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Abstract. — We show that a simple Levi compatibility condition determines stability of WKB solutions to semilinear hyperbolic initial-value problems issued from highly-oscillating initial data with large amplitudes. The compatibility condition involves the hyperbolic operator, the fundamental phase associated with the initial oscillation, and the semilinear source term; it states roughly that hyperbolicity is preserved around resonances.

If the compatibility condition is satisfied, the solutions are defined over time intervals independent of the wavelength, and the associated WKB solutions are stable under a large class of initial perturbations. If the compatibility condition is not satisfied, resonances are exponentially amplified, and arbitrarily small initial perturbations can destabilize the WKB solutions in small time.

In the unstable case, the key observation is that resonances correspond to weakly hyperbolic frequencies; the amplification proof then relies on a short-time Duhamel representation formula for solutions of zeroth-order pseudo-differential equations.

Our examples include coupled Klein-Gordon systems, and systems describing Raman and Brillouin instabilities.

***Résumé* (Un critère de stabilité pour des oscillations haute fréquence)**

Nous prouvons qu'une condition de compatibilité de type Levi détermine la stabilité de solutions WKB de systèmes hyperboliques issues de données rapidement oscillantes. La condition de compatibilité fait intervenir l'opérateur hyperbolique, la phase fondamentale associée aux oscillations initiales, et le terme source semi-linéaire ; la condition de compatibilité est satisfaite quand l'hyperbolicité est préservée au voisinage des résonances.

Si la condition de compatibilité est satisfaite, les solutions sont définies sur des intervalles de temps indépendants de la longueur d'onde, et les solutions WKB associées sont stables sous l'effet d'une grande classe de perturbations initiales. Si la condition de compatibilité n'est pas satisfaite, les résonances sont exponentiellement amplifiées et des perturbations initiales arbitrairement petites peuvent déstabiliser les solutions WKB en temps très court.

Dans le cas instable, nous observons que les résonances correspondent à des fréquences faiblement hyperboliques ; l'analyse de l'amplification se base sur une formule de représentation de Duhamel en temps court pour les solutions d'équations pseudo-différentielles d'ordre zéro.

Nous illustrons nos résultats par des systèmes de Klein-Gordon couplés, et des sytèmes décrivant les amplifications Raman et Brillouin.

CONTENTS

CHAPTER 1

INTRODUCTION

We study highly-oscillating solutions to semi-linear systems of the form

$$(1.1) \qquad \partial_t u + \frac{1}{\varepsilon} A_0 u + \sum_{1 \le j \le d} A_j \partial_{x_j} u = \frac{1}{\sqrt{\varepsilon}} B(u, u),$$

in the small wavelength limit $\varepsilon \to 0$. The unknown u depends on time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}^d$; it takes values in $\mathbb{R}^N$. The first-order differential operator is symmetric hyperbolic, in the sense that $A_0 \in \mathbb{R}^{N \times N}$ is skew-symmetric, and the $A_j \in \mathbb{R}^{N \times N}$ are symmetric. The source term is $B(u, u) \in \mathbb{R}^N$, where $B : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is bilinear; it has a large prefactor $1/\sqrt{\varepsilon}$ which blows up in the limit $\varepsilon \to 0$.

Thus in (1.1) we are considering *large perturbations* of symmetric hyperbolic systems. In other words, the regime in (1.1) is supercritical: we are considering the propagation, over times $O(1)$, of solutions with amplitude $O(1)$ to systems (1.1) with characteristic frequencies $O(1/\varepsilon)$ and large $O(1/\sqrt{\varepsilon})$ source terms.

The underlying physical problems concern the propagation of light, and relevant data are highly oscillating, of the form

$$(1.2) \qquad u(\varepsilon, 0, x) = \Re e \, a(x) e^{ik \cdot x / \varepsilon} + \varepsilon^K \phi(\varepsilon, x),$$

where a has a high Sobolev regularity, k is a given wavenumber in $\mathbb{R}^d$, and $\varepsilon^K \phi$ is a small, real perturbation that is smooth in x and may depend singularly on ε.

In this setting, the existence and uniqueness of local-in-time solutions to (1.1)-(1.2) for fixed $\varepsilon > 0$ in smooth Sobolev spaces H^s, with $s > d/2$, is classical. The a priori existence time is only $O(\varepsilon^{1/2})$. Indeed, the symmetric hyperbolic operator conserves Sobolev norms, so that an energy estimate leads to a differential inequality of the form $(\star)$ $y' \le \varepsilon^{-1/2} y^2$, where $y(0)$ is an appropriate Sobolev norm of the datum, typically a semiclassical norm in which derivatives appear as $\varepsilon \partial_x$, so that the fast oscillations are bounded: $y(0) = O(1)$; from $(\star)$ we deduce an existence time $O(\varepsilon^{1/2})$.

We consider the situation in which (1.1) *admits a family of WKB approximate solutions which are defined over time intervals independent of* ε, *and examine their stability with respect to small initial perturbations.*

That is, given an approximate solution u_a issued from the highly-oscillating datum $u_a(0, x) = \Re e\, a(x)e^{ik \cdot x/\varepsilon}$, with an existence time T_a that is uniformly bounded from below as $\varepsilon \to 0$, given an initial perturbation $\varepsilon^K \phi$, possibly with a very small amplitude ε^K, we examine the question whether the exact solution to (1.1) issued from (1.2) is defined over time intervals independent of ε and stays close to u_a.

Our answer to the above question takes the form of a scalar stability index, which involves the initial wavenumber k, the initial amplitude a, the source term B, and the hyperbolic operator. The associated stability condition is a Levi condition, after E. E. Levi [**33**], in the sense that it involves both the principal and subprincipal symbols (A and B, respectively).

Our examples include systems describing the Raman and Brillouin instabilities, and coupled Klein-Gordon systems.

1.1. Background

The class of problems (1.1) originates in Joly, Métivier and Rauch's article on the Maxwell-Bloch equations (see [**25**], and paragraph 6.3 of Dumas' survey [**17**]). There these authors considered Maxwell-Bloch systems in the critical regime of geometric optics, that is

$$(1.3) \qquad \partial_t u + \frac{1}{\varepsilon} A_0 u + \sum_{1 \leq j \leq d} A_j \partial_{x_j} u = B(u, u), \qquad u(\varepsilon, 0, x) = \Re e\, a(x)e^{ik \cdot x/\varepsilon}.$$

By *critical*, we mean here that (1.3) is a regime in which nonlinear effects ought to be detected in the small wavelength limit $\varepsilon \to 0$ in time $O(1)$. Indeed, the nonlinear source has prefactor $O(1)$, and the amplitude of the datum is $O(1)$.

Joly, Métivier and Rauch observed that for Maxwell-Bloch systems in the scaling (1.3), the limiting equations are *linear* transport equations. They called *transparency* this phenomenon, and explained how it originates in compability conditions involving the hyperbolic operator, the oscillations in the datum, and the source B.

Following Joly, Métivier and Rauch, it was verified by the second author that the Euler-Maxwell equations satisfy a form of transparency [**47, 49**], and by the first author that the Maxwell-Landau-Lifschitz equations also are transparent in one spatial dimension [**34**]. Cheverry, Guès and Métivier showed in [**8**] that for systems of conservation laws, linear degeneracy of a field implies transparency. Jeanne showed in [**22**] that the Yang-Mills equations provide another example of a physical system exhibiting transparency properties.

These results imply in particular that for the aforementioned physical systems, relevant regimes are supercritical, meaning that the appropriate scalings (of the observation time or the amplitudes) lead to systems with large nonlinear source terms, as in (1.1).

Being a compatibility condition bearing on a nonlinear term, transparency is analogous to the *null form* conditions which imply global existence for nonlinear wave equations, as in the classical work of Klainerman [**27**]. The link between transparency and null forms is one of the topics covered by Lannes in his Bourbaki review [**30**].

As formulated in [**25**], the two main questions in the analysis of the high-frequency limit in supercritical regimes are:

(a) does there exist WKB approximate solutions?

(b) are WKB solutions stable with respect to initial perturbations?

If the answer to question (a) is positive, then typically the leading terms of WKB solutions satisfy limiting equations that are much simpler than the original system. If the answer to question (b) is positive, then the limiting equations can be used to describe the original system, in particular in numerical simulations.

The article [**25**] shows existence and stability of WKB solutions to Maxwell-Bloch equations in a supercritical regime (different from (1.1); we briefly comment on the difference in Remarks 6.24 and 6.26 in the Appendix). Later on existence and stability of some supercritical WKB solutions was shown for Yang-Mills in [**22**], for Euler-Maxwell in [**49**], for systems of conservation laws in [**8**], for Maxwell-Landau-Lifshitz in [**34**].

The present work all but completes the analysis of systems in the scaling (1.1), as we exhibit a scalar index, which when positive implies instability and when negative implies stability.

1.2. Resonances, transparency, and WKB solutions

We introduce here the notions of resonance and transparency, which play a preeminent role in Joly, Métivier and Rauch's article [**25**] and the present work.

Consider the initial-value problem (1.1)-(1.2) with $\phi = 0$. Under an appropriate *polarization* condition bearing on the initial amplitude a, the spatial oscillations in the datum are propagated in time by the hyperbolic operator in (1.1), at some temporal frequency $\omega = \omega(k)$. Thus we posit the ansatz

$$(1.4) \qquad u(\varepsilon, t, x) = u_{0,-1}(t,x)e^{-i(k\cdot x - \omega t)/\varepsilon} + u_{0,1}(t,x)e^{i(k\cdot x - \omega t)/\varepsilon} + O(\sqrt{\varepsilon}),$$

for an approximate solution u to (1.1)-(1.2). The bilinear term $B(u,u)$ in (1.1) will create *harmonics* of the fundamental phases $\pm(\omega, k)$, so that the $O(\sqrt{\varepsilon})$ term in (1.4) will likely include oscillations $e^{iq(k\cdot x - \omega t)/\varepsilon}$, with $q \in \{-2, 0, 2\}$, in addition to the

fundamental harmonics $\{-1, 1\}$. A refinement of (1.4) is then

(1.5)
$$u(\varepsilon, t, x) = \sum_{p \in \{-1,1\}} u_{0,p}(t,x) e^{ip(k \cdot x - \omega t)/\varepsilon} + \sqrt{\varepsilon} \sum_{|q| \leq 2} u_{1,q}(t,x) e^{iq(k \cdot x - \omega t)/\varepsilon} + O(\varepsilon).$$

We inject (1.5) into (1.1) and sort out oscillating frequencies and powers of ε. Thus conditions

(1.6)
$$\left(-ip\omega + A_0 + A(ipk) \right) u_{0,p} = 0, \quad p \in \{-1, 1\},$$

and

(1.7) $\left(-iq\omega + A_0 + A(iqk) \right) u_{1,q} + \left(\partial_t + A(\partial_x) \right) u_{0,q} = \displaystyle\sum_{q_1 + q_2 = q} B(u_{0,q_1}, u_{0,q_2}), \quad |q| \leq 2.$

with notation

$$A(\vec{e}) := \sum_j A_j e_j, \qquad \text{for any } \vec{e} = (e_1, \ldots, e_d) \in \mathbb{R}^d,$$

imply that (1.5) is an approximate solution to (1.1), with a remainder of size $O(\varepsilon)$. In the case $A_0 \neq 0$, the family of matrices $-ip\omega + A_0 + A(ipk)$, for $p \in \mathbb{Z}$, is not 1-homogeneous in (ω, k). As a result, only a finite number of these matrices is singular, for instance only those corresponding to $p \in \{-1, 0, 1\}$. Then, equation (1.6) holds with non-trivial $u_{0,p}$ only if for all (t, x), $u_{0,p}(t,x)$ is pointing in the direction of a element of the kernel of $-ip\omega + A_0 + A(ipk)$:

(1.8)
$$u_{0,p}(t,x) \equiv \langle u_{0,p}(t,x), \vec{e}_p \rangle \vec{e}_p, \qquad \vec{e}_p \in \ker \left(-ip\omega + A_0 + A(ipk) \right), \ p \in \{-1, 0, 1\}.$$

Condition (1.8) is the *polarization* condition [1]. The mean mode for the initial datum (1.2) vanishes identically. In our context, no mean mode is created by the nonlinearity [2]: $u_{0,0}(t,x) \equiv 0$. At this stage (1.6) is solved and we turn to (1.7). For $q = 0$, denoting $\mathbf{\Pi}(0)$ the orthogonal projector onto the kernel of the skew-symmetric matrix A_0, we find

$$\mathbf{\Pi}(0)\Big(B(u_{0,1}, u_{0,-1}) + B(u_{0,-1}, u_{0,1}) \Big) = 0.$$

With the polarization (1.8), the above condition takes the form

(1.9)
$$\mathbf{\Pi}(0)\Big(B(\vec{e}_1, \vec{e}_{-1}) + B(\vec{e}_{-1}, \vec{e}_1) \Big) = 0.$$

The *compatibility* condition (1.9) was called *transparency* by Joly, Métivier and Rauch (Assumption 2.1 in [**25**]). This condition is a necessary condition for the existence of WKB solutions for general data (1.2) [3]. Under (1.9), a WKB approximate solution

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1. For Maxwell's equations, with $u = (B, E)$, condition (1.8) takes the explicit form $\omega B_{0,\pm 1} = k \times E_{0,\pm 1}$, corresponding to polarization of light.

2. We show in Appendix 6.6 that this is a consequence of the bilinearity of B, assumption (6.47) on the set of characteristic harmonics, and transparency in the form (6.48). The creation of a mean mode is called rectification; it was studied in depth in [**29, 13**].

3. The case in which (1.9) does not hold is briefly discussed in Remark 6.25 on page 120.

can be constructed, and the leading amplitudes $u_{0,\pm 1}$ are seen to satisfy *nonlinear transport equations*. This is explained in detail in Appendix 6.6.

The central question of the present work is whether such WKB solutions are stable under small initial perturbations. This is a perturbative analysis: the question is whether small data generate solutions to

$$(1.10) \qquad \partial_t v + \frac{1}{\varepsilon} A_0 v + A(\partial_x) v = \frac{1}{\sqrt{\varepsilon}} \big(B(u_a, v) + B(v, u_a) \big) + \varepsilon^{K_a} r_a$$

which grow in time. Here $\varepsilon^{K_a} r_a$ is the consistency error of the WKB solution. In this discussion we assume an infinite order of approximation $K_a = \infty$, so that $\varepsilon^{K_a} r_a \equiv 0$. We denote $B(u_a)v = B(u_a, v) + B(v, u_a)$ in the following. In u_a, the important term is the leading term $\mathbf{u}_0$, so that in (1.10) we may simplify $B(u_a)$ into $B\big(e^{-i\theta} u_{0,-1} + e^{i\theta} u_{0,1} \big)$, with $\theta := (k \cdot x - \omega t)/\varepsilon$. The solution to (1.10) is then given by

$$(1.11) \quad v(t) = \mathcal{L}_\varepsilon(t) v(0) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \mathcal{L}_\varepsilon(t - t') B\big(e^{-i\theta} u_{0,-1}(t') + e^{i\theta} u_{0,1}(t') \big) v(t') \, dt',$$

where $\mathcal{L}_\varepsilon(t) := \exp\Big(-\frac{t}{\varepsilon} \big(A_0 + A(\varepsilon \partial_x) \big) \Big)$. Assuming a smooth spectral decomposition

$$(1.12) \qquad\qquad A_0 + A(i\xi) = \sum_j i\lambda_j(\xi) \Pi_j(\xi),$$

where λ_j are real eigenvalues and Π_j orthogonal projectors, the solution (1.11) then appears as the sum, over p, i, j, of

$$(1.13)$$
$$e^{-it\lambda_i(\varepsilon D)/\varepsilon} \Pi_i(\varepsilon D) v(0) + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{-i(t-t')\lambda_i(\varepsilon D)/\varepsilon} \Pi_i(\varepsilon D) B\big(e^{ip\theta} u_{0,p} \big) \Pi_j(\varepsilon D) v(t') \, dt',$$

in which the first term is the free evolution under the solution operator of the initial perturbation. The goal is to bound the second term in (1.13), that is the Duhamel term encoding the accumulated response of $B(u_a)$, considered as a linear source. In (1.13), the operators $\lambda_j(\varepsilon D)$ and $\Pi_j(\varepsilon D)$ are Fourier multipliers in semi-classical quantization [4]. There holds, by linearity of B,

$$\Pi_i(\varepsilon D) B(e^{ip\theta} u_{0,p}) = e^{ip\theta} \Pi_i(\varepsilon D + pk) B(u_{0,p}),$$

so that the Duhamel term in (1.13) takes the form

$$\frac{1}{\sqrt{\varepsilon}} e^{i(pk \cdot x - t\lambda_i(\varepsilon D + pk))/\varepsilon} \int_0^t e^{it'(-p\omega + \lambda_i(\varepsilon D + pk))/\varepsilon} \Pi_i(\varepsilon D + pk) B(u_{0,p}(t')) \Pi_j(\varepsilon D) v(t') dt'.$$

In the following we overlook the unitary prefactor $e^{i(\cdots)}$ in front of the integral. For short times $t \ll \sqrt{\varepsilon}$, it makes sense to approximate $u_{0,p}$ by its datum a or a^*, and v

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4. Notations pertaining to symbols and pseudo-differential operators are set up in Appendix 6.1, where also classical results on action and composition of such operators are recalled.

by the free evolution term in (1.11) [5]. Thus we are looking at

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t e^{it'(-\omega+\lambda_i(\varepsilon D+k))/\varepsilon} \Pi_i(\varepsilon D + k) B(a) e^{-it'\lambda_j(\varepsilon D)/\varepsilon} \Pi_j(\varepsilon D) v(0)\, dt',$$

where we let $p = 1$ for definiteness. For $t' = O(\sqrt{\varepsilon})$, up to operators which are $O(\sqrt{\varepsilon})$ and regularizing, the function $B(a)$ and the Fourier multiplier $e^{-it'\lambda_j(\varepsilon D)/\varepsilon}$ commute [6], and we arrive at

$$(1.14) \qquad \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{it'(-\omega+\lambda_i(\varepsilon D+k)-\lambda_j(\varepsilon D))/\varepsilon} \Pi_i(\varepsilon D + k) B(a) \Pi_j(\varepsilon D) v(0)\, dt'.$$

The question is whether we can bound (1.14) uniformly in ε. This would provide short-time uniform bounds for the solution v to (1.10), and thus would represent a first step in a proof of stability of the WKB solution.

The key frequencies are ξ such that the phase in (1.14) is stationary. These are the resonances, defined as the solutions $\xi \in \mathbb{R}^d$ to

$$(1.15) \qquad\qquad -\omega + \lambda_i(\xi + k) - \lambda_j(\xi) = 0.$$

Far from these resonant frequencies, we can integrate by parts in time in the Fourier formulation of (1.14) and gain a factor ε. Near resonant frequencies, unless the *interaction coefficient* $\Pi_j(\xi + k) B(a) \Pi_j(\xi)$ is small, the integral is $\sim (1/\sqrt{\varepsilon})$, which could lead to an amplification by $e^{c/\sqrt{\varepsilon}}$ of v.

For systems in $u = (u_1, u_2) \in \mathbb{R}^{N_1 \times (N - N_1)}$ and triangular source terms

$$B(u, u) = \begin{pmatrix} 0 \\ B_2(u_1, u_1) \end{pmatrix},$$

this sketch of analysis was made rigorous in [**12**], following [**25**] [7]. That is, smallness of the interaction coefficients at the resonances was seen as a sufficient condition for stability of the WKB solutions.

1.3. A criterion for stability

As discussed just above, previous analyses [**25, 11, 12, 49, 34**] gave only sufficient conditions for stability of WKB solutions in supercritical regimes. We give here a condition that is almost necessary and sufficient [8]. The first step in our analysis is a *reduction to 2×2 interacting systems*. Then, depending on *symbolic spectrum of the propagator*, we either *symmetrize* the interacting system and prove stability, or use a *Duhamel representation* in order to prove instability.

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5. In other words, we are considering the first Picard iterate for (1.10).

6. For a precise statement, we refer to estimate (6.6) in Appendix 6.1.

7. The scaling in [**12**] is actually slightly different from (1.1), and yet another scaling was considered in [**25**]. Remark 6.24 expands on this point.

8. The degenerate case $\mathbf{\Gamma} = 0$ (with notation introduced in (2.11)) is not covered by our analysis, hence an "almost" necessary and sufficient condition.

Reduction to 2×2 interacting systems. The resonance relation (1.15) appears only implicitly in the sketch of analysis given in Section 1.2 above. We make it play an explicit role by introduction of the variables

$$v_i = \mathrm{op}_\varepsilon(\chi_{ij})\big(e^{-i(k \cdot x - \omega t)/\varepsilon}\mathrm{op}_\varepsilon(\Pi_i)\dot{u}\big), \quad v_j = \mathrm{op}_\varepsilon(\chi_{ij}\Pi_j)\dot{u},$$

where $\dot{u}$ is the perturbation variable, defined by $u =: u_a + \dot{u}$, the Π_j are the spectral projectors introduced in (1.12), and χ_{ij} is a frequency cut-off that is supported in a neigborhood of the resonant set $\{\lambda_i(\cdot + k) = \omega + \lambda_j(\cdot)\}$, which we assume to be bounded.

The question of the stability of u_a reduces to the question of the growth in time of (v_i, v_j), for all relevant couples of indices (i, j). We denote $\tilde{V}$ the total variable, that is the collection of relevant couples (v_i, v_j).

Our first key observation is that under a mild *partial transparency condition* for the resonances (formulated as Assumption 2.8(ii), page 23), the *normal form* of the time-evolution system in (v_i, v_j) has the following features:

  – it is *decoupled* from the system in the other components of the solution (corresponding to resonances (i', j'), with $(i', j') \neq (i, j)$),
  – it has *non-oscillating sources*, and
  – resonances $\{\lambda_i(\cdot + k) = \omega + \lambda_j(\cdot)\}$ appear as the locus of weak hyperbolicity.

This normal form of the system in (v_i, v_j) is

$$(1.16) \qquad \partial_t \breve{U}_{ij} + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon \begin{pmatrix} i(\lambda_i(\cdot + k) - \omega) & -\sqrt{\varepsilon}b_{ij} \\ -\sqrt{\varepsilon}b_{ji} & i\lambda_j \end{pmatrix} \breve{U}_{ij} = f,$$

where $\breve{U}_{ij}$ is the (i, j)-component of the total solution $\breve{U}$ after the change of variable to normal form:

$$\breve{U} = (\mathrm{Id} + \sqrt{\varepsilon}\mathrm{op}_\varepsilon(Q))^{-1}\tilde{V}(t, x), \quad \text{for some appropriate symbol } Q,$$

and the interaction coefficients are

$$b_{ij} := \Pi_i(\xi + k)B(u_{0,1}(t, x))\Pi_j(\xi), \qquad b_{ji} := \Pi_j(\xi)B(u_{0,-1}(t, x))\Pi_i(\xi + k).$$

Here we are using notation $u_{0,\pm 1}, \lambda_i, \lambda_j$ from Section 1.2 and $\mathrm{op}_\varepsilon(\cdot)$ from Appendix 6.1. In equation (1.16), the source $f = f(u)$ is bounded in u.

System (1.16) is nominally $2N \times 2N$. However, if the projectors Π_j, Π_j have rank equal to one, then the matrix of the propagator has rank two, essentially making (1.16) a 2×2 system.

Spectrum of the symbol of the propagator. The eigenvalues of the symbol of the propagator in (1.16), a 2×2 complex matrix, is

$$(1.17) \quad \frac{i}{2}\big(\lambda_i(\xi + k) - \omega - \lambda_j(\xi)\big) \pm \frac{1}{2}\Big(4\varepsilon\mathrm{tr}\, b_{ij}b_{ji} - (\lambda_i(\xi + k) - \omega - \lambda_j(\xi))^2\Big)^{1/2}.$$

Thus it appears that the crucial quantity is the sign of the trace of the product of the interaction coefficients at the resonance:

$$\text{sign tr } b_{ij}b_{ij} \qquad \text{at } \xi \text{ such that } \lambda_i(\xi + k) - \omega - \lambda_j(\xi) = 0.$$

If the sign of positive, then real eigenvalues occur in (1.17), meaning a loss of hyperbolicity around the resonance. Otherwise, eigenvalues are purely imaginary. In the latter case, $\text{sign tr } b_{ij}b_{ji} < 0$, the propagator in (1.16) can be symmetrized. For scalar b_{ij} and b_{ji}, a symmetrizer is indeed $\begin{pmatrix} 1 & 0 \\ 0 & -b_{ij}^*/b_{ji} \end{pmatrix}$. Uniform estimates, hence stability, follow.

Duhamel representation and instability. In the case of real eigenvalues in (1.17), indicating instability, the task ahead is to convert a spectral information at the level of symbols into bounds for the corresponding system of pseudo-differential equations (1.16).

This is achieved with the Duhamel representation formula introduced by the second author in [**46**]. This representation extends the Fourier analysis of the above Section 1.2 (of which a good example is (1.13)) by incorporating the zeroth-order source terms b_{ij} and b_{ji} into the propagator. Since resonances are points of weak hyperbolicity, and since at such points the stability analysis must include lower-order terms, the source terms b_{ij} and b_{ji} indeed belong in the propagator.

The instability occurs in time $O(\sqrt{\varepsilon}|\ln \varepsilon|)$. Indeed, the source term in (1.1) or (1.16) has a $O(1/\sqrt{\varepsilon})$ prefactor. Hence a potential growth $\sim e^{tB/\sqrt{\varepsilon}}$. If we start from a small $\sim \varepsilon^K$ initial perturbation, then the instability is recorded only when the time exponential $e^{tB/\sqrt{\varepsilon}}$ reaches a fraction of the size of the initial perturbation ε^K, meaning an instability time of order $\sqrt{\varepsilon}|\ln \varepsilon|$.

For this reason in the unstable case we rescale in time

$$U_{ij}(t,x) := \check{U}_{ij}(\sqrt{\varepsilon}t, x),$$

so that U_{ij} solves

$$\partial_t U_{ij} + \frac{1}{\sqrt{\varepsilon}}\text{op}_\varepsilon \begin{pmatrix} i(\lambda_i(\cdot + k) - \omega) & -\sqrt{\varepsilon}b_{ij} \\ -\sqrt{\varepsilon}b_{ji} & i\lambda_j \end{pmatrix} U_{ij} = \sqrt{\varepsilon}f,$$

where b_{ij}, b_{ji} and f are evaluated at $(\sqrt{\varepsilon}t, x)$.

We then localize around resonant frequencies. Since the resonant set is assumed to be bounded, this means multiplying the equation to the left by $\text{op}_\varepsilon(\chi)$, where χ is a smooth, compactly supported frequency cut-off that is identically equal to one in a neighborhood of the resonances. Then $V := \text{op}_\varepsilon(U_{ij})$ solves

$$(1.18) \qquad \partial_t V + \frac{1}{\sqrt{\varepsilon}}\text{op}_\varepsilon \left(\chi \begin{pmatrix} i(\lambda_i(\cdot + k) - \omega) & -\sqrt{\varepsilon}b_{ij} \\ -\sqrt{\varepsilon}b_{ji} & i\lambda_j \end{pmatrix} \right) V = \sqrt{\varepsilon}f_V,$$

where b_{ij}, b_{ji} and f_V are evaluated at $(\sqrt{\varepsilon}t, x)$, and f_V enjoys the same bounds as f.

The representation formula of [**46**] states that the solution operator to (1.18) is well approximated, in time $O(|\ln \varepsilon|)$, by the para-differential operator $\mathrm{op}_\varepsilon^\psi(S_0)$, where S_0 is the finite-dimensional solution operator, defined for all (x, ξ) by

$$(1.19) \quad \begin{cases} \partial_t S_0 + \dfrac{1}{\sqrt{\varepsilon}} \chi(\xi) \begin{pmatrix} i(\lambda_i(\xi + k) - \omega) & -\sqrt{\varepsilon} b_{ij}(\sqrt{\varepsilon}t, x, \xi) \\ -\sqrt{\varepsilon} b_{ji}(\sqrt{\varepsilon}t, x, \xi) & i\lambda_j(\xi) \end{pmatrix} S_0 = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad S_0(\tau; \tau, x, \xi) \equiv \mathrm{Id}. \end{cases}$$

That is, the solution to (1.18) admits the representation

$$(1.20) \qquad V = \mathrm{op}_\varepsilon^\psi(S_0(0;t))V(0) + \sqrt{\varepsilon} \int_0^t \mathrm{op}_\varepsilon^\psi(S_0(t';t))\tilde{f}(t')\,dt',$$

where $\tilde{f} \simeq f_V$. Appendix 6.2 is devoted to a proof of (1.20).

A key consequence is that in time $O(|\ln \varepsilon|)$ bounds for (1.18) can be deduced from bounds on S_0 : the approximation result of [**46**] simplifies the analysis of an ordinary differential equation in infinite dimensions (namely, (1.18)[9]) into the analysis of a family of ordinary differential equations in finite dimensions (namely, (1.19)).

Bounds for S_0 do not derive trivially from consideration of the spectrum (1.17), since the resonant locus is at a distance $O(\sqrt{\varepsilon})$ from the singular locus

$$\left\{ \xi \in \mathbb{R}^d, \quad 2\big(\varepsilon \,\mathrm{tr}\, b_{ij} b_{ji}\big)^{1/2} = \lambda_i(\xi + k) - \omega - \lambda_j(\xi) \right\},$$

where eigenvalues (1.17) coalesce. In particular, the eigenprojectors are *not* uniformly bounded in ε near the resonances, and bounds for (1.19) cannot be derived by simply diagonalizing the system. Appendix 6.3 is devoted to a precise derivation of these bounds in the unstable case of a positive trace.

From (1.20), armed with optimal bounds for S_0, meaning a lower rate of exponential growth that is arbitrarily close to the upper rate of growth, the task ahead is to derive *lower bounds* for the free component of the solution $\mathrm{op}_\varepsilon^\psi(S_0(0;t))\check{U}_{ij}(0)$, and *upper bounds* for the time-integral term in (1.20).

Lower bounds for $\mathrm{op}_\varepsilon^\psi(S_0(0;t))U_{ij}(0)$ with a maximal rate of growth are achieved by a careful choice of the initial perturbation $U_{ij}(0)$. Essentially, we choose to initially excite frequencies that grow at the highest rate. This is the purpose of Section 3.1.4 in the proof of Theorem 2.7. Upper bounds for $\displaystyle\int_0^t \mathrm{op}_\varepsilon^\psi(S_0(t';t))\tilde{f}(t')$ derive from bounds for $\mathrm{op}_\varepsilon^\psi(S_0)$, which are deduced from bounds on S_0 via Calderón-Vaillancourt type theorems. Details are given in Sections 3.1.3 and 3.1.5 in the main proof. The comparison of lower bounds with upper bounds in Section 3.1.6 concludes the proof.

There is a specific difficulty associated with the large prefactor $(1/\sqrt{\varepsilon})$ in (1.19). This prefactor implies indeed that S_0 has large variations in ξ : $\partial_\xi S_0 \sim S_0/\sqrt{\varepsilon}$. This

---

9. The propagator in (1.18) is indeed bounded $L^2 \to L^2$; this is a consequence of the Calderón-Vaillancourt theorem [**5, 9**], of which a very simple proof is given in [**21**]. A precise statement is given in Appendix 6.1.

is problematic in view of Calderón-Vaillancourt type theorems, which typically assert boundedness of pseudo-differential operators given boundedness of the symbols *and* their (x, ξ)-derivatives. We overcome this issue by using a result from Hörmander [20] (formulated as Proposition 6.7 in Appendix 6.1) which gives operator bounds involving spatial L^1 norms of the symbols, and no ξ-derivatives. This requires a spatial localization step, since the symbols that we handle are a priori not L^1 in space.

1.4. On the class of initial perturbations

A salient feature of our analysis in the stable case is that we allow for initial perturbations $\phi(\varepsilon, x)$, which do not necessarily depend on (ε, x) periodically through $k \cdot x/\varepsilon$. In particular, we give a geometric optics result for a class of perturbations which is much larger than the class of perturbations allowed in a number of results of the JMR school [23, 15, 24, 25, 29, 13, 11, 16, 12].

In these references, WKB solutions u_a and initial perturbations ϕ which are 2π-periodic in the fast variable $(k \cdot x - \omega t)/\varepsilon$ allow for a representation of the solution in the form of a profile, that is a map $\mathbf{u}$ of (t, x, θ) with a 2π-periodic dependence in θ, the trace of which over $\theta = (k \cdot x - \omega t)/\varepsilon$ is equal to the original solution:

$$u(t, x) = \mathbf{u}\left(t, x, \frac{k \cdot x - \omega t}{\varepsilon}\right).$$

This representation de-singularizes the initial datum, which for profiles appears as

$$\mathbf{u}(0, x, \theta) = \Re e\left(a(x)e^{i\theta}\right) + \varepsilon^K \phi(x, \theta),$$

where ϕ is 2π-periodic in θ, by assumption. In particular, the leading term $\Re e\, ae^{i\theta}$ is bounded in $H^s(\mathbb{R}_x^d \times \mathbb{T}_\theta)$. The drawback is that the equation in $\mathbf{u}$ is more singular than the original system (1.1), since it features the singular differential operator $\frac{1}{\varepsilon}(-\omega\partial_\theta + \sum_j k_j A_j \partial_\theta)$. This operator, however, contributes zero to L^2 estimates in (x, θ), by symmetry.

By contrast, in the present work we do not insist on a periodic dependence in the fast variable $k \cdot x/\varepsilon$ for the initial perturbation ϕ in (1.2). In particular, $\phi(\varepsilon, x)$ may take the form $\phi_0(x/\varepsilon)$, where ϕ_0 is only assumed a high Sobolev regularity. In this context, ε-uniform Sobolev estimates may be derived only for ε-weighted norms, defined as

$$(1.21) \qquad \|u\|_{\varepsilon,s} := \left(\int_{\mathbb{R}^d} (1 + |\varepsilon\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2},$$

and an important tool is the Sobolev product estimate

$$(1.22) \qquad \|uv\|_{\varepsilon,s} \leq C\left(|u|_{L^\infty} \|v\|_{\varepsilon,s} + |(\varepsilon\partial_x)^s u|_{L^\infty} |v|_{L^2}\right),$$

which can be proved by approximating the product uv by the para-product of v by u. Details are given in Appendix 6.1.3. In our use of (1.22), u is the approximate

solution u_a, with a periodic dependence in $(k \cdot x - \omega t)/\varepsilon$, and v is the solution to (1.1), with a priori a singular dependence in x via x/ε, just like the initial perturbation. In particular, $|u_a|_{L^\infty}$ and $|(\varepsilon \partial_x)^s u_a|_{L^\infty}$ are both bounded uniformly in ε, implying the bound

$$\|u_a v\|_{\varepsilon,s} \leq C(u_a) \|v\|_{\varepsilon,s}, \quad \text{for } s \geq 0.$$

By comparison, the Moser-type estimate

$$(1.23) \qquad \|uv\|_{\varepsilon,s} \leq C \big(|u|_{L^\infty} \|v\|_{\varepsilon,s} + |v|_{L^\infty} \|u\|_{\varepsilon,s} \big)$$

would here give only

$$\|u_a v\|_{\varepsilon,s} \leq C(u_a) \big(\|v\|_{\varepsilon,s} + |v|_{L^\infty} \big) \leq \varepsilon^{-d/2} C(u_a) \|v\|_{\varepsilon,s}, \quad \text{for } s > d/2,$$

since the Sobolev embedding $H^s \hookrightarrow L^\infty$, for $s > d/2$, has a large norm when H^s is equipped with (1.21):

$$(1.24) \qquad |u|_{L^\infty} \leq C_{d,s} \varepsilon^{-d/2} \|u\|_{\varepsilon,s}, \qquad s > d/2, \ C_{d,s} > 0.$$

However, for *semi-linear* terms of the type v^2 (or $B(v,v)$) where v is the solution, both (1.22) and (1.23) lead to $\varepsilon^{-d/2}$ losses, via (1.24). This is the main drawback of our approach: while it allows for quite general perturbations, it requires *smallness* of these, typically in the form of the lower bound $K > (1+d)/2$, in order to prove stability.

1.5. Overview of the results

We give five theorems:

• the first, Theorem 2.7 (page 20), states that stability of WKB solutions is determined by a scalar index, which when positive indicates instability, and when negative indicates stability. The degenerate case of a vanishing stability index covers different possible situations with regard to stability, one of them treated in [**49**]. For a discussion of the precise meaning of stability/instability in our context, see in particular Section 2.2. Theorem 2.7 is formulated under the strong assumption that there be only one non-transparent resonance (Assumption 2.6). The reason for this assumption is that it simplifies the exposition of our main ideas by allowing for relatively simple notation.

• In Theorem 2.9 (page 24), we allow for several non-transparent resonances, with the same conclusions as in Theorem 2.7. This is the framework that is encountered in many examples, in particular the coupled Klein-Gordon systems described in Sections 1.7 and 5.2, 5.3. The proof (Section 4.1) relies on the same ideas as the proof of Theorem 2.7, the extra difficulty being only notational.

The last three results are variations on Theorem 2.9 and its proof:

• We first remark, in Theorem 2.11, that all non-transparent resonances are amplified. That is, we can observe an instability even though we initially turn on a resonance

that is not associated with the maximal rate of growth. Here our assumptions are weakest, in particular are essentially only local in frequency, but the amplification is accordingly weaker.

• Next we remark that instability occurs in asymptotically vanishing balls, provided that we give up a little on the amplification rate. This is Theorem 2.12.

• Finally, in Theorem 2.13 we prove that if arbitrarily small perturbations of the WKB datum generate exact solution that persist and are bounded uniformly in (ε, t, x) over time intervals $T\sqrt{\varepsilon}|\ln\varepsilon|$, with T large enough, then the amplification goes from $O(\varepsilon^K)$ to $O(\varepsilon^{K'})$, with K arbitrarily large and K' arbitrarily small, in time $O(\sqrt{\varepsilon}|\ln\varepsilon|)$. Of course, if small perturbations do not generate solutions over such asymptotically small time intervals, or if these solutions are unbounded, then this means instability, in another form, for the WKB solution.

1.6. On related instability results

The article [**25**], cited in Section 1.1 as the main inspiration of the current work, contains limited instability results. In Section 10 and 11 of [**25**], Joly, Métivier and Rauch show that under a condition (Assumption 10.3 in [**25**]) that is very similar to our instability condition $\mathbf{\Gamma} > 0$ below, WKB solutions are unstable. They do so for linear equations, and, most importantly, for constant amplitudes, that is, for WKB solutions of the form $\vec{a}\,e^{ik\cdot x/\varepsilon}$, where $\vec{a} \in \mathbb{R}^N$ is fixed. This allows an analysis by Fourier transform. For the solution u, there holds $|u|_{L^2} \geq |\hat{u}|_{L^2(B_\varepsilon)}$, with $B_\varepsilon = \{\xi \in \mathbb{R}^d, |\varepsilon\xi - \xi_0| \leq h\sqrt{\varepsilon}\}$, where ξ_0 is a distinguished resonant frequency and $h > 0$ is small. This reduces the analysis to our Lemma 6.16.

For systems of conservation laws, under the strong assumption of a constant eigenvalue, Cheverry, Guès and Métivier prove an instability result for high-frequency WKB solutions. This assumption is (6.5) and Hypothèse 6.1 in [**8**]; constancy is in u, in the context of [**8**] eigenvalues are 1-homogeneous in ξ. Then Cheverry studied in [**7**] the viscous relaxation of these instabilities.

We note that our approach to instabilities in nonlinear equations differs fundamentally from the approach of Grenier in his classical work [**18**], in which Grenier formulated spectral assumptions *bearing on linear differential operators.* By contrast, our spectral assumptions are formulated for *symbols of linear pseudo-differential operators.* In particular, our spectral assumptions are, at least theoretically, readily verifiable, since they bear on spectra of matrices. The key Lemma that allows us to transpose the spectral information at the symbolic level into estimates for corresponding systems of partial differential equations is the Duhamel representation Theorem 6.14, drawn from [**46**].

The article [**32**] also uses the Duhamel representation of [**46**], to prove a strong Lax-Mizohata result for weakly non-hyperbolic quasi-linear systems.

1.7. Examples

Our first class of examples (Section 5.1) are three-wave interaction systems, of the form

$$
(1.25) \qquad
\begin{cases}
\partial_t u_1 + c_1 \partial_x u_1 = \dfrac{b_1}{\sqrt{\varepsilon}} \bar{u}_2 u_3, \\[2mm]
\partial_t u_2 + c_2 \partial_x u_2 = \dfrac{b_2}{\sqrt{\varepsilon}} \bar{u}_1 u_3, \\[2mm]
\partial_t u_3 + c_3 \partial_x u_3 = \dfrac{b_3}{\sqrt{\varepsilon}} u_1 u_2,
\end{cases}
$$

and

$$
(1.26) \qquad
\begin{cases}
\partial_t u_1 + \dfrac{c_1}{\varepsilon} \partial_x u_1 = \dfrac{b_1}{\varepsilon} \bar{u}_2 u_3, \\[2mm]
\partial_t u_2 + \dfrac{c_2}{\varepsilon} \partial_x u_2 = \dfrac{b_2}{\varepsilon} \bar{u}_1 u_3, \\[2mm]
\partial_t u_3 + c_3 \partial_x u_3 = b_3 u_1 u_2.
\end{cases}
$$

We show in Section 5.1.2 how these systems are derived in the high-frequency limit from the Euler-Maxwell equations describing laser-plasma interactions. Systems (1.25) and (1.26) can be used to describe Raman and Brillouin scattering, respectively.

In the case $b_2 b_3 > 0$, for any c_1, c_2, c_3, our instability results apply to the reference solutions

$$
\big(a(x - c_1 t), 0, 0 \big) \quad \text{for (1.25),} \quad \text{and} \quad \big(a(\varepsilon x - c_1 t), 0, 0 \big) \quad \text{for (1.26),}
$$

and give a description of the growth of the Raman and Brillouin waves u_3.

