quasi-isomorphism, which is the second claim of Theorem 4.3.2. □

4.4. Push-forward

4.4.1. Preliminary. — We shall freely use the notation in §4.1.5.2. Let $(t_1,\ldots,t_\ell)$ denote the standard coordinate system of $\mathbb{C}^\ell$. We set

$$D_0 := \bigcup_{i=1}^{\ell}\{t_i = 0\}.$$

We have $\widetilde{\mathbb{C}}^\ell(D_0) = \widetilde{\mathbb{C}}^\ell$. Let X be any complex manifold. The projection $X\times\widetilde{\mathbb{C}}^\ell \to X\times\mathbb{C}^\ell$ is denoted by π. We put $H_X := X\times D_0$.

For any closed complex submanifold $Y\subset X$, we have naturally defined morphisms:

$$\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\longrightarrow\widetilde{\imath}_*\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}, \tag{35}$$

$$\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell}\longrightarrow\widetilde{\imath}_*\mathcal{A}^{\mathrm{rapid}}_{Y\times\widetilde{\mathbb{C}}^\ell}. \tag{36}$$

Here, $\widetilde{\imath}: Y\times\widetilde{\mathbb{C}}^\ell\to X\times\widetilde{\mathbb{C}}^\ell$ denotes the map induced by the inclusion $Y\subset X$.

Lemma 4.4.1. — *The morphisms* (35) *and* (36) *are isomorphisms.*

Proof. — Let us prove the claim for (35). The other case can be proved similarly. It is enough to argue it locally around each point of H_X. It is easy to reduce the case $X=\Delta^n=\{(z_1,\ldots,z_n);\,|z_i|<1\}$ and $Y=\{z_1=0\}$. Let F be the endomorphism of $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$ given by $F(x)=z_1x$. The complex $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\xrightarrow{F}\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$ expresses $\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$. Clearly, F is injective. It is enough to prove that the induced map $\rho:\mathrm{Cok}(F)\to\mathcal{A}^{\mathrm{mod}}_{Y\times\mathbb{C}^\ell}$ is an isomorphism. It is clearly surjective. Let f be any section of $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$ on $\mathcal{U}\subset X\times\widetilde{\mathbb{C}}^\ell$ such that $\rho(f)=0$. Then, $z_1^{-1}f$ naturally gives a holomorphic function on $\mathcal{U}\setminus\pi^{-1}(H_X)$. Let us prove that $z_1^{-1}f$ is of moderate growth. We may assume that $\mathcal{U}$ is the product of a multi-sector

$$S_t=\big\{(t_1,\dots,t_\ell)\,;\,|\arg(t_i)-\theta_{0i}|\le\delta_{0i},\,0<|t_i|<r_{0i}\,(i=1,\dots,\ell)\big\}$$

($\theta_{0i}\in\mathbb{R}$, $\delta_{0i}>0$, $r_{0i}>0$) in $(\mathbb{C}^*)^\ell$, and multi-discs $U_1=\{|z_1|\le r_1\}$ and $U=\{(z_2,\dots,z_n)\,;\,|z_i|\le r_2\}$. We put $U_1':=\{\frac{1}{2}r_1\le|z_1|\le r_1\}$. On $U_1'\times U\times S_t$, we have $|z_1^{-1}f|\le C\prod_{i=1}^\ell|t_i|^{-N}$. By using the maximum principle, we obtain the estimate of $z_1^{-1}f$ on $U_1\times U\times S_t$. □

4.4.2. The push-forward of coherent $\mathcal{O}_X$-modules. — For any $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$-module $\mathcal{M}$, we canonically have a standard $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$-flat resolution $\mathcal{N}_\bullet(\mathcal{M})$ of $\mathcal{M}$ given as follows. For any open subset $U\subset X\times\widetilde{\mathbb{C}}^\ell$, let $\mathcal{N}_U$ be the free $\pi^{-1}(\mathcal{O}_{X\times\mathbb{C}^\ell})_{|U}$-module generated by $\mathcal{M}(U)$, and let $\mathcal{N}_U'$ denote its 0-extension on $X\times\widetilde{\mathbb{C}}^\ell$. It is naturally equipped with a morphism $a_U:\mathcal{N}_U'\to\mathcal{M}$. We put $\mathcal{N}_0(\mathcal{M}):=\bigoplus_U\mathcal{N}_U'$, and then $a:=\bigoplus_U a_U$ gives a surjection $\mathcal{N}_0(\mathcal{M})\to\mathcal{M}$. By applying the same procedure to $\operatorname{Ker}a$, we obtain a flat $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$-module $\mathcal{N}_1(\mathcal{M})$ with a surjection $\mathcal{N}_1(\mathcal{M})\to\operatorname{Ker}a$. By the standard inductive procedure, we obtain the flat resolution. In particular, we obtain a canonical flat resolution $\mathcal{N}_\bullet(\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell})$.

Let $\varphi:(Y,g)\to(X,f)$ be a morphism in Cat_ℓ. We have a canonical morphism $\widetilde{\varphi}_1^{-1}\mathcal{N}_\bullet(\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell})\to\mathcal{N}_\bullet(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell})$. Hence, for any $\mathcal{O}_Y$-sheaf M, we obtain the morphism

$$\widetilde{\varphi}_1^{-1}\mathcal{N}_\bullet(\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell})\otimes_{\widetilde{\varphi}_1^{-1}\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\pi^{-1}(\Gamma_{g*}M)\longrightarrow\mathcal{N}_\bullet(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell})\otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}(\Gamma_{g*}M).$$

It induces the morphism

$$\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\pi^{-1}(\Gamma_{f*}R\varphi_!M)\longrightarrow R\widetilde{\varphi}_{1!}(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}\Gamma_{g*}M).\tag{37}$$

Similarly, we have the natural morphism:

$$\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\pi^{-1}(\Gamma_{f*}R\varphi_!M)\longrightarrow R\widetilde{\varphi}_{1!}\big(\mathcal{A}^{\mathrm{rapid}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}(\Gamma_{g*}M)\big).\tag{38}$$

Remark 4.4.2. — Because $\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell}$ is flat over $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$ (Proposition 4.2.4), we may replace $\otimes^L$ in (38) with $\otimes$. Later, we shall prove that $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$ is also flat over $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$ (Theorem 4.6.1). □

THEOREM 4.4.3. — *Suppose that M is $\mathcal{O}_Y$-coherent and that φ is projective. Then, the morphisms* (37) *and* (38) *are isomorphisms.*

Proof. — We shall give details for (37). Because the other case can be argued in a similar way, we give only an indication in the last. It is enough to consider the cases

(i) φ is a closed immersion,
(ii) φ is the projection $Y = \mathbb{P}^n \times X \to X$.

4.4.2.1. The case (i). — The following natural morphisms are isomorphisms:

(39)
$$\begin{aligned}\pi^{-1}(\Gamma_{f*}\varphi_* M) &\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\\ &\simeq \pi^{-1}(\varphi_{1*}\Gamma_{g*}M)\otimes^L_{\pi^{-1}\varphi_{1*}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\big(\pi^{-1}\varphi_{1*}\mathcal{O}_{Y\times\mathbb{C}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\big)\\ &\simeq \widetilde{\varphi}_{1*}\big(\pi^{-1}(\Gamma_{g*}M)\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\big).\end{aligned}$$

Here, we have used Lemma 4.4.1. Thus, we are done in the case (i).

4.4.2.2. The case (ii). — Let us consider the case where $\varphi : Y = \mathbb{P}^n \times X \to X$ is the projection. Let L be a line bundle on $\mathbb{P}^n$. Its pull back to $Y \times \mathbb{C}^\ell = \mathbb{P}^n \times X \times \mathbb{C}^\ell$ is denoted by L_Y.

LEMMA 4.4.4. — *Let $q > 0$. If $H^q(\mathbb{P}^n, L) = 0$, we have*
$$R^q\widetilde{\varphi}_{1*}(\pi^{-1}\Gamma_{g*}L_Y \otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}) = 0.$$

Proof. — We have the natural decomposition $\overline{\partial}_{Y\times\mathbb{C}^\ell} = \overline{\partial}_{\mathbb{P}^n} + \overline{\partial}_X + \overline{\partial}_{\mathbb{C}^\ell}$ into the differentials of the $\mathbb{P}^n$-direction, the X-direction and the $\mathbb{C}^\ell$-direction. Let $\mathcal{B}_{Y\times\widetilde{\mathbb{C}}^\ell}$ be the sheaf of C^∞-functions κ on $Y \times \widetilde{\mathbb{C}}^\ell$ satisfying $(\overline{\partial}_X + \overline{\partial}_{\mathbb{C}^\ell})\kappa = 0$ and the following condition locally:

(Moderate) For any differential operator $\mathcal{R}$ on $\mathbb{P}^n$, there exists $N > 0$ such that $\mathcal{R}(\kappa) = O(\prod_{i=1}^\ell |t_i|^{-N})$.

We naturally have $\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell} \subset \mathcal{B}_{Y\times\widetilde{\mathbb{C}}^\ell}$. We set
$$\mathcal{B}^{0,\bullet}_{Y\times\widetilde{\mathbb{C}}^\ell} := \mathcal{B}_{Y\times\widetilde{\mathbb{C}}^\ell} \otimes \pi^{-1}(\Omega^{0,\bullet}_{Y/X}).$$

The naturally defined morphism $\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell} \to \mathcal{B}^{0,\bullet}_{Y\times\widetilde{\mathbb{C}}^\ell}$ is a quasi isomorphism, which can be proved by a standard argument for Dolbeault's lemma. Hence, we obtain the following $\widetilde{\varphi}_1$-soft resolution of $\pi^{-1}(L_Y) \otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}$:

$$\pi^{-1}(L_Y) \otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell} \longrightarrow \pi^{-1}(L_Y) \otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}} \mathcal{B}^{0,\bullet}_{Y\times\widetilde{\mathbb{C}}^\ell}$$

We take a hermitian metric h_L of L. We fix a Kähler metric $g_{\mathbb{P}^n}$ of $\mathbb{P}^n$.

Let $\overline{\partial}_L^*$ denote the formal adjoint of $\overline{\partial}_L : C^\infty(L \otimes \Omega^{0,\bullet}_{\mathbb{P}^n}) \to C^\infty(L \otimes \Omega^{0,\bullet+1}_{\mathbb{P}^n})$. Let $\Delta_L^{0,\bullet}$ denote the Laplacian on $\Gamma(\mathbb{P}^n, L \otimes \Omega^{0,\bullet}_{\mathbb{P}^n})$ associated to h_L and $g_{\mathbb{P}^n}$. Let $G^{0,\bullet}$ be the Green operator.

By the assumption $H^q(\mathbb{P}^n, L) = 0$ for $q > 0$, we have $\Delta^{0,q} \circ G^{0,q} = G^{0,q} \circ \Delta^{0,q} = \mathrm{id}$ if $q > 0$. We have $[G^{0,\bullet}, \overline{\partial}_L] = [G^{0,\bullet}, \overline{\partial}_L^*] = 0$. In particular, if $\overline{\partial}_L \tau = 0$ for $\tau \in \Gamma(\mathbb{P}^n, L \otimes \Omega^{0,q})$ $(q > 0)$, we have $\overline{\partial}_L \overline{\partial}_L^* G(\tau) = \tau$. Recall the following standard results for elliptic operators:

- ▷ $G^{0,q}$ are integral operators.
- ▷ For any non-negative integer m, there exists $C_m > 0$ such that $\|G^{0,q}(\tau)\|_{L^2_{m+2}} \leq C_m \|\tau\|_{L^2_m}$ for any $\tau \in \Gamma(\mathbb{P}^n, L \otimes \Omega^{0,q})$, where $\|.\|_{L^2_m}$ denotes the Sobolev norm.

Let $P \in \pi^{-1}(H_X)$. Let $\mathcal{U}_P$ be an open neighbourhood of P in $X \times \widetilde{\mathbb{C}}^\ell$. Put

$$\mathcal{U}_P^\circ := \mathcal{U}_P \setminus \pi^{-1}(H_X).$$

We have $\widetilde{\varphi}_1^{-1}(\mathcal{U}_P) = \mathbb{P}^n \times \mathcal{U}_P$. Let $\tau \in \Gamma(\mathbb{P}^n \times \mathcal{U}_P, \pi^{-1} L_Y \otimes_{\pi^{-1}\mathcal{O}_{Y \times \mathbb{C}^\ell}} \mathcal{B}^{0,q})$. We obtain a C^∞-function $G(\tau)$ on $\mathbb{P}^n \times \mathcal{U}_P^\circ$, and we have $\overline{\partial}_{z_i} G(\tau) = 0$ and $\partial_{z_i} G(\tau) = G(\partial_{z_i} \tau)$ for any local coordinate system $(z_1, \ldots, z_n)$ on $X \times \mathbb{C}^\ell$. Then, by the estimate of the Green operator, we obtain that $G(\tau) \in \Gamma(\mathbb{P}^n \times \mathcal{U}_P, \pi^{-1} L_Y \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{B}^{0,q})$. Moreover, if $\overline{\partial}_L \tau = 0$ and $q > 0$, we have $\overline{\partial}_L(\overline{\partial}_L^* G(\tau)) = \tau$. Thus, we obtain Lemma 4.4.4. □

LEMMA 4.4.5. — *We have* $\widetilde{\varphi}_{1*} \mathcal{A}^{\mathrm{mod}}_{Y \times \widetilde{\mathbb{C}}^\ell} \simeq \mathcal{A}^{\mathrm{mod}}_{X \times \widetilde{\mathbb{C}}^\ell}$, *i.e., the morphism* (37) *is an isomorphism for* $\mathcal{O}_Y$.

Proof. — Let $P \in \pi^{-1}(H_X)$. Let $\mathcal{U}_P$ be a small neighbourhood of P in $X \times \widetilde{\mathbb{C}}^\ell$. Let $\kappa \in \Gamma(\mathbb{P}^n \times \mathcal{U}_P, \mathcal{A}^{\mathrm{mod}}_{Y \times \widetilde{\mathbb{C}}^\ell})$. Take any point Q of $\mathbb{P}^n$. We consider the inclusion $\iota_Q : \mathcal{U}_P \simeq \mathcal{U}_P \times \{Q\} \to \mathbb{P}^n \times \mathcal{U}_P$. We have $\mu := \iota_Q^{-1}(\kappa) \in \Gamma(\mathcal{U}_P, \mathcal{A}^{\mathrm{mod}}_{X \times \widetilde{\mathbb{C}}^\ell})$. It is easy to deduce that $\kappa = \widetilde{\varphi}(\mu)$. Then, we obtain Lemma 4.4.5. □

LEMMA 4.4.6. — *Let L be a line bundle on $\mathbb{P}^n$. Then* (37) *is an isomorphism for L_Y.*

Proof. — We use an induction on n. In the case $n = 0$, the claim is trivial. Assume that we have already obtained the claim in the case $n - 1$. Let $L = \mathcal{O}_{\mathbb{P}^n}(m)$. If $m = 0$, the claim follows from Lemma 4.4.5. We fix a hyperplane $\mathbb{P}^{n-1}_\infty \subset \mathbb{P}^n$. If $m > 0$, we can reduce the claim to the case $m - 1$, by using the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_{\mathbb{P}^{n-1}_\infty}(m) \to 0$. If $m < 0$, we can reduce the claim to the case $m + 1$, by using the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_{\mathbb{P}^n}(m+1) \to \mathcal{O}_{\mathbb{P}^{n-1}_\infty}(m+1) \to 0$. □

Let us finish the proof in the case (ii). It is enough to prove that (37) is an isomorphism around any point of $X \times \mathbb{C}^\ell$, which we shall implicitly use. We may assume to have a resolution

$$(\cdots \longrightarrow \mathcal{Q}_p \longrightarrow \mathcal{Q}_{p-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_1 \longrightarrow \mathcal{Q}_0) \simeq M,$$

such that $\mathcal{Q}_p$ are of the form $\bigoplus_{i=1}^{N_p}(L_{p,i})_Y$, where $L_{p,i}$ are line bundles on $\mathbb{P}^n$. By Lemma 4.4.6, the morphisms (37) for $\mathcal{Q}_p$ are isomorphisms. Hence, (37) for M is also an isomorphism. Thus, the proof for (37) is finished.

Let us give an indication to prove that (38) is an isomorphism. We can argue the case (i) in the same way. In the case (ii), we replace the condition (Moderate) in the proof of Lemma 4.4.4 with the following:

(Rapid) Let $\mathcal{R}$ be any differential operators on $\mathbb{P}^n$. Then, $\mathcal{R}(\kappa) = O(\prod |t_i|^N)$ for any N.

Then, we can prove that (38) is an isomorphism in the case (ii). Thus, the proof of Theorem 4.4.3 is finished. □

4.5. Characterization by growth order

4.5.1. Statements

THEOREM 4.5.1. — *Let (X, f) be an object in* Cat_ℓ.

▷ $\mathrm{Tor}_i^{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}(\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}, \pi^{-1}\mathcal{O}_{\Gamma_f(X)}) = 0$ *for $i \neq 0$. Namely,*

$$\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

▷ *Let $\varphi : (Y, g) \to (X, f)$ be a projective birational morphism such that (i) D_Y is normal crossing, (ii) $Y \setminus D_Y \simeq X \setminus D_X$. For the naturally induced map $\rho : \widetilde{Y}(D_Y) \to X \times \widetilde{\mathbb{C}}^\ell$, we have*

(40) $$R\rho_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(D_Y)} \simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)},$$

(41) $$R\rho_*\mathcal{A}^{\mathrm{rapid}}_{\widetilde{Y}(D_Y)} \simeq \mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

▷ *The support of $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}$ and $\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}$ are $\widetilde{X}(f)$.*

REMARK 4.5.2. — Note that $\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell}$ is flat over $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$, according to Proposition 4.2.4. The first claim of the theorem is a special case of the flatness of $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}$ over $\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}$ (Theorem 4.6.1). □

Let us state some consequences. We have the sheaves of algebras $\mathcal{A}^{\mathrm{mod}}_{X,f}$ and $\mathcal{A}^{\mathrm{rapid}}_{X,f}$ on $\widetilde{X}(f)$ determined by the conditions

$$\widetilde{\Gamma}_{f*}\mathcal{A}^{\mathrm{mod}}_{X,f} = \pi^{-1}(\mathcal{O}_{\Gamma_f(X)}) \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell},$$
$$\widetilde{\Gamma}_{f*}\mathcal{A}^{\mathrm{rapid}}_{X,f} = \pi^{-1}(\mathcal{O}_{\Gamma_f(X)}) \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell}.$$

Theorem 4.5.3. — *Let $(X,f)\in \mathrm{Cat}_\ell$.*

- ▷ *For the inclusion $j : X\setminus D_X \to \widetilde{X}(f)$, the natural morphism $\mathcal{A}^{\mathrm{mod}}_{X,f} \to j_*\mathcal{O}_{X\setminus D_X}$ is a monomorphism. The image is $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$.*
- ▷ *The natural morphism $\mathcal{A}^{\mathrm{rapid}}_{X,f} \to j_*\mathcal{O}_{X\setminus D_X}$ is a monomorphism. The image is $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$.*
- ▷ *In particular, if f is submersive, then we naturally have $\mathcal{A}^{\mathrm{mod}}_{X,f} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)}$ and $\mathcal{A}^{\mathrm{rapid}}_{X,f} \simeq \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}$.*

Proof. — It follows from the descriptions (40) and (41). □

Theorem 4.4.3 can be reformulated in terms of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$ and $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$.

Theorem 4.5.4. — *Let $\varphi : (Y,g) \to (X,f)$ be a projective morphism in Cat_ℓ. Let M be any coherent $\mathcal{O}_Y$-module. Then, the following natural morphisms are isomorphisms:*

$$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes^L_{\pi^{-1}\mathcal{O}_X} \pi^{-1}R\varphi_*M \simeq R\widetilde{\varphi}_*(\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)} \otimes^L_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}M), \tag{42}$$

$$\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)} \otimes^L_{\pi^{-1}\mathcal{O}_X} \pi^{-1}R\varphi_*M \simeq R\widetilde{\varphi}_*(\mathcal{A}^{\mathrm{rapid}}_{\widetilde{Y}(g)} \otimes^L_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}M). \tag{43}$$

□

After the flatness results in Proposition 4.2.4 and Theorem 4.6.1 below, we may replace $\otimes^L$ with $\otimes$ in (42) and (43).

4.5.2. Proof of Theorem 4.5.1.

— Let us begin with the simplest case.

Lemma 4.5.5. — *Suppose that f is submersive. For the naturally induced closed immersion $\rho : \widetilde{X}(D_X) \to X \times \widetilde{\mathbb{C}}^\ell$, the following natural morphisms are isomorphisms:*

$$\pi^{-1}\mathcal{O}_{\Gamma_f(X)} \otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \longrightarrow \rho_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)}, \tag{44}$$

$$\pi^{-1}\mathcal{O}_{\Gamma_f(X)} \otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell} \longrightarrow \rho_*\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}. \tag{45}$$

Proof. — It is enough to argue it locally around any point of H_X. We may assume $X = \{(z_1, \dots, z_n)\}$ and $f = (z_1, \dots, z_\ell)$. Let $G : X \times \mathbb{C}^\ell \to \mathbb{C}^n \times \mathbb{C}^\ell$ be given by

$$G(z_1, \dots, z_n, t_1, \dots, t_\ell) = (z_1 - t_1, z_2 - t_2, \dots, z_\ell - t_\ell, z_{\ell+1}, \dots, z_n, t_1, \dots, t_\ell).$$

Then, $G \circ \Gamma_f(z_1, \dots, z_n) = (0, \dots, 0, z_{\ell+1}, \dots, z_n, z_1, \dots, z_\ell)$. By using G, it is easy to prove that the morphisms (44) and (45) are isomorphisms. ☐

Let us consider the case where D_X is normal crossing. We have a naturally defined map $X \setminus D_X \to X \times (\mathbb{C}^*)^\ell$ as the graph. Let us observe that it is extended to $\rho_1 : \widetilde{X}(D_X) \to X \times \widetilde{\mathbb{C}}^\ell$. Let f_i be the composite of $f : X \to \mathbb{C}^\ell$ and the projection $\mathbb{C}^\ell \to \mathbb{C}$ onto the i-th component. It induces a map $g_i : X \setminus D_X \to \mathbb{C}^*$. It is enough to observe that it is extended to a map $\widetilde{X}(D_X) \to \widetilde{\mathbb{C}}$. Let P be any point of D_X. Because $f_i^{-1}(0)$ is contained in the normal crossing hypersurface D_X, we can take a holomorphic coordinate neighbourhood $(X_P; z_1, \dots, z_n)$ around P such that $D_X = \bigcup_{j=1}^p \{z_j = 0\}$ and $f_i = \prod_{j=1}^p z_j^{m_j}$, where $m_j > 0$. Let $z_j = r_i \mathrm{e}^{\sqrt{-1}\,\theta_j}$. Because the map $\widetilde{X}(D_X) \to \mathbb{C}^*$ is described as

$$\big(r_1, \mathrm{e}^{\sqrt{-1}\,\theta_1}, \dots, r_p \mathrm{e}^{\sqrt{-1}\,\theta\ell}, z_{p+1}, \dots, z_n\big) \longmapsto \prod r_i^{m_i} \mathrm{e}^{\sqrt{-1}\,m\theta_i},$$

we obtain that $g_{i|X_P \setminus D_X}$ is extended to $\widetilde{X}_P(D_X \cap X_P) \to \widetilde{\mathbb{C}}$. Then, the claim follows.

We have the naturally defined morphism:

$$\mathcal{A}^{\mathrm{mod}}_{X \times \widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X \times \mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \longrightarrow \rho_{1*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)}. \tag{46}$$

Proposition 4.5.6. — *Suppose that $D_X := f^{-1}(D_0)$ is normal crossing. The morphism* (46) *is an isomorphism. Moreover, we have the following isomorphisms:*

$$R\rho_{1*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)} \simeq \rho_{1*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)},$$
$$\mathcal{A}^{\mathrm{mod}}_{X \times \widetilde{\mathbb{C}}^\ell} \otimes^L_{\pi^{-1}\mathcal{O}_{X \times \mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \simeq \mathcal{A}^{\mathrm{mod}}_{X \times \widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X \times \mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

Proof. — In the proof, we omit to denote π^{-1}. We have the maps

$$\widetilde{\Gamma}_f^{(1)} : \widetilde{X}(D_X) \longrightarrow \widetilde{X}(D_X) \times \widetilde{\mathbb{C}}^\ell \quad \text{and} \quad \widetilde{\Gamma}_f^{(2)} : \widetilde{X}(D_X) \longrightarrow \widetilde{X}(D_X) \times \mathbb{C}^\ell$$

induced by f. We have the projections:

$$\nu_1 : \widetilde{X}(D_X) \times \widetilde{\mathbb{C}}^\ell \longrightarrow X \times \widetilde{\mathbb{C}}^\ell \quad \text{and} \quad \nu_2 : \widetilde{X}(D_X) \times \widetilde{\mathbb{C}}^\ell \longrightarrow \widetilde{X}(D_X) \times \mathbb{C}^\ell.$$

We set $D'_X := D_X \times \mathbb{C}^\ell$. According to §II.1.1 of [**52**], we have the isomorphisms

$$R\nu_{1*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}(*D'_X), \quad R\nu_{2*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell}(*H_X).$$

Hence, we have the natural isomorphisms

$$\begin{aligned}(47)\qquad R\nu_{1*}\big(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}\big) &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}(*D'_X) \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}\\ &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}(*D_{X'})\\ &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}(*H_X)\\ &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}(*H_X) \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}\\ &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}.\end{aligned}$$

We also have

$$\begin{aligned}(48)\quad R\nu_{2*}\big(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}\big) &\simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell}(*H_X) \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}\\ &\simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}.\end{aligned}$$

LEMMA 4.5.7. — *$\widetilde{\Gamma}^{(2)}_f$ is a closed embedding, and that we have*

$$(49)\quad \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)} \simeq \widetilde{\Gamma}^{(2)}_{f*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)}.$$

Proof. — For the expression $f = (f_1, \dots, f_\ell)$, we define $G' : X\times\mathbb{C}^\ell \to X\times\mathbb{C}^\ell$ by $G'(P, t_1, \dots, t_\ell) := (P, t_1 - f_1(P), \dots, t_\ell - f_\ell(P))$. We have $G'\circ\Gamma_f(P) = (P, 0, \dots, 0)$. Then, we can prove (49) by an induction on ℓ. □

LEMMA 4.5.8. — *The support of* $\mathrm{Tor}_*^{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell}, \pi^{-1}\mathcal{O}_{\Gamma_f(X)})$ *is contained in $\widetilde{\Gamma}^{(1)}_f(\widetilde{X}(D_X))$.*

Proof. — Let U denote an ℓ-dimensional vector space with a basis $e_1, \dots, e_\ell$. We set

$$C^{k-\ell} := \bigwedge^k U \otimes \mathcal{O}_{X\times\mathbb{C}^\ell}.$$

Let $\partial : C^m \to C^{m+1}$ be given by

$$\partial\alpha = \sum (t_i - f_i)e_i \wedge \alpha.$$

Then, we obtain a complex of $\mathcal{O}_{X\times\mathbb{C}^\ell}$-modules $C^\bullet$, and it gives a free resolution of $\mathcal{O}_{X\times\mathbb{C}^\ell}$-module $\mathcal{O}_{\Gamma_f(X)}$. If $Q \in \widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell$ is not contained in $\widetilde{\Gamma}^{(1)}_f(X)$, then one of $t_i - f_i$ are invertible in $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell}$ around Q. Hence, the complex $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes C^\bullet$ is acyclic around Q. It implies the claim of Lemma 4.5.8. □

Note that ν_2 induces a homeomorphism $\widetilde{\Gamma}_f^{(1)}(X) \simeq \widetilde{\Gamma}_f^{(2)}(X)$. By Lemma 4.5.8, we obtain that for $p \neq 0$

$$R^p\nu_{2*}\operatorname{Tor}_j^{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}, \pi^{-1}\mathcal{O}_{\Gamma_f(X)}) = 0.$$

By applying the argument of the spectral sequence with (49) to (48), we obtain that

$$\operatorname{Tor}_j^{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}, \pi^{-1}\mathcal{O}_{\Gamma_f(X)}) = 0$$

for $j \neq 0$, i.e.,

$$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}$$

on $\widetilde{X}(D_X) \times \widetilde{\mathbb{C}}^\ell$. We also obtain an isomorphism of sheaves on $\widetilde{X}(D_X) \simeq \widetilde{\Gamma}_f^{(i)}(X)$:

$$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)}.$$

From (47), we obtain

$$\begin{aligned} R\rho_{1*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)} &\simeq R\nu_{1*}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}) \\ &\simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}. \end{aligned} \tag{50}$$

Note $R^p\nu_{1*}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}) = 0$ unless $p \geq 0$, and the p-th cohomology sheaf of $\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes^L_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}$ is 0 unless $p \leq 0$. Hence, (50) implies the claims of Proposition 4.5.6. □

Proposition 4.5.9. — *Suppose that D_X is normal crossing. Then, the natural map*

$$\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)} \simeq \rho_{1*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}$$

is an isomorphism. Moreover, we have $R\rho_{1*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)} \simeq \rho_{1*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}$.

Proof. — It is proved by the arguments in the proof of Proposition 4.5.6. We omit to denote π^{-1}. We have the isomorphisms

$$R\nu_{1*}\mathcal{A}^{<H_X\leq D'_X}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \simeq \mathcal{A}^{<H_X}_{X\times\widetilde{\mathbb{C}}^\ell}(*D'_X) \quad\text{and}\quad R\nu_{2*}\mathcal{A}^{\leq H_X<D'_X}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \simeq \mathcal{A}^{<D'_X}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell}(*H_X).$$

Hence, we have the natural isomorphisms

$$\begin{aligned} R\nu_{1*}(\mathcal{A}^{<H_X\leq D'_X}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}) &\simeq \mathcal{A}^{<H_X}_{X\times\widetilde{\mathbb{C}}^\ell}(*D'_X) \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)} \\ &\simeq \mathcal{A}^{<H_X}_{X\times\widetilde{\mathbb{C}}^\ell} \otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}} \mathcal{O}_{\Gamma_f(X)}, \end{aligned} \tag{51}$$

$$(52)\quad R\nu_{2*}(\mathcal{A}^{\leq H_X<D'_X}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)})\\ \simeq \mathcal{A}^{<D'_X}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell}(*H_X)\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)}\\ \simeq \mathcal{A}^{<D'_X}_{\widetilde{X}(D_X)\times\mathbb{C}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)}\simeq\widetilde{\Gamma}^{(2)}_{f*}\mathcal{A}^{<D_X}_{\widetilde{X}(D_X)}.$$

Let us consider the morphisms

$$(53)\quad \mathcal{A}^{<H_X\leq D'_X}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f}\longleftarrow\mathcal{A}^{<(H_X\cup D'_X)}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f}\\ \longrightarrow\mathcal{A}^{<D'_X\leq H_X}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f}.$$

Because t_i-f_i are invertible on $\mathcal{A}^{<H_X}_{\widehat{\pi^{-1}(D'_X)}}$ and $\mathcal{A}^{<D'_X}_{\widehat{\pi^{-1}(H_X)}}$, we have

$$\mathcal{A}^{<H_X}_{\widehat{\pi^{-1}(D'_X)}}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)}=0\quad\text{and}\quad\mathcal{A}^{<D'_X}_{\widehat{\pi^{-1}(H_X)}}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)}=0.$$

Hence, the morphisms in (53) are isomorphisms. By the argument in the proof of Lemma 4.5.8, we obtain that the support of the sheaves in (53) are contained in $\widetilde{\Gamma}^{(1)}_f(X)$. Because ν_2 gives a homeomorphism $\widetilde{\Gamma}^{(1)}_f(\widetilde{X}(D))\simeq\widetilde{\Gamma}^{(2)}_f(\widetilde{X}(D))$, we identify $\mathcal{A}^{<H_X\leq D'_X}_{\widetilde{X}(D)\times\widetilde{\mathbb{C}}^\ell}\otimes_{\mathcal{O}_{X\times\mathbb{C}^\ell}}\mathcal{O}_{\Gamma_f(X)}$ with $\mathcal{A}^{<D_X}_{\widetilde{X}(D_X)}$ as sheaves on $\widetilde{X}(D_X)$. Then, the claim of Proposition 4.5.9 follows from (51). □

Let us finish the proof of Theorem 4.5.1. Let (X,f) be any object in Cat_ℓ. We take any projective birational morphism $\varphi:(Y,g)\to(X,f)$ such that

(i) D_Y is normal crossing,

(ii) $Y\setminus D_Y\simeq X\setminus D_X$.

We set $D'_Y:=D_Y\times\mathbb{C}^\ell$ and $D'_X:=D_X\times\mathbb{C}^\ell$. We have

$$R\varphi_*\mathcal{O}_Y(*D_Y)\simeq\mathcal{O}_X(*D_X).$$

By using Theorem 4.4.3, we obtain

$$R\widetilde{\varphi}_{1*}\big(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}\Gamma_{g*}\big(\mathcal{O}_Y(*D_Y)\big)\big)\\ \simeq\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^\ell}}\pi^{-1}\Gamma_{f*}\big(\mathcal{O}_X(*D_X)\big).$$

By using Proposition 4.5.6, we obtain

$$(54)\quad R\widetilde{\varphi}_{1*}\big(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}\Gamma_{g*}(\mathcal{O}_Y(*D_Y))\big)\\ \simeq R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes^L_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}\mathcal{O}_{\Gamma_g(Y)})\\ \simeq R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^\ell}\otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^\ell}}\pi^{-1}\mathcal{O}_{\Gamma_g(Y)}).$$

We also have

$$\mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^{\ell}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^{\ell}}} \pi^{-1}\Gamma_{f*}\big(\mathcal{O}_X(*D_X)\big) \simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^{\ell}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^{\ell}}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

We obtain

$$R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{mod}}_{Y\times\widetilde{\mathbb{C}}^{\ell}} \otimes_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}^{\ell}}} \pi^{-1}\mathcal{O}_{\Gamma_g(Y)}) \simeq \mathcal{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}^{\ell}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}^{\ell}}} \pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

It implies that the claims for $\mathcal{A}^{\mathrm{mod}}$ in Theorem 4.5.1. The claims for $\mathcal{A}^{\mathrm{rapid}}$ can be proved similarly. □

4.5.3. Complement for the sheaf of Nilsson type functions (Appendix). — Let us consider an analogue for the sheaves of Nilsson type functions. We restrict ourselves to the case $\ell = 1$. Let $\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}$ denote the sheaf of holomorphic functions of Nilsson type on $X\times\widetilde{\mathbb{C}}$.

Lemma 4.5.10. — *For any complex manifold $i:(Y,g)\subset(X,f)$ in* Cat_1, *the naturally defined morphism*

$$\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}} \pi^{-1}\mathcal{O}_{Y\times\mathbb{C}} \longrightarrow \widetilde{\imath}_*\mathcal{A}^{\mathrm{nil}}_{Y\times\widetilde{\mathbb{C}}}$$

is an isomorphism.

Proof. — As in Lemma 4.4.1, we have an isomorphism

$$\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}} \otimes^{L}_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{Y\times\mathbb{C}} \simeq \mathcal{A}^{\mathrm{rapid}}_{Y\times\widetilde{\mathbb{C}}}.$$

We can check $\mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(H_X)}} \otimes^{L}_{\mathcal{O}_{\widehat{H}_X}} \mathcal{O}_{\widehat{H_Y}} \simeq \mathcal{A}^{\mathrm{nil}}_{\widehat{\pi^{-1}(H_Y)}}$ directly. Then, the claim of the lemma follows. □

Let $\varphi:(Y,g)\to(X,f)$ be a morphism in Cat_1. For any $\mathcal{O}_Y$-coherent sheaf M, we have the following naturally defined morphism

$$\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}} \pi^{-1}(\Gamma_{f*}R\varphi_*M) \longrightarrow R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{nil}}_{Y\times\widetilde{\mathbb{C}}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}}} \pi^{-1}\Gamma_{g*}M). \tag{55}$$

Proposition 4.5.11. — *Suppose that M is $\mathcal{O}_X$-coherent, and that φ is projective. Then, the morphism* (55) *is an isomorphism.*

Proof. — By Theorem 4.4.3, we have an isomorphism

$$\mathcal{A}^{\mathrm{rapid}}_{X\times\widetilde{\mathbb{C}}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}} \pi^{-1}(\Gamma_{f*}R\varphi_*M) \simeq R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{rapid}}_{Y\times\widetilde{\mathbb{C}}} \otimes^{L}_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}}} \pi^{-1}\Gamma_{g*}M).$$

We also have the formal isomorphism

$$\mathcal{A}^{\mathrm{nil}}_{\widehat{H}_X} \otimes^{L}_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}} \pi^{-1}(\Gamma_{f*}R\varphi_*M) \simeq R\widetilde{\varphi}_{1*}(\mathcal{A}^{\mathrm{nil}}_{\widehat{H}_Y} \otimes^{L}_{\pi^{-1}\mathcal{O}_{Y\times\mathbb{C}}} \pi^{-1}\Gamma_{g*}M).$$

Then, the claim of the proposition follows. □

THEOREM 4.5.12. — *Let (X,f) be an object in* Cat_1. *Let $\varphi:(Y,g)\to(X,f)$ be a projective birational morphism such that*

(i) *D_Y is normal crossing,*

(ii) *$Y\setminus D_Y\simeq X\setminus D_X$.*

For the naturally induced map $\rho:\widetilde{Y}(D_Y)\to X\times\widetilde{\mathbb{C}}$, we have

$$R\rho_*\mathcal{A}^{\mathrm{nil}}_{\widetilde{Y}(D_Y)}\simeq\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}}\pi^{-1}\mathcal{O}_{\Gamma_f(X)}.$$

Proof. — As in the proof of Theorem 4.5.1, it is enough to consider the case where $\varphi=\mathrm{id}$. We use the notation in the proof of Proposition 4.5.6. We have the isomorphism $R\nu_{1*}\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}}\simeq\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(*D'_X)$. Hence, we have the following natural isomorphism

$$R\nu_{1*}\big(\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}}\otimes^L_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}\big)\simeq\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}.$$

We have the naturally defined morphism

$$\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}\longrightarrow\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)}.$$

It is enough to prove that the induced morphism is an isomorphism:

$$R\nu_{1*}\big(\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}\big)\longrightarrow R\nu_{1*}\big(\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(D_X)}\big).\tag{56}$$

We have already known that the following is an isomorphism, by Proposition 4.5.9:

$$R\nu_{1*}\big(\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)\times\widetilde{\mathbb{C}}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}\big)\longrightarrow R\nu_{1*}\big(\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}\big).$$

Let $D_X=\bigcup_{i\in\Lambda}D_i$ be the irreducible decomposition. For any $I\subset\Lambda$, we set

$$D_{I0}:=\bigcap_{i\in I}\big(D_i\times\{0\}\big).$$

To prove that (56) is an isomorphism, it is enough to prove that the following natural morphisms are isomorphisms:

$$R\nu_{1*}\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}\mathcal{O}_{\Gamma_f(X)}\longrightarrow R\nu_{1*}\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{<\partial D_I}_{\widehat{\pi_1^{-1}(D_I)}}\tag{57}$$

It is enough to consider the issue locally around any point of $D_X\times\{0\}$. We may assume that $X=\Delta^n$, $D_X=\bigcup_{i=1}^{\ell}\{z_i=0\}$ and $f=\prod_{i=1}^{\ell}z_i^{m_i}$.

LEMMA 4.5.13. — *We may assume that* $\mathrm{g.c.d.}(m_i\mid i\in I)=1$.

Proof. — Let $p:=\mathrm{g.c.d.}(m_i\mid i\in I)$. We set

$$X':=\Delta^n\quad\text{and}\quad D':=\bigcup_{i=1}^{\ell}\{z_i=0\}.$$

We define $D'_I := \bigcap_{i\in I}\{z_i = 0\}$. On X', we set $g := \prod_{i\notin I} z_i^{m_i} \times \prod_{i\in I} z_i^{m_i/p}$. We define $\psi : X \to X'$ by $z_i \mapsto z_i^p$ $(i \in I)$ and $z_i \mapsto z_i$ $(i \notin I)$. We have $f = g \circ \psi$. The map ψ gives

$$D_I \simeq D'_I \quad \text{and} \quad \widetilde{D}_I(\partial D_I) \simeq \widetilde{D}'(\partial D'_I).$$

Let $\widetilde{\Gamma}^{(1)}_g : \widetilde{X}'(D') \to \widetilde{X}'(D') \times \widetilde{\mathbb{C}}$ and $\nu'_1 : \widetilde{X}'(D') \times \widetilde{\mathbb{C}} \to X' \times \widetilde{\mathbb{C}}$ be given similarly to $\widetilde{\Gamma}^{(1)}_f$ and ν_1. We have the following natural commutative diagram of the sheaves on $\widetilde{D}_I(\partial D_I)$:

$$\begin{array}{ccc} R\nu_{1*}\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{\Gamma_f(X)} & \longrightarrow & R\nu_{1*}\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{<\partial D_I}_{\widehat{\pi_1^{-1}(D_I)}} \\ \simeq\uparrow & & \simeq\uparrow \\ R\nu'_{1*}\mathcal{A}^{<\partial D'_{I0}}_{\widehat{\pi^{-1}(D'_{I0})}} \otimes_{\mathcal{O}_{X'\times\mathbb{C}}} \mathcal{O}_{\Gamma_g(X')} & \longrightarrow & R\nu'_{1*}\widetilde{\Gamma}^{(1)}_{g*}\mathcal{A}^{<\partial D'_I}_{\widehat{\pi_1^{-1}(D'_I)}}. \end{array}$$

It is easy to check that the vertical arrows are isomorphisms. Then, we obtain the claim of Lemma 4.5.13. □

Let $\pi_1 : \widetilde{X}(D_X) \to X$, $\pi_2 : \widetilde{X}(f) \to X$ and $\pi : \widetilde{X}(D_X) \times \widetilde{\mathbb{C}} \to X \times \mathbb{C}$ be the projections. We have

$$\pi_1^{-1}(D_I) \simeq \widetilde{D}_I(\partial D_I) \times (S^1)^{|I|},\ \pi^{-1}(D_{I0}) \simeq \widetilde{D}_I(\partial D_I) \times (S^1)^{|I|+1},$$
$$\pi_2^{-1}(D_I) \simeq D_I \times S^1.$$

We decompose the map $\nu_{1|\pi^{-1}(D_{I0})} : \pi^{-1}(D_{I0}) \to \pi_2^{-1}(D_I)$ into

$$\widetilde{D}_I(\partial D_I) \times (S^1)^{|I|+1} \xrightarrow{\mu_1} \widetilde{D}_I(\partial D_I) \times S^1 \xrightarrow{\mu_2} D_I \times S^1.$$

To prove that (57) are isomorphisms, it is enough to prove that

$$R\mu_{1*}\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{\Gamma_f} \longrightarrow R\mu_{1*}\widetilde{\Gamma}^{(1)}_{f*}\mathcal{A}^{<\partial D_I}_{\widehat{\pi_1^{-1}(D_I)}} \tag{58}$$

is an isomorphism. We have the expression

$$\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}} \simeq \varinjlim_{T,N}\big(\mathcal{A}^{\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![t, z_i \mid i \in I]\!] \otimes_{\mathbb{C}[t,z_i|i\in i]} \mathrm{Nil}(t, z_i \mid i \in I)\big).$$

By the argument in Lemma 4.3.9, or by a direct computation of the cohomology of the sheaves on the fiber of μ_1, we obtain

$$R\mu_{1*}\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}} \simeq \varinjlim_{T,N}\big(\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![t, z_i \mid i \in I]\!] \otimes_{\mathbb{C}[t]} \mathrm{Nil}(t)\big).$$

Hence, we obtain the natural isomorphism

$$R\mu_{1*}\mathcal{A}^{<\partial D_{I0}}_{\widehat{\pi^{-1}(D_{I0})}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{\Gamma_f} \simeq \varinjlim_{T,N}\big(\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i \in I]\!] \otimes_{\mathbb{C}[t]} \mathrm{Nil}(t)\big). \tag{59}$$

Here, t acts as f on $\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]$. We have the expression

$$\mathcal{A}^{<\partial D_I}_{\widehat{\pi_1^{-1}(D_I)}} \simeq \varinjlim_{T,N}\big(\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]\otimes_{\mathbb{C}[z_i\mid i\in I]}\mathrm{Nil}(z_i\mid i\in I)\big).$$

We take $T_0\subset\mathbb{C}$ such that $T_0\to\mathbb{C}/\mathbb{Z}$ is bijective. We have the decomposition

$$\mathrm{Nil}(z_i\mid i\in I)=\bigoplus_{\boldsymbol{\alpha}\in T_0^I}\boldsymbol{z}^{\boldsymbol{\alpha}}\mathbb{C}[z_i,\log z_i\mid i\in I].$$

We have the corresponding decomposition:

$$\begin{aligned}(60)\qquad &\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]\otimes_{\mathbb{C}[z_i\mid i\in I]}\mathrm{Nil}(z_i\mid i\in I)\\ &\qquad=\bigoplus_{\boldsymbol{\alpha}\in T_0^I}\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]\boldsymbol{z}^{\boldsymbol{\alpha}}\otimes\mathbb{C}[\log z_i\mid i\in I].\end{aligned}$$

Recall $f=\prod_{i=1}^{\ell} z_i^{m_i}$ with g.c.d.$(m_i\mid i\in I)=1$. Under the assumption, the map $\mathbb{C}/\mathbb{Z}\to(\mathbb{C}/\mathbb{Z})^I$ given by $\beta\longmapsto(\beta m_i\mid i\in I)$ is injective. We have the subsheaf

$$(61)\qquad \bigoplus_{\beta\in T_0}\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]\prod_{i\in I}z_i^{\beta m_i}\otimes\mathbb{C}[\log z_i\mid i\in I].$$

Let $\mathcal{Q}$ be the quotient of (60) by (61). Note that the fibers of $\mu_1\circ\widetilde{\Gamma}^{(1)}_f$ are connected. By a direct computation of the sheaves on the fibers of $\mu_1\circ\widetilde{\Gamma}^{(1)}_f$, we obtain the push-forward of $\mathcal{Q}$ by $\mu_1\circ\widetilde{\Gamma}^{(1)}_f$ is 0. Moreover, we obtain that the push-forward of (61) is naturally isomorphic to

$$(62)\qquad \bigoplus_{\beta\in T_0}\mathcal{A}^{<\partial D_I}_{\widetilde{D}_I(\partial D_I),T,N}[\![z_i \mid i\in I]\!]\prod_{i\in I}z_i^{\beta m_i}\big(\log(\prod_{i=1}^{\ell}z_i^{m_i})\big).$$

Hence, the push-forward of $\mathcal{A}^{<\partial D_I}_{\widehat{\pi_1^{-1}(D_I)}}$ by $\mu_1\circ\widetilde{\Gamma}^{(1)}_f$ is isomorphic to the limit of (62). Together with (59) we obtain Theorem 4.5.12. □

For any object (X,f) in Cat_1, we have the sheaves $\mathcal{A}^{\mathrm{nil}}_{X,f}$ on $\widetilde{X}(f)$ determined by the condition $\widetilde{\Gamma}_{f*}\mathcal{A}^{\mathrm{nil}}_{X,f}=\pi^{-1}\mathcal{O}_{\Gamma_f(X)}\otimes_{\pi^{-1}\mathcal{O}_{X\times\mathbb{C}}}\mathcal{A}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}$. For a morphism $\varphi:(X_1,f_1)\to(X_2,f_2)$ in Cat_1, we naturally have $\widetilde{\varphi}^{-1}\mathcal{A}^{\mathrm{nil}}_{X_2,f_2}\to\mathcal{A}^{\mathrm{nil}}_{X_1,f_1}$.

We obtain the following propositions as in the case of $\mathcal{A}^{\mathrm{rapid}}$ and $\mathcal{A}^{\mathrm{mod}}$.

Proposition 4.5.14. — *For the inclusion $j:X\setminus D_X\to\widetilde{X}(D_X)$, the natural morphism $\mathcal{A}^{\mathrm{nil}}_{X,f}\to j_*\mathcal{O}_{X\setminus D_X}$ is a monomorphism.* □

Proposition 4.5.15. — *Let $\varphi : (Y, g) \to (X, f)$ be a projective morphism in* Cat_1*. Let M be any coherent $\mathcal{O}_Y$-module. Then, the following natural morphism is an isomorphism:*

$$\mathcal{A}^{\mathrm{nil}}_{X,f} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1} R\varphi_* M \simeq R\widetilde{\varphi}_*(\mathcal{A}^{\mathrm{nil}}_{Y,g} \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1} M).$$

□

4.6. Flatness of the sheaf of holomorphic functions with moderate growth

4.6.1. Statement. — Let (X, f) be any object in Cat_ℓ. Let

$$j : X \setminus D_X \longrightarrow \widetilde{X}(f)$$

denote the natural inclusion. For any $\mathcal{O}_X$-module M, we set

$$\pi^*_{\mathrm{mod}} M := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1} M.$$

It is also denoted by $\pi^*_{f\,\mathrm{mod}} M$, when we would like to emphasize the dependence on f. We shall prove the following theorem.

Theorem 4.6.1. — *$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$ is flat over $\pi^{-1}\mathcal{O}_X$, i.e., $\pi^*_{\mathrm{mod}} M \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes^L_{\pi^{-1}\mathcal{O}_X} \pi^{-1} M$ for any coherent $\mathcal{O}_X$-module M. Moreover, the natural morphism $\pi^*_{\mathrm{mod}}(M) \to j_*(M_{|X\setminus D_X})$ is injective.*

Corollary 4.6.2. — *$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$ is faithfully flat over $\pi^{-1}\mathcal{O}_X(*D_X)$.* □

We define $\pi^*_{\mathrm{rapid}} M := \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}_f M$. We can prove the following by a similar argument.

Proposition 4.6.3. — *The natural morphism $\pi^*_{\mathrm{rapid}}(M) \to j_*(M_{|X\setminus D_X})$ is injective.* □

By Theorem 4.3.1, $\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$ is flat over $\pi^{-1}\mathcal{O}_X$. So, we have the following.

Proposition 4.6.4. — *$\mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}$ is faithfully flat over $\pi^{-1}\mathcal{O}_X(*D_X)$.* □

4.6.2. Induction. — We consider the following conditions for any coherent $\mathcal{O}_X$-module M:

($\mathcal{P}1$): $\pi^{-1}M \otimes^L_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \simeq \pi^{-1}M \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$.

($\mathcal{P}2$): $\pi^*_{\mathrm{mod}}(M) \to j_*(M_{|X\setminus D_X})$ is injective.

Let $\mathcal{P}(X)$ denote the class of coherent $\mathcal{O}_X$-modules satisfying the conditions ($\mathcal{P}1$) and ($\mathcal{P}2$). It is our purpose to prove that any coherent $\mathcal{O}_X$-modules are members of $\mathcal{P}(X)$. We shall implicitly use that the conditions are local.

We shall prove the following claim by using an induction on k:

(Q_k): Let (X, f) be any object in Cat_ℓ. Let M be any coherent $\mathcal{O}_X$-module such that $\dim \operatorname{Supp} M \leq k$. Then, M is a member of $\mathcal{P}(X)$.

4.6.3. Preliminary. — The following lemma is easy to prove.

Lemma 4.6.5. — *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of coherent $\mathcal{O}_X$-modules.*

- ▷ *If M_2 and M_3 are members of $\mathcal{P}(X)$, then M_1 is also a member of $\mathcal{P}(X)$.*
- ▷ *If M_1 and M_3 are members of $\mathcal{P}(X)$, then M_2 is also a member of $\mathcal{P}(X)$.* □

The following direct corollary will be used implicitly.

Corollary 4.6.6. — *Let $\rho : M_1 \to M_2$ be any morphism of coherent $\mathcal{O}_X$-modules such that $\operatorname{Cok}(\rho), \operatorname{Ker}(\rho) \in \mathcal{P}(X)$. If M_2 is contained in $\mathcal{P}(X)$, then M_1 is also contained in $\mathcal{P}(X)$.* □

Lemma 4.6.7. — *Let Z be any complex submanifold of X with the inclusion $i_Z : Z \to X$. Let M_Z be any locally free $\mathcal{O}_Z$-module. Then, we have $i_{Z*}M_Z \in \mathcal{P}(X)$.*

Proof. — It follows from Theorem 4.5.1 and Theorem 4.5.4. □

4.6.4. Functoriality for the push-forward. — Let $\varphi : (X', f') \to (X, f)$ be a morphism in Cat_ℓ such that $\varphi : X' \to X$ is projective and birational. We do not assume that $X' \setminus D_{X'}$ is isomorphic to $X \setminus D_X$. Let D'' be the exceptional divisor of φ. Let M be a coherent $\mathcal{O}_{X'}$-module such that $M \in \mathcal{P}(X')$. Assume that $\dim \operatorname{Supp} M = k$ and $\dim \varphi(\operatorname{Supp} M \cap D'') < k$.