Our second class of examples (Sections 5.2 and 5.3) comprises coupled Klein-Gordon systems of the form

$$
(1.27) \qquad
\begin{cases}
\partial_t u + \begin{pmatrix} 0 & \partial_x & 0 \\ \partial_x \cdot & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u + \dfrac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_0 \omega_0 \\ 0 & -\alpha_0 \omega_0 & 0 \end{pmatrix} u = \dfrac{1}{\sqrt{\varepsilon}} B^1(u, v), \\[4mm]
\partial_t v + \begin{pmatrix} 0 & \theta_0 \partial_x & 0 \\ \theta_0 \partial_x \cdot & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \dfrac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & -\omega_0 & 0 \end{pmatrix} v = \dfrac{1}{\sqrt{\varepsilon}} B^2(u, v),
\end{cases}
$$

where $(u, v) = (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^{d+2} \times \mathbb{R}^{d+2}$, $x \in \mathbb{R}^d$. The parameters satisfy $0 < \omega_0$, $0 < \alpha_0$, and $0 < \theta_0 < 1$. The eigenvalues (as in (1.12)) are

$$
\left\{ 0, \pm\sqrt{\alpha_0^2 \omega_0^2 + |\xi|^2}, \pm\sqrt{\omega_0^2 + \theta_0^2 |\xi|^2} \right\}.
$$

The characteristic varieties for $\alpha_0 = 1$ and $\alpha_0 \neq 1$, depicting the branches of eigenvalues as functions of ξ, are pictured on Figure 4 page 89 and Figure 5 page 95, respectively.

If $\alpha_0 = 1$, the masses (corresponding to threshold frequency ω_0) are equal. In the context of laser-plasma interactions, the masses are both equal to the plasma frequency, and systems (1.27) are simplified Euler-Maxwell systems. This case is covered in Section 5.2. The case of different masses is covered in Section 5.3.

In both cases, we give examples of bilinear terms B^1 and B^2 to which our results, stability or instability of WKB solutions, apply.

1.8. Open problems

We conclude this introduction with a list of open problems, listed in what we perceive as an increasing level of difficulty:

Allow for rank-two interaction coefficients. It would be interesting, especially in view of the extension of our results to the Euler-Maxwell equations (see Section 5.1.2), to handle rank-two interaction coefficients. This would mean extending the bounds of Appendix 6.3 to symbolic flows defined by interaction matrices of the form

$$M = \begin{pmatrix} i\mu_1 & 0 & -\sqrt{\varepsilon}b_{ijx}^+ \\ 0 & i\mu_1 & -\sqrt{\varepsilon}b_{ijy}^- \\ -\sqrt{\varepsilon}b_{jix}^- & -\sqrt{\varepsilon}b_{jiy}^- & i\mu_2 \end{pmatrix}.$$

Weaken the partial transparency condition (2.18). In our first class of Klein-Gordon examples (Section 5.2), condition (2.18) is satisfied only in one space dimension. We note that condition (2.18) is used only in the proof of Proposition 4.2 (normal form reduction) which decouples the components of the solution associated with non-transparent resonances. Given a specific set of non-transparent resonances, we could probably make appropriate coordinatization choices so as to forgo, or at least weaken, condition (2.18).

Consider larger initial perturbations. We take into account the presence of high frequencies $\sim 1/\varepsilon$ by measuring L^2 norms of ε-derivatives. The main drawback of this approach is a very poor control of sup norms, as already seen in (1.24). By using (1.24), we are essentially uniformly bounding $a(x)\sin(x/\varepsilon)$ by $C\varepsilon^{-d/2}$, even if a is smooth and compactly supported.

This raises the question: *Does there exist a Banach algebra of distributions in which high-frequency families $\{\varphi(x, x/\varepsilon)\}_{0<\varepsilon<1}$, with $\varphi \in C_c^\infty$, are uniformly bounded, and in which good pseudo-differential bounds are available?*

The minimal requirements for pseudo-differential bounds would be inclusion of the space of pseudo-differential operators of order zero into the space of linear bounded operators from the Banach algebra to itself, and stability under composition.

The space $\mathcal{F}L^1$ of distributions with L^1 Fourier transform satisfies the first two conditions (algebra, uniform bounds for oscillating families), but not the third (pseudo-differential bounds). The Sobolev space H^s equipped with the semi-classical norm $\| \cdot \|_{\varepsilon,s}$ satisfies the last two conditions, but is not a Banach algebra.

In the absence of a positive answer to the above question, we perform in Section 3.1.5 estimates in both $\mathcal{F}L^1$ and H^s, so as to combine the advantages of both functional settings. This gives an existence time T_0 (2.12), that approaches the "optimal" existence time $T_\infty = K/(\gamma|a|_{L^\infty})$, using notation introduced in Section 2. The optimal character of T_∞ is seen on Theorem 2.13: this existence time allows for the amplification exponent K' to be arbitrarily small, hence for the instability to be almost Lyapunov.

Allow for more singular scalings. Laser pulses typically propagate in one spatial dimension x and have large transverse variations in transverse directions y : they have the form

$$a(x, y)\sin((kx - \omega t)/\varepsilon)\sin(y/\sqrt{\varepsilon}),$$

where a is a slowly varying amplitude. A corresponding scaling would be, instead of (1.1), the more singular

$$(1.28) \qquad \partial_t u + \left(\frac{1}{\varepsilon}A_0 + \frac{1}{\sqrt{\varepsilon}}A(\partial_y) + A\partial_x\right)u = \frac{1}{\sqrt{\varepsilon}}B(u, u),$$

with data oscillating in x at frequencies $\sim 1/\varepsilon$. In this scaling, the Zakharov equations with non-zero group velocity were formally derived from the Euler-Maxwell equations in [**48**]. A stability analysis of WKB solutions to (1.28) would lead to consideration of symbolic flows of interaction matrices as in (1.19). The important difference would an $\sqrt{\varepsilon}$-semiclassical scaling of the relevant pseudo-differential operators, with the catastrophic consequence

$$\frac{1}{\sqrt{\varepsilon}}\Big(\mathrm{op}_{\sqrt{\varepsilon}}(MS_0) - \mathrm{op}_{\sqrt{\varepsilon}}(M)\mathrm{op}_{\sqrt{\varepsilon}}(S_0)\Big) = \mathrm{op}_{\sqrt{\varepsilon}}(i\partial_\xi M\partial_y S_0) + \sqrt{\varepsilon}(\dots),$$

meaning an error $O(1)$, instead of $O(\sqrt{\varepsilon})$, in the first step of the construction of a solution operator. Then, $\mathrm{op}_{\sqrt{\varepsilon}}(S_0)$ would not appear as an approximation of the solution operator. What would constitute a good approximation of the solution operator, then ?

CHAPTER 2

ASSUMPTIONS AND RESULTS

For the family of systems (1.1), reproduced here:

$$\partial_t u + \frac{1}{\varepsilon} A_0 u + \sum_{1 \leq j \leq d} A_j \partial_{x_j} u = \frac{1}{\sqrt{\varepsilon}} B(u, u),$$

we make the following assumptions.

ASSUMPTION 2.1 (Smooth spectral decomposition). — *We assume that the matrices A_j, for $1 \leq j \leq d$, are real symmetric, that the matrix A_0 is real skew-symmetric, and that the family of hermitian matrices $\left\{ A_0/i + \sum_{1 \leq j \leq d} \xi_j A_j \right\}_{\xi \in \mathbb{R}^d}$ has the spectral decomposition*

$$(2.1) \qquad A_0/i + \sum_{1 \leq j \leq d} \xi_j A_j := \sum_{1 \leq j \leq J} \lambda_j(\xi) \Pi_j(\xi),$$

for some fixed J, where λ_j are smooth eigenvalues and Π_j are smooth eigenprojectors, satisfying bounds, for all $\beta \in \mathbb{N}^d$, for some $C_\beta > 0$:

$$(2.2) \qquad |\partial_\xi^\beta \lambda_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{(1-|\beta|)/2}, \quad |\partial_\xi^\beta \Pi_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{-|\beta|/2}.$$

We do *not* assume that eigenvalues do not cross; indeed, for physical systems, crossing does typically occur at least for $\xi = 0$: examples are given in Sections 5.1, 5.2, 5.3 of Chapter 5 and Appendix 6.7. The smoothness condition in Assumption 2.1 means that at coalescence points, there is an ordering of the eigenvalues so that regularity is preserved. Bounds (2.2) mean that $\lambda_j \in S^1$ and $\Pi_j \in S^0$. The classical classes S^m of pseudo-differential operators of order m are introduced in Appendix 6.1.

Assumption 2.1 is discussed in Appendix 6.5. There we give, in particular, a sufficient condition for bounds (2.2) to hold.

ASSUMPTION 2.2 (WKB approximate solution). — *For some $K_a \in \mathbb{N}$, some $T_a > 0$, all $\varepsilon > 0$, there exists u_a an approximate solution to (1.1), in the sense that there*

holds in $[0, T_a]$:

$$(2.3) \qquad \partial_t u_a + \frac{1}{\varepsilon} A_0 u_a + \sum_{1 \leq j \leq d} A_j \partial_{x_j} u_a = \frac{1}{\sqrt{\varepsilon}} B(u_a, u_a) + \varepsilon^{K_a} r_a^\varepsilon.$$

The approximate solution has the form of a WKB expansion

$$(2.4)$$
$$u_a(\varepsilon, t, x) = e^{-i(k \cdot x - \omega t)/\varepsilon} u_{0,-1}(t, x) + e^{i(k \cdot x - \omega t)/\varepsilon} u_{0,1}(t, x) + \sqrt{\varepsilon} v_a(\varepsilon, t, x) \ \in \mathbb{R}^N,$$

where

    – the phases (ω, k) and $(-\omega, -k) \in \mathbb{R}^{1+d}$ are characteristic *for the hyperbolic operator, in the sense that*

$$(2.5) \qquad \Big(\partial_t + \frac{1}{\varepsilon} A_0 + \sum_{1 \leq j \leq d} A_j \partial_{x_j}\Big)\big(e^{\pm i(k \cdot x - \omega t)/\varepsilon} \vec{e}_{\pm 1}\big) = 0, \qquad \vec{e}_{-1} = (\vec{e}_1)^*,$$

    where $\vec{e}_{-1}$ and $\vec{e}_1$ are fixed, unit vectors in $\mathbb{C}^N$ and $(\vec{e}_1)^*$ denotes component-by-component complex conjugation, and the leading amplitudes $u_{0,\pm 1}$ are polarized along $\vec{e}_{\pm 1}$, in the sense that

$$(2.6) \qquad u_{0,1}(t, x) = g(t, x) \vec{e}_1, \quad u_{0,-1} = g(t, x)^* \vec{e}_{-1}, \qquad g \in C^1([0, T_a], H^{s_a}(\mathbb{R}^d)),$$

    where g^* denotes complex conjugation of the amplitude $g \in \mathbb{C}$,
    – there holds $v_a, r_a^\varepsilon \in C^0([0, T_a], H^{s_a}(\mathbb{R}^d))$, with

$$(2.7) \qquad \sup_{\varepsilon > 0} \Big(\sup_{|\alpha| \leq s_a} |(\varepsilon \partial_x)^\alpha (v_a, r_a^\varepsilon)|_{L^\infty([0,T], L^2)} + |\mathcal{F}(v_a, r_a^\varepsilon)|_{L^\infty(0, T_a], L^1)} \Big) < \infty.$$

We give in Appendix 6.6 sufficient conditions for Assumption 2.2 to hold true. An example of non-oscillating data, corresponding to $(\omega, k) = (0, 0)$, is described in Section 5.1.

We note that it suffices for v_a and r_a^ε to be trigonometric polynomials in θ, as is typically the case in WKB expansions, in order for the above uniform bound on $|\mathcal{F}(v_a, r_a^\varepsilon)|_{L^1}$ to hold.

NOTATION 2.3. — *Given $\vec{u} \in \mathbb{R}^N$, we denote $B(\vec{u}) : \mathbb{R}^N \to \mathbb{R}^N$ the map defined by*

$$B(\vec{u})v := B(\vec{u}, v) + B(v, \vec{u}).$$

DEFINITION 2.4 (Resonances, interaction coefficients). — *Given $(i, j) \in \{1, \ldots, J\}^2$, with J as in Assumption 2.1, we define the set of resonant frequencies associated with (i, j) by*

$$\mathcal{R}_{ij} := \big\{ \xi \in \mathbb{R}^d, \quad \omega = \lambda_i(\xi + k) - \lambda_j(\xi) \big\}.$$

The families of matrices

$$\Pi_i(\xi + k) B(\vec{e}_1) \Pi_j(\xi) \in \mathbb{C}^{N \times N} \quad \text{and} \quad \Pi_j(\xi) B(\vec{e}_{-1}) \Pi_i(\xi + k) \in \mathbb{C}^{N \times N},$$

indexed by $\xi \in \mathbb{R}^d$, are called the interaction coefficients *associated with (i, j).*

    The scalar function $\xi \to \lambda_i(\xi + k) - \lambda_j(\xi) - \omega$ is called the resonant phase *associated with (i, j).*

We often say *the (i,j) resonance*, and *the (i,j) interaction coefficients*.

We note that auto-resonances, i.e. resonances associated with (i,i), for $1 \le i \le J$, are taken into account in this definition. We also note that $\mathcal{R}_{ij} \ne \mathcal{R}_{ji}$ in general.

We introduce the property of *transparency*:

DEFINITION 2.5 (Transparency). — An interaction coefficient $\Pi_i(\cdot + k)B(\vec{e}_1)\Pi_j$ is said to be *transparent* if the associated resonant phase can be factored out, that is if for some $C > 0$, there holds for all $\xi \in \mathbb{R}^d$ the bound

$$(2.8) \qquad |\Pi_i(\xi + k)B(\vec{e}_1)\Pi_j(\xi)| \le C|\lambda_i(\xi + k) - \omega - \lambda_j(\xi)|.$$

Similarly, an interaction coefficient $\Pi_j B(\vec{e}_{-1})\Pi_i(\cdot + k)$ is said to be transparent if for some $C > 0$, there holds for all ξ

$$(2.9) \qquad |\Pi_j(\xi)B(\vec{e}_{-1})\Pi_i(\xi + k)| \le C|\lambda_i(\xi + k) - \omega - \lambda_j(\xi)|.$$

If both interaction coefficients associated with resonance (i,j) are transparent, then the resonance is said to be transparent.

We can now state our main, and provisory, assumption:

ASSUMPTION 2.6. — *We suppose*
    (i) *(boundedness) the set $\mathcal{R}_{12}$ is nonempty and bounded,*
    (ii) *(partial transparency) for all $(i,j) \ne (1,2)$, the (i,j) resonance is transparent;*
    (iii) *(rank-one coefficients) for all ξ in an open set containing $\mathcal{R}_{12}$, the ranks of the $(1,2)$ interaction coefficients are at most 1.*

Assumption 2.6(i) is discussed in Appendix 6.7; there we show in particular that Assumption 2.6(i) satisfied as soon as the eigendecomposition (2.1) is smooth at infinity, and λ_1 and λ_2 non asymptotic at infinity, that is $|\lambda_j(\xi)| = c_j|\xi| + o(|\xi|)$ with $c_1 \ne c_2$.

Assumption 2.6(ii) is here only to simplify the exposition; we will see in Section 2.3 that our results fully extend to the case of several non-transparent resonances, under a mild partial transparency condition.

Assumption 2.6(iii) is satisfied as soon as (but not only if) λ_1 and λ_2 are simple eigenvalues.

2.1. Main result

We denote Γ the trace of the product of the $(1,2)$ interaction coefficients:

$$(2.10) \qquad \Gamma(\xi) := \operatorname{tr} \Pi_1(\xi + k)B(\vec{e}_1)\Pi_2(\xi)B(\vec{e}_{-1})\Pi_1(\xi + k).$$

In the context of Assumption 2.6, the stability index is

$$(2.11) \qquad \mathbf{\Gamma} := \max\left(\max_{\xi \in \mathcal{R}_{12}} \Re e\, \Gamma(\xi),\ \max_{\xi \in \mathcal{R}_{12}} |\Im m\, \Gamma(\xi)| \right),$$

so that

– if $\boldsymbol{\Gamma} > 0$, then for some $\xi \in \mathcal{R}_{12}$, there holds $\Gamma(\xi) \notin (-\infty, 0]$.

– If $\boldsymbol{\Gamma} < 0$, then for all $\xi \in \mathcal{R}_{12}$, $\Gamma(\xi) \in (-\infty, 0)$.

In the unstable case, corresponding to $\boldsymbol{\Gamma} > 0$, the limiting observation time T_0 and amplification exponent K_0 are

$$(2.12) \quad T_0 := \max\left(\frac{K}{|B|_0 |\hat{a}|_{L^1}}, \frac{K - d/2}{\gamma |a|_{L^\infty}}\right), \qquad K_0 := \min\left(K\left(1 - \frac{\gamma |a|_{L^\infty}}{|B|_0 |\hat{a}|_{L^1}}\right), d/2\right),$$

where

$$(2.13) \qquad |B|_0 := \max_{\xi \in \mathcal{R}_{12}} \max\left(|\Pi_1(\xi + k)B(\vec{e}_1)\Pi_2(\xi)|, |\Pi_2(\xi)B(\vec{e}_{-1})\Pi_1(\xi + k)|\right),$$

and

$$(2.14) \qquad \gamma := \left| \max_{\xi \in \mathcal{R}_{12}} \Re e\left(\Gamma(\xi)^{1/2}\right) \right|.$$

In the definition of $|B|_0$ above, we denote $|Z|$ the norm of a matrix $Z \in \mathbb{C}^{N \times N}$, deriving from the L^∞ norm in $\mathbb{C}^N$: $|Z| := \max_{|u|=1} |Zu|$, where $|u| = |u_1, \ldots, u_N| := \max_i |u_i|$. These norms in finite dimensions are used throughout the paper.

THEOREM 2.7. — *Under Assumptions* 2.1 *(regularity of the spectral decomposition),* 2.2 *(existence of WKB solutions) and* 2.6 *(resonances and transparency), the sign of index* $\boldsymbol{\Gamma}$ *determines stability of the WKB approximate solution* u_a *with respect to initial perturbations, as follows:*

- *If* $\boldsymbol{\Gamma} > 0$, *then for some initial perturbation* $\phi(\varepsilon, \cdot) \in C_c^\infty$ *such that*

$$\sup_{0 < \varepsilon < 1} \left(\| \phi(\varepsilon, \cdot)\|_{\varepsilon, s} + |\phi(\varepsilon, \cdot)|_{L^\infty}\right) < \infty$$

for all s, *the solution* u *to* (1.1) *issued from the initial datum*

$$(2.15) \qquad u(0, x) = u_a(0, x) + \varepsilon^K \phi(\varepsilon, x)$$

satisfies:

for any $K > 0$, *if* $K_a + 1/2 \geq K$, *for any* $K' > K_0$, *for some* $T < T_0$, *some* $\varepsilon_0 > 0$, *for all* $\varepsilon \in (0, \varepsilon_0)$, $u \in C^0([0, T\sqrt{\varepsilon}|\ln \varepsilon|], H^s(\mathbb{R}^d))$ *for* $d/2 < s \leq s_a$, *and*

$$(2.16) \qquad \sup_{0 < \varepsilon < \varepsilon_0} \sup_{0 \leq t \leq T\sqrt{\varepsilon}|\ln \varepsilon|} \varepsilon^{-K'} |(u - u_a)(t)|_{L^2(B(x_0, \rho))} = \infty,$$

for some $x_0 \in \mathbb{R}^d$, *some* $\rho > 0$.

- *If* $\boldsymbol{\Gamma} < 0$, *then for any* $\phi(\varepsilon, \cdot) \in H^s$ *such that* $\sup_{0 < \varepsilon < 1} \|\phi(\varepsilon, \cdot)\|_{\varepsilon, s} < \infty$ *for all* $s \leq s_a$, *the solution* u *to* (1.1) *issued from the initial datum* (2.15) *satisfies:*

for any $K \geq (d+1)/2$, *if* $K_a \geq (d+1)/2$, *for some* $\varepsilon_0 > 0$, *some* $C(T_a) > 0$, *for all* $\varepsilon \in (0, \varepsilon_0)$, $u \in C^0([0, T_a], H^s(\mathbb{R}^d))$, *for* $d/2 < s \leq s_a$, *and*

$$(2.17) \qquad \sup_{t \in [0, T_a]} \|(u - u_a)(t)\|_{\varepsilon, s} \leq C(T_a)\|(u - u_a)(0)\|_{\varepsilon, s}.$$

In the above Theorem, we use the semiclassical Sobolev norm $\|\cdot\|_{\varepsilon,s}$ defined in (1.21) on page 10. The time T_a is any existence time for the WKB solution u_a, such that the bounds stated in Assumption 2.2 hold, the index s_a is the Sobolev index of regularity of u_a, and T_0 is the limiting observation time defined in (2.12).

Theorem 2.7 is proved in Sections 3.1 and 3.2.

2.2. Comments

On our assumptions:

• Assumption 2.1 asserts regularity of the spectral decomposition, and bounds at infinity. We show in Appendix 6.5 that these conditions at infinity follow from smoothness of the spectral decomposition of an associated "short-wave" operator.

• In Assumption 2.2, the polarization condition (2.6) and the bound (2.7) on the correctors and remainder are standard. What is not clear, however, is that system (1.1) admits WKB solutions at all. Indeed, as noted by Joly, Métivier and Rauch [**25**], and briefly discussed on page 4 above, in the context of supercritical geometric optics, for WKB solutions to exist the large source term must satisfy compatibility conditions, which are similar to, and weaker than, transparency in the sense of Definition 2.5. In the context of (1.1), these conditions are given in Appendix 6.6. Also, in Proposition 6.22, we give sufficient conditions for Assumption 2.8 to imply Assumption 2.2.

• Point (i) of Assumption 2.6 is typically easy to check, and discussed in Appendix 6.7. Point (ii) is too strong (indeed in examples there is typically more than one non-transparent resonance); this is remedied in Assumption 2.8. In theory, the verification of a transparency condition is a simple matter; see our computations in Section 5.2 in the case of coupled Klein-Gordon equations. In practice, the computations are sometimes involved, see in particular [**49**] in the case of Euler-Maxwell. Point (iii) simplifies the linear algebra in Appendix 6.3. As mentioned in Section 1.8, it would be interesting to handle two-dimensional eigenspaces, especially in view of the extension of our results to the Euler-Maxwell system in three space dimensions, for which the longitudinal modes are two-dimensional.

On the nature of the instability and parameters T_0 and K_0 :

• The smaller the amplification exponent K_0 defined in (2.12), the stronger the amplification. There holds $K_0 = K - T_0\gamma|a|_{L^\infty}$: that is, the limitation on the amplification exponent is the existence time T_0. An existence proof up to time $K/(\gamma|a|_{L^\infty})$ would allow for $K_0 = 0$, corresponding to an $O(1)$ deviation; this observation is exploited in Theorem 2.13 below.

• For the existence part in Theorem 2.7, two approaches are combined in the proof. This explains why T_0 in (2.12) is the maximum of two quantities.

We first use the constant-coefficient nature of the hyperbolic operator in (1.1), and the semilinear nature of the source term in the right-hand side of (1.1). Such equations are amenable to estimates in $\mathcal{F}L^1$, the Banach algebra of functions with Fourier transforms in L^1. The key is that the $\mathcal{F}L^1$ norm controls the L^∞ norm. Combined a priori estimates in $\mathcal{F}L^1$ and H^s yield the existence time $K/(|B|_0|\hat{a}|_{L^1})$, where $|B|_0$ is defined in (2.13). Details are given in Section 3.1.5.1. Here we note that $\gamma \le |B|_0$. Indeed, if the trace $\Gamma(\xi)$ is positive, then $\Gamma(\xi)$ is the only non-zero eigenvalue of the product of the interaction coefficients. In particular, the modulus $|\Gamma(\xi)|$ is bounded from above by the norm of the product of the interaction coefficients: $|\Gamma(\xi)| \le |B|_0^2$, which implies $\gamma \le |B|_0$. We could hope for the better existence time $K/(\gamma|\hat{a}|_{L^1})$. The issue here is that we are unable to use the precise Duhamel representation of Section 3.1.3. Indeed, this representation introduces pseudo-differential operators, the action of which cannot be easily estimated in $\mathcal{F}L^1$. Remark 3.15 expands on this point.

In a second approach, we perform Sobolev estimates. A control of the L^∞ norm is then given by Sobolev embedding. An issue here is that in semiclassical norms this embedding has a large $\sim \varepsilon^{-d/2}$ norm. Hence, via the Duhamel representation of Section 3.1.3, in which we use Theorem 6.14, the existence time $(K - d/2)/(\gamma|a|_{L^\infty})$.

• In accordance with the above two comments, the amplification exponent K_0 is the minimum of two positive quantities. The first is equal to a fraction of K, which goes to 0 as $|a|_{L^\infty}/|\hat{a}|_{L^1} \to 1$ and $\gamma \to |B|_0$.

In general, however, and in particular as K becomes larger, for a given system and a given initial amplitude, the amplification factor will be equal to $d/2$. That is, even though the deviation from u_a is small ($\sim \varepsilon^{d/2}$), the amplification in Theorem 2.7, on top of being localized in space and asymptotically instantaneous in time, is *absolute*, meaning that the perturbation grows from $O(\varepsilon^K)$ to $O(\varepsilon^{K'})$, with $K/K' \to \infty$ as $K \to \infty$. In other words, the flow at $t = 0$ is not Hölder continuous: the ratio $|u - u_a|_{L^2(B(x_0,\rho))}/|(u - u_a)(0)|_{\varepsilon,s}^\alpha$ is unbounded in the limit $\varepsilon \to 0$, with a Hölder exponent $\alpha = K'/K$ which tends to 0 as $K \to \infty$. Still, the instability is weaker than a Lyapunov instability, which would correspond to $K_0 = 0$.

On the initial perturbation in the unstable case:

In the unstable case, we pick the initial perturbation that will lead to a maximal amplification. In the more general context of Theorem 2.13, these initial perturbations are precisely described in (2.27) below. Essentially, $u(0) - u_a(0)$ has an ε^K prefactor, oscillates at frequency $(\xi_0 + k)/\varepsilon$, where ξ_0 is a distinguished resonant frequency, is spatially localized around a point at which $|a|$ is maximum, and is pointing in an eigendirection of the product of the interaction coefficients.

On the stability result and the class of initial perturbations:

A relative weakness of our stability result is that initial perturbations are small $\sim \varepsilon^K$, with $K \geq (1 + d)/2$, where $1/2$ accounts for the large prefactor in $(1/\sqrt{\varepsilon})B(u, u)$, while $d/2$ accounts for the Sobolev embedding (1.24); this point was briefly discussed in Section 1.8.

A strong point, however, is that stability is meant here with respect to a large class of initial perturbations, as mentioned in Section 1.4 above. We allow indeed initial perturbations of the form $\varphi(x/\varepsilon)$, where φ is smooth, as opposed to perturbations in the form of *profiles,* that is, with a dependence in x/ε that is identical to the one in $u_a(0)$. In particular, the perturbations that we use in the unstable case are allowed in the stable case.

We finally note that, by Sobolev embedding, the stability estimate (2.17) implies the pointwise estimate $|(u - u_a)(t, x)| \leq C(T_a)\varepsilon^{K-d/2}$.

2.3. Extensions

We give here four results that complement Theorem 2.7.

2.3.1. Several non-transparent resonances.
— We announced that the role of Assumption 2.6 was only to simplify the exposition. Here is a more general, and more satisfactory, version of Assumption 2.6, in which we do not assume that the set of non-transparent resonances is reduced to a singleton:

ASSUMPTION 2.8. — *We suppose*

(i) *(boundedness) The resonant set* $\mathfrak{R} = \bigcup\limits_{i,j} \mathcal{R}_{ij}$ *is bounded.*

(ii) *(partial transparency) For some subset* $\mathfrak{R}_0 \subset \mathfrak{R}$: *given* $(i, j) \in \mathfrak{R} \setminus \mathfrak{R}_0$, *the* (i, j) *resonance is transparent; given* $(i, j) \in \mathfrak{R}_0$, *the* (i, j) *interaction coefficients are transparent on a neighborhood of*

(2.18)
$$\mathcal{R}_{ij} \bigcap \Big(\big(\mathcal{R}_{i'i} - k \big) \bigcup \big(\mathcal{R}_{jj'} + k \big) \Big), \quad \text{for all } i', j' \text{ with } (i', i) \in \mathfrak{R}_0 \text{ and } (j, j') \in \mathfrak{R}_0,$$

and on a neighborhood of

(2.19)
$$\mathcal{R}_{ij} \bigcap \Big(\mathcal{R}_{ii'} \bigcup \mathcal{R}_{j'j} \Big), \quad \text{for all } i' \neq j, \text{ all } j' \neq i, \text{ with } (i, i') \in \mathfrak{R}_0 \text{ and } (j', j) \in \mathfrak{R}_0.$$

Besides, $\mathfrak{R}_0$ *does not contain auto-resonances: for all* $1 \leq i \leq J$, $(i, i) \notin \mathfrak{R}_0$.

(iii) *(rank-one coefficients) For all* $(i, j) \in \mathfrak{R}_0$, *for all* ξ *in an open set containing* $\mathcal{R}_{ij}$, *the ranks of the* (i, j) *interaction coefficients are at most 1; except for* $(i, j) \in \mathfrak{R}_0$ *such that one interaction coefficient is identically equal to zero, in which case we make no assumption on the rank of the other coefficient.*

In condition (ii), transparency of an interaction coefficient over a frequency set means factorization of the phase, as in (2.8) or (2.9), for ξ restricted to the frequency set in question.

We note the inclusions

$$\mathcal{R}_{ij} \bigcap \mathcal{R}_{ii'} \subset \{\lambda_j = \lambda_{i'}\}, \qquad \mathcal{R}_{ij} \bigcap \mathcal{R}_{j'j} \subset \{\lambda_i(\cdot + k) = \lambda_{j'}(\cdot + k)\}.$$

Thus condition (ii) in Assumption 2.8 means that, while we allow for an arbitrarily large subset $\mathfrak{R}_0$ of non-transparent resonances, we cannot allow for some (i, j) interaction coefficient to be non-transparent in all of the resonant set $\mathcal{R}_{ij}$: we need to assume transparency at these exceptional frequencies in $\mathcal{R}_{ij}$ which correspond to a translate of another non-transparent resonance involving λ_i or λ_j (condition (2.18)) or to a coalescing point in the spectrum involving λ_i or λ_j and another branch associated with a non-transparent resonance (condition (2.19)).

Given a resonance $(i, j) \in \mathfrak{R}_0$, we let

$$\Gamma_{ij}(\xi) := \operatorname{tr} \Pi_i(\xi + k) B(\vec{e}_1) \Pi_j(\xi) B(\vec{e}_{-1}) \Pi_i(\xi + k).$$

The stability index is

(2.20) $$\mathbf{\Gamma} := \max_{(i,j)\in\mathfrak{R}_0} \max \left(\max_{\xi\in\mathcal{R}_{ij}} \Re e\, \Gamma_{ij}(\xi), \ \max_{\xi\in\mathcal{R}_{ij}} |\Im m\, \Gamma_{ij}(\xi)| \right).$$

The observation time T_0 and amplification rate K_0 are defined as in (2.12), with $|B|_0$ defined by

(2.21) $$|B|_0 := \max_{(i,j)\in\mathfrak{R}_0} \max_{\xi\in\mathcal{R}_{ij}} \max \left(|\Pi_i(\xi+k) B(\vec{e}_1) \Pi_j(\xi)| \ , \ |\Pi_j(\xi) B(\vec{e}_{-1}) \Pi_i(\xi+k)| \right),$$

and γ defined by

(2.22) $$\gamma := \max_{(i,j)\in\mathfrak{R}_0} \gamma_{ij}, \qquad \gamma_{ij} := \left| \max_{\xi\in\mathcal{R}_{ij}} \Re e\, \left(\Gamma_{ij}(\xi)^{1/2} \right) \right|.$$

THEOREM 2.9. — *The conclusions of Theorem 2.7 still hold when* Assumption 2.6 *is replaced by* Assumption 2.8, *with stability index* $\mathbf{\Gamma}$ *defined in* (2.20).

Theorem 2.9 is proved in Section 4.1.

2.3.2. All non-transparent resonances are amplified. — Next we modify the partial transparency condition of Assumption 2.8 into a separation condition. The result (Theorem 2.11 below) states that any non-transparent resonance that is separated from other resonances gives rise to an amplification.

Recall that the notation $\mathfrak{R}$, introduced in Assumption 2.8, denotes the set of resonant indices.

ASSUMPTION 2.10. — *We suppose*
(i) *(boundedness) The resonant set $\mathfrak{R}$ is bounded.*

(ii) *(separation) For any resonant pair (i, j) that is distinct from $(1, 2)$, there holds*

$$(2.23) \qquad \left(\mathcal{R}_{12} + qk \right) \bigcap \mathcal{R}_{ij} = \varnothing, \quad 0 \le |q| \le 1.$$

(iii) *(rank-one coefficients) For all ξ in an open set containing $\mathcal{R}_{12}$, the ranks of the $(1, 2)$ interaction coefficients are at most 1.*

The separation condition (2.23) asserts non-intersection of resonant sets, a much stronger property than the partial transparency condition of Assumption 2.8, but only relative to the $(1, 2)$ resonance. With regard to other resonances, we make no assumption besides boundedness of resonant sets and separation from $\mathcal{R}_{12}$ as prescribed by (2.23).

In the context of Assumption 2.10, the limiting observation time is

$$T_0' := \min \left(\, \max \left(\frac{K - 1/2}{|B||\hat{a}|_{L^1}}, \, \frac{K - (d+1)/2}{|B||a|_{L^\infty}} \right), \, \frac{1}{2(|B| - \gamma_{12})|a|_{L^\infty}} \right),$$

and the amplification exponent is

$$(2.24) \qquad K_0' := K - T_0' \gamma_{12} |a|_{L^\infty},$$

with γ_{12} defined in (2.22).

THEOREM 2.11. — *Under Assumptions 2.1, 2.2 and 2.10, if $\gamma_{12} > 0$, then the WKB solution u_a is unstable, in the sense of Theorem 2.7, where T_0 and K_0 are replaced with T_0' and K_0' defined above.*

Theorem 2.11 is proved in Section 4.2.

The limiting time T_0' is the minimum of two quantities. The first looks very much like (but is smaller than) the limiting time T_0 in Theorems 2.7 and 2.9. The main difference is that we have $|B|$ here in the denominator in T_0', where we had $|B|_0$, possibly much smaller than $|B|$, in the denominator in T_0 (2.12). The second term in the definition of T_0' is independent of K and, for K large enough, smaller than the first. The point is that the instability here is only *relative* to the initial size of the perturbation: there holds for K large enough (depending on B and a):
$$(2.25)$$

$$\sup_{0 < \varepsilon < \varepsilon_0} \, \sup_{0 \le t \le \tilde{T}_0 \sqrt{\varepsilon} |\ln \varepsilon|} \frac{|u(t) - u_a(t)|_{L^2(B(x_0, \rho))}}{\|u(0) - u_a(0)\|_{\varepsilon, s}} = \infty, \qquad \tilde{T}_0 := \frac{1}{2(|B| - \gamma_{12})|a|_{L^\infty}},$$

and the way that the above diverges to ∞ is quantified by K_0'. In other words, the flow is not Lipschitz continuous from $\| \cdot \|_{\varepsilon, s}$ to $\| \cdot \|_{L^2(B(x_0, \rho))}$ at $t = 0$.

The important point in Theorem 2.11 is that we do not assume maximality of the growth rate γ_{12}. That is, in order to record an Hadamard instability, it is not necessary to initially activate unstable frequencies with the larger rate of growth.

An interesting feature of Assumption 2.10 is that, given boundedness of $\mathfrak{R}$, the separation condition (ii) is local in frequency, and bears only on the eigenvalues of the

hyperbolic operator. The assumption of boundedness of $\mathfrak{R}$ is not local, but typically easily verified by asymptotic expansions of the eigenvalues at infinity, as discussed in Appendix 6.7.

2.3.3. Improved spatial localization. — Next we remark that we can improve on the radius of the instability ball in the statement of Theorem 2.7, at the price of a smaller amplification rate. That is, introducing the observation time

$$T_0'' := \max \left(\frac{K - 1/2}{|B|_0 |\hat{a}|_{L^1}}, \frac{K - (d+1)/2}{\gamma |a|_{L^\infty}} \right),$$

and the amplification exponent

$$(2.26) \qquad\qquad K_0'' := K + \frac{\beta d}{2} - T_0'' \gamma |a|_{L^\infty},$$

we have the following result, with stability index $\mathbf{\Gamma}$ defined in (2.20).

THEOREM 2.12. — *Under Assumptions* 2.1, 2.2 *and* 2.8, *in the unstable case* $\mathbf{\Gamma} > 0$, *the conclusion of Theorem* 2.7 *still holds with* $\rho = \varepsilon^\beta$, *for any* $\beta < 1/d$, *with* $\varepsilon_0 = \varepsilon_0(\beta)$, *if* T_0 *and* K_0 *are replaced with* T_0'' *and* K_0'' *as above.*

Theorem 2.12 is proved in Section 4.3. Note that the observation time T_0'' is strictly smaller than T_0 (2.12), hence the amplification exponent K_0'' is strictly greater than K_0. That is, the deviation estimate is better localized, but we are able to follow the solution only on a shorter time interval, meaning a smaller magnitude of the deviation from u_a.

2.3.4. A greater deviation estimate. — As noted in Section 2.2, the limitation on the amplification exponent K_0 in Theorems 2.7 and 2.9 is the upper bound T_0 on the existence time. In our final result, we *assume* a better existence time, and from there deduce a greater deviation estimate.

In the context of Assumption 2.8, the maximum γ of the coefficients γ_{ij}, for (i, j) ranging over the set $\mathfrak{R}_0$ of non-transparent resonances is attained at (i_0, j_0). We consider the same datum as in Theorems 2.7 and 2.9, that is

$$(2.27) \qquad\qquad u(0, x) := u_a(0, x) + \varepsilon^K e^{ix \cdot (\xi_0 + k)/\varepsilon} \varphi_{i_0 j_0}(x) \vec{e}_{i_0 j_0},$$

where

    – ξ_0 is such that $\gamma = \left| \max_{\xi \in \mathcal{R}_{i_0 j_0}} \Re e \left(\Gamma_{i_0 j_0}(\xi)^{1/2} \right) \right|$ is attained at ξ_0.

    – x_0 is such that $|a|_{L^\infty}$ is attained at x_0.

    – $\varphi_{i_0 j_0} \in C_c^\infty(\mathbb{R}^d)$ is a spatial truncation around x_0 (precisely defined in Section 4.1.3).

    – $\vec{e}_{i_0 j_0}$ generates the range of matrix $\Pi_{i_0}(\xi_0 + k) B(\vec{e}_1) \Pi_{j_0}(\xi_0) B(\vec{e}_{-1}) \Pi_{i_0}(\xi_0 + k)$.

Let $T_\infty := \dfrac{K}{\gamma |a|_{L^\infty}}$.

THEOREM 2.13. — *Under Assumptions 2.1, 2.2 and 2.8, in the unstable case* $\boldsymbol{\Gamma} > 0$, *for any $K > 0$, if $K_a + 1/2 \geq K$, for $d/2 < s \leq s_a$:*

    – *either for some $T < T_\infty$, for any ε small enough, the initial-value problem* (1.1)-(2.27) *does not have a solution $u \in C^0([0, T\sqrt{\varepsilon}|\ln\varepsilon|], H^s(\mathbb{R}^d))$,*

    – *or for some $T < T_\infty$, for any $\varepsilon_0 > 0$, the solution u to* (1.1)-(2.27) *satisfies*

$$\sup_{0 < \varepsilon < \varepsilon_0} \ \sup_{0 \leq t \leq T\sqrt{\varepsilon}|\ln\varepsilon|} |u(t, \cdot)|_{L^\infty} = \infty,$$

    – *or for any $K' > 0$, for some $T < T_\infty$, there holds the deviation estimate*

$$(2.28) \qquad \sup_{0 < \varepsilon < \varepsilon_0} \ \sup_{0 \leq t \leq T\sqrt{\varepsilon}|\ln\varepsilon|} \varepsilon^{-K'} |(u - u_a)(t)|_{L^2(B(x_0, \varepsilon^\beta))} = \infty,$$

for some $x_0 \in \mathbb{R}^d$, some $\beta > 0$, some $\varepsilon_0 > 0$.

The proof (Section 4.4) shows that $\beta \to 0$ as $K' \to 0$: the localization becomes less precise as the amplification becomes larger.

Theorem 2.13 states that, with $(u - u_a)(0)$ given in (2.27), in particular compactly supported and $O(\varepsilon^K)$ in L^∞ and $\|\cdot\|_{\varepsilon,s}$ norms, where K and s are arbitrarily large:

    – either the solution to (1.1)-(2.27) is not defined in time $[0, T\sqrt{\varepsilon}|\ln\varepsilon|]$, meaning a catastrophic collapse of the existence time around u_a (if we consider u_a as a solution, with an existence time $T_a > 0$, independent of ε),

    – or the solution is defined in time $[0, T\sqrt{\varepsilon}|\ln\varepsilon|]$ but is unbounded, meaning in particular that $|u - u_a|_{L^\infty}$ is unbounded, in the limit $\varepsilon \to 0$,

    – or the deviation $|u - u_a|$ goes from $O(\varepsilon^K)$, as measured in $\|\cdot\|_{\varepsilon,s}$ norm, to $O(\varepsilon^{K'})$, as measured in an $L^2(B(x_0, \varepsilon^\beta))$ norm, over a time interval of length $O(\sqrt{\varepsilon}|\ln\varepsilon|)$, with K' arbitrarily small.