Lemma 4.6.8. — *Assume that Q_{k-1} holds. Then, we obtain $\varphi_*(M') \in \mathcal{P}(X)$.*

Proof. — According to Theorem 4.5.4, we have the isomorphism:

$$R\widetilde{\varphi}_*(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}'(f')} \otimes^L_{\pi^{-1}\mathcal{O}_{X'}} \pi^{-1}M) \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes^L_{\pi^{-1}\mathcal{O}_X} \pi^{-1}R\varphi_*M. \tag{63}$$

If $i > 0$, we have $R^i\varphi_*M \in \mathcal{P}(X)$, because $\dim \operatorname{Supp} R^i\varphi_*M < k$. By using the degeneration of the spectral sequence, we obtain

$$H^i(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \otimes^L_{\pi^{-1}\mathcal{O}_X} \pi^{-1}R\varphi_*M) \simeq \begin{cases} \mathrm{Tor}^{\pi^{-1}\mathcal{O}_X}_{-i}(\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}, \pi^{-1}\varphi_*M) & (i < 0), \\ \pi^*_{\mathrm{mod}} R^i\varphi_*M & (i \geq 0). \end{cases}$$

By (63) and the isomorphism $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}'(f')} \otimes^L_{\pi^{-1}\mathcal{O}_{X'}} M \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}'(f')} \otimes_{\pi^{-1}\mathcal{O}_{X'}} M$, we have $H^i = 0$ for $i < 0$. Hence, we obtain that φ_*M satisfies $(\mathcal{P}1)$. Because

$$\pi^*_{\mathrm{mod}}\varphi_*M \simeq \widetilde{\varphi}_*(\pi^*_{\mathrm{mod}}M),$$

$(\mathcal{P}2)$ for φ_*M follows from $(\mathcal{P}2)$ for M. □

We have a direct consequence. Let $(X', f') \to (X, f)$ be a morphism in Cat_ℓ such that $\varphi : X' \to X$ is a projective birational morphism. We do not assume that $X' \setminus D_{X'}$ is isomorphic to $X \setminus D_X$. Let $Z' \subset X'$ be a k-dimensional irreducible complex submanifold. We assume that Z' is not contained in the exceptional divisor of φ, in particular, Z' is birational to $\varphi(Z')$. We obtain the following lemma from Lemma 4.6.7 and Lemma 4.6.8.

Corollary 4.6.9. — *Let $M_{Z'}$ be any locally free $\mathcal{O}_{Z'}$-module. Suppose Q_{k-1}. Then, we have $\varphi_*(i_{Z'*}M_{Z'}) \in \mathcal{P}(X)$.* □

4.6.5. Coherent sheaves on submanifolds. — Let Z be any k-dimensional irreducible submanifold of X with the inclusion $i_Z : Z \to X$.

Lemma 4.6.10. — *Let M be any coherent $\mathcal{O}_X$-module such that* $\operatorname{Supp}(M) \subset Z$. *Assume that Q_{k-1} holds. Then, we have $M \in \mathcal{P}(X)$.*

Proof. — It is enough to consider locally around each point P of X. We shall shrink X around P without mention.

First, let us consider the case where $M = i_{Z*}M_Z$. We may assume that M_Z is a torsion-free $\mathcal{O}_Z$-module. We can find a projective birational morphism $\varphi : (X', f') \to (X, f)$ in Cat_ℓ such that

(i) the strict transform Z' of Z is a complex submanifold of X',

(ii) there exists a locally free $\mathcal{O}_{Z'}$-module M' with a morphism $\psi : \varphi^*M \to M'$ such that $\psi_{|X' \setminus D''}$ is an isomorphism.

We obtain a morphism $\psi_1 : M \to \varphi_* M'$, which is an isomorphism on $Z \setminus \varphi(D'')$. By Q_{k-1}, $\operatorname{Ker}\psi_1$ and $\operatorname{Cok}\psi_1$ are contained in $\mathcal{P}(X)$. Then, we obtain $\iota_{Z*}M \in \mathcal{P}(X)$.

In the general case, we have a finite increasing filtration

$$F = \{F_i(M) \,|\, i = 0, \dots, N\}$$

of M by $\mathcal{O}_X$-modules such that each $F_i(M)/F_{i-1}(M)$ comes from an $\mathcal{O}_Z$-module. Then, the claim of the lemma is reduced to the result in the previous paragraph. □

4.6.6. End of the proof of Theorem 4.6.1. — Let Z be any k-dimensional irreducible reduced analytic subset of X such that $Z \not\subset D_X$.

LEMMA 4.6.11. — *Let M be any coherent $\mathcal{O}_X$-module such that* $\operatorname{Supp}(M) \subset Z$. *Assume that Q_{k-1} holds. Then, we have $M \in \mathcal{P}(X)$.*

Proof. — It is enough to consider the issue locally around any point P of X. Hence, we shall shrink X around P without mention. Let Z_1 denote the union of the singular points of Z and $D_X \cap Z$. There exists a projective birational morphism $\varphi_P : (X', f') \to (X, f)$ in Cat_ℓ with the following properties:

- ▷ The induced morphism $X \setminus D'' \to X \setminus (Z_1 \cup D)$ is an isomorphism.
- ▷ The strict transform Z' of Z is a complex submanifold of X'.

We have $M \to \varphi_* \varphi^* M$, which is an isomorphism outside the singular locus of Z. Hence, we obtain $M \in \mathcal{P}(X)$ by Lemma 4.6.8 and Lemma 4.6.10. □

Let M be any coherent $\mathcal{O}_X$-module such that $\dim \operatorname{Supp}(M) \leq k$. If we have a decomposition $\operatorname{Supp}(M) = Z_1 \cup Z_2$ such that $Z_1 \cap Z_2 \subsetneq Z_i$, then we have an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of coherent $\mathcal{O}_X$-modules, such that $\operatorname{Supp}(M_i) \subset Z_i$. Hence, by an easy induction, we obtain $M \in \mathcal{P}(X)$ from Lemma 4.6.11. Thus, our induction can proceed, and the proof of Theorem 4.6.1 is finished. □

4.7. Push-forward of good $\mathcal{D}$-modules and real blow up

4.7.1. Rapid decay and moderate growth. — Let (X, f) be any object in Cat_ℓ. We put

$$\mathcal{D}^{\mathrm{mod}}_{\widetilde{X}(f)} := \pi^{-1}(\mathcal{D}_X) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}.$$

For any $\mathcal{D}_X$-module $\mathcal{M}$, we set

$$\pi^*_{f\,\mathrm{mod}}(\mathcal{M}) := \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)}, \quad \pi^*_{f\,\mathrm{rapid}}(\mathcal{M}) := \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)}.$$

They are naturally $\mathcal{D}^{\mathrm{mod}}_{\widetilde{X}(f)}$-modules.

Let $\varphi : (X,f) \to (Y,g)$ be any morphism in Cat_ℓ. For any $\mathcal{D}^{\mathrm{mod}}_{\widetilde{X}(f)}$-module $\widetilde{\mathcal{M}}$, we put

$$\widetilde{\varphi}_\dagger(\widetilde{\mathcal{M}}) := R\widetilde{\varphi}_!(\pi^{-1}(\mathcal{D}_{Y\leftarrow X}) \otimes^L_{\pi^{-1}\mathcal{D}_X} \widetilde{\mathcal{M}}). \tag{64}$$

Let $\mathcal{M}$ be any $\mathcal{D}_X$-module. We have the naturally defined morphism

$$\widetilde{\varphi}^{-1}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)} \otimes_{\widetilde{\varphi}^{-1}\pi^{-1}\mathcal{O}_Y} \pi^{-1}(\mathcal{M}) \longrightarrow \pi^*_{f\,\mathrm{mod}}(\mathcal{M}).$$

It induces the following morphism in the derived category of $\mathcal{D}_{Y,g}$-modules:

$$R\widetilde{\varphi}_!\big(\pi^{-1}(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}) \otimes_{\widetilde{\varphi}^{-1}\pi^{-1}\mathcal{O}_Y} \widetilde{\varphi}^{-1}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)}\big) \longrightarrow \widetilde{\varphi}_\dagger\pi^*_{f\,\mathrm{mod}}(\mathcal{M}).$$

We also have the isomorphisms:

$$\begin{aligned} R\widetilde{\varphi}_!\big(\pi^{-1}(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}) \otimes_{\widetilde{\varphi}^{-1}\pi^{-1}\mathcal{O}_Y} \widetilde{\varphi}^{-1}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)}\big) \\ \simeq R\widetilde{\varphi}_!\big(\pi^{-1}(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M})\big) \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)} \\ \simeq \pi^{-1}R\varphi_!(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}) \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}(g)} \simeq \pi^*_{g\,\mathrm{mod}}\varphi_\dagger\mathcal{M}. \end{aligned}$$

Hence, we obtain the following morphism in the derived category of $\mathcal{D}_{Y,g}$-modules:

$$\pi^*_{g\,\mathrm{mod}}\varphi_\dagger\mathcal{M} \longrightarrow \widetilde{\varphi}_\dagger\pi^*_{f\,\mathrm{mod}}(\mathcal{M}). \tag{65}$$

Similarly, we obtain the morphism

$$\pi^*_{g\,\mathrm{rapid}}\varphi_\dagger\mathcal{M} \longrightarrow \widetilde{\varphi}_\dagger\pi^*_{f\,\mathrm{rapid}}(\mathcal{M}). \tag{66}$$

PROPOSITION 4.7.1. — *Assume that φ is projective, and that M has a good filtration in the neighbourhood of fibers of φ. Then, the morphisms* (65) *and* (66) *are isomorphisms.*

Proof. — By considering a resolution, it is enough to consider the case $\mathcal{M} = M \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes \Omega^{-1}_X$, and M is an $\mathcal{O}_X$-coherent sheaf. Then, the claim is reduced to Theorem 4.5.4. ∎

Let (X,f) be an object in Cat_ℓ such that D_X is normal crossing. We set

$$\mathcal{D}^{\mathrm{mod}}_{\widetilde{X}(D_X)} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{D}_X.$$

Let $\pi_1 : \widetilde{X}(D_X) \to X$ be the projection. For any $\mathcal{D}_X$-module $\mathcal{M}$, we define

$$\pi^*_{1\,\mathrm{mod}}\mathcal{M} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(D_X)} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{M}, \quad \pi^*_{1\,\mathrm{rapid}}\mathcal{M} := \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(D_X)} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{M}.$$

We have the naturally defined proper map $\rho : \widetilde{X}(D_X) \to \widetilde{X}(f)$.

We obtain the following proposition from Theorem 4.5.1.

PROPOSITION 4.7.2. — *We have the following natural isomorphisms for any coherent $\mathcal{D}_X$-module $\mathcal{M}$:*

$$R\rho_*\pi^*_{1\,\mathrm{mod}}\mathcal{M} \simeq \pi^*_{f\,\mathrm{mod}}\mathcal{M}, \quad R\rho_*\pi^*_{1\,\mathrm{rapid}}\mathcal{M} \simeq \pi^*_{f\,\mathrm{rapid}}\mathcal{M}.$$

□

4.7.2. Compatibility with the de Rham functor. — For any $\mathcal{D}_X$-module $\mathcal{M}$, we put

$$\mathrm{DR}^{\mathrm{mod}}_{X,f}(\mathcal{M}) := \pi^{-1}(\mathrm{DR}_X\,\mathcal{M}) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \simeq \pi^{-1}(\Omega_X) \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{f\,\mathrm{mod}}(\mathcal{M}),$$

$$\mathrm{DR}^{\mathrm{rapid}}_{X,f}(\mathcal{M}) := \pi^{-1}(\mathrm{DR}_X\,\mathcal{M}) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{rapid}}_{\widetilde{X}(f)} \simeq \pi^{-1}(\Omega_X) \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{f\,\mathrm{rapid}}(\mathcal{M}).$$

COROLLARY 4.7.3. — *Suppose that $\mathcal{M}$ has a good filtration in the neighbourhood of fibers of φ. Assume that φ is projective. Then, we have natural isomorphisms:*

$$R\widetilde{\varphi}_!\,\mathrm{DR}^{\mathrm{mod}}_{X,f}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{mod}}_{Y,g}\,\varphi_\dagger(\mathcal{M}), \quad R\widetilde{\varphi}_!\,\mathrm{DR}^{\mathrm{rapid}}_{X,f}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rapid}}_{Y,g}\,\varphi_\dagger(\mathcal{M}).$$

Proof. — From $\widetilde{\varphi}_\dagger\pi^*_{\mathrm{mod}}\mathcal{M} \simeq \pi^*_{\mathrm{mod}}\varphi_\dagger\mathcal{M}$, we obtain the isomorphisms

$$\begin{aligned} R\widetilde{\varphi}_!\,\mathrm{DR}^{\mathrm{mod}}_{X,f}\,\mathcal{M} &\simeq R\widetilde{\varphi}_!(\pi^{-1}\Omega_X \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{f\,\mathrm{mod}}\mathcal{M}) \\ &\simeq \pi^{-1}\Omega_Y \otimes^L_{\pi^{-1}\mathcal{D}_Y} \widetilde{\varphi}_\dagger\pi^*_{f\,\mathrm{mod}}\mathcal{M} \\ &\simeq \pi^{-1}\Omega_Y \otimes^L_{\pi^{-1}\mathcal{D}_Y} (\pi^*_{g\,\mathrm{mod}}\varphi_\dagger\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{mod}}_{Y,g}\,\varphi_\dagger\mathcal{M}. \end{aligned} \tag{67}$$

Thus, we obtain the first isomorphism. We obtain the second one similarly. □

Let (X,f) be an object in Cat_ℓ such that D_X is normal crossing. We consider the real blow up $\pi_1 : \widetilde{X}(D_X) \to X$. We define $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(D_X)}(\mathcal{M})$ and $\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}(\mathcal{M})$ as follows:

$$\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(D_X)}(\mathcal{M}) := \pi^{-1}\Omega \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{1\,\mathrm{mod}}(\mathcal{M}),$$

$$\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}(\mathcal{M}) := \pi^{-1}\Omega \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{1\,\mathrm{rapid}}(\mathcal{M}).$$

We have the naturally defined proper map $\rho : \widetilde{X}(D_X) \to \widetilde{X}(f)$.

PROPOSITION 4.7.4. — *The following natural morphisms are isomorphisms:*

$$R\rho_*\,\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(D_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{mod}}_{X,f}(\mathcal{M}), \quad R\rho_*\,\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rapid}}_{X,f}(\mathcal{M}).$$

Proof. — It immediately follows from Proposition 4.7.2. □

We obtain the following corollary from Corollary 4.7.3 and Proposition 4.7.4.

Corollary 4.7.5. — *Let $\varphi : X \to Y$ be any projective morphism of complex manifolds. Let D_Y be a normal crossing hypersurface of Y such that $D_X := \varphi^{-1}(D_Y)$ is normal crossing. Let $\widetilde{\varphi} : \widetilde{X}(D_X) \to \widetilde{Y}(D_Y)$ be the induced map. Then, for any coherent $\mathcal{D}_X$-module having a good filtration in the neighbourhood of fibers of φ, we have the following natural isomorphisms:*

$$R\widetilde{\varphi}_! \,\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(D_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{mod}}_{\widetilde{Y}(D_Y)}\, \varphi_\dagger \mathcal{M}, \tag{68}$$

$$R\widetilde{\varphi}_! \,\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rapid}}_{\widetilde{Y}(D_Y)}\, \varphi_\dagger \mathcal{M}. \tag{69}$$

□

Remark 4.7.6. — G. Morando informed the author that the isomorphism (68) and its generalizations can be deduced from some results in [**24**]. While the author hopes that the generalization would make the subject more transparent, he also hopes that our direct method would be also significant for our understanding. □

4.7.3. Nilsson type (Appendix). — We have variants in the case of Nilsson type. Let (X, f) be an object in Cat_1. We set

$$\mathcal{D}^{\mathrm{nil}}_{\widetilde{X}(f)} := \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(f)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{D}_X.$$

For any $\mathcal{D}_X$-module $\mathcal{M}$, we set $\pi^*_{\mathrm{nil}}(\mathcal{M}) := \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{nil}}_{\widetilde{X}(f)}$. They are naturally $\mathcal{D}^{\mathrm{nil}}_{\widetilde{X}(f)}$-modules.

Let $\varphi : (X, f) \to (Y, g)$ be a morphism in Cat_1. For any $\mathcal{D}^{\mathrm{nil}}_{\widetilde{X}(f)}$-module $\widetilde{\mathcal{M}}$, we define $\widetilde{\varphi}_\dagger(\widetilde{\mathcal{M}})$ by the formula (64). We also define

$$\mathrm{DR}^{\mathrm{nil}}_{X,f}(\mathcal{M}) := \pi^{-1}\Omega_X \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^*_{f\,\mathrm{nil}}\mathcal{M}.$$

We obtain the following from Proposition 4.5.15.

Proposition 4.7.7. — *Suppose that φ is projective and that $\mathcal{M}$ has a good filtration in the neighbourhood of fibers of φ. Then, the natural morphism*

$$\pi^*_{\mathrm{nil}}\varphi_\dagger\mathcal{M} \longrightarrow \widetilde{\varphi}_\dagger \pi^*_{\mathrm{nil}}(\mathcal{M}) \tag{70}$$

is an isomorphism. In particular, a natural morphism $R\widetilde{\varphi}_! \,\mathrm{DR}^{\mathrm{nil}}_{X,f}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{nil}}_{Y,g}\, \varphi_\dagger\mathcal{M}$ is an isomorphism. □

Let (X, f) be an object in Cat_1 such that D_X is normal crossing. We consider the real blow up $\pi_1 : \widetilde{X}(D_X) \to X$. We define

$$\mathrm{DR}^{\mathrm{nil}}_{\widetilde{X}(D_X)}(\mathcal{M}) := \pi_1^{-1}\Omega \otimes_{\pi_1^{-1}\mathcal{O}_X} \pi^*_{1\,\mathrm{nil}}\mathcal{M}$$

for any $\mathcal{D}_X$-module $\mathcal{M}$. We obtain the following proposition from Theorem 4.5.12.

PROPOSITION 4.7.8. — *Let $\rho : \widetilde{X}(D_X) \to \widetilde{X}(f)$ be the natural map. We have a natural isomorphism*

$$R\rho_*\pi^*_{1\,\mathrm{nil}}(\mathcal{M}) \simeq \pi^*_{\mathrm{nil}}(\mathcal{M}).$$

In particular, we obtain an isomorphism $R\rho_* \mathrm{DR}^{\mathrm{nil}}_{\widetilde{X}(D)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{nil}}_{X,f}(\mathcal{M})$. □

COROLLARY 4.7.9. — *Let $\varphi : X \to Y$ be any projective morphism of complex manifolds. Let D_Y be a smooth hypersurface of Y such that $\varphi^{-1}(D_Y)$ is normal crossing. Let $\widetilde{\varphi} : \widetilde{X}(D_X) \to \widetilde{Y}(D_Y)$ be the induced map. Then, for any coherent $\mathcal{D}_X$-module $\mathcal{M}$ having a good filtration in the neighbourhood of fibers of φ, we have the natural isomorphism*

$$R\widetilde{\varphi}_! \,\mathrm{DR}^{\mathrm{nil}}_{\widetilde{X}(D_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{nil}}_{\widetilde{Y}(D_Y)}(\mathcal{M}). \qquad \square$$

CHAPTER 5

COMPLEXES ON THE REAL BLOW UP ASSOCIATED TO GOOD MEROMORPHIC FLAT BUNDLES

5.1. De Rham complexes

5.1.1. De Rham complex and a description by dual. — Let X be a complex manifold and D be a normal crossing hypersurface with a decomposition $D = D_1 \cup D_2$. (Note that D_i are not necessarily irreducible; see §3.2.1.) We set

$$d_X := \dim X.$$

Let $\pi : \widetilde{X}(D) \to X$ be the real blow up. Let $\Omega^\bullet_X$ denote the sheaf of holomorphic 1-forms on X. We put

$$\Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} := \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega^\bullet_X,$$
$$\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} := \Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega^\bullet_X.$$

For any holonomic $\mathcal{D}$-module $\mathcal{M}$ on X, we define

$$\begin{aligned}\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}) := \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\,\mathrm{DR}_X(\mathcal{M})\\ \simeq \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}[d_X] \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\\ \simeq \mathrm{Tot}\big(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\big)[d_X].\end{aligned}$$

Note $\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}) \simeq \mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}(*D))$ because $\Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}(*D) = \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}$.

We have a natural isomorphism $R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}) \simeq \operatorname{DR}^{<D_1\leq D_2}_X \mathcal{M}$ induced as follows, by Theorem 4.3.1:

$$(71)\qquad R\pi_* \operatorname{Tot}\bigl(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}\bigr)[d_X] \simeq \operatorname{Tot}\bigl(R\pi_*\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\mathcal{O}_X} \mathcal{M}\bigr)[d_X] \simeq \operatorname{Tot}\bigl(\Omega^{\bullet,\bullet\,<D_1}_X(*D_2) \otimes_{\mathcal{O}_X} \mathcal{M}\bigr)[d_X].$$

Lemma 5.1.1. — *We have a natural isomorphism*

$$R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)})[d_X] \simeq \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\boldsymbol{D}\mathcal{M}).$$

Proof. — Since $\mathcal{M}$ is $\mathcal{D}_X$-coherent, we have the isomorphisms

$$(72)\qquad \begin{aligned} &R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M},\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)})[d_X] \\ &\simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M},\, \pi^{-1}\mathcal{D}_X) \otimes^L_{\pi^{-1}\mathcal{D}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}[d_X] \\ &= \pi^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \boldsymbol{D}\mathcal{M}) \otimes^L_{\pi^{-1}\mathcal{D}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \\ &\simeq \bigl(\pi^{-1}\Omega_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \otimes^L_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\boldsymbol{D}\mathcal{M}. \end{aligned}$$

Because $\mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$ is flat over $\pi^{-1}\mathcal{O}_X$ (Theorem 4.3.1), $\pi^{-1}\mathcal{D}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$ is flat over $\pi^{-1}\mathcal{D}_X$. Therefore,

$$\mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)} \simeq \pi^{-1}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta^{-\bullet}_X) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}$$

is a $\pi^{-1}\mathcal{D}_X$-flat resolution. Hence, (72) is quasi-isomorphic to the following:

$$(73)\qquad \bigl(\pi^{-1}(\Omega^\bullet_X \otimes \mathcal{D}_X) \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)}\bigr) \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\boldsymbol{D}\mathcal{M}[d_X] \simeq \Omega^{\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\boldsymbol{D}\mathcal{M}[d_X].$$

Thus, we obtain the desired isomorphism. □

According to Lemma 5.1.1, we have a natural isomorphism

$$(74)\qquad \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\mathcal{M}) \simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\boldsymbol{D}\mathcal{M},\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)})[d_X] \simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\boldsymbol{D}(\mathcal{M}(*D)),\, \mathcal{A}^{<D_1\leq D_2}_{\widetilde{X}(D)})[d_X].$$

We will implicitly identify them in the following argument.

5.1.2. A combinatorial description in the case of good meromorphic flat bundles. — Let X be a complex manifold with a normal crossing hypersurface D. Let $\pi : \widetilde{X}(D) \to X$ be the real blow up. Let V be a good meromorphic flat bundle on (X, D). We have the local system on $X - D$ associated to $V_{|X-D}$. Its prolongment over $\widetilde{X}(D)$ is denoted by $\mathcal{L}$. If V is unramifiedly good, for any $P \in \pi^{-1}(D)$, we have the Stokes filtration $\mathcal{F}^P$ of the stalk $\mathcal{L}_P$ indexed by the set of the irregular values $\mathrm{Irr}(V, \pi(P)) \subset \mathcal{O}_X(*D)_{\pi(P)}/\mathcal{O}_{X,\pi(P)}$ with the order $\leq_P$. The system of filtrations $\{\mathcal{F}^P \mid P \in \pi^{-1}(D)\}$ satisfies some compatibility condition. See [**47**], [**48**] or §3 of [**49**] for more details.

Let $D = D_1 \cup D_2$ be a decomposition. Let us describe $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V)$ in terms of the Stokes filtrations. If V is unramifiedly good, for $P \in \widetilde{X}(D)$, let $\mathcal{L}_P^{<D_1 \leq D_2}$ be the union of the subspaces $\mathcal{F}^P_{\mathfrak{a}}(\mathcal{L}_P) \subset \mathcal{L}_P$ such that

(i) $\mathfrak{a} \leq_P 0$,

(ii) the poles of $\mathfrak{a}$ contain the germ of D_1 at $\pi(P)$.

If V is not unramifiedly good, we take a ramified covering $\varphi : (X', D') \to (X, D)$ such that $V' = \varphi^* V$ is unramifiedly good. We obtain the local system $\mathcal{L}'$ and a sheaf $\mathcal{L}'^{<D'_1 \leq D'_2}$ on $\widetilde{X}'(D')$ associated to V' with the Stokes structure. By taking the descent, we obtain a subsheaf $\mathcal{L}^{<D_1 \leq D_2} \subset \mathcal{L}$.

Lemma 5.1.2. — *The family* $\{\mathcal{L}_P^{<D_1 \leq D_2}\}$ *gives a constructible sheaf* $\mathcal{L}^{<D_1 \leq D_2}$ *on* $\widetilde{X}(D)$.

Proof. — It is enough to consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. We may also assume that V is unramifiedly good. By using a decomposition around P as in Theorem 4.1 of [**49**], it is easy to observe that it is enough to consider the case $V = \mathcal{O}_X(*D)$ with a flat connection $\nabla e = e\, d\mathfrak{a}$, where $\mathfrak{a} = \prod_{i=1}^{m} z_i^{-m_i}$ $(m_i > 0)$ for some $1 \leq m \leq \ell$. We have a decomposition $\underline{\ell} = I_1 \sqcup I_2$ such that $D_j = \bigcup_{i \in I_j} \{z_i = 0\}$. For $P \in \widetilde{X}(D)$, we set $I_j(P) := \{i \in I_j \mid z_i(\pi(P)) = 0\}$. We set

$$F_{\mathfrak{a}} := -|\mathfrak{a}|^{-1} \operatorname{Re} \mathfrak{a}.$$

We put

$$R_0 := \bigcup_{i=1}^{m} \{z_i = 0\} \quad \text{and} \quad R_1 := \bigcup_{i=m+1}^{\ell} \{z_i = 0\} \setminus R_0.$$

▷ For $P \in X - D$, we have $\mathcal{L}_P^{<D_1 \leq D_2} \neq 0$.

▷ For $P \in \pi^{-1}(R_1)$, we have $\mathcal{L}_P^{<D_1 \leq D_2} \neq 0$ if and only if $I_1(P) = \varnothing$.

▷ For $P \in \pi^{-1}(R_0)$, we have $\mathcal{L}_P^{<D_1\leq D_2} \neq 0$ if and only if (i) $F_{\mathfrak{a}}(P) < 0$, (ii) $I_1(P) \subset \underline{m}$.

Then, the claim of the lemma is clear. □

We recall the following proposition. (See [**33**] and [**52**]; see also [**16**].)

PROPOSITION 5.1.3. — *The natural inclusion* $\mathcal{L}^{<D_1\leq D_2}[d_X] \to \mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)$ *is a quasi-isomorphism.*

Proof. — We give a preparation from elementary analysis on multi-sectors. We set

$$Y := \Delta_z \times \Delta^n_{\boldsymbol{w}} \quad \text{and} \quad D_Y = \{z=0\} \cup \bigcup_{i=1}^{\ell} \{w_i = 0\}.$$

Let $\pi : \widetilde{Y}(D_Y) \to Y$ be the real blow up. For $m > 0$ and $\boldsymbol{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k_{>0}$ $(0 \leq k \leq \ell)$, we put

$$\mathfrak{a} = z^{-m} \prod_{i=1}^{k} w_i^{-m_i}.$$

We put $F_{\mathfrak{a}} = -|\mathfrak{a}^{-1}| \operatorname{Re}(\mathfrak{a})$, which naturally gives a C^∞-function on $\widetilde{Y}(D_Y)$. Take a point $P \in \pi^{-1}(O) \subset \widetilde{Y}(D_Y)$. Let $S = S_z \times S_{\boldsymbol{w}}$ be a small multi-sector in $Y - D_Y$ such that P is contained in the interior part of the closure of $\overline{S}$ in $\widetilde{Y}(D_Y)$.

▷ If $F_{\mathfrak{a}}(P) < 0$ (resp. $F_{\mathfrak{a}}(P) > 0$), we assume that $F_{\mathfrak{a}} < 0$ (resp. $F_{\mathfrak{a}} > 0$) on $\overline{S}$.

▷ If $F_{\mathfrak{a}}(P) = 0$, we assume that $F_{\mathfrak{a}}$ is monotonous with respect to θ, where $z = r\mathrm{e}^{\sqrt{-1}\theta}$ is the polar coordinate system. Let θ_i $(i = 1, 2)$ be the arguments of the edges of S_z, i.e., $S_z = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2,\ 0 < r \leq r_0\}$. Let θ_+ be one of θ_i such that $F_{\mathfrak{a}} > 0$ on $\{r\mathrm{e}^{\sqrt{-1}\theta_+}\} \times \overline{S}_{\boldsymbol{w}}$.

Let f be a holomorphic function on S of moderate growth with respect to z and $\boldsymbol{w}$. We set

$$\Phi(f)(z, \boldsymbol{w}) := \int_{\gamma(z,\boldsymbol{w})} \exp\big(-\mathfrak{a}(z, \boldsymbol{w}) + \mathfrak{a}(\zeta, \boldsymbol{w})\big) f(\zeta, \boldsymbol{w}) \mathrm{d}\zeta. \tag{75}$$

Here, $\gamma(z, \boldsymbol{w})$ is a path contained in $S_z \times \{\boldsymbol{w}\}$ taken as follows.

Case $F_{\mathfrak{a}}(P) < 0$. We fix a point $z_0 \in S_z$, and $\gamma(z, \boldsymbol{w})$ is a path from z_0 to z.

Case $F_{\mathfrak{a}}(P) > 0$. Let $\gamma(z, \boldsymbol{w})$ be the segment from 0 to z.

Case $F_{\mathfrak{a}}(P)=0$. Let θ_+ be as above. For the polar coordinate system $z=r\,\mathrm{e}^{\sqrt{-1}\,\theta}$, let $\gamma(z,\boldsymbol{w})$ be the union of the ray $\{\rho\,\mathrm{e}^{\sqrt{-1}\,\theta_+}\mid 0\leq\rho\leq r\}$ and the arc connecting $r\,\mathrm{e}^{\sqrt{-1}\,\theta_+}$ and z.

Lemma 5.1.4. — *For each $N>0$, there exists $C_N>0$ such that*

$$\big|\Phi(f)(z,\boldsymbol{w})\big|\leq C_N\cdot C\,|z|^N\prod_{i=1}^{\ell}|w_i|^{N_i}$$

if $|f(z,\boldsymbol{w})|\leq C\,|z|^N\prod_{i=1}^{\ell}|w_i|^{N_i}$.

Proof. — We give only an outline. Consider the case $F_{\mathfrak{a}}(P)<0$. Let $z_0=r_0\,\mathrm{e}^{\sqrt{-1}\,\theta_0}$ and $z=r\,\mathrm{e}^{\sqrt{-1}\,\theta}$. We may assume that the path γ is the union of

(i) the arc γ_1 connecting z_0 and $z_1=r_0\,\mathrm{e}^{\sqrt{-1}\,\theta}$,

(ii) the segment γ_2 connecting z_1 and z.

The segment γ_2 is divided into

$$\gamma_{2,1}=\gamma_1\cap\big\{|\zeta|>\tfrac{3}{2}|z|\big\}\quad\text{and}\quad\gamma_{2,2}=\gamma_1\cap\big\{|\zeta|\leq\tfrac{3}{2}|z|\big\}.$$

The contributions of γ_1 and $\gamma_{2,1}$ are dominated by

$$\big|\exp(-\mathfrak{a}(z,\boldsymbol{w}))\big|\prod_{i=k+1}^{\ell}|w_i|^{N_i}.$$

The function $\operatorname{Re}\mathfrak{a}$ is monotone on $\gamma_{2,2}$. We also have

$$\big|f(\zeta,\boldsymbol{w})\big|\leq C'\,|z^N|\prod_{i=1}^{\ell}|w_i|^{N_i}$$

on $\gamma_{2,2}$. Hence, the contribution of $\gamma_{2,2}$ is dominated by $|z|^N\prod_{i=1}^{\ell}|w_i|^{N_i}$. Let us consider the case $F_{\mathfrak{a}}(P)\geq 0$. On γ, we have $|f(\zeta,\boldsymbol{w})|\leq C'\,|z^N|\prod_{i=1}^{\ell}|w_i|^{N_i}$, and $\operatorname{Re}(\mathfrak{a})$ is monotone. Hence, it is easy to obtain the desired estimate. □

Let us return to the proof of Proposition 5.1.3. It is enough to consider the case $X=\Delta^n$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. We may assume that V is unramifiedly good. Let $P\in\pi^{-1}(0,\dots,0)$. By using the local decomposition around P as in Theorem 4.1 of [**49**], we can reduce the issue to the case $V=\bigoplus_{i=1}^{M}\mathcal{O}_X(*D)\,e_i$ with a flat connection

$$\nabla\boldsymbol{e}=\boldsymbol{e}\Big(\mathrm{d}\mathfrak{a}+\sum_{i=1}^{\ell}(\alpha_i\,I_M+N_i)\frac{\mathrm{d}z_i}{z_i}\Big),$$

where I_M denotes the identity matrix, N_i ($i = 1, \dots, \ell$) are mutually commuting nilpotent matrices, α_i are complex numbers, and we put

$$\boldsymbol{e} := (e_1, \dots, e_n) \quad \text{and} \quad \mathfrak{a} := \prod_{i=1}^{m} z_i^{-m_i}.$$

Then, it is easy to observe that $\mathcal{L}^{<D_1 \leq D_2}$ is naturally isomorphic to the 0-th cohomology of $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V)[-d_X]$. Hence, it is enough to show the vanishing of the higher cohomology of $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V)[-d_X]$. It is enough to consider the case $\operatorname{rank} V = 1$, and we put $v = e_1$.

First, let us consider the case $D_1 = D$. For a subset $J \subset \{1, \dots, n\}$, we set

$$\mathrm{d}z_J = \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_k}.$$

For a section ω of $\Omega^{\bullet <D}_{\widetilde{X}(D)}$, we have the unique decomposition $\omega = \sum \omega_J \mathrm{d}z_J$, where $\omega_J \in \mathcal{A}^{\leq D}_{\widetilde{X}(D)}$. Let S_i ($i = 1, \dots, \ell$) be a small sector in $\Delta^*_{z_i}$, and let U be a small neighbourhood of $(0, \dots, 0)$ in $\prod_{i=\ell+1}^{n} \Delta_{z_i}$, such that the closure $\overline{S}$ of $S := \prod S_i \times U$ in $\widetilde{X}(D)$ is a neighbourhood of P. In the following, we will shrink S without mention. It is easy to observe that it is enough to consider the case $\alpha_i = 0$ ($i = 1, \dots, \ell$).

Take $h = 1, \dots, n$. Assume $\nabla(\omega\, v) = 0$ for some section ω of $\Omega^{\bullet <D}_{\widetilde{X}(D)}$ on S such that $\omega_J = 0$ unless $J \subset \{1, \dots, h\}$. We have $\mathrm{d}(\exp(\mathfrak{a})\omega) = 0$. For the expression

$$\exp(\mathfrak{a})\, \omega = \sum_{h \notin J} f_J \mathrm{d}z_h \mathrm{d}z_J + \sum_{h \notin J} f_J \mathrm{d}z_J,$$

we set

$$\tau(\boldsymbol{z}) = \sum_{h \notin J} \exp(-\mathfrak{a}) \Big(\int_{\gamma(\boldsymbol{z})} f_J \mathrm{d}z_h \Big) \mathrm{d}z_J,$$

where $\gamma(\boldsymbol{z})$ is a path taken as follows:

- ▷ If $h \leq m$, the condition is similar to that for the path in (75).
- ▷ If $m < h$, γ is a path connecting $(z_1, \dots, z_{h-1}, 0, z_{h+1}, \dots, z_n)$ and $(z_1, \dots, z_n)$.

By using Lemma 5.1.4, we obtain that $\tau \in \Omega^{\bullet <D}_{\widetilde{X}(D)} \otimes V$. By a formal computation, we can show that $\omega\, v - \nabla(\tau\, v)$ does not contain $\mathrm{d}z_j$ for $j \geq h$. Hence, we can show the vanishing of the higher cohomology of $\Omega^{\bullet <D}_{\widetilde{X}(D)} \otimes V$ by an induction.

We have the decomposition $I_1 \sqcup I_2 = \underline{\ell}$ such that $D_j = \bigcup_{i \in I_j} \{z_i = 0\}$. Let us consider $\Omega^{\bullet <D(J^c) \leq D(J)}_{\widehat{\pi^{-1}(D_J)}} \otimes V$ for any subset $J \subset I_2$, where $J^c := \underline{\ell} \setminus J$.

If $\underline{m}\cap J\neq\varnothing$, it is easy to show that $\Omega^{\bullet<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}}\otimes V$ is acyclic by a formal computation. Assume $\underline{m}\cap J=\varnothing$. Let $V_J=\mathcal{O}_{D_J}(*\partial D_J)\,v_J$ be equipped with the flat connection

$$\nabla v_J=v_J\cdot d\mathfrak{a}_{|D_J}$$

on D_J. Let q_J be the projection $\pi^{-1}(D_J)\to\widetilde{D}_J(\partial D_J)$. Then, it is easy to obtain by a formal computation a natural quasi-isomorphism

$$q_J^{-1}(\Omega^{\bullet<\partial D_J}_{\widetilde{D}_J(\partial D_J)}\otimes V_J)\simeq\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}}\otimes V.$$

Hence, we obtain the vanishing of the higher cohomology of $\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}}\otimes V$.

We put $h:=|I_2|$. Let $\mathcal{G}^\bullet_h$ denote the kernel of the surjection

$$\Omega^{\bullet<D_1\leq D_2}_{\widetilde{X}(D)}\otimes V\longrightarrow\Omega^{\bullet\,<D_1\leq D_2}_{\widehat{\pi^{-1}(D_{I_2})}}\otimes V.$$

Inductively, let $\mathcal{G}^\bullet_k$ be the kernel of the surjection

$$\mathcal{G}^\bullet_{k+1}\longrightarrow\bigoplus_{\substack{J\subset I_2\\|J|=k}}\Omega^{\bullet\,<D(J^c)\leq D(J)}_{\widehat{\pi^{-1}(D_J)}}\otimes V.$$

Because $\mathcal{G}^\bullet_1=\Omega^{\leq D}_{\widetilde{X}(D)}\otimes V$, we obtain the vanishing of the higher cohomology by an induction on k. Thus, the proof of Proposition 5.1.3 is finished. □

Similarly, we also obtain the following (see also [**54**]).

PROPOSITION 5.1.5. — *The natural inclusion $\mathcal{L}^{\leq D}[d_X]\to\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(D)}(V)$ is an isomorphism in $D^b_c(\mathbb{C}_{\widetilde{X}(D)})$.* □

5.1.3. Isomorphisms. — Let X and D be as in the beginning of §5.1.1. Let H be hypersurfaces of X contained in D_1. We have the naturally defined projection $\rho:\widetilde{X}(D)\to\widetilde{X}(H)$.

LEMMA 5.1.6. — *For any good meromorphic flat bundle V on (X,D), the following natural morphisms are isomorphisms:*

$$R\rho_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)\xleftarrow{a_1}\mathrm{DR}^{<D_1}_{\widetilde{X}(H)}(V)\xleftarrow{a_2}\mathrm{DR}^{<D_1}_{\widetilde{X}(H)}\big(V(!D_1)\big)\xrightarrow{a_3}\mathrm{DR}^{<H}_{\widetilde{X}(H)}\big(V(!D_1)\big).\tag{76}$$

Proof. — The claim for a_1 follows from Theorem 4.3.2. The claim for a_2 is clear. Let us look at a_3. We use an induction on $\dim X$ and the number of the irreducible components of $D_1\setminus H$. We may assume $X=\Delta^n$ and

$D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. We set $L_i:=\{z_i=0\}$. We may assume $D_1=\bigcup_{i=1}^{\ell_1}L_i$, $H=\bigcup_{i=1}^{m_1}L_i$ and $D_2=\bigcup_{i=\ell_1+1}^{\ell_1+m_1}L_i$. We set

$$D_3:=\bigcup_{i=2}^{\ell_1}\{z_i=0\}.$$

We set $X':=L_1$ and $D_2':=D_2\cap X'$. We set

$$D_3':=X'\cap D_3 \quad\text{and}\quad H':=X'\cap\bigcup_{i=2}^{m_1}L_i.$$

Let $\iota: X'\to X$ denote the inclusion. There exist good meromorphic flat bundles V_3' and V_3'' with the exact sequence

$$0\longrightarrow \iota_\dagger V_3'(!D_3')\longrightarrow V(!D_1)\overset{c}{\to} V(!D_3)\longrightarrow \iota_\dagger V_3''(!D_3')\longrightarrow 0.$$

Let $\mathcal{K}$ denote the image of c. We have the following:

$$\begin{array}{ccccccc}
0\to & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(\iota_\dagger V_3'(!D_3')\big) & \longrightarrow & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(V(!D_1)\big) & \longrightarrow & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}(\mathcal{K}) & \to 0\\
 & \big\downarrow & & \big\downarrow & & \big\downarrow & \\
0\to & \mathrm{DR}_{\widetilde{X}(H)}\big(\iota_\dagger V_3'(!D_3')\big) & \longrightarrow & \mathrm{DR}_{\widetilde{X}(H)}\big(V(!D_1)\big) & \longrightarrow & \mathrm{DR}_{\widetilde{X}(H)}(\mathcal{K}) & \to 0,
\end{array}$$

$$\begin{array}{ccccccc}
0\to & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}(\mathcal{K}) & \longrightarrow & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(V(!D_3)\big) & \longrightarrow & \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(\iota_\dagger V_3''(!D_3')\big) & \to 0\\
 & \big\downarrow & & \big\downarrow & & \big\downarrow & \\
0\to & \mathrm{DR}_{\widetilde{X}(H)}(\mathcal{K}) & \longrightarrow & \mathrm{DR}_{\widetilde{X}(H)}(V(!D_3)) & \longrightarrow & \mathrm{DR}_{\widetilde{X}(H)}\big(\iota_\dagger V_3''(!D_3')\big) & \to 0.
\end{array}$$

By using the inductive assumption, we obtain that

$$\mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(V(!D_1)\big)\longrightarrow \mathrm{DR}_{\widetilde{X}(H)}\big(V(!D_1)\big)$$

is a quasi-isomorphism. Because we have $\mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(V(!D_1)\big)\simeq \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}(V(!L_1))$ and $\mathrm{DR}^{<D_1}_{\widetilde{X}(H)}(V(!D_1))\simeq \mathrm{DR}^{<D_1}_{\widetilde{X}(H)}(V(!L_1))$, it is enough to prove the natural morphism

$$\mathrm{DR}^{<D_1}_{\widetilde{X}(H)}\big(V(!L_1)\big)\longrightarrow \mathrm{DR}^{<D_3}_{\widetilde{X}(H)}\big(V(!L_1)\big) \tag{77}$$

is a quasi-isomorphism.

Let $I\subset\{1,\dots,\ell\}=:\underline{\ell}$ be any subset with $1\in I$. Let $\pi_H:\widetilde{X}(H)\to X$ denote the projection. We set

$$L_I:=\bigcap_{i\in I}L_i \quad\text{and}\quad \partial L_I:=L_I\cap\bigcup_{j\in\underline{\ell}\setminus I}L_j.$$

Lemma 5.1.7. — $\mathrm{DR}^{<\partial L_I}_{\widehat{\pi_I^{-1}(L_I)}}(V(!L_1))=0$.

Proof. — By using the pull back and the push-forward with respect to a ramified covering, we may assume that V is unramifiedly good. Let $\mathcal{I} \subset M(X,D)/H(X)$ denote the set of irregular values of V. We set

$$L(I^c) := \bigcup_{j\in \underline{\ell}\setminus I} L_j.$$

Let $\mathcal{I}_I$ denote the image of $\mathcal{I}$ in $M(X,D)/M(X,L(I^c))$. For each $[\mathfrak{a}] \in \mathcal{I}_I$, we fix a representative $\mathfrak{a}$ in $M(X,D)$. There exist meromorphic $\mathcal{O}_{\widehat{L}_I}(*\partial D)$-subbundles $\widehat{V}_{[\mathfrak{a}]}$ of $V_{|\widehat{L}_I}$ stable by the connection and a decomposition

$$V_{|\widehat{L}_I} = \bigoplus_{[\mathfrak{a}]\in\mathcal{I}_I} \widehat{V}_{[\mathfrak{a}]}$$

compatible with the connection, such that $\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}} := \widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}\,\mathrm{id}_{\widehat{V}_{[\mathfrak{a}]}}$ are regular along L_i ($i \in I$), where $\widehat{\nabla}_{\mathfrak{a}}$ denotes the induced connection on $\widehat{V}_{[\mathfrak{a}]}$.

Let $j \in I$. Suppose $\mathrm{ord}_{z_j}\mathfrak{a} < 0$. We consider the Deligne-Malgrange filtration $\mathcal{P}_*$ on $\widehat{V}_{[\mathfrak{a}]}$. (See [**45**] for a survey.) We have

$$(\partial_j\mathfrak{a})^{-1}\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a},\partial_j}\mathcal{P}_{\boldsymbol{b}}\widehat{V}_{[\mathfrak{a}]} \subset \mathcal{P}_{\boldsymbol{b}}\widehat{V}_{[\mathfrak{a}]}$$

for any $\boldsymbol{b} \in \mathbb{R}^{\ell}$. Hence we obtain that $\widehat{\nabla}_{\mathfrak{a},\partial_j}$ is invertible on $\mathcal{C}^{\infty<\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widehat{V}_{[\mathfrak{a}]}$. Suppose moreover that $j \neq 1$ and that $\mathrm{ord}_{z_1}(\mathfrak{a}) = 0$. Let $\leq$ denote the total order on $\mathbb{C}$ defined by the lexicographic order on $(\mathrm{Re}(\alpha),\mathrm{Im}(\alpha)) \in \mathbb{R}\times\mathbb{R}$. We have the V-filtration $\widetilde{\mathcal{P}}$ of $\widehat{V}_{[\mathfrak{a}]}$ along z_1 indexed by $(\mathbb{C},\leq)$ such that

(i) $z_1\widehat{\nabla}_{\mathfrak{a},\partial_1}$ preserves the filtration $\widetilde{\mathcal{P}}$

(ii) the endomorphisms of $\mathrm{Gr}^{\widetilde{\mathcal{P}}}_{\beta}(\widehat{V}_{[\mathfrak{a}]})$ induced by $-\widehat{\nabla}_{\mathfrak{a},\partial_1}z_1 - \beta$ are nilpotent for any β.

The induced morphisms $\widehat{\nabla}_{\mathfrak{a},\partial_1} : \mathrm{Gr}^{\widetilde{\mathcal{P}}}_{\beta}(\widehat{V}_{[\mathfrak{a}]}) \to \mathrm{Gr}^{\widetilde{\mathcal{P}}}_{\beta+1}(\widehat{V}_{[\mathfrak{a}]})$ are isomorphisms unless $\beta = -1$. We can observe that the filtration $\widetilde{\mathcal{P}}$ is preserved by $\widehat{\nabla}_{[\mathfrak{a}],\partial_j}$ and the multiplication of $\partial_j\mathfrak{a}$. Hence, $\widehat{\nabla}_{\mathfrak{a},\partial_j}$ is invertible on

$$\mathcal{C}^{\infty<\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \mathcal{P}_a\widehat{V}_{[\mathfrak{a}]} \quad\text{and}\quad \mathcal{C}^{\infty<\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \mathrm{Gr}^{\mathcal{P}}_a\,\widehat{V}_{[\mathfrak{a}]}.$$

Suppose $\mathrm{ord}_{z_j}\mathfrak{a} = 0$ for any $j \in I$, i.e., $[\mathfrak{a}] = [0]$. For the Deligne-Malgrange filtration $\mathcal{P}_*$ of $\widehat{V}_{[0]}$, we have

$$\nabla_{[0],\partial_1}\big(\mathcal{P}_{\boldsymbol{b}}\widehat{V}_{[0]}(*\partial L_I)\big) \subset \mathcal{P}_{\boldsymbol{b}+(1,0,\dots,0)}\widehat{V}_{[0]}(*\partial L_I).$$

For the V-filtration $\widetilde{\mathcal{P}}$ along z_1, we obtain that if $\beta < -1$, the following morphism is an isomorphism:

$$\widehat{\nabla}_{0,\partial_1} : \mathcal{C}^{\infty <\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widetilde{\mathcal{P}}_\beta(\widehat{V}_{[0]}) \longrightarrow \mathcal{C}^{\infty <\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widetilde{\mathcal{P}}_{\beta+1}(\widehat{V}_{[0]}).$$

We have the decomposition

$$V(!L_1)_{|\widehat{L}_I} = \bigoplus_{[\mathfrak{a}]} \widehat{V(!L_1)}_{[\mathfrak{a}]},$$

compatible with the decomposition of $V_{|\widehat{L}_I}$. If $\operatorname{ord}_{z_1} \mathfrak{a} < 0$, we have

$$\widehat{V(!L_1)}_{[\mathfrak{a}]} = \widehat{V}_{[\mathfrak{a}]}.$$

The action of $\widehat{\nabla}_{\mathfrak{a},\partial_1}$ on $\mathcal{C}^{\infty <\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widehat{V(!L_1)}_{[\mathfrak{a}]}$ is invertible. If $\operatorname{ord}_{z_1} \mathfrak{a} = 0$, for the V-filtration $\widetilde{\mathcal{P}}$ along z_1, we have $\widetilde{\mathcal{P}}_\beta(\widehat{V(!L_1)}_{[\mathfrak{a}]}) = \widetilde{\mathcal{P}}_\beta(\widehat{V}_{[\mathfrak{a}]})$ for $\beta < 0$, and that

$$\widehat{\nabla}_{\mathfrak{a},\partial_1} : \operatorname{Gr}^{\widetilde{\mathcal{P}}}_\beta(\widehat{V(!L_1)}_{[\mathfrak{a}]}) \longrightarrow \operatorname{Gr}^{\widetilde{\mathcal{P}}}_{\beta+1}(\widehat{V(!L_1)}_{[\mathfrak{a}]})$$

are isomorphisms for $\beta \geq -1$. If $[\mathfrak{a}] \neq [0]$, take $j \in I$ such that $\operatorname{ord}_{z_j} \mathfrak{a} < 0$, and then the action of $\widehat{\nabla}_{\mathfrak{a},\partial_j}$ on $\mathcal{C}^{\infty <\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widehat{V(!L_1)}_{[\mathfrak{a}]}$ is invertible. If $[\mathfrak{a}] = [0]$, the action of $\widehat{\nabla}_{0,\partial_1}$ on $\mathcal{C}^{\infty <\partial L_I}_{\widehat{\pi_H^{-1}(L_I)}} \otimes \widehat{V(!L_1)}_{[\mathfrak{a}]}$ is invertible. Then, the claim of Lemma 5.1.7 follows. □

Then, by an easy inductive argument, we obtain that (77) is a quasi-isomorphism, and the proof of Lemma 5.1.6 is finished. □

Suppose that we are given a holomorphic function $G : X \to \mathbb{C}^\ell$ such that $G^{-1}(D_0) = H$, where $D_0 = \bigcup_{i=1}^{\ell} \{z_i = 0\}$.

Lemma 5.1.8. — *For the naturally defined map $\rho_1 : \widetilde{X}(D) \to \widetilde{X}(G)$, we obtain the natural isomorphism*

$$R\rho_{1*} \operatorname{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V) \simeq \operatorname{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big). \tag{78}$$

Proof. — It follows from Lemma 5.1.6 and Proposition 4.7.4. □

Let $\varphi : X' \to X$ be a projective birational morphism such that:

(i) $D' := \varphi^{-1}(D)$ is normal crossing,

(ii) $X' \setminus D' \simeq X \setminus D$.

We put $D_1' := \varphi^{-1}(D_1)$ and $H_1' := \varphi^{-1}(H_1)$. Let D_2' be the complement of D_1' in D'. We set $G' := G \circ \varphi$. We put $V' := \varphi^* V$. We have the natural commutative diagram:

$$\begin{array}{ccc} \widetilde{X}'(D') & \xrightarrow{\widetilde{\varphi}_1} & \widetilde{X}(D) \\ {\scriptstyle \rho_1'}\big\downarrow & & {\scriptstyle \rho_1}\big\downarrow \\ \widetilde{X}'(G') & \xrightarrow{\widetilde{\varphi}} & \widetilde{X}(G). \end{array}$$

We set $\rho_2 := \widetilde{\varphi} \circ \rho_1'$. Correspondingly, we have the commutative diagram of isomorphisms by the construction:

$$\begin{array}{ccc} R\rho_{2*}\, \mathrm{DR}^{<D_1' \leq D_2'}_{\widetilde{X}'(D')}(V') & \longrightarrow & R\rho_{1*}\, \mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ R\widetilde{\varphi}_*\, \mathrm{DR}^{\mathrm{rapid}}_{X',G'}\big(V'(!D_1')\big) & \xrightarrow{\simeq} & \mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big). \end{array} \tag{79}$$

The lower horizontal arrow is an isomorphism according to Corollary 4.7.5.