In conclusion, under the assumptions of Theorem 2.13, in the case $\boldsymbol{\Gamma} > 0$ the WKB solution can certainly be deemed unstable, although there is some imprecision as to the terms of the instability.

CHAPTER 3

MAIN PROOF

3.1. Proof of Theorem 2.7: instability

Under the assumptions of Theorem 2.7, we suppose $\boldsymbol{\Gamma} > 0$, and proceed to prove instability of u_a.

3.1.1. Overview of the instability proof. — Section 3.1.2.1 contains the first important change of variable, in which resonances appear explicitly as crossing points for the eigenvalues of the propagator. In Section 3.1.2.2, we perform a normal form reduction; this essentially reduces the linear source term $B(u_a)$ to the pair of interaction coefficients associated with resonance $(1, 2)$. Then in Section 3.1.2.3, we localize the analysis around a distinguished point (x_0, ξ_0), with $\xi_0 \in \mathcal{R}_{12}$, in the cotangent space. In Section 3.1.3, we use the Duhamel representation formula of Appendix 6.2 in order to describe semi-explicitly the component of the solution associated with resonance $(1, 2)$. Section 3.1.4 is devoted to the derivation of lower bounds for the action of the solution operator on the datum. In Section 3.1.5, we give existence results and upper bounds in time $O(\sqrt{\varepsilon}|\ln \varepsilon|)$. These are based on the representation formula, and also on $\mathcal{F}L^1$ and Sobolev bounds. The comparison of lower bounds with upper bounds in Section 3.1.6 concludes the proof.

3.1.2. Preparation. — By symmetry of the hyperbolic operator, for $\varepsilon > 0$ the solution u to (1.1) issued from (2.15) is defined over a short time interval $[0, T(\varepsilon)]$, for some $T(\varepsilon) > 0$. It has a high Sobolev regularity: $u \in C^0([0, T(\varepsilon)], H^{s_a}(\mathbb{R}^d))$, where s_a is the Sobolev regularity index introduced in Assumption 2.2. The perturbative unknown $\dot{u}$, defined by

(3.1) $$u =: u_a + \dot{u},$$

solves

(3.2) $$\partial_t \dot{u} + \frac{1}{\varepsilon} A_0 \dot{u} + \sum_{1 \le j \le d} A_j \partial_{x_j} \dot{u} = \frac{1}{\sqrt{\varepsilon}} B(u_a) \dot{u} + \frac{1}{\sqrt{\varepsilon}} B(\dot{u}, \dot{u}) - \varepsilon^{K_a} r_a^\varepsilon,$$

with the datum $\dot{u}(\varepsilon, 0, x) = \varepsilon^K \phi(\varepsilon, x)$. In (3.2), the term r_a^ε is the WKB remainder introduced in Assumption 2.2, and we used Notation 2.3 for $B(\cdot)$.

The goal is to choose ϕ so that $\dot{u}$ grows exponentially in time.

It is understood in this proof that Sobolev indices s are strictly smaller than $s_a - d/2$, where s_a is the Sobolev index of regularity of u_a. We need $s > d/2$ in order to prove short-time existence (since the system is semilinear), but the deviation estimate (2.16) is expressed in a localized L^2 norm.

We frequently use the notation $a \lesssim b$ to indicate that an inequality $a \leq Cb$ holds true, with a constant $C > 0$ depending only on fixed parameters, such as dimensions, in particular not on a, b, nor on ε.

3.1.2.1. Projection and frequency shift. — We decompose $\dot{u}$ according to the eigen-modes of the hyperbolic operator and shift the component associated with Π_1 as we define $U = (U_1, \ldots, U_J) \in \mathbb{R}^{NJ}$ by

$$(3.3) \qquad U_1 := e^{-i\theta} \mathrm{op}_\varepsilon(\Pi_1)\dot{u}, \quad U_j := \mathrm{op}_\varepsilon(\Pi_j)\dot{u}, \ 2 \leq j \leq J,$$

with notation

$$(3.4) \qquad \theta := (k \cdot x - \omega t)/\varepsilon.$$

The projectors $\Pi_j(\xi)$ are eigenprojectors of $A(i\xi) + A_0$ (Assumption 2.1), and $\mathrm{op}_\varepsilon(\Pi_j)$ are the associated Fourier multipliers (6.1). The perturbation unknown $\dot{u}$ can be reconstructed from U via

$$(3.5) \qquad \dot{u} = e^{i\theta} U_1 + \sum_{2 \leq j \leq J} U_j.$$

From (3.2)-(3.3), we find that $U = (U_1, U_2 \,|\, U_3, \ldots, U_J) \in \mathbb{R}^{2N} \times \mathbb{R}^{(J-2)N}$ solves

$$(3.6) \qquad \partial_t U + \frac{1}{\varepsilon} \mathrm{op}_\varepsilon(i\mathcal{A})U = \frac{1}{\sqrt{\varepsilon}} \mathrm{op}_\varepsilon(\mathcal{B})U + F.$$

The symbol of the propagator is the diagonal matrix

$$(3.7) \qquad \mathcal{A} := \mathrm{diag}\left(\lambda_{1,+1} - \omega, \lambda_2 \,|\, \lambda_3, \ldots, \lambda_J\right),$$

with the notation $\lambda_{1,+1}(\xi) := \lambda_1(\xi + k)$, where k is the spatial frequency of the WKB datum. More generally, we will often use the notation

$$(3.8) \qquad \sigma_{+p}(x, \xi) := \sigma(x, \xi + pk).$$

In the symbol $\mathcal{A}$, the frequency shift is caused by the fast spatial oscillation in the definition of U_1: there holds indeed the identity

$$\mathrm{op}_\varepsilon(\sigma)(e^{ip\theta}v) = e^{ip\theta} \mathrm{op}_\varepsilon(\sigma_{+p})v, \qquad \text{for all } \sigma, p, v.$$

The symbol of the singular source term is

$$\mathcal{B} := \left(\begin{array}{c|c} \mathcal{B}_{[1,2]} & \mathcal{B}_{[1,2,J]} \\ \hline \mathcal{B}_{[J,1,2]} & \mathcal{B}_{[J,J]} \end{array} \right),$$

where the top left block is

$$\mathcal{B}_{[1,2]} := \sum_{p=\pm 1} \begin{pmatrix} e^{ip\theta}\Pi_{1,+(p+1)}B_p\Pi_{1,+1} & e^{i(p-1)\theta}\Pi_{1,+p}B_p\Pi_2 \\ e^{i(p+1)\theta}\Pi_{2,+(p+1)}B_p\Pi_{1,+1} & e^{ip\theta}\Pi_{2,+p}B_p\Pi_2 \end{pmatrix} \in \mathbb{R}^{2N\times 2N},$$

using notation $\Pi_{j,+q}(\xi) := \Pi_j(\xi + qk)$, for $q \in \mathbb{Z}$, in accordance with (3.8), and

(3.9) $$B_p := B(u_{0,p}), \qquad p \in \{-1,1\},$$

where $u_{0,\pm 1}$ are the leading amplitudes in the WKB solution, introduced in (2.4). The other blocks in the source are

$$\mathcal{B}_{[1,2,J]} := \sum_{p=\pm 1} \begin{pmatrix} e^{i(p-1)\theta}\Pi_{1,+p}B_p\Pi_3 & \cdots & e^{i(p-1)\theta}\Pi_{1,+p}B_p\Pi_J \\ e^{ip\theta}\Pi_{2,+p}B_p\Pi_3 & \cdots & e^{ip\theta}\Pi_{2,+p}B_p\Pi_J \end{pmatrix} \in \mathbb{R}^{2N\times(J-2)N},$$

$$\mathcal{B}_{[J,1,2]} := \sum_{p=\pm 1} \begin{pmatrix} e^{i(p+1)\theta}\Pi_{3,+(p+1)}B_p\Pi_{1,+1} & e^{ip\theta}\Pi_{3,+p}B_p\Pi_2 \\ \vdots & \vdots \\ e^{i(p+1)\theta}\Pi_{J,+(p+1)}B_p\Pi_{1,+1} & e^{ip\theta}\Pi_{J,+p}B_p\Pi_2 \end{pmatrix} \in \mathbb{R}^{(J-2)N\times 2N},$$

and

$$\mathcal{B}_{[J,J]} := \sum_{p=\pm 1} \left(e^{ip\theta}\Pi_{i,+p}B_p\Pi_j \right)_{3\le i,j,\le J} \in \mathbb{R}^{(J-2)N\times(J-2)N}.$$

In (3.6), the remainder F is the sum of

    – the quadratic term

$$\varepsilon^{-1/2}\left(e^{-i\theta}\mathrm{op}_\varepsilon(\Pi_1)B(\dot u,\dot u), \mathrm{op}_\varepsilon(\Pi_2)B(\dot u,\dot u),\ldots,\mathrm{op}_\varepsilon(\Pi_J)B(\dot u,\dot u)\right),$$

    – the projected WKB remainder

$$\varepsilon^{K_a}\left(e^{-i\theta}\mathrm{op}_\varepsilon(\Pi_1)r_a^\varepsilon, \mathrm{op}_\varepsilon(\Pi_2)r_a^\varepsilon,\ldots,\mathrm{op}_\varepsilon(\Pi_J)r_a^\varepsilon\right),$$

    – the contribution of the higher-order WKB terms v_a^ε :

$$\left(e^{-i\theta}\mathrm{op}_\varepsilon(\Pi_1)B(v_a^\varepsilon)\dot u, \mathrm{op}_\varepsilon(\Pi_2)B(v_a^\varepsilon)\dot u,\ldots,\mathrm{op}_\varepsilon(\Pi_J)B(v_a^\varepsilon)\dot u\right),$$

    – and remainder terms arising from compositions of pseudo-differential operators; these terms have the form

$$\varepsilon^{-1/2}e^{iq_1\theta}\left(\mathrm{op}_\varepsilon(\Pi_{i,+q_2})B_{q_3}\mathrm{op}_\varepsilon(\Pi_{j,+q_4})\dot u - \mathrm{op}_\varepsilon\left(\Pi_{i,+q_2}B_{q_3}\Pi_{j,+q_4}\right)\dot u\right),$$

    where $q_j \in \mathbb{Z}$ and B_{q_3} is defined in (3.9).

From this description of F, we deduce the following bound:

LEMMA 3.1. — *There holds for $s \ge 0$ the bound*

$$\|F\|_{\varepsilon,s} \lesssim (1 + \varepsilon^{-1/2}|\dot u|_{L^\infty})\|\dot u\|_{\varepsilon,s} + \varepsilon^{K_a},$$

and the bound

$$|\hat F|_{L^1} \lesssim (1 + \varepsilon^{-1/2}|\mathcal{F}\dot u|_{L^1})|\mathcal{F}\dot u|_{L^1} + \varepsilon^{K_a}.$$

In Lemma 3.1 we are using the semi-classical Sobolev norms $\|\cdot\|_{\varepsilon,s}$ introduced in (6.2).

Proof. — By (6.3) and (6.14), the quadratic terms satisfy for $s \geq 0$:

$$\varepsilon^{-1/2} \big\| \mathrm{op}_\varepsilon(\Pi_j) B(\dot{u}, \dot{u}) \big\|_{\varepsilon,s} \lesssim \varepsilon^{-1/2} |\dot{u}|_{L^\infty} \|\dot{u}\|_{\varepsilon,s}.$$

By (6.3) and the product law (6.15),

$$\big\| \mathrm{op}_\varepsilon(\Pi_j) B(v_a^\varepsilon) \dot{u} \big\|_{\varepsilon,s} \lesssim |v_a^\varepsilon|_{L^\infty} \|\dot{u}\|_{\varepsilon,s} + |(\varepsilon \partial_x^s) v_a^\varepsilon|_{L^\infty} \|\dot{u}\|_{L^2},$$

and with (2.7), this gives

$$\|\mathrm{op}_\varepsilon(\Pi_j) B(v_a^\varepsilon) \dot{u}\|_{\varepsilon,s} \lesssim \|\dot{u}\|_{\varepsilon,s}.$$

By the commutator estimate (6.6), given $v \in H^s$, if s_a is large enough then

$$\big\| \mathrm{op}_\varepsilon(\Pi_{i,+q_2}) B_{q_3} v - \mathrm{op}_\varepsilon\big(\Pi_{i,+q_2} B_{q_3}\big) v \big\|_{\varepsilon,s} \lesssim \varepsilon |u_{0,\pm 1}|_{H^{s_a}} \|v\|_{\varepsilon,s},$$

We apply this bound to $v = \mathrm{op}_\varepsilon(\Pi_{j,+q_4}) \dot{u}$, satisfying $\|v\|_{\varepsilon,s} \leq \|\dot{u}\|_{\varepsilon,s}$ and find

$$\Big\| e^{iq_1\theta} \varepsilon^{-1/2} \Big(\mathrm{op}_\varepsilon(\Pi_{i,+q_2}) B_{q_3} \mathrm{op}_\varepsilon(\Pi_{j,+q_4}) \dot{u} - \mathrm{op}_\varepsilon\big(\Pi_{i,+q_2} B_{q_3} \Pi_{j,+q_4}\big) \dot{u} \Big) \Big\|_{\varepsilon,s} \lesssim \varepsilon^{1/2} \|u\|_{\varepsilon,s}.$$

The bound in $\mathcal{F}L^1$ is found similarly, using $|B(\dot{u}, \dot{u})|_{\mathcal{F}L^1} \lesssim |\mathcal{F}\dot{u}|_{L^1}^2$, and (6.7). ☐

The point of the change of variable (3.3) is that $\mathcal{R}_{12} = \{\omega = \lambda_{1,+1} - \lambda_2\}$ is now included in the locus of coalescing eigenvalues of symbol $\mathcal{A}$.

We will see in the next paragraph that, under Assumption 2.6, in the source term $\mathcal{B}$ only the top left block $\mathcal{B}_{[1,2]}$ matters.

3.1.2.2. Normal form reduction. — The resonant set $\mathcal{R}_{12}$ introduced in Definition 2.4 is bounded by Assumption 2.6, hence compact by continuity of the eigenvalues. For $h > 0$, to be chosen small enough below [1], we consider the neighborhood $\mathcal{R}_{12}^h$ of $\mathcal{R}_{12}$, corresponding to resonant phases bounded by h:

$$(3.10) \qquad \mathcal{R}_{12}^h := \big\{ \xi \in \mathbb{R}^d, \quad |\lambda_1(\xi + k) - \lambda_2(\xi) - \omega| \leq h \big\},$$

We let χ_0 be a smooth cut-off function in frequency space, such that $0 \leq \chi_0 \leq 1$, with $\chi_0 \equiv 1$ on a neigborhood of $\mathcal{R}_{12}^h$, and $\chi_0 \equiv 0$ further away from the $(1,2)$ resonant locus, for instance on $\mathbb{R}^d \setminus \mathcal{R}_{12}^{2h}$.

The top left block in the symbol $\mathcal{B}$ of the singular source term decomposes as

$$\mathcal{B}_{[1,2]} = \mathcal{B}^r + \mathcal{B}^{nr},$$

with the notation

$$(3.11) \qquad \mathcal{B}^r := \begin{pmatrix} 0 & \Pi_{1,+1} B_1 \Pi_2 \\ \Pi_2 B_{-1} \Pi_{1,+1} & 0 \end{pmatrix}.$$

The following proposition will imply that the operator with symbol

$$\mathcal{D} := \begin{pmatrix} (1-\chi_0)\mathcal{B}^r + \mathcal{B}^{nr} & \mathcal{B}_{[1,2,J]} \\ \mathcal{B}_{[J,1,2]} & \mathcal{B}_{[J,J]} \end{pmatrix},$$

---

1. This will be done in Section 3.1.6, at the end of this proof; see Remark 3.18 on page 51.

can be eliminated from the evolution equation (3.6) in U, up to negligible *uniform remainders*, defined as follows:

DEFINITION 3.2. — By uniform remainder, we mean any family $R_0 = R_0(\varepsilon, t)$ of linear bounded operators $H^s \to H^s$, for $t \in [0, T_a]$, where T_a is an existence time for u_a, with semi-classical Sobolev norms that are bounded in ε and t: for some $C > 0$, for all $v \in H^s$, all $\varepsilon > 0$, all $t \in [0, T_a]$, there holds $\|R_0 v\|_{\varepsilon,s} \le C \|v\|_{\varepsilon,s}$.

PROPOSITION 3.3. — *Under Assumption 2.6, there exists $Q \in S^0$, with $\partial_t Q \in S^0$, such that*

$$(3.12) \qquad \varepsilon[\partial_t, \mathrm{op}_\varepsilon(Q)] + [\mathrm{op}_\varepsilon(i\mathcal{A}), \mathrm{op}_\varepsilon(Q)] = \mathrm{op}_\varepsilon(\mathcal{D}) + \varepsilon R_0,$$

where R_0 is a uniform remainder.

Proof. — The source $\mathcal{D}$ contains oscillations in $e^{i\ell\theta}$, with $|\ell| \le 2$. We denote $\mathcal{D} = \sum_{|\ell|\le 2} e^{i\ell\theta}\mathcal{D}_\ell$. Accordingly, we look for Q in the form

$$(3.13) \qquad Q(\varepsilon, t, x, \xi) = \sum_{|\ell|\le 2} e^{i\ell\theta} Q_\ell(t, x, \xi), \qquad Q_\ell(t, x, \xi) \in \mathbb{R}^{JN \times JN},$$

where the symbols Q_ℓ are tensor products:

$$(3.14) \qquad Q_\ell(t, x, \xi) = \kappa_\ell(t, x)\tilde{Q}_\ell(\xi), \quad \kappa_\ell \text{ scalar}, \ \kappa_\ell \in C^1([0, T_a], H^{s_a}), \ \tilde{Q}_\ell \in S^0.$$

Such symbols Q satisfy $Q \in S^0$, $\partial_t Q \in S^0$, and, by (6.3) and (6.15), the associated operators $\mathrm{op}_\varepsilon(Q)$ are uniform remainders.

In the coordinatization (3.6), the variable U belongs to $\mathbb{C}^{NJ}$, so that we are looking for the Fourier coefficients Q_ℓ of Q in the form of $\mathbb{C}^{NJ \times NJ}$ matrices, which depend on (ε, t, x, ξ). We will denote $Z_{(i,j)} \in \mathbb{C}^{N \times N}$ the (i, j) block of a matrix $Z \in \mathbb{C}^{NJ \times NJ}$. In particular, we will use notation $(Q_\ell)_{(i,j)} \in \mathbb{C}^{N \times N}$ to denote block (i, j) of Q_ℓ.

With Q in the form (3.13), there holds

$$\varepsilon[\partial_t, \mathrm{op}_\varepsilon(Q)] = \sum_{|\ell|\le 2} e^{i\ell\theta}\Big(-i\ell\omega Q_\ell + \varepsilon\,\mathrm{op}_\varepsilon(\partial_t Q_\ell)\Big).$$

The symbol $\mathcal{A}$ being diagonal, there holds

$$[\mathrm{op}_\varepsilon(\mathcal{A}), \mathrm{op}_\varepsilon(Q)]_{(i,j)} = \mathrm{op}_\varepsilon(\mu_i)\mathrm{op}_\varepsilon(Q_{(i,j)}) - \mathrm{op}_\varepsilon(Q_{(i,j)})\mathrm{op}_\varepsilon(\mu_j)$$

$$= \sum_{|\ell|\le 2} e^{i\ell\theta}\Big(\mathrm{op}_\varepsilon(\mu_{i,+\ell})\mathrm{op}_\varepsilon((Q_\ell)_{i,j}) - \mathrm{op}_\varepsilon((Q_\ell)_{i,j})\mathrm{op}_\varepsilon(\mu_j)\Big),$$

where the μ_j are the diagonal entries of $\mathcal{A}$, so that

$$(3.15) \qquad \mu_1 := \lambda_{1,+1} - \omega, \qquad \mu_j := \lambda_j, \quad \text{for } j \ge 2.$$

By (3.14),

$$\mathrm{op}_\varepsilon(\mu_{i,+\ell})\mathrm{op}_\varepsilon((Q_\ell)_{i,j}) = \mathrm{op}_\varepsilon(\mu_{i,+\ell})\big(\kappa_\ell \mathrm{op}_\varepsilon\big((\tilde{Q}_\ell)_{(i,j)}\big)\big),$$

and with the commutator estimate (6.6) and the assumed regularity of κ_ℓ,

$$\mathrm{op}_\varepsilon(\mu_{i,+\ell})\big(\kappa_\ell \mathrm{op}_\varepsilon(\tilde{Q}_\ell))_{(i,j)}\big) = \mathrm{op}_\varepsilon(\mu_{i,+\ell}(Q_\ell)_{i,j}) + \varepsilon R_0,$$

where R_0 is a uniform remainder. This implies

$$[\mathrm{op}_\varepsilon(\mathcal{A}), \mathrm{op}_\varepsilon(Q)]_{(i,j)} = \sum_{|q| \le 2} e^{iq\theta} \mathrm{op}_\varepsilon\Big((\mu_{i,+q} - \mu_j)(Q_q)_{(i,j)}\Big) + \varepsilon R_0,$$

where R_0 is a uniform remainder.

From the above, we deduce that in order to solve (3.12), it is sufficient to solve

$$(3.16) \qquad i\big(-\ell\omega + \mu_{i,+\ell} - \mu_j\big)(Q_\ell)_{(i,j)} = (\mathcal{D}_\ell)_{(i,j)}, \qquad |\ell| \le 2, \quad 1 \le i,j \le NJ.$$

We consider equation (3.16) for all possible values of ℓ, using the definition of $\mathcal{B}$ in Section 3.1.2.1:

- For $\ell = 0$, equation (3.16) reduces to the system

$$i\big(\lambda_{1,+1} - \omega - \lambda_2\big)(Q_0)_{(1,2)} = (1 - \chi_0)\Pi_{1,+1}B_1\Pi_2 \quad \text{(top left block)},$$
$$i\big(\lambda_2 - \lambda_{1,+1} + \omega\big)(Q_0)_{(2,1)} = (1 - \chi_0)\Pi_2 B_{-1}\Pi_{1,+1} \quad \text{(top left block)},$$
$$i\big(\lambda_{1,+1} - \omega - \lambda_j\big)(Q_0)_{(1,j)} = \Pi_{1,+1}B_1\Pi_j, \qquad 3 \le j \le J \quad \text{(top right block)},$$
$$i\big(\lambda_j - \lambda_{1,+1} + \omega\big)(Q_0)_{(j,1)} = \Pi_j B_{-1}\Pi_{1,+1}, \qquad 3 \le j \le J \quad \text{(bottom left block)}.$$

The first two equations involve the $(1,2)$ resonance relation. On the support of $1 - \chi_0$, the phase $\lambda_{1,+1} - \omega - \lambda_2$ is bounded away from zero. Thus we can divide the right-hand sides by the phase and thereby define $(Q_0)_{(1,2)}$ and $(Q_0)_{(2,1)}$ as an element of S^0. The last two equations involve the $(1,j)$ resonance relation. The phase $\lambda_{1,+1} - \omega - \lambda_j$ might vanish for some $\xi \in \mathbb{R}^d$, but the transparency assumption (Assumption 2.6(ii)) ensures that this phase factors out in the right-hand sides, so that we can solve for $(Q_0)_{(1,j)}$ and $(Q_0)_{(j,1)}$ in S^0.

- For $|\ell| = 1$, equation (3.16) reduces to

$$i\big(\lambda_{1,+(\ell+1)} - \ell\omega - \lambda_{1,+1}\big)(Q_\ell)_{(1,1)} = \Pi_{1,+(\ell+1)}B_q\Pi_{1,+1} \quad \text{(top left block)},$$
$$i\big(\lambda_{i,+\ell} - \ell\omega - \lambda_j\big)(Q_\ell)_{(i,j)} = \Pi_{i,+\ell}B_\ell\Pi_j, \qquad 2 \le i,j \le J \quad \text{(all blocks)}$$

The first equation involves the $(1,1)$ resonance relation, and the second involves the (i,j) resonance relation. We use again Assumption 2.6 to solve for the corresponding coefficient of Q in S^0.

- For $|\ell| = 2$, equation (3.16) reduces to

$$i(\lambda_{1,-1} + \omega - \lambda_j)(Q_{-1})_{(1,j)} = \Pi_{1,-1}B_{-1}\Pi_j, \quad 2 \le j \le J,$$
$$i(\lambda_{j,+2} - \omega - \lambda_{1,+1})(Q_1)_{(j,1)} = \Pi_{j,+2}B_1\Pi_{1,+1}, \quad 2 \le j \le J.$$

In both equation we find the $(j,1)$ resonant phase as a prefactor in the left-hand side and an interaction coefficient associated with $(j,1)$ in the right-hand side; both phase

and interaction coefficient are translated by $\pm\omega$. For instance, the first equation can be written

$$-i\big(\lambda_j(\xi+\omega)-\lambda_1(\xi)-\omega\big)(Q_{-1}(t,x,\xi+k))_{(1,j)} = \Pi_1(\xi)B(u_{0,-1}(t,x))\Pi_j(\xi+k),$$

which we can solve for $(Q_{-1})_{(1,j)} \in S^0$, by Assumption 2.6. $\square$

REMARK 3.4. — With Q defined in the above Proposition, there holds $Q_{(i,j)} \in S^{-1}$, unless λ_i and λ_j are bounded, or $i = j$. In both these cases, $Q_{(i,j)} \in S^0$. Indeed, the symbol $\lambda_i(\xi+k)-\lambda_i(\xi)$ is typically bounded in ξ, as shown in Lemma 6.20.

By estimates (6.3) and (6.15), with the symbol Q given in Proposition 3.3 is associated an operator $\mathrm{op}_\varepsilon(Q)$, which is a uniform remainder, in particular satisfies $\|\mathrm{op}_\varepsilon(Q)v\|_{\varepsilon,s} \lesssim \|v\|_{\varepsilon,s}$, for all $v \in H^s$. For ε small enough, $\mathrm{Id} + \sqrt{\varepsilon}\mathrm{op}_\varepsilon(Q)$ is invertible for all t. We consider the change of variable

$$(3.17) \qquad \check{U}(t) := \Big(\mathrm{Id} + \sqrt{\varepsilon}\mathrm{op}_\varepsilon\big(Q(\sqrt{\varepsilon}t)\big)\Big)^{-1} U(\sqrt{\varepsilon}t),$$

corresponding to a normal form reduction and a rescaling in time.

COROLLARY 3.5. — *The equation in $\check{U}$ is*

$$(3.18)\ \ \partial_t\check{U} + \frac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(i\mathcal{A})\check{U} = \mathrm{op}_\varepsilon(\check{\mathcal{B}})\check{U} + \sqrt{\varepsilon}\check{F}, \qquad \check{\mathcal{B}} := \begin{pmatrix} \chi_0(\xi)\mathcal{B}^r(\sqrt{\varepsilon}t,x,\xi) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\mathcal{B}^r$ is defined in (3.11), and $\check{F}$ satisfies the same bounds as F in Lemma 3.1.

Proof. — By definition of $\check{U}$ and (3.6), there holds

$$\partial_t\check{U} = (\mathrm{Id}+\mathbf{Q})^{-1}\Big(-\frac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(i\mathcal{A}) + \mathrm{op}_\varepsilon(\mathcal{B}(\sqrt{\varepsilon}t))\Big)(\mathrm{Id}+\mathbf{Q})\check{U}$$

$$- (\mathrm{Id}+\mathbf{Q})^{-1}\Big(\varepsilon\mathrm{op}_\varepsilon\big((\partial_tQ)(\sqrt{\varepsilon}t)\big)\check{U} + \sqrt{\varepsilon}F\Big),$$

where $\mathbf{Q}$ is short for $\sqrt{\varepsilon}\mathrm{op}_\varepsilon\big(Q(\sqrt{\varepsilon}t)\big)$. There holds

$$\mathrm{op}_\varepsilon(i\mathcal{A})(\mathrm{Id}+\mathbf{Q}) = (\mathrm{Id}+\mathbf{Q})\mathrm{op}_\varepsilon(i\mathcal{A}) + \sqrt{\varepsilon}\big[\mathrm{op}_\varepsilon(i\mathcal{A}), \mathrm{op}_\varepsilon(Q(\sqrt{\varepsilon}t))\big],$$

so that

$$(\mathrm{Id}+\mathbf{Q})^{-1}\mathrm{op}_\varepsilon(i\mathcal{A})(\mathrm{Id}+\mathbf{Q}) = \mathrm{op}_\varepsilon(i\mathcal{A}) + \sqrt{\varepsilon}(\mathrm{Id}+\mathbf{Q})^{-1}\big[\mathrm{op}_\varepsilon(i\mathcal{A}), \mathrm{op}_\varepsilon(Q(\sqrt{\varepsilon}t))\big].$$

Besides,

$$(\mathrm{Id}+\mathbf{Q})^{-1}\mathrm{op}_\varepsilon(\mathcal{B}(\sqrt{\varepsilon}t))(\mathrm{Id}+\mathbf{Q}) = (\mathrm{Id}+\mathbf{Q})^{-1}\mathrm{op}_\varepsilon(\mathcal{B}(\sqrt{\varepsilon}t)) + \sqrt{\varepsilon}R_0,$$

where R_0 denotes a uniform remainder, in the sense of Definition 3.2. The equation in $\check{U}$ thus appears as

$$\partial_t\check{U} + \frac{i}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(\mathcal{A})\check{U}$$

$$= (\mathrm{Id}+\mathbf{Q})^{-1}\Big(\mathrm{op}_\varepsilon(\mathcal{B}(\sqrt{\varepsilon}t)) - [\mathrm{op}_\varepsilon(i\mathcal{A}), \mathrm{op}_\varepsilon(Q(\sqrt{\varepsilon}t))] - \varepsilon\mathrm{op}_\varepsilon(\partial_tQ(\sqrt{\varepsilon}t))\Big)\check{U} + \sqrt{\varepsilon}\check{F},$$

with the source

$$\check{F} := R_0 \check{U} - (\mathrm{Id} + \mathbf{Q})^{-1} F.$$

By Proposition 3.3, the leading term in the right-hand side in the equation in $\check{U}$ reduces to $(\mathrm{Id} + \mathbf{Q})^{-1}\mathrm{op}_\varepsilon(\check{\mathcal{B}})$. Now, expanding the inverse of $\mathrm{Id} + \mathbf{Q}$ in Neumann series and using the fact that $\mathrm{op}_\varepsilon(Q)$ is a uniform remainder, we see that there holds the identity $(\mathrm{Id} + \mathbf{Q})^{-1} = \mathrm{Id} + \sqrt{\varepsilon} R_0$, so that

$$(\mathrm{Id} + \mathbf{Q})^{-1}\mathrm{op}_\varepsilon(\check{\mathcal{B}}) = \mathrm{op}_\varepsilon(\check{\mathcal{B}}) + \sqrt{\varepsilon} R_0.$$

Thus the above equation in $\check{U}$ reduces to (3.18). Since $\mathrm{op}_\varepsilon(Q)$ is a uniform remainder, there holds $\|\check{U}\|_{\varepsilon,s} \lesssim \|U\|_{\varepsilon,s} \lesssim \|\dot{u}\|_{\varepsilon,s}$ and $\|(\mathrm{Id} + \mathbf{Q})^{-1}F\|_{\varepsilon,s} \lesssim \|F\|_{\varepsilon,s}$, hence the bound $\|\check{F}\|_{\varepsilon,s} \lesssim \|F\|_{\varepsilon,s}$. $\qquad\square$

REMARK 3.6. — Note that we do not really need Q bounded, only $\sqrt{\varepsilon} Q$ small. In this sense we could approach the resonance much closer. We will do exactly so in Appendix 6.3, specifically in the proof of Lemma 6.17, where we derive bounds for the symbolic flow.

3.1.2.3. Space-frequency localization. — By the assumed polarization condition (see equation (2.6) in Assumption 2.2), there holds

$$\mathrm{tr}\left(\Pi_1(\xi + k)B(u_{01}(0,x))\Pi_2(\xi)B(u_{0,-1}(0,x))\Pi_1(\xi + k)\right) = |a(x)|^2\Gamma(\xi),$$

where a is the leading amplitude in the initial datum (1.2), and Γ is introduced in (2.10).

By continuity and decay of a at spatial infinity, there exists $x_0 \in \mathbb{R}^d$ such that

$$(3.19) \qquad\qquad 0 < |a(x_0)| = \sup_{x\in\mathbb{R}^d} |a(x)|.$$

By compactness of $\mathcal{R}_{12}$ and positivity of the stability index $\mathbf{\Gamma}$ (defined in (2.11)), the function $\Re e\,\Gamma(\xi)^{1/2}$ is not identically zero on $\mathcal{R}_{12}$. Then, for some $\xi_0 \in \mathcal{R}_{12}$,

$$(3.20) \qquad\qquad 0 < \gamma = \left|\max_{\xi\in\mathcal{R}_{12}} \Re e\left(\Gamma(\xi)^{1/2}\right)\right| \quad \text{is attained at } \xi_0.$$

NOTATION 3.7. — *Given two cut-offs $\theta_1, \theta_2 \in C_c^\infty(\mathbb{R}^d)$, with $0 \le \theta_1, \theta_2 \le 1$, we denote*

$$\theta_1 \prec \theta_2$$

to indicate that θ_2 is an extension of θ_1, in the sense that $(1 - \theta_2)\theta_1 \equiv 0$. In other words: $\theta_2 \equiv 1$ on the support of θ_1.

We denote $\varphi_0, \varphi, \varphi_1$ smooth spatial cut-offs, and χ_0, χ, χ_1 smooth frequency cut-offs, such that $\varphi_j \in C_c^\infty(\mathbb{R}_x^d)$, $\chi_j \in C_c^\infty(\mathbb{R}_\xi^d)$, with $0 \le \varphi_j \le 1$, $0 \le \chi_j \le 1$, $\varphi_j \equiv 1$ on a neighborhood of x_0, $\chi_j \equiv 1$ on the neighborhood $\mathcal{R}_{12}^h$ of the resonant set $\mathcal{R}_{12}$, and

$$\varphi_0 \prec \varphi \prec \varphi_1, \quad \chi_0 \prec \chi \prec \chi_1.$$

We will further (see in particular Propositions 3.14 and 3.17 and Remark 3.18) choose the support of χ_1 to be *small enough*, and the support of φ_0 to be *large enough*. Corresponding small parameters are $h > 0$ for the frequency truncations (as in the first paragraph of Section 3.1.2.2) and $\delta_{\varphi_0} > 0$ for the spatial truncations (see the proof of Proposition 3.14 below).

We let

$$(3.21) \qquad V := \mathrm{op}_\varepsilon(\chi)(\varphi\check{U}),$$

and

$$(3.22) \qquad W = (W_1, W_2) := \Big(\mathrm{op}_\varepsilon(\chi)\big((1-\varphi)\check{U}\big), \ \big(1 - \mathrm{op}_\varepsilon(\chi)\big)\check{U}\Big),$$

so that

$$(3.23) \qquad \check{U} = V + W_1 + W_2.$$

LEMMA 3.8. — *The system in* (V, W) *is*

$$(3.24) \qquad \begin{cases} \partial_t V + \dfrac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon^\psi(M)V = \sqrt{\varepsilon}F_V, \\[2mm] \partial_t W + \dfrac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(i\mathbf{A})W = \mathrm{op}_\varepsilon(D)W + \sqrt{\varepsilon}F_W, \end{cases}$$

with symbols

$$M := i\chi_1\mathcal{A} - \sqrt{\varepsilon}\varphi_1\check{\mathcal{B}}, \qquad \mathbf{A} := \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix}, \qquad D := \begin{pmatrix} (1-\varphi_0)\chi_1\check{\mathcal{B}} & 0 \\ 0 & 0 \end{pmatrix},$$

and source terms F_V, F_W *satisfying the same bound as* F *in Lemma* 3.1.

Proof. — There holds

$$\partial_t V = -\frac{i}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(\chi)\big(\varphi\mathrm{op}_\varepsilon(\mathcal{A})\check{U}\big) + \mathrm{op}_\varepsilon(\chi)\big(\varphi\mathrm{op}_\varepsilon(\check{\mathcal{B}})\check{U}\big) + \sqrt{\varepsilon}\mathrm{op}_\varepsilon(\chi)\big(\varphi\check{F}\big).$$

Using the identity $\mathrm{op}_\varepsilon(\chi) \equiv \mathrm{op}_\varepsilon(\chi_1)\mathrm{op}_\varepsilon(\chi)$, we compute

$$\mathrm{op}_\varepsilon(\chi)\big(\varphi\mathrm{op}_\varepsilon(\mathcal{A})\check{U}\big) = \mathrm{op}_\varepsilon(\chi)[\varphi, \mathrm{op}_\varepsilon(\mathcal{A})]\check{U} + [\mathrm{op}_\varepsilon(\chi), \mathrm{op}_\varepsilon(\mathcal{A})](\varphi\check{U}) + \mathrm{op}_\varepsilon(\chi_1\mathcal{A})V.$$

Similarly, using the identity $\varphi_1\varphi \equiv \varphi$,

$$\mathrm{op}_\varepsilon(\chi)\big(\varphi\mathrm{op}_\varepsilon(\check{\mathcal{B}})\check{U}\big) = \mathrm{op}_\varepsilon(\chi)\big(\varphi_1[\varphi, \mathrm{op}_\varepsilon(\check{\mathcal{B}})]\check{U}\big) + [\mathrm{op}_\varepsilon(\chi), \varphi_1]\mathrm{op}_\varepsilon(\check{\mathcal{B}})(\varphi\check{U})$$
$$+ \varphi_1[\mathrm{op}_\varepsilon(\chi), \mathrm{op}_\varepsilon(\check{\mathcal{B}})](\varphi\check{U}) + \varphi_1\mathrm{op}_\varepsilon(\check{\mathcal{B}})V.$$

Commutators being $O(\varepsilon)$ (in the sense of Proposition 6.6), the leading order term in the above right-hand side is the fourth term $\varphi_1\mathrm{op}_\varepsilon(\check{\mathcal{B}})$.

We now use the tensor product structure of every entry of $\check{\mathcal{B}}$ in order to express the operator $\mathrm{op}_\varepsilon(\varphi_1\check{\mathcal{B}})$ as a *para*-differential operator. Going back to the definition of symbol $\check{\mathcal{B}}$ in (3.18) and $\mathcal{B}^r$ in (3.11), we denote

$$\mathcal{B}_{12}^r = g(t, x)\Pi_{1,+1}B(\vec{e}_1)\Pi_2$$

the top right entry of $\mathcal{B}^r$, and similarly $\check{\mathcal{B}}_{12}$ the corresponding entry in $\check{\mathcal{B}}$:

$$\varphi_1 \mathrm{op}_\varepsilon(\check{\mathcal{B}}_{12}) = \varphi_1(x) g(\sqrt{\varepsilon} t, x) \mathrm{op}_\varepsilon(\chi_0 \Pi_{1,+1} B(\vec{e}_1) \Pi_2).$$

By Remark 6.4,

$$\varphi_1(x) g(\sqrt{\varepsilon} t, x) \mathrm{op}_\varepsilon(\chi_0 \Pi_{1,+1} B(\vec{e}_1) \Pi_2) - \mathrm{op}_\varepsilon^\psi \Big(\varphi_1 g(\sqrt{\varepsilon} t) \chi_0 \Pi_{1,+1} B(\vec{e}_1) \Pi_2 \Big)$$

$$= \Big(\varphi_1(x) g(\sqrt{\varepsilon} t, x) - \mathrm{op}_\varepsilon^\psi(\varphi_1 g(\sqrt{\varepsilon} t, x)) \Big) \mathrm{op}_\varepsilon(\chi_0 \Pi_{1,+1} B(\vec{e}_1) \Pi_2),$$

hence, by Proposition 6.8,

$$\Big\| \Big(\varphi_1 \mathrm{op}_\varepsilon(\check{\mathcal{B}}_{12}) - \mathrm{op}_\varepsilon^\psi(\varphi_1 \check{\mathcal{B}}_{12}) \Big) V \Big\|_{\varepsilon,s} \lesssim \varepsilon \| \varphi_1 g(\sqrt{\varepsilon} t) \|_{\varepsilon,s} |\tilde{V}|_{L^\infty},$$

with $\tilde{V} := \mathrm{op}_\varepsilon(\chi_0 \Pi_{1,+1} B(\vec{e}_1) \Pi_2) V$. By (6.5), $|\tilde{V}|_{L^\infty} \lesssim |V|_{L^2}$, since χ_0 is smooth and compactly supported.

The same is true of course for the other entry of $\check{\mathcal{B}}$, and we arrive at

$$(3.25) \qquad \Big\| \Big(\varphi_1 \mathrm{op}_\varepsilon(\check{\mathcal{B}}) - \mathrm{op}_\varepsilon^\psi(\varphi_1 \check{\mathcal{B}}) \Big) V \Big\|_{\varepsilon,s} \lesssim \varepsilon |V|_{L^2}.$$

Gathering the above results, and using $\mathrm{op}_\varepsilon(\chi_1 \mathcal{A}) \equiv \mathrm{op}_\varepsilon^\psi(\chi_1 \mathcal{A})$ (Remark 6.4), we obtain (3.24)(i), with the source term

$$F_V := \sqrt{\varepsilon} \mathrm{op}_\varepsilon(\chi) (\varphi \check{F}) - \frac{i}{\sqrt{\varepsilon}} \Big(\mathrm{op}_\varepsilon(\chi) [\varphi, \mathrm{op}_\varepsilon(\mathcal{A})] \check{U} + [\mathrm{op}_\varepsilon(\chi), \mathrm{op}_\varepsilon(\mathcal{A})] (\varphi \check{U}) \Big)$$

$$+ \mathrm{op}_\varepsilon(\chi) \big(\varphi_1 [\varphi, \mathrm{op}_\varepsilon(\check{\mathcal{B}})] \check{U} \big) + [\mathrm{op}_\varepsilon(\chi), \varphi_1] \mathrm{op}_\varepsilon(\check{\mathcal{B}}) (\varphi \check{U}) + \varphi_1 [\mathrm{op}_\varepsilon(\chi), \mathrm{op}_\varepsilon(\check{\mathcal{B}})] (\varphi \check{U})$$

$$+ \varepsilon^{-1/2} \Big(\varphi_1 \mathrm{op}_\varepsilon(\check{\mathcal{B}}) - \mathrm{op}_\varepsilon^\psi(\varphi_1 \check{\mathcal{B}}) \Big) V.$$

The fact that F_V satisfies the same bound as $\check{F}$ and F follows from (3.25) and the elementary results of Appendix 6.1.1.