5.2. Duality

5.2.1. Duality morphisms. — Let X, D and $\mathcal{M}$ be as in §5.1.1. We have the following natural morphism given in a way parallel to that of (14):

$$\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(\boldsymbol{D}\mathcal{M}) \longrightarrow \boldsymbol{D}\,\mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}(\mathcal{M}). \tag{80}$$

Namely, we take a $\pi^{-1}(\mathcal{D}_X)$-injective resolution $\widetilde{\mathcal{I}}_1^\bullet$ of $\Omega^{0,\bullet<D_1 \leq D_2}_{\widetilde{X}(D)}[d_X]$, and a $\mathbb{C}_{\widetilde{X}(D)}$-injective resolution $\widetilde{\mathcal{I}}_2^\bullet$ of $\mathrm{Tot}\,\Omega^{\bullet,\bullet<D}_{\widetilde{X}(D)}[2d_X]$ with a morphism $\mathrm{DR}^{\leq D_1 < D_2}_{\widetilde{X}(D)}\widetilde{\mathcal{I}}_1^\bullet \to \widetilde{\mathcal{I}}_2^\bullet$ extending a natural morphism

$$\mathrm{DR}^{\leq D_1 < D_2}_{\widetilde{X}(D)}(\Omega^{0,\bullet<D_1 \leq D_2}_{\widetilde{X}(D)}[d_X]) \longrightarrow \mathrm{Tot}\,\Omega^{\bullet,\bullet<D}_{\widetilde{X}(D)}[2d_X].$$

Then, (80) is given as the composite of the morphisms

$$\begin{aligned} &\mathcal{H}om_{\pi^{-1}(\mathcal{D}_X)}(\pi^{-1}\mathcal{M}, \widetilde{\mathcal{I}}_1^\bullet) \\ &\qquad \longrightarrow \mathcal{H}om_{\mathbb{C}_{\widetilde{X}(D)}}(\mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}\mathcal{M}, \mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}\widetilde{\mathcal{I}}_1) \\ &\qquad\qquad \longrightarrow \mathcal{H}om_{\mathbb{C}_{\widetilde{X}(D)}}(\mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}\mathcal{M}, \widetilde{\mathcal{I}}_2). \end{aligned} \tag{81}$$

PROPOSITION 5.2.1. — *The following diagram is commutative:*

$$
\begin{array}{ccc}
R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\boldsymbol{D}\mathcal{M}) & \longrightarrow & R\pi_*\boldsymbol{D}\operatorname{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(\mathcal{M}) \\
\simeq\downarrow & & \simeq\downarrow \\
\operatorname{DR}^{<D_1\leq D_2}_{X}(\boldsymbol{D}\mathcal{M}) & \longrightarrow & \boldsymbol{D}\operatorname{DR}^{<D_2\leq D_1}_{X}(\mathcal{M}).
\end{array} \tag{82}
$$

The upper horizontal arrow is induced by (80)*, the lower horizontal arrow is given as in* (14)*, the left vertical arrow is given in* (71)*, and the right vertical arrow is given by*

$$R\pi_*\boldsymbol{D}\operatorname{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\mathcal{M} \simeq \boldsymbol{D}R\pi_*\operatorname{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\mathcal{M} \simeq \boldsymbol{D}\operatorname{DR}^{<D_2\leq D_1}_{X}(\mathcal{M}).$$

Proof. — We have a morphism $R\pi_* \operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(\boldsymbol{D}\mathcal{M}) \to \operatorname{DR}^{<D_1\leq D_2}_{X}(\boldsymbol{D}\mathcal{M})$ given as follows, by Lemma 5.1.1:

$$
\begin{aligned}
R\pi_* R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)})[d_X] &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\pi_*\Omega^{0,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)})[d_X] \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega^{0,\bullet}_{X}(*D_2)^{<D_1})[d_X].
\end{aligned} \tag{83}
$$

It is equal to the morphism obtained as in (71). Then, the claim of the proposition can be checked easily. ∎

5.2.2. The case of good meromorphic flat bundles. — Let us consider the case where $\mathcal{M}$ is a good meromorphic flat bundle V on (X, D).

THEOREM 5.2.2. — *The duality morphism*

$$\operatorname{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\boldsymbol{D}V \longrightarrow \boldsymbol{D}\operatorname{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}V$$

is an isomorphism.

Proof. — We begin with elementary preparations. Let $\mathbb{R}^2 = S_0 \cup S_1 \cup S_2$ be a decomposition given as follows:

$$S_0 := \big\{(x,y)\,\big|\, y \geq 0\big\},\ \ S_1 := \big\{(x,y)\,\big|\, y \leq 0,\ x \leq 0\big\},$$
$$S_2 := \big\{(x,y)\,\big|\, y \leq 0,\ x \geq 0\big\}.$$

We put

$$X_1 := (\mathbb{R}\times S_1)\cup(\mathbb{R}_{\geq 0}\times S_0),\quad X_2 := (\mathbb{R}\times S_2)\cup(\mathbb{R}_{\leq 0}\times S_0).$$

The following lemma is easy to see.

LEMMA 5.2.3. — *$X_i \subset \mathbb{R}^3$ $(i = 1, 2)$ are closed C^0-submanifolds with boundaries. We have $X_1 \cup X_2 = \mathbb{R}^3$ and $X_1 \cap X_2 = \partial X_i$.* ∎

We put $\mathcal{J} :=]-1,1[$, $\mathcal{J}_+ := [0,1[$, $\mathcal{J}_- :=]-1,0]$, and $\mathcal{I}_i := [0,1[$ $(i=1,2,3)$. We have a homeomorphism $\partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3) \simeq \mathbb{R}^2$, and we can identify the decomposition

$$\partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3) = (\partial\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3) \cup (\mathcal{I}_1 \times \partial\mathcal{I}_2 \times \mathcal{I}_3) \cup (\mathcal{I}_1 \times \mathcal{I}_2 \times \partial\mathcal{I}_3)$$

with $\mathbb{R}^2 = S_0 \cup S_1 \cup S_2$. We put

$$X'_1 := (\mathcal{J} \times \mathcal{I}_1 \times \partial\mathcal{I}_2 \times \mathcal{I}_3) \cup (\mathcal{J}_+ \times \partial\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3),$$
$$X'_2 := (\mathcal{J} \times \mathcal{I}_1 \times \mathcal{I}_2 \times \partial\mathcal{I}_3) \cup (\mathcal{J}_- \times \partial\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3).$$

They are closed subsets of $\mathcal{J} \times \partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3)$. We obtain the following lemma from Lemma 5.2.3.

Lemma 5.2.4. — *$X'_i \subset \mathcal{J} \times \partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3)$ are C^0-submanifolds with boundaries. We have $X'_1 \cup X'_2 = \mathcal{J} \times \partial(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3)$ and $X'_1 \cap X'_2 = \partial X'_i$.* □

We recall some elementary facts on constructible sheaves. Let Y be an oriented ℓ-dimensional C^0-manifold with the boundary ∂Y. For a closed C^0-submanifold $W \subset \partial Y$ with boundary such that $\dim W = \ell - 1$, let $j_W : Y - W \to Y$ denote the inclusion. We have the natural isomorphisms

$$R\mathcal{H}om_{\mathbb{C}_Y}(j_{W!}\mathbb{C}_{Y-W}, K) \simeq Rj_{W*}R\mathcal{H}om_{\mathbb{C}_{Y-W}}(\mathbb{C}_{Y-W}, Rj_W^! K) \simeq Rj_{W*}j_W^* K.$$

The dualizing complex ω_Y of Y is given by $j_{\partial Y!}\mathbb{C}_{Y-\partial Y}[\ell]$.

Lemma 5.2.5. — *Let $Y_i \subset \partial Y$ be closed C^0-submanifolds with boundaries such that $Y_1 \cup Y_2 = Y$ and $Y_1 \cap Y_2 = \partial Y_i$. Then, we have*

$$\boldsymbol{D} j_{Y_1!}\mathbb{C}_{Y-Y_1} \simeq j_{Y_2!}\mathbb{C}_{Y-Y_2}.$$

Proof. — The left hand side is naturally isomorphic to

$$j_{Y_1*}j_{Y_1}^*\omega_Y \simeq j_{Y_1*}j_{0!}\mathbb{C}_{Y-\partial Y}[\ell],$$

where j_0 denotes the inclusion $Y - \partial Y \to Y - Y_1$. Then, we can check the claim directly. □

Let us return to the proof of Theorem 5.2.2. It is enough to consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. As in the proof of Proposition 5.1.3, we can reduce the issue to the case where $V = \mathcal{O}_X(*D)\,v$ with a meromorphic flat connection $\nabla v = v\,d\mathfrak{a}$, where $\mathfrak{a} = \prod_{i=1}^m z_i^{-m_i}$ $(m_i > 0)$. We put

$$F_{\mathfrak{a}} := -|\mathfrak{a}|^{-1}\,\mathrm{Re}\,\mathfrak{a}.$$

We have the decomposition $I_1 \sqcup I_2 = \underline{\ell}$ such that $D_j = \bigcup_{i \in I_j}\{z_i = 0\}$ $(j = 1,2)$. We set

$$I_j(>m) := \bigl\{i \in I_j \bigm| i > m\bigr\}$$

and
$$D(>m):=\bigcup_{i=m+1}^{\ell}\{z_i=0\},\quad D(\leq m):=\bigcup_{i=1}^{m}\{z_i=0\}.$$
We consider the closed subsets $W_i\subset\pi^{-1}(D)$ $(i=1,2)$ given as follows:
$$W_1:=\pi^{-1}\big(D_1\cap D(>m)\big)\cup\big[\pi^{-1}(D(\leq m))\cap\{F_{\mathfrak{a}}\geq 0\}\big],$$
$$W_2:=\pi^{-1}\big(D_2\cap D(>m)\big)\cup\big[\pi^{-1}(D(\leq m))\cap\{F_{\mathfrak{a}}\leq 0\}\big].$$

Lemma 5.2.6. — *$W_i\subset\pi^{-1}(D)$ are closed C^0-submanifolds with boundaries, and we have $W_1\cup W_2=\pi^{-1}(D)$ and $W_1\cap W_2=\partial W_i$.*

Proof. — It is easy to observe that it is enough to consider the case $n=\ell$. We have the natural identification $\widetilde{X}(D)\simeq(S^1)^\ell\times\mathbb{R}^\ell_{\geq 0}$. By the decomposition
$$\underline{\ell}=\underline{m}\sqcup I_1(>m)\sqcup I_2(>m),$$
we identify $\mathbb{R}^\ell_{\geq 0}=\mathbb{R}^m_{\geq 0}\times\mathbb{R}^{I_1(>m)}_{\geq 0}\times\mathbb{R}^{I_2(>m)}_{\geq 0}$.

We argue the case $I_j(>m)\neq\varnothing$ $(j=1,2)$. The other cases are easier. We fix homeomorphisms
$$\mathbb{R}^m_{\geq 0}\simeq\mathcal{I}_1\times\mathbb{R}^{m-1},\ \ \mathbb{R}^{I_1(>m)}_{\geq 0}\simeq\mathcal{I}_2\times\mathbb{R}^{|I_1(>m)|-1},\ \ \mathbb{R}^{I_2(>m)}_{\geq 0}\simeq\mathcal{I}_3\times\mathbb{R}^{|I_2(>m)|-1}.$$
We put $N:=m+|I_1(>m)|+|I_2(>m)|-3$. Let $H_\pm$ be the subsets of $(S^1)^\ell$ given as
$$H_+:=\Big\{\cos\Big(\sum m_i\theta_i\Big)\geq 0\Big\}\quad\text{and}\quad H_-:=\Big\{\cos\Big(\sum m_i\theta_i\Big)\leq 0\Big\}.$$
Then, $\pi^{-1}(D)$ is identified with $(S^1)^\ell\times\partial(\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)\times\mathbb{R}^N$, under which we have
$$W_1\simeq\big(((S^1)^\ell\times\mathcal{I}_1\times\partial\mathcal{I}_2\times\mathcal{I}_3)\cup(H_-\times\partial\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)\big)\times\mathbb{R}^N,$$
$$W_2=\big(((S^1)^\ell\times\mathcal{I}_1\times\mathcal{I}_2\times\partial\mathcal{I}_3)\cup(H_+\times\partial\mathcal{I}_1\times\mathcal{I}_2\times\mathcal{I}_3)\big)\times\mathbb{R}^N.$$
For $Q\in H_+\cap H_-$, we can take a neighbourhood U_Q such that $U\simeq\mathcal{J}\times\mathbb{R}^{\ell-1}$ under which $H_\pm\cap U_Q=\mathcal{J}_\pm\times\mathbb{R}^{\ell-1}$. Then, we obtain Lemma 5.2.6 from Lemma 5.2.4. □

Let $j_{W_i}:\widetilde{X}(D)\setminus W_i\to\widetilde{X}(D)$ be the inclusion. Let $\mathcal{L}$ and $\mathcal{L}^\vee$ be the local systems on $\widetilde{X}(D)$ associated to V and $V^\vee$, respectively. According to the description of $\mathcal{L}^{<D_1\leq D_2}$ and $\mathcal{L}^{\vee<D_2\leq D_1}$, we have the natural isomorphisms:
$$\mathcal{L}^{<D_1\leq D_2}\simeq j_{W_1!}(\mathcal{L}_{\widetilde{X}(D)\setminus W_1}),\quad\mathcal{L}^{\vee<D_2\leq D_1}\simeq j_{W_2!}(\mathcal{L}^\vee_{\widetilde{X}(D)\setminus W_2}).$$

Lemma 5.2.5 gives an isomorphism $\boldsymbol{D}(\mathcal{L}^{<D_1\leq D_2}[d_X]) \simeq \mathcal{L}^{\vee\,<D_2\leq D_1}[d_X]$. It is uniquely determined by its restriction to $X-D$. Then, we can deduce that

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\,\boldsymbol{D}V \longrightarrow \boldsymbol{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\,V$$

is an isomorphism. Thus, the proof of Theorem 5.2.2 is finished. □

Corollary 5.2.7. — *For any good meromorphic flat bundle V on (X,D), we have the commutative diagram of the isomorphisms:*

$$\begin{array}{ccc} R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}\,\boldsymbol{D}V & \xrightarrow{\ \simeq\ } & R\pi_*\boldsymbol{D}\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}\,V \\ {\scriptstyle\simeq}\big\downarrow & & {\scriptstyle\simeq}\big\downarrow \\ \mathrm{DR}_X\,V^{\vee}(!D_1) & \xrightarrow{\ \simeq\ } & \boldsymbol{D}\,\mathrm{DR}_X\,V(!D_2). \end{array}$$

Proof. — It follows from Theorem 3.2.4, Proposition 5.2.1 and Theorem 5.2.2. □

5.3. Functoriality

Let X be a complex manifold, and let D be a normal crossing hypersurface with a decomposition $D = D_1 \cup D_2$. Let D_3 be a hypersurface of X. Let $\varphi : X' \to X$ be a proper birational morphism such that

(i) $D' := \varphi^{-1}(D \cup D_3)$ is normal crossing,

(ii) $X' \setminus D' \simeq X \setminus (D \cup D_3)$.

Let $\widetilde{X}(D) \to X$ and $\widetilde{X}'(D') \to X'$ be the real blow up. Both the projections are denoted by π. Let $\widetilde{\varphi} : \widetilde{X}'(D') \to \widetilde{X}(D)$ be the induced map. We put $D'_1 := \varphi^{-1}(D_1)$. We have $D'_2 \subset D'$ such that $D' = D'_1 \cup D'_2$ is a decomposition. Let V be a meromorphic flat bundle on (X,D). We set $V' := \varphi^*(V) \otimes \mathcal{O}_{X'}(*D')$.

Theorem 5.3.1. — *We have in $D^b_c(\mathbb{C}_{\widetilde{X}(D)})$ a morphism*

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V) \longrightarrow R\widetilde{\varphi}_*\,\mathrm{DR}^{<D'_1\leq D'_2}_{\widetilde{X}'(D')}(V')$$

such that the following diagram of perverse sheaves is commutative:

$$\begin{array}{ccc} R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V) & \longrightarrow & R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D'_1\leq D'_2}_{\widetilde{X}(D')}(V') \\ {\scriptstyle\simeq}\big\downarrow & & {\scriptstyle\simeq}\big\downarrow \\ \mathrm{DR}_X\big(V(!D_1)\big) & \longrightarrow & R\varphi_*\,\mathrm{DR}_{X'}\big(V'(!D'_1)\big). \end{array} \tag{84}$$

Here, the vertical isomorphisms are given by (71) *and* (12), *and the lower horizontal arrow is induced by the morphism of $\mathcal{D}$-modules $V(!D_1) \to \varphi_\dagger V'(!D'_1)$.*

Similarly, we have a morphism

$$R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_2'\leq D_1'}_{\widetilde{X}'(D')}(V')\longrightarrow \mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(V)$$

such that the following diagram of perverse sheaves is commutative:

$$\begin{array}{ccc} R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_2'\leq D_1'}_{\widetilde{X}'(D')}(V') & \longrightarrow & R\pi_*\,\mathrm{DR}^{<D_2\leq D_1}_{\widetilde{X}(D)}(V) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ R\varphi_*\,\mathrm{DR}_X\big(V'(!D_2')\big) & \longrightarrow & \mathrm{DR}_X\big(V(!D_2)\big). \end{array} \tag{85}$$

Proof. — We have a naturally induced morphism

$$\widetilde{\varphi}^{-1}(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\otimes\pi^{-1}V)\longrightarrow\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{\widetilde{X}'(D')}\otimes\pi^{-1}V'. \tag{86}$$

It induces a morphism of cohomologically constructible complexes

$$\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V)\longrightarrow\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}'(D')}(V'). \tag{87}$$

We can directly check the commutativity of the diagram:

$$\begin{array}{ccc} \Omega^{\bullet,\bullet\,<D_1\leq D_2}_X\otimes V & \longrightarrow & \varphi_*(\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{X'}\otimes V') \\ \big\downarrow & & \big\downarrow \\ \pi_*(\Omega^{\bullet,\bullet\,<D_1\leq D_2}_{\widetilde{X}(D)}\otimes\pi^{-1}V) & \longrightarrow & \pi_*(\widetilde{\varphi}_*\Omega^{\bullet,\bullet\,<D_1'\leq D_2'}_{\widetilde{X}'(D')}\otimes\pi^{-1}V'). \end{array}$$

It implies the commutativity of the diagram

$$\begin{array}{ccc} R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V) & \longrightarrow & R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}'(D')}(V') \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathrm{DR}^{<D_1\leq D_2}_X(V) & \longrightarrow & R\varphi_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{X'}(V'). \end{array} \tag{88}$$

Then, we obtain the commutativity of (84) from Theorem 3.2.5.

Considering the dual of (87) with $V^\vee$ (see Theorem 5.2.2), we obtain the morphism:

$$R\widetilde{\varphi}_*\,\mathrm{DR}^{\leq D_1'<D_3'}_{\widetilde{X}'(D')}(V')\longrightarrow\mathrm{DR}^{\leq D_1<D_2}_{\widetilde{X}(D)}(V). \tag{89}$$

Let us prove the commutativity of the diagram (85). From (88) for $V^\vee$, we obtain the commutative diagram:

$$\begin{array}{ccc} \boldsymbol{D}R\pi_*R\widetilde{\varphi}_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{\widetilde{X}'(D')}(V'^\vee) & \longrightarrow & \boldsymbol{D}R\pi_*\,\mathrm{DR}^{<D_1\leq D_2}_{\widetilde{X}(D)}(V^\vee) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \boldsymbol{D}R\varphi_*\,\mathrm{DR}^{<D_1'\leq D_2'}_{X'}(V'^\vee) & \longrightarrow & \boldsymbol{D}\,\mathrm{DR}^{<D_1\leq D_2}_X(V^\vee). \end{array}$$

By Proposition 5.2.1 and Theorem 5.2.2, we have the following commutative diagram:

$$\begin{array}{ccc} \boldsymbol{D} R\pi_* \operatorname{DR}^{<D_1\le D_2}_{\widetilde{X}(D)}(V^\vee) & \xrightarrow{\simeq} & R\pi_* \operatorname{DR}^{\le D_1 < D_2}_{\widetilde{X}(D)}(V) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \boldsymbol{D} \operatorname{DR}^{<D_1\le D_2}(V^\vee) & \xrightarrow{\simeq} & \operatorname{DR}^{<D_2\le D_1}_X(V). \end{array}$$

We have a similar diagram for V'. Then, we obtain the commutativity of (85) from the constructions of (89) and (20). □

5.4. A rigidity property (Appendix)

The author originally used Theorem 5.4.1 below for the functoriality of the Betti structure by projective morphisms. After the improvement, it is now not necessary. But, it seems interesting to the author, so we keep it. The reader can skip this subsection.

5.4.1. Statement. — We set $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let V be a good meromorphic flat bundle on (X, D). Let $\mathcal{L}$ be the associated local system on $\widetilde{X}(D)$. Let g be a holomorphic function on X such that $g^{-1}(0) = D$. We have the naturally defined morphisms:

$$\widetilde{X}(D) \xrightarrow{\pi_1} \widetilde{X}(g) \xrightarrow{\pi_0} X.$$

We put $\pi_2 := \pi_0 \circ \pi_1$. We set $\mathcal{K} := R\pi_{1*}\mathcal{L}^{\le D}$. In this subsection, we will work on the derived category of cohomologically constructible sheaves.

Theorem 5.4.1. — *The restriction* $\operatorname{Hom}(\mathcal{K},\mathcal{K}) \to \operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-D)}, \mathcal{K}_{|\pi_0^{-1}(X-D)})$ *is injective.*

We will give a consequence in §5.4.6.

5.4.2. Reduction. — We put $D^{[m]} := \bigcup_{\substack{I\subset \underline{\ell}\\ |I|=m}} D_I$. It is easy to see that

$$\operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-D^{[2]})}, \mathcal{K}_{|\pi_0^{-1}(X-D^{[2]})}) \longrightarrow \operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-D)}, \mathcal{K}_{|\pi_0^{-1}(X-D)})$$

is injective. Hence, it is enough to show for $m \ge 2$ the injectivity of the morphisms

$$\operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-D^{[m+1]})}, \mathcal{K}_{|\pi_0^{-1}(X-D^{[m+1]})}) \longrightarrow \operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-D^{[m]})}, \mathcal{K}_{|\pi_0^{-1}(X-D^{[m]})}).$$

Then, it is easy to observe that it is enough to consider the case $\ell = n$ and the morphism

$$\operatorname{Hom}(\mathcal{K},\mathcal{K}) \longrightarrow \operatorname{Hom}(\mathcal{K}_{|\pi_0^{-1}(X-O)}, \mathcal{K}_{|\pi_0^{-1}(X-O)}).$$

By the adjunction $\mathrm{Hom}(\pi_1^*\mathcal{K}, \mathcal{L}^{\leq D}) \simeq \mathrm{Hom}(\mathcal{K}, \mathcal{K})$, it is enough to show the injectivity of the morphism

$$\mathrm{Hom}(\pi_1^*\mathcal{K},\, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Hom}(\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)},\, \mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}).$$

We have $R^i\pi_{1*}\mathcal{L}^{\leq D} = 0$ unless $0 \leq i \leq n-1$, because the real dimension of the fiber is less than $n-1$. We set

$$\mathcal{K}^i := \pi_1^* R^i \pi_{1*}\mathcal{L}^{\leq D}.$$

Let $\mathfrak{j} : \pi_2^{-1}(X-O) \to \widetilde{X}(D)$ and $\mathfrak{i} : \pi_2^{-1}(O) \to \widetilde{X}(D)$.

Lemma 5.4.2. — *To prove Theorem* 5.4.1, *it is enough to prove*

$$\mathcal{E}xt^j(\mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i, \mathcal{L}^{\leq D}) = 0 \qquad (i, j \leq n-1). \tag{90}$$

Proof. — From the distinguished triangle $\mathcal{K}^i[-i] \to \tau_{\geq i}\pi_1^*\mathcal{K} \to \tau_{\geq i+1}\pi_1^*\mathcal{K} \xrightarrow{+1}$, we obtain the long exact sequence

$$\begin{aligned} \mathrm{Ext}^{i-1}(\mathcal{K}^i, \mathcal{L}^{\leq D}) &\longrightarrow \mathrm{Hom}(\tau_{\geq i+1}\pi_1^*\mathcal{K}, \mathcal{L}^{\leq D}) \\ &\longrightarrow \mathrm{Hom}(\tau_{\geq i}\pi_1^*\mathcal{K}, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Ext}^i(\mathcal{K}^i, \mathcal{L}^{\leq D}) \end{aligned} \tag{91}$$

and the corresponding long exact sequences for the restrictions to $\pi_2^{-1}(X-O)$. The injectivity of $\mathrm{Hom}(\tau_{\geq i}\pi_1^*\mathcal{K},\, \mathcal{L}^{\leq D}) \to \mathrm{Hom}(\tau_{\geq i}\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)}, \mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)})$ can follow from the injectivity of

$$\mathrm{Ext}^i(\mathcal{K}^i, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Ext}^i(\mathcal{K}^i_{|\pi_2^{-1}(X-O)},\, \mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}), \tag{92}$$

$$\mathrm{Hom}(\tau_{\geq i+1}\pi_1^*\mathcal{K},\, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Hom}(\tau_{\geq i+1}\pi_1^*\mathcal{K}_{|\pi_2^{-1}(X-O)}, \mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}), \tag{93}$$

and the surjectivity of

$$\mathrm{Ext}^{i-1}(\mathcal{K}^i, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Ext}^{i-1}(\mathcal{K}^i_{|\pi_2^{-1}(X-O)}, \mathcal{L}^{\leq D}_{|\pi_2^{-1}(X-O)}). \tag{94}$$

By an easy inductive argument, we can reduce Theorem 5.4.1 to the injectivity of (92) and the surjectivity of (94) for any $i \leq n-1$.

From the exact sequence $0 \to \mathfrak{j}_!\mathfrak{j}^*\mathcal{K}^i \to \mathcal{K}^i \to \mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i \to 0$ and the adjunction $\mathrm{Ext}^i(\mathfrak{j}_!\mathfrak{j}^*\mathcal{K}^i, \mathcal{L}^{\leq D}) \simeq \mathrm{Ext}^i(\mathfrak{j}^*\mathcal{K}^i,\, \mathfrak{j}^*\mathcal{L}^{\leq D})$, we obtain the exact sequence

$$\begin{aligned} \mathrm{Ext}^{i-1}(\mathcal{K}^i,\, \mathcal{L}^{\leq D}) &\longrightarrow \mathrm{Ext}^{i-1}(\mathfrak{j}^*\mathcal{K}^i,\, \mathfrak{j}^*\mathcal{L}^{\leq D}) \to \mathrm{Ext}^i(\mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i,\, \mathcal{L}^{\leq D}) \\ &\longrightarrow \mathrm{Ext}^i(\mathcal{K}^i, \mathcal{L}^{\leq D}) \longrightarrow \mathrm{Ext}^i(\mathfrak{j}^*\mathcal{K}^i,\, \mathfrak{j}^*\mathcal{L}^{\leq D}). \end{aligned} \tag{95}$$

Hence, the proof of Theorem 5.4.1 is reduced to the vanishing

$$\mathrm{Ext}^i(\mathfrak{i}_*\mathfrak{i}^*\mathcal{K}^i,\, \mathcal{L}^{\leq D}) = 0$$

for any $0 \leq i \leq n-1$. For that purpose, it is enough to prove (90). Thus, the proof of Lemma 5.4.2 is finished. □

In the following, we will prove $\mathcal{E}xt^i(\pi_1^{-1}(I), \mathcal{L}^{\leq D}) = 0$ $(i = 0, \dots, n-1)$ for any constructible sheaf I on $\pi_0^{-1}(O) \simeq S^1$.

5.4.3. Local form of $\pi_1^{-1}(I)$. — Let $(z_1, \dots, z_n)$ be a coordinate system with $z_i^{-1}(0) = D_i$. It induces a coordinate system $(\theta_1, \dots, \theta_n)$ of $\pi_2^{-1}(O)$, which is independent of the choice of $(z_1, \dots, z_n)$ up to parallel transport. We take a coordinate system t of $\mathbb{C}$, which induces a coordinate system θ of $\pi_0^{-1}(O)$. The induced map $\pi_2^{-1}(O) \to \pi_0^{-1}(O)$ is affine with respect to the coordinate systems $(\theta_1, \dots, \theta_n)$ and θ.

Let us consider the behaviour of $\pi_1^{-1}(I)$ around $P \in \pi_2^{-1}(O)$, where I is a constructible sheaf on $\pi_0^{-1}(O)$. We may assume $P = (0, \dots, 0)$. The map $\pi_2^{-1}(O) \to \pi_0^{-1}(O)$ is of the form $(\theta_1, \dots, \theta_n) \mapsto \sum \alpha_i \theta_i + \beta$, where $\beta = \pi_1(P)$. The sheaf I is the direct sum of sheaves of the following forms:

- ▷ the constant sheaf around β;
- ▷ $j_!\mathbb{C}_J$ or $j_*\mathbb{C}_J$, where J is an open interval such that one of the end points is β, and j denotes the inclusion $J \to \pi^{-1}(O)$.

Hence, $\pi_1^{-1}(I)$ around P is described as the direct sum of sheaves of the following forms:

- ▷ the constant sheaf $\mathbb{C}_{\pi_0^{-1}(O)}$;
- ▷ $j_*\mathbb{C}_H$ or $j_!\mathbb{C}_H$, where H is an open half space such that ∂H contains P, and $j : H \to \pi_0^{-1}(O)$. They are denoted by $\mathbb{C}_{H*}$ and $\mathbb{C}_{H!}$.

5.4.4. Local form of $\mathcal{L}^{\leq D}$ and $\mathcal{L}/\mathcal{L}^{\leq D}$. — Let $P \in \pi_0^{-1}(O)$. We have a decomposition around P:

$$\mathcal{L} = \bigoplus_{\mathfrak{a} \in \operatorname{Irr}(\nabla)} \mathcal{L}_{\mathfrak{a}}, \quad \mathcal{L}^{\leq D} = \bigoplus_{\mathfrak{a} \in \operatorname{Irr}(\nabla)} \mathcal{L}_{\mathfrak{a}}^{\leq D}.$$

Let us describe $\mathcal{L}_{\mathfrak{a}}$ and $\mathcal{L}/\mathcal{L}_{\mathfrak{a}}^{\leq D}$ around P. For an appropriate coordinate system, $\mathfrak{a} = z_1^{-m_1} \cdots z_n^{-m_n}$ for some $m_i \geq 0$. Let

$$q_{\mathfrak{a}} : \Delta^n \longrightarrow \Delta, \quad (z_1, \dots, z_n) \longmapsto \prod z_i^{m_i}.$$

Let $\pi_\Delta : \widetilde{\Delta}(0) \to \Delta$ be the real blow up. We have the induced map

$$q_{\mathfrak{a}} : \widetilde{X}(D) \longrightarrow \widetilde{\Delta}(0), \quad (r_i, \theta_i) \longmapsto \Big(\prod_{i=1}^{n} r_i^{m_i}, \sum m_i \theta_i \Big).$$

Let $\mathcal{Q}$ be the local system on $\widetilde{\Delta}(0)$ with Stokes structure, corresponding to the meromorphic flat bundle $(\mathcal{O}_\Delta(*0), \mathrm{d}+\mathrm{d}(1/z))$. Note that $\mathcal{Q}/\mathcal{Q}^{\leq 0}$ is the constructible sheaf $j_*\mathbb{C}_J$ on $\pi_\Delta^{-1}(0)$, where $j: J=(-\pi,\pi)\to\pi_\Delta^{-1}(0)$. Let $r(\mathfrak{a})$ be the rank of $\mathcal{L}_\mathfrak{a}$. We have isomorphisms:

$$\mathcal{L}_\mathfrak{a}\simeq q_\mathfrak{a}^*\mathcal{Q}^{\oplus r(\mathfrak{a})},\quad \mathcal{L}_\mathfrak{a}^{\leq D}\simeq q_\mathfrak{a}^*(\mathcal{Q}^{\leq 0})^{\oplus r(\mathfrak{a})},\quad \mathcal{L}_\mathfrak{a}/\mathcal{L}_\mathfrak{a}^{\leq D}\simeq q_\mathfrak{a}^*(\mathcal{Q}/\mathcal{Q}^{\leq 0})^{\oplus r(\mathfrak{a})}.$$

Around P, we have an isomorphism $q_\mathfrak{a}^*(\mathcal{Q}/\mathcal{Q}^{\leq 0})\simeq\iota_*\mathbb{C}$, where $Z:=q_\mathfrak{a}^{-1}(J)$ and $\iota: Z\to(S^1)^n\times\mathbb{R}^n_{\geq 0}$. Note that Z is of the form $Z_0\times\partial\mathbb{R}^n_{\geq 0}$, where Z_0 is the inverse image of J via the induced map $(S^1)^n\times\{0\}\to S^1\times\{0\}$. Hence, $q_\mathfrak{a}^*(\mathcal{Q}/\mathcal{Q}^{\leq 0})$ is isomorphic to one of the following, around P:

- ▷ the constant sheaf $\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}$;
- ▷ $j_{K*}\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}}$, where K is an open half space such that ∂K contains P, and $j_K: K\times\partial\mathbb{R}^n_{\geq 0}\to(S^1)^n\times\mathbb{R}^n_{\geq 0}$. It is denoted by $\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}$.

5.4.5. Proof of Theorem 5.4.1. — We reduce the proof of the theorem to the computation of $\mathcal{E}xt^i(\pi_1^{-1}I, q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0}))$ for $i\leq n-2$, where I is a constructible sheaf on $\pi_0^{-1}(O)$.

LEMMA 5.4.3. — *We have $\mathcal{E}xt^i(\pi_1^{-1}I, q_\mathfrak{a}^{-1}\mathcal{Q})=0$ for any i. In particular, we have isomorphisms*

$$\mathcal{E}xt^i(\pi_1^{-1}I,\, q_\mathfrak{a}^{-1}\mathcal{Q}^{\leq 0})\simeq\mathcal{E}xt^{i-1}\big(\pi_1^{-1}I, q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\big).$$

Proof. — Let $\iota: (S^1)^n\times\{0\}\to(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}$ denote the inclusion. There exists a constructible sheaf $\mathcal{F}$ on $(S^1)^n$ such that $\pi_1^{-1}I\simeq\iota_*\mathcal{F}$. We have the adjunction

$$\mathcal{E}xt^i(\iota_*\mathcal{F},\, q_\mathfrak{a}^{-1}\mathcal{Q})=\iota_*\mathcal{E}xt^i(\mathcal{F}, i^! q_\mathfrak{a}^{-1}\mathcal{Q}).$$

Note $\iota^! q_\mathfrak{a}^{-1}\mathcal{Q}=\boldsymbol{D}\iota^{-1}\boldsymbol{D}(q_\mathfrak{a}^{-1}\mathcal{Q})=0$, because $\boldsymbol{D}q_\mathfrak{a}^{-1}\mathcal{Q}$ is 0-extension of a constant sheaf on $(S^1)^n\times\mathbb{R}^n_{>0}$ by $(S^1)^n\times\mathbb{R}^n_{>0}\to(S^1)^n\times\mathbb{R}^n_{\geq 0}$. Hence, we obtain $\mathcal{E}xt^i(\iota_*\mathcal{F}, q_\mathfrak{a}^{-1}\mathcal{Q})=0$, and the proof of Lemma 5.4.3 is finished. □

Now, let us prove the following vanishing of the stalks at P:

$$\mathcal{E}xt^j\big(\pi_1^{-1}I, q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\big)_P=0,\quad (j\leq n-2). \tag{96}$$

It can be computed on $(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}$. We have the following cases, divided by the local forms of $\pi_1^{-1}(I)$ and $q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})$ around P:

(I): $\pi_1^{-1}I\simeq\mathbb{C}_{(S^1)^n}$ and $q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\simeq\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}$;

(II): $\pi_1^{-1}I\simeq\mathbb{C}_{(S^1)^n}$ and $q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\simeq\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}$;

(III): $\pi_1^{-1}I=\mathbb{C}_{H\star}$ and $q_\mathfrak{a}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0})\simeq\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}}$, where $\star=*,!$;

(IV): $\pi_1^{-1}I \simeq \mathbb{C}_{H\star}$ and $q_{\mathfrak{a}}^{-1}(\mathcal{Q}/\mathcal{Q}^{\leq 0}) \simeq \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}$, where $\star = *, !$.

Moreover, this last case is divided into three subcases:

(IV-1) ∂H and ∂K are transversal,

(IV-2) $H = K$,

(IV-3) $H = -K$.

In the following, for a given $i : Y_1 \subset Y_2$ and $\star = *, !$, let $\mathbb{C}_{Y_1\star} := i_\star \mathbb{C}_{Y_1}$ on Y_2. It is also denoted just by $\mathbb{C}_{Y_1}$ if there is no risk of confusion.

5.4.5.1. The case (I). — Instead of $(S^1)^n \times \{0\} \to (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$, it is enough to consider the inclusion $\{0\} \to \partial\mathbb{R}^n_{\geq 0} \simeq \mathbb{R}^{n-1}$. We obtain (96) from the following standard result:

$$\mathcal{E}xt^j(\mathbb{C}_0, \mathbb{C}_{\mathbb{R}^{n-1}})_0 \simeq \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

5.4.5.2. The case (II). — We have the exact sequence

$$0 \longrightarrow \mathbb{C}_{(S^1)^n \setminus K!} \longrightarrow \mathbb{C}_{(S^1)^n} \longrightarrow \mathbb{C}_{K*} \longrightarrow 0.$$

Let ι denote the inclusion $((S^1)^n \setminus K) \times \partial\mathbb{R}^n_{\geq 0} \to (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$. Note $\iota^* = \iota^!$, and hence $\iota^! \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*} = 0$. We have

$$\begin{aligned} &\mathcal{E}xt^j\big(\mathbb{C}_{((S^1)^n\setminus K)\times\{0\}\,!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P \\ &\qquad \simeq \iota_*\mathcal{E}xt^j\big(\mathbb{C}_{((S^1)^n\setminus K)\times\{0\}}, \iota^!\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P = 0. \end{aligned}$$

Hence, we obtain

$$\mathcal{E}xt^j\big(\mathbb{C}_{(S^1)^n}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P \simeq \mathcal{E}xt^j\big(\mathbb{C}_{K*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*}\big)_P = \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

5.4.5.3. The case (III). — Let us consider the case $\star = *$. We have the exact sequence:

$$0 \longrightarrow \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!} \longrightarrow \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}} \longrightarrow \mathbb{C}_{H*} \longrightarrow 0.$$

Let k_1 denote the inclusion $H \times \{0\} \to (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$, and let k_2 denote the open embedding of the complement. Because $k_1^* \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!} = 0$, we have the isomorphisms

$$\begin{aligned} (97)\qquad & R\mathcal{H}om(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}\,!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P \\ &\quad \simeq R\mathcal{H}om(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}!})_P \\ &\qquad \simeq k_{2*}(\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}\setminus H\times\{0\}})_P \simeq (\mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P. \end{aligned}$$

We obtain $R\mathcal{H}om(\mathbb{C}_{H*}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P = 0$. In particular, $\mathcal{E}xt^j(\mathbb{C}_{H*}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P$ is null for any j.

Let us consider the case $\star = !$. From the exact sequence

$$0 \longrightarrow \mathbb{C}_{H!} \longrightarrow \mathbb{C}_{(S^1)^n} \longrightarrow \mathbb{C}_{(S^1)^n\setminus H\,*} \longrightarrow 0,$$

we obtain the isomorphisms

$$\mathcal{E}xt^j(\mathbb{C}_{H!}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P = \mathcal{E}xt^j(\mathbb{C}_{(S^1)^n}, \mathbb{C}_{(S^1)^n\times\partial\mathbb{R}^n_{\geq 0}})_P = \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

5.4.5.4. The case (IV-1). — Let us consider the case $\star = *$. Let $\mathcal{N}$ be the kernel of $\mathbb{C}_{H*} \to \mathbb{C}_{H\cap K*}$.

Lemma 5.4.4. — *We have $R\mathcal{H}om(\mathcal{N}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*})_P = 0$.*

Proof. — Let ι be the inclusion $((S^1)^n \setminus K) \times \partial\mathbb{R}^n_{\geq 0} \to (S^1)^n \times \partial\mathbb{R}^n_{\geq 0}$. Then, $\mathcal{N}$ is of the form $\iota_!\mathcal{N}_1$. Then, the claim follows from $\iota^!\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*} = 0$. ☐

We have the exact sequence:

$$0 \longrightarrow \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}!} \longrightarrow \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}} \longrightarrow \mathbb{C}_{(H\cap K)\times\{0\}*} \longrightarrow 0.$$

Let $k : K \times \partial\mathbb{R}^n_{\geq 0} \setminus (H \cap K) \times \{0\} \to K \times \partial\mathbb{R}^n_{\geq 0}$ denote the inclusion. We have the isomorphisms

$$\begin{aligned} (98)\quad & R\mathcal{H}om(\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}})_P \\ &\simeq Rk_*R\mathcal{H}om(\mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\setminus(H\cap K)\times\{0\}})_P \\ &\simeq \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0},P}. \end{aligned}$$

Hence, we obtain $R\mathcal{H}om(\mathbb{C}_{(H\cap K)\times\{0\}\,*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*})_P = 0$. In particular, we have $\mathcal{E}xt^j(\mathbb{C}_{H*}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}\,*})_P = 0$ for any j.

Let us consider the case $\star = !$. We have an exact sequence

$$0 \to \mathbb{C}_{H!} \to \mathbb{C}_{(S^1)^n} \to \mathbb{C}_{(S^1)^n\setminus H*} \to 0$$

on $(S^1)^n$. By using the previous results, we obtain

$$\mathcal{E}xt^j(\mathbb{C}_{H!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}*})_P = \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

5.4.5.5. The case (IV-2). — Let us consider the case $\star = *$. By considering $0 \to \partial\mathbb{R}^n_{\geq 0}$, we obtain

$$\mathcal{E}xt^j(\mathbb{C}_{H*}, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}*})_P \simeq \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

Let us consider the case $\star = !$. We have an exact sequence

$$0 \longrightarrow \mathbb{C}_{H!} \longrightarrow \mathbb{C}_{H*} \longrightarrow \mathbb{C}_{\partial H*} \longrightarrow 0.$$

Let us look at $\mathcal{E}xt^j(\mathbb{C}_{\partial H*}, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}})_P$. For $0 \to [0,1[\times\mathbb{R}^{n-1}$, we have

$$\mathcal{E}xt^j(\mathbb{C}_0, \mathbb{C}_{[0,1[\times\mathbb{R}^{n-1}}) = 0$$

for any j. Hence, we obtain

$$\mathcal{E}xt^j(\mathbb{C}_{H!}, \mathbb{C}_{H\times\partial\mathbb{R}^n_{\geq 0}})_P = \begin{cases} 0 & (j \leq n-2), \\ \mathbb{C} & (j = n-1). \end{cases}$$

5.4.5.6. The case (IV-3). — It is easy to show $\mathcal{E}xt^j(\mathbb{C}_{H!}, \mathbb{C}_{K\times\partial\mathbb{R}^n_{\geq 0}}) = 0$ for any j. By using the argument in (IV-2), we can show $\mathcal{E}xt^j(\mathbb{C}_{H*}, \mathbb{C}_{K\times\partial\mathbb{R}^n}) = 0$ for any j. Thus, the proof of Theorem 5.4.1 is finished. □

5.4.6. A uniqueness result on the K-structure. — We use the notation in §5.4.1. Let V be a good meromorphic flat bundle on (X, D). Let g be a holomorphic function on X such that $g^{-1}(0) = D$, and let i_g be the graph $X \to X \times \mathbb{C}$. We regard $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ as a cohomologically constructible sheaf on $\widetilde{X}(g)$.

Let K be a subfield of $\mathbb{C}$. A K-structure of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ is defined to be a K-cohomologically constructible complex $\mathcal{F}$ on $\widetilde{X}(g)$ with an isomorphism $\alpha : \mathcal{F}\otimes\mathbb{C} \simeq \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ in the derived category. Two K-structures $(\mathcal{F}_i, \alpha_i)$ $(i = 1, 2)$ are called equivalent if there exists an isomorphism $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}_1\otimes\mathbb{C} & \xrightarrow{\beta\otimes\mathbb{C}} & \mathcal{F}_2\otimes\mathbb{C} \\ {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\alpha_2}\big\downarrow \\ \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V) & \xrightarrow{=} & \mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V). \end{array}$$

LEMMA 5.4.5. — *Let $(\mathcal{F}_i, \alpha_i)$ $(i = 1, 2)$ be K-structures of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$. If their restriction to $\pi_1^{-1}(X - D)$ are equivalent, then they are equivalent on $\widetilde{X}(g)$.*

Proof. — We put $\mathcal{F}_i^{\mathbb{C}} := \mathcal{F}_i \otimes \mathbb{C}$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \otimes \mathbb{C} & \longrightarrow & \mathrm{Hom}(\mathcal{F}_{1|\pi_1^{-1}(X-D)}, \mathcal{F}_{2|\pi_1^{-1}(X-D)}) \otimes \mathbb{C} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}(\mathcal{F}_1^{\mathbb{C}}, \mathcal{F}_2^{\mathbb{C}}) & \longrightarrow & \mathrm{Hom}(\mathcal{F}^{\mathbb{C}}_{1|\pi_1^{-1}(X-D)}, \mathcal{F}^{\mathbb{C}}_{2|\pi_1^{-1}(X-D)}). \end{array}$$

According to Theorem 5.4.1, the horizontal arrows are injective. Hence, we obtain the equality

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = \mathrm{Hom}(\mathcal{F}_{1|\pi_1^{-1}(X-D)}, \mathcal{F}_{2|\pi_1^{-1}(X-D)}) \cap \mathrm{Hom}(\mathcal{F}_1^{\mathbb{C}}, \mathcal{F}_2^{\mathbb{C}})$$

in $\mathrm{Hom}(\mathcal{F}^{\mathbb{C}}_{1|\pi_1^{-1}(X-D)}, \mathcal{F}^{\mathbb{C}}_{2|\pi_1^{-1}(X-D)})$. Then, the element of $\mathrm{Hom}(\mathcal{F}_1^{\mathbb{C}}, \mathcal{F}_2^{\mathbb{C}})$ corresponding to the identity of $\mathrm{DR}^{\mathrm{nil}}_{X\times\widetilde{\mathbb{C}}}(i_{g\dagger}V)$ comes from $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$. □

CHAPTER 6

GOOD K-STRUCTURE

6.1. Good meromorphic flat bundles

6.1.1. Good K-structure of good meromorphic flat bundles. — Let $K \subset \mathbb{C}$ be a subfield. Let X be a complex manifold with a normal crossing hypersurface D.

Definition 6.1.1. — Let V be a good meromorphic flat bundle on (X, D).

- ▷ A K-structure of V is a pre-K-Betti structure of the flat bundle $V_{|X-D}$.
- ▷ A K-structure of V is good if the Stokes structures are defined over K.

Later (see §6.4), we shall extend the definition to the case where V is not necessarily good.

Let $D = D_1 \cup D_2$ be a decomposition. Let $\mathcal{L}$ be the local system with the Stokes structure on $\widetilde{X}(D)$ associated to V. Recall that the complex $\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V)$ is quasi-isomorphic to $\mathcal{L}^{<D_1 \leq D_2}[\dim X]$. (See §5.1.2.)

If V has a good K-structure, it is naturally equipped with a K-structure $\mathcal{L}^{<D_1 \leq D_2}_K[\dim X]$. By the isomorphisms (12) and (71), we obtain a pre-K-Betti structure

$$\mathcal{F}^{<D_1 \leq D_2}_V := R\pi_* \mathcal{L}^{<D_1 \leq D_2}_K[\dim X]$$

of the holonomic $\mathcal{D}$-module $V(!D_1)$. This pre-K-Betti structure is called canonical. Let $D_1' \cup D_2' = D$ be another decomposition such that $D_1 \subset D_1'$. The natural morphism $V(!D_1') \to V(!D_1)$ is compatible with the pre-K-Betti structures. We use the symbols $\mathcal{F}_{V*}$ and $\mathcal{F}_{V!}$ to denote $\mathcal{F}^{\leq D}_V$ and $\mathcal{F}^{<D}_V$, respectively. We also use the symbol $\mathcal{F}_V$ to denote $\mathcal{F}_{V*}$ for simplicity.

More generally, let $\iota : Z \subset X$ be a complex submanifold with a normal crossing hypersurface D_Z. Let V_Z be a good meromorphic flat bundle on (Z, D_Z).

We say that $\iota_\dagger V_Z$ has a good K-structure if V_Z has a good K-structure in the above sense. The canonical pre-K-Betti structures for $\iota_\dagger V_Z(!D_{Z,1})$ are also defined in a similar way for a decomposition $D_Z = D_{Z,1} \cup D_{Z,2}$.

6.1.2. Some basic property

6.1.2.1. Some functoriality. — Let X be any complex manifold with a normal crossing hypersurface D. The following lemma is clear.

Lemma 6.1.2. — *Let V_i $(i = 1, 2)$ be good meromorphic flat bundles on (X, D) with a good K-structure. If $V_1 \oplus V_2$ is good, then the induced K-structure is good. Similar claims hold for $V_1 \otimes V_2$ and $\mathcal{H}om(V_1, V_2)$.* □

Let V be a good meromorphic flat bundle on (X, D). Let $\varphi : X' \to X$ be a morphism of complex manifolds such that $D' := \varphi^{-1}(D)$ is normal crossing. We obtain a good meromorphic flat bundle $V' := \varphi^* V$ on (X', D'). Suppose that V is equipped with a K-structure, which induces a K-structure of V'.

Lemma 6.1.3. — *If the K-structure of V is good, the K-structure of V' is also good. Conversely, suppose that φ is surjective and that the K-structure of V' is good. Then, the K-structure of V is good.*

Proof. — Let P' be any point of D'. Let $P := \varphi(P')$. We take a small neighbourhood X_P with a coordinate $(z_1, \ldots, z_n)$ around P in X such that $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$, and a ramified covering

$$\kappa_P : (X_P^{(1)}, D_P^{(1)}) \longrightarrow (X_P, D \cap X_P)$$

such that $V_P^{(1)} := \kappa_P^*(V)$ is unramifiedly good. Let e_i $(i = 1, \ldots, \ell)$ denote the ramification index of κ_P along $z_i = 0$. We take a small neighbourhood $X'_{P'}$ of P'. Because $(z_i \circ \varphi)^{-1}(0)$ $(i = 1, \ldots, \ell)$ are contained in $D' \cap X'_P$, we can take a ramified covering

$$\kappa'_{P'} : (X'^{(1)}_{P'}, D'^{(1)}_{P'}) \longrightarrow (X'_{P'}, D' \cap X'_{P'})$$

such that there exist functions $(z_i \circ \varphi \circ \kappa'_{P'})^{1/e_i}$ $(i = 1, \ldots, \ell)$ on $X'^{(1)}_{P'}$. Then, we have a morphism $\rho : X'^{(1)}_{P'} \to X_P^{(1)}$ such that $\kappa_P \circ \rho = \varphi \circ \kappa'_{P'}$. Then,

$$V'^{(1)} := (\kappa'_{P'})^* V' = \rho^* \kappa_P^*(V)$$

is unramifiedly good. Let $\mathcal{L}$ be the local system on $\widetilde{X}_P^{(1)}(D_P^{(1)})$ associated to $V^{(1)}$. Let $\mathcal{L}'$ be the local system on $\widetilde{X}'^{(1)}_{P'}(D'^{(1)}_P)$ associated to $V'^{(1)}$. The map induced by ρ is denoted

$$\widetilde{\rho} : \widetilde{X}'^{(1)}_P(D'^{(1)}_P) \longrightarrow \widetilde{X}_P^{(1)}(D_P^{(1)}).$$

We have $\mathcal{L}' = \widetilde{\varphi}^{-1}(\mathcal{L})$. Let

$$\pi^{(1)} : \widetilde{X}_P^{(1)}(D_P^{(1)}) \longrightarrow X_P^{(1)} \quad \text{and} \quad \pi'^{(1)} : \widetilde{X}_{P'}'^{(1)}(D_{P'}'^{(1)}) \longrightarrow X_{P'}'^{(1)}$$

denote the projections. Let Q_1' be any point of $(\pi'^{(1)})^{-1}(D_P'^{(1)})$. We set

$$Q_1 := \widetilde{\rho}(Q_1').$$

Let $P_1' := \pi'^{(1)}(Q')$ and $P_1 := \pi^{(1)}(Q)$. The set of the irregular values of $V'^{(1)}$ at P_1' is the pull back of the set of the irregular values of $V^{(1)}$ at P_1. The partial order $\leq_{Q_1'}$ on the set is equal to $\leq_{Q_1}$. The Stokes filtration $\mathcal{F}^{Q_1'}$ is obtained as the pull back of $\mathcal{F}^{Q_1}$. Hence, $\mathcal{F}^{Q_1}$ is defined over K if and only if $\mathcal{F}^{Q_1'}$ is defined over K. $\square$

6.1.2.2. Curve test. — Let us consider the case $X = \Delta^n$, $D_i := \{z_i = 0\}$ and $D = \bigcup_{i=1}^{\ell} D_i$. We set $D_i^\circ := D_i \setminus \bigcup_{j\neq i} D_j$. Let $p_i : X \to D_i$ denote the projection.

Proposition 6.1.4. — *Let V be a good meromorphic flat bundle on (X, D) with a K-structure with the property:*

(C1) *Let P be any point of D_i° for $i = 1, \dots, \ell$. Then, the induced K-structure of $V_{|p_i^{-1}(P)}$ is good.*

Then, the K-structure of V is good.

Proof. — We may assume that V is unramifiedly good. Let $\pi : \widetilde{X}(D) \to X$ denote the projection. Let $\mathcal{L}$ be the local system on $\widetilde{X}(D)$ with the induced K-structure. Let Q be any point of $\pi^{-1}(D)$. It is enough to prove that the Stokes filtration $\mathcal{F}^Q(\mathcal{L}_Q)$ is defined over K. It is enough to consider the case $\pi(Q) = (0, \dots, 0)$. We set

$$S := \bigl\{(\mathfrak{a}, \mathfrak{b}) \in \operatorname{Irr}(V)^2 \mid \mathfrak{a} \neq \mathfrak{b}\bigr\}.$$

We have i such that $\operatorname{ord}_{z_i}(\mathfrak{a} - \mathfrak{b}) < 0$ for any $(\mathfrak{a}, \mathfrak{b}) \in S$. For any $(\mathfrak{a}, \mathfrak{b}) \in S$, let $H(\mathfrak{a}, \mathfrak{b})$ be denote the intersection of $\pi^{-1}(D_i)$ and the closure of

$$\bigl\{R \in X \setminus D \mid \operatorname{Re}(\mathfrak{a} - \mathfrak{b})(R) = 0\bigr\}$$

in $\widetilde{X}(D)$. Let $\mathcal{U}$ be a small neighbourhood of Q in $\pi^{-1}(D_i)$. Then, for any $(\mathfrak{a}, \mathfrak{b}) \in S$, we have $\mathfrak{a} <_Q \mathfrak{b}$ if and only if we have $\mathfrak{a} <_{Q'} \mathfrak{b}$ for any

$$Q' \in \mathcal{U}' := \pi^{-1}(D_i^\circ) \cap \mathcal{U} \setminus \bigcup_{(\mathfrak{a},\mathfrak{b})\in S} H(\mathfrak{a}, \mathfrak{b}).$$

We have natural identifications of $\mathcal{L}_Q$ and $\mathcal{L}_{Q'}$ for $Q' \in \mathcal{U}$. We have

$$\mathcal{F}^Q_{\mathfrak{a}} = \bigcap_{Q' \in \mathcal{U}'} \mathcal{F}^{Q'}_{\mathfrak{a}}.$$

Under the assumption **(C1)**, $\mathcal{F}^{Q'}_{\mathfrak{a}}$ are defined over K for any $Q' \in \mathcal{U}'$. Hence, we obtain that $\mathcal{F}^Q_{\mathfrak{a}}$ are defined over K. □

6.1.2.3. Sub-quotients. — Let X be any complex manifold with a normal crossing hypersurface D. Let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence of good meromorphic flat bundles on (X, D). Suppose that V and V_i are equipped with K-structures which are compatible with the morphisms.

LEMMA 6.1.5. — *If the K-structure of V is good, then the K-structures of V_i $(i = 1, 2)$ are good.*

Proof. — We may assume that V is unramifiedly good. We may assume that $X = \Delta$ and $D = \{0\}$. Let $\mathcal{L}_i$ and $\mathcal{L}$ be the local systems on $\widetilde{X}(D)$ corresponding to V_i and V, respectively. For any point $P \in \widetilde{X}(D)$, the stalks $\mathcal{L}_{1P}$ and $\mathcal{L}_P$ are equipped with the Stokes filtrations $\mathcal{F}^P$. Note that the Stokes filtrations are characterized by the growth order. Hence, $\mathcal{L}_{1P} \to \mathcal{L}_P$ is strict with respect to the filtrations, i.e., $\mathcal{F}^P(\mathcal{L}_{1P})$ is equal to the filtration obtained as the restriction of $\mathcal{F}^P(\mathcal{L}_P)$. Then, if $\mathcal{L}_{1P}$ and $\mathcal{F}^P(\mathcal{L}_P)$ are defined over K, the filtration $\mathcal{F}^P(\mathcal{L}_{1P})$ is also defined over K. □

LEMMA 6.1.6. — *Let V_i $(i = 1, 2)$ be good meromorphic flat bundles on (X, D). Let $f : V_1 \to V_2$ be a morphism of meromorphic flat bundles.*

- ▷ $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{Cok}(f)$ *are also good.*
- ▷ *Suppose that V_i are equipped with good K-structures, and that f is compatible with the K-structures. Then, the induced K-structures of* $\mathrm{Ker}(f)$, $\mathrm{Cok}(f)$ *and* $\mathrm{Im}(f)$ *are good.*

Proof. — It is enough to check the claims locally around any point of D. We may assume that V_i are unramifiedly good. Let P be any point of D. Let $f_{|\widehat{P}}$ denote the induced morphism $V_{1|\widehat{P}} \to V_{2|\widehat{P}}$. Because the formal completion is exact, we have $\mathrm{Ker}(f)_{|\widehat{P}} \simeq \mathrm{Ker}(f_{|\widehat{P}})$ and similar isomorphisms for Im and Cok. We have the decompositions $V_{i|\widehat{P}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(V_i, P)} V_{i,\widehat{P},\mathfrak{a}}$. It is easy to check that $f_{|\widehat{P}}$ is compatible with the decompositions. Then, the first claim follows. The second claim follows from the first claim and Lemma 6.1.5. □

If V_i are unramifiedly good in Lemma 6.1.6, we have

$$\mathrm{Irr}(\mathrm{Ker}\, f, P) \subset \mathrm{Irr}(V_1, P), \quad \mathrm{Irr}(\mathrm{Cok}\, f, P) \subset \mathrm{Irr}(V_2, P),$$

$$\mathrm{Irr}(\mathrm{Im}\, f, P) \subset \mathrm{Irr}(V_1, P) \cap \mathrm{Irr}(V_2, P).$$

6.1.3. Functoriality for projective birational morphisms. — Let D_3 be a hypersurface of X. Let $\varphi : X' \to X$ be a projective birational morphism such that $D' := \varphi^{-1}(D \cup D_3)$ is normal crossing, and that

$$X' \setminus D' \simeq X \setminus (D_3 \cup D).$$

Let V be a good meromorphic flat bundle on (X, D). Suppose that V is equipped with a good K-structure. We put

$$V' := \varphi^* V \otimes \mathcal{O}_{X'}(*D').$$

The induced K-structure of V' is good. Let $D_1 \cup D_2$ be a decomposition of D. We set $D_1' := \varphi^{-1}(D_1)$. We take $D_2' \subset D'$ such that $D_1' \cup D_2'$ is a decomposition of D'.

Proposition 6.1.7. — *The natural morphisms*

$$V(!D_1) \longrightarrow \varphi_\dagger V'(!D_1'), \quad \varphi_\dagger V'(!D_2') \longrightarrow V(!D_2)$$

are compatible with the canonical pre-K-Betti structures.

Proof. — Let us prove the second claim. We use the notation introduced in §5.3. Let $\widetilde{\varphi} : \widetilde{X}'(D') \to \widetilde{X}(D)$ be the induced map. By construction, it is easy to see that the morphisms

$$\mathrm{DR}^{<D_1 \leq D_2}_{\widetilde{X}(D)}(V) \longrightarrow R\widetilde{\varphi}_* \, \mathrm{DR}^{<D_1' \leq D_2'}_{\widetilde{X}'(D')}(V'),$$

$$R\widetilde{\varphi}_* \, \mathrm{DR}^{<D_2' \leq D_1'}_{\widetilde{X}'(D')}(V') \longrightarrow \mathrm{DR}^{<D_2 \leq D_1}_{\widetilde{X}(D)}(V)$$

are compatible with the induced K-structures. Then, the second claim follows from Theorem 5.3.1. □

6.1.4. A characterization of compatibility with Stokes filtrations

Let $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let V be an unramifiedly good meromorphic flat bundle on (X, D). Its good set of irregular values is denoted by $\mathrm{Irr}(V)$. For each $\mathfrak{a} \in \mathrm{Irr}(V)$, put

$$L(-\mathfrak{a}) = \mathcal{O}_X(*D)\, e$$

with the meromorphic flat connection $\nabla e = e\,d(-\mathfrak{a})$. We fix a K-structure of $L(-\mathfrak{a})$ by the trivialization $\exp(\mathfrak{a})\,e$. We have on $\widetilde{X}(D)$ a constructible sheaf

$$\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D)}\big(V\otimes L(-\mathfrak{a})\big).$$

The following lemma will be useful to check that a K-structure is good.

LEMMA 6.1.8. — *Suppose that V has a K-structure with the property:*

- ▷ *For each $\mathfrak{a}\in\mathrm{Irr}(V)$, the induced K-structure of $(V\otimes L(-\mathfrak{a}))_{|X-D}$ is extended to a K-structure of* $\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D)}(V\otimes L(-\mathfrak{a}))$.

Then, the K-structure of V is good.

Proof. — Let $\mathcal{L}$ be the local system with the Stokes structure on $\widetilde{X}(D)$ associated to $V_{|X\setminus D}$. It is equipped with the Stokes structure i.e., for each $P\in\pi^{-1}(D)$, the stalk $\mathcal{L}_P$ has the Stokes filtration $\mathcal{F}^P$. By the assumption, the local system $\mathcal{L}$ has a K-structure. Let $O=(0,\dots,0)\in X$. Let π denote the projection $\widetilde{X}(D)\to X$. It is enough to prove that the Stokes filtrations $\mathcal{F}^P$ of $\mathcal{L}_P$ are defined over K for $P\in\pi^{-1}(O)$.

Let S denote the set of pairs $(\mathfrak{a},\mathfrak{b})$ in $\mathrm{Irr}(V)$ with $\mathfrak{a}\neq\mathfrak{b}$. For any $(\mathfrak{a},\mathfrak{b})\in S$, let $H(\mathfrak{a},\mathfrak{b})$ denote the closure of the set $\{\mathrm{Re}(\mathfrak{a}-\mathfrak{b})\}$ in $\widetilde{X}(D)$. Take a small neighbourhood U_1 of P in $\pi^{-1}(O)$ such that for any $(\mathfrak{a},\mathfrak{b})\in S$, we have $H(\mathfrak{a},\mathfrak{b})\cap U_1\neq\varnothing$ if and only if $P\in H(\mathfrak{a},\mathfrak{b})$. Let

$$U_1' := U_1\setminus\bigcup_{(\mathfrak{a},\mathfrak{b})\in S} H(\mathfrak{a},\mathfrak{b}).$$

We have $\mathfrak{a}<_P\mathfrak{b}$ if and only if $\mathfrak{a}<_{P'}\mathfrak{b}$ for any $P'\in U_1'$. We have natural identifications $\mathcal{L}_P\simeq\mathcal{L}_{P'}$ for any $P'\in U_1$. Under the identifications, we have

$$\mathcal{F}^P_{\mathfrak{a}}=\bigcap_{P'\in U_1'}\mathcal{F}^{P'}_{\mathfrak{a}}.$$

So, if $\mathcal{F}^{P'}_{\mathfrak{a}}$ are defined over K for any $P'\in U_1'$, $\mathcal{F}^P_{\mathfrak{a}}$ is also defined over K. For the points $P'\in U_1'$, the order $\leq_{P'}$ is totally ordered. So, it is enough to prove that $\mathcal{F}^{P'}_{<\mathfrak{a}}$ are defined over K for any $\mathfrak{a}\in\mathrm{Irr}(V)$ and for any $P'\in U_1'$. But, it follows from the assumption of the lemma. □

6.1.5. The behaviour of the pre-K-Betti structure by the nearby cycle functor and the maximal functor. — We set

$$X:=\Delta^n\quad\text{and}\quad D:=\bigcup_{i=1}^{\ell}\{z_i=0\}.$$

Let V be a good meromorphic flat bundle on (X,D) with a good K-structure. For each $I\subset\underline{\ell}$, we set $I_{!i}:=I\cup\{i\}$ and $I_{*i}:=I\setminus\{i\}$. The $\mathcal{D}$-module

$$\Pi^{a,b}_{i\star}\big(V\big(!D(I)\big)\big)=(V\otimes\mathfrak{I}^{a,b}_{z_i})\big(!D(I_{\star i})\big)$$

has the canonical pre-K-Betti structure, where $\star=*,!$. Hence, $\psi_i^{(a)}(V(!D(I)))$ and $\Xi_i^{(a)}(V(!D(I)))$ have the induced pre-K-Betti structures.

Lemma 6.1.9. — *The induced K structure of $\psi_i^{(a)}(V)$ is good, i.e., it is compatible with the Stokes filtrations. The induced pre-K-Betti structure of $\psi_i^{(a)}(V(!D(I)))$ is canonical for each $I\subset\underline{\ell}$.*

Proof. — It is enough to consider the case $a=0$ and $i=1$. We give a preparation. We set $\Pi^{-\infty,a}_{f\star}V:=\varinjlim_N\Pi^{-N,a}_{f\star}(V)$. By Lemma 3.2.3, we have the commutative diagram

$$\begin{array}{ccc}\mathrm{DR}_X\big(\Pi^{-\infty,0}_{1!}(V(!D(I)))\big) & \longrightarrow & \mathrm{DR}_X\big(\Pi^{-\infty,0}_{1*}(V(!D(I)))\big)\\ \simeq\uparrow & & \simeq\uparrow\\ \mathrm{DR}_X^{<D(I_{*1})}(\Pi^{-\infty,0}_{1!}V) & \longrightarrow & \mathrm{DR}_X^{<D(I_{*1})}(\Pi^{-\infty,0}_{1*}V)\\ \simeq\uparrow & & \simeq\uparrow\\ \mathrm{DR}_X^{<D(I_{!1})}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}) & \longrightarrow & \mathrm{DR}_X^{<D(I_{*1})}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}).\end{array}\tag{99}$$

By the upper square, the induced K-structure of $\mathrm{DR}_X\,\psi_1^{(0)}(V(!D(I)))$ can be identified with the K-structure of

$$\begin{aligned}(100)\qquad &\mathrm{DR}_X^{<D(I_{*1})}\,\psi_1^{(0)}(V)\\ &\simeq\mathrm{Cone}\,\big(\mathrm{DR}_X^{<D(I_{*1})}(\Pi^{-\infty,0}_{1!}V)\to\mathrm{DR}_X^{<D(I_{*1})}(\Pi^{-\infty,0}_{1*}V)\big).\end{aligned}$$

We set $D':=\bigcup_{i=2}^{\ell}D_i$. Let $\pi_1:\widetilde{X}(D')\to X$ be the real blow up. We obtain (100) as the push-forward of the following on $\widetilde{X}(D')$:

$$\begin{aligned}(101)\quad &\mathrm{DR}_{\widetilde{X}(D')}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}\,\psi_1^{(0)}(V)\\ &\simeq\mathrm{Cone}\,\big(\mathrm{DR}_{\widetilde{X}(D')}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}(\Pi^{-\infty,0}_{1!}V)\to\mathrm{DR}_{\widetilde{X}(D')}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}(\Pi^{-\infty,0}_{1*}V)\big).\end{aligned}$$

We prepare some commutative diagram in a general setting. For any holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we put

$$\mathrm{DR}_{\widetilde{X}(D')}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}\mathcal{M}:=\mathrm{Tot}\,\Omega^{\bullet,\bullet,<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\otimes_{\pi_1^{-1}\mathcal{O}_X}\pi_1^{-1}\mathcal{M}[\dim X],$$

$$\mathrm{DR}_{\widetilde{X}(D')}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}\mathcal{M}:=\mathrm{Tot}\,\Omega^{\bullet,\bullet,<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}(*D_1)\otimes_{\pi_1^{-1}\mathcal{O}_X}\pi_1^{-1}\mathcal{M}[\dim X].$$

We have the commutative diagram

$$\begin{array}{ccc}
\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M}(!D_1) & \longrightarrow & \mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M}(*D_1) \\
\Big\uparrow & & =\Big\uparrow \\
\mathrm{DR}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M} & \longrightarrow & \mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D')}\mathcal{M}.
\end{array}$$

If $\mathcal{M}$ is a good meromorphic flat bundle, the left vertical arrow is also a quasi-isomorphism, which follows from Lemma 5.1.6.

Let $\rho:\widetilde{X}(D)\to\widetilde{X}(D')$ be the induced map. We have the natural commutative diagram, where the vertical arrows are quasi-isomorphisms by Theorem 4.3.2:

$$\begin{array}{ccc}
\mathrm{DR}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\mathcal{M} & \longrightarrow & \mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D')}\mathcal{M} \\
\simeq\Big\downarrow & & \simeq\Big\downarrow \\
\rho_*\,\mathrm{DR}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}\mathcal{M} & \longrightarrow & \rho_*\,\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}\mathcal{M}.
\end{array}$$

Thus, we obtain the commutative diagram, in which the vertical arrows are quasi-isomorphisms:

$$\begin{array}{ccc}
\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}(\Pi^{-\infty,0}_{1!}V) & \longrightarrow & \mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}(\Pi^{-\infty,0}_{1*}V) \\
\simeq\Big\uparrow & & \simeq\Big\uparrow \\
\rho_*\,\mathrm{DR}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}) & \longrightarrow & \rho_*\,\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1}).
\end{array}\tag{102}$$

Because $\mathrm{DR}^{<D(I_{!1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D)}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1})$ and $\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{*1})}_{\widetilde{X}(D)}(V\otimes\mathfrak{I}^{-\infty,0}_{z_1})$ are equipped with K-structures compatible with the morphism, we obtain a K-structure of $\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi^{(0)}_1(V)$ from (101) and (102). The lower square in (99) is obtained as the push-forward of (102). Hence, the K-structure of $\mathrm{DR}_X\,\psi^{(0)}_1(V(!D(I)))$ is obtained as the push-forward of the K-structure of $\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi^{(0)}_1(V)$.

Let us consider the case $I=\{1,\dots,\ell\}$. By the above consideration, we obtain that $\mathcal{F}^P_{<0}$ is compatible with the K-structure, where $\mathcal{F}^P$ denotes the Stokes filtration of $\psi^{(0)}_1(V)$ at each point $P\in\pi_1^{-1}(\partial D_1)$. By considering the tensor product with meromorphic flat bundles with rank one, we can deduce that $\mathcal{F}^P$ is defined over K, as in Lemma 6.1.8. Since the pre-K-Betti structure of $\psi^{(0)}_1(V(!D(I)))$ comes from the K-structure of $\mathrm{DR}^{<D(I_{*1})\le D(\underline{\ell}-I_{!1})}_{\widetilde{X}(D')}\psi^{(0)}_1(V)$, it is canonical. □

6.2. Good holonomic $\mathcal{D}$-modules with good K-structure (Local case)

6.2.1. Definition. — Let $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Set $\underline{\ell} := \{1,\dots,\ell\}$. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (X,D).

Definition 6.2.1. — We say that $\mathcal{M}$ has a good K-structure if

(i) for each $I \subset \underline{\ell}$, $\phi_I(\mathcal{M})(*D(I^c))$ is equipped with a good K-structure (put $\phi_{\varnothing}(\mathcal{M}) := \mathcal{M}$),

(ii) for $i \notin I$, the induced morphisms

$$\psi_i^{(1)}\big(\phi_I(\mathcal{M})\big(*D(I^c)\big)\big) \longrightarrow \big(\phi_i\phi_I(\mathcal{M})\big)\big(*D(I^c_{!i})\big) \longrightarrow \psi_i^{(0)}\big(\phi_I(\mathcal{M})\big(*D(I^c)\big)\big) \tag{103}$$

are compatible with the K-structures, where $I_{!i} := I \sqcup \{i\}$. □

Morphisms of good holonomic $\mathcal{D}$-modules with a good K-structure

$$f : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$$

are morphisms of $\mathcal{D}$-modules such that $\phi_I(f)$ are compatible with K-structures for any $I \subset \underline{\ell}$.

Let $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ denote the category of good holonomic $\mathcal{D}_X$-modules with a good K-structure on (X,D).

Lemma 6.2.2. — *Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$. Then, the $\mathcal{D}$-modules $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ and $\mathrm{Cok}(f)$ are naturally objects in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.*

Proof. — It follows from Lemma 6.1.6. (See also the reconstruction of a good holonomic $\mathcal{D}$-module $\mathcal{M}$ from $\phi_I^{(\mathbf{0})}(\mathcal{M})$ in §6.3.) □

6.2.2. Cells. — Let V be any good meromorphic flat bundle on X with a good K-structure. Let us observe that we have natural objects in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ associated to V.

Lemma 6.2.3. — *Let $D^{(1)}$ be a hypersurface of X contained in D.*

- ▷ *We can naturally regard $V(!D^{(1)})$ as an object in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.*
- ▷ *Suppose that we are given an object $\mathcal{M}$ in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ such that*
  - (i) *the underlying $\mathcal{D}_X$-module is isomorphic to $V(!D^{(1)})$,*
  - (ii) *the K-structure on $X \setminus D$ is equal to that of $V(!D^{(1)})$ under the isomorphism.*

  Then, $\mathcal{M}$ is isomorphic to $V(!D^{(1)})$ in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.

Proof. — We have $I \subset \underline{\ell}$ such that $D^{(1)} = D(I)$. We have for any $J \subset \underline{\ell}$ a natural isomorphism

$$\phi_J^{(\mathbf{0}_J)}\big(V\big(!D(I)\big)\big)(*D(J^c)) \simeq \psi_{J\cap I}^{(\boldsymbol{\delta}_{J\cap I})}\psi_{J\setminus I}^{(\mathbf{0}_{J\setminus I})}(V),$$

where $\boldsymbol{\delta}_{J\cap I} = (1,\dots,1) \in \mathbb{Z}^{J\cap I}$ and $\mathbf{0}_{J\setminus I} = (0,\dots,0) \in \mathbb{Z}^{J\setminus I}$. They are equipped with good K-structures, satisfying the compatibility condition (103). Via these K-structures, we regard $V(!D(I)) \in \mathrm{Hol}^{\mathrm{good}}(X,D,K)$. Thus, we obtain the first claim.

Let us prove the second claim. We are given the isomorphism of $\mathcal{D}_X$-modules $V(!D^{(1)}) \simeq \mathcal{M}$ under which the K-structures on $X \setminus D$ are equal. Suppose that we have already known that $\phi_I^{(\mathbf{0})}(V(!D^{(1)})) \simeq \phi_I^{(\mathbf{0})}(\mathcal{M})$ preserves the K-structures. Set $V_1 := V(!D^{(1)})$ and $V_2 := \mathcal{M}$. Because one of

$$\psi_i^{(1)}\phi_I^{(\mathbf{0})}(V_i) \longrightarrow \phi_i^{(0)}\phi_I^{(\mathbf{0})}(V_i) \quad \text{or} \quad \psi_i^{(1)}\phi_I^{(\mathbf{0})}(V_i) \longrightarrow \phi_i^{(0)}\phi_I^{(\mathbf{0})}(V_i)$$

is an isomorphism compatible with K-structures. Hence, we obtain that $\phi_i^{(0)}\phi_I^{\mathbf{0}}(V_1) \to \phi_i^{(0)}\phi_I^{\mathbf{0}}(V_2)$ is also compatible with the K-structures. □

More generally, take $J \sqcup I \subset \underline{\ell}$. Let V_J be a good meromorphic flat bundle on D_J with a good K-structure. Then, we can naturally regard $\iota_\dagger V_J(!D(I))$ as an object in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.

Let g be a meromorphic function on (X,D) such that $g^{-1}(0) \subset D$. Let $D = D_1 \cup D_2$ be a decomposition such that $D_1 \supset g^{-1}(\infty)$ and $D_2 \subset g^{-1}(0)$. (Note that D_i are not necessarily irreducible.) Because $\Xi_g^{(0)}(V,*D_1)$ and $\psi_g^{(0)}(V,*D_1)$ are the kernel of

$$\big(V \otimes \mathfrak{I}_g^{-\infty,a}(!D_2)\big)(*D_1) \longrightarrow V \otimes \mathfrak{I}_g^{-\infty,0}(*D)$$

for $a = 1,0$, they are naturally objects in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.

6.2.3. Some operations. — Let us observe that some operations on $\mathrm{Hol}(X)$ are naturally lifted on $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$. Let Forget denote the forgetful functor from $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ to $\mathrm{Hol}(X)$.

LEMMA 6.2.4. — *We have a naturally defined dual functor* $\boldsymbol{D}$ *on* $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ *such that*

$$\boldsymbol{D} \circ \mathrm{Forget} = \mathrm{Forget} \circ \boldsymbol{D}.$$

Proof. — Let $\mathcal{M} \in \mathrm{Hol}^{\mathrm{good}}(X,D,K)$. For each $I \subset \{1,\dots,\ell\}$,

$$\phi_I^{(\boldsymbol{a})}\big(\boldsymbol{D}\mathcal{M})(*D(I^c)\big) \simeq \boldsymbol{D}\phi_I^{(-\boldsymbol{a}-\boldsymbol{\delta})}(\mathcal{M})\big(*D(I^c)\big)$$

has an induced K-structure. For $I_0 := I \sqcup \{i\}$, the morphisms

$$\psi_i^{(1)}\phi_I^{(\mathbf{0})}(\boldsymbol{D}\mathcal{M})\big(*D(I_0^c)\big) \longrightarrow \phi_i^{(0)}\phi_I^{(\mathbf{0})}(\boldsymbol{D}\mathcal{M})\big(*D(I_0^c)\big) \\ \longrightarrow \psi_i^{(0)}\phi_I^{(\mathbf{0})}(\boldsymbol{D}\mathcal{M})\big(*D(I_0^c)\big)$$

are obtained as the dual of

$$\psi_i^{(0)}\phi_I^{(-\boldsymbol{\delta})}(\mathcal{M})\big(*D(I_0^c)\big) \longrightarrow \phi_i^{(-1)}\phi_I^{(-\boldsymbol{\delta})}(\mathcal{M})\big(*D(I_0^c)\big) \\ \longrightarrow \psi_i^{(-1)}\phi_I^{(-\boldsymbol{\delta})}(\mathcal{M})\big(*D(I_0^c)\big),$$

they are compatible with the K-structure. Hence, they give a good K-structure on $\boldsymbol{D}\mathcal{M}$. The construction gives a contravariant functor $\boldsymbol{D}$ on $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. ☐

LEMMA 6.2.5. — *Let $D^{(1)} \subset D$ be a hypersurface of X. We have a functor*

$$\Phi_{*D^{(1)}} : \mathrm{Hol}^{\mathrm{good}}(X, D, K) \longrightarrow \mathrm{Hol}^{\mathrm{good}}(X, D, K)$$

such that

$$\mathrm{Forget} \circ \Phi_{*D^{(1)}}(\mathcal{M}) = \mathrm{Forget}(\mathcal{M})(*D^{(1)})$$

for any $\mathcal{M}$ in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. We also have a natural transformation

$$\mathcal{M} \longrightarrow \Phi_{*D^{(1)}}(\mathcal{M}).$$

Such a functor is unique.

Proof. — First, let us observe the uniqueness. Let $\mathcal{M} \in \mathrm{Hol}^{\mathrm{good}}(X, D, K)$. We have $I \subset \underline{\ell}$ such that $D^{(1)} = D(I)$. For any $J \subset \underline{\ell}$, the following isomorphism is compatible with the K-structure:

$$\phi_{J\setminus I}^{(\mathbf{0})}(\mathcal{M})\big(*D((J\setminus I)^c)\big) \xrightarrow{\ \alpha\ } \phi_{J\setminus I}^{(\mathbf{0})}(\Phi_{*D^{(1)}}\mathcal{M})\big(*D((J\setminus I)^c)\big).$$

The following induced isomorphism is compatible with the K-structure:

$$\psi_{J\cap I}^{(\mathbf{0})}\phi_{J\setminus I}^{(\mathbf{0})}(\mathcal{M})\big(D(J^c)\big) \xrightarrow{\ \psi_{J\cap I}^{(\mathbf{0})}(\alpha)\ } \psi_{J\cap I}^{(\mathbf{0})}\phi_{J\setminus I}^{(\mathbf{0})}(\Phi_{*D^{(1)}}\mathcal{M})\big(D(J^c)\big).$$

Note that the following natural morphism is an isomorphism:

$$\phi_J^{(\mathbf{0})}(\Phi_{*D^{(1)}}\mathcal{M})\big(*D(J^c)\big) \longrightarrow \psi_{J\cap I}^{\mathbf{0}}\phi_{J\setminus I}^{(\mathbf{0})}(\Phi_{*D^{(1)}}\mathcal{M})\big(*D(J^c)\big).$$

It is compatible with the K-structure by the condition for $\Phi_{*D^{(1)}}\mathcal{M}$. Hence, the good K-structure of

$$\phi_{J\setminus I}^{(\mathbf{0})}(\mathcal{M})\big(*D((J\setminus I)^c)\big)$$

uniquely determines the K-structure of $\phi_J^{(\mathbf{0})}(\Phi_{*D^{(1)}}\mathcal{M})(*D(J^c))$. It means the uniqueness of $\Phi_{*D^{(1)}}$.

As for the existence of $\Phi_{*D^{(1)}}$, it is enough to consider the case $I = \{1\}$.

If $i \in J$, we have

$$\phi_J^{(\mathbf{0})}\big(\mathcal{M}(*D^{(1)})\big) \simeq \psi_1^{(0)}\phi^{(\mathbf{0})}_{J\setminus\{1\}}(\mathcal{M}).$$

If $i \notin J$, we have

$$\phi_J^{(\mathbf{0})}\big(\mathcal{M}(*D^{(1)})\big) \simeq \phi_J^{(\mathbf{0})}\big(\mathcal{M}\big)(*D^{(1)}).$$

The induced K-structures on $\phi_J^{(\mathbf{0})}\big(\mathcal{M}(*D^{(1)})\big)(*D(J^c))$ give a good K-structure of $\mathcal{M}(*D^{(1)})$, for which the natural morphism $\mathcal{M} \to \mathcal{M}(*D^{(1)})$ is a morphism in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$. □

Lemma 6.2.6

▷ *For any hypersurface $D^{(1)}$ of X contained in D, we have a unique functor*

$$\Phi_{!D^{(1)}} : \mathrm{Hol}^{\mathrm{good}}(X, D, K) \longrightarrow \mathrm{Hol}^{\mathrm{good}}(X, D, K)$$

such that, for any $\mathcal{M}$ in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$,

$$\mathrm{Forget} \circ \Phi_{!D^{(1)}}(\mathcal{M}) = \mathrm{Forget}(\mathcal{M})(!D^{(1)})$$

with a natural transformation $\Phi_{!D^{(1)}} \to \mathrm{id}$.

▷ *We have $\Phi_{\star D^{(1)}} \circ \Phi_{\star D^{(2)}} = \Phi_{\star(D^{(1)} \cup D^{(2)})}$.*

▷ *If $\dim(D^{(1)} \cap D^{(2)}) < n-1$, then $\Phi_{!D^{(1)}} \circ \Phi_{*D^{(2)}} = \Phi_{*D^{(2)}} \circ \Phi_{!D^{(1)}}$.*

Proof. — The first claim follows from Lemma 6.2.5 as the dual. The second claim follows from the uniqueness. For $\mathcal{M} \in \mathrm{Hol}^{\mathrm{good}}(X, D, K)$, the underlying $\mathcal{D}_X$-modules of $\Phi_{!D^{(1)}} \circ \Phi_{*D^{(2)}}(\mathcal{M})$ and $\Phi_{*D^{(2)}} \circ \Phi_{!D^{(1)}}(\mathcal{M})$ are

$$\mathcal{M}(!D^{(1)} * D^{(2)}) = \mathcal{M}(*D^{(2)}!D^{(1)}).$$

We have in $\mathrm{Hol}^{\mathrm{good}}(X, D, K)$ the natural morphisms

$$\Phi_{!D^{(1)}}(\mathcal{M}) \longrightarrow \Phi_{!D^{(1)}} \circ \Phi_{*D^{(2)}}(\mathcal{M}), \quad \Phi_{!D^{(1)}}(\mathcal{M}) \longrightarrow \Phi_{*D^{(2)}} \circ \Phi_{!D^{(1)}}(\mathcal{M}).$$

Then, by the argument for the uniqueness in the proof of Lemma 6.2.5, we obtain that the K-structures are the same. □

We denote $\Phi_{\star D^{(1)}}(\mathcal{M})$ by $\mathcal{M}(\star D^{(1)})$ for $\star = *, !$.

6.3. Good pre-K-holonomic $\mathcal{D}$-modules

6.3.1. Statements. — Let $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. Let $\mathrm{Hol}^{\mathrm{pre}}(X, K)$ denote the category of pre-K-holonomic $\mathcal{D}_X$-modules.

Proposition 6.3.1. — *We have a naturally defined exact fully faithful functor* $\Upsilon : \mathrm{Hol}(X, D, K) \to \mathrm{Hol}^{\mathrm{pre}}(X, K)$ *over* $\mathrm{Hol}(X)$*. We have* $\Upsilon \circ \boldsymbol{D} = \boldsymbol{D} \circ \Upsilon$*. The essential image of* Υ *is independent of the choice of a holomorphic coordinate system.*

Definition 6.3.2. — Any object in the essential image of Υ is called a good pre-K-holonomic $\mathcal{D}$-module on (X, D). The pre-K-Betti structure is called a good pre-K-Betti structure. (The definition will be globalized in Definition 6.3.4 below.) □

Let V be a good meromorphic flat bundle on (X, D) with a good K-structure. Let $D^{(1)} \subset D$ be a hypersurface of X.

Proposition 6.3.3. — *The canonical pre-K-Betti structure of* $V(!D^{(1)})$ *is associated to the good K-structure of* $V(!D^{(1)})$ *by* Υ.

We shall construct the functor in §6.3.3–§6.3.5. We shall prove the full faithfulness in §6.3.7. The independence from the coordinate system will be proved in §6.3.8. Proposition 6.3.3 will be proved in §6.3.6.

6.3.2. Some consequences. — Before going to the proof of Proposition 6.3.1, we give some consequences. The full faithfulness and the independence on the coordinate system in Proposition 6.3.1 ensure that we can globalize the notion of good pre-K-holonomic $\mathcal{D}$-modules in Definition 6.3.2.

Definition 6.3.4. — Let Y be any complex manifold with a normal crossing hypersurface D_Y. Let $\mathcal{M}$ be a good holonomic $\mathcal{D}$-module on (Y, D_Y) with a pre-K-Betti structure $\mathcal{F}$. It is called a good pre-K-holonomic $\mathcal{D}$-module if its restriction to any holomorphic coordinate neighbourhood is a good pre-K-holonomic $\mathcal{D}$-module. In that case, $\mathcal{F}$ is called a good pre-K-Betti structure. □

The category of good pre-K-holonomic $\mathcal{D}$-modules on (Y, D_Y) is not abelian (see §3.1.6). If we would like to work on abelian categories, for example, the full subcategory of $\boldsymbol{\mathcal{I}}$-good pre-K-holonomic $\mathcal{D}$-modules is abelian, where $\boldsymbol{\mathcal{I}}$ is any good system of ramified irregular values on (Y, D_Y).

Let Y be any complex manifold with a normal crossing hypersurface D. Let V be a good meromorphic flat bundle on (Y, D) with a good K-structure. Let g be any meromorphic function on (Y, D) such that it is invertible on $Y \setminus D$. We take a hypersurface $D^{(1)} \subset D$ such that $g^{-1}(0) \subset D^{(1)}$. We obtain a good meromorphic flat bundle $V \otimes \mathfrak{I}_g^{a,b}$ with a good K-structure on (Y, D). It induces pre-K-holonomic $\mathcal{D}$-modules $\Pi^{a,b}_{g\star}(V)(*D^{(1)})$, $\Xi^{(a)}_g(V,*D^{(1)})$ and $\psi^{(a)}_g(V,*D^{(1)})$ with the canonical pre-K-Betti structures. We obtain the following proposition from Proposition 6.3.3.

PROPOSITION 6.3.5. — *The holonomic $\mathcal{D}_Y$-modules*

$$\Pi^{a,b}_{g\star}(V)(*D^{(1)}),\quad \Xi^{(a)}_g(V,*D^{(1)}),\quad \psi^{(a)}_g(V,*D^{(1)}),\quad \phi^{(a)}_g(V,*D^{(1)})$$

are naturally good pre-K-holonomic $\mathcal{D}$-modules on (Y,D). □

The claims for $\psi^{(a)}_g(V,*D^{(1)})$ and $\phi^{(a)}_g(V,*D^{(1)})$ will be particularly useful.

6.3.3. Induced pre-K-Betti structures of $\Xi^{(\boldsymbol{a})}_I\psi^{\boldsymbol{b}}_J(\iota_\dagger V_I)$. — In the following, we shall prove Proposition 6.3.1 and Proposition 6.3.3.

Let $K\sqcup J\sqcup I=L\subset\underline{\ell}$. Let V_I be an $\mathcal{I}$-good meromorphic flat bundle on $(D_I,\partial D_I)$. Let $\iota:D_I\to X$. For a map $f:K\sqcup J\to\{0,1\}$, we set $K_0(f):=f^{-1}(0)\cap K$. We put

$$\mathcal{C}_f(J,K,\iota_\dagger V_I):=\Big(\iota_\dagger V_I\otimes\bigotimes_{k\in K_0(f)}\mathfrak{I}^{-\infty,1}_{z_k}\otimes\bigotimes_{k\notin K_0(f)}\mathfrak{I}^{-\infty,0}_{z_k}\Big)\big(!D\big(f^{-1}(0)\big)\big).$$

Let $\mathbf{0}$ denote the constant map valued in $\{0\}$. Let $\boldsymbol{\delta}_i$ denote the map such that $\boldsymbol{\delta}_i(j)=0$ $(j\neq i)$ and $\boldsymbol{\delta}_i(i)=1$. We can identify $\Xi^{(\mathbf{0})}_K\psi^{(\mathbf{0})}_J\big(\iota_\dagger V_I\big)$ as the kernel of the following morphism:

$$\mathcal{C}_{\mathbf{0}}\big(J,K,\iota_\dagger V_I\big)\longrightarrow\bigoplus_{i\in K\sqcup J}\mathcal{C}_{\boldsymbol{\delta}_i}\big(J,K,\iota_\dagger V_I\big).\tag{104}$$

If V_I has a good K-structure, we obtain a pre-K-Betti structure of $\Xi^{(\mathbf{0})}_K\psi^{(\mathbf{0})}_J(\iota_\dagger V_I)$ by (104). By taking the tensor product with $\mathfrak{I}^{a,a+1}$ appropriately, we also obtain an induced pre-K-Betti structure of $\Xi^{(\boldsymbol{a})}_K\psi^{(\boldsymbol{b})}_J(\iota_\dagger V_I)$.

LEMMA 6.3.6. — *The following morphisms are compatible with the pre-K-Betti structures:*

$$\Xi^{(\boldsymbol{a})}_K\psi^{(\boldsymbol{b})}_J\psi^{(1)}_i\big(\iota_\dagger V_I\big)\longrightarrow\Xi^{(\boldsymbol{a})}_K\psi^{(\boldsymbol{b})}_J\Xi^{(0)}_i\big(\iota_\dagger V_I\big)\longrightarrow\Xi^{(\boldsymbol{a})}_K\psi^{(\boldsymbol{b})}_J\psi^{(0)}\big(\iota_\dagger V_I\big).$$

Proof. — It is clear by construction. □

Recall that we have the naturally induced good K-structure on $\psi^{(0)}_i\big(\iota_\dagger V_I\big)$ for $i\notin I$ (Lemma 6.1.9).

LEMMA 6.3.7. — *For any $i\notin L$, the natural isomorphism*

$$\Xi^{(\mathbf{0})}_K\psi^{(\mathbf{0})}_{Ji}\big(\iota_\dagger V_I\big)\simeq\Xi^{(\mathbf{0})}_K\psi^{(\mathbf{0})}_J\big(\psi^{(0)}_i(\iota_\dagger V_I)\big)$$

is compatible with the induced K-structures.

Proof. — Both the K-structures are obtained as the kernel of the morphism (104) for (Ji, K). □

6.3.4. ℓ-squares of complexes. — For small categories A_i $(i = 1, \dots, \ell)$, let $\prod_{i=1}^{\ell} A_i$ denote their product, i.e., the category whose objects and morphisms are given by $\mathrm{ob}\Big(\prod_{i=1}^{\ell} A_i\Big) = \prod_{i=1}^{\ell} \mathrm{ob}(A_i)$ and $\mathrm{Mor}(\boldsymbol{a}, \boldsymbol{b}) = \prod \mathrm{Mor}(a_i, b_i)$. Let Γ be a small category given by the following commutative diagram:

$$\begin{array}{ccc} (0,0) & \xrightarrow{a} & (0,1) \\ {\scriptstyle b}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ (1,0) & \xrightarrow{d} & (1,1) \end{array} \qquad c \circ a = d \circ b.$$

Let A be an abelian category. Let $C(A)$ be the category of complexes in A. A square in $C(A)$ is a functor $F : \Gamma \to C(A)$. For a given square F, let $H(F)$ be the total complex of the following double complex:

$$F(0,0)[1] \xrightarrow{F(a)+F(b)} F(0,1) \oplus F(1,0) \xrightarrow{F(c)-F(d)} F(1,1)[-1].$$

An ℓ-square in $C(A)$ is a functor $F : \Gamma^{\ell} \to C(A)$. Let $\pi_i : \Gamma^{\ell} \to \Gamma^{\ell-1}$ be the projection forgetting the i-th component. For a given ℓ-square F, we obtain an $(\ell-1)$-square $\pi_{i*}F$ by $\pi_{i*}F(\boldsymbol{a}) = H\big(F_{|\pi_i^{-1}(\boldsymbol{a})}\big)$.

Lemma 6.3.8. — *For $i < j$, we have an isomorphism $\pi_{i*}\pi_{j*}F \simeq \pi_{j-1*}\pi_{i*}F$.*

Proof. — It is enough to consider the case $\ell = 2$, $(i, j) = (1, 2)$. The i-th terms of the both complexes are given by

$$\bigoplus_{a_1+a_2+b_1+b_2=i-2} F(a_1, a_2, b_1, b_2).$$

The multiplication of -1 on $F(0,0,0,0) \oplus F(1,1,0,0) \oplus F(0,0,1,1) \oplus F(1,1,1,1)$ interpolates the differentials for $\pi_{i*}\pi_{j*}F$ and $\pi_{j-1*}\pi_{i*}F$. □

More generally, for any subset $I \subset \underline{\ell}$, I-square in $C(A)$ is a functor $\Gamma^I \to C(A)$. For the naturally defined projection $\pi_I : \Gamma^{\ell} \to \Gamma^I$, we take $I = I_0 \subset I_1 \subset \cdots \subset I_m = \underline{\ell}$, which induces the factorization $\pi_I = \pi^{(1)} \circ \pi^{(2)} \circ \cdots \circ \pi^{(m)}$, where $\pi^{(i)} : \Gamma^{I_i} \to \Gamma^{I_{i-1}}$. Then, we obtain an I-square $\pi_{I*}F := \pi_*^{(1)} \circ \cdots \circ \pi_*^{(m)} F$. It is well defined up to isomorphisms as above.

6.3.5. A construction of the functor Υ. — Let m be any positive integer. Let $\mathcal{I}\subset M(X^{(m)},D^{(m)})/H(X^{(m)})$ be any good set of ramified irregular values as in §3.1.1. Let $\mathcal{M}$ be any $\mathcal{I}$-good holonomic $\mathcal{D}$-module on (X,D).

Let $H\subset\underline{\ell}$. Let us construct an H-square in the category of $\mathcal{I}$-good holonomic $\mathcal{D}$-modules on (X,D). For $(\boldsymbol{i},\boldsymbol{j})=\big((i_k,j_k)\,\big|\,k\in H\big)\in\operatorname{ob}\Gamma^H$, we have the following subsets of H:

$$I(\boldsymbol{i},\boldsymbol{j})=\big\{k\,\big|\,(i_k,j_k)=(0,1)\big\},\qquad K(\boldsymbol{i},\boldsymbol{j})=\big\{k\,\big|\,(i_k,j_k)=(1,0)\big\},$$

$$J_0(\boldsymbol{i},\boldsymbol{j})=\big\{k\,\big|\,(i_k,j_k)=(0,0)\big\},\qquad J_1(\boldsymbol{i},\boldsymbol{j})=\big\{k\,\big|\,(i_k,j_k)=(1,1)\big\}.$$

Then, we put

$$\mathcal{Q}^H(\mathcal{M},\boldsymbol{i},\boldsymbol{j}):=\Xi^{(\boldsymbol{0})}_{I(\boldsymbol{i},\boldsymbol{j})}\psi^{(\boldsymbol{\delta}_{J_0(\boldsymbol{i},\boldsymbol{j})})}_{J_0(\boldsymbol{i},\boldsymbol{j})}\psi^{(\boldsymbol{0})}_{J_1(\boldsymbol{i},\boldsymbol{j})}\phi^{(\boldsymbol{0})}_{K(\boldsymbol{i},\boldsymbol{j})}\mathcal{M}.$$

For $k_0\notin H$, we have the following naturally induced diagram:

$$\begin{array}{ccc}\psi^{(1)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M} & \longrightarrow & \Xi^{(0)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}\\ \big\downarrow & & \big\downarrow\\ \phi^{(0)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M} & \longrightarrow & \psi^{(0)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}.\end{array}\tag{105}$$

For each decomposition $H=\{h\}\cup(H-\{h\})$, we have a similar diagram. Thus, we obtain an H-square $\mathcal{Q}^H(\mathcal{M})$ of good holonomic $\mathcal{D}$-modules. The cohomology of the complex associated to (105) is naturally isomorphic to $\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}$. Hence, we have a natural quasi-isomorphism $\pi_{H*}\mathcal{Q}^{\underline{\ell}}(\mathcal{M})\simeq\mathcal{Q}^H(\mathcal{M})$. In particular, we have a natural quasi-isomorphism $\pi_{\underline{\ell}*}\mathcal{Q}^{\underline{\ell}}(\mathcal{M})\simeq\mathcal{M}$.

If $\mathcal{M}$ has a good K-structure, each $\mathcal{Q}^{\underline{\ell}}(\mathcal{M},\boldsymbol{i},\boldsymbol{j})$ is equipped with the pre-K-Betti structure $\mathcal{F}^{\underline{\ell}}_{\mathcal{M}}(\boldsymbol{i},\boldsymbol{j})$ given as in §6.3.3.

Lemma 6.3.9. — *The morphisms in* (105) *are compatible with the induced pre-K-Betti structures.*

Proof. — The morphisms

$$\psi^{(1)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}\to\Xi^{(0)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}\to\psi^{(0)}_{k_0}\Xi^{(\boldsymbol{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\boldsymbol{0})}_{J_1}\phi^{(\boldsymbol{0})}_K\mathcal{M}$$

are compatible with the pre-K-Betti structures by construction, as remarked in Lemma 6.3.6. Let $K':=\underline{\ell}-(K\sqcup k_0)$. By definition, the morphisms

$$\psi^{(1)}_{k_0}\phi^{(\boldsymbol{0})}_K\mathcal{M}\big(*D(K')\big)\to\phi^{(0)}_{k_0}\phi^{(\boldsymbol{0})}_K\mathcal{M}\big(*D(K')\big)\to\psi^{(0)}_{k_0}\phi^{(\boldsymbol{0})}_K\mathcal{M}\big(*D(K')\big)$$

are compatible with the K-structures. We remark Lemma 6.3.7, and then it follows that the morphisms

$$\psi^{(1)}_{k_0}\Xi^{(\mathbf{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\mathbf{0})}_{J_1}\phi^{(\mathbf{0})}_K\mathcal{M}\to\phi^{(0)}_{k_0}\Xi^{(\mathbf{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\mathbf{0})}_{J_1}\phi^{(\mathbf{0})}_K\mathcal{M}\to\psi^{(0)}_{k_0}\Xi^{(\mathbf{0})}_I\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\mathbf{0})}_{J_1}\phi^{(\mathbf{0})}_K\mathcal{M}$$

are compatible with the pre-K-Betti structures. □

Thus, we obtain a pre-K-Betti structure of $\pi_{\underline{\ell}*}\mathcal{Q}^{\underline{\ell}}(\mathcal{M})\simeq\mathcal{M}$, which is independent of the choice of a factorization of $\pi_{\underline{\ell}}$. It is called the pre-K-Betti structure of $\mathcal{M}$ associated to the good K-structure, and denoted by $\mathcal{F}_{\mathcal{M}}$. We obtain a pre-$K$-holonomic $\mathcal{D}_X$-module $\Upsilon(\mathcal{M}):=(\mathcal{M},\mathcal{F}_{\mathcal{M}})$. Thus, we obtain the desired exact functor $\Upsilon:\mathrm{Hol}^{\mathrm{good}}(X,D,K)\to\mathrm{Hol}^{\mathrm{pre}}(X,K)$. It is clearly exact.

6.3.6. Proof of Proposition 6.3.3. — If $\mathcal{M}\bigl(*D(H^c)\bigr)=\mathcal{M}$, any $\mathcal{Q}^H(\mathcal{M},\boldsymbol{i},\boldsymbol{j})$ are equipped with the pre-K-Betti structures, which induce a pre-K-Betti structure of $\mathcal{M}$.

Lemma 6.3.10. — *The associated pre-K-Betti structures of $\mathcal{M}$ are the same.*

Proof. — The naturally defined morphisms

$$\Xi^{(\mathbf{0})}_{H^c}\Xi^{(\mathbf{0})}_K\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\mathbf{0})}_{J_1}\phi^{(\mathbf{0})}_I(\mathcal{M})\longrightarrow\Xi^{(\mathbf{0})}_K\psi^{(\boldsymbol{\delta}_{J_0})}_{J_0}\psi^{(\mathbf{0})}_{J_1}\phi^{(\mathbf{0})}_I(\mathcal{M})$$

induce the quasi-isomorphism $\pi_{\underline{\ell}*}\mathcal{Q}^{\underline{\ell}}(\mathcal{M})\to\pi_{H*}\mathcal{Q}^H(\mathcal{M})$, which is compatible with the pre-K-Betti structures. □

Let us prove Proposition 6.3.3. By the above consideration, the following isomorphisms are compatible with the pre-K-Betti structures:

$$V\bigl(!D(H)\bigr)\xrightarrow{\simeq}\mathcal{Q}^H\bigl(V\bigl(!D(H)\bigr)\bigr)\xleftarrow{\simeq}\mathcal{Q}^{\underline{\ell}}\bigl(V\bigl(!D(H)\bigr)\bigr).$$

Thus, we obtain Proposition 6.3.3. □

6.3.7. Full faithfulness. — Let us prove that the functor Υ is fully faithful. We denote $\Upsilon(\mathcal{M}_i)$ by $\mathcal{M}_i$ to simplify the notation. Let $\mathcal{M}_i\in\mathrm{Hol}^{\mathrm{good}}(X,D,K)$ $(i=1,2)$. Suppose we are given a morphism $\varphi:\mathcal{M}_1\to\mathcal{M}_2$ in $\mathrm{Hol}^{\mathrm{pre}}(X,K)$. We would like to prove that φ gives a morphism in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$.

We use an induction on $\rho(\mathcal{M}_1\oplus\mathcal{M}_2)$. (See §3.1.2 for ρ.) We take a subset $J\subset\underline{\ell}$ such that $|J|=n-\dim\mathrm{Supp}(\mathcal{M}_1\oplus\mathcal{M}_2)$ and $(\mathcal{M}_1\oplus\mathcal{M}_2)\bigl(*D(J^c)\bigr)\neq 0$. Let g be a holomorphic function such that $g^{-1}(0)=D(J^c)$. Then, $\mathcal{M}_i(*g)$ and

$\mathcal{M}_i \otimes \mathfrak{I}_g^{a,b}$ come from good meromorphic flat bundles with good K-structures on $(D_J, D_J(J^c))$. We have the following morphisms in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$:

$$\mathcal{M}_i(!g) \longrightarrow \Xi_g^{(0)}\big(\mathcal{M}_i(*g)\big) \longrightarrow \mathcal{M}_i(*g).$$

They are compatible with the associated pre-K-Betti structures. By the localization in Lemma 6.2.5 and Lemma 6.2.6, we obtain the following in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$:

$$\mathcal{M}_i(!g) \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{M}_i(*g).$$

Note the uniqueness of good K-structure on $\mathcal{M}_i(\star g)$ in Lemma 6.2.3. We obtain the following diagram of the pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc} \mathcal{M}_1(!g) & \longrightarrow & \Xi_g^{(0)}(\mathcal{M}_1(*g)) \oplus \mathcal{M}_1 & \longrightarrow & \mathcal{M}_1(*g) \\ \big\downarrow{\scriptstyle \varphi(!g)} & & \big\downarrow{\scriptstyle \Xi_g^{(0)}(\varphi)\oplus\varphi} & & \big\downarrow{\scriptstyle \varphi(*g)} \\ \mathcal{M}_2(!g) & \longrightarrow & \Xi_g^{(0)}(\mathcal{M}_2(*g)) \oplus \mathcal{M}_2 & \longrightarrow & \mathcal{M}_2(*g). \end{array}$$

We obtain a morphism $\phi_g^{(0)}(\varphi) : \phi_g^{(0)}(\mathcal{M}_1) \to \phi_g^{(0)}(\mathcal{M}_2)$ in $\mathrm{Hol}^{\mathrm{pre}}(X,K)$. By using the inductive assumption, $\phi_g^{(0)}(\varphi)$ is a morphism in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$. Then, φ is obtained as the cohomology of the following:

$$\begin{array}{ccccc} \psi_g^{(1)}\big(\mathcal{M}_1(*g)\big) & \longrightarrow & \Xi_g^{(0)}\big(\mathcal{M}_1(*g)\big) \oplus \phi_g^{(0)}(\mathcal{M}_1) & \longrightarrow & \psi_g^{(0)}\big(\mathcal{M}_1(*g)\big) \\ \big\downarrow{\scriptstyle \psi_g^{(1)}\varphi} & & \big\downarrow{\scriptstyle \Xi_g^{(0)}(\varphi)\oplus\phi_g^{(0)}\varphi} & & \big\downarrow{\scriptstyle \psi_g^{(0)}\varphi} \\ \psi_g^{(1)}\big(\mathcal{M}_2(*g)\big) & \longrightarrow & \Xi_g^{(0)}\big(\mathcal{M}_2(*g)\big) \oplus \phi_g^{(0)}(\mathcal{M}_2) & \longrightarrow & \psi_g^{(0)}\big(\mathcal{M}_2(*g)\big). \end{array} \tag{106}$$

The morphisms in (106) are morphisms in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$. Therefore, we obtain that φ is also a morphism in $\mathrm{Hol}^{\mathrm{good}}(X,D,K)$. □

6.3.8. Independence from the coordinate system. — Let us prove that the essential image of Υ is independent of the choice of a coordinate system. Let $(w_1,\dots,w_n)$ be another holomorphic coordinate system such that $w_i^{-1}(0) = z_i^{-1}(0)$. It is enough to prove the following lemma.