The equation in W_1 is derived in the same way, the only difference being the use of $(1 - \varphi_0)(1 - \varphi) \equiv (1 - \varphi)$ in place of $\varphi_1 \varphi \equiv \varphi$.

Finally, the equation in W_2 involves the source term $\mathrm{op}_\varepsilon(1 - \chi) \mathrm{op}_\varepsilon(\check{\mathcal{B}}) \check{U}$. The symbol $(1 - \chi) \check{\mathcal{B}}$ vanishes identically, by $(1 - \chi) \chi_0 = 0$ and definition of $\check{\mathcal{B}}$ in (3.18). Hence, by estimate (6.6), there holds $\mathrm{op}_\varepsilon(1 - \chi) \mathrm{op}_\varepsilon(\check{\mathcal{B}}) = \varepsilon R_0$, where R_0 is a uniform remainder (in the sense of Definition 3.2 page 33). $\qquad \square$

System (3.24) is the *prepared system*, in which
   – the symbol M is the key term; it involves the diagonal hyperbolic operator $\mathcal{A}$ in a neighborhood of the $(1, 2)$ resonance, and the interaction coefficients associated with $(1, 2)$ via $\mathcal{B}^r$,
   – the source term in the right-hand side of the equation in W_1 will be made small, by choice of a spatial cut-off φ_0 with a *large* support, exploiting decay at infinity of the leading profile of the WKB solution;
   – the equation in W_2 is non-singular, a consequence of the normal form reduction of Section 3.1.2.2.

3.1.3. Duhamel representation. — In this Section we use Theorem 6.14 from Appendix 6.2 and write an integral representation formula for the variable V introduced in (3.21). From this representation we derive an upper bound for $\|V\|_{\varepsilon,s}$.

The symbol M of the propagator in the equation (3.24)(i) in V is

$$(3.26) \quad M(\varepsilon,t,x,\xi) = i\chi_1 \mathcal{A} - \sqrt{\varepsilon}\varphi_1 \check{\mathcal{B}} = \begin{pmatrix} i\chi_1\mu_1 & -\sqrt{\varepsilon}\,\tilde{b}_{12} & 0 & \cdots & 0 \\ -\sqrt{\varepsilon}\,\tilde{b}_{21} & i\chi_1\mu_2 & 0 & \cdots & 0 \\ 0 & 0 & i\chi_1\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & i\chi_1\lambda_J \end{pmatrix}$$

where

 – the cut-offs functions φ_1, χ_1 were introduced just below Notation 3.7 on page 36,

 – the shifted eigenvalues $\mu_1 = \lambda_1(\cdot + k) - \omega$ and $\mu_2 = \lambda_2$ were introduced in (3.15),

 – the $N \times N$ extra-diagonal blocks are

$$(3.27) \quad \tilde{b}_{12} := \chi_0(\xi)\varphi_1(x)g(\sqrt{\varepsilon}t,x)b_{12}^+(\xi), \quad \tilde{b}_{21} = \chi_0(\xi_0)\varphi_1(x)g(\sqrt{\varepsilon}t,x)^* b_{21}^-(\xi),$$

where

$$(3.28) \quad b_{12}^+(\xi) := \Pi_1(\xi + k)B(\vec{e}_1)\Pi_2(\xi), \qquad b_{21}^-(\xi) := \Pi_2(\xi)B(\vec{e}_{-1})\Pi_1(\xi + k),$$

are the interaction coefficients associated with resonance $(1,2)$, in the sense of Definition 2.5.

By regularity of the eigenprojectors (Assumption 2.1) and the approximate solution (Assumption 2.2), Assumption 6.9 is satisfied, where $x_\star$ is the maximum of all $|x|$ with x in the support of φ_1.

The symbolic flow S_0 of M is defined as the solution to the initial-value problem

$$(3.29) \qquad \partial_t S_0 + \frac{1}{\sqrt{\varepsilon}}M S_0 = 0, \qquad S_0(\tau,\tau) = \mathrm{Id}.$$

PROPOSITION 3.9. — *For all $T > 0$, all $0 \leq \tau \leq t \leq T|\ln\varepsilon|$, $\alpha \in \mathbb{N}^d$, there holds*

$$|\partial_x^\alpha S_0(\tau,t)| \lesssim |\ln\varepsilon|^* \exp\big((t-\tau)\gamma^+\big),$$

where

$$(3.30) \qquad \gamma^+ := |a|_{L^\infty}\big|\max_{\xi \in \mathcal{R}_{12}^h} \Re e\,\big(\Gamma(\xi)^{1/2}\big)\big|,$$

and $|\ln\varepsilon|^*$ denotes $|\ln\varepsilon|^{N^*}$ for some large constant $N^* > 0$ depending on all parameters, but not on ε nor on τ, t.

Note that γ^+ is 1-homogeneous in a, while γ, defined in (2.14), does not depend on a.

Proof. — The proof is postponed to Appendix 6.3. It uses elementary linear alge-
bra, rendered non-trivial by the fact that the resonant frequencies are asymptotically
close to crossing points in the spectrum of M, and a non-stationary phase argument
analogous (and complementary) to the normal form reduction of Section 3.1.2.2. $\qquad\square$

Proposition 3.9 verifies that the flow of M defined in (3.26) satisfies Assump-
tion 6.10.

Thus, Theorem 6.14 from Appendix 6.2 page 106 applies, and the unique solution
to (3.24)(i) satisfies the representation

$$(3.31) \qquad V = \mathrm{op}_\varepsilon^\psi(S(0;t))V(0) + \sqrt{\varepsilon} \int_0^t \mathrm{op}_\varepsilon^\psi(S(t';t))\tilde{F}_V(t')\,dt',$$

where $S(t';t) := \sum_{0\leq q\leq q_0} S_q$, the leading term S_0 being the symbolic flow (3.29),
and the correctors S_q, for $1 \leq q \leq q_0$, being defined in (6.19). The order q_0 of the
expansion is a function of Γ and T_0, as seen on equation (6.29) page 107. The source
term $\tilde{F}_V$ can be expressed in terms of F_V and the datum $V(0)$, as in (6.26). The
bound (6.27) implies

$$(3.32) \qquad \|\tilde{F}_V\|_{\varepsilon,s} \leq \|F_V\|_{\varepsilon,s} + \|V(0)\|_{\varepsilon,s}.$$

Proposition 3.9 and Lemma 6.12 imply that $\mathrm{op}_\varepsilon^\psi(S(t';t))$ satisfies the bound

$$(3.33) \qquad \|\mathrm{op}_\varepsilon^\psi(S(t';t))v\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^* e^{(t-t')\gamma^+}\|v\|_{\varepsilon,s}, \qquad v \in H^s.$$

From there, we deduce the bound, for $s \geq 0$:

$$\|V(t)\|_{\varepsilon,s} \lesssim e^{t\gamma^+}|\ln \varepsilon|^*\|V(0)\|_{\varepsilon,s} + \varepsilon^{1/2}|\ln \varepsilon|^* \int_0^t e^{(t-t')\gamma^+}\|F_V(t')\|_{\varepsilon,s}\,dt'.$$

According to Lemmas 3.1 and 3.8, there holds

$$(3.34) \qquad \|F_V\|_{\varepsilon,s} \lesssim \big(1 + \varepsilon^{-1/2}|\dot{u}(\varepsilon^{1/2}t)|_{L^\infty}\big)\|\dot{u}(\sqrt{\varepsilon}t)\|_{\varepsilon,s} + \varepsilon^{K_a}.$$

Going up the chain of changes of variables (3.21)-(3.22), (3.17), (3.5), we see that

$$(3.35) \qquad \|\dot{u}(\sqrt{\varepsilon}t)\|_{\varepsilon,s} \lesssim \|(V,W)(t)\|_{\varepsilon,s}.$$

Since by assumption $K \leq K_a + 1/2$, we conclude that
$$(3.36)$$
$$\|V(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln \varepsilon|^* e^{t\gamma^+}$$
$$+ \varepsilon^{1/2}|\ln \varepsilon|^* \int_0^t e^{(t-t')\gamma^+}\big(1 + \varepsilon^{-1/2}|\dot{u}(\varepsilon^{1/2}t')|_{L^\infty}\big)\|(V,W)(t')\|_{\varepsilon,s}\,dt'.$$

3.1.4. Lower bound. — We now choose the datum

$$(3.37) \qquad \dot{u}(0,x) := \varepsilon^K e^{ix\cdot(\xi_0+k)/\varepsilon}\varphi_0(x)\vec{e}_0,$$

where φ_0 is the spatial truncation introduced just below Notation 3.7 on page 36, and
the fixed vector $\vec{e}_0$ satisfies

$$(3.38) \qquad \vec{e}_0 = \Pi_1(\xi_0+k)B(\vec{e}_1)\Pi_2(\xi_0)B(\vec{e}_{-1})\Pi_1(\xi_0+k)\vec{e}_0, \quad |\vec{e}_0| = 1.$$

The matrix $\Pi_1(\xi_0+k)B(\vec{e}_1)\Pi_2(\xi_0)B(\vec{e}_{-1})\Pi_1(\xi_0+k)$ has rank one by assumption (rank at most one by Assumption 2.6(iii), and at least one by $\Gamma \neq 0$), so that by (3.38) the vector $\vec{e}_0$ is defined as the unitary generator of its image.

LEMMA 3.10. — *With the choice (3.37), the datum for V is*

$$V(0) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}V_0 + \varepsilon^{K+1/2}\tilde{V}_0^\varepsilon, \quad V_0 := \varphi_0(x)\big(\vec{e}_0, 0, \dots, 0\big), \quad \sup_{0<\varepsilon<\varepsilon_0} \|\tilde{V}_0^\varepsilon\|_{\varepsilon,s} < \infty.$$

Proof. — We denote in this proof f_0^ε any family of H^s maps such that

$$\sup_{0<\varepsilon<\varepsilon_0} \|f_0^\varepsilon\|_{\varepsilon,s} < \infty.$$

With this notation, given a Fourier multiplier $P \in S^0$ and $f \in C_c^\infty$, there holds

$$(3.39) \qquad\qquad \mathrm{op}_\varepsilon(P)f = P(0)f + \varepsilon f_0^\varepsilon.$$

With the choice (3.37), there holds

$$U_1(0) = \varepsilon^K e^{-ik\cdot x/\varepsilon}\mathrm{op}_\varepsilon(\Pi_1)\big(e^{ix\cdot(\xi_0+k)/\varepsilon}\varphi_0\vec{e}_0\big) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}\mathrm{op}_\varepsilon(\Pi_{1,+(\xi_0+k)})\varphi_0\vec{e}_0.$$

This implies, by (3.39),

$$U_1(0) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}\varphi_0(x)\Pi_1(\xi_0+k)\vec{e}_0 + \varepsilon f_0^\varepsilon = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}\varphi_0(x)\vec{e}_0 + \varepsilon f_0^\varepsilon,$$

the second equality by $\Pi_1(\xi_0+k)\vec{e}_0 = \vec{e}_0$. Next we compute, for $j \geq 2$, using (3.39) again,

$$\mathrm{op}_\varepsilon(\Pi_j)\big(e^{ix\cdot(\xi_0+k)/\varepsilon}\varphi_0\vec{e}_0\big) = e^{ix\cdot(\xi_0+k)/\varepsilon}\varphi_0(x)\Pi_j(\xi_0+k)\vec{e}_0 + \varepsilon f_0^\varepsilon.$$

This gives $U_j(0) = \varepsilon f_0^\varepsilon$, since $\Pi_j(\xi_0+k)\vec{e}_0 = 0$ for $j \geq 2$. From there, we obtain

$$\check{U}_1(0) = U_1(0) - \sqrt{\varepsilon}R_0 U(0) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}\varphi_0(x)\vec{e}_0 + \sqrt{\varepsilon}f_0^\varepsilon,$$

and

$$V_1(0) = \varepsilon^K\mathrm{op}_\varepsilon(\chi)\big(\varphi e^{ix\cdot\xi_0/\varepsilon}\varphi_0\big)\vec{e}_0 + \varepsilon^{K+1/2}f_0^\varepsilon = \varepsilon^K\mathrm{op}_\varepsilon(\chi_{+\xi_0})\varphi_0\vec{e}_0,$$

since $\varphi\varphi_0 \equiv \varphi_0$, and then with (3.39),

$$V_1(0) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon}\chi(\xi_0)\varphi_0(x)\vec{e}_0 + \varepsilon^{K+1/2}f_0^\varepsilon.$$

Since $\chi(\xi_0) = 1$, we obtained the first component of $V(0)$. We conclude with

$$V_j(0) = \mathrm{op}_\varepsilon(\chi)\big(\varphi(\varepsilon f_0^\varepsilon - \sqrt{\varepsilon}R_0 U(0))\big) = \sqrt{\varepsilon}f_0^\varepsilon, \quad j \geq 2.$$

$\square$

LEMMA 3.11. — *For the datum $V(0)$ described in the above Lemma, there holds for small enough $\rho > 0$, for some $C(\rho) > 0$:*

$$(3.40) \qquad \big|\mathrm{op}_\varepsilon^\psi(S(0;t))V(0)\big|_{L^2(B(x_0,\rho))} \geq C(\rho)\varepsilon^K\Big(e^{t\gamma^-} - \varepsilon^{1/2}|\ln\varepsilon|^* e^{t\gamma^+}\Big),$$

where $\gamma^- := \gamma \min\limits_{|x-x_0|\leq\rho} |a(x)|$, with γ as in (2.14).

We recall that notation $|\ln\varepsilon|^*$, introduced in the statement of Proposition 3.9, denotes $|\ln\varepsilon|^{N^*}$, for some $N^* > 0$ independent of ε, t.

Note that the lower rate of growth γ^- in (3.40) is 1-homogeneous in a, just like the upper rate of growth γ^+ (3.30) and unlike coefficient γ (2.14).

Proof of Lemma 3.11. — By Lemma 3.10, the datum $V(0)$ decomposes as a leading term and a remainder $\varepsilon^{K+1/2}\tilde{V}_0^\varepsilon$. By (3.33), there holds

$$\|\mathrm{op}_\varepsilon^\psi(S(0;t))(\varepsilon^{K+1/2}\tilde{V}_0^\varepsilon)\|_{L^2} \lesssim \varepsilon^{K+1/2}|\ln\varepsilon|^*e^{t\gamma^+}.$$

We turn to the action of $\mathrm{op}_\varepsilon^\psi(S)$ on the leading term $e^{ix\cdot\xi_0/\varepsilon}V_0$ in $V(0)$. By Remark 6.2,

$$\mathrm{op}_\varepsilon^\psi(S(0;t))\big(e^{ix\cdot\xi_0/\varepsilon}V_0\big) = e^{ix\cdot\xi_0/\varepsilon}\int e^{ix\cdot\xi}\mathbf{S}(0;t,x,\xi_0+\varepsilon\xi)\,\hat{V}_0(\xi)\,d\xi,$$

where $\mathbf{S}(0;t,x,\xi) := \left(\mathcal{F}^{-1}\psi \star \tilde{S}(0;t)\right)\left(\dfrac{x}{\varepsilon},\xi\right)$, with $\tilde{S}(0;t,x,\xi) = S(0;t,\varepsilon x,\xi)$. This gives

$$\mathrm{op}_\varepsilon^\psi(S(0;t))\big(e^{ix\cdot\xi_0/\varepsilon}V_0\big) = e^{ix\cdot\xi_0/\varepsilon}S(0;t,x,\xi_0)V_0(x) + \varepsilon\tilde{V}_0,$$

where the remainder $\tilde{V}_0$ is the sum $\tilde{V}_0 = \tilde{V}_{01} + \tilde{V}_{02}$:

$$\tilde{V}_{01} := \sum_{|\alpha|=1}\int e^{ix\cdot\xi}\left(\int_0^1 \left(\partial_\xi^\alpha\mathbf{S}\right)(0;t,x,\xi_0+\varepsilon\tau\xi)\,d\tau\right)\widehat{\partial_x^\alpha V_0}(\xi)\,d\xi,$$

$$\tilde{V}_{02} := \sum_{|\alpha|=1}\int_{\mathbb{R}^d}\mathcal{F}^{-1}(\partial_\eta^\alpha\psi)(y,\xi_0)\left(\int_0^1 \partial_x^\alpha S(0;t,x-\varepsilon\tau y,\xi_0)\,d\tau\right)dy\,V_0(x).$$

There holds

$$|\tilde{V}_{01}|_{L^2(B(x_0,\rho))} \leq \rho^{d/2}\sup_{\xi\in\mathbb{R}^d}|\partial_\xi\mathbf{S}(\cdot,\xi)|_{L^\infty(B(x_0,\rho))}\|V_0\|_{H^{1+d/2+}}.$$

By Remark 6.3 and Lemma 6.11,

$$|\partial_\xi\mathbf{S}|_{L^\infty} \lesssim |S|_{L^\infty} + |\partial_\xi S|_{L^\infty} \lesssim \varepsilon^{-1/2}|\ln\varepsilon|^*e^{t\gamma^+}.$$

By Remark 6.3 and Proposition 3.9,

$$|\tilde{V}_{02}|_{L^2(B(x_0,\rho))} \lesssim \sup_{x\in\mathbb{R}^d}|\partial_x S(0;t,\cdot,\xi_0)||V_0|_{L^2(B(x_0,\rho))} \lesssim |\ln\varepsilon|^*e^{t\gamma^+}.$$

It remains to bound from below the function $S(0;t,x,\xi_0)V_0$ on $B(x_0,\rho)$. Note that here ξ is frozen at ξ_0, so that the regularity issues of Appendix 6.3 do not come into play.

The symbolic flow S is defined just above Lemma 6.13 on page 105 as

$$S(0;t) = S_0 + \varepsilon^{1/2}\big(S_1 + \cdots + \varepsilon^{q_0-1/2}S_{q_0}\big).$$

Lemma 6.11 implies the uniform bound

$$\varepsilon^{1/2}\big|S_1 + \cdots + \varepsilon^{q_0-1/2}S_{q_0}\big| \lesssim \varepsilon^{1/2}|\ln\varepsilon|^*e^{t\gamma^+}.$$

According to Section 6.3.2, and especially (6.37) on page 111, the leading term S_0 decomposes as

$$S_0(0; t, x, \xi_0) = \exp\left(t M(0, x, \xi_0)/\sqrt{\varepsilon}\right) + \varepsilon^{1/2}\Sigma, \qquad \text{where } |\Sigma| \lesssim e^{t\gamma^+}.$$

We are left with the matrix exponential $\exp\left(t M(0, x, \xi_0)/\sqrt{\varepsilon}\right)$, where M is given explicitly in (3.26). At $\xi = \xi_0$, there holds $\mu_1 = \mu_2$, so that

$$\exp\left(t M(0, x, \xi_0)/\sqrt{\varepsilon}\right) = \text{diag}\left(e^{it\lambda_2(\xi_0)/\sqrt{\varepsilon}} \exp\left(t\tilde{M}(x)\right), \left(e^{it\lambda_j(\xi_0)/\sqrt{\varepsilon}}\right)_{3 \leq j \leq J}\right).$$

The matrix $\tilde{M}$ is

$$\tilde{M}(x) := \begin{pmatrix} 0 & \tilde{b}_{12}(0, x, \xi_0) \\ \tilde{b}_{21}(0, x, \xi_0) & 0 \end{pmatrix},$$

where $\tilde{b}_{12}$ and $\tilde{b}_{21}$ are defined in (3.27). It has rank two, by Assumption 2.6(iii), and spectrum

$$\left\{0, \pm\text{tr}\left(\tilde{b}_{12}(0, x, \xi_0)\tilde{b}_{21}(0, x, \xi_0)^{1/2}\right\} = \left\{0, \pm|a(x)|(\gamma + i\alpha)\right\},$$

where $\alpha := \Im m\left(\Gamma(\xi_0)^{1/2}\right)$. (For a detailed computation, see the paragraphs just above the statement of Lemma 6.16 in Appendix 6.3 on page 109.) By definition of x_0 and ξ_0 in Section 3.1.2.3 on page 36, there holds $\gamma \neq 0$, and $|a(x)| \neq 0$ locally around x_0. As a consequence, locally around x_0 the matrix $\tilde{M}$ has a smooth spectral decomposition

$$\tilde{M}(x) = a(x)(\gamma + i\alpha)\left(P_+(x) - P_-(x)\right),$$

with rank-one projectors. The ranges of the eigenprojectors (eigenspaces of $\tilde{M}$) are

$$\text{Ran } P_\pm(x) = \left\{c\left(\vec{e}_0, \pm\frac{\tilde{b}_{21}(0, x, \xi_0)}{|a(x)|\gamma}\vec{e}_0\right), \quad c \in \mathbb{C}\right\}.$$

In particular,

$$(3.41) \qquad P_\pm(x)\left(\vec{e}_0, 0\right) \equiv \left(1 + \left(\tilde{b}_{21}(0, x, \xi_0)\gamma^{-1}|a(x)|^{-1}\right)^2\right)^{-1/2}(\vec{e}_0, 0).$$

This gives for x close to x_0:

$$\left|\exp\left(t\tilde{M}(x)\right)V_0(x)\right| \geq C(x)e^{ta(x)\gamma} - \left|P_-(x)(\vec{e}_0, 0)\right|,$$

where $C(x) \neq 0$ in a neighborhood of x_0, and the result follows by

$$(3.42) \qquad \left(\int_{B(x_0, \rho)} e^{2ta(x)\gamma}\, dx\right)^{1/2} \geq C(\rho)e^{t\gamma^-},$$

with $C(\rho) = O(\rho^d)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

REMARK 3.12. — In (3.41) we see that $P_\pm V_0 = (\star, 0)$. For $\text{op}_\varepsilon^\psi(S(0; t))V$, which, at a given (t, x), is a vector in $\mathbb{C}^{NJ}$:

$$\text{op}_\varepsilon^\psi(S(0; t)V(0) = \left((\text{op}_\varepsilon^\psi(S(0; t)V(0))_1, \text{op}_\varepsilon^\psi(S(0; t)V(0))_2, \dots\right) \in \mathbb{C}^{N \times N \times \cdots},$$

this implies that the leading term is $(\text{op}_\varepsilon^\psi(S(0; t)V(0))_1$. This observation will be useful at the end of Section 3.1.6.

3.1.5. Existence over logarithmic times and upper bound. — We denote
(3.43)

$(V(0), W(0))$, with $V(0)$ as described by Lemma 3.10, and $\|W(0)\|_{\varepsilon,s} = O(\varepsilon^K)$,

the datum derived from $\dot{u}(0)$ (3.37) in the coordinates (V, W) of the prepared system (3.24).

We prove here existence and uniqueness of a solution (V, W) to (3.24) issued from (3.43), over the interval $[0, T_0\sqrt{\varepsilon}|\ln\varepsilon|)$, where the limiting time T_0 is defined in (2.12).

The difficulty is in the treatment of the L^∞ norm.

In a first part (Section 3.1.5.1), we perform estimates in $\mathcal{F}L^1$ norm on the prepared system (3.24) and combine these with Sobolev estimates on W from (3.24) and the upper bound (3.36) for V that we derived from the Duhamel representation (3.31); here we use the bound $|u|_{L^\infty} \leq |\hat{u}|_{L^1}$.

In a second part (Section 3.1.5.2), we only use Sobolev estimates on W from (3.24) and the upper bound (3.36); there we use the Sobolev embedding $|u|_{L^\infty} \lesssim \varepsilon^{-d/2}\|u\|_{\varepsilon,s}$, for $s > d/2$.

3.1.5.1. In $\mathcal{F}L^1$ and H^s. — An observation time T_1 is given, such that

$$(3.44) \qquad\qquad T_1 < \frac{K}{|B|_0|\hat{a}|_{L^1}},$$

where notation $|B|_0$ is introduced in (2.13).

LEMMA 3.13. — *If $\varphi_0 \equiv 1$ on a large enough ball around x_0, and if ε is small enough, then the initial value problem (3.24)-(3.43) is well-posed in $\mathcal{F}L^1$ over the interval $[0, T_1|\ln\varepsilon|]$, and there holds the bound*

$$(3.45) \qquad\qquad \sup_{0 \leq t \leq T_1|\ln\varepsilon|} |\mathcal{F}(V, W)(t)|_{L^1} \leq \varepsilon^{\eta_1},$$

for some $\eta_1 = \eta_1(\varepsilon, T_1) > 0$, with $\eta_1 \to 0$ as $T_1 \to K/(|B|_0|\hat{a}|_{L^1})$ and $\varepsilon \to 0$.

Above, $\mathcal{F}L^1$ is the Banach algebra of maps u with Fourier transform $\hat{u}$ in L^1.

Proof. — While a para-differential formulation of (3.24) was useful for the Duhamel representation of Section 3.1.3, we return here to a purely *pseudo*-differential formulation of (3.24). This simply means changing $\mathrm{op}_\varepsilon^\psi(M)$ into $\mathrm{op}_\varepsilon(M)$ in the left-hand side, an operation that takes one term out of the source F_V (namely, the term in the third line of the definition of F_V in the proof of Lemma 3.8).

Being symmetric hyperbolic and semilinear, the initial-value problem (3.2)-(3.37) is locally well-posed in $\mathcal{F}L^1$. Since the prepared system (3.24) derives from (3.2) via Fourier multipliers, and since Fourier multipliers operate in $\mathcal{F}L^1$ (as evidenced by (6.4)), the initial-value problem (3.24)-(3.43) is also locally well-posed in time.

From (3.37), we infer, via (6.4), that there holds $|(V, W)(0)|_{\mathcal{F}L^1} \lesssim \varepsilon^K$, on top of the bounds given in Lemma 3.10 and (3.43).

The local-in-time existence theory (based on the Cauchy-Lipschitz theorem) gives a notion of maximal existence time, which we denote $T_\star(\varepsilon)$. The function $t \to |\mathcal{F}(V,W)(t)|_{L^1}$ is continuous over $[0, T_\star(\varepsilon))$. Consider the set

$$J := \left\{ t \in (0, T_\star(\varepsilon)) \cap (0, T_1\sqrt{\varepsilon}|\ln \varepsilon|], \quad \forall t' \in (0, t), \quad |\mathcal{F}(V,W)(t')|_{L^1} \le \varepsilon^{\eta_1} \right\},$$

where $0 < \eta_1 < K$ will be appropriately chosen below, depending on T_1.

We are going to prove that, for ε small enough, J is non-empty, open and closed in $(0, T_1\sqrt{\varepsilon}|\ln \varepsilon|]$. This will prove well-posedness over $[0, T_1\sqrt{\varepsilon}|\ln \varepsilon|]$, by connectedness.

By $|\mathcal{F}(V,W)(0)|_{L^1} = O(\varepsilon^K) \ll \varepsilon^{\eta_1}$ and continuity of $t \to |\mathcal{F}(V,W)(t)|_{L^1}$, we see that J is not empty. The fact that J is closed follows immediately from its definition.

Now given $t \in J$, there certainly holds $(t - \zeta, t] \subset J$ for some $\zeta > 0$. Therefore we only have to prove that $[t, t + \zeta) \cap (0, T_1\sqrt{\varepsilon}|\ln \varepsilon|] \subset J$ for some $\zeta > 0$.

After applying the Fourier transform to both equations in (3.24) and factorizing the oscillations, we find, for a given $t \in J$:

$$\hat{V}(t) = e^{-it(\chi_1 \mathcal{A})(\varepsilon \xi)/\sqrt{\varepsilon}} \hat{V}(0) + \int_0^t e^{-i(t-t')(\chi_1 \mathcal{A})(\varepsilon \partial_x)/\sqrt{\varepsilon}} \left(\mathcal{F}\left(\mathrm{op}_\varepsilon(\varphi_1 \check{\mathcal{B}}) V\right) + \sqrt{\varepsilon} \hat{F}_V \right) dt'.$$

and

$$\hat{W}(t) = e^{-it\mathbf{A}(\varepsilon \xi)/\varepsilon} \hat{W}(0) + \int_0^t e^{-i(t-t')\mathbf{A}(\varepsilon \xi)/\varepsilon} \left(\mathcal{F}\left(\mathrm{op}_\varepsilon(D) W\right) + \hat{F}_W \right) dt'.$$

The symbols $\mathcal{A}$ and $\mathbf{A}$ are diagonal and real, so that $|e^{it\mathcal{A}(\xi)}| \le 1$, $|e^{it\mathbf{A}(\xi)}| \le 1$. Besides, by Young's convolution inequality, and recalling that the norm in use in $\mathbb{C}^N$ is the sup norm,

$$\left| \mathrm{op}_\varepsilon(\varphi_1 \check{\mathcal{B}}) V \right|_{\mathcal{F}L^1} \le |B|_0 |\hat{\varphi}_1|_{L^1} |\hat{u}_0(\sqrt{\varepsilon}t)|_{L^1} |\hat{V}|_{L^1},$$

where $|B|_0$ is defined in (2.13), and similarly

$$\left| \mathrm{op}_\varepsilon(D) W \right|_{\mathcal{F}L^1} \le |(1 - \varphi_0) u_0(\sqrt{\varepsilon}t)|_{\mathcal{F}L^1} |B| |\hat{W}|_{L^1}.$$

There holds over $[0, T_1|\ln \varepsilon|]$:

$$|\hat{u}_0(\sqrt{\varepsilon}t)|_{L^1} \le |\hat{a}|_{L^1} + C_0 \sqrt{\varepsilon}|\ln \varepsilon|, \quad |(1-\varphi_0)u_0(\sqrt{\varepsilon}t)|_{\mathcal{F}L^1} \le |(1-\varphi_0)a|_{\mathcal{F}L^1} + C_0 \sqrt{\varepsilon}|\ln \varepsilon|,$$

where $C_0 > 0$ is independent of ε, t, and depends on $\partial_t u_0$, which according to Assumption 2.2 belongs to $C^0 H^{s_a}$, hence to $\mathcal{F}L^1$. As $\varphi_0 \to 1$ (the function identically equal to 1), there holds $|\hat{\varphi}_1|_{L^1} \to 1$ and $|(1 - \varphi_0)a|_{\mathcal{F}L^1} \to 0$. In particular, for any $\delta_{\varphi_0} > 0$, we can choose φ_0, φ_1 such that

$$|\hat{\varphi}_1|_{L^1} \le 1 + \delta_{\varphi_0}, \qquad |(1 - \varphi_0)a|_{\mathcal{F}L^1} \le \delta_{\varphi_0}.$$

Thus we obtain

$$|\hat{V}(t)|_{L^1} \lesssim \varepsilon^K + |B|_0(1 + \delta_{\varphi_0})(|\hat{a}|_{L^1} + C_0\sqrt{\varepsilon}|\ln \varepsilon|) \int_0^t |\hat{V}(t')|_{L^1} dt' + \sqrt{\varepsilon} \int_0^t |\hat{F}_V|_{L^1} dt',$$

$$|\hat{W}(t)|_{L^1} \lesssim \varepsilon^K + |B|(\delta_{\varphi_0} + C_0\sqrt{\varepsilon}|\ln \varepsilon|) \int_0^t |\hat{W}(t')|_{L^1} dt' + \sqrt{\varepsilon} \int_0^t |\hat{F}_W|_{L^1} dt'.$$

With the $\mathcal{F}L^1$ bound for (F_V, F_W) derived from Lemma 3.1 and $|\dot{u}|_{\mathcal{F}L^1} \lesssim |V, W|_{\mathcal{F}L^1}$, a consequence of (6.4), this yields, using $t \in J$ and $K_a + 1/2 \geq K$:

$$|(\hat{V}, \hat{W})(t)|_{L^1} \lesssim \varepsilon^K |\ln \varepsilon|^* + \Big(|B|_0(1 + \delta_{\varphi_0})(|\hat{a}|_{L^1} + C_0\sqrt{\varepsilon}|\ln \varepsilon|) + \varepsilon^{\eta_1} \Big)$$
$$\times \int_0^t |(\hat{V}, \hat{W})(t')|_{L^1} dt',$$

for δ_{φ_0} small enough (depending on B and $|\hat{a}|_{L^1}$). We now let

(3.46) $2\eta_1 := K - |B|_0|\hat{a}|_{L^1}T_1.$

Then, for ε small enough, and δ_{φ_0} small enough, depending in particular on T_1, there holds

$$\frac{3}{2}\eta_1 < K - |B|_0|\hat{a}|_{L^1}T_1 - \Big(\varepsilon^{\eta_1} + C_0\sqrt{\varepsilon}|\ln \varepsilon||B|_0(1 + \delta_{\varphi_0}) + \delta_{\varphi_0}|B|_0|\hat{a}|_{L^1} \Big)T_1.$$

With the above bound in $|(\hat{V}, \hat{W})|_{L^1}$ and Gronwall's lemma, this implies, for ε small enough,

(3.47) $|\mathcal{F}(V, W)(t)|_{L^1} \leq \varepsilon^{(3/2)\eta_1}, \quad t \in J.$

Then, by continuity of $t \to |\mathcal{F}(V, W)(t)|_{L^1}$, we obtain $|\mathcal{F}(V, W)(t + \zeta)|_{L^1} \leq \varepsilon^{\eta_1}$ if ζ and ε are small enough. This concludes the verification that J is open in the interval $(0, T_1\sqrt{\varepsilon}|\ln \varepsilon|)$. The bound (3.47) is then valid over $[0, T_1\sqrt{\varepsilon}|\ln \varepsilon|]$, and this is (3.45). $\square$

PROPOSITION 3.14. — *If $\varphi_0 \equiv 1$ on a large enough ball around x_0, and if ε is small enough, the solution (V, W) to system (3.24) issued from (3.43) is defined over the interval $[0, T_1|\ln \varepsilon|]$, and there holds the bound*

(3.48) $\|(V, W)(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln \varepsilon|^* e^{t\gamma^+}.$

The observation time T_1 is introduced at the beginning of this Section, and the amplification rate γ^+ is defined in (3.30). The spatial cut-off φ_0 is introduced just below Notation 3.7 and intervenes in the equation (3.24)(ii) in W.

Proof. — We compute $2\Re e\,(\Lambda^s \partial_t W, \Lambda^s W)_{L^2}$, where Λ^s is the Fourier multiplier defined by $\Lambda^s := \mathrm{op}_\varepsilon\big(1 + |\cdot|^2\big)^{s/2}$, and W solves (3.24)(ii). By symmetry, the contribution of $\mathbf{A}$ is zero, so that

(3.49) $\partial_t\big(\|W\|_{\varepsilon,s}^2\big) \leq 2\big(|\Lambda^s\mathrm{op}_\varepsilon(D)W|_{L^2} + \varepsilon^{1/2}\|F_W\|_{\varepsilon,s}\big)\|W\|_{\varepsilon,s}.$

By (6.6),

(3.50) $|\mathrm{op}_\varepsilon(D)\Lambda^s W|_{L^2} \lesssim |(1 - \varphi_0)u_0|_{L^\infty}\|W\|_{\varepsilon,s} \leq \delta_{\varphi_0}\|W\|_{\varepsilon,s},$

where $\delta_{\varphi_0} > 0$ can be made arbitrarily small by letting $\varphi_0 \equiv 1$ on a very large ball around x_0, since u_0 decays at spatial infinity. The commutator is estimated by (6.8) (here, we use again the fact that every entry of D is a tensor product $D_1(x)D_2(\xi)$):

(3.51) $\big|[\Lambda^s, \mathrm{op}_\varepsilon(D)]W\big|_{L^2} \lesssim \varepsilon\|u_a\|_{H^{s_a}}\|W\|_{\varepsilon,s-1}.$

By Lemma 3.13, there holds for $t \leq T_1 |\ln \varepsilon|$:

$$(3.52) \qquad |\dot{u}(\sqrt{\varepsilon}t)|_{L^\infty} \leq |\mathcal{F}\dot{u}(\sqrt{\varepsilon}t)|_{L^1} \lesssim |\mathcal{F}(V,W)(t)|_{L^1} \leq \varepsilon^{\eta_1},$$

where η_1 is defined in (3.46). A bound for F_W is given in Lemmas 3.1 and 3.8. With (3.52), this bound is

$$\|F_W\|_{\varepsilon,s} \lesssim (1 + \varepsilon^{-1/2+\eta_1})\|V,W\|_{\varepsilon,s} + \varepsilon^{K_a}.$$

We obtained, for ε small enough,

$$(3.53) \qquad \|W(t)\|_{\varepsilon,s}^2 \lesssim \varepsilon^{2K} + \delta_{\varphi_0} \int_0^t \|W(t')\|_{\varepsilon,s}^2 \, dt' + \varepsilon^{\eta_1} \int_0^t \|(V,W)(t')\|_{\varepsilon,s}^2 \, dt'$$
$$+ \varepsilon^{K_a+1/2} \int_0^t \|W(t')\|_{\varepsilon,s} \, dt'.$$

Going back to the upper bound (3.36) for V and exploiting (3.52), we see that there also holds

$$(3.54) \qquad \|V(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln \varepsilon|^* e^{t\gamma^+} + \varepsilon^{\eta_1} |\ln \varepsilon|^* \int_0^t e^{(t-t')\gamma^+} \|(V,W)(t')\|_{\varepsilon,s} \, dt'.$$

From (3.53) and (3.54), and $K_a + 1/2 \geq K$, we find that the scalar quantity $y(t) := \max_{t' \in [0,t]} \|V(t')\|_{\varepsilon,s} + \|W(t')\|_{\varepsilon,s}$ satisfies the bound

$$y(t) \lesssim \varepsilon^K \varepsilon^{t\gamma^+} |\ln \varepsilon|^* + \int_0^t \delta_{\varphi_0} y(t') \, dt' + \varepsilon^{\eta_1} |\ln \varepsilon|^* \int_0^t e^{(t-t')\gamma^+} y(t') \, dt'.$$

By application of the Gronwall Lemma of Appendix 6.4, this gives (3.48), under extra conditions on the small constant δ_{φ_0}, implying conditions on the support of φ_0, which are $\eta_1 - \delta_{\varphi_0} T_1 > 0$ and $\gamma |a|_{L^\infty} > \delta_{\varphi_0}$.

The fact that the a priori bound (3.48) translates into a bound from below for the existence time follows from a classical continuation argument, similar to the one detailed in the proof of Lemma 3.13. □

REMARK 3.15. — It would be tempting to use $\mathcal{F}L^1$ bounds in conjunction with the Duhamel representation (3.31) of V, instead of $\mathcal{F}L^1$ bounds for the equation (3.24)(i) in V, in the hope of obtaining a better estimate on the existence time, one that would involve $\gamma|\hat{a}|_{L^1}$ instead of $|B|_0|\hat{a}|_{L^1}$. This would require $\mathcal{F}L^1 \to \mathcal{F}L^1$ estimates on pseudo-differential operators which, as far as we know, are not known to exist.

REMARK 3.16. — Going back to the proof of Proposition 3.14 and using in (3.53) the bound (3.48), we see that W enjoys the better upper bound

$$\|W(t)\|_{\varepsilon,s} \lesssim \varepsilon^{K-T_1\delta_{\varphi_0}/2} + \varepsilon^{K+(\eta_1-T_1\delta_{\varphi_0})/2} |\ln \varepsilon|^* e^{t\gamma^+}, \qquad t < T_1 |\ln \varepsilon|.$$

This will be useful in Section 3.1.6.

3.1.5.2. In H^s. — We revisit here the estimates of the proof of Proposition 3.14, and give a slightly different existence result. We now consider an observation time T_2 such that

$$(3.55) \qquad T_2 < \frac{K - d/2}{\gamma |a|_{L^\infty}}.$$

Recall that $h > 0$ intervenes in the upper rate γ^+ defined in (3.30); it plays the role of a security distance from the resonance. By continuity, there holds $\gamma^+ \to |a|_{L^\infty}\gamma$ in the limit $h \to 0$.

PROPOSITION 3.17. — *If $\varphi_0 \equiv 1$ on a large enough ball around x_0, and if ε is small enough, the solution (V, W) to (3.24) issued from (3.43) is defined over $[0, T_2|\ln\varepsilon|]$, and there holds the bound*

$$(3.56) \qquad \|(V, W)(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln\varepsilon|^* e^{t\gamma^+}.$$

Proof. — We go back to the proof of Proposition 3.14. Instead of appealing to Lemma 3.13 to gain control of the L^∞ norm of $\dot u$, we use the Sobolev embedding

$$(3.57) \qquad |v|_{L^\infty} \le C_{s,d}\varepsilon^{-d/2}\|v\|_{\varepsilon,s}, \qquad C_{s,d} > 0, \quad v \in H^s, \ s > d/2,$$

and control of $\|\dot u\|_{\varepsilon,s}$ by $\|V, W\|_{\varepsilon,s}$, as in (3.35), as follows:

Local-in-time well-posedness in H^s is granted by symmetric hyperbolicity, the semi-linear nature of the nonlinearity, and $s > d/2$. Let $T_*(\varepsilon)$ be the maximal existence time [2] in H^s. Consider the set

$$J := \Big\{ t \in [0, T_*(\varepsilon)) \cap (0, T_2\sqrt{\varepsilon}|\ln\varepsilon|], \ \ \forall t' \in (0, t), \ \ |\dot u(t')|_{L^\infty} \le \varepsilon^{\eta_2} \Big\},$$

where $\eta_2 > 0$ will be chosen appropriately below, depending on T_2.