LEMMA 6.3.11. — *If $\mathcal{M}$ has a good K-structure with respect to the coordinate system $(z_1,\dots,z_n)$, it has an induced good K-structure with respect to $(w_1,\dots,w_n)$ such that the associated pre-K-Betti structures are the same.*

Proof. — We use symbols $\phi_{\boldsymbol{z},I}^{(\mathbf{0})}$ and $\phi_{\boldsymbol{w},I}^{(\mathbf{0})}$ to distinguish the dependence on the coordinate systems. As remarked in §2.2.7, we have the natural isomorphisms (10). They induce isomorphisms $\phi_{\boldsymbol{z},I}^{(\mathbf{0})}(\mathcal{M}) \simeq \phi_{\boldsymbol{w},I}^{(\mathbf{0})}(\mathcal{M})$ and $\psi_i^{(a)}\phi_{\boldsymbol{z},I}^{(\mathbf{0})}(\mathcal{M}) \simeq \psi_i^{(a)}\phi_{\boldsymbol{z},I}^{(\mathbf{0})}(\mathcal{M})$. Hence, we obtain good K-structure of $\mathcal{M}$ with respect to

$(w_1,\ldots,w_n)$. Let $\mathcal{Q}^{\ell}_{\underline{\boldsymbol{z}}}(\mathcal{M})$ and $\mathcal{Q}^{\ell}_{\underline{\boldsymbol{w}}}(\mathcal{M})$ denote the $\underline{\ell}$-square associated to $\mathcal{M}$ with respect to the coordinate systems $(z_1,\ldots,z_n)$ and $(w_1,\ldots,w_n)$, respectively. It is easy to observe that isomorphisms (10) induce $\pi_{\underline{\ell}*}\mathcal{Q}^{\ell}_{\underline{\boldsymbol{z}}}(\mathcal{M}) \simeq \pi_{\underline{\ell}*}\mathcal{Q}^{\ell}_{\underline{\boldsymbol{w}}}(\mathcal{M})$ compatible with pre-K-Betti structures, and they induce the identity on $\mathcal{M}$. Hence, the associated pre-K-Betti structures on $\mathcal{M}$ are the same. Thus, the proof of Lemma 6.3.11 and Proposition 6.3.1 are finished. □

6.4. Meromorphic flat connections with good K-structure

6.4.1. Good K-structure of meromorphic flat connections. — Let X be a complex manifold with a hypersurface D. Let V be a meromorphic flat connection on (X,D), i.e., V is a reflexive $\mathcal{O}_X(*D)$-coherent sheaf with a flat connection. We do not assume that V is good.

Definition 6.4.1. — As in the case of good meromorphic flat bundles, a K-structure of V means a pre-K-Betti structure of the flat bundle $V_{|X\setminus D}$.

Recall that, according to K. Kedlaya (see [**26**], Theorem 8.2.2 of [**27**]), for any point $P\in X$, there exist a neighbourhood $X_P\subset X$ and a projective birational morphism $\lambda_P:\check{X}_P\to X_P$ such that

(i) $\lambda_P:\check{X}_P\setminus\lambda_P^{-1}(D)\simeq X_P\setminus D$,

(ii) $\check{D}_P:=\lambda_P^{-1}(D)$ is normal crossing,

(iii) λ_P^*V is a good meromorphic flat bundle.

(See also [**44**] and Theorem 16.2.1 of [**47**] for the algebraic case.)

Such (X_P,λ_P) is called a local resolution of V in this paper. In the situation, we set $D_P:=D\cap X_P$.

Definition 6.4.2. — A K-structure of V is called good at P if the following holds:

- ▷ For any local resolution (X_P,λ_P) around P, the induced pre-K-Betti structure of $\lambda_P^*(V_{|X_P\setminus D})$ is a good K-structure of λ_P^*V.

A K-structure of V is called good if it is good at any point of X. □

If a K-structure of V is good, the induced K-structure on the dual $V^{\vee}$ is also good. The following lemma is easy to see.

Lemma 6.4.3. — *Let V_i $(i=1,2)$ be meromorphic flat bundles on (X,D) with a good K-structure.*

- ▷ *The naturally induced K-structures on $V_1 \oplus V_2$, $V_1 \otimes V_2$ and $\mathcal{H}om(V_1, V_2)$ are good.*
- ▷ *Let $f : V_1 \to V_2$ be a flat morphism which is compatible with the K-structures. Then, the naturally induced K-structures of* $\operatorname{Ker} f$, $\operatorname{Cok} f$ *and* $\operatorname{Im}(f)$ *are good.* □

Let $\varphi : X' \to X$ be a morphism of complex manifolds such that $D' := \varphi^{-1}(D)$ is normal crossing. We have the induced good meromorphic flat bundle $V' = \varphi^* V$. A K-structure of V induces a K-structure of V'.

Lemma 6.4.4. — *If the K-structure of V is good, the K-structure of V' is also good. Conversely, suppose that the K-structure of V' is good and that φ is surjective. Then, the K-structure of V is also good.*

Proof. — Let (X_P, λ_P) be a local resolution for V around $P \in X$. We take a projective birational morphism $\lambda : \check{X}'_P \to \check{X}_P \times_X X'$ such that:

(i) $\check{X}'_P$ is smooth,

(ii) the induced morphism $\varphi_P : \check{X}'_P \to \check{X}_P$ gives $\check{X}'_P \setminus \check{D}'_P \simeq \check{X}_P \setminus \check{D}_P$, where $\check{D}'_P := \lambda^{-1}(\check{X}_P \times_X D')$.

The induced map $\lambda'_P : \check{X}'_P \to X'$ gives a local resolution for V'. Then, the claim follows from Lemma 6.1.3. □

We obtain the following lemma from Proposition 6.1.4.

Lemma 6.4.5. — *Let V be a meromorphic flat connection on (X, D) with a K-structure. Suppose that, for any morphism $\Delta \to X$ with $\varphi(\Delta) \cap D = \{\varphi(0)\}$, the induced K-structure of $\varphi^*(V)$ is good. Then, the K-structure of V is also good.* □

We obtain the following lemma from Lemma 6.1.5.

Lemma 6.4.6. — *Let V be a meromorphic flat connection with a good K-structure. Let $V_1 \subset V$ be a sub-connection such that $V_{1|X \setminus D}$ is compatible with the K-structure. Then, the induced K-structure of V_1 is good. A similar claim holds for quotients of V.* □

6.4.2. Canonical pre-K-Betti structures. — Let V be a meromorphic flat connection on (X, D) with a good K-structure. Let

$$D = D_1 \cup D_2$$

be a decomposition, i.e., D_i are unions of irreducible components of D such that $\operatorname{codim}_X(D_1 \cap D_2) > 1$. Let (X_P, λ_P) be any local resolution of V around $P \in X$. Put

$$D_{P1} = D_1 \cap X_P \quad \text{and} \quad \check{D}_{P1} := \lambda_P^{-1}(D_1).$$

We have the decomposition $\check{D}_P = \check{D}_{P1} \cup \check{D}_{P2}$. We set

$$V_P := V_{|X_P} \quad \text{and} \quad \check{V}_P := \lambda_P^* V.$$

The canonical pre-K-Betti structure $\mathcal{V}_{\check{V}_P}^{<\check{D}_{P1} \leq \check{D}_{P2}}$ of $\check{V}_P(!\check{D}_{P1})$ induces a pre-K-Betti structure $\mathcal{G}$ of $V_P(!D_{P1})$. Let $(X_P^{(1)}, \lambda_P^{(1)})$ be another local resolution of V around $P \in X$. It induces a pre-K-Betti structure $\mathcal{G}^{(1)}$ of $V_{|X_P^{(1)}}$. We have $\mathcal{G}^{(1)} = \mathcal{G}$ on $X_P \cap X_P^{(1)}$. Indeed, we can find a local resolution $(X_P^{(2)}, \lambda_P^{(2)})$ with morphisms $a : \check{X}_P^{(2)} \to \check{X}_P^{(1)}$ and $b : \check{X}_P^{(2)} \to \check{X}_P$ such that

$$\lambda_P^{(2)} = \lambda_P^{(1)} \circ a = \lambda_P \circ b.$$

By using $(X_P^{(2)}, \lambda_P^{(2)})$ with Proposition 6.1.7, we can prove that the pre-K-Betti structures are equal. Therefore, by gluing the pre-K-Betti structures around any $P \in X$, we obtain a pre-K-Betti structure of $V(!D_1)$. (See Proposition 10.2.9 of [**23**].)

We denote it by $\mathcal{F}_V^{<D_1}$. It is called the canonical pre-K-Betti structure of $V(!D_1)$.

By taking the dual of $(V^\vee)(!D_1)$, we obtain a pre-K-Betti structure of $(V(!D))(*D_1)$, denoted by $\mathcal{F}_V^{<D \leq D_1}$.

Let D_3 be a hypersurface of X. Let $\varphi : X' \to X$ be a projective birational morphism such that:

(i) $X' \setminus D' \simeq X \setminus (D \cup D_3)$ where $D' := \varphi^{-1}(D \cup D_3)$,

(ii) D' is normal crossing.

We set $D'_1 := \varphi^{-1}(D_1)$. We have D'_2 such that $D' = D'_2 \cup D'_1$ is a decomposition. We set $V' = \varphi^* V(*D')$.

Proposition 6.4.7. — *The natural morphisms*

$$V(!D_1) \longrightarrow \varphi_\dagger V'(!D'_1), \quad \varphi_\dagger\big(V'(!D')(*D'_1)\big) \longrightarrow V(!D)(*D_1)$$

are compatible with the canonical pre-K-Betti structures.

Proof. — Let (X_P, λ_P) be a local resolution for V around $P \in X$. We take a projective birational morphism $\lambda : \check{X}'_P \to \check{X}_P \times_X X'$ such that:

(i) $\check{X}'_P$ is smooth,

(ii) the induced morphism $\varphi_P : \check{X}'_P \to \check{X}_P$ gives $\check{X}'_P \setminus \check{D}'_P \simeq \check{X}_P \setminus \check{D}_P$, where $\check{D}'_P := \lambda^{-1}(\check{X}_P \times_X D')$.

The induced map $\lambda'_P : \check{X}'_P \to X'$ gives a local resolution for V'. By Proposition 6.1.7, $\lambda_P^*(V)(!\check{D}_{P1}) \to \varphi_{P\dagger}(\lambda_P'^*V'(!\check{D}'_{P1}))$ is compatible with the pre-K-Betti structures. Then, we obtain that

$$V(!D_1) \longrightarrow \varphi_\dagger V'(!D'_1)$$

is compatible with the pre-K-Betti structures. We obtain the claim for the other as the dual. □

6.4.3. Pre-K-Betti structure on the real blow up. — Let X, D and V be as in the beginning of §6.4.2. Let $G : X \to \mathbb{C}^\ell$ be a holomorphic function such that $G^{-1}(D_0) \subset D_1$, where $D_0 = \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We obtain an object (X, G) in Cat_ℓ. Let $\pi : \widetilde{X}(G) \to X$ denote the real blow up.

Lemma 6.4.8. — *The natural morphism*

$$R\pi_* \, \mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big) \longrightarrow \mathrm{DR}_X\big(V(!D_1)\big)$$

is an isomorphism in $D^b(\mathbb{C}_X)$.

Proof. — It is enough to check the claim locally around each $P \in X$. Let (X_P, λ_P) be a local resolution of V around P. We set $G_P := G_{|X_P}$ and $\check{G}_P := G \circ \lambda_P$. We obtain a morphism $\lambda_P : (\check{X}_P, \check{G}_P) \to (X_P, G_P)$ in Cat_ℓ. We set $\check{\mathcal{M}}_P := \check{V}_P(!\check{D}_1)$. By Corollary 4.7.3, we have in $D^b(\mathbb{C}_{\widetilde{X}_P(G_P)})$ the isomorphism

$$R\widetilde{\lambda}_{P*} \, \mathrm{DR}^{\mathrm{rapid}}_{\check{X}_P,\check{G}_P}(\check{\mathcal{M}}_P) \simeq \mathrm{DR}^{\mathrm{rapid}}_{X_P,G_P}(\lambda_{P\dagger}\check{\mathcal{M}}_P) = \mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big)_{|\widetilde{X}_P(G_P)}.$$

By using $R\pi_{\check{G}_P *} \, \mathrm{DR}^{\mathrm{rapid}}_{\check{X}_P,\check{G}_P}(\check{\mathcal{M}}_P) \simeq \mathrm{DR}_{\check{X}_P}(\check{\mathcal{M}}_P)$, we obtain the claim. □

In the situation of the proof of Lemma 6.4.8, let $\widetilde{\widetilde{X}}_P(\check{D}_P)$ be the real blow up along $\check{D}_P$. We have the natural map $\rho : \widetilde{\widetilde{X}}_P(\check{D}_P) \to \widetilde{\widetilde{X}}_P(\check{G}_P)$. As in Lemma 5.1.8, we have the following natural isomorphism:

$$R\rho_* \, \mathrm{DR}^{\leq \check{D}_{P2} < \check{D}_{P1}}_{\widetilde{\widetilde{X}}_P(\check{D}_P)}(\check{V}_P) \simeq \mathrm{DR}^{\mathrm{rapid}}_{\check{X}_P,\check{G}_P}\big(\check{V}_P(!\check{D}_{P1})\big).$$

In particular, a good K-structure of $\check{V}_P$ induces a K-structure of

$$\mathrm{DR}^{\mathrm{rapid}}_{\check{X}_P,\check{G}_P}\big(\check{V}_P(!\check{D}_{P1})\big).$$

We would like to glue them.

Lemma 6.4.9. — *Suppose that there exists a finite family*

$$\big\{(\mathcal{U}_i,\lambda_i)\mid i\in\Lambda\big\}\quad\big(|\Lambda|<\infty\big)$$

of local resolutions of V such that $X=\bigcup\mathcal{U}_i$. Then, there exists an object $\mathcal{K}$ in $D^b(K_{\widetilde{X}(G)})$ with isomorphisms

$$c_1:\mathcal{K}\otimes\mathbb{C}\simeq \mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big)\qquad \text{in } D^b(\mathbb{C}_{\widetilde{X}(G)}),$$
$$c_2:R\pi_*\mathcal{K}\simeq\mathcal{F}_V^{<D_1}\qquad \text{in } D^b(K_X),$$

such that $c_2\otimes\mathbb{C}$ is equal to $R\pi_*c_1$.

Proof. — We shall construct a K-complex $\mathcal{K}$ on $\widetilde{X}(G)$ as follows. For $I\subset\Lambda$, we set $\mathcal{U}_I:=\bigcap_{i\in I}\mathcal{U}_i$. Let $\iota_I:\mathcal{U}_I\to X$ denote the inclusion. We set

$$G_I:=G_{|\mathcal{U}_I}.$$

Take local resolutions $\lambda_I:\check{\mathcal{U}}_I\to\mathcal{U}_I$ of V. We may assume to have

$$\lambda_{IJ}:\check{\mathcal{U}}_J\longrightarrow\check{\mathcal{U}}_I$$

such that $\iota_I\circ\lambda_I\circ\lambda_{IJ}=\iota_J\circ\lambda_J$ for any $I\subset J$. We have

$$\lambda_{I_1I_2}\circ\lambda_{I_2I_3}=\lambda_{I_1I_3}.$$

We put $V_I:=\lambda_I^*V$.

We set $\check{D}_I:=\lambda_I^{-1}(D)$, and $\check{D}_{I1}:=\lambda_I^{-1}(D_1)$. Let $\check{D}_{I2}$ denote the complement of $\check{D}_{I1}$ in $\check{D}_I$. Let $\check{\pi}_I:\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)\to\check{\mathcal{U}}_I$ denote the real blow up. We have the induced morphisms

$$\widetilde{\lambda}_{IJ}:\widetilde{\check{\mathcal{U}}}_J(\check{D}_J)\longrightarrow\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)$$

and the induced morphisms

$$\widetilde{\lambda}_I:\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)\longrightarrow\widetilde{\mathcal{U}}_I(G_I).$$

Let $\widetilde{\iota}_I:\widetilde{\mathcal{U}}_I(G_I)\to\widetilde{X}(G)$ denote the inclusion.

Let $\mathcal{L}_{K,I}$ denote the K-local system on $\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)$ with the Stokes structure associated to V_I with good K-structure. We have the constructible sheaves $\mathcal{L}_{K,I}^{<\check{D}_{I1}\leq\check{D}_{I2}}$ on $\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)$, and natural morphisms

$$\widetilde{\lambda}_{IJ}^{-1}\mathcal{L}_{K,I}^{<\check{D}_{I1}\leq\check{D}_{I2}}\longrightarrow\mathcal{L}_{K,J}^{<\check{D}_{J1}\leq\check{D}_{J2}}.$$

For any sheaf $\mathcal{F}$, let $\mathrm{Gd}(\mathcal{F})$ denote its Godement resolution. By the construction, we have natural morphisms

$$\widetilde{\lambda}_{IJ}^{-1}\,\mathrm{Gd}(\mathcal{L}_{K,I}^{<\check{D}_{I1}\leq\check{D}_{I2}})\longrightarrow\mathrm{Gd}(\widetilde{\lambda}_{IJ}^{-1}\mathcal{L}_{K,I}^{<\check{D}_{I1}\leq\check{D}_{I2}})\longrightarrow\mathrm{Gd}(\mathcal{L}_{K,J}^{<\check{D}_{J1}\leq\check{D}_{J2}}).\tag{107}$$

We set

$$\mathcal{G}^\bullet_{K,I} := \widetilde{\iota}_{I*}\widetilde{\lambda}_{I*}\,\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq \check{D}_{I2}}_{K,I})[d_X]$$

on $\widetilde{X}(G)$. The morphisms (107) induce $\lambda_{JI}:\mathcal{G}^\bullet_{K,I}\to\mathcal{G}^\bullet_{K,J}$. They satisfy

$$\lambda_{I_1I_2}\circ\lambda_{I_2I_3}=\lambda_{I_1I_3}.$$

We take a K-vector space U_K with a basis $\{e_i\mid i\in\Lambda\}$. Let $U_{K,I}$ denote the subspace in $\bigwedge^\bullet U_K$ generated by $e_{i_1}\wedge\cdots\wedge e_{i_m}$ where $I=(i_1,\dots,i_m)$. For $m\in\mathbb{Z}_{\geq 0}$, we set

$$\mathcal{K}^{m,\bullet}_K := \bigoplus_{|I|=m+1}\mathcal{G}^\bullet_{I,K}\otimes U_{K,I}.$$

We have the morphism $\mathcal{K}^{m,\bullet}_K\to\mathcal{K}^{m+1,\bullet}_K$ induced by the morphisms

$$\lambda_{I,I\cup\{j\}}\otimes(e_j\wedge\bullet).$$

They give a double complex $\mathcal{K}^{\bullet,\bullet}_K$ of $K_{\widetilde{X}(G)}$-modules. The total complex is denoted by $\mathcal{K}^\bullet_K$.

We have the $\mathbb{C}$-local systems $\mathcal{L}_I$ with the Stokes structure on $\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)$ associated to V. Using $\mathcal{L}_I$ with the same construction, we obtain complexes $\mathcal{G}^\bullet_{\mathbb{C},I}$, a double complex $\mathcal{K}^{\bullet,\bullet}_{\mathbb{C}}$ and a complex $\mathcal{K}^\bullet_{\mathbb{C}}$.

We have naturally defined isomorphisms

$$\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{K,I}\otimes\mathbb{C}\longrightarrow\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{I}.$$

The natural morphisms

$$\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{K,I})\otimes\mathbb{C}\longrightarrow\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{I})$$

are quasi-isomorphisms. By the projection formula, we have the natural isomorphisms

$$\widetilde{\iota}_{I*}\widetilde{\lambda}_{I*}\big(\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{K,I})[d_X]\otimes\mathbb{C}\big)\simeq\mathcal{G}_{K,I}\otimes\mathbb{C}.$$

It also implies that the complex $(\widetilde{\iota}\circ\widetilde{\lambda}_I)_*(\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{K,I})\otimes\mathbb{C})$ represents

$$R(\widetilde{\iota}\circ\widetilde{\lambda}_I)_*\big(\mathrm{Gd}(\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{K,I})\otimes\mathbb{C}\big).$$

Hence, the natural morphism $\mathcal{G}_{K,I}\otimes\mathbb{C}\to\mathcal{G}_{\mathbb{C},I}$ is a quasi-isomorphism. Then, it is easy to deduce that the natural morphism $\mathcal{K}^\bullet_K\otimes_K\mathbb{C}\to\mathcal{K}^\bullet_{\mathbb{C}}$ is a quasi-isomorphism.

We have the natural quasi-isomorphism

$$\mathcal{L}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{I}[d_X]\longrightarrow\mathrm{DR}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)}(\check{V}_I).$$

We have morphisms

$$\widetilde{\lambda}_{JI}^{-1}\,\mathrm{DR}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)}(\check{V}_I)\longrightarrow \mathrm{DR}^{<\check{D}_{J1}\leq\check{D}_{J2}}_{\widetilde{\check{\mathcal{U}}}_J(\check{D}_J)}(\check{V}_J).$$

By applying the above construction to $\mathrm{DR}^{<\check{D}_{I1}\leq\check{D}_{I2}}_{\widetilde{\check{\mathcal{U}}}_I(\check{D}_I)}(\check{V}_I)$ instead of $\mathcal{L}_I^{<\check{D}_{I1}\leq\check{D}_{I2}}[d_X]$, we obtain double complexes $\mathcal{G}^{\bullet;\bullet}_{I,\mathrm{DR}}$ on $\widetilde{X}(G)$, and a complex $\mathcal{K}^{\bullet}_{\mathrm{DR}}$ on $\widetilde{X}(G)$. The natural morphism $\mathcal{K}^{\bullet}_{\mathbb{C}}\to\mathcal{K}^{\bullet}_{\mathrm{DR}}$ is a quasi-isomorphism.

Set $\check{H}_I:=\check{G}_I^{-1}(0)$. We have on $\widetilde{\check{\mathcal{U}}}_I(\check{H}_I)$ the complexes

$$\mathrm{DR}^{<\check{D}_{I1}}_{\widetilde{\check{\mathcal{U}}}_I(\check{H}_I)}(\check{V}_I)\quad\text{and}\quad \mathrm{DR}^{<\check{H}_I}_{\widetilde{\check{\mathcal{U}}}_I(\check{H}_I)}\big(\check{V}_I(!\check{D}_{I1})\big).$$

By applying the above construction to them, we obtain double complexes $\mathcal{G}^{\bullet;\bullet}_{I,a}$ $(a=1,2)$, and complexes $\mathcal{K}_a$ $(a=1,2)$ on $\widetilde{X}(G)$. We have the following natural quasi-isomorphisms of complexes, as in Lemma 5.1.6:

$$\mathcal{G}^{\bullet}_{I,\mathrm{DR}}\longleftarrow\mathcal{G}^{\bullet}_{I,1}\longrightarrow\mathcal{G}^{\bullet}_{I,2}.$$

Hence, we have the natural quasi-isomorphisms of complexes

$$\mathcal{K}^{\bullet}_{\mathrm{DR}}\longleftarrow\mathcal{K}^{\bullet}_{1}\longrightarrow\mathcal{K}^{\bullet}_{2}.$$

We set

$$\mathcal{G}^{\bullet}_{I,3}:=\widetilde{\iota}_{I*}\,\mathrm{Gd}\left(\widetilde{\iota}_I^{\,-1}\,\mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big)\right).$$

As before, by the Čech construction we obtain a complex $\mathcal{K}^{\bullet}_3$. We have natural quasi-isomorphism $\mathcal{G}_{I,3}\to\mathcal{G}_{I,2}$, which induce $\mathcal{K}^{\bullet}_3\to\mathcal{K}^{\bullet}_2$. By construction, we have natural quasi-isomorphisms

$$\mathrm{Gd}\,\mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big)\longrightarrow\mathcal{K}^{\bullet}_3.$$

(See Proposition 2.8.4 of [**23**].) In all, we obtain the sequence of quasi-isomorphisms

$$\mathcal{K}^{\bullet}_K\otimes\mathbb{C}\longrightarrow\mathcal{K}^{\bullet}_{\mathrm{DR}}\longleftarrow\mathcal{K}^{\bullet}_1\longrightarrow\mathcal{K}^{\bullet}_2\longleftarrow\mathrm{Gd}\,\mathrm{DR}^{\mathrm{rapid}}_{X,G}\big(V(!D_1)\big).\tag{108}$$

We define c_1 as the composite of the morphisms.

The projections $\varphi_i:\mathcal{K}^{\bullet}_{K|\widetilde{\mathcal{U}}_i(G_i)}\to\mathcal{G}^{\bullet}_{K,i|\widetilde{\mathcal{U}}_i(G_i)}$ are quasi-isomorphisms. It is easy to see that

$$\lambda_{\{ij\},i|\widetilde{\mathcal{U}}_{ij}(G_{ij})}\circ\varphi_{i|\widetilde{\mathcal{U}}_{ij}(G_{ij})}\quad\text{and}\quad\lambda_{\{ij\},j|\widetilde{\mathcal{U}}_{ij}(G_{ij})}\circ\varphi_{j|\widetilde{\mathcal{U}}_{ij}(G_{ij})}$$

are chain homotopic. Hence, $\pi_*\mathcal{K}^{\bullet}$ is a K-perverse sheaf obtained as the gluing of $\pi_*\mathcal{G}_{K,i|\widetilde{\mathcal{U}}_i(G_i)}$. We obtain an isomorphism of K-perverse sheaves

$$\mathcal{F}_V^{<D_1}\simeq\pi_*\mathcal{K}^{\bullet},$$

which is c_2. We can easily compare $(c_2\otimes\mathbb{C})_{|\mathcal{U}_i}$ and $R\pi_*(c_1)_{|\mathcal{U}_i}$, and we obtain $c_2\otimes\mathbb{C}=R\pi_*(c_1)$. □

6.4.4. Sequence of hypersurface pairs. — Let X be a complex manifold. Let $\boldsymbol{H}=(H_!,H_*)$ be an ordered pair of (possibly empty) hypersurfaces of X. Such a pair is called a hypersurface pair in the following. For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, we define

$$\mathfrak{P}_{\boldsymbol{H}}(\mathcal{M}):=\big(\mathcal{M}(*H_*)\big)(!H_!)\quad\text{and}\quad\mathfrak{P}'_{\boldsymbol{H}}(\mathcal{M}):=\big(\mathcal{M}(!H_!)\big)(*H_*).$$

We set $\boldsymbol{DH}=(H_*,H_!)$. Then, we have natural isomorphisms

$$\boldsymbol{D}(\mathfrak{P}_{\boldsymbol{H}}(\mathcal{M}))\simeq\mathfrak{P}'_{\boldsymbol{DH}}(\boldsymbol{D}\mathcal{M}).$$

If we are given a sequence of hypersurface pairs $\mathfrak{H}=(\boldsymbol{H}_1,\boldsymbol{H}_2,\dots,\boldsymbol{H}_N)$, we set

$$\mathfrak{P}_{\mathfrak{H}}:=\mathfrak{P}_{\boldsymbol{H}_N}\circ\cdots\circ\mathfrak{P}_{\boldsymbol{H}_2}\circ\mathfrak{P}_{\boldsymbol{H}_1}\quad\text{and}\quad\mathfrak{P}'_{\mathfrak{H}}:=\mathfrak{P}'_{\boldsymbol{H}_N}\circ\cdots\circ\mathfrak{P}'_{\boldsymbol{H}_2}\circ\mathfrak{P}'_{\boldsymbol{H}_1}.$$

Clearly, $\mathfrak{P}_{\mathfrak{H}}$ can be described as $\mathfrak{P}'_{\mathfrak{H}_1}$ for an appropriate $\mathfrak{H}_1$. We shall use a special case of this operation in §8.5.

6.4.5. Generalization. — Let X, D and V be as in the beginning of §6.4.2. Suppose that we are given a sequence of hypersurface pairs $\mathfrak{H}=(\boldsymbol{H}_1,\dots,\boldsymbol{H}_N)$ contained in D. Let us observe that $\mathfrak{P}_{\mathfrak{H}}(V)$ and $\mathfrak{P}'_{\mathfrak{H}}(V)$ are naturally equipped with pre-K-Betti structures.

Let P be any point of X. We take a local resolution (X_P,λ_P) of V around P. By taking the pull back, we obtain a sequence of hypersurface pairs

$$\check{\mathfrak{H}}_P:=\lambda_P^*(\mathfrak{H})$$

contained in $\check{D}_P$. For the irreducible decomposition $\check{D}_P=\bigcup_{j\in\Lambda_P}\check{D}_{Pj}$, there uniquely exists a subset $I_P\subset\Lambda_P$ such that

$$\mathfrak{P}_{\check{\mathfrak{H}}_P}(\check{V}_P)\simeq\check{V}_P\big(!\check{D}_P(I_P)\big),$$

where $\check{D}_P(I_P)=\bigcup_{j\in I_P}\check{D}_{Pj}$. Hence, we have the canonical pre-K-Betti structure $\check{V}_P(!\check{D}_P(I_P))$ induced by the good K-structure of $\check{V}_P$. By the natural isomorphism

$$\lambda_{P\dagger}\mathfrak{P}_{\check{\mathfrak{H}}_P}(\check{V}_P)\simeq\mathfrak{P}_{\mathfrak{H}}(V)_{|X_P},$$

we obtain a pre-K-Betti structure of $\mathfrak{P}_{\mathfrak{H}}(V)_{|X_P}$.

Suppose that we are given other local resolutions $(X_P^{(i)},\lambda_P^{(i)})$ $(i=1,2)$ as in §6.4.2. We put $\check{V}_P^{(2)}:=\lambda_P^{(2)*}V$. We have the expression

$$\mathfrak{P}_{\check{\mathfrak{H}}_P^{(2)}}(\check{V}_P^{(2)})\simeq\check{V}_P^{(2)}\big(!\check{D}_P^{(2)}(I_P^{(2)})\big).$$

For the morphism $a : \check{X}_P^{(2)} \to \check{X}_P$, we have

$$\check{D}_P(I_P) = a\big(\check{D}_P^{(2)}(I_P^{(2)})\big).$$

We have the natural isomorphisms of holonomic $\mathcal{D}$-modules

$$a_\dagger \mathfrak{P}_{\check{\mathfrak{H}}_P^{(2)}}(\check{V}_P^{(2)}) \simeq a_\dagger\big(\check{V}_P^{(2)}\big(!a^{-1}(\check{D}_P(I_P))\big)\big) \simeq \mathfrak{P}_{\check{\mathfrak{H}}_P}(\check{V}_P)$$

which are compatible with the pre-K-Betti structures. Therefore, we obtain the pre-K-Betti structures of $\mathfrak{P}_{\mathfrak{H}}(V)$ by gluing the locally given pre-K-Betti structures. We obtain a pre-K-Betti structure of $\mathfrak{P}'_{\mathfrak{H}}(V)$ in the same way.

They are called the canonical pre-K-Betti structure of $\mathfrak{P}_{\mathfrak{H}}(V)$ and $\mathfrak{P}'_{\mathfrak{H}}(V)$, denoted by $\mathcal{F}_{\mathfrak{H},V}$ and $\mathcal{F}'_{\mathfrak{H},V}$.

Lemma 6.4.10. — *Let $\mathfrak{H}^\circ = (\boldsymbol{H}_1^\circ, \dots, \boldsymbol{H}_N^\circ)$ be a sequence of hypersurface pairs such that $H_{i*}^\circ \subset H_{i*}$ and $H_{i!}^\circ \supset H_{i!}$ for any i. The natural morphisms $\mathfrak{P}_{\mathfrak{H}^\circ}(V) \to \mathfrak{P}_{\mathfrak{H}}(V)$ and $\mathfrak{P}'_{\mathfrak{H}^\circ}(V) \to \mathfrak{P}'_{\mathfrak{H}}(V)$ are compatible with the K-Betti structures.*

Proof. — It is reduced to the easy case where V is good. □

Let $G : X \to \mathbb{C}^\ell$ be a holomorphic function. The following lemma can be shown by the same arguments as those in the proof of Lemma 6.4.8 and Lemma 6.4.9.

Proposition 6.4.11. — *Suppose that $G^{-1}(D_0) \subset H_{N!}$ for $\boldsymbol{H}_N = (H_{N!}, H_{N*})$. Then, the natural morphism*

$$R\pi_* \operatorname{DR}_{X,G}^{\mathrm{rapid}}\big(\mathfrak{P}_{\mathfrak{H}}(V)\big) \longrightarrow \operatorname{DR}_X\big(\mathfrak{P}_{\mathfrak{H}}(V)\big)$$

is an isomorphism. If we are given a finite family of local resolutions of V as in Lemma 6.4.9, then there exists an object $\mathcal{K}$ in $D^b(K_{\widetilde{X}(G)})$ with isomorphisms

$$\begin{aligned} &c_1 : \mathcal{K} \otimes \mathbb{C} \simeq \operatorname{DR}_{X,G}^{\mathrm{rapid}}\big(\mathfrak{P}_{\mathfrak{H}}(V)\big) \qquad \textit{in } D^b(\mathbb{C}_{\widetilde{X}(G)}),\\ &c_2 : R\pi_*\mathcal{K} \simeq \mathcal{F}_{\mathfrak{H},V} \qquad \textit{in } D^b(K_X), \end{aligned}$$

such that $c_2 \otimes \mathbb{C}$ is equal to $R\pi_* c_1$. □

Let D_3, $\varphi : X' \to X$ and V' be as in Proposition 6.4.7. By the pull back, we obtain a sequence of hypersurface pairs $\mathfrak{H}' := \varphi^{-1}\mathfrak{H}$.

Proposition 6.4.12. — *The natural morphisms*

$$\varphi_\dagger \mathfrak{P}_{\mathfrak{H}'}\big(V'(!D')\big) \longrightarrow \mathfrak{P}_{\mathfrak{H}}\big(V(!D)\big) \quad \textit{and} \quad \mathfrak{P}_{\mathfrak{H}}(V) \longrightarrow \varphi_\dagger \mathfrak{P}_{\mathfrak{H}'}(V')$$

are compatible with the canonical pre-K-Betti structures. The natural morphisms

$$\varphi_\dagger \mathfrak{P}'_{\mathfrak{H}'}\big(V'(!D')\big) \longrightarrow \mathfrak{P}'_{\mathfrak{H}}\big(V(!D)\big) \quad \textit{and} \quad \mathfrak{P}'_{\mathfrak{H}}(V) \longrightarrow \varphi_\dagger \mathfrak{P}'_{\mathfrak{H}'}(V')$$

are also compatible with the canonical pre-K-Betti structures.

Proof. — It is reduced to the case where V is good. We have

$$\mathfrak{P}_{\mathfrak{H}}(V) = V(!D^{(1)}) \quad \text{and} \quad \mathfrak{P}_{\mathfrak{H}'}(V') = V'(!D'^{(1)})$$

for some $D^{(1)} \subset D$ and $D'^{(1)} \subset D'$. We have $\varphi(D'^{(1)}) = D^{(1)}$. We set $L^{(1)} := \varphi^{-1}(D^{(1)})$. Then, the natural morphisms

$$\mathfrak{P}_{\mathfrak{H}}(V) \simeq \varphi_\dagger V'(!L^{(1)}) \longrightarrow \varphi_\dagger \mathfrak{P}_{\mathfrak{H}'}(V')$$

are compatible with the pre-K-Betti structures. Similarly, we obtain that

$$\mathfrak{P}'_{\mathfrak{H}}(V) \longrightarrow \varphi_\dagger \mathfrak{P}'_{\mathfrak{H}'}(V')$$

is compatible with the pre-K-Betti structure. We obtain the others by the dual. □

6.5. Preliminary for push-forward

Let Y be a complex manifold with a hypersurface D_Y. Let $G : X \to Y$ be a projective morphism of complex manifolds. We set

$$D_{X0} := G^{-1}(D_Y).$$

Let D_X be a hypersurface of X with a decomposition $D_X = D_{X1} \cup D_{X2}$ such that $D_{X0} \subset D_{X2}$.

Let V be a meromorphic flat connection on (X, D_X) with a good K-structure. Put $\mathcal{M} := V(!D_{X2})$. Let $\mathcal{F}_{\mathcal{M}}$ be the canonical pre-K-Betti structure. Assume the following:

▷ $G^i_\dagger \mathcal{M} = 0$ for any $i \neq 0$, and $V_1 := G^0_\dagger(\mathcal{M})(*D_Y)$ is a meromorphic flat connection on (Y, D_Y).

We put

$$\mathcal{G} := RG_*(\mathcal{F}_{\mathcal{M}})_{|Y-D_Y},$$

which gives a pre-K-Betti structure of $G^0_\dagger(\mathcal{M})_{|Y-D_Y}$.

The following theorem will be used in the proof of Theorem 8.1.1. (See §8.5.1.)

THEOREM 6.5.1. — *The K-structure $\mathcal{G}$ of V_1 is good, i.e., it is compatible with the Stokes filtrations. Moreover, $RG_*\mathcal{F}_{\mathcal{M}}$ is the canonical pre-K-Betti structure of $G^0_\dagger(\mathcal{M})$.*

Proof. — It is enough to consider the issues locally around any point P of Y. Let (Y_P, λ_P) be a local resolution of V_1. We take a projective birational morphism $\lambda : X' \to \check{Y}_P \times_Y X$ such that:

(i) X' is smooth,

(ii) $D'_X := \check{X}_P \times_X D_X$ is normal crossing,

(iii) the induced morphism $X' \setminus D'_X \to X \setminus D_X$ is an isomorphism.

Let $\mu : X' \to X$ and $G' : X' \to \check{Y}_P$ be the induced maps. We obtain a meromorphic flat connection $V' = \mu^*V$ with a good K-structure. We set $D'_{X2} := \mu^{-1}(D_{X2})$. We have

$$\mu_\dagger\big(V'(!D'_{X2})\big) = V(!D_{X2}),\ \ G'_\dagger\big(V'(!D'_X)\big)(*D_P) = \lambda_P^* V_1,$$
$$\lambda_{P\dagger} G'_\dagger\big(V'(!D'_X)\big) \simeq \mathcal{M}_{|Y_P}.$$

It is enough to prove the claims on $\check{Y}_P$. Hence, we may and will assume that D_Y is normal crossing, and that V_1 is a good meromorphic flat bundle.

It is enough to consider the case where $Y := \Delta^n$ and $D_Y := \bigcup_{i=1}^{\ell}\{z_i = 0\}$. We have

$$G^0_\dagger(\mathcal{M}) = V_1(!D_Y).$$

Let $F : Y \to \mathbb{C}^\ell$ be given by $(z_1, \dots, z_\ell)$. We set $F_X := F \circ G$. We obtain in Cat_ℓ a projective morphism

$$G : (X, F_X) \longrightarrow (Y, F).$$

We have $\widetilde{Y}(F) = \widetilde{Y}(D_Y)$. According to Corollary 4.7.5, we have the following isomorphism in $D^b_c(\widetilde{Y}(D_Y))$:

$$R\widetilde{G}_* \,\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(F_X)}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rapid}}_{\widetilde{Y}(D)}(G^0_\dagger \mathcal{M}).$$

The good K-structure of V induces a K-structure of $\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D_{X2})}(\mathcal{M})$ on $\widetilde{X}(D_{X_2})$ (Lemma 6.4.9). It induces a K-structure of $R\widetilde{G}_* \,\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(F_X)}(\mathcal{M})$, which is compatible with the natural K-structure of $G^0_\dagger(\mathcal{M})_{|Y \setminus D_Y}$.

Let us prove that the K-structure of V_1 is good. First, we consider the case where V_1 is unramifiedly good. Take $\mathfrak{a} \in \mathrm{Irr}(V_1)$. Let $L(-\mathfrak{a})$ be a meromorphic flat bundle with a K-structure as in §6.1.4. Then,

$$V \otimes G^* L(-\mathfrak{a})$$

has a good K-structure. By applying the previous argument, we obtain that

$$\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{Y}(D_Y)}\big(V_1\otimes L(-\mathfrak{a})\big)$$

has a K-structure, whose restriction to $Y\setminus D_Y$ is the same as one induced by the K-structure of V_1 and $L(-\mathfrak{a})$. Hence, by Lemma 6.1.8, we obtain that the K-structure of V_1 is good if V_1 is unramifiedly good.

Let us consider the case where V_1 is not necessarily unramified. Let

$$\kappa : Y' \longrightarrow Y$$

be a ramified covering such that κ^*V_1 is unramifiedly good. We put

$$D'_Y := \kappa^{-1}(D_Y).$$

We take a projective birational map $\mu : X' \to X\times_Y Y'$ such that:

(i) X' is smooth,

(ii) $X' - \mu^{-1}(X\times_Y D') \simeq X - (X\times_Y D')$.

We set $D'_X := \mu^{-1}(D_X\times_Y Y')$. Let $\mu_1 : X'\to X$ and $G' : X'\to Y'$ be the induced morphisms. We have the decomposition $D'_X = D'_{X1}\cup D'_{X2}$ such that

$$D'_{X2} := \mu_1^{-1}(D_{X2}).$$

Let $\mathcal{M}' := \mu_1^*(V)(!D'_{X2})$. Applying the previous argument to $G'^0_\dagger(\mathcal{M}')$, we obtain that the K-structure of V_1 is good even in the ramified case.

Because the pre-K-Betti structure $\mathcal{G}$ of $G^0_\dagger\mathcal{M}$ is induced by the K-structure of $\mathrm{DR}^{\mathrm{rapid}}_{\widetilde{X}(D)}(G^0_\dagger\mathcal{M})$, it is canonical. Thus, the proof of Theorem 6.5.1 is finished. □

Corollary 6.5.2. — *Under the assumption, the induced K-structure of a meromorphic flat connection $G^0_\dagger(\boldsymbol{D}\mathcal{M})$ is good, and $RG_*\boldsymbol{D}\mathcal{F}_{\mathcal{M}}$ gives the canonical pre-K-Betti structure of $G^0_\dagger\boldsymbol{D}\mathcal{M}$.* □

We have a variant of Theorem 6.5.1 and Corollary 6.5.2. Let

$$\mathfrak{H} = (\boldsymbol{H}_1,\dots,\boldsymbol{H}_N)$$

be a sequence of hypersurface pairs of X contained in D_X.

Theorem 6.5.3. — *Suppose either*

(i) $D_{X0}\subset H_{N!}$*; or*

(ii) $H_{N!}=\varnothing$ *and* $D_{X0}\subset H_{N*}$.

We also assume that $G^i_{\dagger}\mathfrak{P}_{\mathfrak{H}}(V) = 0$ *unless* $i = 0$. *Then, the induced* K*-structure of* $G^0_{\dagger}\mathfrak{P}_{\mathfrak{H}}(V)(*D_Y)$ *is good, and the induced pre-*K*-Betti structure* $RG_*(\mathcal{F}_{\mathfrak{H},V})$ *is the canonical pre-*K*-Betti structure of* $G^0_{\dagger}\mathfrak{P}_{\mathfrak{H}}(V)$.

Proof. — The case (i) can be proved by Proposition 6.4.11 and the argument in the proof of Theorem 6.5.1. The case (ii) can be obtained as the dual. □

CHAPTER 7

K-HOLONOMIC $\mathcal{D}$-MODULES

7.1. Preliminary

7.1.1. Cells and cell functions. — Let X be a complex manifold or a smooth complex algebraic variety. In the complex analytic case, we use ordinary topology. In the algebraic case, we consider Zariski topology. In the algebraic setting, $\mathcal{D}$-modules are assumed to be algebraic. An open subset $\mathcal{U}$ is called principal if it is the complement of a hypersurface. Let P be a point of X. For any closed subvariety W of X, let $\dim_P W$ denote the dimension of the germ of W at P. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X with $\dim_P \operatorname{Supp} \mathcal{M} \leq n$. An n-dimensional cell of $\mathcal{M}$ at P is a tuple $\mathcal{C} = (Z, U, \varphi, V)$ as follows:

(Cell 1) $\varphi : Z \to X$ is a morphism of complex manifolds or smooth complex algebraic varieties, such that $P \in \varphi(Z)$ and $\dim Z = n$. We assume that there exists a neighbourhood of X_P of P in X such that $\varphi : \varphi^{-1}(X_P) \to X_P$ is projective. We permit that Z may be non-connected or empty.

(Cell 2) $U \subset Z$ is a principal open subset with the complementary hypersurface denoted by D_Z. We assume that the restriction $\varphi_{|U}$ is an immersion, and that there exists a hypersurface H of X_P such that $\varphi^{-1}(H) = D_Z \cap \varphi^{-1}(X_P)$.

(Cell 3) V is a meromorphic flat connection on (Z, D_Z) with morphisms

$$\varphi_\dagger(V_!)_P \longrightarrow \mathcal{M}_P \longrightarrow \varphi_\dagger(V)_P$$

such that $\mathcal{M}_P(*H) \simeq \varphi_\dagger(V)_P$ for the hypersurface H in (Cell 2), where the subscript "P" means the restriction to X_P. Note that we have $\mathcal{M}_P(!H) \simeq \varphi_\dagger(V_!)_P$, where $V_! := V(!D_Z)$. The restriction of V to some connected components of Z may be 0.

The cell $\mathcal{C}$ is called good if

(i) D_Z is normal crossing,

(ii) V is good on (Z, D_Z).

For a given holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and $P \in \operatorname{Supp}\mathcal{M}$, there always exists a cell for $\mathcal{M}$ at P. If $\dim_P \mathcal{M} = 1$, any cell is good. If $\dim_P \mathcal{M} = 2$, there always exists a good cell for $\mathcal{M}$ at P, due to Kedlaya [**26**]. (See also [**44**] for the algebraic case.) In the algebraic case, there always exists a good cell for $\mathcal{M}$ at P (see [**27**], [**44**] and [**47**]).

Remark 7.1.1. — Let (Z, U, φ) be a tuple satisfying (Cell 1) and (Cell 2). If we are given a meromorphic flat connection V on (Z, D_Z), the tuple (Z, U, φ, V) is called a cell at P. □

Let g be a holomorphic or algebraic function on X_P. It is called a cell function for $\mathcal{C}$ if $U = \varphi(\operatorname{Supp}\mathcal{M}_P \setminus g^{-1}(0))$. For such g, we obtain a description of $\mathcal{M}_P$ as the cohomology of the complex in the category of analytic or algebraic holonomic $\mathcal{D}_{X_P}$-modules:

$$\psi_g^{(1)}\big(\varphi_\dagger(V)_P\big) \longrightarrow \Xi_g^{(0)}\big(\varphi_\dagger(V)_P\big) \oplus \phi_g^{(0)}(\mathcal{M}_P) \longrightarrow \psi_g^{(0)}\big(\varphi_\dagger(V)_P\big).$$

For a given cell, a cell function always exists after we shrink X_P and Z appropriately.

Remark 7.1.2. — Let $\mathcal{C}$ be a cell of $\mathcal{M}$ at P. If we have a neighbourhood X_P of P for which (Cell 1–3) are satisfied, they are also satisfied for any neighbourhood $X'_P \subset X_P$. Hence, we do not have to be careful with a choice of X_P. □

7.1.2. Refinement and enhancement. — Let $\mathcal{C}' = (Z', \varphi', U', V')$ and $\mathcal{C} = (Z, \varphi, U, V)$ be n-cells of $\mathcal{M}$ at P. We say that $\mathcal{C}'$ is a refinement of $\mathcal{C}$, and denote $\mathcal{C}' \prec \mathcal{C}$ if the following holds:

▷ φ' factors through φ in the sense that there exists $\varphi_1 : Z' \to Z$ such that
  (i) $\varphi' = \varphi \circ \varphi_1$,
  (ii) $\varphi_1(U') \subset U$.

▷ $V' = \varphi_1^* V \otimes \mathcal{O}_{Z'}(*D_{Z'})$, where $D_{Z'} := Z' - U'$.

In that situation, there exist naturally induced morphisms

$$\varphi'_\dagger(V'_!)_P \longrightarrow \varphi_\dagger(V_!)_P \longrightarrow \mathcal{M}_P \longrightarrow \varphi_\dagger(V)_P \longrightarrow \varphi'_\dagger(V')_P. \tag{109}$$

We say that $\mathcal{C}'$ is a dominant refinement of $\mathcal{C}$ if U' is dense in U.

Let $\mathcal{C} = (Z, U, \varphi, V)$ be an n-cell of $\mathcal{M}$ at P. We take an n-dimensional closed subvariety $Z' \subset X$ such that $\dim(\varphi(Z) \cap Z') < n$. We take a refinement of $\mathcal{C}$ such that $\varphi(U) \cap Z' = \varnothing$. Let Z_1 be a complex manifold with a projective birational morphism $\varphi_1 : Z_1 \to Z'$ and a smooth open subset $U_1 \subset Z_1$ such that

(i) $\varphi_{1|U_1}$ is an immersion,

(ii) $Z_1 - U_1$ is normal crossing and the pull back of a hypersurface in X around P.

We set $\widetilde{Z} := Z \sqcup Z_1$ and $\widetilde{U} := U \sqcup U_1$. We have the induced map $\widetilde{\varphi} : \widetilde{Z} \to X$. Let $\widetilde{V}$ be a meromorphic flat connection on $\widetilde{Z}$ such that $\widetilde{V}_{|Z} = V$ and $\widetilde{V}_{|Z_1} = 0$. Then, it is easy to observe that $\widetilde{\mathcal{C}} := (\widetilde{Z}, \widetilde{U}, \widetilde{\varphi}, \widetilde{V})$ is an n-cell of $\mathcal{M}$, which is called an enhancement of $\mathcal{C}$.

In the following, for a cell $\mathcal{C} = (Z, U, \varphi, V)$, we implicitly assume $\varphi^{-1}(X_P) = Z$ by taking a refinement of $\mathcal{C}$. So we omit the subscript 'P' in $\varphi_\dagger(V_!)_P$ and $\varphi_\dagger(V)_P$.

7.1.3. K-cells and the induced pre-K-Betti structure on the nearby cycle sheaves. — Let $\mathcal{F}$ be a pre-K-Betti structure of $\mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be an n-cell of $\mathcal{M}$ at P.

Definition 7.1.3. — We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible if the following holds:

- ▷ The induced K-structure of $V_{|U}$ is good. (We do not assume that V is a good meromorphic flat bundle. See §6.4.)
- ▷ The induced morphisms $\varphi_\dagger(V_!) \to \mathcal{M}_P \to \varphi_\dagger(V)$ are compatible with the pre-K-Betti structures. (See §6.4.2 for the canonical pre-K-Betti structures of $V_!$ and V.)

Such a cell $\mathcal{C}$ is called a K-cell of $(\mathcal{M}, \mathcal{F})$. □

It is not difficult to construct an example of a pre-K-holonomic $\mathcal{D}$-module, for which there does not exist a K-cell at some point.

Lemma 7.1.4. — *Let $\mathcal{C} = (Z, U, \varphi, V)$ be a K-cell of $(\mathcal{M}, \mathcal{F})$ at P. Any refinement $\mathcal{C}' = (Z', U', \varphi', V')$ of $\mathcal{C}$ is also a K-cell. Moreover, the induced morphisms in* (109) *are compatible with pre-K-Betti structures.*

Proof. — It follows from Proposition 6.4.7. □

Let g be any cell function for a K-cell $\mathcal{C}$. We observe that $\Xi_g^{(a)}(\varphi_\dagger(V))$, $\psi_g^{(a)}(\varphi_\dagger(V))$ and $\phi_g^{(a)}(\mathcal{M}_P)$ are equipped with induced pre-K-Betti structures. We set $V_{g\star}^{a,b} := \Pi_{\varphi^{-1}(g)\star}^{a,b} V$ for $\star = *, !$. Note that $\varphi_\dagger(V_{g\star}^{a,b})$ have the canonical pre-K-Betti structures. Since $\Xi_g^{(a)}(\varphi_\dagger V)$ and $\psi_g^{(a)}(\varphi_\dagger V)$ are of the form

$$\mathrm{Ker}\,\big(\varphi_\dagger(V_{g!}^{a,b}) \to \varphi_\dagger(V_{g*}^{a',b'})\big),$$

they are equipped with induced pre-K-Betti structures, denoted by ${}^D\Xi_g^{(a)}(\varphi_*\mathcal{F}_V)$ and ${}^D\psi_g^{(a)}(\varphi_*\mathcal{F}_V)$. We will use the following obvious lemma implicitly.

Lemma 7.1.5. — *The natural isomorphisms*

$$\Xi_g^{(a)}\big(\varphi_\dagger(V)\big) \simeq \varphi_\dagger\big(\Xi_{g\circ\varphi}^{(a)}(V)\big), \quad \psi_g^{(a)}(\varphi_\dagger V) \simeq \varphi_\dagger \psi_{g\circ\varphi}^{(a)}(V)$$

are compatible with the induced pre-K-Betti structures. □

Since $\phi_g^{(0)}(\mathcal{M}_P)$ is the cohomology of the complex

$$\varphi_\dagger V_! \longrightarrow \Xi_g^{(0)}(\varphi_\dagger V) \oplus \mathcal{M} \longrightarrow \varphi_\dagger V,$$

we obtain a pre-K-Betti structure of $\phi_g^{(0)}(\mathcal{M}_P)$, denoted by ${}^D\phi_g^{(0)}(\mathcal{F})$. The tuples

$$\big(\Xi_g^{(a)}(\varphi_\dagger V), {}^D\Xi_g^{(a)}(\varphi_*\mathcal{F}_V)\big), \ \big(\psi_g^{(a)}(\varphi_\dagger V), {}^D\psi_g^{(a)}(\varphi_*\mathcal{F}_V)\big), \ \big(\phi_g^{(a)}(\mathcal{M}), {}^D\phi_g^{(a)}(\mathcal{F})\big)$$

are also denoted by $\Xi_g^{(a)}\varphi_\dagger(V,\mathcal{F}_V)$, $\psi_g^{(a)}\varphi_\dagger(V,\mathcal{F}_V)$ and $\phi_g^{(a)}(\mathcal{M},\mathcal{F})$. We will often omit to denote the pre-K-Betti structures if there is no risk of confusion.

7.2. K-Betti structure

7.2.1. Definition of K-Betti structure. — Let X be any complex manifold, and P be any point of X. Let $(\mathcal{M},\mathcal{F})$ be a pre-K-holonomic $\mathcal{D}$-module on X. Let us define the notion of K-Betti structure of $\mathcal{M}$ at P, inductively on the dimension of $\mathrm{Supp}\,\mathcal{M}$ at P.

Definition 7.2.1. — In the case $\dim_P \mathrm{Supp}\,\mathcal{M} = 0$, a K-Betti structure is defined to be a pre-K-Betti structure.

Let us consider the case $\dim_P \mathrm{Supp}\,\mathcal{M} \leq n$. We say that $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$ at P if there exists an n-dimensional K-cell $\mathcal{C}_0 = (Z_0, \varphi_0, U_0, V_0)$ of $(\mathcal{M},\mathcal{F})$ at P with the properties:

- ▷ $\dim_P((\mathrm{Supp}\,\mathcal{M} \cap X_P) \setminus \varphi_0(Z_0)) < n$ for some neighbourhood X_P of P in X;

▷ for any dominant refinement $\mathcal{C} \prec \mathcal{C}_0$ and any cell function g for $\mathcal{C}$, the induced pre-K-Betti structure ${}^D\phi_g^{(0)}(\mathcal{F})$ is a K-Betti structure of $\phi_g^{(0)}(\mathcal{M}_P)$ at P. Note that $\dim_P \phi_g^{(0)}(\mathcal{M}) < n$.

Such an n-cell $\mathcal{C}_0$ is called a bounding n-cell of $\mathcal{M}$ at P. □

If $\mathcal{C}_0$ is a bounding n-cell of $\mathcal{M}$, any dominant refinement and enhancement are also bounding n-cells of $\mathcal{M}$.

DEFINITION 7.2.2. — If $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$ at any point of X, it is called a K-Betti structure of $\mathcal{M}$. A holonomic $\mathcal{D}$-module with a K-Betti structure is called a K-holonomic $\mathcal{D}$-module. □

Morphisms of K-holonomic $\mathcal{D}$-modules $(\mathcal{M}_1, \mathcal{F}_1) \to (\mathcal{M}_2, \mathcal{F}_2)$ are defined to be morphisms of pre-K-holonomic $\mathcal{D}$-modules. The category of K-holonomic $\mathcal{D}_X$-modules is denoted by $\mathrm{Hol}(X, K)$. It is a full subcategory of the category of pre-K-holonomic $\mathcal{D}_X$-modules $\mathrm{Hol}^{\mathrm{pre}}(X, K)$ by definition.

REMARK 7.2.3. — As we will see later in §8, for any K-cell $\mathcal{C} = (Z, U, \varphi, V)$ with a cell function g at P, the pre-K-holonomic $\mathcal{D}$-modules $\varphi_\dagger(V)$, $\varphi_\dagger(V_!)$, $\varphi_\dagger \Xi^{(a)}_{g\circ\varphi}(V)$, and $\varphi_\dagger \psi^{(a)}(V)$ on a neighbourhood of P are K-holonomic. We will see that $\mathrm{Hol}(X, K)$ is an abelian category in Proposition 7.2.4 below. So, we may replace the condition in the higher dimensional case in Definition 7.2.1 with the following, which is easier to check:

▷ We say that $\mathcal{F}$ is a K-Betti structure of $\mathcal{M}$ at P if there exists an n-dimensional K-cell $\mathcal{C} = (Z, \varphi, V, U)$ with a cell function g at P such that the induced pre-K-Betti structure ${}^D\phi_g^{(0)}(\mathcal{F})$ is a K-Betti structure of $\phi_g^{(0)}(\mathcal{M})$ at P.

It seems convenient for the author to begin with a stronger condition as in Definition 7.2.1 for the development of the theory. □

7.2.2. Abelian category. — It is basic to obtain the following.

PROPOSITION 7.2.4. — $\mathrm{Hol}(X, K)$ *is abelian.*

Proof. — Let P be any point of X. We use an induction on the dimension of $\mathrm{Supp}_P \mathcal{M}$. Let $(f_{\mathcal{D}}, f_{\mathcal{P}}) : (\mathcal{M}_1, \mathcal{F}_1) \to (\mathcal{M}_2, \mathcal{F}_2)$ be a morphism of K-holonomic $\mathcal{D}$-modules. Let us prove that $\mathrm{Ker}(f_{\mathcal{P}})$ is a K-Betti structure of $\mathrm{Ker}\, f_{\mathcal{D}}$.

Let $n \geq \max\{\dim \mathrm{Supp}_P \mathcal{M}_i\}$. Let $\mathcal{C}_{i,0} = (Z_{i,0}, U_{i,0}, \varphi_{i,0}, V_{i,0})$ $(i = 1, 2)$ be bounding n-cells for $\mathcal{M}_i$ at P. By considering refinement and enhancement, we may assume that $(Z_{1,0}, U_{1,0}, \varphi_{1,0}) = (Z_{2,0}, U_{2,0}, \varphi_{2,0})$, which is denoted by (Z_0, U_0, φ_0). We have an induced morphism $f_{Z_0} : V_{1,0} \to V_{2,0}$. We obtain a cell $\mathcal{C}_0(\mathrm{Ker}) = (Z_0, U_0, \varphi_0, \mathrm{Ker}\, f_{Z_0})$ of $\mathrm{Ker}\, f_{\mathcal{D}}$. The K-structure of $\mathrm{Ker}\, f_{\mathcal{D}}$ is good by Lemma 6.4.3.

Let $\mathcal{C}(\mathrm{Ker}) = (Z, U, \varphi, K_Z)$ be a dominant refinement of $\mathcal{C}_0(\mathrm{Ker})$.

We have refinements $\mathcal{C}_i = (Z, U, \varphi, V_i)$ of $\mathcal{C}_{i,0}$ with the induced morphism $f_Z : V_1 \to V_2$. We have $\mathrm{Ker}\, f_Z \simeq K_Z$. We obtain the commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccccc} \varphi_\dagger V_{1!} & \longrightarrow & \mathcal{M}_{1\,P} & \longrightarrow & \varphi_\dagger V_1 \\ \downarrow & & \downarrow & & \downarrow \\ \varphi_\dagger V_{2!} & \longrightarrow & \mathcal{M}_{2\,P} & \longrightarrow & \varphi_\dagger V_2. \end{array}$$

Hence, the induced morphisms

$$\varphi_\dagger K_{Z!} \longrightarrow \mathrm{Ker}(f_{\mathcal{D}})_P \longrightarrow \varphi_\dagger K_Z$$

are compatible with the pre-K-Betti structures. We have the commutative diagram of pre-K-holonomic $\mathcal{D}$-modules:

$$\begin{array}{ccc} \varphi_\dagger(V^{a,b}_{1,g!}) & \longrightarrow & \varphi_\dagger(V^{a,b}_{1,g*}) \\ \downarrow & & \downarrow \\ \varphi_\dagger(V^{a,b}_{2,g!}) & \longrightarrow & \varphi_\dagger(V^{a,b}_{2,g*}). \end{array}$$

Hence, the morphisms

$$\Xi^{(0)}_g(\varphi_\dagger V_1) \longrightarrow \Xi^{(0)}_g(\varphi_\dagger V_2), \quad \psi^{(0)}_g(\varphi_\dagger V_1) \longrightarrow \psi^{(0)}_g(\varphi_\dagger V_2)$$

preserve the pre-K-Betti structures. Therefore, $\phi^{(0)}_g(f_{\mathcal{D}})$ preserves the pre-K-Betti structures, i.e.,

$${}^D\phi^{(0)}_g(f_{\mathcal{P}}) : {}^D\phi^{(0)}_g(\mathcal{F}_1) \longrightarrow {}^D\phi^{(0)}_g(\mathcal{F}_2)$$

is induced. By the assumption of the induction, $\mathrm{Ker}\, {}^D\phi^{(0)}_g(f_{\mathcal{P}})$ is a K-Betti structure. It is easy to obtain that

$${}^D\phi^{(0)}_g\, \mathrm{Ker}\, f_{\mathcal{P}} = \mathrm{Ker}\, {}^D\phi^{(0)}_g(f_{\mathcal{P}}).$$

Then, we can conclude that $(\mathrm{Ker}\, f_{\mathcal{D}}, \mathrm{Ker}\, f_{\mathcal{P}})$ is a K-holonomic $\mathcal{D}$-module. The claims for the cokernel and the image can be proved similarly. □

7.2.3. Dual

PROPOSITION 7.2.5. — *Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}_X$-module. Then, the dual $\boldsymbol{D}(\mathcal{M},\mathcal{F}) := (\boldsymbol{D}\mathcal{M},\boldsymbol{D}\mathcal{F})$ is also K-holonomic.*

Proof. — Let P be any point of $\operatorname{Supp}\mathcal{M}$, and let $\mathcal{C}_0$ be a bounding n-cell at P. Let $\mathcal{C}=(Z,U,\varphi,V)$ be any refinement of $\mathcal{C}_0$. Let $\mathcal{F}_V$ and $\mathcal{F}_{V!}$ be the canonical pre-K-Betti structures of V and $V_!$. Let $\mathcal{C}^\vee := (Z,U,\varphi,V^\vee)$. We have the induced K-structure of $V^\vee$.

By using Proposition 5.2.1 and Theorem 5.2.2, we obtain that $\boldsymbol{D}\mathcal{F}_{V!}$ and $\boldsymbol{D}\mathcal{F}_V$ are the canonical pre-K-Betti structures of $V^\vee$ and $V_!^\vee$. Hence, we obtain that $\mathcal{C}^\vee$ and $\boldsymbol{D}\mathcal{F}$ are compatible. We also obtain that $\boldsymbol{D}\,{}^D\Xi_g^{(a)}\varphi_*\mathcal{F}_V$ is equal to the canonical pre-K-Betti structure of $\Xi_g^{(-a-1)}\varphi_*V^\vee$. Moreover, the induced K-structure of $\phi_g^{(a)}(\boldsymbol{D}\mathcal{M}_P)$ is equal to $\boldsymbol{D}\,{}^D\phi_g^{(-a-1)}\mathcal{F}$ under the isomorphism $\phi_g^{(a)}\boldsymbol{D}\mathcal{M}_P\simeq\boldsymbol{D}\phi_g^{(-a-1)}\mathcal{M}_P$. By the inductive assumption, it is K-Betti structure. Thus, we obtain that $\boldsymbol{D}(\mathcal{M},\mathcal{F})$ is K-holonomic. $\square$

7.2.4. Sub-objects and quotient objects. — Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}$-module.

PROPOSITION 7.2.6. — *If $(\mathcal{M}_1,\mathcal{F}_1)$ is a subobject of $(\mathcal{M},\mathcal{F})$ in $\operatorname{Hol}^{\mathrm{pre}}(X,K)$, it is also K-holonomic. A similar claim holds for quotients.*

Proof. — Let P be any point of X. We use an induction on the dimension of the support of $\mathcal{M}$. Let $n\geq\dim_P\operatorname{Supp}\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a bounding n-cell of $\mathcal{M}$ at P. Let $V_1\subset V$ denote the sub-connection induced by $\mathcal{M}_1$. Then, $\mathcal{C}_1=(Z,U,\varphi,V_1)$ is an n-cell of $\mathcal{M}_1$ at P. Let us prove that $\mathcal{C}_1$ and $\mathcal{F}_1$ are compatible. By Lemma 6.4.6, the K-structure of V_1 is good. Let $\mathcal{F}_*$ and $\mathcal{F}_!$ denote the canonical K-structures of $\varphi_\dagger V$ and $\varphi_\dagger V_!$. Let $\mathcal{F}_{1*}$ and $\mathcal{F}_{1!}$ denote the canonical K-structures of $\varphi_\dagger V_1$ and $\varphi_\dagger V_{1!}$. We have the morphisms:

$$\begin{array}{ccccc} \varphi_\dagger(V_!) & \longrightarrow & \mathcal{M} & \longrightarrow & \varphi_\dagger(V) \\ \uparrow & & \uparrow & & \uparrow \\ \varphi_\dagger(V_{1!}) & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \varphi_\dagger(V_1), \end{array} \qquad \begin{array}{ccccc} \mathcal{F}_! & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_* \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{F}_{1!} & & \mathcal{F}_1 & & \mathcal{F}_{1*}. \end{array}$$

Because the morphism $\varphi_\dagger(V_{1!})\to\mathcal{M}/\mathcal{M}_1$ is 0, the morphism $\mathcal{F}_{1!}\to\mathcal{F}/\mathcal{F}_1$ is also 0, i.e., $\mathcal{F}_{1!}\to\mathcal{F}$ factors through $\mathcal{F}_1$. Similarly, we obtain that $\mathcal{F}_1\to\mathcal{F}_*$ factors through $\mathcal{F}_{1*}$. Hence, $\mathcal{C}_1$ is compatible with $\mathcal{F}_1$.

Let f be a cell function for $\mathcal{C}$. We have

$${}^D\Xi_f^{(a)}(\mathcal{F})\supset{}^D\Xi_f^{(a)}(\mathcal{F}_1)\quad\text{and}\quad{}^D\psi_f^{(a)}(\mathcal{F})\supset{}^D\psi_f^{(a)}\mathcal{F}_1.$$

Hence, we obtain ${}^D\phi_f^{(a)}(\mathcal{F}) \supset {}^D\phi_f^{(a)}(\mathcal{F}_1)$, which are pre-$K$-Betti structures of $\phi_f^{(a)}\mathcal{M}$ and $\phi_f^{(a)}\mathcal{M}_1$. By the assumption of the induction, we obtain that ${}^D\phi_f^{(a)}(\mathcal{F}_1)$ is a K-Betti structure of $\phi_f\mathcal{M}_1$. □

7.2.5. Twist. — Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}$-module on X. Let $\mathcal{V}$ be a flat bundle on X with a K-structure, i.e., we have a K-local system $\mathcal{F}_{\mathcal{V}}$ such that $\mathcal{F}_{\mathcal{V}} \otimes \mathbb{C}[\dim X] \simeq \mathrm{DR}_X(V)$. Then, we obtain a pre-$K$-Betti structure $\mathcal{F}\otimes\mathcal{F}_{\mathcal{V}}$ of $\mathcal{M}\otimes\mathcal{V}$.

Lemma 7.2.7. — *$\mathcal{F}\otimes\mathcal{F}_{\mathcal{V}}$ is a K-Betti structure of $\mathcal{M}\otimes\mathcal{V}$.*

Proof. — Let P be any point of X. We use an induction on $\dim_P \operatorname{Supp}\mathcal{M}$. Let $\mathcal{C}=(Z,U,\varphi,V)$ be a K-cell of $\mathcal{M}$ at P. Then, $\mathcal{C}'=(Z,U,\varphi,V\otimes\varphi^*\mathcal{V})$ is a K-cell of $\mathcal{M}\otimes\mathcal{V}$ at P. Let g be a cell function of $\mathcal{C}$. Then, we have natural isomorphism of pre-K-holonomic $\mathcal{D}_X$-modules

$$\psi_g^{(a)}\big(\varphi_\dagger(V\otimes\varphi^*\mathcal{V})\big) \simeq \psi_g^{(a)}\big(\varphi_\dagger(V)\big)\otimes\mathcal{V},$$
$$\Xi_g^{(a)}\big(\varphi_\dagger(V\otimes\varphi^*\mathcal{V})\big) \simeq \Xi_g^{(a)}\big(\varphi_\dagger(V)\big)\otimes\mathcal{V}.$$

Hence, we obtain an isomorphism of pre-K-holonomic $\mathcal{D}$-modules

$$\phi_g^{(a)}(\mathcal{M}\otimes\mathcal{V}) \simeq \phi_g^{(a)}(\mathcal{M})\otimes\mathcal{V}.$$

By using the inductive assumption, we obtain that $\phi_g^{(a)}(\mathcal{M}\otimes\mathcal{V})$ is K-holonomic. Hence, we obtain that $\mathcal{M}\otimes\mathcal{V}$ is K-holonomic at P. □

7.2.6. K-cells

Proposition 7.2.8. — *Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}$-module. Then, any cell $\mathcal{C}=(Z,U,\varphi,V)$ of $\mathcal{M}$ is a K-cell.*

Proof. — Let P be any point of $\operatorname{Supp}(\mathcal{M})$. Let $\mathcal{C}'_P=(Z'_P,U'_P,\varphi'_P,V'_P)$ be a bounding K-cell of $\mathcal{M}$ at P, which is a refinement of $\mathcal{C}$. By Lemma 6.4.4, we obtain that the induced K-structure of V is good around $\varphi^{-1}(P)$. By varying P, we obtain that the K-structure of V is good. Moreover, for P and $\mathcal{C}'_P$ as above, the induced morphisms

$$\mathcal{M}_P \longrightarrow \varphi'_{P\dagger}V'_P \quad\text{and}\quad \varphi_\dagger(V)_P \longrightarrow \varphi'_{P\dagger}V'_P$$

are compatible with pre-K-Betti structures, where $\varphi_\dagger(V)_P$ denotes the restriction to a small neighbourhood of P. Because $\varphi_\dagger(V)_P\to\varphi'_{P\dagger}V'_P$ is a monomorphism, we obtain that $\mathcal{M}_P\to\varphi_\dagger(V)_P$ is also compatible with pre-K-Betti structures. By varying P in X, we obtain that $\mathcal{M}_P \to \varphi_\dagger(V)$ is also compatible with pre-K-Betti structures. We can prove that $\varphi_! V \to \mathcal{M}$ is also compatible with pre-K-Betti structures with a similar argument. □

7.3. $K(*D)$-Betti structure

We introduce a variant notion of $K(*D)$-Betti structure of holonomic $\mathcal{D}_{X(*D)}$-modules, where D is a hypersurface. It is rather auxiliary. Indeed, as proved in §8, it is equivalent to K-Betti structure for holonomic $\mathcal{D}_{X(*D)}$-modules, although it will be convenient in some arguments.

7.3.1. Cells and cell functions for holonomic $\mathcal{D}_{X(*D)}$-modules. — Let X be any complex manifold or smooth complex algebraic variety, and D be any hypersurface of X. Let $\mathcal{M}$ be any holonomic $\mathcal{D}_{X(*D)}$-module, i.e., $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module such that $\mathcal{M}(*D) = \mathcal{M}$. A cell of a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$ is defined to be a cell of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. The notions of refinement and enhancement of a cell of a holonomic $\mathcal{D}_{X(*D)}$-module are defined in the same way. However, we will be interested in the morphisms

$$\varphi_\dagger(V_!)(*D) \longrightarrow \mathcal{M}_P \longrightarrow \varphi_\dagger V.$$

The notion of cell functions is modified. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a cell of a holonomic $\mathcal{D}_{X(*D)}$-module $\mathcal{M}$. A cell function g of $\mathcal{C}$ is a meromorphic function on X whose poles are contained in D, such that $U = \operatorname{Supp} \mathcal{M} \setminus (g^{-1}(0) \cup D)$.

7.3.2. $K(*D)$-cell. — Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{X(*D)}$-module. Let $\mathcal{F}$ be a pre-K-Betti structure of $\mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be an n-cell of $\mathcal{M}$ at P. We say that $\mathcal{F}$ and $\mathcal{C}$ are compatible if

(i) the induced K-structure of V is good,

(ii) the induced morphisms $\varphi_\dagger(V_!)(*D) \to \mathcal{M}_P \to \varphi_\dagger(V)$ are compatible with the pre-K-Betti structures.

Such a cell $\mathcal{C}$ is called a $K(*D)$-cell of $(\mathcal{M}, \mathcal{F})$. Note that condition (i) implies that $\varphi_\dagger(V_!)(*D)$ and $\varphi_\dagger(V)$ are equipped with the canonical pre-K-Betti structure.

Let g be a cell function for a $K(*D)$-cell $\mathcal{C}$. For $\star = *, !$, we set

$$V^{a,b}_{g\star}(*D) := (V \otimes \mathfrak{I}^{a,b}_{\varphi^{-1}(g)})(*\varphi^{-1}D).$$

Note that $\varphi_\dagger(V^{a,b}_{g\star}(*D))$ have the canonical pre-K-Betti structures. Since $\Xi^{(c)}_g(\varphi_\dagger V,*D)$ and $\psi^{(c)}_g(\varphi_\dagger V,*D)$ are of the form

$$\operatorname{Ker}\big(\varphi_\dagger\big(V^{a,b}_{g!}(*D)\big)\longrightarrow\varphi_\dagger\big(V^{a',b'}_{g*}(*D)\big)\big),$$

they are equipped with the induced pre-K-Betti structures ${}^D\Xi^{(c)}_g(\varphi_*\mathcal{F}_V,*D)$ and ${}^D\psi^{(c)}_g(\varphi_*\mathcal{F}_V,*D)$. The tuples

$$\big(\Xi^{(c)}_g(\varphi_\dagger V,*D),{}^D\Xi_g(\varphi_*\mathcal{F}_V,*D)\big),\quad\big(\psi^{(c)}_g(\varphi_\dagger V,*D),{}^D\psi_g(\varphi_*\mathcal{F}_V,*D)\big)$$

are also denoted by $\Xi^{(c)}_g\varphi_\dagger(V,\mathcal{F}_V,*D)$ and $\psi^{(c)}_g\varphi_\dagger(V,\mathcal{F}_V,*D)$. We will often omit to denote the pre-K-Betti structures. We will use the following obvious lemma implicitly.

Lemma 7.3.1. — *The natural isomorphisms*

$$\Xi^{(a)}_g(\varphi_\dagger V,*D)\simeq\varphi_\dagger\Xi^{(a)}_g(V,*\varphi^{-1}D),\quad\psi^{(a)}_g(\varphi_\dagger V,*D)\simeq\varphi_\dagger\psi^{(a)}_g(V,*\varphi^{-1}D)$$

are compatible with the induced pre-K-Betti structures. □

Since $\phi^{(0)}_g(\mathcal{M}_P,*D)$ is the cohomology of the complex in the category of pre-K-holonomic $\mathcal{D}_X$-modules

$$\varphi_\dagger(V_!)(*D)\longrightarrow\Xi^{(0)}_g(\varphi_\dagger V,*D)\oplus\mathcal{M}_P\longrightarrow\varphi_\dagger(V)(*D),$$

we obtain a pre-K-Betti structure of $\phi^{(a)}_g(\mathcal{M}_P,*D)$ denoted by ${}^D\phi^{(a)}_g(\mathcal{F},*D)$. Let $\phi^{(a)}_g(\mathcal{M}_P,\mathcal{F},*D)$ denote the tuple

$$\big(\phi^{(a)}_g(\mathcal{M}_P,*D),{}^D\phi^{(a)}_g(\mathcal{F},*D)\big).$$

We will often omit to denote the pre-K-Betti structure.

7.3.3. Definition of $K(*D)$-Betti structure. — Let us define the notion of $K(*D)$-Betti structure at any point of D, inductively on the dimension of the support of $\mathcal{D}_{X(*D)}$-modules. Let $(\mathcal{M},\mathcal{F})$ be a pre-K-holonomic $\mathcal{D}_{X(*D)}$-module.

Note that we have $\mathcal{M}=0$ around $P\in D$ in the case $\dim_P\operatorname{Supp}\mathcal{M}=0$.

Definition 7.3.2. — Let P be any point of D. Suppose $\dim_P\operatorname{Supp}\mathcal{M}\leq n$. We say that $\mathcal{F}$ is a $K(*D)$-Betti structure of $\mathcal{M}$ at P if there exists an n-dimensional $K(*D)$-cell $\mathcal{C}_0=(Z_0,\varphi_0,U_0,V_0)$ at P with the following properties:

- ▷ $\dim_P((\operatorname{Supp}\mathcal{M}\cap X_P)\setminus\varphi_0(Z_0))<n$ for some neighbourhood X_P of P in X;

▷ for any dominant refinement $\mathcal{C} \prec \mathcal{C}_0$ and any cell function g for $\mathcal{C}$ as a $\mathcal{D}_{X(*D)}$-module, the induced pre-K-Betti structure ${}^D\phi_g^{(0)}(\mathcal{F}, *D)$ is a $K(*D)$-Betti structure at P.

Such an n-cell $\mathcal{C}_0$ is called a bounding n-cell of $\mathcal{M}$ at P. □

If $\mathcal{C}_0$ is a bounding n-cell of $\mathcal{M}$, its dominant refinements and enhancements are also bounding n-cells of $\mathcal{M}$.

Definition 7.3.3. — A pre-K-Betti structure $\mathcal{F}$ of $\mathcal{M}$ is called a $K(*D)$-Betti structure if it is K-Betti structure of $\mathcal{M}$ at any points of $X \setminus D$, and if it is $K(*D)$-Betti structure of $\mathcal{M}$ at any points of D. A holonomic $\mathcal{D}_{X(*D)}$-module with a $K(*D)$-Betti structure is called a $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module. □

Let $\mathrm{Hol}(X, *D, K) \subset \mathrm{Hol}^{\mathrm{pre}}(X, K)$ denote the full subcategory of $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules. The following lemma is similar to Proposition 7.2.4.

Lemma 7.3.4. — *The category* $\mathrm{Hol}(X, *D, K)$ *is abelian.* □

The following lemma is similar to Proposition 7.2.6.

Lemma 7.3.5. — *Let* $(\mathcal{M}, \mathcal{F})$ *be any* $K(*D)$*-holonomic* $\mathcal{D}_X$*-module. Any subobject of* $(\mathcal{M}, \mathcal{F})$ *in* $\mathrm{Hol}^{\mathrm{pre}}(X, K)$ *is also* $K(*D)$*-holonomic. A similar claim holds for quotients.* □

The following lemma is analogue of Proposition 7.2.8.

Lemma 7.3.6. — *Let* $(\mathcal{M}, \mathcal{F})$ *be a* $K(*D)$*-holonomic* $\mathcal{D}_{X(*D)}$*-module. Then, any cell* $\mathcal{C} = (Z, U, \varphi, V)$ *of* $\mathcal{M}$ *is a* $K(*D)$*-cell.* □

7.3.4. Uniqueness. — We have the following uniqueness.

Proposition 7.3.7. — *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}_{X(*D)}$*-module with* $K(*D)$*-Betti structures* $\mathcal{F}_i$ $(i = 1, 2)$*. If* $\mathcal{F}_{1|X-D} = \mathcal{F}_{2|X-D}$*, then we have* $\mathcal{F}_1 = \mathcal{F}_2$*.*

Proof. — It is enough to consider the issue locally around any point $P \in D$. We use an induction on $\dim_P \operatorname{Supp} \mathcal{M}$. In the case $\dim_P \operatorname{Supp} \mathcal{M} = 0$, the claim is clear. Suppose $\dim_P \operatorname{Supp} \mathcal{M} \leq n$. Let $\mathcal{C}$ be any bounding cell at P, and let g be any cell function of $\mathcal{C}$. Let ${}^D\phi_g^{(0)}(\mathcal{F}_i, *D)$ be the induced pre-$K(*D)$-Betti structures of $\phi_g^{(0)}(\mathcal{M}, *D)$. By the assumption of the induction, we have

$$ {}^D\phi_g^{(0)}(\mathcal{F}_1, *D) = {}^D\phi_g^{(0)}(\mathcal{F}_2, *D). $$

Because $\mathcal{F}_i$ can be reconstructed from ${}^D\phi_g^{(0)}(\mathcal{F}_i, *D)$ and the canonical pre-$K(*D)$-Betti structures of $\psi_g^{(a)}(\varphi_* V, *D)$ and $\Xi_g^{(a)}(\varphi_* V, *D)$, we obtain $\mathcal{F}_1 = \mathcal{F}_2$. □

7.3.5. Independence from a compactification. — Let $F : X' \to X$ be a projective birational morphism of complex manifolds such that

$$X' - D' \simeq X - D,$$

where $D' := F^{-1}(D)$. Recall that $F_\dagger$ denotes the push-forward of pre-K-holonomic $\mathcal{D}$-modules.

PROPOSITION 7.3.8. — *The functor $F_\dagger$ induces an equivalence of the categories* $\mathrm{Hol}(X, *D, K)$ *and* $\mathrm{Hol}(X', *D', K)$.

Proof. — It is enough to check the claims locally around any $P \in D$. We begin with a remark. Let $\mathcal{M}'$ be a holonomic $\mathcal{D}_{X'(*D')}$-module. We set $\mathcal{M} := F_\dagger \mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a cell of $\mathcal{M}$ at P. By taking a refinement, we may assume that φ factors through F, i.e., $\varphi = F \circ \varphi'$, and that $\mathcal{C}' = (Z, U, \varphi', V)$ is a cell of $\mathcal{M}'$. Let g be a cell function for $\mathcal{C}$ as a $\mathcal{D}_{X(*D)}$-module. Note that $g' = g \circ F$ is a cell function for $\mathcal{C}'$. We have a description of $\mathcal{M}'$ as the cohomology of the complex

$$\psi_{g'}^{(1)}(\varphi'_\dagger V, *D') \longrightarrow \Xi_{g'}^{(0)}(\varphi'_\dagger V, *D') \oplus \phi_{g'}^{(0)}(\mathcal{M}', *D') \longrightarrow \psi_{g'}^{(0)}(\varphi'_\dagger V, *D'). \tag{110}$$

By the push-forward $F_\dagger$, it induces a description of $\mathcal{M}$ as the cohomology of the complex

$$\psi_{g}^{(1)}(\varphi_\dagger V, *D) \longrightarrow \Xi_{g}^{(0)}(\varphi_\dagger V, *D) \oplus \phi_{g}^{(0)}(\mathcal{M}, *D) \longrightarrow \psi_{g}^{(0)}(\varphi_\dagger V, *D). \tag{111}$$

Suppose that $\mathcal{F}'$ is a $K(*D)$-Betti structure of $\mathcal{M}'$. Let us prove that $F_\dagger \mathcal{F}'$ is a $K(*D)$-Betti structure of $\mathcal{M}$. By Lemma 7.3.6, $\mathcal{C}'$ is a $K(*D)$-cell of $\mathcal{M}'$. We obtain that $\mathcal{C}$ is a $K(*D)$-cell of $\mathcal{M}$. Because the pre-K-holonomic $\mathcal{D}$-module $\phi_g^{(0)}(\mathcal{M}, *D)$ is obtained as $F_\dagger \phi_{g'}^{(0)}(\mathcal{M}', *D)$, we obtain that $\phi_g^{(0)}(\mathcal{M}, *D)$ is $K(*D)$-holonomic by the inductive assumption. Hence, $\mathcal{F}$ is also a $K(*D)$-Betti structure. Thus, $F_\dagger$ induces a functor

$$\mathrm{Hol}(X', *D', K) \longrightarrow \mathrm{Hol}(X, *D, K).$$

It is clearly faithful.

Let us prove that it is full. We use an induction on the dimensions of the supports of the holonomic $\mathcal{D}$-modules. Let $(\mathcal{M}'_i, \mathcal{F}'_i)$ $(i = 1, 2)$ be objects in $\mathrm{Hol}(X', *D', K)$. Let

$$f : F_\dagger(\mathcal{M}'_1, \mathcal{F}'_1) \longrightarrow F_\dagger(\mathcal{M}'_2, \mathcal{F}'_2)$$

be a morphism in $\mathrm{Hol}(X,*D,K)$. We have a morphism $f' : \mathcal{M}'_1 \to \mathcal{M}'_2$ of holonomic $\mathcal{D}_{X'(*D')}$-modules. It is enough to show that it is compatible with the $K(*D)$-Betti structures. For the cohomological descriptions (110) for $\mathcal{M}'_i$, $\psi^{(a)}_{g'}(f')$ and $\Xi^{(a)}_{g'}(f')$ are compatible with the pre-K-Betti structures. Because $\phi^{(a)}_g(f)$ is compatible with the $K(*D)$-Betti structures, we obtain that $\phi^{(a)}_{g'}(f')$ is compatible with the $K(*D')$-Betti structures. Thus, we obtain that f' is compatible with the $K(*D')$-Betti structures.

Let us prove the essential surjectivity. We use an induction on the dimension of the support. Let $\mathcal{M}$ and $\mathcal{M}'$ be as above. Let $\mathcal{F}$ be a $K(*D)$-Betti structure of $\mathcal{M}$. By the inductive assumption, the $K(*D)$-Betti structure of $\psi^{(a)}_g(\varphi_\dagger(V),*D)$ and $\phi^{(a)}_g(\mathcal{M},*D)$ induce $K(*D)$-Betti structures of $\psi^{(a)}_{g'}(\varphi'_\dagger(V),*D')$ and $\phi^{(a)}_{g'}(\mathcal{M}',*D')$, which are compatible with the natural morphisms. We also have the canonical K-Betti structures of $\psi^{(a)}_{g'}(\varphi'_\dagger(V),*D')$ and $\Xi^{(a)}_{g'}(\varphi'_\dagger V,*D')$. By Proposition 7.3.7, the induced $K(*D)$-Betti structures on $\psi^{(a)}_{g'}(\varphi'_\dagger(V),*D')$ are the same. Hence, (110) is a complex of $K(*D)$-holonomic $\mathcal{D}(*D)$-modules. Hence, we have an induced $K(*D)$-Betti structure of $\mathcal{M}'$. The functoriality is clear from the above construction. □

CHAPTER 8

FUNCTORIALITY PROPERTIES

8.1. Statements

We give several statements.

THEOREM 8.1.1. — *Let $F : X \to Y$ be any projective morphism of complex manifolds. For any K-holonomic $\mathcal{D}_X$-module $(\mathcal{M}, \mathcal{F})$, the push-forward $F_\dagger^i(\mathcal{M}, \mathcal{F}) := (F_\dagger^i \mathcal{M}, F_\dagger^i \mathcal{F})$ are also K-holonomic for any i.*

Here, $F_\dagger^i \mathcal{F}$ denotes the i-th cohomology of $RF_* \mathcal{F}$ with respect to the middle perversity.

THEOREM 8.1.2. — *Let X be any complex manifold with a normal crossing hypersurface D. Any good pre-K-holonomic $\mathcal{D}$-module on (X, D) is K-holonomic.*

See Definition 6.3.4 for good pre-K-holonomic $\mathcal{D}$-modules.

THEOREM 8.1.3. — *Let X be a complex manifold with a hypersurface D. Let $\mathfrak{H}$ be a sequence of hypersurface pairs contained in D. Let V be any meromorphic flat connection on (X, D) with a good K-structure. Then, the pre-K-holonomic $\mathcal{D}$-module $\mathfrak{P}_{\mathfrak{H}}(V)$ is K-holonomic.*

See §6.4 for hypersurface pairs and $\mathfrak{P}_{\mathfrak{H}}(V)$.

THEOREM 8.1.4. — *Let X be any complex manifold with a hypersurface D. We have a unique functor $\mathrm{Hol}(X, K) \to \mathrm{Hol}(X, *D, K)$ with the following properties:*

- ▷ *It is an enhancement of the functor*

$$\mathrm{Hol}(X) \longrightarrow \mathrm{Hol}(X, *D), \quad \mathcal{M} \longmapsto \mathcal{M}(*D).$$

▷ *For any* $(\mathcal{M},\mathcal{F})\in \mathrm{Hol}(X,K)$, *the natural morphism* $\mathcal{M}\to\mathcal{M}(*D)$ *is compatible with the induced pre-*K*-Betti structures.*

8.1.1. Auxiliary statements. — We will use an induction on the dimension of the supports of $\mathcal{D}$-modules for the proof. Let $SI(\leq n)$ denote the statement of Theorem 8.1.1 in the case $\dim\operatorname{Supp}\mathcal{M}\leq n$.

Let GOOD($\leq n$) means the following:

▷ the claim of Theorem 8.1.2 holds if $\dim\operatorname{Supp}\mathcal{M}\leq n$;

▷ the claim of Theorem 8.1.3 holds if $\dim X\leq n$.

For any complex manifold X with a hypersurface D, let

$$\mathrm{Hol}_{\leq n}(X,K)\subset \mathrm{Hol}(X,K)$$

denote the full subcategory of K-holonomic $\mathcal{D}_X$-modules $(\mathcal{M},\mathcal{F})$ with $\dim\operatorname{Supp}\mathcal{M}\leq n$.

We use the symbols $\mathrm{Hol}_{\leq n}(X)$, $\mathrm{Hol}_{\leq n}(X,*D)$ and $\mathrm{Hol}_{\leq n}(X,*D,K)$ with a similar meaning.

Let LOC($\leq n$) means the following:

▷ The claim of Theorem 8.1.4 holds if we replace $\mathrm{Hol}(X,K)$, $\mathrm{Hol}(X,*D,K)$, etc., by $\mathrm{Hol}_{\leq n}(X,K)$, $\mathrm{Hol}_{\leq n}(X,*D,K)$, etc.

Our induction will proceed as follows:

▷ SI($<n$) + GOOD($<n$) $\Longrightarrow$ GOOD($\leq n$) (§8.2.3 and §8.2.4);

▷ SI($<n$) + GOOD($\leq n$) + LOC($<n$) $\Longrightarrow$ LOC($\leq n$) (§8.3.3);

▷ SI($<n$) + GOOD($\leq n$) + LOC($\leq n$) $\Longrightarrow$ SI($\leq n$) (§8.5).

Remark 8.1.5. — In the proof, we will observe the equivalence of $K(*D)$-Betti structure and K-Betti structure. (See Lemma 8.3.1.) □

8.2. Step 1

8.2.1. K-cell. — Let $\varphi: Z\to X$ be a projective morphism of complex manifolds such that $\dim Z=n$. Let D_Z be a hypersurface of Z. Assume that $\varphi_{|Z-D_Z}$ is an immersion. Let V be a meromorphic flat connection on (Z,D_Z) with a good K-structure. We have the canonical pre-K-Betti structures $\mathcal{F}_V$ and $\mathcal{F}_{V!}$ of V and $V(!D_Z)$, respectively. More generally, for any sequence of hypersurface pairs $\mathfrak{H}$ contained in D_Z, we obtain the canonical pre-K-holonomic $\mathcal{D}_Z$-modules $\mathfrak{P}_{\mathfrak{H}}(V)$. Note that the natural morphisms

$$V(!D_Z)\longrightarrow \mathfrak{P}_{\mathfrak{H}}(V)\longrightarrow V$$

are compatible with the pre-K-Betti structures. Hence, we can regard (Z, U, id, V) as a K-cell of $\mathfrak{P}_{\mathfrak{H}}(V)$.

Lemma 8.2.1. — *Suppose* SI($< n$) *and* GOOD($< n$). *Let g be any cell function for $\mathcal{C}_0 = (Z, U, \varphi, V)$. We set*

$$g_Z := g \circ \varphi.$$

The pre-K-holonomic $\phi^{(a)}_{g_Z}(\mathfrak{P}_{\mathfrak{H}}(V))$ and $\phi^{(a)}_{g}(\varphi_\dagger \mathfrak{P}_{\mathfrak{H}}(V))$ are K-holonomic. In particular, $\psi^{(a)}_{g_Z}(V)$ and $\psi^{(a)}_{g}\varphi_\dagger(V)$ are K-holonomic.

Proof. — By SI($< n$), it is enough to prove that $\phi^{(0)}_{g_Z}(\mathfrak{P}_{\mathfrak{H}}(V))$ is K-holonomic. It is enough to consider the issue locally around any point $P \in D_Z$. We take a local resolution (Z_P, λ_P) of V. We put

$$\check{g}_P := g_Z \circ \lambda_P \quad \text{and} \quad \check{\mathfrak{H}}_P := \lambda_P^{-1}(\mathfrak{H}), \quad \check{V}_P := \lambda_P^* V.$$

We have the good pre-K-holonomic $\mathcal{D}_{\check{Z}_P}$-module $\phi^{(0)}_{\check{g}_P}(\mathfrak{P}_{\check{\mathfrak{H}}_P}(\check{V}_P))$ (Proposition 6.3.5). By GOOD($< n$), it is K-holonomic. By SI($< n$), $\lambda_{P\dagger}\phi^{(0)}_{\check{g}_P}(\mathfrak{P}_{\check{\mathfrak{H}}_P}(\check{V}_P))$ is K-holonomic, which means that $\phi^{(0)}_{g_Z}(\mathfrak{P}_{\mathfrak{H}}(V))$ is K-holonomic at P. □

Proposition 8.2.2. — *Suppose that* SI($< n$) *and* GOOD($< n$) *hold. Then, the pre-K-holonomic $\mathcal{D}$-modules $\varphi_\dagger(V, \mathcal{F}_V)$ and $\varphi_\dagger(V_!, \mathcal{F}_{V!})$ are K-holonomic.*

Proof. — Let us prove the claim for $\varphi_\dagger(V, \mathcal{F}_V)$. The other can be proved as the dual. Let us prove that $\mathcal{C}_0 = (Z, U, \varphi, V)$ is a bounding n-cell for $\varphi_\dagger(V, \mathcal{F}_V)$. Let P be any point of X. Let $\mathcal{C}' = (Z', U', \varphi', V')$ be a dominant refinement at P with a cell function g. We have a factorization $\varphi' = \varphi \circ \varphi_1$, where $\varphi_1 : Z' \to Z$. We put

$$g' := g \circ \varphi.$$

We have $V' = \varphi_1^{-1} V \otimes \mathcal{O}_{Z'}(*g')$. We have the canonical pre-K-Betti structures $\mathcal{F}_{V'}$ and $\mathcal{F}_{V'!}$ of V' and $V'_!$, respectively. According to Proposition 6.4.7, the morphisms

$$\varphi_{1\dagger} V'_! \longrightarrow \varphi_\dagger V \longrightarrow \varphi_{1\dagger} V'$$

are compatible with pre-K-Betti structures. Hence, $\mathcal{C}'$ is a K-cell. We obtain a monomorphism

$$\phi^{(0)}_g(\varphi_\dagger V) \longrightarrow \phi^{(0)}_g(\varphi'_\dagger V')$$

of pre-K-holonomic $\mathcal{D}_X$-modules. By Lemma 8.2.1, $\phi^{(0)}_g(\varphi'_\dagger V')$ is K-holonomic. Then, we obtain that $\phi^{(0)}_g(\varphi_\dagger V)$ is K-holonomic by Proposition 7.2.6. □

Corollary 8.2.3. — *Assume that* SI($<n$) *and* GOOD($<n$). *Let* f *be a cell function of* $\mathcal{C}=(Z,U,\varphi,V)$. *Then,* $\Xi_f^{(a)}(\varphi_\dagger V)$ *with the canonical pre-*K*-Betti structures are* K*-holonomic.*

Proof. — Applying the previous results to $\varphi_\dagger(\Pi_{f\star}^{a,b}V)$ $(\star=!,*)$, we obtain that they are K-holonomic. Then, we obtain the corollary. □

8.2.2. Gluing. — By Lemma 8.2.1 and Corollary 8.2.3, we have a gluing construction of K-holonomic $\mathcal{D}$-modules. Let X be a complex manifold, $\mathcal{C}=(Z,U,\varphi,V)$ be a K-cell as in §8.2.1. Let f be a cell function for $\mathcal{C}$ on X. Let $\mathcal{Q}$ be a K-holonomic $\mathcal{D}$-module whose support is contained in $f^{-1}(0)$. Assume that we are given morphisms of K-holonomic $\mathcal{D}$-modules

$$\psi_f^{(1)}(\varphi_\dagger V)\longrightarrow\mathcal{Q}\longrightarrow\psi_f^{(0)}(\varphi_\dagger V),$$

such that the composite is equal to the canonical map $\psi_f^{(1)}(\varphi_\dagger V)\to\psi_f^{(0)}(\varphi_\dagger V)$. Then, we obtain a K-holonomic $\mathcal{D}$-module as the cohomology of the following complex:

$$\psi_f^{(1)}(\varphi_\dagger V)\longrightarrow\Xi_f^{(0)}(\varphi_\dagger V)\oplus\mathcal{Q}\longrightarrow\psi_f^{(0)}(\varphi_\dagger V).$$

8.2.3. Good holonomic $\mathcal{D}$-module with good K-structure

Suppose SI($<n$) and GOOD($<n$). Let X be a complex manifold with a simply normal crossing hypersurface D. Let $\mathcal{M}$ be a good pre-K-holonomic $\mathcal{D}$-module on (X,D) such that $\dim\operatorname{Supp}\mathcal{M}=n$. Let us prove that $\mathcal{M}$ is K-holonomic. We may assume that $X=\Delta^N$ and $D=\bigcup_{i=1}^{\ell}\{z_i=0\}$. Let $\rho(\mathcal{M})\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{>0}$ denote the pair of $\dim\operatorname{Supp}\mathcal{M}$ and the number of the irreducible components of $\operatorname{Supp}\mathcal{M}$ with the maximal dimension. We use the lexicographic order on $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{>0}$. For a good holonomic $\mathcal{D}$-module $\mathcal{M}$ on (X,D), there exists $J\subset\underline{\ell}$ with $n=N-|J|$ such that $\mathcal{M}(*g)\neq 0$ comes from a meromorphic flat bundle V on D_J, where $g:=\prod_{\substack{j\notin J\\ j\leq\ell}}z_j$. Let $\iota:D_J\to X$ denote the inclusion. We have a description of $\mathcal{M}$ as the cohomology of the complex of pre-K-holonomic $\mathcal{D}$-modules

$$\psi_g^{(1)}(\iota_\dagger V)\longrightarrow\Xi_g^{(0)}(\iota_\dagger V)\oplus\phi_g^{(0)}(\mathcal{M})\longrightarrow\psi_g^{(0)}(\iota_\dagger V).$$

They are good pre-K-holonomic $\mathcal{D}$-modules. By Lemma 8.2.1 and Corollary 8.2.3, $\psi_g^{(a)}(V)$ and $\Xi_g^{(a)}(V)$ are K-holonomic. Because $\rho(\phi_g^{(0)}(\mathcal{M}))<\rho(\mathcal{M})$, we obtain that $\phi_g^{(0)}(\mathcal{M})$ is K-holonomic. Hence, we obtain that $\mathcal{M}$ is also K-holonomic.

8.2.4. Generalization. — We use the notation introduced in §8.2.1.

Proposition 8.2.4. — *Suppose that* SI($< n$) *and* GOOD($< n$). *Then, the pre-K-holonomic $\mathcal{D}_X$-module $\varphi_\dagger\mathfrak{P}_{\mathfrak{H}}(V)$ is K-holonomic.*

Proof. — It is enough to consider the issue locally around any point $P \in X$. We will shrink X around P without mention. Let $\mathcal{C}' = (Z', U', \varphi', V')$ be a dominant refinement of $\mathcal{C}$ with a cell function g for $\mathcal{C}'$. We set $\mathfrak{H}' := (\varphi')^{-1}(\mathfrak{H})$.

Lemma 8.2.5. — *Under the assumptions* SI($< n$) *and* GOOD($< n$), *the pre-K-holonomic $\mathcal{D}$-module $\varphi'_\dagger(\mathfrak{P}_{\mathfrak{H}'}(V'))$ is K-holonomic.*

Proof. — We have the expression of $\varphi'_\dagger(\mathfrak{P}_{\mathfrak{H}'}(V'))$ as the cohomology of the following complex of pre-K-holonomic $\mathcal{D}$-modules:

$$\psi_g^{(1)}\big(\varphi'_\dagger(V')\big) \longrightarrow \phi_g^{(0)}\varphi'_\dagger\big(\mathfrak{P}_{\mathfrak{H}'}(V')\big) \oplus \Xi_g^{(0)}\varphi'_\dagger(V') \longrightarrow \psi_g^{(0)}\big(\varphi'_\dagger(V')\big).$$

By Lemma 8.2.1 and Corollary 8.2.3, we obtain that the pre-K-holonomic $\mathcal{D}$-modules $\psi_g^{(a)}(\varphi'_\dagger(V'))$ and $\Xi_g^{(a)}\varphi'_\dagger(V')$ are K-holonomic. By Lemma 8.2.1, $\phi_g^{(0)}\varphi'_\dagger(\mathfrak{P}_{\mathfrak{H}'}(V'))$ is K-holonomic. Hence, we obtain that $\varphi'_\dagger(\mathfrak{P}_{\mathfrak{H}'}(V'))$ is K-holonomic. Thus, we obtain Lemma 8.2.5. □

We have a natural monomorphism of pre-K-holonomic $\mathcal{D}$-modules

$$\varphi_\dagger\big(\mathfrak{P}_{\mathfrak{H}}(V)\big) \longrightarrow \varphi'_\dagger\big(\mathfrak{P}_{\mathfrak{H}'}(V')\big),$$

as remarked in Proposition 6.4.12. Then, by Proposition 7.2.6, we obtain that $\varphi_\dagger(\mathfrak{P}_{\mathfrak{H}}(V))$ is K-holonomic. □

8.2.5. $K(*D)$-cell. — We use the notation introduced in §8.2.1. Let D be a hypersurface of X such that $D_{Z1} := \varphi^{-1}(D) \subset D_Z$. We have the pre-$K$-holonomic $\mathcal{D}_Z$-module $V(!D_{Z1})$. We obtain the following proposition as a special case of Proposition 8.2.4.

Proposition 8.2.6. — $\varphi_\dagger(V(!D_{Z1}))$ *is K-holonomic.* □

8.3. Step 2

8.3.1. Equivalence of $K(*D)$-Betti structure and K-Betti structure

Let X be any complex manifold with a hypersurface D. Let $(\mathcal{M}, \mathcal{F})$ be any pre-K-holonomic $\mathcal{D}_{X(*D)}$-module with $\dim \operatorname{Supp} \mathcal{M} \leq n$.

Lemma 8.3.1

- ▷ *Assume* SI($< n$) *and* GOOD($< n$). *If* $\mathcal{F}$ *is a* $K(*D)$*-Betti structure, then it is a* K*-Betti structure.*
- ▷ *Assume* LOC($\leq n$). *If* $\mathcal{F}$ *is a* K*-Betti structure, then it is a* $K(*D)$*-Betti structure.*

Proof. — Let us prove the first claim. We use an induction on the dimension of the support. Let P be any point of $D \cap \operatorname{Supp} \mathcal{M}$. We take a bounding cell $\mathcal{C} = (Z, U, \varphi, V)$ of $(\mathcal{M}, \mathcal{F})$ at P, and a cell function g of $\mathcal{C}$ as $\mathcal{D}_{X(*D)}$-module. We have a description of $\mathcal{M}$ as the cohomology of the complex of $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-modules

$$\psi_g^{(1)}(\varphi_\dagger(V), *D) \longrightarrow \Xi_g^{(0)}(\varphi_\dagger V, *D) \oplus \phi_g^{(0)}(\mathcal{M}, *D) \longrightarrow \psi_g^{(0)}(\varphi_\dagger(V), *D).$$

By the inductive assumption, $\phi_g^{(0)}(\mathcal{M}, *D)$ is K-holonomic. By Proposition 8.2.6, $\psi_g^{(a)}(\varphi_\dagger(V), *D)$ and $\Xi_g^{(a)}(\mathcal{M}, *D)$ are K-holonomic. Hence, we obtain that $\mathcal{M}$ is also K-holonomic.

Let us prove the second claim. By the assumption $LOC(\leq n)$, we obtain a $K(*D)$-holonomic $\mathcal{D}_{X(*D)}$-module $(\mathcal{M}(*D), \mathcal{F}(*D))$ with a morphism of pre-K-holonomic $\mathcal{D}$-modules

$$(\mathcal{M}, \mathcal{F}) \longrightarrow \big(\mathcal{M}(*D), \mathcal{F}(*D)\big).$$

Because $\mathcal{M} = \mathcal{M}(*D)$, we obtain $\mathcal{F} = \mathcal{F}(*D)$, and hence $\mathcal{F}$ is a $K(*D)$-Betti structure. □

We reformulate the uniqueness (Proposition 7.3.7) as follows.

Corollary 8.3.2. — *Let* $\star$ *be* $*$ *or* $!$. *Assume* SI($< n$), GOOD($< n$) *and* LOC($\leq n$). *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}$*-module on* X *such that* $\mathcal{M}(\star D) = \mathcal{M}$. *Let* $\mathcal{F}_i$ $(i = 1, 2)$ *be* K*-Betti structures on* $\mathcal{M}$. *If* $\mathcal{F}_{1|X-D} = \mathcal{F}_{2|X-D}$, *then* $\mathcal{F}_1 = \mathcal{F}_2$.

Proof. — The claim for $\star = *$ follows from Lemma 8.3.1 and Proposition 7.3.7. We obtain the claim for $\star = !$ by using the dual with Proposition 7.2.5. □

Corollary 8.3.3. — *Suppose* SI($< n$), GOOD($< n$) *and* LOC($\leq n$). *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}_X$*-module. Assume that one of the following holds;*

(i) $\mathcal{M}(!D) \to \mathcal{M}$ *is surjective,*

(ii) $\mathcal{M} \to \mathcal{M}(*D)$ *is injective.*

Let $\mathcal{F}_i$ $(i = 1, 2)$ *be* K*-Betti structures on* $\mathcal{M}$. *If* $\mathcal{F}_{1|X-D} = \mathcal{F}_{2|X-D}$, *then* $\mathcal{F}_1 = \mathcal{F}_2$. □

We reformulate the independence from a compactification (Proposition 7.3.8). Let $F : X' \to X$ be a projective birational morphism of complex manifolds. Let D be a hypersurface, and we put

$$D' := F^{-1}(D).$$

Assume $X' - D' \simeq X - D$.

Proposition 8.3.4. — *Assume that* $\mathrm{SI}(< n)$, $\mathrm{GOOD}(< n)$ *and* $\mathrm{LOC}(\leq n)$ *hold. Let* $\mathcal{M}'$ *be a holonomic* $\mathcal{D}_{X'(*D')}$*-module. We set* $\mathcal{M} := F_\dagger \mathcal{M}'$*.*

- ▷ *If* $\mathcal{F}'$ *is a* K*-Betti structure of* $\mathcal{M}'$*, then* $F_*\mathcal{F}'$ *is a* K*-Betti structure of* $\mathcal{M}$*.*
- ▷ *If* $\mathcal{F}$ *is a* K*-Betti structure of* $\mathcal{M}$*, then* $\mathcal{M}'$ *is equipped with a* K*-Betti structure* $\mathcal{F}'$ *such that* $\mathcal{F}'_{|X'-D'} = \mathcal{F}_{|X-D}$ *under the isomorphism* $\mathcal{M}'_{|X'-D'} \simeq \mathcal{M}_{|X-D}$*. It is functorial.* □

8.3.2. $K(*D)$**-cell.** — Let $\varphi : Z \to X$ be a projective morphism of complex manifolds such that $\dim Z = n$. Let D_Z be a normal crossing hypersurface of Z. Assume that $\varphi_{|Z-D_Z}$ is an immersion. We suppose

$$D_1 := \varphi^{-1}(D) \subset D_Z.$$

Let V be a meromorphic flat connection on (Z, D_Z) with a good K-structure. We obtain the pre-K-holonomic $\mathcal{D}_Z$-modules V and $V_!(*D_1)$.

Proposition 8.3.5. — *Assume that* $\mathrm{SI}(< n)$, $\mathrm{GOOD}(< n)$ *and* $\mathrm{LOC}(< n)$ *hold. Then,* $\varphi_\dagger V_!(*D_1)$ *and* $\varphi_\dagger V$ *are* $K(*D)$*-holonomic.*

Proof. — Let us prove that $\mathcal{C}_0 = (Z, U, \varphi, V)$ is a bounding n-cell at any $P \in D \cap \varphi(Z)$. It is enough to consider the issue locally. We shall shrink X without mention.