We now prove that, given $t \in J$, for some $\zeta > 0$ there holds $[t, t+\zeta) \subset J$. Just like in the proof of Lemma 3.13, this will imply $T_*(\varepsilon) > T_2\sqrt{\varepsilon}|\ln\varepsilon|$ by a connectedness argument.

From (3.57) and Lemmas 3.1 and 3.8, for $t \in J$ we deduce for the source term F_W the bound

$$\|F_W\|_{\varepsilon,s} \lesssim \big(1 + \varepsilon^{-1/2+\eta_2}\big)\|V, W\|_{\varepsilon,s} + \varepsilon^{K_a}.$$

Combined with estimates (3.49)-(3.50)-(3.51) in W, this gives the bound, for $t \in J$:

$$\|W(t)\|_{\varepsilon,s}^2 \lesssim \varepsilon^{2K} + \delta_{\varphi_0}\int_0^t \|W(t')\|_{\varepsilon,s}^2\, dt' + \varepsilon^{\eta_2}\int_0^t \|(V, W)(t')\|_{\varepsilon,s}^2\, dt'$$

$$+ \varepsilon^{K_a+1/2}\int_0^t \|W(t')\|_{\varepsilon,s}\, dt'.$$

---

2. The fact that notation $T_*(\varepsilon)$ was already used, with a different meaning, in the proof of Lemma 3.13 should not be a factor of confusion, since this use of $T_*(\varepsilon)$ is confined to the present proof. Same for J below.

By (3.36), for $t \in J$:

$$\|V(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln \varepsilon|^* e^{t\gamma^+} + C\varepsilon^{\eta_2}|\ln \varepsilon|^* \int_0^t e^{(t-t')\gamma^+} \|(V,W)(t')\|_{\varepsilon,s} \, dt'.$$

By application of Lemma 6.18 we deduce from the above bounds the inequality

$$(3.58) \qquad \|V,W(t)\|_{\varepsilon,s} \lesssim \varepsilon^K |\ln \varepsilon|^* e^{t\gamma^+}, \qquad t \in J,$$

for δ_{φ_0} small enough (depending on η_2, T_2 and γ). By (3.57) and (3.35), this implies

$$|\dot{u}(t)|_{L^\infty} \lesssim \varepsilon^{K-d/2}|\ln \varepsilon|^* e^{t\gamma^+}, \quad t \in J.$$

We let

$$2\eta_2 := K - d/2 - T_2\gamma|a|_{L^\infty}.$$

Then, if h is small enough, there holds $(3/2)\eta_2 < K - d/2 - T_2\gamma^+$. This implies, for ε small enough, the upper bound $|\dot{u}(t)|_{L^\infty} \le \varepsilon^{(3/2)\eta_2}$, and we conclude as in the proof of Proposition 3.14: the bound (3.58), which we now know to be valid over the interval $[0, T_2\sqrt{\varepsilon}|\ln \varepsilon|]$, is the bound (3.56). $\qquad\square$

3.1.6. Endgame: proof of the deviation estimate (2.16). — Let $T < T_0$ be given, where the limiting observation time T_0 is defined in (2.12). By Propositions 3.14 and 3.17, the solution (V,W) to the prepared system (3.24) is defined over $[0, T|\ln \varepsilon|]$.

Consider first the case

$$(3.59) \qquad T_0 = \frac{K}{|B|_0 |\hat{a}|_{L^1}}, \quad \text{so that} \quad K_0 = K\left(1 - \frac{\gamma|a|_{L^\infty}}{|B|_0 |\hat{a}|_{L^1}}\right),$$

and the bounds of Section 3.1.5.1 apply.

From the Duhamel representation (3.31) and the bound (3.33) for the action of $\mathrm{op}_\varepsilon(S)$, we find the lower bound

$$(3.60) \qquad \begin{aligned} \left|V(T|\ln \varepsilon|)\right|_{L^2(B(x_0,\rho))} &\ge \left|\mathrm{op}_\varepsilon(S(0; T|\ln \varepsilon|))V(0)\right|_{L^2(B(x_0,\rho))} \\ &\quad - C\sqrt{\varepsilon}|\ln \varepsilon|^* \int_0^{T|\ln \varepsilon|} e^{(T|\ln \varepsilon|-t')\gamma^+} |\tilde{F}_V(t')|_{L^2} dt'. \end{aligned}$$

From the upper bounds (3.32) and (3.34) for F_V, and (3.35) and Proposition 3.14, we deduce

$$\sqrt{\varepsilon}|\tilde{F}_V(t)|_{L^2} \lesssim \varepsilon^{K+1/2} + \varepsilon^{K_a+1/2} + (\varepsilon^{1/2} + |\dot{u}(\sqrt{\varepsilon}t)|_{L^\infty})\varepsilon^K e^{t\gamma^+}.$$

By Lemma 3.13, there holds $|\dot{u}(\sqrt{\varepsilon}t)|_{L^\infty} \le \varepsilon^{\eta_1}$ for $t < T$, under condition (3.59). Together with $K_a + 1/2 \ge K$, this implies that the above upper bound in $\tilde{F}_V$ takes the form

$$(3.61) \qquad \sqrt{\varepsilon}|\tilde{F}_V(t)|_{L^2} \lesssim \varepsilon^K + \varepsilon^{K+\eta_1} e^{t\gamma^+}.$$

We now use in (3.60) the lower bound for $\mathrm{op}_\varepsilon(S(0; t))V(0)$ given in Lemma 3.11 and bound (3.61). This shows that $|V(T|\ln \varepsilon|)|_{L^2(B(x_0,\rho))}$ is bounded from below by

$$C(\rho)\varepsilon^{K-T\gamma^-} - C|\ln \varepsilon|^* \varepsilon^{K+1/2-T\gamma^+} - CT|\ln \varepsilon|^* \varepsilon^{K+\eta_1-T\gamma^+}.$$

Up to a multiplicative constant, we can rewrite this lower bound

$$\varepsilon^{K-T\gamma^-}\left(1-C|\ln\varepsilon|^*\varepsilon^{\eta_1-T(\gamma^+-\gamma^-)}\right).$$

The smaller the exponent $K-T\gamma^-$, the better the above lower bound. Under (3.59), there certainly holds

$$K_0 \leq K - T\gamma^-,$$

since $\gamma^- \leq \gamma|a|_{L^\infty}$. However, for any $K' > K_0$, by choosing ρ small enough in Lemma 3.11, and by choosing $T_0 - T$ small enough, we can achieve

$$(3.62) \qquad\qquad K_0 < K - T\gamma^- < K'.$$

Besides, given $T < T_0$, we can choose h in (3.10) and ρ small enough, possibly even smaller than above, so that, for ε small enough, the minimal amplification rate γ^- defined just below (3.40) and the maximal amplification rate γ^+ defined in (3.30) are close enough so that

$$(3.63) \qquad\qquad \gamma^+ - \gamma^- < \frac{\eta_1}{T},$$

where η_1 is defined in (3.46). This implies

$$(3.64) \qquad\qquad \big|V\big(T|\ln\varepsilon|\big)\big|_{L^2(B(x_0,\rho))} \geq \frac{1}{2}C(\rho)\varepsilon^{K-T\gamma^-},$$

for $T_0 - T$, h, ρ, and ε small enough. Now with (3.23),

$$\big|\check{U}\big(T|\ln\varepsilon|\big)\big|_{L^2(B(x_0,\rho))} \geq \big|V\big(T|\ln\varepsilon|\big)\big|_{L^2(B(x_0,\rho))} - |W(T|\ln\varepsilon|)|_{L^2(\mathbb{R}^d)}.$$

By Remark 3.16 page 47, there holds

$$\|W(T|\ln\varepsilon|)|_{L^2} \lesssim \varepsilon^{K+(\eta_1-T_1\delta_{\varphi_0})/2-T\gamma_+}.$$

If we now make sure, by choice of h, ρ, that

$$(3.65) \qquad\qquad \gamma^+ - \gamma^- < \frac{\eta_1 - \delta_{\varphi_0}T_1}{2T_1},$$

then the exponent $K-T\gamma^-$ in the lower bound for $V(T|\ln\varepsilon|)$ is strictly smaller than the exponent $K+(\eta_1-\delta_{\varphi_0}T_1)/2-T\gamma^+$ in the upper bound for $W(T|\ln\varepsilon|)$, and the above shows that $|\check{U}(T|\ln\varepsilon|)|_{L^2(B(x_0,\rho))}$ enjoys the same lower bound (3.64) as V.

Still going up the chain of changes of variables, we arrive by (3.17) at

$$|U|_{L^2(B(x_0,\rho))} \geq |\check{U}|_{L^2(B(x_0,\rho))} - \varepsilon^{1/2}|R_0\check{U}|_{L^2(\mathbb{R}^d)}.$$

Since $|R_0\check{U}|_{L^2} \lesssim |V,W|_{L^2}$, it suffices to use Proposition 3.14 again. This time there is no need to further shrink our parameters, and we obtain that U satisfies the same lower bound as V.

Finally, by Remark 3.12, the leading term in $V(T|\ln\varepsilon|)$ is the first component $V_1(T|\ln\varepsilon|) \in \mathbb{C}^N$, in the sense that all other components are smaller by a

factor $\varepsilon^{1/2}$, so that $V_1(T|\ln\varepsilon|)$ enjoys the lower bound (3.64). The same is true for $\check{U}_1(T|\ln\varepsilon|)$, and finally for U_1, which shows that

$$(3.66) \qquad |\dot{u}(\sqrt{\varepsilon}t)|_{L^2(B(x_0,\rho))} \geq C e^{K-T\gamma^-}.$$

The lower bound (3.66) implies the deviation estimate (2.16), and concludes the proof of the instability statement in Theorem 2.7, in the case (3.59).

In the case

$$\frac{K}{|B|_0|\hat{a}|_{L^1}} < \frac{K-d/2}{\gamma|a|_{L^\infty}}, \quad \text{so that} \quad K_0 = d/2,$$

we use the bounds of Section 3.1.5.2 instead of the bounds of Section 3.1.5.1, and arrive at (2.16) in exactly the same fashion as above.

REMARK 3.18. — The choice of parameters is made in the following order: under (3.59), given $K' > K_0$, we choose T_1 so that (3.62) holds for all $\rho < \rho_0(T_1)$. Associated with this T_1, we have η_1 defined in (3.46). Depending on η_1, T_1 and γ, we choose φ_0 so that the conditions on δ_{φ_0} that are formulated in the proofs of Propositions 3.14 and 3.17 hold. Then, we choose h and $\rho < \rho_0(T_1)$, so that (3.63) and (3.65) hold. From there, the final deviation estimate (2.16) holds if ε is small enough, depending on all the other parameters.

3.2. Proof of Theorem 2.7: stability

We assume $\boldsymbol{\Gamma} < 0$ and define a perturbation unknown by

$$(3.67) \qquad u =: u_a + \varepsilon^\kappa \dot{u}, \quad \text{with } (1+d)/2 \leq \kappa \leq \min(K, K_a).$$

In a first step, we follow closely the analysis of Section 3.1. The unknown $\dot{u}$ satisfies

$$\partial_t \dot{u} + \frac{1}{\varepsilon}A_0\dot{u} + \sum_{1 \leq j \leq d} A_j \partial_{x_j} \dot{u} = \frac{1}{\sqrt{\varepsilon}}B(u_a)\dot{u} + \varepsilon^{\kappa-1/2}B(\dot{u},\dot{u}) - \varepsilon^{K_a-\kappa}r_a^\varepsilon.$$

By the change of variables (3.3), we arrive at (3.6), and verify as in the proof of Lemma 3.1 that the source term F satisfies the bound

$$\|F\|_{\varepsilon,s} \lesssim \left(1 + \varepsilon^{\kappa-1/2}|\dot{u}|_{L^\infty}\right)\|\dot{u}\|_{\varepsilon,s} + \varepsilon^{K_a-\kappa}.$$

By the Sobolev embedding $|\dot{u}|_{L^\infty} \leq C\varepsilon^{-d/2}\|\dot{u}\|_{\varepsilon,s}$, and $(1+d)/2 \leq \kappa$, this yields

$$(3.68) \qquad \|F\|_{\varepsilon,s} \lesssim (1 + \|\dot{u}\|_{\varepsilon,s})\|\dot{u}\|_{\varepsilon,s} + \varepsilon^{K_a-\kappa}.$$

Then we perform a normal form reduction as in Section 3.1.2.2. By Assumption 2.6, Proposition 3.3 page 33 holds true. This gives a symbol Q, by which we define

$$\check{U}(t) := \left(\text{Id} + \sqrt{\varepsilon}\text{op}_\varepsilon(Q(t))\right)^{-1}U(t),$$

corresponding to (3.17) without the rescaling in time. Indeed, we prove here stability in time $O(1)$, whereas the instability analysis of Section 3.1 takes place in short time $O(\sqrt{\varepsilon}|\ln\varepsilon|)$.

As in Corollary 3.5, we find that the equation in $\check{U}$ is

$$\partial_t \check{U} + \frac{i}{\varepsilon}\text{op}_\varepsilon(\mathcal{A})\check{U} = \frac{1}{\sqrt{\varepsilon}}\text{op}_\varepsilon(\check{\mathcal{B}})\check{U} + \check{F}, \qquad \check{\mathcal{B}} := \begin{pmatrix} \chi_0\mathcal{B}^r & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathcal{A}$ is defined in (3.7) page 30, $\mathcal{B}^r$ in (3.11) page 32, and $\check{F}$ satisfies bound (3.68). The compactly supported frequency cut-off χ_0, introduced in Section 3.1.2.2 page 32, is identically equal to one in a neighborhood of the resonant set $\mathcal{R}_{12}$.

3.2.1. Symmetrizer. — By Assumption 2.2, there holds $u_{0,1} = g(t,x)\vec{e}_1$, and $u_{0,-1} = g(t,x)^*\vec{e}_{-1}$, with $g \in \mathbb{C}$ and constant vectors $\vec{e}_{\pm 1}$. In particular, the symbol $\mathcal{B}^r$, defined in (3.11) page 32, appears as

$$\mathcal{B}^r = \begin{pmatrix} 0 & g(t,x)b_{12}^+(\xi) \\ g(t,x)^*b_{21}^-(\xi) & 0 \end{pmatrix},$$

where b_{12}^+ and b_{21}^- (3.28) are the interaction coefficient associated with resonance $(1,2)$. Since $\mathbf{\Gamma} < 0$, going back to the definition of $\mathbf{\Gamma}$ in (2.10)-(2.11), we see that $\Gamma(\xi) < 0$ on the whole resonant set $\mathcal{R}_{12}$. In particular, if the support of χ_1 is small enough (that is, contains $\mathcal{R}_{12}$ and not much more), there holds

$$(3.69) \qquad \Gamma(\xi) = \text{tr}\, b_{12}^+b_{21}^-(\xi) < 0, \qquad \text{for all } \xi \text{ in the support of } \chi_1.$$

LEMMA 3.19. — *Let $C_{12}, C_{21} : \xi \in \Omega \to C_{12}(\xi), C_{21}(\xi) \in \mathbb{C}^{N\times N}$ be smooth families of matrices defined in a bounded open set $\Omega \subset \mathbb{R}^d$, and such that*

$$(3.70) \qquad \text{rank}\, C_{12} \equiv \text{rank}\, C_{21} \equiv 1 \quad and \quad \text{tr}\, C_{12}C_{21} \neq 0, \qquad \xi \in \bar{\Omega}.$$

Then, there exists smooth scalar maps c_{12}, c_{21} and a block-diagonal, smooth family of change of basis $P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22,} \end{pmatrix}$ such that $\text{tr}\, C_{12}C_{21} = c_{12}c_{21}$, and, given complex numbers $\nu_{12}, \nu_{21} \in \mathbb{C}$:
$$(3.71)$$
$$\begin{pmatrix} 0 & \nu_{12}C_{12} \\ \nu_{21}C_{21} & 0 \end{pmatrix} = P^{-1}\begin{pmatrix} 0 & \tilde{C}_{12} \\ \tilde{C}_{21} & 0 \end{pmatrix}P, \qquad \tilde{C}_{ij} = \begin{pmatrix} \nu_{ij}c_{ij} & 0 \\ 0 & 0_{\mathbb{C}^{(N-1)\times(N-1)}} \end{pmatrix},$$

with

$$(3.72) \qquad \sup_{\xi\in\Omega}|\partial_\xi^\alpha P(\xi)| + |\partial_\xi^\alpha P^{-1}(\xi)| < \infty, \qquad \alpha \in \mathbb{N}^d.$$

Proof. — In a first step, we work with fixed $\xi \in \Omega$. Given a vector $x \in \mathbb{C}^N$, we denote $x^\sharp = (x,0) \in \mathbb{C}^{2N}$ and $x_\sharp = (0,x) \in \mathbb{C}^{2N}$.

By $\text{rank}\, C_{12}C_{21} \leq \text{rank}\, C_{21} = 1$ and $\text{tr}\, C_{12}C_{21} \neq 0$, there holds $\text{rank}\, C_{12}C_{21} = 1$, and for some $\lambda_{12} \neq 0$ and some $e \in \mathbb{C}^N$, there holds $C_{12}C_{21}e = \lambda_{12}e$. The vector e generates the range of C_{12}.

By symmetry, the same holds for $C_{21}C_{12}$: for some vector $f \in \mathbb{C}^N$, there holds $C_{21}C_{12}f = \lambda_{21}f$, with $\lambda_{12} \neq 0$. The vector f generates the range of C_{21}.

Besides, $\operatorname{rank} C_{12}C_{21} = \operatorname{rank} C_{21}$ implies $\dim \ker C_{12}C_{21} = \dim \ker C_{21} = N - 1$, hence equality of the kernels: $\ker C_{21} = \ker C_{12}C_{21}$. Denoting $\{a_1, \cdots, a_{N-1}\}$ a basis of $\ker C_{21}$, we find that $\{e, a_1, \cdots, a_{N-1}\}$ is a basis of $\mathbb{C}^N$, since $e \notin \ker C_{21}$. Then the family $\{e^\sharp, a_1^\sharp, \cdots, a_{N-1}^\sharp\}$ is a basis of $\mathbb{C}^N \times \{0\}$.

Similarly, denoting $\{b_1, \cdots, b_{N-1}\}$ a basis of $\ker C_{12} = \ker C_{21}C_{12}$, since f does not belong to $\ker C_{12}$, the family $\{f, b_1, \ldots, b_{N-1}\}$ is a basis of $\mathbb{C}^N$, and the family $\{f_\sharp, b_{1,\sharp}, \cdots, b_{n-1,\sharp}\}$ is a basis of $\{0\} \times \mathbb{C}^N$.

Consider now $C_{21}e$ and $C_{12}f \in \mathbb{C}^N$. The vector $C_{21}e$ belongs to the range of C_{21}, hence it is colinear to f. There cannot be $C_{21}e = 0$, since then $C_{12}C_{21}e = 0$, which does not hold. Hence $C_{21}e = c_{21}f$, for some $c_{21} \neq 0$. Similarly, $C_{12}f = c_{12}e$, with $c_{12} \neq 0$.

In particular, $C_{12}C_{21}e = c_{12}c_{21}e$, so that $\operatorname{tr} C_{12}C_{21} = c_{12}c_{21}$.

Then, given the matrix $\underline{C} := \begin{pmatrix} 0 & \nu_{12}C_{12} \\ \nu_{21}C_{21} & 0 \end{pmatrix}$, there holds the identity $\underline{C}a_i^\sharp = \nu_{21}(0, C_{21}a_i) = 0$, and similarly, $\underline{C}b_{i\sharp} = \nu_{12}(C_{12}b_i, 0) = 0$. Besides, there holds $\underline{C}e^\sharp = \nu_{21}(0, C_{21}e) = \nu_{21}c_{21}f_\sharp$, and $\underline{C}f_\sharp = \nu_{12}(C_{12}f, 0) = \nu_{12}c_{12}e^\sharp$.

The above implies that for the matrix P defined by columns as

$$(3.73) \qquad P = \operatorname{col}\left(e^\sharp, a_1^\sharp, \cdots, a_{n-1}^\sharp, f_\sharp, b_{1\sharp}, \cdots, b_{n-1\sharp}\right)$$

there holds (3.71).

The trace of $C_{12}C_{21}$ is bounded away from 0 on the compact $\bar{\Omega}$. It is also equal to λ_{12}. This means that the image and kernel of $C_{12}C_{21}$ are strictly separated over the domain $\bar{\Omega}$, implying smoothness of the projection onto the kernel and parallel to the image. This, in turn, implies existence of a smooth basis of the kernel (see Kato's treatise [**26**], Section II.4.2). Since $\ker C_{12}C_{21} = \ker C_{21}$, this means that we can choose the a_i to vary smoothly over Ω. Similarly, we can choose the b_i to vary smoothly over Ω.

We can also choose e and f to vary smoothly in ξ, since these are eigenvectors associated to simple eigenvalues. Then P is smooth, and everywhere invertible, with determinant bounded away from 0 on $\bar{\Omega}$. This gives regularity of P^{-1} by the comatrix formula, which translates into estimate (3.72). Finally, the c_{ij} are smooth by consideration of (3.71). $\qquad \square$

Assumption 2.6(iii) ensures that the rank condition (3.70) is satisfied by the matrices $C_{12} = b_{12}^+$, $C_{21} = b_{21}^-$. Thus we apply Lemma 3.19 to these matrices, with coefficients $\nu_{12} = g(t, x)$, $\nu_{21} = g(t, x)^*$, and $\operatorname{supp} \chi_1 = \bar{\Omega}$. This gives a change of basis P. We coordinatize

$$\check{U} =: (\check{U}_{12}, W_2^{(\mathrm{s})}) \in \mathbb{C}^{2N} \times \mathbb{C}^{(J-2)N},$$

and let

$$V^{(\mathrm{s})} := \operatorname{op}_\varepsilon(\chi_1 P^{-1})\check{U}_{12}, \quad W_1^{(\mathrm{s})} := \operatorname{op}_\varepsilon(1 - \chi_1)\check{U}_{12}, \quad W^{(\mathrm{s})} = (W_1^{(\mathrm{s})}, W_2^{(\mathrm{s})}).$$

so that
$$\check{U}_{12} = \mathrm{op}_\varepsilon(P)V^{(\mathrm{s})} + W_1^{(\mathrm{s})}.$$

Here the exponent (s) indicates that these local unknowns are used only in this stability proof, and distinguishes these from unknowns V, W_1, W_2 in the instability proof. This is relatively heavy notation, but we will not carry it very far.

LEMMA 3.20. — *There holds*

$$(3.74) \qquad \begin{cases} \partial_t V^{(\mathrm{s})} + \dfrac{1}{\varepsilon}\mathrm{op}_\varepsilon(\mathbf{M}_{12})V^{(\mathrm{s})} = F_V, \\[2mm] \partial_t W^{(\mathrm{s})} + \dfrac{1}{\varepsilon}\mathrm{op}_\varepsilon(i\mathbf{A})W^{(\mathrm{s})} = F_W, \end{cases}$$

where

$$\mathbf{M}_{12} := \begin{pmatrix} i(\lambda_{1,+1} - \omega) & -\sqrt{\varepsilon}g\chi_0\mathbf{b}_{12}(\xi) \\ -\sqrt{\varepsilon}g^*\chi_0\mathbf{b}_{21}(\xi) & i\lambda_2 \end{pmatrix}, \qquad \mathbf{b}_{ij} := \begin{pmatrix} m_{ij} & 0 \\ 0 & 0_{\mathbb{C}^{(N-1)\times(N-1)}} \end{pmatrix},$$

with $m_{ij}(\xi)$ such that

$$\Gamma(\xi) = \mathrm{tr}\,\Pi_1(\xi+k)B(\vec{e}_1)\Pi_2(\xi)B(\vec{e}_{-1})\Pi_1(\xi+k) = m_{12}(\xi)m_{21}(\xi).$$

The diagonal Fourier multiplier $\mathbf{A}_0$ is

$$\mathbf{A}_0 := \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A}_{3,J} \end{pmatrix}, \qquad \mathcal{A}_{3,J} := \mathrm{diag}(\lambda_3, \ldots, \lambda_J).$$

The sources F_V and F_W satisfy bound (3.68).

Proof. — The change of basis P being block-diagonal (3.73), there holds

$$P\begin{pmatrix} \lambda_{1,+1} - \omega & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_{1,+1} - \omega & 0 \\ 0 & \lambda_2 \end{pmatrix}P.$$

Besides, by estimate (6.6):

$$\mathrm{op}_\varepsilon(\chi_1 P)\mathrm{op}_\varepsilon(\chi_0\mathcal{B}^r) = \mathrm{op}_\varepsilon(\chi_0 P\mathcal{B}^r) + \varepsilon R_0,$$

where R_0 is a uniform remainder in the sense of Definition 3.2, so that

$$\mathrm{op}_\varepsilon(\chi_1 P)\mathrm{op}_\varepsilon(\chi_0\mathcal{B}^r)\check{U}_{12} = \mathrm{op}_\varepsilon(\chi_0 P\mathcal{B}^r P^{-1})V^{(\mathrm{s})} + \mathrm{op}_\varepsilon(\chi_0 P\mathcal{B}^r)W_1^{(\mathrm{s})} + \varepsilon R_0(V^{(\mathrm{s})} + W_1^{(\mathrm{s})}).$$

But then

$$\mathrm{op}_\varepsilon(\chi_0 P\mathcal{B}^r)W_1^{(\mathrm{s})} = \mathrm{op}_\varepsilon(\chi_0 P\mathcal{B}^r)\mathrm{op}_\varepsilon(1 - \chi_1)\check{U}_{12} = \varepsilon R_0\check{U}_{12},$$

since $\chi_0(1 - \chi_1) \equiv 0$. By Lemma 3.19, there holds for ξ in the support of χ_1 the identity

$$P\mathcal{B}^r P^{-1} = \begin{pmatrix} 0 & g\mathbf{b}_{12} \\ g^*\mathbf{b}_{21} & 0 \end{pmatrix}.$$

The above verifies the form of the equation in $V^{(\mathrm{s})}$. For the equation in $W_1^{(\mathrm{s})}$, we use

$$\mathrm{op}_\varepsilon(1 - \chi_1)\mathrm{op}_\varepsilon(\chi_0\mathcal{B}^r) = \mathrm{op}_\varepsilon\big((1 - \chi_1)\chi_0\mathcal{B}^r\big) + \varepsilon R_0 = \varepsilon R_0,$$

by (6.6) and $(1 - \chi_1)\chi_0 \equiv 0$. $\qquad\square$

The symmetrizer is defined as the Fourier multiplier

$$\mathbf{S}(\xi) := \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & -m_{12}^*(\xi)m_{21}(\xi)^{-1}\mathrm{Id}_N \end{pmatrix}, \quad \text{for } \xi \in \mathrm{supp}\,\chi_1,$$

where m_{12}^* denotes complex conjugate of m_{12}. We then choose any extension to all of $\mathbb{R}_\xi^d$ so that $\mathbf{S}$ is smooth, real diagonal, with diagonal entries bounded and bounded away from zero. We note that since $m_{12}m_{21} \in \mathbb{R}$, there holds $m_{12}^*/m_{21} \in \mathbb{R}$, so that, in particular, $\mathbf{S}^* \equiv \mathbf{S}$.

The coefficients m_{ij} are bounded on the support of χ_1. Besides, by (3.69) and the fact that $\Gamma = m_{12}m_{21}$, they are bounded away from zero on the support of χ_1. This implies the bounds

$$(3.75) \qquad |u|_{L^2} \lesssim |\mathrm{op}_\varepsilon(\mathbf{S}^{1/2})u|_{L^2}, \qquad u \in L^2,$$

and

$$(3.76) \qquad \|\mathrm{op}_\varepsilon(\mathbf{S})u\|_{\varepsilon,s} \lesssim \|u\|_{\varepsilon,s}, \quad u \in H^s.$$

The fact that $\mathbf{S}$ is a symetrizer is expressed in the following lemma:

LEMMA 3.21. — *There holds for all $u \in H^s$ the bound*

$$(3.77) \qquad \Re e\left(\mathrm{op}_\varepsilon(\mathbf{S})\mathrm{op}_\varepsilon(\mathbf{M}_{12})u + \left(\mathrm{op}_\varepsilon(\mathbf{S})\mathrm{op}_\varepsilon(\mathbf{M}_{12})\right)^* u,\ u\right)_{L^2} \lesssim \varepsilon\|u\|_{L^2}^2.$$

Recall that if $z \in \mathbb{C}$, then z^* denotes complex conjugate, if $z \in \mathbb{C}^{N \times N}$, then z^* denotes complex transpose, and if z is linear bounded $L^2 \to L^2$, then z^* denotes the adjoint operator. We use the latter in (3.77), and all three in the forthcoming proof.

Proof. — We compute

$$\mathrm{op}_\varepsilon(\mathbf{S})\mathrm{op}_\varepsilon \begin{pmatrix} i(\lambda_{1,+1} - \omega) & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathrm{op}_\varepsilon \begin{pmatrix} i(\lambda_{1,+1} - \omega) & 0 \\ 0 & i\lambda_2 \end{pmatrix} \mathrm{op}_\varepsilon(\mathbf{S}),$$

and

$$\mathrm{op}_\varepsilon \begin{pmatrix} i(\lambda_{1,+1} - \omega) & 0 \\ 0 & \lambda_2 \end{pmatrix}^* = -\mathrm{op}_\varepsilon \begin{pmatrix} i(\lambda_{1,+1} - \omega) & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \mathrm{op}_\varepsilon(\mathbf{S})^* = \mathrm{op}_\varepsilon(\mathbf{S}),$$

so that the diagonal entries of $\mathbf{M}_{12}$ contribute nothing to (3.77). Next we compute

$$\mathrm{op}_\varepsilon(-m_{12}^*m_{21}^{-1})\mathrm{op}_\varepsilon(\chi_0 g^*\mathbf{b}_{21}) = g^*\mathrm{op}_\varepsilon(\chi_0\mathbf{b}_{12}^*) + \varepsilon R_0,$$

via (6.6) and definition of $\mathbf{b}_{ij}$. This implies

$$(3.78) \quad \mathrm{op}_\varepsilon(\mathbf{S})\mathrm{op}_\varepsilon \begin{pmatrix} 0 & g\chi_0\mathbf{b}_{12} \\ g^*\chi_0\mathbf{b}_{21} & 0 \end{pmatrix} = \mathrm{op}_\varepsilon \begin{pmatrix} 0 & g\chi_0\mathbf{b}_{12} \\ -g^*\chi_0\mathbf{b}_{12}^* & 0 \end{pmatrix} + \varepsilon R_0.$$

Besides, using (6.6) once more,

$$\mathrm{op}_\varepsilon \begin{pmatrix} 0 & g\chi_0\mathbf{b}_{12} \\ g^*\chi_0\mathbf{b}_{21} & 0 \end{pmatrix}^* = \mathrm{op}_\varepsilon \begin{pmatrix} 0 & g\chi_0\mathbf{b}_{21}^* \\ g^*\chi_0\mathbf{b}_{12}^* & 0 \end{pmatrix} + \varepsilon R_0,$$

so that

$$(3.79) \quad \mathrm{op}_\varepsilon \begin{pmatrix} 0 & g\chi_0 \mathbf{b}_{12} \\ g^*\chi_0 \mathbf{b}_{21} & 0 \end{pmatrix}^* \mathrm{op}_\varepsilon(\mathbf{S})^* = \mathrm{op}_\varepsilon \begin{pmatrix} 0 & -g\chi_0 \mathbf{b}_{12} \\ g^*\chi_0 \mathbf{b}_{12}^* & 0 \end{pmatrix} + \varepsilon R_0.$$

With (3.78) and (3.79), the contribution of the extra-diagonal entries of $\mathbf{M}_{12}$ to the left-hand side of (3.77) has the form $\varepsilon \Re e \, (R_0 u, u)_{L^2}$. This concludes the proof. □

3.2.2. Uniform bounds. — We use the Fourier multiplier $\Lambda^s = \mathrm{op}_\varepsilon \big((1 + |\cdot|^2)^{s/2} \big)$, and compute

$$\partial_t \Big(\big| \mathrm{op}_\varepsilon(\mathbf{S}^{1/2}) \Lambda^s V^{(\mathrm{s})} \big|_{L^2}^2 \Big) = 2 \Re e \, \big(\mathrm{op}_\varepsilon(\mathbf{S}) \Lambda^s \partial_t V^{(\mathrm{s})}, \Lambda^s V^{(\mathrm{s})} \big)_{L^2}.$$

Following (3.74), the above right-hand side decomposes into two terms. The first is

$$-\frac{2}{\varepsilon} \Re e \, \Big(\mathrm{op}_\varepsilon(\mathbf{S}) \mathrm{op}_\varepsilon(\mathbf{M}_{12}) \Lambda^s V^{(\mathrm{s})}, \Lambda^s V^{(\mathrm{s})} \Big)_{L^2}$$

$$= -\frac{2}{\varepsilon} \Big(\mathrm{op}_\varepsilon(\mathbf{S}) \mathrm{op}_\varepsilon(\mathbf{M}_{12}) \Lambda^s V^{(\mathrm{s})} + \big(\mathrm{op}_\varepsilon(\mathbf{S}) \mathrm{op}_\varepsilon(\mathbf{M}_{12}) \big)^* \Lambda^s V^{(\mathrm{s})}, \Lambda^s V^{(\mathrm{s})} \Big)_{L^2},$$

and, by Lemma 3.21, is controlled by $\|V^{(\mathrm{s})}\|_{\varepsilon,s}^2$. The second term is

$$\Big| 2 \Re e \, \big(\mathrm{op}_\varepsilon(\mathbf{S} \Lambda^s F_V, \Lambda^s V^{(\mathrm{s})} \big)_{L^2} \Big| \lesssim \|F_V\|_{\varepsilon,s} \|V^{(\mathrm{s})}\|_{\varepsilon,s}$$

$$\lesssim \Big((1 + \|\dot u\|_{\varepsilon,s}) \|\dot u\|_{\varepsilon,s} + \varepsilon^{K_a - \kappa} \Big) \|V^{(\mathrm{s})}\|_{\varepsilon,s},$$

the first inequality by (3.76) and the second by estimate (3.68). Gathering the above estimates and using $\|\dot u\|_{\varepsilon,s} \lesssim \|V^{(\mathrm{s})}, W^{(\mathrm{s})}\|_{\varepsilon,s}$ and $K_a - \kappa \geq 0$, we obtain

$$(3.80) \qquad \partial_t \big(\big| \mathrm{op}_\varepsilon(\mathbf{S}^{1/2}) \Lambda^s V^{(\mathrm{s})} \big|_{L^2}^2 \big) \lesssim \|V^{(\mathrm{s})}\|_{\varepsilon,s} + \|V^{(\mathrm{s})}, W^{(\mathrm{s})}\|_{\varepsilon,s}^2.$$

Besides, from (3.74) we deduce

$$(3.81) \qquad \partial_t (\|W^{(\mathrm{s})}\|_{\varepsilon,s}^2) \lesssim \|F_W\|_{\varepsilon,s} \|W^{(\mathrm{s})}\|_{\varepsilon,s} \lesssim \|W^{(\mathrm{s})}\|_{\varepsilon,s} + \|V^{(\mathrm{s})}, W^{(\mathrm{s})}\|_{\varepsilon,s}^2.$$

From (3.80)-(3.81) and the lower bound (3.75), we deduce the stability estimate

$$\|V^{(\mathrm{s})}, W^{(\mathrm{s})}\|_{\varepsilon,s} \leq \varepsilon^{K-\kappa} C(T_a), \qquad 0 \leq t \leq T_a.$$

The stability estimate (2.17) then follows from $\|\dot u\|_{\varepsilon,s} \lesssim \|V^{(\mathrm{s})}, W^{(\mathrm{s})}\|_{\varepsilon,s}$, and definition of $\dot u$ in (3.67).

CHAPTER 4

OTHER PROOFS

4.1. Proof of Theorem 2.9

The differences with the main proof (Sections 3.1 and 3.2) are essentially notational.

An issue that we face right away as we consider more than one non-transparent resonance is the definition of the frequency-shifted and projected variable U. Consider indeed the case of a set $\mathfrak{R}_0$ of non-transparent resonances equal to

$$\mathfrak{R}_0 = \{(1,2), (2,3), (3,1)\}.$$

It is easy to see that shifted and projected variables, following (3.3), are not appropriate. That is, if we let $U_1 = e^{-i\theta}\mathrm{op}_\varepsilon(\Pi_1)\dot{u}$ and $U_2 = \mathrm{op}_\varepsilon(\Pi_2)\dot{u}$ to account for the $(1,2)$ resonance, and then $U_3 = e^{i\theta}\mathrm{op}_\varepsilon(\Pi_3)\dot{u}$ to account for the $(2,3)$ resonance, then the frequency shifts in U_1 and U_3 are not suitable for the $(3,1)$ resonance.

We overcome this issue by localizing the definitions of the projected variables (Section 4.1.1). This is relatively straightforward, but notations are heavy.

Then, by taking advantage of the partial transparency hypothesis (Assumption 2.8(ii)), all couplings are eliminated, except for those describing non-transparent resonances. This is done in Section 4.1.2.

No further difficulty arises, and the estimates, both in the stable and unstable case, are similar to the estimates in Sections 3.1 and 3.2.

4.1.1. Coordinatization. — The perturbation variable $\dot{u}$ is defined by

$$u =: u_a + \varepsilon^\kappa \dot{u},$$

with $\kappa = 0$ in the unstable case, as in (3.1), and $(1+d)/2 \leq \kappa \leq \min(K, K_a)$ in the stable case, as in (3.67).

Associated with $(i,j) \in \mathfrak{R}_0$, we define smooth frequency cut-offs χ_{ij}, $\chi_{ij}^\sharp$, such that $\chi_{ij} \prec \chi_{ij}^\sharp$ (in the sense of Notation 3.7 page 36), with $\chi_{ij} \equiv 1$ on $\mathcal{R}_{ij}$, and such that the support of $\chi_{ij}^\sharp$ is a small neighborhood of $\mathcal{R}_{ij}$. All truncations are compactly

supported (owing to Assumption 2.8(i)) and take values in $[0,1]$. The normal form reduction of Section 4.1.2 will require the supports of $\chi_{ij}^{\sharp}$ to be not much larger than $\mathcal{R}_{ij}$.

Let i such that $(i,j) \in \mathfrak{R}_0$ for some j. We let $I_i^+ = \{j \in [1,J], (i,j) \in \mathfrak{R}_0\}$. A local variable associated with resonance (i,j) is defined by

$$(4.1) \qquad u_{ij}^+ := \mathrm{op}_\varepsilon(\chi_{ij}) e^{-i\theta} \mathrm{op}_\varepsilon(\Pi_i)\dot{u}, \qquad j \in I_i^+.$$

We let $I_i^- = \{j' \in [1,J], (j',i) \in \mathfrak{R}_0\}$, so that if i is such that $(j',i) \in \mathfrak{R}_0$ for some index j', then I_i^- is not empty. A local variable is defined by

$$(4.2) \qquad u_{ij'}^- := \mathrm{op}_\varepsilon(\chi_{j'i}) \mathrm{op}_\varepsilon(\Pi_i)\dot{u}, \qquad j' \in I_i^-.$$

We now let

$$(4.3) \qquad v_i := \mathrm{op}_\varepsilon\Big(1 - \sum_{j \in I_i^+}(\chi_{ij})_{-1} - \sum_{j' \in I_i^-}\chi_{j'i}\Big)\mathrm{op}_\varepsilon(\Pi_i)\dot{u},$$

where $(\chi_{ij})_{-1} := \chi_{ij}(\cdot - k)$, in accordance with (3.8), so that

$$(4.4) \qquad \mathrm{op}_\varepsilon(\Pi_i)\dot{u} = v_i + \sum_{j \in I_i^+} e^{i\theta} u_{ij}^+ + \sum_{j' \in I_i^-} u_{ij'}^-,$$

and of course $\dot{u}$ is then reconstructed by summation of (4.4) over $i \in [1,J]$. By convention, sums over empty sets are equal to zero.

REMARK 4.1. — If $(1,2) \in \mathcal{R}_0$, then the variables that describe the $(1,2)$ resonance are u_{12}^+ and u_{21}^-. Indeed, $2 \in I_1^+$ and $1 \in I_2^-$. Thus u_{12}^+ and u_{21}^- will play here the role played by variables U_1 and U_2 in Section 3.1.2.1.

We now introduce a more compact notation. Variables are indexed by resonant indices (i,j), the position $+$ or $-$ in the resonance (meaning first or second term: i or j in (i,j)), and also the nature of the variable: "inner" variables are denoted by the letter u and "outer" variables are denoted v.