Let $\mathcal{C}' = (Z', U', \varphi', V')$ be a dominant refinement at P with a cell function g as $\mathcal{D}_X(*D)$-modules. We have a factorization $\varphi' = \varphi \circ \varphi_1$, where $\varphi_1 : Z' \to Z$. We put

$$g' := g \circ \varphi' \quad \text{and} \quad D_1' := (\varphi')^{-1} D.$$

We have $V' = \varphi_1^{-1} V \otimes \mathcal{O}_{Z'}(*g')$. According to Proposition 6.4.7, the morphisms $\varphi'_\dagger(V'_!)(*D) \to \varphi_\dagger(V_!)(*D) \to \varphi_\dagger V \to \varphi'_\dagger V'$ are compatible with the canonical pre-K-Betti structures. We obtain the induced pre-K-Betti structures of $\phi_g^{(a)}(\varphi_\dagger(V), *D)$ and $\phi_g^{(a)}(\varphi_\dagger(V_!), *D)$.

We obtain pre-K-holonomic $\mathcal{D}$-modules $\phi_{g'}^{(a)}(V'_!, *D'_1)$ and $\phi_{g'}^{(a)}(V', *D'_1)$ on Z'. They are K-holonomic, which can be proved by the argument in the proof of Lemma 8.2.1. We obtain that

$$\phi_g^{(a)}(\varphi'_\dagger V', *D) \quad \text{and} \quad \phi_g^{(a)}\big(\varphi'_\dagger(V'_!), *D\big)$$

are K-holonomic by the assumption SI($< n$).

By Lemma 8.3.1 and assumption LOC($< n$),

$$\phi_g^{(a)}(\varphi'_\dagger V', *D) \quad \text{and} \quad \phi_g^{(a)}(\varphi'_\dagger V'_!, *D)$$

are $K(*D)$-holonomic. Because $\phi_g^{(a)}(\varphi_\dagger V, *D) \subset \phi_g^{(a)}(\varphi'_\dagger V', *D)$ is compatible with the pre-K-Betti structures, $\phi_g^{(a)}(\varphi_\dagger V, *D)$ is also a $K(*D)$-holonomic by Lemma 7.3.5. Since the surjection

$$\phi_g^{(a)}(\varphi'_\dagger V'_!, *D) \longrightarrow \phi_g^{(a)}(\varphi_\dagger V_!, *D)$$

is compatible with the pre-K-Betti structures, $\phi_g^{(a)}(\varphi_\dagger V_!, *D)$ is also $K(*D)$-holonomic by Lemma 7.3.5. ☐

Corollary 8.3.6. — *Assume that* SI($< n$), GOOD($< n$) *and* LOC($< n$) *hold. Let f be a cell function of an n-dimensional cell $\mathcal{C} = (Z, U, \varphi, V)$ as $\mathcal{D}_{X(*D)}$-module. Then, $\psi_f^{(a)}(\varphi_\dagger V, *D)$ and $\Xi_f^{(a)}(\varphi_\dagger V, *D)$ with the canonical pre-K-Betti structures are $K(*D)$-holonomic.*

Proof. — Applying the previous results to $\Pi_{f\star}^{a,b}(\varphi_\dagger V, *D)$ for $\star = *, !$, we obtain that they are $K(*D)$-holonomic. Then, we obtain the corollary. ☐

8.3.3. Localization. — Let us prove LOC($\leq n$) by assuming SI($< n$), GOOD($< n$) and $LOC(< n)$. By Proposition 7.3.7, the problem is local. Let $\mathcal{M}$ be a K-holonomic $\mathcal{D}_X$-module with $\dim \operatorname{Supp} \mathcal{M} \leq n$.

Let P be any point of D. Let (Z, U, φ, V) be a bounding cell of $\mathcal{M}$ at P with a cell function g as K-holonomic $\mathcal{D}$-modules. By taking a refinement, we may assume $U \cap D = \varnothing$. We put $g_1 := \varphi^{-1}(g)$ and $D_1 := \varphi^{-1}(D)$. We have the expression of $\mathcal{M}$ as the cohomology of the complex of the K-holonomic $\mathcal{D}$-modules

$$\psi_g^{(1)}\varphi_\dagger(V_!) \longrightarrow \Xi_g^{(0)}\varphi_\dagger(V) \oplus \phi_g^{(0)}(\mathcal{M}) \longrightarrow \psi_g^{(0)}\varphi_\dagger(V). \tag{112}$$

By the assumption of the induction, $\psi_g^{(a)}(\varphi_\dagger V_!, *D)$ and $\phi_g^{(a)}(\mathcal{M}, *D)$ are equipped with the induced $K(*D)$-Betti structures. We also have the commutative diagram of pre-K-holonomic $\mathcal{D}$-modules

$$\begin{array}{ccccc} \psi_g^{(1)}(\varphi_\dagger V) & \longrightarrow & \phi_g^{(0)}(\mathcal{M}) & \longrightarrow & \psi_g^{(0)}(\varphi_\dagger V) \\ \downarrow & & \downarrow & & \downarrow \\ \psi_g^{(1)}(\varphi_\dagger V_!, *D) & \longrightarrow & \phi_g^{(0)}(\mathcal{M}, *D) & \longrightarrow & \psi_g^{(0)}(\varphi_\dagger V_!, *D). \end{array}$$

We have the canonical pre-K-Betti structures of $\psi_{g_1}^{(a)}(V, *D_1)$ and $\Xi_{g_1}^{(a)}(V, *D_1)$. According to Corollary 8.3.6, their push-forward $\varphi_\dagger\psi_{g_1}^{(a)}(V, *D_1)$ and $\varphi_\dagger\Xi_{g_1}^{(a)}(V, *D_1)$ are $K(*D)$-holonomic.

We also have the commutative diagram of pre-K-holonomic $\mathcal{D}$-modules

$$\begin{array}{ccccc} \varphi_\dagger\psi_{g_1}^{(1)}(V) & \longrightarrow & \varphi_\dagger\Xi_{g_1}^{(0)}(V) & \longrightarrow & \varphi_\dagger\psi_{g_1}^{(0)}(V) \\ \downarrow & & \downarrow & & \downarrow \\ \varphi_\dagger\psi_{g_1}^{(1)}(V, *D_1) & \longrightarrow & \varphi_\dagger\Xi_{g_1}^{(0)}(V, *D_1) & \longrightarrow & \varphi_\dagger\psi_{g_1}^{(0)}(V, *D_1). \end{array}$$

By Proposition 7.3.7, the identification

$$\varphi_\dagger\psi_{g_1}^{(a)}(V, *D_1) \simeq \psi_g^{(a)}(\varphi_\dagger V, *D)$$

is compatible with the pre-K-Betti structures. Hence, we obtain a $K(*D)$-Betti structure of $\mathcal{M}(*D)$ with a morphism of pre-K-holonomic $\mathcal{D}$-modules $\mathcal{M} \to \mathcal{M}(*D)$ whose restriction to $X - D$ is an isomorphism. The functoriality is clear from the above construction. □

8.3.4. Twist. — Let $(\mathcal{M}, \mathcal{F})$ be any $K(*D)$-holonomic $\mathcal{D}(*D)$-module such that $\dim \operatorname{Supp} \mathcal{M} \leq n$. Let $\mathcal{V}$ be a meromorphic flat connection on (X, D) with a good K-structure $\mathcal{F}_{\mathcal{V}}$. According to Lemma 7.2.7, $\mathcal{F}_{\mathcal{M}|X-D} \otimes \mathcal{F}_{\mathcal{V}|X-D}$ is a K-Betti structure of $(\mathcal{M} \otimes \mathcal{V})_{|X-D}$.

Proposition 8.3.7. — *Assume that* SI($< n$), GOOD($< n$) *and* LOC($< n$) *hold. There exists a $K(*D)$-Betti structure $\mathcal{F}_{\mathcal{M}\otimes\mathcal{V}}$ of $\mathcal{M} \otimes \mathcal{V}$ such that*

$$\mathcal{F}_{\mathcal{M}\otimes\mathcal{V}|X-D} \simeq \mathcal{F}_{\mathcal{M}|X-D} \otimes \mathcal{F}_{\mathcal{V}|X-D}.$$

It is functorial with respect to $\mathcal{M}$ and $\mathcal{V}$.

Proof. — Let $P \in D$. It is enough to consider the issue locally around P. We use an induction on $\dim_P \operatorname{Supp} \mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a dominating cell of $\mathcal{M}$ at P. Let g be a cell function for $\mathcal{C}$ as $\mathcal{D}_{X(*D)}$-module. By the inductive assumption, we have the $K(*D)$-Betti structures of

$$\psi_g^{(a)}(\varphi_\dagger V, *D) \otimes \mathcal{V} \quad \text{and} \quad \phi_g^{(a)}(\varphi_\dagger V, *D) \otimes \mathcal{V}.$$

According to Corollary 8.3.6, we have the $K(*D)$-Betti structures of

$$\psi_g^{(a)}(\varphi_\dagger V, *D) \otimes \mathcal{V} \quad \text{and} \quad \Xi_g^{(a)}(\varphi_\dagger V, *D) \otimes \mathcal{V}$$

induced by the isomorphisms

$$\psi_g^{(a)}(\mathcal{M}, *D) \otimes \mathcal{V} \simeq \psi_g^{(a)}(\mathcal{M} \otimes \mathcal{V}, *D), \;\; \Xi_g^{(a)}(\mathcal{M}, *D) \otimes \mathcal{V} \simeq \Xi_g^{(a)}(\mathcal{M} \otimes \mathcal{V}, *D).$$

By the uniqueness, the induced $K(*D)$-Betti structures on $\psi_g^{(a)}(\mathcal{M}, *D) \otimes \mathcal{V}$ are equal. Because $\mathcal{M} \otimes \mathcal{V}$ is expressed as the cohomology of the complex

$$\psi_g^{(1)}(\mathcal{M}, *D) \otimes \mathcal{V} \longrightarrow \Xi_g^{(0)}(\mathcal{M}, *D) \otimes \mathcal{V} \oplus \phi_g^{(0)}(\mathcal{M}, *D) \otimes \mathcal{V} \longrightarrow \psi_g^{(0)}(\mathcal{M}, *D) \otimes \mathcal{V},$$

we obtain a $K(*D)$-Betti structure on $\mathcal{M} \otimes \mathcal{V}$ with the desired property. □

8.3.5. Nearby, vanishing and maximal functors. — Suppose that $\mathrm{SI}(< n)$, $\mathrm{GOOD}(\leq n)$ and $\mathrm{LOC}(< n)$ hold. Let $(\mathcal{M}, \mathcal{F})$ be a K-holonomic $\mathcal{D}_X$-module with $\dim \operatorname{Supp} \mathcal{M} \leq n$. Let f be any holomorphic function on X. As proved in §8.3.3, we obtain a morphism $\mathcal{M} \to \mathcal{M}(*f)$ of K-holonomic $\mathcal{D}_X$-modules. By considering the dual, we also obtain a morphism of K-holonomic $\mathcal{D}_X$-modules $\mathcal{M}(!f) \to \mathcal{M}$.

By Proposition 8.3.7, for any $a \leq b$, we have K-holonomic $\mathcal{D}_X$-modules

$$\Pi_{f\star}^{a,b}(\mathcal{M}) \quad (\star = *, !).$$

Hence, we obtain K-holonomic $\mathcal{D}_X$-modules $\Pi_{f\star!}^{a,b}(\mathcal{M})$. In particular, we obtain K-holonomic $\mathcal{D}_X$-modules $\Xi_f^{(a)}(\mathcal{M})$ and $\psi_f^{(a)}(\mathcal{M})$ with morphisms

$$\mathcal{M}(!f) \longrightarrow \Xi_f^{(0)}(\mathcal{M}) \longrightarrow \mathcal{M}(*f) \quad \text{and} \quad \psi_f^{(1)}(\mathcal{M}) \longrightarrow \Xi_f^{(0)}(\mathcal{M}) \longrightarrow \psi_f^{(0)}(\mathcal{M})$$

in $\mathrm{Hol}(X, K)$. We obtain a K-holonomic $\mathcal{D}_X$-module $\phi_f^{(0)}(\mathcal{M})$ as the cohomology of the complex

$$\mathcal{M}(!f) \longrightarrow \Xi_f^{(0)}(\mathcal{M}) \oplus \mathcal{M} \longrightarrow \mathcal{M}(*f)$$

in $\mathrm{Hol}(X, K)$. We can recover $\mathcal{M}$ as the cohomology of the complex

$$\psi_f^{(1)}(\mathcal{M}) \longrightarrow \Xi_f^{(0)}(\mathcal{M}) \oplus \phi_f^{(0)}(\mathcal{M}) \longrightarrow \psi_f^{(0)}(\mathcal{M})$$

in $\mathrm{Hol}(X, K)$.

8.4. Some resolutions

This subsection is a preliminary for the proof of Theorem 8.1.1.

8.4.1. Non-characteristic condition. — Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex manifold X. There exists a stratification

$$\mathrm{Supp}(\mathcal{M}) = \coprod_{i\in\Lambda} Z_i$$

such that

(i) each Z_i is a smooth locally closed analytic subset of X,

(ii) $\mathrm{Ch}(\mathcal{M}) = \coprod_{i\in\Lambda} T^*_{Z_i}X$.

Lemma 8.4.1. — *A complex submanifold $W \subset X$ is non-characteristic with respect to $\mathcal{M}$ if and only if W and Z_i are transversal for any $i \in \Lambda$. In that case, for the inclusion $\iota : W \to X$, we have $\mathrm{Ch}(\iota_*\iota^*\mathcal{M}) = \coprod_{i\in\Lambda} T^*_{Z_i\cap W}X$.*

Proof. — We have subspaces $(T^*_{Z_i}X)_{|P}$ and $(T^*_W X)_{|P}$ of $(T^*X)_{|P}$ for any $P \in W \cap Z_i$. Then, W and Z_i are transversal at P if and only if

$$(T^*_W X)_{|P} \cap (T^*_{Z_i}X)_{|P} = \{0\}.$$

The first claim of the lemma is clear. The second claim follows from general formulas of the characteristic varieties for the pull back by a non-characteristic closed immersion and the push-forward by a closed immersion. □

Lemma 8.4.2. — *Let D be a smooth hypersurface of X. If D is non-characteristic with respect to $\mathcal{M}$, the natural morphism $\mathcal{M}(!D) \to \mathcal{M}\otimes\mathcal{O}(!D)$ is an isomorphism*

Proof. — Let $i : D \to X$ be the closed immersion. Because D is non-characteristic with respect to $\mathcal{M}$, we have the exact sequence

$$0 \longrightarrow i_*i^*\mathcal{M} \longrightarrow \mathcal{M}(!D) \longrightarrow \mathcal{M} \longrightarrow 0.$$

We have

$$0 \longrightarrow i_*i^*\mathcal{O}_X \longrightarrow \mathcal{O}_X(!D) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

By the non-characteristic condition and the projection formula, we obtain

$$0 \longrightarrow i_*i^*\mathcal{M} \longrightarrow \mathcal{M}\otimes\mathcal{O}_X(!D) \longrightarrow \mathcal{M} \longrightarrow 0.$$

Then, we obtain the claim of the lemma. □

Lemma 8.4.3. — *Let D_i $(i = 1, 2)$ be smooth hypersurfaces of X such that*

(i) *D_1 and D_2 are transversal,*

(ii) *D_1, D_2 and $D_1 \cap D_2$ are non-characteristic with respect to $\mathcal{M}$.*

Then, D_2 is non-characteristic with respect to $\mathcal{M}(*D_1)$, and we have

$$\big(\mathcal{M}(*D_1)\big)(!D_2) \simeq \big(\mathcal{M}(!D_2)\big)(*D_1) \simeq \mathcal{M}\otimes\mathcal{O}(!D_2)\otimes\mathcal{O}(*D_2). \tag{113}$$

Proof. — By the assumption, D_i $(i = 1, 2)$ and $D_1 \cap D_2$ are transversal to Z_j for $j \in \Lambda$. It is elementary to check that D_2 is transversal to $D_1 \cap Z_j$ $(j \in \Lambda)$. We obtain that D_2 is non-characteristic with respect to $\mathcal{M}(*D_1)$. We obtain the isomorphisms (113) from Lemma 8.4.2. □

8.4.2. Non-characteristic tuple of hyperplane subbundles. — Let $\mathcal{E}$ be a locally free sheaf on any complex manifold Y. Let X be its projectivization with the projection $G : X \to Y$. If a section s of $\mathcal{O}_{\mathbb{P}(\mathcal{E})/Y}(1)$ gives a nowhere vanishing section of $G_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})/Y}(1))$, the zero set of s is called a hyperplane subbundle of X. For any hyperplane subbundle H of X and $P \in Y$, let $H_{|P}$ denote the fiber over P.

Let $\mathcal{M}$ be any holonomic $\mathcal{D}_X$-module. Let $\boldsymbol{H} := (H_1, \dots, H_N)$ be a tuple of hyperplane subbundles of X such that, for each $P \in Y$, the tuple of hyperplanes $(H_{1|P}, H_{2|P}, \dots, H_{N|P})$ is of general position.

We say that $\boldsymbol{H}$ is non-characteristic with respect to $\mathcal{M}$ if $H_I := \bigcap_{i \in I} H_i$ are non-characteristic with respect to $\mathcal{M}$ for any $I \subset \{1, \dots, N\}$.

We can prove the following lemma by a standard argument of genericity.

Lemma 8.4.4. — *Suppose that $(H_1, \dots, H_N)$ is non-characteristic with respect to $\mathcal{M}$. Let P be any point of Y. Then, if we shrink Y around P, we can take a hyperplane subbundle H_{N+1} such that $(H_1, \dots, H_N, H_{N+1})$ is also non-characteristic with respect to $\mathcal{M}$.* □

Recall the following general lemma.

Lemma 8.4.5. — *Let (H_1, H_2) be a tuple of hyperplane bundles of X, which is non-characteristic with respect to $\mathcal{M}$. Then, for any $i \neq 0$,*

$$G_\dagger^i\big(\mathcal{M}(*H_1!H_2)\big) = 0.$$

Proof. — Let $\mathcal{M}_i$ $(i = 1, 2)$ be holonomic $\mathcal{D}_X$-modules, and let H_i be hypersurfaces which is non-characteristic with respect to $\mathcal{M}_i$. Because $\mathcal{M}_i$ has a global good filtration according to [**39**], we have an exhaustive filtration $\mathcal{G}_a$ $(a = 1, 2, \dots)$ by coherent $\mathcal{O}_X$-submodules of $\mathcal{M}_1$. We have

$$R^b G_*(\mathcal{G}_a(*H_1) \otimes \Omega^j_{X/Y}) = 0$$

for any $b > 0$. Hence, we have

$$R^b G_* \mathcal{M}_1(*H_1) \otimes \Omega^j_{X/Y} = 0.$$

Then, we obtain $G_{\dagger}^{i}\mathcal{M}_1(*H_1) = 0$ for any $i > 0$. By using the duality, we obtain that

$$G_{\dagger}^{i}(\mathcal{M}_2(!H_2)) = 0$$

for any $i < 0$. Then, the claim follows from Lemma 8.4.3. □

8.4.3. Resolutions. — Let X, Y, $\mathcal{M}$ be as in §8.4.2 and $\boldsymbol{H} = (H_1, \dots, H_N)$ be a tuple of hyperplane subbundles of X, non-characteristic with respect to $\mathcal{M}$. Let $\underline{i} := \{1, \dots, i\}$, and let $\iota_{H_{\underline{i}}}$ denote the inclusion $H_{\underline{i}} \subset X$. We put

$$\mathcal{N}_0 := \mathcal{M}(*H_1), \quad \mathcal{C}_i := \iota_{H_{\underline{i}}\dagger}\iota^*_{H_{\underline{i}}}\mathcal{M} \quad \text{and} \quad \mathcal{N}_i := \mathcal{C}_i(*H_{i+1}).$$

We have the natural exact sequences

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{C}_1 \longrightarrow 0, \quad 0 \longrightarrow \mathcal{C}_i \longrightarrow \mathcal{N}_i \longrightarrow \mathcal{C}_{i+1} \longrightarrow 0. \tag{114}$$

Hence, we obtain the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{N}_1 \longrightarrow \cdots \longrightarrow \mathcal{N}_n \longrightarrow \cdots \tag{115}$$

Let $\boldsymbol{H}' = (H'_j \mid j = 1, \dots, N')$ be a tuple of hyperplane subbundles of X such that $\boldsymbol{H} \sqcup \boldsymbol{H}'$ is non-characteristic with respect to $\mathcal{M}$. We set

$$\mathcal{Q}_{i,0} := \mathcal{N}_i(!H'_1), \quad \mathcal{K}_{i,-j} := \iota_{H'_{\underline{j}}\dagger}\iota^*_{H'_{\underline{j}}}\mathcal{N}_i \quad \text{and} \quad \mathcal{Q}_{i,-j} := \mathcal{K}_{i,-j}(!H_{j+1}).$$

We have the natural exact sequences

$$0 \longrightarrow \mathcal{K}_{i,-1} \longrightarrow \mathcal{Q}_{i,0} \longrightarrow \mathcal{N}_i \longrightarrow 0, \quad 0 \longrightarrow \mathcal{K}_{i,-j-1} \longrightarrow \mathcal{Q}_{i,-j} \longrightarrow \mathcal{K}_{i,-j} \longrightarrow 0.$$

Hence, we obtain the exact sequences

$$0 \longleftarrow \mathcal{N}_i \longleftarrow \mathcal{Q}_{i,0} \longleftarrow \mathcal{Q}_{i,-1} \longleftarrow \mathcal{Q}_{i,-2} \longleftarrow \cdots$$

By construction, we have the naturally defined morphisms $\mathcal{Q}_{i,-j} \to \mathcal{Q}_{i+1,-j}$ and the commutative diagram:

$$\begin{array}{ccc} \mathcal{Q}_{i,-j} & \longrightarrow & \mathcal{Q}_{i+1,-j} \\ \big\downarrow & & \big\downarrow \\ \mathcal{Q}_{i,-j+1} & \longrightarrow & \mathcal{Q}_{i+1,-j+1}. \end{array}$$

Let $\mathrm{Tot}(\mathcal{Q}_{\bullet,\bullet})$ denote the total complex of the double complex $\mathcal{Q}_{\bullet,\bullet}$. We have natural quasi-isomorphisms

$$\mathrm{Tot}(\mathcal{Q}_{\bullet,\bullet}) \xrightarrow{\simeq} \mathcal{N}_\bullet \xleftarrow{\simeq} \mathcal{M}.$$

By the construction, for each $\mathcal{Q}_{i,-j}$, there exists a holonomic $\mathcal{D}$-module $\mathcal{P}_{i,-j}$ such that

(i) (H_{i+1}, H'_{j+1}) is non-characteristic with respect to $\mathcal{P}_{i,-j}$,

(ii) $\mathcal{Q}_{i,-j} = \mathcal{P}_{i,-j}(*H_{i+1}!H'_{j+1})$.

8.5. Step 3

Let us prove that SI($< n$), GOOD($\leq n$) and LOC($\leq n$) imply SI($\leq n$). The following argument is inspired by [**3**].

8.5.1. Special case I. — Let $G : X \to Y$ be any projective morphism of complex manifolds with $\dim X \leq n$. Let D be a hypersurface of X. Let V be a meromorphic flat connection on (X, D) with a good K-structure. Suppose that we are given a sequence of hypersurface pairs $\mathfrak{H}$ contained in D. We obtain a K-holonomic $\mathcal{D}_X$-module $\mathcal{M} := \mathfrak{P}_{\mathfrak{H}}(V)$ with the canonical K-Betti structure $\mathcal{F}$.

Proposition 8.5.1. — *If $G_\dagger^i \mathcal{M} = 0$ for $i \neq 0$, then $RG_* \mathcal{F}$ is a K-Betti structure of $G_\dagger^0 \mathcal{M}$.*

Proof. — It is enough to argue the issue locally around any points of Y. Let us consider the case $\operatorname{Supp} G_\dagger^0 \mathcal{M} \subsetneq G(X)$. We take a holomorphic function f such that $\operatorname{Supp} G_\dagger^0 \mathcal{M} \subset f^{-1}(0)$ and $G(X) \not\subset f^{-1}(0)$. We set

$$f_X := f \circ G.$$

As remarked in §8.3.5, we have a description of the K-holonomic $\mathcal{D}$-module $\phi_{f_X}^{(0)} \mathcal{M}$ as the cohomology of

$$\mathcal{M}(!f_X) \longrightarrow \Xi_{f_X}^{(0)} \mathcal{M}(*f_X) \oplus \mathcal{M} \longrightarrow \mathcal{M}(*f_X).$$

By the assumption,

$$G_\dagger \mathcal{M}(!f_X) = G_\dagger \mathcal{M}(*f_X) = G_\dagger \Xi_{f_X}^{(0)} \mathcal{M}(*f_X) = 0.$$

Hence, we obtain

$$G_\dagger(\mathcal{M}, \mathcal{F}) \simeq G_\dagger \phi_{f_X}^{(0)}(\mathcal{M}, \mathcal{F})$$

as pre-K-holonomic $\mathcal{D}$-modules. By SI($< n$), we obtain that $RG_* \mathcal{F}$ is a K-Betti structure of $G_\dagger^0 \mathcal{M}$.

Let us consider the case $G(X) = \operatorname{Supp} G_\dagger^0 \mathcal{M}$. Let $P \in \operatorname{Supp} G_\dagger^0 \mathcal{M}$. Let $\mathcal{C} = (Z, U, \varphi, E)$ be a cell of $G_\dagger^0 \mathcal{M}$ at P with a cell function g. We set

$$g_Z := \varphi^{-1} g \quad \text{and} \quad g_X := G^{-1} g.$$

We have the K-Betti structures $\mathcal{F}(*g_X)$ of $\mathcal{M}(*g_X)$ by LOC($\leq n$). By considering the dual, we obtain the K-Betti structure $\mathcal{F}(!g_X)$ of $\mathcal{M}(!g_X)$.

LEMMA 8.5.2. — *The K-structure of E is good, and the natural isomorphisms*

$$\varphi_\dagger E(\star g_Z) \simeq G_\dagger(\mathcal{M})(\star g)$$

are compatible with the pre-K-Betti structures for $\star = *, !$.

Proof. — We argue the case $\star = !$. The case $\star = *$ can be argued similarly. We take a projective birational morphism $\kappa : X' \to X$ such that

(i) X' is smooth,

(ii) $X' - (g_X \circ \kappa)^{-1}(0) \simeq X - g_X^{-1}(0)$,

(iii) the induced morphism $X' \to Y$ factors into $X' \overset{G_Z}{\to} Z \overset{\varphi}{\to} Y$.

We set

$$g_{X'} := g_X \circ \kappa \quad \text{and} \quad \mathfrak{H}' := \varphi^{-1}(\mathfrak{H}),$$
$$V' := \kappa^* V \otimes \mathcal{O}(*g_{X'}) \quad \text{and} \quad \mathcal{M}' := \mathfrak{P}_{\mathfrak{H}'}(V')(!g_{X'}).$$

Note that $\kappa_\dagger \mathcal{M}' \simeq \mathcal{M}(!g_X)$ and $G_{Z\dagger}\mathcal{M}' = E(!g_Z)$.

We have the canonical pre-K-Betti structure $\mathcal{F}'$ of $\mathcal{M}'$. We have

$$R\kappa_* \mathcal{F}' = \mathcal{F}(!g_X).$$

By Theorem 6.5.1, we obtain that the K-structure of E is compatible with the Stokes structures, and that $RG_{Z*}\mathcal{F}'$ is the canonical K-Betti structure of $G_{Z\dagger}\mathcal{M}'$. Hence, we obtain that $RG_*\mathcal{F}(!g_X)$ is the canonical K-Betti structure of $G_\dagger(\mathcal{M})(!g) = \varphi_\dagger E(!g_Z)$. Thus, we obtain Lemma 8.5.2. □

LEMMA 8.5.3. — *The natural isomorphisms*

$$G_\dagger \Xi^{(a)}_{g_X}\big(\mathcal{M}(*g_X)\big) \simeq \Xi^{(a)}_g(\varphi_\dagger E) \quad \textit{and} \quad G_\dagger \psi^{(a)}_{g_X}\big(\mathcal{M}(*g_X)\big) \simeq \psi^{(a)}_g(\varphi_\dagger E)$$

are compatible with the induced pre-K-Betti structures.

Proof. — By Lemma 8.5.2, the natural isomorphisms

$$G_\dagger\big(\mathcal{M}(*g_X) \otimes \mathfrak{I}^{a,b}_{g_X}\big)(\star g_X) \simeq \varphi_\dagger E \otimes \mathfrak{I}^{a,b}_{g_Z}(\star g_Z)$$

are compatible with the induced pre-K-Betti structures. Hence, the Lemma 8.5.3. □

By Lemma 8.5.2, the morphisms $\varphi_\dagger E_! \to G_\dagger \mathcal{M} \to \varphi_\dagger E$ are compatible with the induced pre-K-Betti structures, i.e., $\mathcal{C}$ is a K-cell. Hence, we have an induced pre-K-Betti structure ${}^D\phi^{(0)}_g(RG_*\mathcal{F})$ of $\phi^{(0)}_g(G^0_\dagger\mathcal{M})$. We also have the induced K-Betti structure ${}^D\phi^{(0)}_{g_X}(\mathcal{F})$ of $\phi^{(0)}_{g_X}\mathcal{M}$. By using Lemma 8.5.3, we obtain ${}^D\phi^{(0)}_g(RG_*\mathcal{F}) = RG_*\,{}^D\phi^{(0)}_{g_X}(\mathcal{F})$ under the isomorphism

$$\phi^{(0)}_g(G^0_\dagger\mathcal{M}) \simeq G^0_\dagger \phi^{(0)}_{g_X}\mathcal{M}.$$

By the assumption SI($<\dim X$), we obtain that ${}^{D}\phi_g^{(0)}(RG_*\mathcal{F})$ is a K-Betti structure of $\phi_g^{(0)}(G_\dagger\mathcal{M})$. Thus, we obtain Proposition 8.5.1. □

8.5.2. Special case II. — Let $G : X \to Y$ be a projective morphism of complex manifolds. Let $\varphi : Z \to X$ be a projective morphism. Let D_Z be a hypersurface of Z. Assume that $\varphi_{|Z-D_Z}$ is an immersion. Let V be a meromorphic flat connection on (Z, D_Z) with a good K-Betti structure.

Suppose that we are given a sequence of hypersurface pairs $\mathfrak{H}_Z$ of Z contained in D_Z. We obtain the K-holonomic $\mathcal{D}_Z$-modules $\mathcal{M} := \varphi_\dagger\mathfrak{P}_{\mathfrak{H}_Z}(V)$.

Lemma 8.5.4. — *Suppose $G_\dagger^i\mathcal{M} = 0$ unless $i = 0$. Then, the pre-K-holonomic $\mathcal{D}_Y$-module $G_\dagger^0\mathcal{M}$ is K-holonomic.*

Proof. — It follows from Proposition 8.5.1. □

8.5.3. Special case III. — Let $\mathcal{E}$ be a locally free sheaf on a complex manifold Y. Let X be its projectivization. Let H_i $(i = 0, 1, 2)$ be hyperplane subbundles. Let $\mathcal{N}$ be a K-holonomic $\mathcal{D}$-module on X such that $\mathcal{N}(*H_0) = \mathcal{N}$. By shrinking Y, we may assume that $X = Y \times \mathbb{P}^n$ for some n.

Lemma 8.5.5. — *Let $A \subsetneq X$ be any closed complex analytic subset. If we shrink Y appropriately, there exists a meromorphic function g on X such that*

(i) *the poles of g are contained in H_0,*
(ii) *A is contained in $H_0 \cup g^{-1}(0)$.*

Proof. — Let $\mathcal{I}_A$ denote the ideal sheaf of A on X. If m is sufficiently large, we have a non-zero section of $\mathcal{I}_A(mH_0)$ for m. □

Lemma 8.5.6. — *We can take a meromorphic function g on X such that*

(i) *the poles of g are contained in H_0,*
(ii) *$\mathcal{N}(*g)$ is obtained as $\varphi_\dagger V$ for a cell $\mathcal{C} = (Z, U, \varphi, V)$.*

(*Note that we do not assume that V is a good meromorphic flat bundle on Z.*)

Proof. — We have a decomposition of $\mathrm{Supp}(\mathcal{N})$ into the locally closed complex analytic subsets $\coprod A_i$ such that the characteristic variety of $\mathcal{N}$ is $\coprod T^*_{A_i}X$. Applying the previous lemma to the lower dimensional strata, we find a meromorphic function g on X such that

(i) the poles are contained in H_0,
(ii) $A_i \subset H_0 \cup g^{-1}(0)$ if $\dim A_i < \dim \mathrm{Supp}(\mathcal{N})$.

By using the resolution of singularity to the irreducible components of $\mathrm{Supp}(\mathcal{N})$ with the maximal dimension, we obtain the cell. □

Suppose that $\boldsymbol{H}=(H_1,H_2)$ is non-characteristic with respect to $\mathcal{N}$, $\mathcal{N}(*g)$, $\mathcal{N}(!g)(*H_0)$, $\psi_g^{(a)}(\mathcal{N},*H_0)$, $\Xi_g^{(a)}(\mathcal{N},*H_0)$ and $\phi_g^{(a)}(\mathcal{N},*H_0)$. In this case, $\boldsymbol{H}$ is non-characteristic with respect to $\Pi_{g!}^{a,b}(\mathcal{N},*H_0)$ and $\Pi_{g*}^{a,b}(\mathcal{N})$ for any a,b.

Lemma 8.5.7. — *The induced pre-K-Betti structure of $G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\mathcal{N}$ is a K-Betti structure.*

Proof. — By $\mathrm{LOC}(\leq n)$,

$$\mathfrak{P}_{\boldsymbol{H}}(\Pi_{g!}^{a,b}(\mathcal{N},*H_0)) \quad \text{and} \quad \mathfrak{P}_{\boldsymbol{H}}\big(\Pi_{g*}^{a,b}\mathcal{N}\big)$$

are naturally K-holonomic $\mathcal{D}$-modules. By Lemma 8.4.5, we have

$$G_{\dagger}^i\mathfrak{P}_{\boldsymbol{H}}\big(\Pi_{g!}^{a,b}(\mathcal{N},*H_0)\big)=0, \quad G_{\dagger}^i\mathfrak{P}_{\boldsymbol{H}}(\Pi_{g*}^{a,b}\mathcal{N})=0$$

unless $i=0$. According to Lemma 8.5.4,

$$G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\big(\Pi_{g!}^{a,b}(\mathcal{N},*H_0)\big) \quad \text{and} \quad G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}(\Pi_{g*}^{a,b}\mathcal{N})$$

are K-holonomic. Hence, we obtain that

$$G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\Xi_g^{(a)}(\mathcal{N},*H_0) \quad \text{and} \quad G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\psi_g^{(a)}(\mathcal{N},*H_0)$$

are K-holonomic. We have the description of $G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\mathcal{N}$ as the cohomology of the complex of pre-K-holonomic $\mathcal{D}_Y$-modules

$$G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\psi_g^{(1)}(\mathcal{N},*H_0)\longrightarrow G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\Xi_g^{(0)}(\mathcal{N},*H_0)\oplus G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\phi_g^{(0)}(\mathcal{N},*H_0) \\ \longrightarrow G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\psi_g^{(0)}(\mathcal{N},*H_0).$$

By $\mathrm{SI}(<n)$, we obtain that $G_{\dagger}^0\mathfrak{P}_{\boldsymbol{H}}\phi_g^{(0)}(\mathcal{N},*H_0)$ is K-holonomic. Then, we obtain Lemma 8.5.7. □

8.5.4. Proof of Theorem 8.1.1. — It is enough to consider the case $X=\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on Y. Let $(\mathcal{M},\mathcal{F})$ be a K-holonomic $\mathcal{D}_X$-module with $\dim\mathrm{Supp}\,\mathcal{M}\leq n$. Let us prove that $F_{\dagger}^i(\mathcal{M},\mathcal{F})$ are K-holonomic.

We take a resolution $\mathcal{N}_\bullet$ of $\mathcal{M}$ as in (115) of §8.4.3. Then, by applying the construction $\mathcal{Q}_{\bullet,\bullet}$ in §8.4.2 to each $\mathcal{N}_i$, we take a resolution $\mathrm{Tot}(\mathcal{Q}(\mathcal{N}_\bullet)_{\bullet,\bullet})$ of $\mathcal{M}$. It is naturally equipped with the K-Betti structure $\mathrm{Tot}(\mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet,\bullet})$. Then, $F_{\dagger}^i(\mathcal{M},\mathcal{F})$ is described as the i-th cohomology of

$$\mathrm{Tot}\left(F_{\dagger}^0\big(\mathcal{Q}(\mathcal{N}_\bullet)_{\bullet,\bullet},\mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet,\bullet}\big)\right).$$

Hence, it is enough to show that $F^0_\dagger(\mathcal{Q}(\mathcal{N}_\bullet)_{\bullet,\bullet}, \mathcal{F}^{\mathcal{Q}}_{\bullet,\bullet,\bullet})$ are K-holonomic. By the construction, we have $\dim \operatorname{Supp} \mathcal{Q}(\mathcal{N}_k)_{i,j} < \dim \operatorname{Supp} \mathcal{M}$ for $(k,i,j) \neq (0,0,0)$, to which we can apply the inductive assumption. Hence, it is enough to show that $F^0_\dagger(\mathcal{Q}(\mathcal{N}_0)_{0,0}, \mathcal{F}^{\mathcal{Q}}_{0,0,0})$ is K-holonomic, which follows from Lemma 8.5.7. □

CHAPTER 9

DERIVED CATEGORY OF ALGEBRAIC K-HOLONOMIC $\mathcal{D}$-MODULES

We study the standard functors on the derived category of algebraic K-holonomic $\mathcal{D}$-modules. It is enough to follow very closely the arguments in [**3**], [**4**], [**5**] and [**57**], [**58**]. This section is included for a rather expository purpose.

9.1. Standard exact functors

Let X be a smooth complex quasi-projective variety. We take a smooth projective completion $X \subset \overline{X}$ such that $D = \overline{X} - X$ is a hypersurface. We set

$$\mathrm{Hol}(X, K) := \mathrm{Hol}(\overline{X}, *D, K),$$

which is independent of the choice of a completion $\overline{X}$ (Proposition 8.3.4). Let $D^b(\mathrm{Hol}(X, K))$ denote the derived category of $\mathrm{Hol}(X, K)$. We will implicitly use the following obvious lemma. (Later, we will prove a stronger version in Theorem 9.4.1.)

Lemma 9.1.1. — *The forgetful functor* $\mathrm{Hol}(X, K) \to \mathrm{Hol}(X)$ *is faithful.* □

9.1.1. Dual. — For any $\mathcal{M} \in \mathrm{Hol}(\overline{X}, *D, K)$, we have the K-holonomic $\mathcal{D}_{\overline{X}(*D)}$-module $\boldsymbol{D}_X\mathcal{M} := \boldsymbol{D}_{\overline{X}}(\mathcal{M})(*D)$.

Lemma 9.1.2. — $\boldsymbol{D}_X(\mathcal{M})$ *is well defined in* $\mathrm{Hol}(X, K)$.

Proof. — Let $\overline{X}'$ be another smooth projective compactification of X. Put

$$D' := \overline{X}' - X.$$

We may assume to have a projective morphism

$$\varphi : \overline{X}' \longrightarrow \overline{X}$$

such that $\varphi_{|X} = \mathrm{id}_X$. We have a K-holonomic $\mathcal{D}_{\overline{X}'(*D')}$-module $\mathcal{M}'$ such that $\varphi_\dagger \mathcal{M}' = \mathcal{M}$, which is unique up to canonical isomorphisms. Then, the natural isomorphism

$$\varphi_\dagger(\boldsymbol{D}\mathcal{M}')(*D') \simeq \boldsymbol{D}(\mathcal{M})(*D)$$

preserves the K-Betti structure by the uniqueness (Corollary 8.3.2). It implies the claim of the lemma. □

Corollary 9.1.3. — *There exists a functor $\boldsymbol{D}_X$ on $\mathrm{Hol}(X,K)$ which is compatible with the standard duality functors on $\mathrm{Hol}(X)$ and the category of K-perverse sheaves. We also have a functor $\boldsymbol{D}_X$ on $D^b(\mathrm{Hol}(X,K))$, compatible with the standard duality functors on $D^b_{\mathrm{hol}}(X)$ and $D^b_c(K_X)$. They are unique up to natural equivalences.* □

We use the symbol ${}^K\boldsymbol{D}_X$ if we would like to emphasize that it is a functor for K-holonomic $\mathcal{D}$-modules.

Lemma 9.1.4. — *For $\mathcal{M}, \mathcal{N} \in \mathrm{Hol}(X,K)$, we have a natural isomorphism:*

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(\mathcal{M},\mathcal{N}) \simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}({}^K\boldsymbol{D}_X\mathcal{N}, {}^K\boldsymbol{D}_X\mathcal{M}).$$

Proof. — It follows from the comparison of Yoneda extensions. □

9.1.2. Localization. — Let H be a hypersurface of X. As is shown in Theorem 8.1.4 and Proposition 8.3.4, we have the localization

$$*H : \mathrm{Hol}(X,K) \longrightarrow \mathrm{Hol}(X,K), \quad \mathcal{M} \longmapsto \mathcal{M}(*H).$$

It is an exact functor. By considering the dual, we obtain an exact functor

$$!H : \mathrm{Hol}(X,K) \longmapsto \mathrm{Hol}(X,K), \quad \mathcal{M} \longmapsto \mathcal{M}(!H).$$

They induce exact functors $*H$ and $!H$ on $D^b(\mathrm{Hol}(X,K))$.

Lemma 9.1.5. — *For $\mathcal{M}, \mathcal{N} \in \mathrm{Hol}(X,K)$, we have the natural isomorphisms:*

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{M},\mathcal{N}(*D)\big) \simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{M}(*D),\mathcal{N}(*D)\big),$$

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(\mathcal{M}(!D),\mathcal{N}) \simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{M}(!D),\mathcal{N}(!D)\big).$$

Proof. — It follows from comparisons of Yoneda extensions. □

9.1.3. Nearby cycle, vanishing cycle and maximal functors. — Let g be an algebraic function on X. By Proposition 8.3.7, we have the exact functors $\Pi^{a,b}_{g\star}$ ($\star = *, !$) on $\mathrm{Hol}(X,K)$ given by

$$\Pi^{a,b}_{g\star}(\mathcal{M}) := (\mathcal{M} \otimes \mathfrak{I}^{a,b}_g)(\star g) \quad (a,b \in \mathbb{Z}).$$

Hence, we obtain the exact functors $\Xi^{(a)}_g$, $\psi^{(a)}_g$ and $\phi^{(a)}_g$ on $\mathrm{Hol}(X,K)$. They induce the corresponding exact functors on $D^b(\mathrm{Hol}(X,K))$. We use the symbols ${}^K\Xi^{(a)}_g$, ${}^K\psi^{(a)}_g$ and ${}^K\phi^{(a)}_g$, when we would like to emphasize that they are functors for K-holonomic $\mathcal{D}$-modules. We remark that the functors are not compatible with the forgetful functor $D^b(\mathrm{Hol}(X,K)) \to D^b_c(K_X)$.

The K-Betti structure of ${}^K\psi^{(a)}_g(\mathcal{M},\mathcal{F})$ is denoted by ${}^D\psi^{(a)}_g(\mathcal{F})$ for the distinction, when we would like to emphasize it. Similar notations such as ${}^D\Xi^{(a)}_g$ and ${}^D\phi^{(a)}_g$ are used.

9.2. Push-forward and pull-back

9.2.1. Statements. — Let $f : X \to Y$ be an algebraic morphism of quasi-projective varieties. We take a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a_1}\downarrow & & {\scriptstyle a_2}\downarrow \\ \overline{X} & \xrightarrow{\bar{f}} & \overline{Y} \end{array}$$

where

(i) a_i are open immersions,

(ii) $\overline{X}$ and $\overline{Y}$ are smooth projective,

(iii) $H_X = \overline{X} - X$ and $H_Y := \overline{Y} - Y$ are hypersurfaces.

We have a natural equivalence between $\mathrm{Hol}(\overline{X}, *H_X, K)$ and $\mathrm{Hol}(X,K)$. Let $\overline{\mathcal{M}} \in \mathrm{Hol}(\overline{X}, *H_X, K)$ correspond to $\mathcal{M} \in \mathrm{Hol}(X,K)$.

According to Theorem 8.1.1, we obtain the following objects in $\mathrm{Hol}(Y,K)$:

$${}^Kf^i_*(\mathcal{M}) := f^i_\dagger \overline{\mathcal{M}} \quad \text{and} \quad {}^Kf^i_!(\mathcal{M}) := f^i_\dagger\big(\overline{\mathcal{M}}(!H_X)\big)(*H_Y).$$

They are independent of the choice of $\overline{X}$ up to natural isomorphisms. Thus, we obtain cohomological functors ${}^Kf^i_*, {}^Kf^i_! : \mathrm{Hol}(X,K) \to \mathrm{Hol}(Y,K)$ for $i \in \mathbb{Z}$.

Proposition 9.2.1. — *For $\star = !, *$, there exists a functor of triangulated categories*

$${}^Kf_\star : D^b\big(\mathrm{Hol}(X,K)\big) \longrightarrow D^b\big(\mathrm{Hol}(Y,K)\big)$$

such that

(i) *it is compatible with the standard functor* $f_\star : D^b_{\mathrm{hol}}(X) \to D^b_{\mathrm{hol}}(Y)$,

(ii) *the induced functor* $H^i({}^K f_\star) : \mathrm{Hol}(X,K) \to \mathrm{Hol}(Y,K)$ *is isomorphic to* ${}^K f^i_\star$.

It is characterized by the property (i) *and* (ii) *up to natural equivalences.*

As in §4 of [**57**], the pull back is defined to be the adjoint of the push-forward.

PROPOSITION 9.2.2. — *The functor* ${}^K f_!$ *has the right adjoint* ${}^K f^!$, *and* ${}^K f_*$ *has the left adjoint* ${}^K f^*$. *Thus, we obtain the functors*

$$ {}^K f^\star : D^b(\mathrm{Hol}(Y,K)) \longrightarrow D^b\big(\mathrm{Hol}(X,K)\big) \quad (\star = !, *). $$

They are compatible with the corresponding functors of holonomic $\mathcal{D}$-modules with respect to the forgetful functor.

Let us consider the case where f is a closed immersion, via which X is regarded as a submanifold of Y. Let $D^b_X(\mathrm{Hol}(Y,K))$ be the full subcategory of $D^b(\mathrm{Hol}(Y,K))$ which consists of the objects $\mathcal{M}^\bullet$ such that the supports of the cohomology $\bigoplus_i \mathcal{H}^i\mathcal{M}^\bullet$ are contained in X.

PROPOSITION 9.2.3. — *The natural functor* ${}^K f_! : D^b\,\mathrm{Hol}(X,K) \to D^b_X\,\mathrm{Hol}(Y,K)$ *is an equivalence.*

REMARK 9.2.4. — It is a deep theorem[1] of Z. Mebkhout that the irregularity sheaf of any holonomic $\mathcal{D}$-module $\mathcal{M}$ is a perverse sheaf. See [**43**]. By using the above functors, in the algebraic case, we obtain that the irregularity sheaf of a K-holonomic $\mathcal{D}$-module is equipped with an induced K-structure which is clear by the definition of the irregularity sheaf. We may apply the argument even in the analytic case. □

9.2.2. Preliminary. — Let X be a smooth complex projective variety with a hypersurface D. Let

$$ D^b\big(\mathrm{Hol}(X,*D,K)\big) $$

denote the derived category of $\mathrm{Hol}(X,*D,K)$. Similarly, let

$$ D^b\big(\mathrm{Hol}(X,*D)\big) $$

denote the derived category of $\mathrm{Hol}(X,*D)$.

(1) This remark is due to the referee.

Let $f : X \to Y$ be a morphism of smooth projective varieties. Let D_X and D_Y be hypersurfaces of X and Y respectively, such that $D_X \supset f^{-1}(D_Y)$. We have the functor

$$^K f^i_* : \mathrm{Hol}(X, *D_X, K) \longrightarrow \mathrm{Hol}(Y, *D_Y, K),$$

naturally given by $f^i_\dagger$. We have a decomposition $D_X = D_{X1} \cup D_{X2}$ such that $D_{X2} = f^{-1}(D_Y)$. We set $\boldsymbol{D}_X := (D_{X1}, D_{X2})$. We have the functor

$$^K f^i_! : \mathrm{Hol}(X, *D_X, K) \longrightarrow \mathrm{Hol}(Y, *D_Y, K), \quad ^K f^i_!(\mathcal{M}, \mathcal{F}) = f^i_\dagger \mathfrak{P}'_{\boldsymbol{D}_X} \mathcal{M}.$$

Lemma 9.2.5. — *For $\star = *, !$, there exist functors*

$$^K f_\star : D^b\big(\mathrm{Hol}(X, *D_X, K)\big) \longrightarrow D^b\big(\mathrm{Hol}(Y, *D_Y, K)\big)$$

such that

(i) *they are compatible with the standard functors $f_\star : D^b(\mathrm{Hol}(X, *D_X)) \to D^b(\mathrm{Hol}(Y, *D_Y))$ by the forgetful functors,*

(ii) *the induced functor $H^i(^K f_\star) : \mathrm{Hol}(X, *D_X, K) \to \mathrm{Hol}(Y, *D_Y, K)$ are isomorphic to $^K f^i_\star$.*

It is characterized by (i) *and* (ii) *up to natural equivalences.*

Proof. — Let us consider the case $\star = *$. Let $\mathcal{M}$ be a K-holonomic $\mathcal{D}_{X(*D_X)}$-module. Let $\boldsymbol{H} = (H_1, \dots, H_M)$ be a tuple of hypersurfaces of X. We put

$$H_I := \bigcup_{i \in I} H_i.$$

We take a K-vector space U with a base $(e_1, \dots, e_M)$. For $I = (i_1, \dots, i_m) \subset \{1, \dots, M\}$, let U_I denote the subspace of $\bigwedge^\bullet U$ generated by $e_{i_1} \wedge \cdots \wedge e_{i_m}$. For $m \geq 0$, we set

$$\mathcal{C}^m_{*\boldsymbol{H}}(\mathcal{M}) := \bigoplus_{|I|=m+1} \mathcal{M}(*H_I) \otimes U_I.$$

For $Ii := I \sqcup \{i\} \subset \{1, \dots, M\}$, the natural morphism $\mathcal{M}(*H_I) \to \mathcal{M}(*H_{Ii})$ and the multiplication of e_i induce

$$\mathcal{M}(*H_I) \otimes U_I \longrightarrow \mathcal{M}(*H_{Ii}) \otimes U_{Ii}.$$

They give a complex $(\mathcal{C}^\bullet_{*\boldsymbol{H}}(\mathcal{M}), \partial_{*\boldsymbol{H}})$. We have a natural morphism of complexes

$$\mathcal{M} \longrightarrow \mathcal{C}^\bullet_{*\boldsymbol{H}}(\mathcal{M}).$$

If $\bigcap H_i = \varnothing$, it is a quasi-isomorphism.

Suppose we are given a tuple of hypersurfaces $\boldsymbol{L} = (L_1, \dots, L_N)$. We put $\boldsymbol{HL} = (H_1, \dots, H_M, L_1, \dots, L_N)$. The natural projection

$$\mathcal{C}^{\bullet}_{*\boldsymbol{HL}}(\mathcal{M}) \longrightarrow \mathcal{C}^{\bullet}_{*\boldsymbol{H}}(\mathcal{M})$$

gives a complex of morphisms.

Let $\boldsymbol{H}' = (H'_1, \dots, H'_N)$ be a tuple of hypersurfaces on X. We take a K-vector space U' with a base $(e'_1, \dots, e'_N)$. For $J = (j_1, \dots, j_n) \subset \{1, \dots, N\}$, let U'_J be the subspace of $\bigwedge U'$ generated by $e'_{j_1} \wedge \dots \wedge e'_{j_n}$. For $n \leq 0$, we set

$$\mathcal{C}^n_{!\boldsymbol{H}'}(\mathcal{M}) := \bigoplus_{|J|=-n+1} \mathcal{M}(!H'_J) \otimes U'_J.$$

Let $e'^{\vee}_j$ denote the dual base. For $Jj = J \sqcup \{j\} \subset \{1, \dots, N\}$, the natural morphism $\mathcal{M}(H'_{Jj}) \to \mathcal{M}(H'_J)$ and the inner product of $e'^{\vee}_j$ induce

$$\mathcal{M}(H'_{Jj}) \otimes U'_{Jj} \longrightarrow \mathcal{M}(H'_J) \otimes U'_J.$$

They give a complex $(\mathcal{C}^{\bullet}_{!\boldsymbol{H}'}(\mathcal{M}), \partial_{!\boldsymbol{H}'})$. We have a natural morphism of complexes

$$\mathcal{C}_{!\boldsymbol{H}'}(\mathcal{M}) \longrightarrow \mathcal{M}.$$

If $\bigcap H'_i = \varnothing$, it is a quasi-isomorphism.

Suppose that we are given a tuple of hypersurfaces $\boldsymbol{L}' = (L'_1, \dots, L'_M)$. We put $\boldsymbol{H}'\boldsymbol{L}' = (H'_1, \dots, H'_N, L'_1, \dots, L'_M)$. The natural inclusion

$$\mathcal{C}^{\bullet}_{!\boldsymbol{H}'}(\mathcal{M}) \longrightarrow \mathcal{C}^{\bullet}_{!\boldsymbol{H}'\boldsymbol{L}'}(\mathcal{M})$$

gives a quasi-isomorphism.