We introduce the set of indices

$$(4.5) \qquad \mathbb{A} := \big\{(i,\mathrm{out}),\ i \in [1,J]\big\} \bigcup \big\{(i,j,p,\mathrm{in}),\ (i,j) \in \mathfrak{R}_0,\ p \in \{+,-\}\big\}.$$

The associated local variables are $(\mathbf{u}_\alpha)_{\alpha \in \mathbb{A}}$:

$$\mathbf{u}_\alpha := \begin{cases} u_{ij}^{\pm}, & \alpha = (i,j,\pm,\mathrm{in}) \\ v_i, & \alpha = (i,\mathrm{out}). \end{cases}$$

We denote $\boldsymbol{\chi}_\alpha$ the truncation in $\mathbf{u}_\alpha$; this is the truncation that appears in (4.1) or (4.2) or (4.3), explicitly:

$$\boldsymbol{\chi}_\alpha := \begin{cases} \chi_{ij}, & \alpha = (i,j,+,\mathrm{in}), \\ \chi_{ji}, & \alpha = (i,j,-,\mathrm{in}), \\ \displaystyle 1 - \sum_{j \in I_i^+}(\chi_{ij})_{-1} - \sum_{j' \in I_i^-}\chi_{j'i}, & \alpha = (i,\mathrm{out}). \end{cases}$$

With the notation

$$\delta_\alpha := \begin{cases} 1, & \alpha = (i, j, +, \mathrm{in}), \text{ for some } (i, j), \\ 0, & \text{otherwise}, \end{cases}$$

it appears from (4.1)-(4.2)-(4.3) that there holds

(4.6) $$\mathbf{u}_\alpha = \mathrm{op}_\varepsilon\big(\chi_\alpha^\sharp \Pi_{i, +\delta_\alpha}\big)\mathbf{u}_\alpha, \qquad \chi_\alpha \prec \chi_\alpha^\sharp.$$

The derivation of the coupled system satisfied by the $\mathbf{u}_\alpha$ is essentially identical to the computations of Section 3.1.2.1. We find

(4.7) $$\partial_t \mathbf{u}_\alpha + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon(i\boldsymbol{\mu}_\alpha)\mathbf{u}_\alpha = \frac{1}{\sqrt{\varepsilon}} \sum_{p=\pm 1} \sum_{\beta \in \mathbb{A}} \mathrm{op}_\varepsilon(\mathbf{B}_{p\alpha\beta})\mathbf{u}_\beta + \mathbf{F}_\alpha,$$

where

 – the symbols of the propagators are

$$\boldsymbol{\mu}_\alpha := \begin{cases} \lambda_{i, +1} - \omega, & \alpha = (i, j, +, \mathrm{in}), \\ \lambda_i, & \alpha \in \{(i, j, -, \mathrm{in}), (i, \mathrm{out})\}; \end{cases}$$

 – using (4.4) and (4.6), we find that the symbols of the coupling terms are

(4.8) $\mathbf{B}_{p\alpha\beta} := e^{i(p-\delta_\alpha+\delta_\beta)\theta}B_{p\alpha\beta}, \qquad B_{p\alpha\beta} := \chi_{\alpha, +(p-\delta_\alpha+\delta_\beta)}\chi_\beta^\sharp \Pi_{i, +(p+\delta_\beta)}B_p\Pi_{i', +\delta_\beta},$

for $p \in \{-1, 1\}$, $(\alpha, \beta) \in \mathbb{A} \times \mathbb{A}$, where B_p is defined in (3.9) page 31;

 – as in Lemma 3.1 page 31, the source term $\mathbf{F}_\alpha$ satisfies

(4.9) $$\|\mathbf{F}_\alpha\|_{\varepsilon, s} \lesssim (1 + \varepsilon^{\kappa-1/2}|\dot{u}|_{L^\infty})\|\dot{u}\|_{\varepsilon, s} + \varepsilon^{K_a - \kappa}.$$

4.1.2. Normal form reduction. — The goal is to derive from (4.7) a reduced system in which non-resonant and non-transparent coupling terms do not appear. This is done by a change of variables that is very much similar to the one conducted in Section 3.1.2.2, and now described in some detail. The difference with Section 3.1.2.2 is that in the current context of several non-transparent resonances, a partial transparency assumption bearing on non-transparent resonances (Assumption 2.8(ii)) has to be introduced in order for the normal form reduction to go through as before.

Recall that R_0 denotes uniform remainders in the sense of Definition 3.2. Here in a slight misuse of notation we also denote R_0 any *symbol* such that the associated pseudo-differential operators are uniform remainders in the sense of Definition 3.2.

PROPOSITION 4.2. — *Under Assumption 2.8(ii), given $(\alpha, \beta) \in \mathbb{A} \times \mathbb{A}$, $p \in \{-1, 1\}$, equation*

(4.10) $$i\big(-\ell\omega + \boldsymbol{\mu}_{\alpha, +\ell} - \boldsymbol{\mu}_\beta\big)Q_{\ell\alpha\beta} = B_{p\alpha\beta} + \varepsilon R_0, \qquad \ell = p - \delta_\alpha + \delta_\beta$$

has a solution $Q_{\ell\alpha\beta} \in S^0$ except if (a) $p = 1$, $\alpha = (i, j, 1, in)$, $\beta = (j, i, -1, in)$ *with $(i, j) \in \mathfrak{R}_0$, or* (b) $p = -1$, $\alpha = (i, j, -1, in)$, $\beta = (j, i, 1, in)$, *with $(j, i) \in \mathfrak{R}_0$.*

Proof. — Following (3.28), we denote b_{ij}^+ and b_{ji}^- the interaction coefficients associated with a given resonance (i, j) :

$$(4.11) \qquad b_{ij}^+ = \Pi_{i+1} B(\vec{e}_1) \Pi_j, \qquad b_{ji}^- = \Pi_j B(\vec{e}_{-1}) \Pi_{i,+1}.$$

We let $\alpha = (i, j, q, \text{in})$ or $\alpha = (i, \text{out})$, and $\beta = (i', j', q', \text{in})$ or $\beta = (i', \text{out})$. On the complement of the support of $B_{p\alpha\beta}$ a trivial solution to (4.10) is $Q_{\ell\alpha\beta} = 0$, so that it is sufficient to consider (4.10) for frequencies such that $\chi_{\alpha, +(p-\delta_\alpha+\delta_\beta)} \neq 0$ and $\chi_\beta^\sharp \neq 0$. We go over most cases in detail. In all cases Assumption 2.8(ii) is invoked. We call "phase" the scalar $-\ell\omega + \boldsymbol{\mu}_{\alpha, +\ell} - \boldsymbol{\mu}\beta$ in factor of the unknown $Q_{\ell\alpha\beta}$, and "source" the right-hand side $B_{p\alpha\beta}$ in the homological equation (4.10).

- *"in-in" coupling terms:* here $\alpha = (i, j, q, \text{in})$, $\beta = (i', j', q', \text{in})$. There are four subcases.

-- If $q = q' = 1$, then the source terms and phases for $p = 1$ and $p = -1$ are

$$B_{1\alpha\beta} = g(t, x) \chi_{ij, +1} \chi_{i'j'}^\sharp b_{ii', +1}^+, \quad \text{with phase } (\lambda_{i, +1} - \omega - \lambda_{i'})_{+1}, \text{ and}$$

$$B_{-1\alpha\beta} = g(t, x)^* \chi_{ij, -1} \chi_{i'j'}^\sharp b_{ii'}^-, \quad \text{with phase } -(\lambda_{i', +1} - \omega - \lambda_i),$$

respectively. We recall notation from (2.6): $u_{0,1} = g(t, x)\vec{e}_1$, $u_{0,-1} = g(t, x)^* \vec{e}_{-1}$.

If $(i, i') \notin \mathfrak{R}_0$, meaning that (i, i') is a transparent resonance (or is not a resonant pair), then by definition the phase $(\lambda_{i, +1} - \omega - \lambda_{i'})_{+1}$ factorizes in the interaction coefficient $b_{ii', +1}^+$. This implies that the symbol

$$(4.12) \qquad g(t, x) (\lambda_{i, +1} - \omega - \lambda_{i'})_{+1}^{-1} \chi_{ij, +1} \chi_{i'j'}^\sharp b_{ii', +1}^+$$

is bounded, belongs to S^0, and provides a solution to (4.10).

Otherwise $(i, i') \in \mathfrak{R}_0$. By definition of the truncations associated with non-transparent resonances, the phase $\lambda_{i, +1} - \omega - \lambda_{i'}$ is bounded away from zero over the support of $1 - \chi_{ii'}$. Thus for $p = 1$ it suffices to solve (4.10) for the source terms

$$\chi_{ii', +1} B_{1\alpha\beta} = g \chi_{ii', +1} \chi_{ij, +1} \chi_{i'j'}^\sharp b_{ii', +1}^+, \qquad \text{with phase } (\lambda_{i, +1} - \omega - \lambda_{i'})_{+1}.$$

We now invoke the transparency condition (2.18) from Assumption 2.8. Indeed, by (2.18), the interaction coefficient $b_{ii', +1}^+$ is transparent on the support of $\chi_{ii', +1} \chi_{i'j'}^\sharp$, if the supports of these cut-offs are tightly cut around the corresponding resonance sets. Again, this implies that (4.12) is bounded, belongs to S^0 and solves (4.10).

In the case $p = -1$, the same argument applies.

-- If $q = 1$, $q' = -1$, then the source terms are

$$B_{1\alpha\beta} = g \chi_{ij} \chi_{j'i'}^\sharp b_{ii'}^+, \qquad B_{-1\alpha\beta} = g^* \chi_{ij, -2} \chi_{j'i'}^\sharp b_{ii', -1}^-.$$

Again, if $(i, i') \notin \mathfrak{R}_0$ (for $p = -1$, if $(i', i) \notin \mathfrak{R}_0$), the corresponding equation (4.10) is solved by dividing the source by the phase. Thus it suffices to solve (4.10) for

$$\chi_{ii'} B_{1\alpha\beta} = g \chi_{ii'} \chi_{ij} \chi_{j'i'}^\sharp b_{ii'}^+, \qquad\qquad\qquad \text{with phase } \lambda_{i, +1} - \omega - \lambda_{i'};$$

$$\chi_{i'i, -1} B_{-1\alpha\beta} = g \chi_{i'i, -1} \chi_{ij, -2} \chi_{j'i'}^\sharp b_{ii', -1}^-, \qquad \text{with phase } -(\lambda_{i', +1} - \omega - \lambda_i)_{-1},$$

assuming $(i, i') \in \mathfrak{R}_0$ in the case $p = 1$ and $(i', i) \in \mathfrak{R}_0$ in the case $p = -1$.

For $B_{1\alpha\beta}$ $(p = 1)$, we use (2.19) if $j' \neq i$ or $i' \neq j$. Indeed, given $\xi \in \mathcal{R}_{ii'}$, if in addition ξ belongs the support of χ_{ij}, then ξ belongs to a neighborhood of $\mathcal{R}_{ij}$. Since frequencies that belong to $\mathcal{R}_{ii'} \cap \mathcal{R}_{ij}$ necessarily belong to $\{\lambda_{i'} = \lambda_j\}$, we can indeed use the partial transparency condition (2.19).

The remaining case for $p = 1$ is $(i', j') = (j, i)$. Then, there holds $\alpha = (i, j, 1, \mathrm{in})$ and $\beta = (j, i, -1, \mathrm{in})$. But then $(i, i') \notin \mathfrak{R}_0$, otherwise we would be in excluded case (a).

For $B_{-1\alpha\beta}$, we use (2.18) again.

In the remainder of this proof, upon consideration of an interaction coefficient, we will always assume that the relevant resonance, that is (i, i') if $p = 1$ and (i', i) if $p = -1$, is non-transparent. Otherwise the reduction is trivial, meaning that we do not need to appeal to the partial transparency conditions (2.18) and (2.19) and may simply solve (4.10) by dividing the source by the phase, without having to consider the form of the truncation functions.

•• If $q = -1$, $q' = 1$, then the source terms are

$$B_{1\alpha\beta} = g\chi_{ji,+2}\chi_{i'j'}^{\sharp}b_{ii',+1}^{+}, \quad B_{-1\alpha\beta} = g^{*}\chi_{ji}\chi_{i'j'}^{\sharp}b_{ii'}^{-}.$$

It suffices to solve for

$$\chi_{ii',+1}B_{1\alpha\beta} = g\chi_{ii',+1}\chi_{ji,+2}\chi_{i'j'}^{\sharp}b_{ii',+1}^{+}, \qquad \text{with phase } (\lambda_{i,+1} - \omega - \lambda_{i'})_{+1};$$

$$\chi_{i'i}B_{-1\alpha\beta} = g^{*}\chi_{i'i}\chi_{ji}\chi_{i'j'}^{\sharp}b_{ii'}^{-}, \qquad \text{with phase } -(\lambda_{i',+1} - \omega - \lambda_{i}),$$

For the first source term $(p = 1)$, we use (2.18). For the other source term $(p = -1)$ we use (2.19) unless $(i', j') = (j, i)$, in which case $\alpha = (i, j, -1, \mathrm{in})$ and $\beta = (j, i, 1, \mathrm{in})$. Then, $(j, i) \notin \mathfrak{R}_0$, otherwise we would be in excluded case (b). Hence the interaction coefficient $b_{i'i}^{-} = b_{ji}^{-}$ is transparent on the support of $\chi_{i'i}$.

•• If $q = q' = -1$, then the source terms are

$$B_{1\alpha\beta} = g\chi_{ji,+1}\chi_{j'i'}^{\sharp}b_{ii'}^{+}, \quad B_{-1\alpha\beta} = g^{*}\chi_{ji}\chi_{j'i'}^{\sharp}b_{ii',-1}^{-}.$$

It suffices to solve for

$$\chi_{ii'}B_{1\alpha\beta} = g\chi_{ii'}\chi_{ji,+1}\chi_{j'i'}^{\sharp}b_{ii'}^{+}, \qquad \text{with phase } \lambda_{i,+1} - \omega - \lambda_{i'};$$

$$\chi_{i'i,-1}B_{-1\alpha\beta} = g^{*}\chi_{i'i,-1}\chi_{ji,-1}\chi_{j'i'}^{\sharp}b_{ii',-1}^{-}, \qquad \text{with phase } -(\lambda_{i',+1} - \omega - \lambda_{i})_{-1},$$

For both terms (2.18) applies.

• *"in-out" coupling terms:* we consider first $u_{ij}^{\pm}/v_{i'}$ coupling terms, that is corresponding to $\alpha = (i, j, q, \mathrm{in})$, $\beta = (i', \mathrm{out})$.

•• If $q = 1$ and $p = 1$, the source and phase are

$$g\chi_{ij}\Big(1 - \sum_{j' \in I_{i'}^{+}} (\chi_{i'j'})_{-1} - \sum_{j' \in I_{i'}^{-}} \chi_{j'i'}\Big)^{\sharp}b_{ii'}^{+} \quad \text{and} \quad \lambda_{i,+1} - p\omega - \lambda_{i'}.$$

Since we may assume $(i, i') \in \mathfrak{R}_0$, there holds $i \in I_{i'}^-$, and the phase is bounded away from zero on $(1 - \chi_{ii'})^\sharp$. Therefore it suffices to solve for the source terms

$$g \chi_{ii'} \chi_{ij} \chi_{i'j',-1}^\sharp b_{ii'}^+, \ j' \in I_{i'}^+, \quad \text{and} \quad g \chi_{ii'} \chi_{ij} \chi_{j'i'}^\sharp b_{ii'}^+, \ j' \in I_{i'}^-, \ j' \neq i.$$

Conditions (2.18) and (2.19) apply to the first and second terms above, respectively.

•• If $q = 1$ and $p = -1$, the source and phase are

$$g^* \chi_{ij,-2} \Big(1 - \sum_{j' \in I_{i'}^+} \chi_{i'j',-1} - \sum_{j' \in I_{i'}^-} \chi_{j'i'} \Big)^\sharp b_{ii',-1}^- \quad \text{and} \quad -(\lambda_{i',+1} - \omega - \lambda_i)_{-1}.$$

We apply the same reasoning as in the previous case: we may assume $i \in I_{i'}^+$, so that the phase factorizes in the source term over the support of $1 - \chi_{i'i}$, and as a consequence it suffices to solve for the source terms

$$g^* \chi_{i'i,-1} \chi_{ij,-2} \chi_{i'j',-1}^\sharp b_{ii',-1}^-, \ j' \in I_{i'}^+, \ j' \neq i, \text{ and } g^* \chi_{i'i,-1} \chi_{ij,-2} \chi_{j'i'}^\sharp b_{ii',-1}^-, \ j' \in I_{i'}^-.$$

Conditions (2.19) and (2.18) apply to the first and second terms above, respectively.

•• If $q = -1$ and $p = 1$, the source and phase are

$$g \chi_{ji,+1} \Big(1 - \sum_{j' \in I_{i'}^+} (\chi_{i'j'})_{-1} - \sum_{j' \in I_{i'}^-} \chi_{j'i'} \Big)^\sharp b_{ii'}^+ \quad \text{and} \quad \lambda_{i,+1} - \omega - \lambda_{i'}.$$

Here $i \in I_{i'}^-$, so that in the above source term, we can multiply by $\chi_{ii'}$ and neglect the term $(1 - \chi_{ii'})^\sharp$. The remaining source terms are

$$g \chi_{ii'} \chi_{ji,+1} \chi_{i'j',-1}^\sharp b_{ii'}^+, \ j' \in I_{i'}^+, \quad \text{and} \quad g \chi_{ii'} \chi_{ji,+1} \chi_{j'i'}^\sharp b_{ii'}^+, \ j' \in I_{i'}^-, \ j' \neq i.$$

Conditions (2.18) and (2.19) apply to the first and second terms above, respectively.

•• If $q = -1$ and $p = -1$, the source and phase are

$$g^* \chi_{ji,-1} \Big(1 - \sum_{j' \in I_{i'}^+} (\chi_{i'j'})_{-1} - \sum_{j' \in I_{i'}^-} \chi_{j'i'} \Big)^\sharp b_{ii',-1}^- \quad \text{and} \quad -(\lambda_{i',+1} - \omega - \lambda_i)_{-1}.$$

By the same arguments as above, it suffices to handle the source terms

$$g^* \chi_{i'i,-1} \chi_{ji,-1} \chi_{i'j',-1}^\sharp b_{ii',-1}^-, \ j' \in I_{i'}^+ \ j' \neq i, \text{ and } g^* \chi_{i'i,-1} \chi_{ji,-1} \chi_{j'i'}^\sharp b_{ii',-1}^-, \ j' \in I_{i'}^-.$$

Conditions (2.19) and (2.18) apply to the first and second terms above, respectively.

Next we turn to $v_i / u_{i'j'}^\pm$ coupling terms, corresponding to indices $\alpha = (i, \text{out})$ and $\beta = (i', j', q', \text{in})$:

•• If $q' = 1$ and $p = 1$, the source and phase are

$$g \Big(1 - \sum_{j \in I_i^+} \chi_{ij,+1} - \sum_{j \in I_i^-} \chi_{ji,+2} \Big) \chi_{i'j'}^\sharp b_{ii',+1}^+ \quad \text{and} \quad (\lambda_{i,+1} - \omega - \lambda_{i'})_{+1}.$$

By the same arguments as above, it suffices to handle the source terms

$$g \chi_{ii',+1} \chi_{ij,+1} \chi_{i'j'}^\sharp b_{ii',+1}^+, \ j \in I_i^+, \ j \neq i', \quad \text{and} \quad g \chi_{ii',+1} \chi_{ji,+2} \chi_{i'j'}^\sharp b_{ii',+1}^+, \quad j \in I_i^-.$$

Conditions (2.19) and (2.18) apply to the first and second terms above, respectively.

• • The other three terms associated with $v_i/u^{\pm}_{i'j'}$ couplings are entirely similar.

• *"out-out" coupling terms* and phases are, in the case $p = 1$:

$$g\big(1 - \sum_{j\in I_i^+}\chi_{ij} - \sum_{j\in I_i^-}\chi_{ji,+1}\big)\big(1 - \sum_{j'\in I_{i'}^+}\chi_{i'j,-1} - \sum_{j'\in I_{i'}^-}\chi_{j'i'}\big)b^+_{ii'}, \quad \lambda_{i,+1} - \omega - \lambda_{i'},$$

and in the case $p = -1$:

$$g^*\big(1 - \sum_{j\in I_i^+}\chi_{ij,-2} - \sum_{j\in I_i^-}\chi_{ji,-1}\big)\big(1 - \sum_{j'\in I_{i'}^+}\chi_{ij,-1} - \sum_{j'\in I_{i'}^-}\chi_{j'i'}\big)^\sharp b^-_{ii',-1}, \quad (\lambda_{i',+1} - \omega - \lambda_i)_{-1}.$$

Employing the same arguments as in the "in-out" case, we are reduced to considering the source terms, for $p = 1$:

$$g\chi_{ii'}\big(\chi_{ij_1} + \chi_{j_2 i,+1}\big)\big(\chi_{i'j'_1,-1} + \chi_{j'_2 i'}\big)b^+_{ii'}, \ j_1 \in I_i^+\backslash\{i'\}, \ j_2 \in I_i^-, \ j'_1 \in I_{i'}^+, \ j'_2 \in I_{i'}^-\backslash\{i\}.$$

To the terms involving j_1 and j'_2, condition (2.19) applies. To the terms involving j_2 and j'_1, condition (2.18) applies. The case $p = -1$ is handled in the same way. $\qquad\square$

REMARK 4.3. — At first sight it might like look condition (2.18) is too strong for our purposes, since it involves intersections of only two resonant sets, while the interaction coefficients $B_{p\alpha\beta}$ involve three frequency cut-offs. A look at (4.12) shows however that we cannot do with less than (2.18). Indeed, given $(i, i') \in \mathfrak{R}_0$, for the normal form reduction to go through the symbol in (4.12) has to be bounded for all values of j, j' (such that (i, j) and $(i', j') \in \mathfrak{R}_0$), including $j = i'$.

System (4.7) has size $N \times |\mathbb{A}|$, where $|\mathbb{A}|$ is the cardinal of $\mathbb{A}$ defined in (4.5). We let

$$\mathbf{Q}_{(\alpha,\beta)} = \sum_{p\in\pm 1} e^{i(p+\delta_\beta - \delta_\alpha)\theta} Q_{\ell\alpha\beta}, \qquad \ell = p + \delta_\beta - \delta_\alpha,$$

where $Q_{\ell\alpha\beta}$ is given by Proposition 4.2 for relevant indices, meaning all p, α, β at the exclusion of cases (a) and (b), and $Q_{\ell\alpha\beta} := 0$ otherwise. We then form a large matrix $\mathbf{Q} \in \mathbb{C}^{N|\mathbb{A}|\times N|\mathbb{A}|}$ by assembling the $N \times N$ blocks $\mathbf{Q}_{(\alpha,\beta)}$, and, similarly to (3.17), let

$$(4.13) \quad \check{\mathbf{U}} := \Big(\mathrm{Id} + \sqrt{\varepsilon}\mathrm{op}_\varepsilon\big(\mathbf{Q}(\sqrt{\varepsilon}t)\big)\Big)^{-1}\mathbf{U}(\sqrt{\varepsilon}t), \quad \mathbf{U} := (\mathbf{u}_\alpha)_{\alpha\in\mathbb{A}}, \quad \check{\mathbf{U}} =: (\check{\mathbf{u}}_\alpha)_{\alpha\in\mathbb{A}},$$

It can then be checked, exactly as in the proof of Corollary 3.5, that $\check{\mathbf{U}}$ solves the reduced system

$$(4.14) \qquad\qquad \partial_t\check{\mathbf{u}}_\alpha + \frac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon(i\boldsymbol{\mu}_\alpha)\check{\mathbf{u}}_\alpha = \mathrm{op}_\varepsilon(\check{\mathbf{B}}_{\alpha\beta})\check{\mathbf{u}}_\beta + \sqrt{\varepsilon}\check{\mathbf{F}}_\alpha,$$

where the only remaining source terms correspond to constructive interactions between non-transparent resonances, that is

$$(4.15) \quad \check{\mathbf{B}}_{\alpha\beta} := \begin{cases} \mathbf{B}_{1\alpha\beta}, & \text{if } \alpha = (i, j, 1, \text{in}), \ \beta = (j, i, -1, \text{in}), \text{ with } (i, j) \in \mathfrak{R}_0, \\ \mathbf{B}_{-1\alpha\beta}, & \text{if } \alpha = (i, j, -1, \text{in}), \ \beta = (j, i, 1, \text{in}), \text{ with } (j, i) \in \mathfrak{R}_0, \\ 0, & \text{otherwise.} \end{cases}$$

According to (4.8), if $\alpha = (i, j, 1, \text{in})$ and $\beta = (j, i, -1, \text{in})$, then $\mathbf{B}_{1\alpha\beta}$ simplifies into

$$\mathbf{B}_{1\alpha\beta} = \chi_{ij}^{\sharp} \Pi_{i,+1} B_1 \Pi_j,$$

and if $\alpha = (i, j, -1, \text{in})$ and $\beta = (j, i, 1, \text{in})$, then $\mathbf{B}_{-1\alpha\beta}$ simplifies into

$$\mathbf{B}_{-1\alpha\beta} = \chi_{ji}^{\sharp} \Pi_i B_{-1} \Pi_{j,+1}.$$

In particular, in (4.14) there are no fast oscillations in the source. System (4.14) is the reduced system, analogous to system (3.18) in the case of one non-transparent resonance.

4.1.3. Space-frequency localization. — We now isolate the family, indexed by the set $\mathfrak{R}_0$, of $2N \times 2N$ subsystems in (4.14) that correspond to non-zero coupling terms $\check{\mathbf{B}}_{\alpha\beta}$, just like the subsystem corresponding to resonance $(1, 2)$ was naturally isolated from the rest of (3.18).

We let $\check{U}_{ij} := (\check{u}_{ij}^+, \check{u}_{ji}^-)$, and
$$(4.16)$$
$$V_{ij} := \text{op}_\varepsilon(\chi_{ij})(\varphi_{ij}\check{U}_{ij}), \ W_{ij1} := \text{op}_\varepsilon(\chi_{ij})\big((1 - \varphi_{ij})\check{U}_{ij}\big), \ W_{ij2} := \big(1 - \text{op}_\varepsilon(\chi_{ij})\big)\check{U}_{ij},$$

so that, just like in (3.23), $\check{U}_{ij} = V_{ij} + W_{ij1} + W_{ij2}$. The spatial cut-off φ_{ij} is identically equal to one in a large neighborhood of x_0; it is associated with truncations $\varphi_{ij}^{\flat}$, $\varphi_{ij}^{\sharp}$ such that $\varphi_{ij}^{\flat} \prec \varphi_{ij} \prec \varphi_{ij}^{\sharp}$.

We then verify exactly as in Lemma 3.8 that, with this further coordinatization, the reduced system (4.14) takes the form of the prepared system

$$(4.17) \quad \begin{cases} \partial_t V_{ij} + \dfrac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon^\psi(M_{ij}) V_{ij} = \sqrt{\varepsilon} F_{V_{ij}}, & (i, j) \in \mathfrak{R}_0, \\[2mm] \partial_t \mathbf{W} + \dfrac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon(i\mathbf{A}) \mathbf{W} = \text{op}_\varepsilon(D) \mathbf{W} + \sqrt{\varepsilon} F_{\mathbf{W}}, \end{cases}$$

where
– the interaction matrices M_{ij} are

$$(4.18) \quad M_{ij}(\varepsilon, t, x, \xi) := \chi_{ij}^{\sharp} \begin{pmatrix} i(\lambda_{i,+1} - \omega) & -\sqrt{\varepsilon}\varphi_{ij}^{\sharp} g \Pi_{i,+1} B_1 \Pi_j \\ -\sqrt{\varepsilon}\varphi_{ij}^{\sharp} g^* \Pi_j B_{-1} \Pi_{i,+1} & i\lambda_j \end{pmatrix}.$$

– the variable $\mathbf{W}$ is the collection of all $\mathbf{u}_\alpha$ for which the source terms $\check{\mathbf{B}}_{\alpha\beta}$ are all equal to zero [1], and all (W_{ij1}, W_{ij2}), for $(i,j) \in \mathfrak{R}_0$,

– the Fourier multiplier $\mathbf{A}$ is diagonal and purely imaginary,

– the source term D depends linearly on $(1 - \varphi_{ij}^\flat)u_{0,\pm1}$. In particular it can be made arbitrarily small, by choosing the support of $\varphi_{ij}^\flat$ large enough, since $u_{0,\pm1}$ are decaying at infinity,

– the source terms $F_{V_{ij}}$ and $F_\mathbf{W}$ are bounded as was F_α in (4.9).

In (4.17), we see $2N \times 2N$ interaction systems, indexed by $(i,j) \in \mathfrak{R}_0$, that are weakly coupled with a large system in $\mathbf{W}$. The system in $\mathbf{W}$ is symmetric hyperbolic with very small linear and nonlinear source terms. The coupling terms between the subsystem indexed by $\mathfrak{R}_0$ and the system in $\mathbf{W}$ are the source terms F in the right-hand sides. We call these coupling terms weak because of the $\varepsilon^{1/2}$ prefactors. System (4.17) is the prepared system, analogous to system (3.24).

4.1.4. Conclusion. — From (4.17), the estimates are as in the case of one non-transparent resonance.

Assumption 2.8(iii) allows for large-rank interaction coefficients for those resonances $(i,j) \in \mathfrak{R}_0$ for which one interaction coefficient is identically zero. In such cases, the bounds for the symbolic flow are trivial. Indeed, equation (3.29) page 39 reduces to

$$\partial_t S + \frac{1}{\sqrt{\varepsilon}} \begin{pmatrix} i\chi_i\mu_i & -\sqrt{\varepsilon}\tilde{b}_{ij} \\ 0 & i\chi_1\mu_j \end{pmatrix} S = 0,$$

a triangular system of ordinary differential equations. The corresponding bounds are

(4.19)
$$|\partial_x^\alpha S(\tau;t)| \lesssim 1 + (t-\tau)^r,$$

where r is the rank of b_{ij}^+.

For the other pairs $(i,j) \in \mathfrak{R}_0$, we have an amplification coefficient γ_{ij} defined in (2.22). With γ_{ij} are associated an upper rate of growth γ_{ij}^+, defined as in (3.30), and, unless $\gamma_{ij} = 0$, a lower rate of growth γ_{ij}^-, defined as in the statement of Lemma 3.11. Both depend on how tightly we cut around the resonance. This is quantified by parameters $h > 0$ (in frequency space) and $\rho > 0$ (in physical space).

In the unstable case $\mathbf{\Gamma} > 0$, we isolate (i_0, j_0) such that

$$\gamma = \gamma_{i_0 j_0} = \max_{(i,j)\in\mathfrak{R}_0} \gamma_{ij} > 0.$$

The other components have slower (maybe not strictly slower) rates of growth.

Given $(i,j) \in \mathfrak{R}_0$, if $\gamma_{ij} > 0$ we use the Duhamel representation formula as in Section 3.1.3, and this gives an upper bound for V_{ij} that is analogous to (3.36).

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1. Corresponding, by definition of $\check{\mathbf{B}}_{\alpha\beta}$ in (4.15), to all α such that for all β, there holds condition $(\alpha,\beta) \neq ((i,j,1,\text{in}),(j,i,-1,\text{in}))$, for any $(i,j) \in \mathfrak{R}_0$, and $(\alpha,\beta) \neq ((i,j,-1,\text{in}),(j,i,1,\text{in}))$, for any $(j,i) \in \mathfrak{R}_0$.

Given $(i,j) \in \mathfrak{R}_0$, if $\gamma_{ij} < 0$, then we use a symmetrizer to estimate V_{ij} as in Section 3.2, and if $\gamma_{ij} = 0$, we simply use bound (4.19).

This gives upper bounds as in Propositions 3.14 and 3.17:

$$(4.20) \qquad \|(V_{ij})_{(i,j)\in\mathfrak{R}_0}, \mathbf{W})(t')\|_{\varepsilon,s} \lesssim \varepsilon^{K-\kappa} |\ln \varepsilon|^* e^{t\gamma^+},$$

where κ intervenes in the definition of $\dot{u}$ in the first line of Section 4.1.1. The upper rate of growth γ^+ is defined from γ by (3.30). The above estimate is valid for $T < T_0$, where T_0 is defined in (2.12).

By an appropriate choice of the initial datum (namely, (2.27)), we derive a lower bound for the distinguished variable $V_{i_0 j_0}$ as in Section 3.1.4, and conclude as in Section 3.1.6.

In the stable case $\mathbf{\Gamma} < 0$, all resonances $(i,j) \in \mathfrak{R}_0$ are symmetrizable. We define a symmetrizer by blocks, and proceed as in Section 3.2.

4.2. Proof of Theorem 2.11

4.2.1. Preparation. — The perturbation variable is defined by

$$(4.21) \qquad u =: u_a + \varepsilon^{1/2}\dot{u},$$

The $\varepsilon^{1/2}$ prefactor will provide the necessary cushion as we follow the growth of a component of the solution that might not have maximal growth. There holds the bound $\|\dot{u}(0)\|_{\varepsilon,s} = O(\varepsilon^{K-1/2})$.

We use the coordinatization of Section 4.1.1, leading to system (4.7) in the variable $\mathbf{U} := (\mathbf{u}_\alpha)_{\alpha \in \mathbb{A}}$. The source $\mathbf{F}_\alpha$ satisfies (4.9) with $\kappa = 1/2$; explicitly

$$(4.22) \qquad |\mathbf{F}_\alpha|_{L^2} \lesssim (1 + |\dot{u}|_{L^\infty})|\dot{u}|_{L^2} + \varepsilon^{K_a - 1/2}.$$

In the present context, the non-transparent resonance $(1,2)$ plays a distinguished role. The associated variables are $(u_{12}^+, u_{21}^-) \in \mathbb{C}^{2N}$, as defined in (4.1)-(4.2). We now eliminate from the system in $\mathbf{U}$ all the coupling terms $\mathbf{B}_{p\alpha\beta}$ which involve u_{12}^+ or u_{21}^-, at the exception of the crucial interaction coefficients b_{12}^+ and b_{21}^- :

PROPOSITION 4.4. — *Under Assumption* 2.10(ii), *equation*

$$(4.23) \qquad i\big(-\ell\omega + \boldsymbol{\mu}_{\alpha,+\ell} - \boldsymbol{\mu}_\beta\big)Q_{\ell\alpha\beta} = B_{p\alpha\beta} + \varepsilon R_0, \qquad \ell = p - \delta_\alpha + \delta_\beta$$

has a solution $Q_{\ell\alpha\beta} \in S^0$ *for* $\alpha = (1,2,+,in)$ *and all* (p,β) *unless* $B_{p\alpha\beta} = \chi_{12}^\sharp b_{12}^+$, *and also for* $\alpha = (2,1,-,in)$ *and all* (p,β) *unless* $B_{p\alpha\beta} = \chi_{12}^\sharp b_{21}^-$. *In* (4.23), R_0 *is such that* $\mathrm{op}_\varepsilon(R_0)$ *is a uniform remainder in the sense of Definition* 3.2.

Proof. — We revisit the proof of Proposition 4.2. We are going to use the separation condition (2.23).

In a first step, $\alpha = (1,2,+,\mathrm{in})$ and $\beta = (i',j',q',\mathrm{in})$ are given. There are four corresponding coupling terms. For $p = 1$ and $q = 1$, the source has prefactor $\chi_{ij,+1}\chi_{i'j'}\chi_{ii',+1}$, which vanishes identically by the separation condition (2.23).

The same holds, with a different combination of cut-offs, for indices $p = 1$ and $q = -1$, and for indices $p = -1$ and $q = -1$. The remaining term is $\chi_{12}^{\sharp} b_{12}^{+}$.

Next $\beta = (i', \text{out})$ is given, with the same α. Here the reduction is non trivial: we use the fact that any phase is bounded away from its corresponding resonant set to reduce the analysis to frequency sets which are intersections of supports of cut-offs functions, as in the proof of Proposition 4.2. After this is done, we find in the case $p = 1$ products $\chi_{ii'} \chi_{ij} (\chi_{i'j',-1} + \chi_{j''i'})$, with $j'' \neq i$, in factor of the source. By separation, these products vanish identically. The case $p = -1$ is similar.

The case $\alpha = (2, 1, -, \text{in})$ is treated in the same way, by examination of the corresponding cases in the proof of Proposition 4.2: all "in-in" coupling terms are trivial, save for b_{21}^{-}, and all "in-out" coupling terms are trivial, except on frequency sets over which the phase is bounded away from zero. $\qquad\square$

With Proposition 4.4, we define a change of variable $\mathbf{Q}$ as we did in Section 4.1.2, and from there define $\check{\mathbf{U}}$ by (4.13). The variable $\check{U}_{12} = (\check{u}_{12}^{+}, \check{u}_{21}^{-})$ describing resonance $(1, 2)$ satisfies

$$(4.24) \qquad \partial_t \check{U}_{12} + \frac{1}{\sqrt{\varepsilon}} \begin{pmatrix} i\mu_1 & -\chi_{12}^{\sharp} g(\sqrt{\varepsilon}t, x) b_{12}^{+} \\ -\chi_{12}^{\sharp} g(\sqrt{\varepsilon}t, x)^* b_{21}^{-} & i\mu_2 \end{pmatrix} \check{U}_{12} = \sqrt{\varepsilon} \check{F}_{12},$$

while the other variables in $\check{\mathbf{U}}$, which we denote $(\mathbf{u}_{\alpha'})_{\alpha' \in \mathbb{A}'}$, satisfy

$$(4.25) \qquad \partial_t \mathbf{u}_{\alpha'} + \frac{1}{\sqrt{\varepsilon}} \mathrm{op}_\varepsilon(i\boldsymbol{\mu}_{\alpha'}) \check{\mathbf{u}}_{\alpha'} = \sum_{p=\pm 1} \sum_{\beta' \in \mathbb{A}'} \mathrm{op}_\varepsilon(\mathbf{B}_{p\alpha'\beta'}) \check{\mathbf{u}}_{\beta'} + \sqrt{\varepsilon} \check{\mathbf{F}}_{\alpha'}.$$

where $\mathbb{A}'$ is the set of indices alien to the $(1, 2)$ resonance:

$$\mathbb{A}' := \mathbb{A} \setminus \big\{ (1, 2, +, \text{in}), (2, 1, -, \text{in}) \big\}.$$

Equations (4.24) and (4.25) are coupled only via the source terms F.

Next we introduce unknowns $V_{12}, W_{12,1}, W_{12,2}$ that are local to the $(1, 2)$ resonance, as in (4.16), and arrive at the prepared system:

$$(4.26) \qquad \begin{cases} \partial_t V_{12} + \dfrac{1}{\sqrt{\varepsilon}} \mathrm{op}_\varepsilon^\psi(M_{12}) V_{12} = \sqrt{\varepsilon} F_{V_{12}}, \\[2mm] \partial_t \mathbf{W} + \dfrac{1}{\sqrt{\varepsilon}} \mathrm{op}_\varepsilon(i\mathbf{A}) \mathbf{W} = \mathrm{op}_\varepsilon(D) \mathbf{W} + \sqrt{\varepsilon} F_{\mathbf{W}}, \end{cases}$$

analogous to (4.17), where
  - $\mathbf{W} = \big((\mathbf{u}_{\alpha'})_{\alpha' \in \mathbb{A}'}, W_{12,1}, W_{12,2}\big)$, so that $\mathbf{U}$ can be reconstructed from $(V_{12}, \mathbf{W})$ (as in (3.17) and (3.23)), and there holds $\|V_{12}\|_{\varepsilon,s} + \|\mathbf{W}\|_{\varepsilon,s} \lesssim \|\mathbf{U}\|_{\varepsilon,s}$;
  - the interaction matrix M_{12} is defined in (4.18), with $(i, j) = (1, 2)$;
  - the Fourier multiplier $\mathbf{A}$ is diagonal and purely imaginary;
  - the important difference with (4.17) is in the source D, which here is

$$D = \begin{pmatrix} D_{12} & 0 \\ 0 & \sum_{p=\pm 1} (\mathbf{B}_{p\alpha'\beta'})_{\alpha',\beta' \in \mathbb{A}'} \end{pmatrix}, \qquad D_{12} = \begin{pmatrix} (1 - \varphi_{12}^{\flat}) \check{\mathcal{B}}_{12} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\check{\mathcal{B}}_{12}$ is defined as $\check{\mathcal{B}}$ in (3.18). The point is that D is *not* small: there holds[2]

$$(4.27) \qquad\qquad |D(t,x,\xi)| \leq |u_0(\sqrt{\varepsilon}t,x)||B|.$$

– the source terms $F_{V_{12}}$ and $F_{\mathbf{W}}$ satisfy bound (4.22).

4.2.2. Upper bounds. — The estimates for (4.26) differ from the analogous ones conducted in Section 3.1.5 only in the estimate (4.27) for D.

PROPOSITION 4.5. — *Given*

$$(4.28) \qquad\qquad T < \max\left(\frac{K - 1/2}{|B||\hat{a}|_{L^1}}, \ \frac{K - (d+1)/2}{|B||a|_{L^\infty}} \right),$$

if ε is small enough, then the solution to (4.26) issued from $\|(V_{12},\mathbf{W})(0)\|_{\varepsilon,s} = O(\varepsilon^K)$, with $s > d/2$, is defined over $[0, T|\ln\varepsilon|]$, with the estimate

$$(4.29) \qquad\qquad \|(V_{12},\mathbf{W})(t)\|_{\varepsilon,s} \lesssim \varepsilon^{K-1/2} |\ln\varepsilon|^* e^{t|B||a|_{L^\infty}}.$$

Proof. — We follow Section 3.1.5.

First step: $\mathcal{F}L^1$ estimates. We follow the proof of Lemma 3.13. Modulo differences in notation, the only change is in the bound for D. Here we have, using (4.27):

$$|\text{op}_\varepsilon(D)\mathbf{W}|_{\mathcal{F}L^1} \leq |B||\hat{u}_0(\sqrt{\varepsilon}t)|_{L^1}|\hat{\mathbf{W}}|_{L^1}.$$

This leads to

$$|(\hat{V}_{12},\hat{\mathbf{W}})(t)|_{L^1} \lesssim \varepsilon^{K-1/2}|\ln\varepsilon|^* + \left(|B|(1+\delta_{\varphi_0})(|\hat{a}|_{L^1} + O(\varepsilon^{\eta_1})) \right) \int_0^t |(\hat{V}_{12},\hat{\mathbf{W}})(t')|_{L^1}\, dt',$$

having assumed the bound $|\hat{V}_{12},\hat{\mathbf{W}}|_{L^1} \leq \varepsilon^{\eta_1}$ up to time t. Above $\delta_{\varphi_0} > 0$ can be made arbitrarily small. By the same argument as in the proof of Lemma 3.13, this shows that the above bound propagates: there holds $\sup_{0\leq t\leq T_1|\ln\varepsilon|} |(V_{12},\mathbf{W})(t)|_{\mathcal{F}L^1} \leq \varepsilon^{\eta_1}$, for any $T_1 < (K - 1/2)/(|B||\hat{a}|_{L^1})$, for some $\eta_1 = \eta_1(T_1)$.