Let $\mathcal{M}^{\bullet}$ be a complex of K-holonomic $\mathcal{D}_{X(*D_X)}$-modules. Let $\boldsymbol{H}$ and $\boldsymbol{H}'$ be tuples of hypersurfaces. The total complex of $\mathcal{C}^{\bullet}_{*\boldsymbol{H}}\mathcal{C}^{\bullet}_{!\boldsymbol{H}'}(\mathcal{M}^{\bullet})$ is denoted by

$$\mathcal{C}^{\bullet}_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}^{\bullet}).$$

The total complexes of $\mathcal{C}^{\bullet}_{*\boldsymbol{H}}(\mathcal{M}^{\bullet})$ and $\mathcal{C}^{\bullet}_{!\boldsymbol{H}'}(\mathcal{M}^{\bullet})$ are also denoted by the same notation. We assume $\bigcap H_i = \bigcap H'_j = \varnothing$.

We have the natural quasi-isomorphisms of complexes

$$\mathcal{C}^{\bullet}_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}^{\bullet}) \longrightarrow \mathcal{C}^{\bullet}_{*\boldsymbol{H}}(\mathcal{M}^{\bullet}) \longleftarrow \mathcal{M}^{\bullet}.$$

Let $(\boldsymbol{H}_i, \boldsymbol{H}'_i)$ $(i = 1, 2)$ be tuples of hypersurfaces as above. We say that we have a morphism $(\boldsymbol{H}_1, \boldsymbol{H}'_1) \to (\boldsymbol{H}_2, \boldsymbol{H}'_2)$ if $\boldsymbol{H}_1 \supset \boldsymbol{H}_2$ and $\boldsymbol{H}'_1 \subset \boldsymbol{H}'_2$ are satisfied. Then, we have a naturally defined quasi-isomorphism of complexes:

$$\mathcal{C}^{\bullet}_{*\boldsymbol{H}_1!\boldsymbol{H}'_1}(\mathcal{M}^{\bullet}) \longrightarrow \mathcal{C}^{\bullet}_{*\boldsymbol{H}_2!\boldsymbol{H}'_2}(\mathcal{M}^{\bullet}).$$

For a tuple of ample hypersurfaces $(\boldsymbol{H}, \boldsymbol{H}')$ which is non-characteristic with respect to $\mathcal{M}^{\bullet}$ (§8.4.2), we have $f^i_{\dagger}\mathcal{M}^p(*H_I!H_J) = 0$ unless $i = 0$. For each $\mathcal{M}^\bullet$, we choose such $(\boldsymbol{H}(\mathcal{M}^\bullet), \boldsymbol{H}'(\mathcal{M}^\bullet))$. We obtain a complex of K-holonomic $\mathcal{D}_{Y(*D_Y)}$-modules

$$ {}^K f_*(\mathcal{M}^\bullet) := f_\dagger^0 \mathcal{C}^\bullet_{*\boldsymbol{H}(\mathcal{M}^\bullet)!\boldsymbol{H}'(\mathcal{M}^\bullet)}(\mathcal{M}^\bullet). $$

Let $\mathcal{M}_1^\bullet \overset{a}{\longleftarrow} \mathcal{M}_1'^\bullet \overset{b}{\to} \mathcal{M}_2^\bullet$ be morphisms, where a is a quasi-isomorphism. We take a tuple of ample hypersurfaces $(\boldsymbol{H}, \boldsymbol{H}')$ such that

(i) the tuple $(\boldsymbol{H}, \boldsymbol{H}')$ is non-characteristic with respect to $\mathcal{M}_i^\bullet$ and $\mathcal{M}_1'^\bullet$,

(ii) the tuple $(\boldsymbol{H}, \boldsymbol{H}(\mathcal{M}_i^\bullet), \boldsymbol{H}', \boldsymbol{H}'(\mathcal{M}_i'))$ is non-characteristic with respect to $\mathcal{M}_i^\bullet$.

We have the morphism of complexes

$$ \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_1^\bullet) \overset{a_0}{\longleftarrow} \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_1'^\bullet) \longrightarrow \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_2^\bullet). $$

Here, a_0 is a quasi-isomorphism. We set $\boldsymbol{H}_i = \boldsymbol{H}(\mathcal{M}_i^\bullet)$ and $\boldsymbol{H}_i' = \boldsymbol{H}'(\mathcal{M}_i^\bullet)$. We have the quasi-isomorphisms

$$ \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_i^\bullet) \xrightarrow{a_{i1}} \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}\mathcal{C}^\bullet_{*\boldsymbol{H}_i}(\mathcal{M}_i^\bullet) \overset{a_{i2}}{\longleftarrow} \mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}\mathcal{C}^\bullet_{*\boldsymbol{H}_i!\boldsymbol{H}_i'}(\mathcal{M}_i^\bullet). $$

Note that $\mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}\mathcal{C}^\bullet_{*\boldsymbol{H}_i!\boldsymbol{H}_i'}(\mathcal{M}_i^\bullet)$ and $\mathcal{C}^\bullet_{*\boldsymbol{H}_i!\boldsymbol{H}_i'}\mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_i^\bullet)$ are naturally isomorphic. We also have the quasi-isomorphisms

$$ \mathcal{C}^\bullet_{*\boldsymbol{H}_i!\boldsymbol{H}_i'}\mathcal{C}^\bullet_{*\boldsymbol{H}!\boldsymbol{H}'}(\mathcal{M}_i^\bullet) \xrightarrow{a_{i3}} \mathcal{C}^\bullet_{*\boldsymbol{H}_i!\boldsymbol{H}_i'}\mathcal{C}^\bullet_{*\boldsymbol{H}}(\mathcal{M}_i^\bullet) \overset{a_{i4}}{\longleftarrow} \mathcal{C}^\bullet_{*\boldsymbol{H}_i\boldsymbol{H}_i'}(\mathcal{M}_i^\bullet). $$

Note that $f_\dagger^0(a_0)$ and $f_\dagger^0(a_{ij})$ are quasi-isomorphisms. They induce a morphism in $D^b(\mathrm{Hol}(Y, *D_Y, K))$:

$$ {}^K f_*^0(\mathcal{M}_1^\bullet) \longrightarrow {}^K f_*^0(\mathcal{M}_2^\bullet) \tag{116} $$

If we are given morphisms $\mathcal{M}_1^\bullet \overset{a}{\longleftarrow} \mathcal{M}_1'^\bullet \overset{b}{\to} \mathcal{M}_2^\bullet$ such that a' and b' are chain homotopic to a and b respectively, it is easy to check that the induced morphisms (116) in $D^b(\mathrm{Hol}(Y, *D_Y, K))$ are the same.

Let us check that (116) is independent from the choice of $(\boldsymbol{H}, \boldsymbol{H}')$. Let $(\boldsymbol{L}, \boldsymbol{L}')$ be other choice. Take a sequence of sufficiently generic ample hypersurfaces $(\boldsymbol{H}^{(j)}, \boldsymbol{H}'^{(j)})$ $(j = 1, \ldots, 2L)$ satisfying the above conditions, such that

(i) $(\boldsymbol{H}^{(1)}, \boldsymbol{H}'^{(1)}) = (\boldsymbol{H}, \boldsymbol{H}')$ and $(\boldsymbol{H}^{(2L)}, \boldsymbol{H}'^{(2L)}) = (\boldsymbol{L}, \boldsymbol{L}')$,

(ii) we have morphisms

$$ (\boldsymbol{H}^{(2m-1)}, \boldsymbol{H}'^{(2m-1)}) \longleftarrow (\boldsymbol{H}^{(2m)}, \boldsymbol{H}'^{(2m)}) \longrightarrow (\boldsymbol{H}^{(2m+1)}, \boldsymbol{H}'^{(2m+1)}). $$

Then, it is easy to check that $(\boldsymbol{H}, \boldsymbol{H}')$ and $(\boldsymbol{L}, \boldsymbol{L}')$ induce the same morphism (116) in $D^b(\mathrm{Hol}(Y, *D_Y, K))$. Hence, the morphism (116) depends only on the morphism in $D^b(\mathrm{Hol}(X,*D_X,K))$ determined by (a,b), i.e., we obtain a morphism

$$\mathrm{Hom}_{D^b(\mathrm{Hol}(X,K))}(\mathcal{M}_1^\bullet,\mathcal{M}_2^\bullet) \longrightarrow \mathrm{Hom}_{D^b(\mathrm{Hol}(Y,K))}({}^Kf_*\mathcal{M}_1^\bullet,\ {}^Kf_*\mathcal{M}_2^\bullet).$$

Thus, we obtain a functor

$$D^b\big(\mathrm{Hol}(X,*D_X,K)\big) \longrightarrow D^b\big(\mathrm{Hol}(Y,*D_Y,K)\big).$$

We set

$${}^Kf_! := {}^K\boldsymbol{D}_Y \circ {}^Kf_* \circ {}^K\boldsymbol{D}_X.$$

By the construction, they satisfy the conditions (i) and (ii). The uniqueness follows from the existence of a resolution by K-holonomic $\mathcal{D}$-modules $\mathcal{N}$ such that $f_\dagger^i\mathcal{N}=0$ unless $i=0$. □

9.2.3. Proof of Proposition 9.2.1. — We take projective completions $X\subset\overline{X}$ and $Y\subset\overline{Y}$ with the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ \subset\ } & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ Y & \xrightarrow{\ \subset\ } & \overline{Y}. \end{array} \tag{117}$$

Set $D_X:=\overline{X}-X$ and $D_Y:=\overline{Y}-Y$. The functor ${}^Kf_\star: D^b(\mathrm{Hol}(\overline{X},*D_X,K))\to D^b(\mathrm{Hol}(\overline{Y},*D_Y,K))$ induces ${}^Kf_\star: D^b(\mathrm{Hol}(X,K))\to D^b(\mathrm{Hol}(Y,K))$.

Let $X\subset\overline{X}'$ and $Y\subset\overline{Y}'$ be other projective completions with a commutative diagram as in (117). We set $D'_X:=\overline{X}'-X$ and $D'_Y:=\overline{Y}-Y$. Let us prove that the induced morphisms ${}^Kf_\star: D^b(\mathrm{Hol}(X,K))\to D^b(\mathrm{Hol}(Y,K))$ are equal up to natural equivalences. It is enough to consider the case where we have the commutative diagram:

$$\begin{array}{ccc} \overline{X}' & \xrightarrow{\ \bar f'\ } & \overline{Y}' \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ \overline{X} & \xrightarrow{\ \bar f\ } & \overline{Y}. \end{array}$$

Here, φ_X and φ_Y are projective and birational such that $\varphi_X^{-1}(D_X)=D'_X$ and $\varphi_Y^{-1}(D_Y)=D'_Y$. The following diagrams are commutative up to equivalences:

$$\begin{array}{ccc} D^b\big(\mathrm{Hol}(\overline{X}',*D'_X,K)\big) & \xrightarrow{\ {}^Kf_\star\ } & D^b\big(\mathrm{Hol}(\overline{Y}',*D'_Y,K)\big) \\ {\scriptstyle {}^K\varphi_{X\star}}\downarrow & & \downarrow{\scriptstyle {}^K\varphi_{Y\star}} \\ D^b\big(\mathrm{Hol}(\overline{X},*D_X,K)\big) & \xrightarrow{\ {}^Kf_\star\ } & D^b\big(\mathrm{Hol}(\overline{Y},*D_Y,K)\big). \end{array}$$

It implies that ${}^K f_\star : D^b(\mathrm{Hol}(X,K)) \to D^b(\mathrm{Hol}(Y,K))$ are independent of the choice of projective completions up to equivalences. Thus, the proof of Proposition 9.2.1 is finished. □

9.2.4. Proof of Proposition 9.2.3. — Let $\mathcal{M},\mathcal{N} \in \mathrm{Hol}(X,K)$. According to Proposition 3.1.16 of [**5**], it is enough to check the following effaceability:

▷ For any $f \in \mathrm{Ext}^i_{\mathrm{Hol}(Y,K)}(\mathcal{M},\mathcal{N})$, there exists a monomorphism $\mathcal{N} \to \mathcal{N}'$ in $\mathrm{Hol}(X,K)$ such that the image of f in $\mathrm{Ext}^i_{\mathrm{Hol}(Y,K)}(\mathcal{M},\mathcal{N}')$ is 0.

We can prove it by using the arguments in §2.2.1 and §2.2.2 in [**3**]. □

9.2.5. Proof of Proposition 9.2.2. — It is enough to consider the cases

(i) f is a closed immersion,

(ii) f is a projection $X \times Y \to Y$.

We closely follow the arguments in §2.19 and §4.4 of [**57**].

9.2.5.1. Closed immersion. — Let $f : X \to Y$ be a closed immersion. Let $\mathcal{M}^\bullet$ be a complex of K-holonomic $\mathcal{D}_Y$-modules. Let H_i $(i = 1,\dots,N)$ be sufficiently general ample hypersurfaces of Y such that

(i) $H_i \supset X$,

(ii) $\mathcal{M}^\bullet \to \mathcal{M}^\bullet(*H_i)$ are monomorphisms,

(iii) $\bigcap_{i=1}^N H_i = X$.

For any subset $I = (i_1,\dots,i_m) \subset \{1,\dots,N\}$, let $\mathbb{C}_I$ be the subspace of $\bigwedge^m \mathbb{C}^N$ generated by $e_{i_1} \wedge \cdots \wedge e_{i_m}$, where $e_i \in \mathbb{C}^N$ denotes an element whose k-th entry is 1 $(k = i)$ or 0 $(k \neq i)$. For $I = I_0 \sqcup \{i\}$, we set $H_I = \bigcup_{i\in I} H_i$. The inclusion $\mathcal{M}^p(*H_{I_0}) \to \mathcal{M}^p(*H_I)$ and the multiplication of e_i induces

$$\mathcal{M}^p(*H_{I_0}) \otimes \mathbb{C}_{I_0} \longrightarrow \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I.$$

For $m \geq 0$, we put

$$\mathcal{C}^m(\mathcal{M}^p, *\boldsymbol{H}) := \bigoplus_{|I|=m} \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I,$$

and we obtain the double complex $\mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. The total complex is denoted by $\mathrm{Tot}\,\mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. It is easy to observe that the support of the cohomology of $\mathrm{Tot}\,\mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$ is contained in X. According to Proposition 9.2.3, we obtain

$${}^K f^! \mathcal{M}^\bullet := \mathrm{Tot}\,\mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$$

in $D^b(\mathrm{Hol}(X,K))$. We obtain a functor

$${}^K f^! : D^b\big(\mathrm{Hol}(Y,K)\big) \longrightarrow D^b\big(\mathrm{Hol}(X,K)\big)$$

as in Lemma 9.2.5. Note that the underlying $\mathcal{D}_Y$-complex is naturally quasi-isomorphic to $f^!\mathcal{M}^\bullet$, where $f^!$ is the left adjoint of $f_\dagger : D^b_{\mathrm{hol}}(X) \to D^b_{\mathrm{hol}}(Y)$.

We have the naturally defined morphism

$$\alpha : \operatorname{Tot} \mathcal{C}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H}) \longrightarrow \mathcal{M}^\bullet.$$

We put $\mathcal{K}^\bullet := \operatorname{Cone}(\alpha)$. We have another description. For $m \geq 0$, we put

$$\overline{\mathcal{C}}^m(\mathcal{M}^p, *\boldsymbol{H}) := \bigoplus_{|I|=m+1} \mathcal{M}^p(*H_I) \otimes \mathbb{C}_I,$$

and we obtain the double complex $\overline{\mathcal{C}}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. We have a natural quasi-isomorphism $\mathcal{K}^\bullet \simeq \operatorname{Tot} \overline{\mathcal{C}}^\bullet(\mathcal{M}^\bullet, *\boldsymbol{H})$. By using the second description and Lemma 9.1.5, we obtain the following vanishing for any $\mathcal{N}^\bullet \in D^b(\mathrm{Hol}(X, K))$:

$$\operatorname{Hom}_{D^b(\mathrm{Hol}(Y,K))}({}^K f_! \mathcal{N}^\bullet, \mathcal{K}^\bullet) = 0.$$

Hence, we have the following isomorphisms for any K-holonomic $\mathcal{D}_X$-complex $\mathcal{N}^\bullet$:

$$\begin{aligned}\operatorname{Hom}_{D^b(\mathrm{Hol}(Y,K))}({}^K f_! \mathcal{N}^\bullet, \mathcal{M}^\bullet) &\simeq \operatorname{Hom}_{D^b(\mathrm{Hol}(Y,K))}({}^K f_! \mathcal{N}^\bullet, {}^K f_! {}^K f^! \mathcal{M}^\bullet) \\ &\simeq \operatorname{Hom}_{D^b(\mathrm{Hol}(X,K))}(\mathcal{N}^\bullet, {}^K f^! \mathcal{M}^\bullet)\end{aligned}$$

Hence, we obtain that the above functor ${}^K f^!$ is the right adjoint of ${}^K f_!$. By taking the dual, we obtain the left adjoint ${}^K f^*$ of ${}^K f_*$.

9.2.5.2. Projection. — Let $f : Z \times Y \to Y$ be the natural projection. Let $(\mathcal{M}, \mathcal{F})$ be a K-holonomic $\mathcal{D}_Y$-module. We put

$${}^K f^*(\mathcal{M}, \mathcal{F}) := \big(\mathcal{O}_Z \boxtimes \mathcal{M}[-\dim Z], K_Z \boxtimes \mathcal{F}\big).$$

It is easy to check that ${}^K f^*(\mathcal{M}, \mathcal{F})$ is K-holonomic. Thus, we obtain the exact functor

$${}^K f^* : D^b\big(\mathrm{Hol}(Y, K)\big) \longrightarrow D^b\big(\mathrm{Hol}(Z \times Y, K)\big).$$

Let us prove that ${}^K f^*$ is the left adjoint of ${}^K f_*$. It is enough to repeat the argument in §4.4 of [**57**], which we include for the convenience of readers. It is enough to construct natural transformations

$$\alpha : \mathrm{id} \longrightarrow {}^K f_* {}^K f^* \quad \text{and} \quad \beta : {}^K f^* {}^K f_* \longrightarrow \mathrm{id}$$

such that

$$\beta \circ {}^K f^* \alpha : {}^K f^* \mathcal{M}^\bullet \longrightarrow {}^K f^* {}^K f_* {}^K f^* \mathcal{M}^\bullet \longrightarrow {}^K f^* \mathcal{M}^\bullet,$$

$${}^K f_* \beta \circ \alpha : {}^K f_* \mathcal{N}^\bullet \longrightarrow {}^K f_* {}^K f^* {}^K f_* \mathcal{N}^\bullet \longrightarrow {}^K f_* \mathcal{N}^\bullet$$

are the identities. We define α as the external tensor product with the natural map $(\mathbb{C},K)\to(H^0_{DR}(Z),H^0(Z,K))$. For the construction of β, the following diagram is used:

$$\begin{array}{ccccc} Z\times Y & \xrightarrow{\ i\ } & Z\times Z\times Y & \xrightarrow{\ q_1\ } & Z\times Y \\ & & {\scriptstyle q_2}\big\downarrow & & {\scriptstyle p_1}\big\downarrow \\ & & Z\times Y & \xrightarrow{\ p_2\ } & Y. \end{array}$$

Here, i is induced by the diagonal $Z\to Z\times Z$, q_j are induced by the projection $Z\times Z\to Z$ onto the j-th component, and p_j are the projections. We have the morphisms of K-holonomic $\mathcal{D}$-complexes

$$\begin{aligned} (118)\qquad {}^Kf^{*K}f_*\mathcal{M}^\bullet &= {}^Kp_2^{*K}p_{1*}\mathcal{M}^\bullet \\ &\simeq {}^Kq_{2*}{}^Kq_1^*\mathcal{M}^\bullet \longrightarrow {}^Kq_{2*}({}^Ki_*{}^Ki^{*K}q_1^*\mathcal{M}^\bullet)\simeq {}^Ki^{*K}q_1^*\mathcal{M}^\bullet. \end{aligned}$$

LEMMA 9.2.6. — *We have in $D^b(\mathrm{Hol}(Z\times Y,K))$ a natural isomorphism*

$${}^Ki^{*K}q_1^*\mathcal{M}^\bullet\simeq\mathcal{M}^\bullet.$$

Proof. — We have the following morphism of K-holonomic $\mathcal{D}$-complexes:

$$\mathcal{M}^\bullet\overset{\alpha}{\to}{}^Kq_{1*}{}^Kq_1^*\mathcal{M}^\bullet\longrightarrow{}^Kq_{1*}{}^Ki_{1*}{}^Ki_1^{*K}q_1^*\mathcal{M}^\bullet\simeq{}^Ki_1^{*K}q_1^*\mathcal{M}^\bullet.$$

It is enough to check that the composite of the morphisms is an isomorphism for the underlying $\mathcal{D}_Y$-modules. It is enough to consider the issue locally around any point of $Z\times Y$. Then, it can be checked by a direct computation. □

We define β as the composite of (118) with the isomorphism in Lemma 9.2.6. Let us look at ${}^Kf_*\beta\circ\alpha$, which is the composite of the morphisms

$$\begin{aligned} (119)\quad {}^Kf_*\mathcal{M}^\bullet={}^Kp_{1*}\mathcal{M}^\bullet &\longrightarrow{}^Kp_{2*}{}^Kp_2^{*K}p_{1*}\mathcal{M}^\bullet\longrightarrow{}^Kp_{2*}{}^Kq_{2*}{}^Kq_1^*\mathcal{M}^\bullet \\ &\longrightarrow{}^Kp_{2*}{}^Kq_{2*}{}^Ki_*{}^Ki^{*K}q_1^*\mathcal{M}^\bullet\longrightarrow{}^Kf_*{}^Ki^{*K}q_1^*\mathcal{M}^\bullet \\ &\simeq{}^Kf_*\mathcal{M}^\bullet. \end{aligned}$$

We have a natural identification $p_{2*}q_2^*q_1^*\simeq p_{1*}q_{1*}q_1^*$, and $p_{1*}\to p_{2*}q_2^*q_1^*$ in (119) is induced by α for q_1 under the identification. Then, it is easy to see that the composite is the identity by the construction. As for $\beta\circ{}^Kf^*\alpha$, it is expressed as follows:

$$\begin{aligned} (120)\qquad {}^Kf^*\mathcal{N}^\bullet={}^Kp_2^*\mathcal{N}^\bullet &\longrightarrow{}^Kp_2^{*K}p_{1*}{}^Kp_1^*\mathcal{N}^\bullet\longrightarrow{}^Kq_{2*}{}^Kq_1^{*K}p_1^*\mathcal{N}^\bullet \\ &\longrightarrow{}^Kq_{2*}{}^Ki_*{}^Ki^{*K}q_2^{*K}p_2^*\mathcal{N}^\bullet\simeq{}^Kp_2^*\mathcal{N}^\bullet={}^Kf^*\mathcal{N}^\bullet. \end{aligned}$$

We have a natural identification $p_2^* p_1^* p_{1*} \simeq q_{2*} q_2^* p_2^*$, and $p_2^* \to p_2^* p_1^* p_{1*}$ in (120) is induced by α for q_2. Then, it is easy to observe that the composite is the identity. Thus, the proof of Proposition 9.2.2 is finished. □

9.3. Tensor product and inner homomorphism

9.3.1. Statement. — Let $(\mathcal{M}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be K-holonomic $\mathcal{D}$-modules on X_i.

PROPOSITION 9.3.1. — *$\mathcal{F}_1 \boxtimes \mathcal{F}_2$ is a K-Betti structure of $\mathcal{M}_1 \boxtimes \mathcal{M}_2$. As a result, we obtain a natural functor*

$$\boxtimes : \mathrm{Hol}(X_1, K) \times \mathrm{Hol}(X_2, K) \longrightarrow \mathrm{Hol}(X_1 \times X_2, K),$$

compatible with the standard external products

$$\boxtimes : \mathrm{Hol}(X_1) \times \mathrm{Hol}(X_2) \longrightarrow \mathrm{Hol}(X_1 \times X_2)$$

and

$$D_c^b(K_{X_1}) \times D_c^b(K_{X_2}) \longrightarrow D_c^b(K_{X_1 \times X_2}).$$

Before going into the proof of Proposition 9.3.1, We give a standard consequence. Let X be an algebraic variety. Let $\delta_X : X \to X \times X$ be the diagonal morphism. We obtain the functors $\otimes$ and $R\mathcal{H}om$ on $D^b(\mathrm{Hol}(X, K))$ in standard ways:

$$\mathcal{M} \otimes \mathcal{N} := {}^K\delta_X^*(\mathcal{M} \boxtimes \mathcal{N}), \quad R\mathcal{H}om(\mathcal{M}, \mathcal{N}) := {}^K\delta_X^!(\boldsymbol{D}_X \mathcal{M} \boxtimes \mathcal{N})$$

They are compatible with the corresponding functors on $D^b_{\mathrm{hol}}(X)$.

9.3.2. Preliminary. — Let $(\mathcal{M}, \mathcal{F}_{\mathcal{M}})$ be a K-holonomic $\mathcal{D}_X$-module. Let $\mathcal{V}$ be a meromorphic flat connection on (Y, D_Y) with a good K-structure. Let $\mathcal{F}_{\mathcal{V}}$ and $\mathcal{F}_{\mathcal{V}!}$ denote the canonical K-Betti structures of $\mathcal{V}$ and $\mathcal{V}_!$, respectively.

LEMMA 9.3.2. — *$\mathcal{F}_{\mathcal{V}} \boxtimes \mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{V}!} \boxtimes \mathcal{F}_{\mathcal{M}}$ are K-Betti structures of $\mathcal{V} \boxtimes \mathcal{M}$ and $\mathcal{V}_! \boxtimes \mathcal{M}$, respectively.*

Proof. — We use an induction on the dimension of the support of $\mathcal{M}$. Let P be any point of X. It is enough to consider locally around $Y \times \{P\}$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a K-cell of $\mathcal{M}$ at P with a cell function g. The pre-K-holonomic $\mathcal{D}$-module $\mathcal{V} \otimes \mathcal{M}$ is expressed as the cohomology of the following complex of pre-K-holonomic $\mathcal{D}$-modules:

$$\mathcal{V} \boxtimes \psi_g(\varphi_\dagger V) \longrightarrow \mathcal{V} \boxtimes \Xi_g(\varphi_\dagger V) \oplus \mathcal{V} \boxtimes \phi_g(\mathcal{M}) \longrightarrow \mathcal{V} \boxtimes \psi_g(\varphi_\dagger V).$$

By the inductive assumption, $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\psi_g(\varphi_*\mathcal{F}_V)$ and $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\phi_g(\varphi_*\mathcal{F}_V)$ are K-Betti structures of $\mathcal{V} \boxtimes \psi_g(\varphi_\dagger V)$ and $\mathcal{V} \boxtimes \phi_g(\varphi_\dagger V)$, respectively. We put

$$g_Z := \varphi^* g.$$

By using Theorem 8.1.2, we obtain that $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\Xi_{g_Z}(\mathcal{F}_V)$ and $\mathcal{F}_{\mathcal{V}} \boxtimes {}^D\psi_{g_Z}(\mathcal{F}_V)$ are K-Betti structures of $\mathcal{V} \boxtimes \Xi_{g_Z}(V)$ and $\mathcal{V} \boxtimes \psi_{g_Z}(V)$, respectively. By construction, the isomorphism

$$\mathcal{V} \boxtimes \varphi_\dagger\big(\psi_{g_Z}(V)\big) \simeq \mathcal{V} \boxtimes \psi_g(\varphi_\dagger V)$$

preserves K-Betti structures. Hence, we obtain that $\mathcal{F}_{\mathcal{M}} \boxtimes \mathcal{F}_{\mathcal{V}}$ is a K-Betti structure. Thus, we obtain the first claim. By considering the dual, we obtain the second claim. □

Let g be a holomorphic function on Y such that $g^{-1}(0) = D_Y$. We obtain the following corollary from Lemma 9.3.2.

Corollary 9.3.3. — *${}^D\psi_g(\mathcal{F}_{\mathcal{V}}) \boxtimes \mathcal{F}_{\mathcal{M}}$ and ${}^D\Xi_g(\mathcal{F}_{\mathcal{V}}) \boxtimes \mathcal{F}_{\mathcal{M}}$ are K-Betti structures of $\psi_g(\mathcal{V}) \boxtimes \mathcal{M}$ and $\Xi_g(\mathcal{V}) \boxtimes \mathcal{M}$, respectively.* □

9.3.3. Proof of Proposition 9.3.1. — Let P be any point of X_1. It is enough to consider locally around $\{P\} \times X_2$. We use an induction on $\dim_P \operatorname{Supp} \mathcal{M}_1$. Let $\mathcal{C} = (Z, U, \varphi, V)$ be a K-cell of $\mathcal{M}_1$. The pre-K-holonomic $\mathcal{D}$-module $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is expressed as the cohomology of the following complex:

$$\psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2 \longrightarrow \Xi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2 \oplus \phi_g(\mathcal{M}_1) \boxtimes \mathcal{M}_2 \longrightarrow \psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$$

By the inductive assumption, $\psi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ and $\phi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ are K-holonomic. According to Theorem 8.1.1 and Corollary 9.3.3, $\Xi_g(\varphi_\dagger V) \boxtimes \mathcal{M}_2$ is K-holonomic. Hence, we obtain that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is also K-holonomic. Thus, we obtain Proposition 9.3.1. □

9.4. K-structure of the space of morphisms

9.4.1. Statements

Theorem 9.4.1. — *For $M^\bullet, N^\bullet \in D^b(\mathrm{Hol}(X, K))$, the induced morphism*

$$\operatorname{Hom}_{D^b(\mathrm{Hol}(X,K))}(M^\bullet, N^\bullet) \otimes \mathbb{C} \longrightarrow \operatorname{Hom}_{D^b_{\mathrm{hol}}(X)}(M^\bullet, N^\bullet) \tag{121}$$

is an isomorphism. In other words, the forgetful functor

$$D^b(\mathrm{Hol}(X, K)) \otimes \mathbb{C} \longrightarrow D^b_{\mathrm{hol}}(X)$$

is fully faithful.

We closely follow Beilinson's argument in [**3**] for the proof.

THEOREM 9.4.2. — *We have the natural isomorphism*

$$\operatorname{Hom}_{D^b_{\mathrm{hol}}(X,K)}(M^\bullet, N^\bullet) \simeq \operatorname{Hom}_{D^b(\mathrm{Hol}(X,K))}\big(\mathcal{O}_X,\ R\mathcal{H}om(M^\bullet, N^\bullet)[d_X]\big).$$

We essentially use a commutative diagram due to Saito in [**58**].

9.4.2. Homomorphisms and extensions for meromorphic flat connections with a good K-structure.

— Let X be a smooth complex projective variety with a hypersurface D.

LEMMA 9.4.3. — *Let V be a meromorphic flat connection on (X,D) with a good K-structure. Let $\mathcal{F}_V$ be the canonical K-Betti structure of V. We have the following natural isomorphisms for $i=0,1$:*

$$\operatorname{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), V\big) \simeq H^i\big(X, \mathcal{F}_V[-d_X]\big).$$

Proof. — By taking a global resolution of turning points in the algebraic situation (see [**27**], [**47**]), we may assume that V is a good meromorphic flat bundle. Let $\mathcal{L}(V)$ be the associated local system with the Stokes structure on $\widetilde{X}(D)$. It is naturally equipped with a K-structure $\mathcal{L}_K(V)$. If we are given an extension

$$0 \longrightarrow V \longrightarrow P \longrightarrow \mathcal{O}_X(*D) \longrightarrow 0$$

as K-holonomic $\mathcal{D}_X$-modules, P is also a good meromorphic flat bundle with a good K-structure, and it induces an extension

$$0 \longrightarrow \mathcal{L}_K(V)^{\leq D} \longrightarrow \mathcal{L}_K(P)^{\leq D} \longrightarrow K_{\widetilde{X}(D)} \longrightarrow 0$$

of K-constructible sheaves. Conversely, assume that we are given an extension of K-constructible sheaves

$$0 \longrightarrow \mathcal{L}_K(V)^{\leq D} \longrightarrow \mathcal{G}_K \longrightarrow K_{\widetilde{X}(D)} \longrightarrow 0.$$

We obtain a K-local system $\widetilde{\mathcal{G}}_K := \widetilde{\iota}_* \mathcal{G}_{|X\setminus D}$, where $\iota : X\setminus D \to X$. The $\mathbb{C}$-local system $\widetilde{\mathcal{G}}_K \otimes \mathbb{C}$ is naturally equipped with a Stokes structure compatible with the K-structure. Hence, we obtain an extension of K-holonomic $\mathcal{D}_X$-modules

$$0 \longrightarrow V \longrightarrow P \longrightarrow \mathcal{O}_X(*D) \longrightarrow 0.$$

The above procedures are mutually inverse. Thus, we obtain a bijection

$$\operatorname{Ext}^1_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), V\big) \simeq \operatorname{Ext}^1_{K_{\widetilde{X}(D)}}\big(K_{\widetilde{X}(D)}, \mathcal{L}_K(V)^{\leq D}\big) \simeq H^1\big(X, \mathcal{F}_V[-d_X]\big).$$

Similarly, we have a natural isomorphism

$$\operatorname{Ext}^0_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), V\big) \simeq H^0\big(X, \mathcal{F}_V[-d_X]\big).$$ □

Let V, W be meromorphic flat connections on (X, D) with good K-structures. We have a natural bijection

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(W, V) \simeq \mathrm{Ext}^i_{\mathrm{Hol}(X,K)}\big(\mathcal{O}_X(*D), W^\vee \otimes V\big)$$

for any i. We obtain the natural isomorphisms

$$\mathrm{Ext}^i_{\mathrm{Hol}(X,K)}(W, V) \simeq H^i\big(X, \mathcal{F}_{W^\vee \otimes V}[-d_X]\big)$$

for $i = 0, 1$. Because

$$H^i\big(X, \mathcal{F}_{W^\vee\otimes V}[-d_X]\big)\otimes_K \mathbb{C} \simeq H^i\big(X, \mathrm{DR}_X(W^\vee \otimes V)[-d_X]\big) =: H^i_{\mathrm{DR}}(X, W^\vee \otimes V),$$

the vector spaces $H^i_{\mathrm{DR}}(X, W^\vee \otimes V)$ have the natural K-structure. We say that an element $f \in H^i_{\mathrm{DR}}(X, W^\vee \otimes V)$ is compatible with K-structure if it comes from $H^i(X, \mathcal{F}_{W^\vee\otimes V}[-d_X])$. An element $f \in H^1_{\mathrm{DR}}(X, W^\vee \otimes V)$ induces an extension

$$0 \longrightarrow V \longrightarrow P \longrightarrow W \longrightarrow 0$$

in $\mathrm{Hol}(X, K)$ as observed above.

9.4.3. Some extensions. — Let X be a smooth complex quasi-projective variety. Let V_i ($i = 1, 2$) be algebraic flat bundles on X with a good K-structure, i.e., there exists a projective variety $\overline{X} \supset X$ such that

(i) $D := \overline{X} - X$ is normal crossing,

(ii) V_i are good meromorphic flat bundles on $(\overline{X}, D)$ with a good K-structure.

According to [**3**], we have

$$\mathrm{Ext}^i_{\mathrm{Hol}(X)}(V_1, V_2) \simeq H^i_{DR}(X, V_1^\vee \otimes V_2).$$

Lemma 9.4.4. — *There exist a Zariski open subset $U \subset X$ and an extension $V_3 \supset V_{2|U}$ on U of algebraic flat bundles with a good K-structure, such that the induced morphisms*

$$\mathrm{Ext}^i_{\mathrm{Hol}(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{\mathrm{Hol}(U)}(V_{1|U}, V_3)$$

are 0 *for* $i > 0$.

Proof. — We use an induction on $\dim X$. In the case $\dim X = 0$, the claim is trivial. Let us consider the case $\dim X > 0$. We take a Zariski open subset $X_1 \subset X$ with a smooth affine fibration $\rho : X_1 \to Z_1$ such that the relative dimension is 1. For any algebraic flat bundle $\mathcal{V}$ on X_1, we put

$$\rho^q_*(\mathcal{V}) := R^q\rho_*(\mathcal{V} \otimes \Omega^\bullet_{X_1/Z_1}).$$

For a Zariski open subset $Z_1' \subset Z_1$, the induced morphism $\rho^{-1}(Z_1') \to Z_1'$ is also denoted by ρ.

We may assume that $L_q := \rho_*^q(V_1^\vee \otimes V_2)$ are algebraic flat bundles on Z_1, which is equipped with the induced good K-structure. We have $L_q = 0$ unless $q = 0, 1$. By the argument in §2.1 of [**3**], we can reduce Lemma 9.4.4 to Lemma 9.4.5 below which is Lemma 2.1.2 of [**3**] with a minor enhancement.

Lemma 9.4.5

(a) *There exist a Zariski open subset $Z_2 \subset Z_1$ and an extension $P \supset V_{2|X_2}$ of algebraic flat bundles with good K-structure on $X_2 := \rho^{-1}(Z_2)$, such that the following induced morphism is* 0*:*

$$\rho_*^1(V_1^\vee \otimes V_{2|X_2}) \longrightarrow \rho_*^1(V_1^\vee \otimes P).$$

(b) *There exists a Zariski open subset $Z_3 \subset Z_1$ and an extension $Q \supset V_{2|X_3}$ of algebraic flat bundles with good K-structure on $X_3 := \rho^{-1}(Z_3)$, such that the following induced maps are* 0 *for any $p > 0$:*

$$H^p_{\mathrm{DR}}\big(Z_3, \rho_*^0(V_1^\vee \otimes V_{2|X_3})\big) \longrightarrow H^p_{\mathrm{DR}}\big(Z_3, \rho_*^0(V_1^\vee \otimes Q)\big).$$

Proof. — It is enough to use the argument in the proof of Lemma 2.1.2 of [**3**]. We give only an indication. Let

$$\alpha \in H^0_{\mathrm{DR}}(Z_1, L_1^\vee \otimes L_1) = H^0_{\mathrm{DR}}\big(Z_1, \rho_*^1\big((\rho^* L_1 \otimes V_1)^\vee \otimes V_2\big)\big)$$

be the element corresponding to the identity of L_1, which is compatible with K-structure. We have the exact sequence compatible with K-structures

$$\begin{aligned} H^1_{\mathrm{DR}}\big(X_1, (\rho^* L_1 \otimes V_1)^\vee \otimes V_2\big) &\longrightarrow H^0_{\mathrm{DR}}\big(Z_1, \rho_*^1\big((\rho^* L_1 \otimes V_1)^\vee \otimes V_2\big)\big) \\ &\xrightarrow{\ \partial\ } H^2_{\mathrm{DR}}\big(Z_1, \rho_*^0((\rho^* L_1 \otimes V_1)^\vee \otimes V_2)\big) \\ &= H^2_{\mathrm{DR}}(Z_1, L_1^\vee \otimes L_0). \end{aligned}$$

Applying the inductive assumption to $L_0^\vee$ and $L_1^\vee$, we have a Zariski open subset $Z_2 \subset Z_1$ and an extension $\varphi : L_1^\vee \subset R$ of algebraic flat bundles with a good K-structures on Z_2, such that the induced morphism

$$H^2(Z, L_1^\vee \otimes L_0) \longrightarrow H^2(Z_1, R \otimes L_0)$$

is 0. In particular, $\varphi(\partial\alpha) = 0$. We obtain the element

$$\varphi(\alpha) \in H^0_{\mathrm{DR}}(Z_1, R \otimes L_1) = H^0_{\mathrm{DR}}\big(Z_1, \rho_*^1((\rho^* R^\vee \otimes V_1)^\vee \otimes V_2)\big)$$

which is compatible with K-structure. By construction, we have a lift

$$\widetilde{\varphi(\alpha)} \in H^1_{\mathrm{DR}}\big(X, (\rho^* R^\vee \otimes V_1)^\vee \otimes V_2\big)$$

compatible with K-structure. It induces an extension

$$0 \longrightarrow V_{2|X_2} \longrightarrow P \longrightarrow \rho^* R^\vee \otimes V_{1|X_2} \longrightarrow 0$$

of algebraic flat bundles with good K-structure on X_2. (See §9.4.2.) It is easy to observe that P is the desired one. Thus, we obtain the claim (a). The claim (b) can also be proved by the argument in [**3**]. □

9.4.4. Vanishing and lifting. — Let X be a smooth quasi-projective variety. We put $C_1(X) := \mathrm{Hol}(X)$ and $C_2(X) := \mathrm{Hol}(X, K) \otimes \mathbb{C}$. Let V_i $(i = 1, 2)$ be algebraic flat bundles on X with good K-structure. Let us consider the natural morphism:

$$g_X : \mathrm{Ext}^i_{C_2(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{C_1(X)}(V_1, V_2)$$

They are isomorphisms in the cases $i = 0, 1$ (§9.4.2).

Lemma 9.4.6. — *Let $i > 0$.*

- ▷ *Let $a \in \mathrm{Ext}^i_{C_2(X)}(V_1, V_2)$ such that $g_X(a) = 0$. There exists $U \subset X$ such that $a = 0$ in $\mathrm{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U})$.*
- ▷ *Let $a \in \mathrm{Ext}^i_{C_1(X)}(V_1, V_2)$. There exist $U \subset X$ and $b \in \mathrm{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U})$ such that $a_{|U} = g_U(b)$.*

Proof. — We give only an outline. We use an induction on i. We have already known the case $i = 1$. Let $a \in \mathrm{Ext}^i_{C_2(X)}(V_1, V_2)$ such that $g_X(a) = 0$.

We have an extension $V_2 \subset V_3$ of a meromorphic flat bundle with a good K-structure such that the image of a is mapped to 0 via

$$\mathrm{Ext}^i_{C_2(X)}(V_1, V_2) \longrightarrow \mathrm{Ext}^i_{C_2(X)}(V_1, V_3).$$

Let $\mathcal{K} := V_3/V_2$. We have $c \in \mathrm{Ext}^{i-1}_{C_2(X)}(V_1, \mathcal{K})$ which is mapped to a via

$$\mathrm{Ext}^{i-1}_{C_2(X)}(V_1, \mathcal{K}) \longrightarrow \mathrm{Ext}^i_{C_2(X)}(V_1, V_2).$$

We have $d \in \mathrm{Ext}^{i-1}_{C_1(X)}(V_1, V_3)$ which is mapped to $g_X(c)$ via

$$\mathrm{Ext}^{i-1}_{C_1(X)}(V_1, V_3) \longrightarrow \mathrm{Ext}^{i-1}_{C_1(X)}(V_1, \mathcal{K}).$$

By using the inductive assumption, we can find $U \subset X$ and an element $e \in \mathrm{Ext}^{i-1}_{C_2(U)}(V_1, V_3)$ such that $g_U(e) = d_{|U}$. By using the inductive assumption, and by shrinking U, we may assume that e is mapped to $c_{|U}$ via

$$\mathrm{Ext}^{i-1}_{C_2(X)}(V_1, V_3) \longrightarrow \mathrm{Ext}^{i-1}_{C_2(X)}(V_1, \mathcal{K}).$$

Hence, we obtain $a_{|U} = 0$.

Let $a \in \operatorname{Ext}^i_{C_1(X)}(V_1, V_2)$. According to Lemma 9.4.4, we can find $U \subset X$ and an extension $V_{2|U} \subset V_3$ of meromorphic flat bundles with good K-structures such that the induced map

$$\operatorname{Ext}^j_{C_1(U)}(V_{1|U}, V_{2|U}) \longrightarrow \operatorname{Ext}^j_{C_1(U)}(V_{1|U}, V_3)$$

is 0 for any $j > 0$. We put $\mathcal{K} := V_3/V_{2|U}$ We can find $c \in \operatorname{Ext}^{i-1}_{C_1(U)}(V_{1|U}, \mathcal{K})$ which is mapped to a via

$$\operatorname{Ext}^{i-1}_{C_1(U)}(V_{1|U}, \mathcal{K}) \longrightarrow \operatorname{Ext}^i_{C_1(U)}(V_{1|U}, V_{2|U}).$$

By using the inductive assumption and by shrinking U, we can find an element $d \in \operatorname{Ext}^{i-1}_{C_2(U)}(V_{1|U}, \mathcal{K})$ such that $g_U(d) = c$. Let b be the image of d via

$$\operatorname{Ext}^{i-1}_{C_2(U)}(V_{1|U}, \mathcal{K}) \longrightarrow \operatorname{Ext}^i_{C_2(U)}(V_{1|U}, V_{2|U}).$$

Then, it has the desired property. □

9.4.5. Support. — Let X be a smooth quasi-projective variety. For any subvariety $Z \subset X$, let $D^b_{j,Z}(X)$ $(j = 1, 2)$ denote the derived category of bounded complexes $M^\bullet$ in $C_j(X)$ such that the supports of $\mathcal{H}^\bullet(M^\bullet)$ are contained in Z. For any $M^\bullet, N^\bullet$ in $D^b_{j,Z}(X)$, we set

$$\operatorname{Hom}^k_{j,Z}(M^\bullet, N^\bullet) := \operatorname{Hom}_{D^b_{j,Z}(X)}\bigl(M^\bullet, N^\bullet[k]\bigr).$$

If $Z = X$, we omit to denote Z. If Z is smooth, then $D^b_{j,Z}(X)$ is equivalent to the derived category of $C_j(Z)$. (See Proposition 9.2.3.)

Let $i : Z \to X$ denote the inclusion. The natural exact functor

$$D^b_{j,Z}(X) \longrightarrow D^b_j(X)$$

is denoted by i_*. As in §9.2.5, we have a functor

$$i^! : D^b_j(X) \longrightarrow D^b_{j,Z}(X).$$

We set $i^* := \boldsymbol{D}_X \circ i^! \circ \boldsymbol{D}_X$.

9.4.6. Proof of Theorem 9.4.1. — Let X be a smooth quasi-projective variety. Let $M^\bullet, N^\bullet \in D^b_2(X)$. Let us prove that (121) is an isomorphism. We use an induction on $\dim X$.

It is enough to prove that (121) is an isomorphism when $M, N \in C_2(X)$. Take any hypersurface $D \subset X$. Let $i : D \to X$ denote the inclusion. We have the distinguished triangles

$$i_*i^!N \longrightarrow N \longrightarrow N(*D) \xrightarrow{+1} \quad \text{and} \quad M(!D) \longrightarrow M \longrightarrow i_*i^*M \xrightarrow{+1}.$$

For $j=1,2$, we obtain the exact sequence

$$\begin{aligned}(122)\qquad \operatorname{Ext}^{k-1}_{C_j(X)}\big(M(!D),N(*D)\big) &\longrightarrow \operatorname{Hom}^k_{j,D}(i_*i^*M,\, i_*i^!N)\\ &\longrightarrow \operatorname{Ext}^k_{C_j(X)}(M,N)\\ &\longrightarrow \operatorname{Ext}^k_{C_j(X)}\big(M(!D),N(*D)\big)\\ &\longrightarrow \operatorname{Hom}^{k+1}_{j,D}(i_*i^*M,\, i_*i^!N).\end{aligned}$$

We naturally have

$$\operatorname{Ext}^i_{C_j(X)}\big(M(!D),N(*D)\big)\simeq \operatorname{Ext}^i_{C_j(X)}\big(M(*D),N(*D)\big),$$

as remarked in Lemma 9.1.5.

By using the exact sequences (122) in the case where D is smooth, and by using the inductive assumption, we can reduce the issue to the case where X is affine, which we will assume in the following.

We use an induction on the dimension of the support of $M\oplus N$. We take a projective birational morphism

$$\varphi: Z\longrightarrow \operatorname{Supp}(M\oplus N)$$

such that Z is smooth. There exist an open subset $U\subset Z$, flat bundles V_N and V_M on U with morphisms

$$M\longrightarrow \varphi_\dagger V_M \quad\text{and}\quad N\longrightarrow \varphi_\dagger V_N$$

which is an isomorphism on generic points of $\operatorname{Supp}(M\oplus N)$. If we shrink U appropriately, there exists a hypersurface $D\subset X$ such that $\varphi^{-1}(D)=Z\setminus U$. In that case, we have

$$M(*D)=\varphi_\dagger V_M \quad\text{and}\quad N(*D)=\varphi_\dagger V_N.$$

In the exact sequence (122), the dimension of the supports of the cohomology sheaves of i_*i^*M and $i_*i^!N$ are strictly smaller than $\dim\operatorname{Supp}(M\oplus N)$. Then, it is easy to obtain that (121) for i_*i^*M and $i_*i^!N$ is an isomorphism. By using Proposition 9.2.3, we obtain

$$\operatorname{Ext}^k_{C_j(X)}\big(M(!D),N(*D)\big)\simeq \operatorname{Ext}^k_{C_j(X)}\big(M(*D),N(*D)\big)\simeq \operatorname{Ext}^k_{C_j(U)}(V_M,V_N).$$

For $D_1\subset D_2$, we have the commutative diagram

$$\begin{array}{ccccccc} M(!D_1) & \longrightarrow & M & \qquad & N & \longrightarrow & N(*D_1)\\ \uparrow & & \| & & \| & & \downarrow\\ M(!D_2) & \longrightarrow & M, & & N & \longrightarrow & N(*D_2).\end{array}$$

Let $i_a : D_a \to X$ denote the inclusions. We set $U_a := Z \setminus \varphi^{-1}(D_a)$. Hence, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}^i_{j,D_1}(i_{1*}i_1^* M, i_{1*}i_1^! N) & \longrightarrow & \mathrm{Ext}^i_{C_j(X)}(M,N) & \longrightarrow & \mathrm{Ext}^i_{C_j(U_1)}(V_M,V_N) \\ \downarrow & & =\downarrow & & \downarrow \\ \mathrm{Hom}^i_{j,D_2}(i_{2*}i_2^* M, i_{2*}i_2^! N) & \longrightarrow & \mathrm{Ext}^i_{C_j(X)}(M,N) & \longrightarrow & \mathrm{Ext}^i_{C_j(U_2)}(V_M,V_N). \end{array}$$

Then, it is easy to prove that

$$\mathrm{Ext}^i_{C_2(X)}(M,N) \longrightarrow \mathrm{Ext}^i_{C_1(X)}(M,N)$$

is an isomorphism by using Lemma 9.4.6. □

9.4.7. Proof of Theorem 9.4.2. — Recall a commutative diagram in Proposition 4.6 of [**58**]. For $M^\bullet, N^\bullet \in D^b(\mathcal{D}_X)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{D(\mathcal{D}_X)}(M^\bullet, N^\bullet) & \xrightarrow{\simeq} & \mathrm{Hom}_{D(\mathcal{D}_{X\times X})}\big(M^\bullet \boxtimes \boldsymbol{D}N^\bullet,\, \delta_\dagger \mathcal{O}_X[d_X]\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D(\mathbb{C}_X)}(\mathrm{DR}_X\, M^\bullet,\, \mathrm{DR}_X\, N^\bullet) & \xrightarrow{\simeq} & \mathrm{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M^\bullet \otimes \boldsymbol{D}\,\mathrm{DR}_X\, N^\bullet,\, \delta_* \mathbb{C}_X[2d_X]\big). \end{array} \tag{123}$$

Let M be a holonomic $\mathcal{D}_X$-module with a K-Betti structure $\mathcal{F}$. We have

$$\mathrm{Hom}_{D(\mathcal{D}_X)}(M,M) \simeq \mathrm{Hom}_{\mathrm{Hol}(X)}(M,M) \simeq \mathrm{Hom}_{\mathrm{Hol}(X,K)}(M,M)\otimes\mathbb{C}.$$

We have similar isomorphisms for $\mathrm{Hom}_{D(\mathcal{D}_X)}(M \boxtimes \boldsymbol{D}M,\, \delta_\dagger\mathcal{O}_X[d_X])$. Hence, we obtain the following diagram from (123):

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Hol}(X,K)}(M,M)\otimes\mathbb{C} & \underset{\simeq}{\xrightarrow{c}} & \mathrm{Hom}_{\mathrm{Hol}(X\times X,K)}\big(M\boxtimes\boldsymbol{D}M,\, \delta_\dagger\mathcal{O}_X[d_X]\big)\otimes\mathbb{C} \\ a\downarrow & & b\downarrow \\ \mathrm{Hom}_{D(\mathbb{C}_X)}(\mathrm{DR}_X\, M,\, \mathrm{DR}_X\, M) & \xrightarrow{\simeq} & \mathrm{Hom}_{D(\mathbb{C}_X)}\big(\mathrm{DR}_X\, M\otimes\boldsymbol{D}\,\mathrm{DR}_X\, M,\, \delta_*\mathbb{C}_X[2d_X]\big) \\ \simeq\uparrow & & \simeq\uparrow \\ \mathrm{Hom}_{D(K_X)}\big(\mathcal{F},\mathcal{F}\big)\otimes\mathbb{C} & \xrightarrow{\simeq} & \mathrm{Hom}_{D(K_X)}(\mathcal{F}\boxtimes\boldsymbol{D}\mathcal{F},\, \delta_*K_X[2d_X])\otimes\mathbb{C}. \end{array}$$

Note that a is injective. Hence, b is also injective. Since a and b are compatible with K-structures, c is also compatible with K-structures. Let

$$C : M\boxtimes\boldsymbol{D}M \longrightarrow \delta_*\mathcal{O}_X[d_X]$$

correspond to $1 : M \to M$. It is compatible with K-Betti structures.

For $M^\bullet \in D^b(\mathrm{Hol}(X,K))$, let

$$C : M^\bullet\boxtimes\boldsymbol{D}M^\bullet \longrightarrow \delta_\dagger\mathcal{O}_X[d_X]$$

correspond to $1 : M^\bullet \to M^\bullet$. We obtain that C is compatible with K-Betti structures. Then, we obtain that the isomorphism

$$\operatorname{Hom}_{D(\mathcal{D}_X)}(M^\bullet, N^\bullet) \longrightarrow \operatorname{Hom}_{D(\mathcal{D}_{X\times X})}\big(M^\bullet \boxtimes \boldsymbol{D}N^\bullet,\, \delta_\dagger \mathcal{O}_X[d_X]\big)$$

is compatible with K-Betti structures for any $M^\bullet, N^\bullet \in D_{\mathrm{hol}}(X, K)$. By taking the dual, we obtain Theorem 9.4.2. □

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