Second step: Sobolev estimates based on $\mathcal{F}L^1$ estimates. We follow the proof of Proposition 3.14. In the present context, the only significant difference is that instead of (3.50), we have here

$$|\text{op}_\varepsilon^\psi(D)\Lambda^s\mathbf{W}|_{L^2} \leq |B||u_0(\sqrt{\varepsilon}t)|_{L^\infty}\|\mathbf{W}\|_{\varepsilon,s}.$$

With the above first step, the source satisfies

$$(4.30) \qquad\qquad \|F_{\mathbf{W}}\|_{\varepsilon,s} \lesssim \|V_{12},\mathbf{W}\|_{\varepsilon,s} + \varepsilon^{K_a - 1/2}.$$

This implies the upper bound

$$\|\mathbf{W}\|_{\varepsilon,s}^2 \lesssim \varepsilon^{2(K-1/2)} + \left(|B||a|_{L^\infty} + \varepsilon^{1/2}|\ln\varepsilon| \right) \int_0^t \|V_{12},\mathbf{W}\|_{\varepsilon,s}^2 + \varepsilon^{K_a} \int_0^t \|\mathbf{W}(t')\|_{\varepsilon,s}\, dt'.$$

---

2. Multiplicative constants do matter here, since upper bounds translate into growth rates. In (4.27), $|\cdot|$ denotes the sup norms in $\mathbb{C}^N$ and $\mathbb{C}^{N\times N}$, in accordance with notation set up on page 20.

The estimate for V_{12} is identical to the estimate in V (3.54). Just like in the proof of Proposition 3.14, this implies, via Lemma 6.18, the bound (4.29) for $t \leq \dfrac{K - 1/2}{|B||\hat{a}|_{L^1}}$. Here we are using Lemma 6.18 with $\delta_0 = |B||a|_{L^\infty}$, $\gamma_0 = \gamma_{12}$, and with $\max(\delta_0, \gamma_0) = |B| \, |a|_{L^\infty}$.

Third step: Sobolev estimates based on the $L^\infty \hookrightarrow H^s$ embedding. In the proof of Proposition 3.17, we replace ε^{η_2} with $C(T_2) > 0$ in the definition of J. Indeed, in Proposition 3.17, the role of $\eta_2 > 0$ was to guarantee smallness of the nonlinear terms. Here so long as $\dot{u}$ is bounded, the nonlinear terms are small, since (4.21) implied the better estimate (4.30). This gives, exactly as in the proof of Proposition 3.17, the bound (4.29) for $t \leq \dfrac{K - (d+1)/2}{|B||a|_{L^\infty}}$. $\qquad\square$

4.2.3. Conclusion.

The lower bound is exactly as in the main proof (Section 3.1.4). Starting from (3.60) with V_{12} in place of V, this gives

$$
(4.31) \quad
\begin{aligned}
|V_{12}(T|\ln\varepsilon|)|_{L^2(B(x_0,\rho))} &\geq C(\rho)\varepsilon^{K - T\gamma_{12}^-} - C|\ln\varepsilon|^* \varepsilon^{K+1/2 - T\gamma_{12}^+} \\
&\quad - C\sqrt{\varepsilon}|\ln\varepsilon|^* \int_0^{T|\ln\varepsilon|} e^{(T|\ln\varepsilon| - t')\gamma_{12}^+}|F_{V_{12}}(t')|_{L^2} dt'.
\end{aligned}
$$

The source $F_{V_{12}}$ satisfies estimate (4.30). With Proposition 4.5 and its proof, this gives

$$
|F_{V_{12}}(t)|_{L^2} \leq \varepsilon^{K-1/2}|\ln\varepsilon|^* \varepsilon^{t|B||a|_{L^\infty}} + \varepsilon^{K_a - 1/2},
$$

so that $|V_{12}(T|\ln\varepsilon|)|_{L^2(B(x_0,\rho))}$ is bounded from below by

$$
\varepsilon^{K-1/2 - T\gamma_{12}^-} - C|\ln\varepsilon|^* \varepsilon^{K - T\gamma_{12}^+} - C|\ln\varepsilon|^* \varepsilon^{K - T|B||a|_{L^\infty}},
$$

up to a multiplicative constant. We now impose for the final observation time the upper bound

$$
(4.32) \qquad\qquad \frac{1}{2} - T(|B| - \gamma_{12})|a|_{L^\infty} > 0,
$$

corresponding to $T < \tilde{T}_0$, with notation introduced in (2.25). For T satisfying (4.28) and (4.32), for h, ρ, c and ε small enough, there holds the lower bound

$$
|V_{12}(T|\ln\varepsilon|)|_{L^2(B(x_0,\rho))} \geq C\varepsilon^{K-1/2 - T\gamma_{12}^-}.
$$

Hence the limiting amplification exponent K_0' satisfies $K_0' \geq K - T\gamma_{12}|a|_{L^\infty}$, corresponding to (2.24). The end of the proof, going up the chain of changes of variables from V_{12} to $\dot{u}$, is identical to Section 3.1.6.

4.3. Proof of Theorem 2.12

We posit here $u =: u_a + \varepsilon^{1/2}\dot{u}$, as in (4.21), and then follow closely the proof of Theorem 2.9. The $\varepsilon^{1/2}$ factor in the definition of $\dot{u}$ modifies the estimates for the source terms F, as in the proof of Theorem 2.11. This gives an existence time

$$T_0'' = \max\Big(\frac{K - 1/2}{|B||\hat{a}|_{L^1}}, \ \frac{K - (d+1)/2}{\gamma|a|_{L^\infty}}\Big),$$

with γ defined in (2.22). Indeed, the numerators are as in (4.28), since the ansatz is as in (4.21), and the denominators are as in T_0 (2.12), since the assumptions are the same as in Theorem 2.9. On $[0, T\sqrt{\varepsilon}|\ln\varepsilon|]$, the upper bounds are identical to (4.20), except for the size $O(\varepsilon^{K-1/2})$ of the initial datum:

$$(4.33) \qquad \|(V_{ij})_{(i,j)\in\mathfrak{R}_0}, \mathbf{W})(t')\|_{\varepsilon,s} \lesssim \varepsilon^{K-1/2}|\ln\varepsilon|^* e^{t\gamma^+},$$

with $\gamma = \gamma_{i_0 j_0}$ (2.22), where V_{ij} and $\mathbf{W}$ are defined exactly as in Section 4.1.3. The estimate for the source term $F_{V_{i_0 j_0}}$ is identical to (4.30):

$$(4.34) \qquad |F_{V_{i_0 j_0}}|_{L^2} \lesssim \|(V_{ij})_{(i,j)\in\mathfrak{R}_0}, \mathbf{W}\|_{L^2} + \varepsilon^{K_a}.$$

Now the difference is in the lower bound, which we consider on a small ball $B(x_0, \varepsilon^\beta)$. We follow the proof of Lemma 3.11, until we arrive at (3.42). In the present context, we replace (3.42) with

$$\Big(\int_{B(x_0,\varepsilon^\beta)} e^{2ta(x)\gamma}\, dx\Big)^{1/2} \geq C\varepsilon^{\beta d/2} e^{t\gamma|a|_{L^\infty}},$$

for ε small enough. Since by assumption $\beta d/2 < 1/2$, this term is indeed the leading term in the lower bound for the action of the solution operator on the initial datum, and there holds

$$(4.35) \qquad \big|\mathrm{op}_\varepsilon^\psi(S(0;t))V_{i_0 j_0}(0)\big|_{L^2(B(x_0,\varepsilon^\beta))} \geq C\varepsilon^K\Big(\varepsilon^{\beta d/2} e^{t\gamma^-} - \varepsilon^{1/2}|\ln\varepsilon|^* e^{t\gamma^+}\Big).$$

From (4.33), (4.34) and (4.35), via an integral representation of $V_{i_0 j_0}$ (Section 3.1.3), we deduce as in (3.60) and (4.31) the lower bound

$$|V_{i_0 j_0}(T|\ln\varepsilon|)|_{L^2(B(x_0,\varepsilon^\beta))} \geq C\varepsilon^{K+\beta d/2-1/2-T\gamma|a|_{L^\infty}} - C|\ln\varepsilon|^*\varepsilon^{K-T\gamma^+}.$$

Since $\beta < 1/d$, given an observation time $T < T_0''$, as soon as h is small enough, the exponent $K + \beta d/2 - 1/2 - T\gamma|a|_{L^\infty}$ is strictly smaller than the exponent $K - T\gamma^+$, yielding a deviation estimate. The lower bound for the amplification exponent is

$$K + \frac{\beta d}{2} - T_0''\gamma|a|_{L^\infty},$$

corresponding to K_0'' given in (2.26).

4.4. Proof of Theorem 2.13

We assume that given $T < T_\infty$, for ε small enough, the solution to (1.1)-(2.27) is defined over $[0, T\sqrt{\varepsilon}|\ln\varepsilon|]$, belongs to $C^0([0, T\sqrt{\varepsilon}|\ln\varepsilon|], H^s(\mathbb{R}^d))$, and is uniformly bounded in (ε, t, x). From there, we proceed to prove the deviation estimate (2.28).

Let $K' > 0$ be given. The smaller K', the better the amplification in (2.28). In particular we may assume K' to be smaller than $1/2$. We posit

$$u =: u_a + \varepsilon^{K'}\dot{u},$$

and, following the proof of Theorem 2.9, we arrive at a system that is identical to (4.17). The upper bound for the source terms is here

$$\|F_{V_{ij}}, F_W\|_{\varepsilon,s} \le (1 + \varepsilon^{K'-1/2}|\dot{u}|_{L^\infty})\|V_{ij}, W\|_{\varepsilon,s} + \varepsilon^{K_a - K'},$$

for $t \in [0, T\sqrt{\varepsilon}|\ln\varepsilon|]$, for all $T < T_\infty$.

Next we follow the proof of Proposition 3.17 in order to derive an upper bound. The control of L^∞ is here a priori given, so that we can forgo the Sobolev embedding. In particular, the above estimate for the source term simplifies into

$$\|F_{V_{ij}}, F_W\|_{\varepsilon,s} \le (1 + \varepsilon^{-1/2+K'}C(T))|\dot{u}|_{L^2} + \varepsilon^{K_a - K'},$$

for some $C(T) > 0$. This gives

$$\|(V_{ij})_{(i,j)\in\mathfrak{R}_0}, \mathbf{W})(t')\|_{\varepsilon,s} \lesssim \varepsilon^{K-K'}|\ln\varepsilon|^* e^{t\gamma^+}, \qquad t \in [0, T\sqrt{\varepsilon}|\ln\varepsilon|], \text{ for all } T < T_\infty,$$

where $\gamma = \gamma_{i_0 j_0}$ (2.22). For the lower bound, we consider small balls with radius ε^β, as in Section 4.3. We arrive at

$$|V_{i_0 j_0}(T|\ln\varepsilon|)|_{L^2(B(x_0, \varepsilon^\beta))} \ge C\varepsilon^{K-K'+\beta d/2 - T\gamma|a|_{L^\infty}} - C|\ln\varepsilon|^* \varepsilon^{K-K'+1/2-T\gamma^+}$$
$$- C\sqrt{\varepsilon}|\ln\varepsilon|^* \int_0^{T|\ln\varepsilon|} e^{(T|\ln\varepsilon|-t')\gamma^+}|F_{V_{i_0 j_0}}(t')|_{L^2}dt'.$$

With the above upper bound for $F_{V_{12}}$, this gives

$$|V_{i_0 j_0}(T|\ln\varepsilon|)|_{L^2(B(x_0, \varepsilon^\beta))} \ge C\varepsilon^{K-K'+\beta d/2 - T\gamma|a|_{L^\infty}} - C|\ln\varepsilon|^* \varepsilon^{K-K'+1/2-T\gamma^+}$$
$$- C|\ln\varepsilon|^* \varepsilon^{K-T\gamma^+}.$$

Since $K' < 1/2$, among the last two terms in the above upper bound, the biggest is the second one. For $T_\infty - T$ small enough, and ρ, h and β small enough, there holds

$$0 < K - K' + \beta d/2 - T\gamma|a|_{L^\infty} < K - T\gamma^+,$$

and we conclude as in the other cases.

CHAPTER 5

EXAMPLES

5.1. Raman and Brillouin instabilities

We consider here systems

$$(5.1) \qquad \partial_t u + A(\partial_x)u = \frac{1}{\sqrt{\varepsilon}}B(u,u), \qquad A(\partial_x) = \sum_{1 \leq j \leq d} A_j \partial_{x_j},$$

where A satisfies Assumption 2.1, and B is bilinear $\mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$. We analyze the stability of special solutions of the form

$$(5.2) \qquad u_a(t,x) = \mathbf{a}(\eta \cdot x - \tau t) \in \mathbb{C}^N,$$

with

$$(5.3) \qquad \tau \in \mathrm{sp}\, A(\eta), \quad \mathbf{a}(y) \equiv (\mathbf{a}(y), \vec{e})\vec{e}, \quad \vec{e} \in \mathrm{Ker}\, A(\eta) - \tau \mathrm{Id}, \quad B(\vec{e}, \vec{e}) = 0,$$

where $(\cdot, \cdot)$ denotes the Hermitian scalar product in $\mathbb{C}^N$.

The systems (5.1) and solutions (5.2) that we are considering here are not exactly typical of our main framework (1.1)-(1.2). We chose however to present our application to the Raman and Brillouin instabilities first, since these are connected to the Euler-Maxwell equations, which later on will be relevant for Klein-Gordon systems (Sections 5.2 and 5.3).

The untypical features of (5.1)-(5.3) are as follows:

– the hyperbolic operator in (5.1) is non-dispersive, in the sense that $A_0 = 0$. As a consequence, the symbol $A(\xi)$ is 1-homogeneous in ξ.

– The datum for (5.2) is not highly-oscillating, and the reference solution is complex-valued. The reason is that the solution u to (5.1) describes a vector of complex *envelopes* of highly-oscillating fields. That is, systems (5.1) and solutions (5.2) can be thought of as resulting from a WKB approximation, and the stability analysis is performed on the limiting system.

The solutions (5.2)-(5.3) satisfy Assumption 2.2, with $v_a = 0$, $r_a^\varepsilon = 0$, if $\mathbf{a}$ has a large Sobolev regularity. Here $(\omega, k) = (0,0)$: the reference solution is not highly

oscillating. In particular, in this context, resonances (Definition 2.4 page 18) are crossing points on the variety:

$$\mathcal{R}_{ij} = \{\xi \in \mathbb{R}^d, \lambda_i(\xi) = \lambda_j(\xi)\},$$

where λ_i and λ_j are eigenvalues of A. By homogeneity of A, the eigenvalues are 1-homogeneous in ξ. In particular, if $\xi \in \mathbb{S}^{d-1}$ is a resonant frequency, then the whole line defined by ξ is resonant. As a consequence, the resonant set is bounded (Assumption 2.8(i)) only if eigenvalues cross only at $\xi = 0$.

A variant of (5.1) is given by systems

$$(5.4) \qquad \partial_t u + \frac{1}{\varepsilon} A(\partial_x) u = \frac{1}{\varepsilon} B(u, u),$$

with the same assumptions on A and B. We posit the ansatz

$$(5.5) \qquad u(\varepsilon, t, x) = v\big(\varepsilon, \varepsilon^{-1/2} t, \varepsilon^{1/2} x\big).$$

Then, u solves (5.4) if and only if v solves (5.1). Special solutions to (5.1) of the form (5.2) correspond to special solutions of (5.4) of the form

$$(5.6) \qquad u_a(\varepsilon, t, x) = \mathbf{a}\Big(\varepsilon^{1/2}\big(\eta \cdot x - \frac{\tau t}{\varepsilon}\big)\Big).$$

Note that an instability in short time $O(\sqrt{\varepsilon}|\ln \varepsilon|)$ for (5.1) translates into an instability in shorter time $O(\varepsilon|\ln \varepsilon|)$ for (5.4), expressing the fact that (5.4) is more singular than (5.1).

5.1.1. Three-wave interaction systems. — We consider here specifically

$$(5.7) \qquad \begin{cases} \partial_t u_1 + c_1 \partial_x u_1 = \dfrac{b_1}{\sqrt{\varepsilon}} \bar{u}_2 u_3, \\[2mm] \partial_t u_2 + c_2 \partial_x u_2 = \dfrac{b_2}{\sqrt{\varepsilon}} \bar{u}_1 u_3, \\[2mm] \partial_t u_3 + c_3 \partial_x u_3 = \dfrac{b_3}{\sqrt{\varepsilon}} u_1 u_2, \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $u_i \in \mathbb{C}$, with velocities $c_i \in \mathbb{R}$ and coefficients $b_i \in \mathbb{R}$. The reference solution (5.2)-(5.3) is explicitly

$$(5.8) \qquad u_a(t, x) = \big(a(x - c_1 t), 0, 0\big) \in \mathbb{R}^3,$$

corresponding to $\vec{e} = (1, 0, 0) \in \mathbb{C}^3$, $(\tau, \eta) = (c_1, 1)$.

THEOREM 5.1. — *The assumptions of Theorem 2.7 are satisfied by (5.7)-(5.8), with stability index* $\mathrm{sgn}\,\mathbf{\Gamma} = \mathrm{sgn}\, b_2 b_3$. *In the case* $b_2 b_3 > 0$, *this implies instability of arbitrarily small initial perturbations of* $(a(x), 0, 0)$ *in time* $O(\sqrt{\varepsilon}|\ln \varepsilon|)$, *in the sense of Theorems 2.7, 2.12 and 2.13.*

Proof. — Assumption 2.1 is trivially satisfied, since the hyperbolic operator is in diagonal form. Assumption 2.2 is obviously satisfied by the exact solution (5.8). Eigenvalues cross only at $\xi = 0$. As noted above, this implies that the only resonance is $\xi = 0$, and Assumption 2.6(i) is satisfied. The linearized source $B(u_a)$ is block-diagonal

$$B(u_a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_2\bar{a} \\ 0 & b_3 a & 0 \end{pmatrix},$$

and $(2,3)$ is the only non-transparent resonance, so that Assumption 2.6(ii) is satisfied. The trace of the product of the (rank-one, satisfying Assumption 2.6(iii)) interaction coefficients is $\Gamma_{23} = b_2 b_3 |a|^2$, implying the result. □

As an example of systems (5.4), we consider

$$(5.9) \quad \begin{cases} \partial_t u_1 + \dfrac{c_1}{\varepsilon}\partial_x u_1 = \dfrac{b_1}{\varepsilon}\bar{u}_2 u_3, \\[2mm] \partial_t u_2 + \dfrac{c_2}{\varepsilon}\partial_x u_2 = \dfrac{b_2}{\varepsilon}\bar{u}_1 u_3, \\[2mm] \partial_t u_3 + c_3 \partial_x u_3 = b_3 u_1 u_2, \end{cases}$$

corresponding to scaling (5.4), where, as in (5.7), $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $c_i \in \mathbb{R}$, $b_i \in \mathbb{R}$, and $u_i \in \mathbb{C}$, with reference solution

$$(5.10) \qquad u_a(\varepsilon, t, x) = \sqrt{\varepsilon}\Big(a\big(\varepsilon^{1/2}\big(x - c_1 t/\varepsilon\big)\big), 0, 0\Big) \in \mathbb{C}^3,$$

corresponding to (5.6) with $\vec{e} = (1, 0, 0) \in \mathbb{C}^3$, $(\tau, \eta) = (c_1, 1)$.

THEOREM 5.2. — *If $b_2 b_3 > 0$, then the solution (5.10) to system (5.9) is unstable under small initial perturbations of the form $\varepsilon^{K+1/2}\phi(\varepsilon, \varepsilon^{1/2}x)$, satisfying bound $\sup_{0 < \varepsilon < 1} \|\phi(\varepsilon, \cdot)\|_{\varepsilon,s} < \infty$. The amplification occurs in time $O(\varepsilon|\ln \varepsilon|)$, in $L^2(\mathbb{R}^d)$ norm.*

Naturally, the above "amplification" is meant in the sense of Theorems 2.7 and 2.13. There are three notable differences with Theorem 5.1: the form of the admissible perturbations and the time intervals and radii of the balls over which the instability may be recorded. From our general analysis, we can also deduce a stability result in the case $b_2 b_3 < 0$; its time range, however, is only $O(\sqrt{\varepsilon})$.

Proof. — Given the datum

$$\varepsilon^{1/2}\big(a(\varepsilon^{1/2}x), 0, 0\big) + \varepsilon^{K+1/2}\phi(\varepsilon, \varepsilon^{1/2}x),$$

we posit the ansatz, similar to (5.5):

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \varepsilon^{1/2} v_1 \\ \varepsilon^{1/2} v_2 \\ v_3 \end{pmatrix} (\varepsilon^{-1/2}t, \varepsilon^{1/2}x).$$

Then (v_1, v_2, v_3) solves (5.7), and Theorem 5.1 applies. The instability occurs in time $O(\sqrt{\varepsilon}|\ln\varepsilon|)$ for the corresponding system in the scaling (5.1), hence in time $O(\varepsilon|\ln\varepsilon|)$ for (5.9). It occurs in small balls $B(x_0, \rho)$ or $B(x_0, \varepsilon^\beta)$ for (v_1, v_2, v_3), hence in large balls $B(x_0, \varepsilon^{-1/2}\rho)$ in the scaling (5.9). $\qquad\square$

We can find more than one change of variables leading from (5.9) to (5.7). In particular, we can prove instability of reference solutions of amplitude $O(1)$ to (5.9), assuming slow variations in space, as follows. Consider the family of solutions to (5.9):

$$(5.11) \qquad u_a(\varepsilon, t, x) = \Big(a(\varepsilon x - c_1 t), 0, 0\Big)$$

THEOREM 5.3. — *The stability of solution* (5.11) *to* (5.9) *under perturbations of the form* $\varepsilon^K \phi(\varepsilon, \varepsilon x)$, *with* $\sup_{0<\varepsilon<1} \|\phi(\varepsilon, \cdot)\|_{\varepsilon, s} < \infty$, *and* $\phi = (\phi_1, \phi_2, \varepsilon^{1/2}\phi_3)$, *is determined by the sign of* $b_2 b_3$. *Instability occurs in time* $O(\sqrt{\varepsilon}|\ln\varepsilon|)$ *and is measured in norm* $L^2(\mathbb{R}^d)$.

(In)stability is meant in the sense of Theorems 2.7 and 2.13.

Proof. — Given the datum

$$\Big(a(\varepsilon x) + \phi_1(\varepsilon, \varepsilon x), \phi_2(\varepsilon, \varepsilon x), \varepsilon^{1/2}\phi_3(\varepsilon, \varepsilon x)\Big),$$

we posit the ansatz

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \varepsilon^{1/2} v_3 \end{pmatrix} (t, \varepsilon x).$$

Then (v_1, v_2, v_3) solves (5.7), and Theorem 5.1 applies. The instability occurs in small balls for (v_1, v_2, v_3), hence in large balls in the scaling (5.9). $\qquad\square$

REMARK 5.4. — We note that for homogeneous (x-independent) fields u_i, the condition $b_2 b_3 > 0$ implies spectral instability of the equilibrium $(a, 0, 0)$. In Section 9.2 of [**38**], Rauch observes that for (5.7) or (5.9) with $\varepsilon = 1$, in the case of coefficients satisfying $b_1 < 0$, $b_2 > 0$, $b_3 > 0$, the quantity $a_1|u_1|_{L^2}^2 + a_2|u_2|_{L^2}^2 + a_3|u_3|_{L^2}^2$, with $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$, is conserved, implying Lyapunov *s*tability of the trivial solution, in contrast with instability of the progressing wave (5.8) as given by Theorem 5.1. Section 9.2 of [**38**] also contains a blow-up result in the case $b_i > 0$.

5.1.2. Derivation of three-wave interaction systems from Euler-Maxwell. — The Euler-Maxwell equations describe laser-plasma interactions [**6, 14, 44**]. In

the non-dimensional form introduced in [48], they are

$$(EM) \begin{cases} \partial_t B + \nabla \times E = 0, \\[2mm] \partial_t E - \nabla \times B = \dfrac{1}{\epsilon}((1 + n_e + f(n_e))v_e - \dfrac{1}{\epsilon}\dfrac{\theta_i}{\theta_e}(1 + n_i + f(n_i))v_i, \\[2mm] \partial_t v_e + \theta_e(v_e \cdot \nabla)v_e = -\theta_e \nabla n_e - \dfrac{1}{\epsilon}(E + \theta_e v_e \times B), \\[2mm] \partial_t n_e + \theta_e \nabla \cdot v_e + \theta_e(v_e \cdot \nabla)n_e = 0, \\[2mm] \partial_t v_i + \theta_i(v_i \cdot \nabla)v_i = -\alpha^2 \theta_i \nabla n_i + \dfrac{1}{\epsilon}\dfrac{\theta_i}{\theta_e}(E + \theta_i v_i \times B), \\[2mm] \partial_t n_i + \theta_i \nabla \cdot v_i + \theta_i(v_i \cdot \nabla)n_i = 0, \end{cases}$$

where $(B, E) \in \mathbb{R}^6$ is the electromagnetic field, $(v_e, v_i) \in \mathbb{R}^6$ are the electronic and ionic velocities, and $(n_e, n_i) \in \mathbb{R}^2$ are the electronic and ionic fluctuations of density from a constant background. The function f is $f(x) = e^x - 1 - x$. The small parameter with respect to which the WKB will be performed is

$$\epsilon = \frac{1}{\omega_{pe} t_0},$$

where ω_{pe} is the electronic plasma frequency $\omega_{pe} = \sqrt{\dfrac{4\pi e^2 n_0}{m_e}}$ and t_0 is the duration of the laser pulse. The other parameters are

$$\theta_e = \frac{1}{c}\sqrt{\frac{\gamma_e T_e}{m_e}}, \quad \theta_i = \frac{1}{c}\sqrt{\frac{\gamma_i T_i}{m_i}}, \quad \alpha = \frac{T_i}{T_e},$$

with $-e$ the charge of the electrons, $+e$ the charge of the ions, m_e and m_i the masses, n_0 the background density, c the speed of light, γ_e and γ_i the specific heat ratios, T_e and T_i the temperatures. For plasmas created by lasers, the parameters θ_e, θ_i and α are typically small, and ϵ is even smaller. Since the mass of the ions is much larger than the mass of the electrons, there holds $\theta_i \ll \theta_e$.

In the right-hand side of the Ampère equation in (EM), we find the current density, and in the right-hand sides of the equations of conservation of momentum, we find the pressure terms in ∇n and the Lorentz forces in $v \times B$.

We denote $x \in \mathbb{R}$ the direction of propagation of the laser pulse, and coordinatize

$$(x, y) = (x, y_1, y_2) \in \mathbb{R}^3, \qquad M = (M_x, M_y) = (M_x, M_{y_1}, M_{y_2}) \in \mathbb{C}^3.$$

The hyperbolic operator in (EM) splits into *transverse* and *longitudinal* components. We denote

$$u_\perp = (B_{y_1}, B_{y_2}, E_{y_1}, E_{y_2}, v_{ey_1}, v_{ey_2}), \quad u_\parallel = (B_x, E_x, v_{ex}, n_e, v_{ix}, n_i).$$

Then, (EM) takes the form

$$\begin{cases} \big(\partial_t + A_\perp + \dfrac{1}{\boldsymbol{\epsilon}} A_{\perp 0}\big) u_\perp = F_\perp(u), \\[2mm] \big(\partial_t + A_\| + \dfrac{1}{\boldsymbol{\epsilon}} A_{\|0}\big) u_\| = F_\|(u), \end{cases}$$

where $F_\perp, F_\|$ contain all nonlinear terms: convection, current density and Lorentz force. The transverse and longitudinal hyperbolic operators have the form (1.1) with $A_0 \neq 0$. The *transverse* eigenvalues $i\omega$ are the eigenvalues of $A_\perp(i\vec{k}) + A_{\perp 0}$, where

$$\vec{k} = (k, 0, 0) \in \mathbb{R}^3;$$

they satisfy

$$(5.12) \qquad\qquad (-i\omega)^4 \Big(\omega^2 - k^2 - 1 - \frac{\theta_i^2}{\theta_e^2}\Big)^2 = 0.$$

The *longitudinal* eigenvalues $i\omega$ are the eigenvalues of $A_\|(i\vec{k}) + A_{\|0}$; they satisfy

$$(5.13) \qquad (-i\omega)^2 \Big((\omega^2 - \alpha^2 k^2 \theta_i^2)(\omega^2 - 1 - k^2 \theta_e^2) - (\omega^2 - k^2 \theta_e^2)\frac{\theta_i^2}{\theta_e^2}\Big) = 0.$$

The above longitudinal dispersion relation takes the alternate form

$$(5.14) \qquad\qquad 1 = \frac{1}{\omega^2 - k^2\theta_e^2} + \frac{\theta_i^2}{\theta_e^2}\frac{1}{\omega^2 - k^2\alpha^2\theta_i^2},$$

which we will find useful in Section 5.1.2.5.

We denote (t) the branches of non-trivial solutions of (5.12). We denote (ℓ) and (s) the branches of non-trivial solutions to (5.13). The (t) branches correspond to electromagnetic waves with a transverse polarization, typically light sources. The waves corresponding to the (ℓ) branches are called *electronic plasma waves*. They encode part of the response of the plasma to an incident light source. The (s) branches comprise acoustic waves.

These satisfy the following expansions in the limit $\theta_i \to 0$:

$$(5.15) \qquad \omega_{(\ell)}^2 = 1 + k^2\theta_e^2 + O(\theta_i)^2, \quad \omega_{(s)}^2 = k^2\theta_i^2\Big(\alpha^2 + \frac{1}{1 + k^2\theta_e^2}\Big) + O(\theta_i^4),$$

locally uniformly in k. For large k,

$$(5.16) \qquad\qquad \omega_{(\ell)} = \pm\theta_e k + O\Big(\frac{1}{k}\Big), \qquad \omega_{(s)} = \pm\alpha\theta_i k + O\Big(\frac{1}{k}\Big).$$

The characteristic variety for the Euler-Maxwell equations, meaning the collection of all branches of solutions $\omega(k)$ to (5.12) and (5.13) is pictured on Figure 1.

REMARK 5.5. — The proof by Guo, Ionescu and Pausader [19] of existence of small, global solutions to the Euler-Maxwell equations is based on an interesting reformulation of (EM) as three coupled dispersive equations (system (3.9) in [19]) with dispersion relations given by two Klein-Gordon modes and an acoustic mode, in agreement with Figure 1.
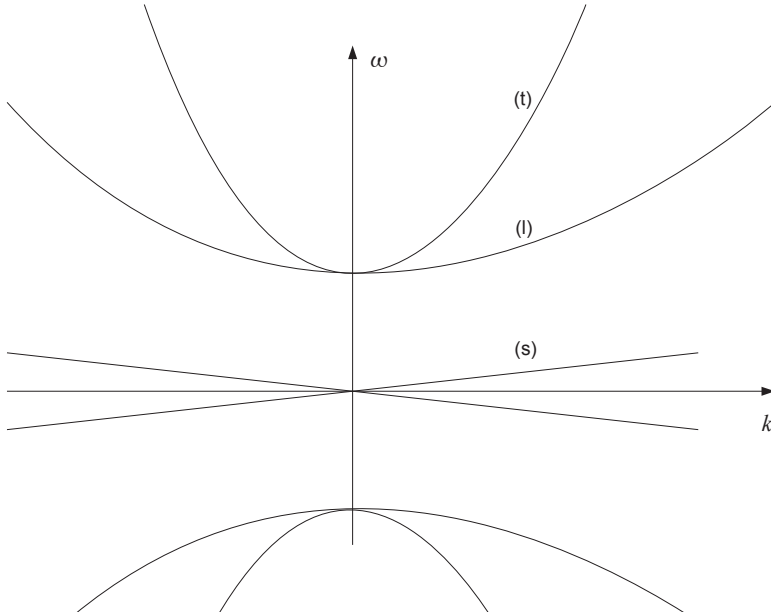
FIGURE 1. The characteristic variety for the Euler-Maxwell equations.

5.1.2.1. *Polarization and compatibility conditions.* — Given $\beta = (\omega, k)$, we denote $\mathbf{\Pi}(\beta)$ the orthogonal projector onto the kernel of the total operator

$$\begin{pmatrix} -i\omega + A_\perp(i\vec{k}) + A_{\perp 0} & 0 \\ 0 & -i\omega + A_\parallel(i\vec{k}) + A_{\parallel 0} \end{pmatrix}.$$

We denote $\beta_\perp$ a transverse phase, and $\beta_\parallel$ a longitudinal phase. The transverse polarization condition $\mathbf{\Pi}(\beta_\perp)u = u$ is explicitly (with $\beta_\perp = (\omega, k)$):

(5.17)

$$u_\parallel = 0, \quad (B_{y_1}, B_{y_2}) = k\omega^{-1}(E_{y_2}, -E_{y_1}), \quad v_{ey_j} = \frac{1}{ip\omega}E_{y_j}, \quad v_{iy_j} = -\frac{1}{ip\omega}\frac{\theta_i}{\theta_e}E_{y_j}.$$

The longitudinal polarization condition $\mathbf{\Pi}(\beta_\parallel)u = u$ is explicitly (with $\beta_\parallel = (\omega, k)$):

$$u_\perp = 0, \qquad B_x = 0,$$

$$v_{ex} = -\left(-i\omega + i\omega^{-1}k^2\theta_e^2\right)^{-1}E_x, \qquad v_{iz} = \left(-i\omega + i\omega^{-1}\alpha^2 k^2\theta_i^2\right)^{-1}\frac{\theta_i}{\theta_e}E_x,$$

(5.18)
$$n_e = -\frac{k\theta_e}{\omega}\left(-i\omega + i\omega^{-1}k^2\theta_e^2\right)^{-1}E_x,$$

$$n_i = \frac{\alpha^2 k\theta_i}{\omega}\frac{\theta_i}{\theta_e}\left(-i\omega + i\omega^{-1}\alpha^2 k^2\theta_i^2\right)^{-1}E_x.$$

The transverse compatibility condition $\mathbf{\Pi}(\beta_\perp)u = 0$ is explicitly (with $\beta_\perp = (\omega, k)$):

$$(5.19) \qquad \frac{k}{\omega}B_{y_2} = E_{y_1} - \frac{1}{i\omega}\left(v_e - \frac{\theta_i}{\theta_e}v_i\right)_{y_1}, \qquad -\frac{k}{\omega}B_{y_1} = E_{y_2} - \frac{1}{i\omega}\left(v_e - \frac{\theta_i}{\theta_e}v_i\right)_{y_2}.$$

The longitudinal compatibility condition $\mathbf{\Pi}(\beta_\parallel)u = 0$ is explicitly (with $\beta_\parallel = (\omega, k)$):

$$(5.20) \qquad \frac{i\omega}{\omega^2 - k^2\theta_e^2}\left(\frac{k\theta_e}{\omega}n_e + v_{ex}\right) - \frac{\theta_i}{\theta_e}\frac{i\omega}{\omega^2 - k^2\alpha^2\theta_i^2}\left(\frac{k\alpha^2\theta_i}{\omega}n_i + v_{ix}\right) + E_x = 0.$$

5.1.2.2. Initial data and ansatz. — We consider initial data with two oscillating phases:

$$(5.21) \qquad u(\epsilon, 0, x, y) = \Re e\left(a_1(x, y)e^{ik_1x/\epsilon} + a_2(x, y)e^{ik_2x/\epsilon}\right),$$

where the amplitudes a_1 and a_2 satisfy polarization conditions

$$\mathbf{\Pi}(\beta_1)a_1 = a_1, \quad \mathbf{\Pi}(\beta_2)a_2 = a_2,$$

where β_1 and β_2 are transverse phases. In accordance with (5.21), we consider the two-phase ansatz

$$u(\epsilon, t, x, y) = \mathbf{u}\left(\epsilon, t, x, y, \frac{k_1x - \omega_1 t}{\epsilon}, \frac{k_2x - \omega_2 t}{\epsilon}\right),$$

where $\mathbf{u}$ is 2π-periodic in both fast variables:

$$\mathbf{u}(\epsilon, t, x, y, \theta_1, \theta_2) = \sum_{\substack{p_1 \in \mathbb{Z} \\ p_2 \in \mathbb{Z}}} e^{ip_1\theta_1 + ip_2\theta_2}u_{p_1p_2}(\epsilon, t, x, y).$$

Each amplitude $u_{p_1p_2}$ is decomposed into powers of ε :

$$(5.22) \qquad \mathbf{u}(\epsilon, t, x, y, \theta_1, \theta_2) = \epsilon\sum_{j \geq 0}\sum_{\substack{p_1 \in \mathbb{Z} \\ p_2 \in \mathbb{Z}}} e^{ip_1\theta_1 + ip_2\theta_2}\epsilon^j u_{j,p_1p_2}(t, x, y).$$

We inject (5.22) into the equation in $\mathbf{u}$ and sort out powers of ϵ. We will see that a formal solution that is consistent at order 2 has leading terms which solve the three-wave interaction systems.

REMARK 5.6. — It was shown in [**48**] that for the Euler-Maxwell equations, the weakly nonlinear regime of geometric optics leads to *linear* transport equations. The weakly nonlinear regime is precisely the one we consider with the ansatz (5.22). (By constrast, the scaling in (1.1) corresponds to higher amplitudes for Euler-Maxwell, namely $u = O(\sqrt{\epsilon})$.) However, we derive in Section 5.1.2.5 below nonlinear transport equations, namely the three-wave interaction systems of Section 5.1.1. The discrepancy with [**48**] simply comes from the fact that we are looking at two-phase expansions here, while the result of [**48**] holds for single-phase expansions. The Euler-Maxwell equations are *less transparent* when two phases are considered, since there are more potential couplings. For instance, the fundamental phase corresponding to Fourier modes $(1, 0)$ is produced by the bilinear interaction $(1, 0) = (0, 0) + (1, 0)$, but also by $(1, 0) = (1, 1) + (0, -1)$. As seen in Section 5.1.2.5 below, transparency implies
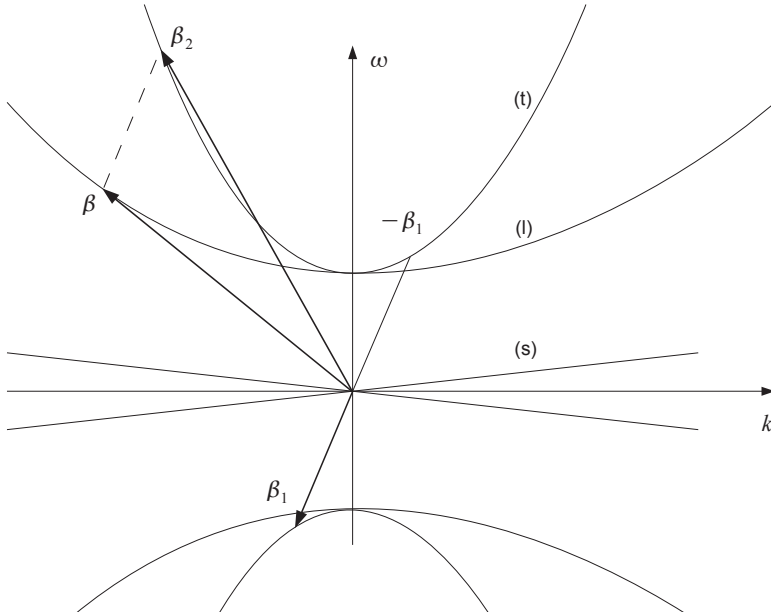
FIGURE 2. Resonance associated with the Raman instability.

the cancellation $u_{0,(0,0)} = 0$, so that the interaction $(1,0) = (0,0) + (1,0)$ is not constructive, but the interaction $(1,0) = (1,1) + (0,-1)$ is constructive, as seen on equations (5.28)-(5.29).

5.1.2.3. Choice of phases. — The phases β_1 and β_2 are chosen to be transverse, and such that

$$\beta = \beta_1 + \beta_2 \tag{5.23}$$

is a longitudinal characteristic phase. We let

$$\beta_1 = (\omega_1, k_1), \quad \beta_2 = (\omega_2, k_2), \quad \beta = (\omega, k).$$

In particular, in the following the notation β does not denote a generic characteristic phase, but *the* longitudinal characteristic phase resulting from $\beta_1 + \beta_2$.

By symmetry, $-\beta_1, -\beta_2$ and $-\beta$ also are characteristic phases. We assume that the phases $p\beta$, $p\beta_1$ and $p\beta_2$, for $p \notin \{-1, 0, 1\}$, are not characteristic, corresponding to the typical situation for a given (k_1, k_2).

If β is an electronic plasma wave, then the resonance (5.23) is associated with the phenomenon known as Raman instability ("scattering of light from optical phonons" [**3**], paragraph 8.1). This case is examined in Section 5.1.3. The corresponding (t)(t)(ℓ) resonance is pictured on Figure 2.

FIGURE 3. Resonance associated with the Brillouin instability.

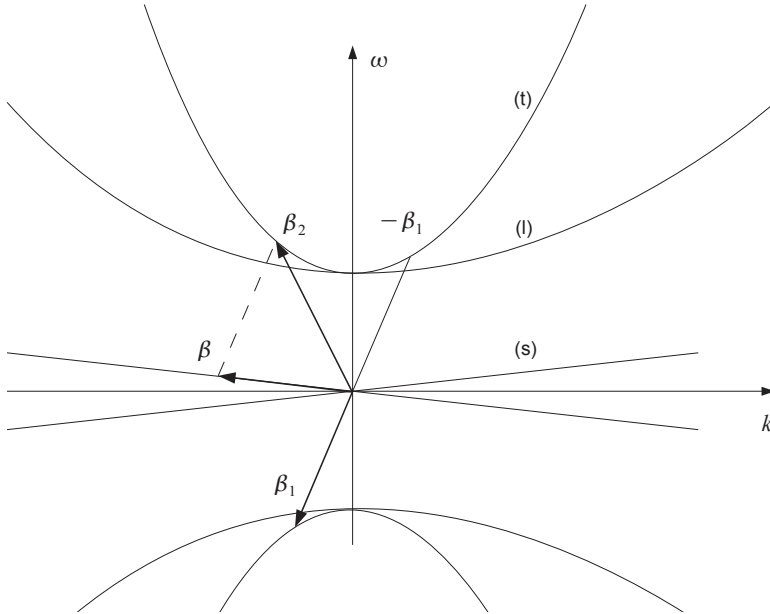If β is an acoustic wave, then in (5.23) we are looking at the Brillouin instability ("scattering of light from acoustic phonons" [3], paragraph 8.1). This case is examined in Section 5.1.4. The corresponding (t)(t)(s) resonance is pictured on Figure 3.

The notation u_{j,p_1,p_2}, introduced in (5.22), denotes the (p_1, p_2) Fourier mode of the profile $\mathbf{u}_j$, which is the $O(\epsilon^j)$ term in the expansion of $\mathbf{u}$ into powers of ϵ. In the following, we often use the slightly different notation

$$(5.24) \quad u_{j,p\beta_1} := u_{j,p,0}, \qquad u_{j,p\beta_2} := u_{j,0,p}, \qquad u_{j,p\beta} := u_{j,p,p}, \qquad p \in \{-1, 0, 1\}.$$

5.1.2.4. WKB equations: $O(1)$. — The equations for the terms $O(1)$ are

$$(-i\omega + A_\perp(ik) + A_{\perp 0})\mathbf{u}_{0\perp} = 0, \quad (-i\omega + A_\parallel(ik) + A_{\parallel 0})\mathbf{u}_{0\parallel} = 0,$$

corresponding to the dispersion relations for the phases β, β_1, β_2 and the polarization conditions for the amplitudes:

$$(5.25) \quad \mathbf{\Pi}(p\beta_j)u_{0,p\beta_j} = u_{0,p\beta_j}, \quad \mathbf{\Pi}(p\beta)u_{0,p\beta} = u_{0,p\beta}, \quad j \in \{1, 2\}, \ p \in \{-1, 0, 1\},$$

explicitly given by (5.17) and (5.18) for $p \in \{-1, 1\}$. In accordance with Section 5.1.2.3, these are the only non-trivial harmonics. In (5.25) we used notation (5.24).

We note that (5.25) for $p = 0$ implies $E_{0,0} = 0$: the mean mode of the leading amplitude in the electric field vanishes identically.

5.1.2.5. WKB equations: $O(\epsilon)$. — The equations for the terms $O(\epsilon)$ are

$$(-i\omega + A_\perp(ik) + A_{\perp 0})\mathbf{u}_{1\perp} = \mathbf{F}_{1\perp}, \quad (-i\omega + A_\parallel(ik) + A_{\parallel 0})\mathbf{u}_{0\parallel} = \mathbf{F}_{1\parallel}.$$

We project onto the kernels. The compatibility conditions

$$(5.26) \qquad \mathbf{\Pi}(p\beta_1)F_{1\perp,p\beta_1} = 0, \qquad \mathbf{\Pi}(p\beta_2)\mathbf{F}_{1\perp,p\beta_2} = 0,$$

give the evolution equations in $u_{0,p\beta_1}$ and $u_{0,p\beta_2}$, and the compatibility condition

$$(5.27) \qquad \mathbf{\Pi}(p\beta)F_{1\parallel,p\beta} = 0$$

gives the evolution equations in $u_{0,p\beta}$. There are two types of terms in these evolution equations: transport operators at the group velocities, and bilinear coupling terms. We consider in succession the mean mode $(0,0)$, the transverse modes $p\beta_1$ and $p\beta_2$, and the longitudinal modes $p\beta$, with $p = \pm 1$.

Mean mode. The Lorentz force terms are transparent: there holds, given $\mathbf{u}_0$ and phases as described in Sections 5.1.2.2 to 5.1.2.4:

$$\left(\mathbf{v}_{e0} \times \mathbf{B}_0\right)_{(0,0)} = 0, \qquad \left(\mathbf{v}_{i0} \times \mathbf{B}_0\right)_{(0,0)} = 0.$$

From there, and $E_{0,0} = 0$ (Section 5.1.2.4), we find by direct computation on the Euler-Maxwell equations that the mean mode is constant. Given the form of the datum, we infer $u_{0,(0,0)} = 0$: the mean mode of the leading amplitude vanishes identically.

Transverse modes. The nonlinear terms satisfy a form of transparency. This was first observed in [48] (see Proposition 2.1 from that reference). Given $\mathbf{u}_0$ and phases as described in Sections 5.1.2.2 to 5.1.2.4:

$$\left(\mathbf{v}_{e0z}ik\mathbf{v}_{e0y} + (\mathbf{v}_{e0} \times \mathbf{B}_0)_y\right)_{p\beta_j} = 0, \qquad j \in \{1,2\}, \ p \in \{-1,1\},$$

$$\left(\mathbf{v}_{i0z}ik\mathbf{v}_{i0y} - \frac{\theta_i}{\theta_e}(\mathbf{v}_{i0} \times \mathbf{B}_0)_y\right)_{p\beta_j} = 0, \qquad j \in \{1,2\}, \ p \in \{-1,1\}.$$

This implies that only the current density contributes to the evolution equations (5.26) for the transverse amplitudes $u_{0,p\beta_j}$. With (5.19), and the specific form of $\mathbf{F}_{1\perp}$ and $\mathbf{F}_{1\parallel}$ as given by the Euler-Maxwell equations, we find that these equations are

$$\partial_t E_{0y,p\beta_j} + \frac{k_j}{\omega_j}\partial_x E_{0y,p\beta_j} = \left(\mathbf{n}_{e0}\mathbf{v}_{e0y} - \frac{\theta_i}{\theta_e}\mathbf{n}_{i0}\mathbf{v}_{i0y}\right)_{p\beta_j}, \qquad p \in \{-1,1\}.$$

By the form of the transverse dispersion relation (5.12), the ratio k_j/ω_j is the group velocity [1]:

$$\frac{k_j}{\omega_j} = \frac{d\omega_{(\mathrm{t})}}{dk}\big|_{k=k_j}.$$

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1. We further comment on the form of the transport equations in geometric optics on page 118.

The coupling terms are made explicit, in terms of the electrical amplitudes, by use of the polarization conditions (5.17) and (5.18). We obtain

$$(5.28) \quad \partial_t E_{0y,\beta_1} + \frac{k_1}{\omega_1}\partial_z E_{0y,\beta_1} = \frac{1}{\omega_2}\Big(\frac{k\theta_e}{\omega^2 - k^2\theta_e^2} - \frac{\theta_i^3}{\theta_e^3}\frac{\alpha^2 k\theta_i}{\omega^2 - \alpha^2 k^2\theta_i^2}\Big)E_{0z,\beta}\bar{E}_{0y,\beta_1}.$$

and

$$(5.29) \quad \partial_t E_{0y,\beta_2} + \frac{k_2}{\omega_2}\partial_x E_{0y,\beta_2} = \frac{1}{\omega_1}\Big(\frac{k\theta_e}{\omega^2 - k^2\theta_e^2} - \frac{\theta_i^3}{\theta_e^3}\frac{\alpha^2 k\theta_i}{\omega^2 - \alpha^2 k^2\theta_i^2}\Big)E_{0z,\beta}\bar{E}_{0y,\beta_2}.$$

By symmetry, $E_{0y,-\beta_j} \equiv \bar{E}_{0y,\beta_j}$.

Longitudinal modes. Next we turn to the equation (5.27) in $u_{0,p\beta}$, for $|p| = 1$. The convective terms are transparent:

$$\Big(v_{e0x}\cdot ik\begin{pmatrix}\mathbf{n}_{e0}\\\mathbf{v}_{e0x}\end{pmatrix}\Big)_{p\beta} = 0, \qquad \Big(v_{i0x}\cdot ik\begin{pmatrix}\mathbf{n}_{i0}\\\mathbf{v}_{i0x}\end{pmatrix}\Big)_{p\beta} = 0.$$

The current density terms also are transparent:

$$\Big(\mathbf{n}_{e0}\mathbf{v}_{e0x} - \frac{\theta_i}{\theta_e}\mathbf{n}_{i0}\mathbf{v}_{i0x}\Big)_{p\beta} = 0.$$

Thus the only nonlinear term in the longitudinal equation comes from the Lorentz force. Just like for the transverse equations, in order to spell out equation (5.27), we use the compatibility condition (5.20) together with the explicit expression of $\mathbf{F}_{1\parallel}$ as read on the Euler-Maxwell equations. We find

$$(5.30) \quad \Big(1 + \frac{\omega^2 + k^2\theta_e^2}{\mathbf{e}^2} + \frac{\theta_i^2}{\theta_e^2}\frac{\omega^2 + \alpha^2 k^2\theta_i^2}{\mathbf{i}^2}\Big)\partial_t E_{0x,\beta} + 2k\Big(\frac{\theta_e^2\omega}{\mathbf{e}^2} + \frac{\theta_i^2}{\theta_e^2}\frac{\alpha^2\theta_i^2\omega}{\mathbf{i}^2}\Big)\partial_x E_{0x,\beta}$$
$$= \text{Lorentz force term},$$

with notation

$$\mathbf{e} := \omega^2 - k^2\theta_e^2, \qquad \mathbf{i} := \omega^2 - k^2\alpha^2\theta_i^2.$$

From (5.14), we compute by term-by-term differentiation

$$0 = 2\omega(k)\omega'(k)\Big(\frac{1}{\mathbf{e}^2} + \frac{\theta_i^2}{\theta_e^2}\frac{1}{\mathbf{i}^2}\Big) - 2k\Big(\frac{\theta_e^2}{\mathbf{e}^2} + \frac{\theta_i^2}{\theta_e^2}\frac{\alpha^2\theta_i^2}{\mathbf{i}^2}\Big), \qquad \omega' = \frac{d\omega_{(\ell)}}{dk} \text{ or } \omega' = \frac{d\omega_{(s)}}{dk}.$$

Together with (5.14), this shows that, for $p = 1$, (5.30) is a transport equation at group velocity:

$$\partial_t E_{0x,\beta} + \omega'(k)\partial_x E_{0x,\beta} = \Big(1 + \frac{\omega^2 + k^2\theta_e^2}{\mathbf{e}^2} + \frac{\theta_i^2}{\theta_e^2}\frac{\omega^2 + \alpha^2 k^2\theta_i^2}{\mathbf{i}^2}\Big)^{-1}\times\big(\text{Lorentz force term}\big).$$

The β-harmonics of the electronic Lorentz force term is

$$(\mathbf{v}_{e0}\times\mathbf{B}_0)_\beta = v_{e0,\beta_1}\times B_{0,\beta_2} + v_{e0,\beta_2}\times B_{0,\beta_1} = \begin{pmatrix}v_{e0y_1,\beta_1}\times B_{0y_2,\beta_2} - v_{e0y_2,\beta_1}B_{0y_1,\beta_2}\\0\\0\end{pmatrix}.$$

With the polarization (5.17), the x component of the β-harmonics of the electronic Lorentz force appears as

$$\frac{k}{i\omega_1\omega_2}\big(E_{0y_1,\beta_1}E_{0y_2,\beta_2} + E_{0y_1,\beta_1}E_{0y_2,\beta_2}\big).$$

Taking into account the ionic component of the Lorentz force, and the compatibility condition (5.20), the longitudinal transport equation finally takes the form
(5.31)
$$\partial_t E_{0x,\beta} + \omega'(k)\partial_x E_{0x,\beta}$$
$$= \frac{k\omega}{\omega_1\omega_2}\Big(\frac{\theta_e}{\omega^2 - k^2\theta_e^2} + \frac{\theta_i^2}{\theta_e^2}\frac{\theta_i}{\omega^2 - k^2\alpha^2\theta_i^2}\Big)\Big(E_{0y_1,\beta_1}E_{0y_1,\beta_2} + E_{0y_2,\beta_1}E_{0y_2,\beta_2}\Big).$$

In the case of one spatial transverse dimension $y \in \mathbb{R}$, or $E_{0y_2} \equiv 0$, the system of equations (5.28)-(5.29)-(5.31) fall into the category of three-wave interaction systems as described in Section 5.1.1.

REMARK 5.7. — What about stability of the WKB expansion that was sketched here? Correctors can be constructed in a classical way, implying consistency of the WKB approximation in the sense of (2.3). Short-time stability of small initial perturbations then follows by Sobolev estimates for the singular equations satisfied by the profiles, since the regime is weakly nonlinear.

REMARK 5.8. — In paragraph 2.2 of Boyd's treatise [3], three-wave interactions systems such as (5.28)-(5.29)-(5.31) are derived from Maxwell's equations, under the assumption of an ad hoc expansion for the nonlinear polarization encoding the nonlinear response of the medium. Another derivation, this time from the three-level Maxwell-Bloch equations, is given in Section 12 of [25], by means of a similar two-phase expansion in a weakly nonlinear regime. In [2], Schrödinger-Bloch systems were derived in the high-frequency limit from three-level Maxwell-Bloch systems, and formal arguments were given to further derive three-wave interaction systems from Schrödinger-Bloch systems. In [41], three-wave interaction systems are rigorously derived from the gravity-capillary water-wave system. Another derivation from Euler-Maxwell is given in [10].

5.1.3. Raman. — The Raman instability corresponds to a growth of the electronic plasma waves. Here β belongs to the (ℓ) branch of the variety. With (5.15), if the spatial transverse dimension is equal to one, or $E_{0y_2} \equiv 0$, the three-wave interaction

system (5.28)-(5.29)-(5.31) is

$$
(5.32) \quad
\begin{cases}
\partial_t E_{\beta_1} + \dfrac{k_1}{\omega_1} \partial_x E_{\beta_1} = \left(\dfrac{k\theta_e}{\omega_2} + O(\theta_i^2) \right) \bar{E}_{\beta_2} E_\beta, \\[2ex]
\partial_t E_{\beta_2} + \dfrac{k_2}{\omega_2} \partial_x E_{\beta_2} = \left(\dfrac{k\theta_e}{\omega_1} + O(\theta_i^2) \right) \bar{E}_{\beta_1} E_\beta, \\[2ex]
\partial_t E_\beta + \left(\dfrac{k\theta_e^2}{\omega} + O(\theta_i^2) \right) \partial_x E_\beta = \left(\dfrac{k\omega\theta_e}{\omega_1\omega_2} + O(\theta_i^2) \right) E_{\beta_1} E_{\beta_2},
\end{cases}
$$

locally uniformly in k, where E_{β_j} stands for E_{0y_1,β_j}, and similarly $E_\beta = E_{0x,\beta}$.

We consider in (5.32) the parameters k and $\omega, \omega_1, \omega_2$ to be fixed, θ_e to be small, and θ_i to be even smaller than θ_e. For the solution $U = (E_{\beta_1}, E_{\beta_2}, E_\beta)$ to (5.32), we posit the ansatz

$$
U(\theta_e, t, x) = V(\theta_e^{3/2} t, \theta_e^{3/2} x).
$$

Then, V solves the 3-wave interaction system (5.7), with $\varepsilon = \theta_e$, and there holds $\operatorname{sgn} b_2 b_3 = \operatorname{sgn} \dfrac{\omega}{\omega_2} > 0$. Theorem 5.1 asserts instability of the reference solution $V_a(t,x) = (a(x - c_1 t), 0, 0)$, with $c_1 = \dfrac{k_1}{\omega_1}$, under initial perturbations of the form $\varepsilon^K \phi(\varepsilon, x)$, in time $O(\sqrt{\varepsilon} |\ln \varepsilon|)$, in small balls $B(x_0, \rho)$.

In the scaling of (5.32), this translates as instability, in the sense of Theorem 2.7, of the reference solution

$$
U_a(t,x) = \left(E_{\beta_1}\left(\theta_e^{3/2}\left(x - \dfrac{k_1}{\omega_1} t \right) \right), 0, 0 \right),
$$

under initial perturbations of the form $\theta_e^K \phi(\theta_e, \theta_e^{3/2} x)$, satisfying the uniform bound $\sup_{0 < \theta_e < 1} \|\phi(\theta_e, \cdot)\|_{\theta_e, s} < \infty$. The instability occurs in long time $O(\theta_e^{-1} |\ln \theta_e|)$ and is recorded in norm $L^2(\mathbb{R}^d)$.

The coupling in (5.32) is *weak*, implying that instability are recorded only in *long time*, for which our analysis applies only if initial perturbations are *slowly varying* in x.

As shown in particular in Section 3.1.6, the amplification is maximal for the components of the solution associated with the unstable resonance. Here, this means in particular that small perturbations of the initially null electromagnetic plasma field E_β are amplified, corresponding to the Raman instability.

REMARK 5.9. — The two-dimensional model of Colin and Colin [**10**] is a refinement of system (5.32) for the description of Raman scattering.

5.1.4. Brillouin. — The Brillouin instability corresponds to a growth of the acoustic waves. Here β belongs to the (s) branch of the variety. With (5.15), if the spatial

transverse dimension is equal to one, or $E_{0y_2} \equiv 0$, the three-wave interaction system (5.28)-(5.29)-(5.31) is

$$(5.33) \quad \begin{cases} \partial_t E_{\beta_1} + \dfrac{k_1}{\omega_1} \partial_x E_{\beta_1} = \left(\dfrac{-1}{\omega_2 k \theta_e} + O(\theta_i^2) \right) \bar{E}_{\beta_2} E_\beta, \\[2ex] \partial_t E_{\beta_2} + \dfrac{k_2}{\omega_2} \partial_x E_{\beta_2} = \left(\dfrac{-1}{\omega_1 k \theta_e} + O(\theta_i^2) \right) \bar{E}_{\beta_1} E_\beta, \\[2ex] \partial_t E_\beta + \left(\theta_i \boldsymbol{\alpha} + O(\theta_i^2) \right) \partial_x E_\beta = \left(\dfrac{-1}{\omega_1 \omega_2} \dfrac{\theta_i}{\theta_e} \boldsymbol{\alpha} + O(\theta_i^2) \right) E_{\beta_1} E_{\beta_2}, \end{cases}$$

where $\boldsymbol{\alpha} := \left(\alpha^2 + \dfrac{1}{1 + k^2 \theta_e^2} \right)^{1/2}$, and E_{β_j} stands for E_{0y_1,β_j}, similarly $E_\beta = E_{0z,\beta}$.

We consider in (5.33) the parameters k and $\omega, \omega_1, \omega_2$ to be fixed, θ_e to be small, and θ_i to be even smaller than θ_e; for instance

$$(5.34) \qquad\qquad \theta_e = \varepsilon^{1/2}, \qquad \theta_i = \varepsilon.$$

For the solution $U = (E_{\beta_1}, E_{\beta_2}, E_\beta)$ to (5.32), we posit the ansatz

$$U(\varepsilon, t, x) = V(\varepsilon^{1/2} t, \varepsilon^{-1/2} x).$$

Then, V solves the 3-wave interaction system (5.9), with ε defined in (5.34), and

$$\operatorname{sgn} b_2 b_3 = \operatorname{sgn} \frac{\theta_i \boldsymbol{\alpha}}{k \omega_2} = \operatorname{sgn} \left(\frac{\omega}{\omega_2} \frac{\boldsymbol{\alpha}}{\alpha} + O(\theta_i) \right) > 0.$$

Theorem 5.2 applies. In the scales of (5.33), it asserts instability of the reference solution

$$\sqrt{\varepsilon} \Big(E_{\beta_1}(x - \frac{k_1}{\omega_1} t), 0, 0 \Big)$$

under initial perturbations of the form $\varepsilon^K \phi(\varepsilon, x)$, in time $O(\sqrt{\varepsilon} |\ln \varepsilon|)$, in $L^2(B(x_0, \rho))$ or $L^2(B(x_0, \varepsilon^\beta))$ norms.

Theorem 5.3 also applies. It asserts instability of the reference solution

$$\Big(E_{\beta_1} \Big(\sqrt{\varepsilon}(x - \frac{k_1}{\omega_1} t) \Big), 0, 0 \Big)$$

under initial perturbations of the form $\varepsilon^K \phi(\varepsilon, \varepsilon^{1/2} x)$, with $\phi_3 = O(\varepsilon^{1/2})$, in time $O(|\ln \varepsilon|)$, as measured in $L^2(\mathbb{R}^d)$.

The proof of Section 3.1 shows that small perturbations of the initially null acoustic field E_β are amplified, corresponding to the Brillouin instability.

5.2. Coupled Klein-Gordon systems with equal masses

Our second class of examples comprises coupled Klein-Gordon systems in $\mathbb{R}^d$, with equal masses and different velocities. Our motivation here is the Euler-Maxwell system describing laser-plasma interactions, which, when linearized around zero, precisely gives two such Klein-Gordon systems and an acoustic system, as we saw in

Section 5.1.2 above. This form of the linearized Euler-Maxwell system induces us to think that high-frequency instabilities in the full Euler-Maxwell could be captured by the model systems that we now describe.

We denote

$$\partial_t + A_1(\partial_x) + \frac{1}{\varepsilon}L_0, \qquad \partial_t + A_1(\theta_0 \partial_x) + \frac{1}{\varepsilon}L_0$$

the Klein-Gordon operators, with $0 < \theta_0 < 1$, implying different velocities, and

(5.35) $$A_1(\partial_x) = \begin{pmatrix} 0 & -\partial_x & 0 \\ \partial_x \cdot & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & -\omega_0 & 0 \end{pmatrix},$$

where $\omega_0 > 0$, and $x \in \mathbb{R}^d$. Expressed in terms of the unknown $U = (u, v) \in \mathbb{R}^{2(d+2)}$ with $u = (u_1, u_2, u_3)$ where $u_1 \in \mathbb{R}^d$, $u_2 \in \mathbb{R}$, $u_3 \in \mathbb{R}$, and $v = (v_1, v_2, v_3) \in \mathbb{R}^{d+1+1}$, the coupled systems have the form

(5.36) $$\begin{cases} \left(\partial_t + A_1(\partial_x) + L_0\right)u = \dfrac{1}{\sqrt{\varepsilon}}B^1(U, U), \\[2mm] \left(\partial_t + A_1(\theta_0 \partial_x) + L_0\right)v = \dfrac{1}{\sqrt{\varepsilon}}B^2(U, U), \end{cases}$$

where B^1 and B^2 are bilinear $\mathbb{R}^{2(d+2)} \times \mathbb{R}^{2(d+2)} \to \mathbb{R}^{2(d+2)}$. The eigenvalues $\lambda(\xi)$ of matrix $A_1(\xi) + L_0/i$ are

(5.37) $$\lambda_1(\xi) = \sqrt{\omega_0^2 + |\xi|^2} = -\lambda_4(\xi),$$

and a multiplicity-d null branch. Similarly, the eigenvalues of matrix $A_1(\theta_0 \xi) + L_0/i$ are

(5.38) $$\lambda_2(\xi) = \sqrt{\omega_0^2 + \theta_0^2 |\xi|^2} = -\lambda_3(\xi),$$

and a multiplicity-d null branch. We denote $\lambda_5 \equiv 0$ the null branch for the whole system, with total multiplicity $2d$. The eigenvalues are depicted on Figure 4. There are two fast Klein-Gordon branches, corresponding to (5.37), and two slow Klein-Gordon branches (slow since $\theta_0 < 1$), corresponding to (5.38).

Thus we see that in approximating Euler-Maxwell by a system of the form (5.36), disregarding the specific form of the right-hand side of (5.36), we are simply approximating the speed of sound by zero and neglecting convective terms.

We will show that for these Klein-Gordon operators, some bilinear coupling terms B^1, B^2 allow for non-trivial WKB solutions that are unstable, which we believe to be the situation for Euler-Maxwell (in the case of highly-oscillating data; non-oscillating data are known to generate stable WKB solutions [49]). In Euler-Maxwell, it is the current density in the Ampère equation and the Maxwell-Lorentz force in the equation of conservation of momentum that couple the Klein-Gordon and acoustic systems, as we saw in Section 5.1.2.
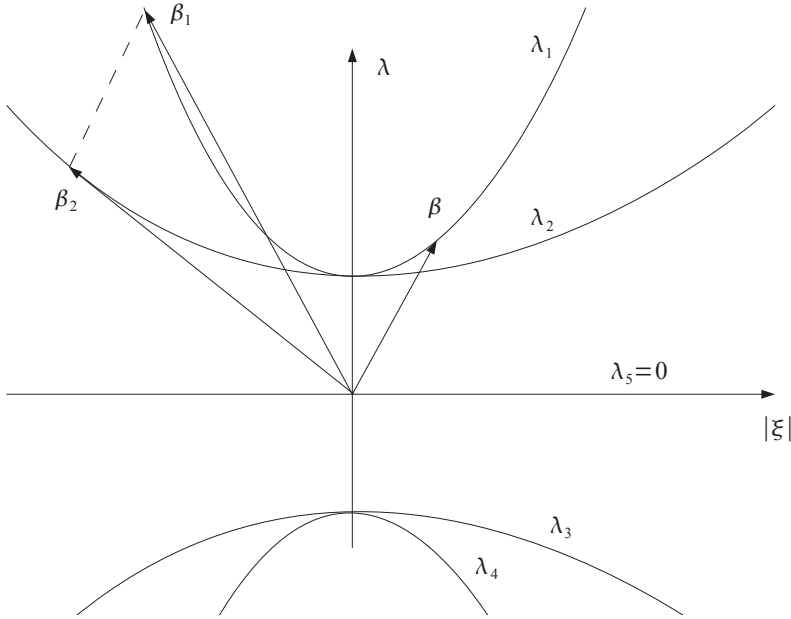
FIGURE 4. Coupled Klein-Gordon with equal masses.

For definiteness we will perform computations on the following explicit expressions for these coupling terms: coordinatizing

$$U = (u_1, u_2, u_3, v_1, v_2, v_3), \qquad U' = (u'_1, u'_2, u'_3, v'_1, v'_2, v'_3),$$

we let

(5.39)
$$B^1(U, U') = \frac{1}{2} \begin{pmatrix} 0 \\ u_3 v'_3 + v_3 u'_3 + v_3 v'_3 \\ 0 \end{pmatrix},$$

$$B^2(U, U') = \frac{1}{2} \begin{pmatrix} 0 \\ -u_2 u'_2 + v_2 v'_3 + v_3 v'_2 \\ 0 \end{pmatrix}.$$

We denote

$$A(\partial_x) := \begin{pmatrix} A_1(\partial_x) & 0 \\ 0 & A_1(\theta_0 \partial_x) \end{pmatrix}, \quad A_0 := \begin{pmatrix} L_0 & 0 \\ 0 & L_0 \end{pmatrix}, \quad B(\cdot, \cdot) := \begin{pmatrix} B^1(\cdot, \cdot) \\ B^2(\cdot, \cdot) \end{pmatrix}.$$

Then equation (5.36) takes the form (1.1).

For system (5.36), we check that Assumption 2.1 is satisfied (a simple property of the Klein-Gordon operators), that for most initial wavenumbers Assumption 2.2 is satisfied as well (the result of WKB computations performed in Section 5.2.2), and

finally, in Section 5.2.3, that Assumption 2.8 is satisfied in one space dimension $d = 1$, with $\mathbf{\Gamma} > 0$.

Thus the conclusions of Theorems 2.9, 2.12 and 2.13 apply, proving instability of the WKB solutions described below, if $d = 1$.

5.2.1. Verification of Assumption 2.1: smooth spectral decomposition. —
The spectral decomposition is

$$A_1(\xi) + L_0/i = \lambda_1(\xi)P_1(\xi) - \lambda_1(\xi)P_3(\xi) + 0 \cdot P_5(\xi).$$

The eigenvalues are separated, implying regularity of both eigenvalues and eigenprojectors (see for instance Theorem 1.8 in Chapter 2 of [26]).

It remains to prove bounds (2.2). By Lemma 6.19 in Appendix 6.5, these follow from a regularity result at infinity. The associated symbol at infinity is

$$A_\infty(\bar{\omega}, x) = A_1(\bar{\omega}) - ixL_0 = \begin{pmatrix} 0 & \bar{\omega} & 0 \\ \bar{\omega}\cdot & 0 & -ix\omega_0 \\ 0 & ix\omega_0 & 0 \end{pmatrix}, \qquad (\bar{\omega}, x) \in \mathbb{S}^{d-1} \times \mathbb{R}.$$

It has simple eigenvalues $\pm(\bar{\omega}^2 + x^2\omega_0^2)^{1/2}$, and a multiplicity-three null eigenvalue. Given $\bar{\omega} \in \mathbb{S}^{d-1}$, the eigenvalues are separated in a neighborhood of $x = 0$. Hence the spectral decomposition of A_∞ is smooth at $(\bar{\omega}, 0)$, for all $\bar{\omega} \in \mathbb{S}^{d-1}$, and Lemma 6.19 applies, implying bounds (2.2) for $A(\partial_x) + L_0$. Naturally, this also applies to the operator $A_1(\theta_0\partial_x) + L_0$, hence to the total operator $A(\partial_x) + A_0$.

5.2.2. Verification of Assumption 2.2: WKB expansion. —
We select a characteristic temporal frequency $\omega \in \mathbb{R}$ associated with the initial wavenumber $k \in \mathbb{R}^d$, such that the following conditions are satisfied:

– The phase $\beta = (\omega, k)$ belongs to the fast positive Klein-Gordon branch on the variety:

$$\omega = \sqrt{\omega_0^2 + |k|^2}.$$

– The only harmonics of β on the fast Klein-Gordon branches are $p \in \{-1, 1\}$:

$$p^2\omega^2 = \omega_0^2 + p^2|k|^2 \implies p \in \{-1, 1\}.$$

– No harmonics of β belongs to the slow branches on the variety:

$$p^2\omega^2 \neq \omega_0^2 + p^2\theta_0^2|k|^2, \quad \text{for all } p \in \mathbb{Z}.$$

– There are no auto-resonances: the equations in $\xi \in \mathbb{R}^d$

$$\lambda_1(\xi + k) = \pm\omega + \lambda_1(\xi), \quad \lambda_2(\xi + k) = \pm\omega + \lambda_2(\xi)$$

have no solution.

While this may seem like a lot of requirements on the fundamental phase, a look at Figure 4 should suffice to convince the reader that for most phases β on the fast positive Klein-Gordon branch, these assumptions are satisfied.

The zeroth harmonics $p = 0$ belongs to the variety. With the above, this implies in particular that condition (6.47) on page 115, describing $\{-1, 0, 1\}$ as the set of characteristic harmonics of the fundamental phase (ω, k), is satisfied.

By Proposition 6.21 in Appendix 6.6, in order to verify Assumption 2.2, it then suffices to check that the weak transparency condition (6.48) is satisfied.

In this view, borrowing notation from Appendix 6.6, we denote $\mathbf{\Pi}(p\beta)$ the orthogonal projector onto $\ker(-ip\omega + A(ipk) + A_0)$. For $|p| = 1$, these kernels are one-dimensional, generated by $\vec{e}_1$ and $\vec{e}_{-1} = (\vec{e}_1)^*$, with notation

$$(5.40) \qquad \vec{e}_1 := \frac{1}{\sqrt{2}}\Big(-\frac{k}{\omega}, 1, \frac{i\omega_0}{\omega}, 0_{\mathbb{C}^{d+2}}\Big) \in \mathbb{C}^{2(d+2)},$$

so that

$$\mathbf{\Pi}(\beta)U = (U, \vec{e}_1)\vec{e}_1, \qquad \mathbf{\Pi}(-\beta)U = (U, \vec{e}_1^*)\vec{e}_1^*,$$

denoting $(\cdot, \cdot)$ the Hermitian scalar product in $\mathbb{C}^{2(d+2)}$. The orthogonal projector $\mathbf{\Pi}(0)$ onto the six-dimensional kernel of $A(0) + A_0/i$ is

$$\mathbf{\Pi}(0)U = (u_1, 0, 0, v_1, 0, 0),$$

implying, for B given by (5.39), the identities

$$\mathbf{\Pi}(0)B \equiv 0, \quad B(\mathbf{\Pi}(0)\cdot, \cdot) \equiv 0, \quad B(\cdot, \mathbf{\Pi}(0)\cdot) \equiv 0,$$

which yield (6.48).

5.2.3. Verification of Assumption 2.8: resonances and transparency. — We verify here the conditions (i) boundedness, (ii) partial transparency, and (iii) rank-one interaction coefficients of Assumption 2.8.

By the form of the characteristic variety, and the choice of β, resonant pairs are

$$\mathfrak{R} := \big\{(1, 2),\ (1, 5),\ (2, 5),\ (3, 4),\ (5, 3),\ (5, 4)\big\}.$$

A resonant frequency is pictured on Figure 4, corresponding to a $(1, 2)$ resonance, which we may denote $\beta_1 = \beta + \beta_2$, or $\lambda_1(\xi + k) = \omega + \lambda_2(\xi)$.

(i) Boundedness of $\mathfrak{R}$. Here we apply Lemma 6.27 page 121, as we may since the assumptions of Lemma 6.19 have been verified in Section 5.2.1. The asymptotic branches on the variety are obviously distinct (Figure 4).

(ii) Partial transparency. From the definition of B in (5.39) and $\vec{e}_1$ in (5.40), we see that

$$B(\vec{e}_1)U = \frac{1}{\sqrt{2}}\Big(0_{\mathbb{C}^{d+2}}, \frac{i\omega_0}{\omega}v_3, 0, 0_{\mathbb{C}^d}, -u_2, 0\Big),$$

$$B(\vec{e}_{-1})U = \frac{1}{\sqrt{2}}\Big(0_{\mathbb{C}^{d+2}}, \frac{-i\omega_0}{\omega}v_3, 0, 0_{\mathbb{C}^d}, -u_2, 0\Big).$$

An element in the image of $\Pi_5(\xi)$, the orthogonal projector onto the kernel of the matrix $A(\xi) + A_0/i$, has the form

$$U_5(\xi) = \left(u_1, 0, \frac{-i\xi \cdot u_1}{\omega_0}, v_1, 0, \frac{-i\theta_0\xi \cdot v_1}{\omega_0} \right), \qquad (u_1, v_1) \in \mathbb{C}^6.$$

In particular, for all $U \in \mathbb{C}^{2(d+2)}$, $B(\vec{e}_{\pm 1})U$ belongs to the orthogonal of the range of $\Pi_5(\xi)$, so that

$$(5.41) \qquad\qquad \Pi_5(\cdot)B(\vec{e}_{\pm 1}) \equiv 0.$$

The other projectors are

$$\Pi_j(\xi)U = \frac{1}{\sqrt{2}} \big(U, \Omega_j(\xi) \big) \, \Omega_j(\xi),$$

where

$$\Omega_j(\xi) := \frac{1}{\sqrt{2}} \Big(-\frac{\xi}{\lambda_j}, 1, \frac{i\omega_0}{\lambda_j}, 0_{\mathbb{C}^{d+2}} \Big), \ j = 1, 4;$$

and

$$\Omega_{j'}(\xi) := \Big(0_{\mathbb{C}^{d+2}}, -\frac{\theta_0\xi}{\lambda_{j'}}, 1, \frac{i\omega_0}{\lambda_{j'}} \Big), \ j' = 2, 3.$$

From there, we compute

$$\Pi_2(\xi)B(e_{\pm 1})\Pi_5(\xi') \equiv 0, \quad \Pi_3(\xi)B(\vec{e}_{\pm 1})\Pi_5(\xi') \equiv 0.$$

Together with (5.41) this implies that resonances $(2, 5)$ and $(5, 3)$ are transparent. Besides,

$$(5.42) \qquad
\begin{aligned}
\Big(\Omega_1(\xi'), \, B(\vec{e}_1)\Omega_2(\xi) \Big) &= \frac{-\omega_0^2}{2\omega\lambda_2(\xi)}, \\
\Big(\Omega_2(\xi), \, B(\vec{e}_{-1})\Omega_1(\xi') \Big) &= \Big(\Omega_3(\xi), \, B(\vec{e}_1)\Omega_4(\xi') \Big) = \frac{-1}{2}, \\
\Big(\Omega_1(\xi + k), \, B(\vec{e}_1)U_5(\xi) \Big) &= \frac{\theta_0\xi \cdot v_1}{2\omega}, \\
\Big(\Omega_4(\xi), \, B(\vec{e}_{-1})U_5(\xi + k) \Big) &= -\frac{\theta_0(\xi + k) \cdot v_1}{2\omega}.
\end{aligned}$$

Thus

$$\mathfrak{R}_0 := \big\{ (1, 2), \, (1, 5), \, (3, 4), \, (5, 4) \big\},$$

a subset of $\mathfrak{R}$ which does not contain auto-resonances. The associated resonant sets are

$$(5.43) \qquad
\begin{aligned}
\mathcal{R}_{12} &= \left\{ \xi : \sqrt{\omega_0^2 + |\xi + k|^2} = \omega + \sqrt{\omega_0^2 + \theta_0^2|\xi|^2} \right\}, \\
\mathcal{R}_{15} &= \{ \xi : |\xi + k| = |k| \}, \\
\mathcal{R}_{34} &= \left\{ \xi : \sqrt{\omega_0^2 + |\xi|^2} = \omega + \sqrt{\omega_0^2 + \theta_0^2|\xi + k|^2} \right\}, \\
\mathcal{R}_{54} &= \{ \xi : |\xi| = |k| \},
\end{aligned}$$

where we recall $\omega = \sqrt{\omega_0^2 + |k|^2}$.

We now turn to the verification of condition (2.18) in the partial transparency condition Assumption 2.8(ii). The only relevant intersection here is

$$\mathcal{R}_{15} \bigcap \Big(\mathcal{R}_{54} + k \Big) = \{\xi : |\xi + k| = |k|\} \bigcap \big(\{\xi : |\xi - k| = |k|\} \big) = \{0\}.$$

The ratio interaction coefficient over phase is

$$\frac{\Big(\Omega_1(\xi + k),\, B(\vec{e}_1) U_5(\xi) \Big)}{\lambda_1(\xi + k) - \omega} = \frac{\theta_0 \xi \cdot v_1}{4\omega} \frac{\sqrt{\omega_0^2 + |\xi + k|^2} + \sqrt{\omega_0^2 + |k|^2}}{|\xi|^2 + 2\xi \cdot k}.$$

This ratio is bounded in a neighborhood of $\xi = 0$ only in one space dimension. Hence Assumption 2.8(i) holds in one space dimension only.

We finally turn to the verification of condition (2.19). There holds

$$\mathcal{R}_{12} \bigcap \mathcal{R}_{15} \subset \{\lambda_2 = \lambda_5\} = \varnothing.$$

Similarly, by (5.43)(iii)(-iv),

$$\mathcal{R}_{34} \bigcap \mathcal{R}_{54} = \varnothing.$$

(iii) Rank-one coefficients: the eigenvalues λ_j, for $j \in \{1, 2, 3, 4\}$, are simple eigenvalues, implying that the interaction coefficients have rank at most one. The kernel, associated with $\lambda_5 \equiv 0$, has multiplicity 5, but by (5.41) the interaction coefficients b_{51}^- and b_{54}^+ are identically zero. This verifies Assumption 2.8(iii).

5.2.4. Stability index. — There holds

$$\Pi_1(\xi + k) B(\vec{e}_1) \Pi_2(\xi) B(\vec{e}_{-1}) \Pi_1(\xi + k)$$
$$= \Big(\Omega_1(\xi + k),\, B(\vec{e}_1) \Omega_2(\xi) \Big) \Big(\Omega_2(\xi),\, B(\vec{e}_{-1}) \Omega_1(\xi + k) \Big) \Pi_1(\xi + k),$$

so that the trace of the product of the $(1, 2)$ interaction coefficients is equal to

$$\Gamma_{12}(\xi) = \Big(\Omega_1(\xi + k),\, B(\vec{e}_1) \Omega_2(\xi) \Big) \Big(\Omega_2(\xi),\, B(\vec{e}_{-1}) \Omega_1(\xi + k) \Big),$$

and with (5.42)(i) and (5.42)(ii) we find

$$\Gamma_{12}(\xi) = \frac{\omega_0^2}{4\omega \lambda_2(\xi)}.$$

Besides,

$$\Gamma_{34}(\xi) = \frac{\omega_0^2}{4\omega \lambda_2(\xi + k)}, \quad \Gamma_{15}(\xi) = \Gamma_{54}(\xi) \equiv 0.$$

This gives $\mathbf{\Gamma} > 0$, implying instability.

5.3. Coupled Klein-Gordon systems with different masses

Our third class of examples is made up of coupled Klein-Gordon systems with different velocities and different masses. As we will see, the assumption that masses are different implies a much smaller resonant set than in our previous example.

Borrowing notation from Section 5.2, in particular (5.35), we consider systems

$$\begin{cases} \partial_t u + A_1(\partial_x)u + \dfrac{1}{\varepsilon}\alpha_0 L_0 u = \dfrac{1}{\sqrt{\varepsilon}} B^3(U, U), \\[2mm] \partial_t v + A_1(\theta_0 \partial_x)v + \dfrac{1}{\varepsilon} L_0 v = \dfrac{1}{\sqrt{\varepsilon}} B^4(U, U). \end{cases}$$

We assume here $\alpha_0 > 1$, in contrast with (5.36). We consider the bilinear forms B^3 and B^4 defined by

(5.44)
$$B^3(U, U') = \frac{1}{2} \begin{pmatrix} 0 \\ u_3 v_3' + v_3 u_3' + v_3 v_3' \\ 0 \end{pmatrix},$$

$$B^4(U, U') = \frac{-\iota}{2} \begin{pmatrix} 0 \\ u_2 u_2' + u_2 v_2' + v_2 u_2' \\ 0 \end{pmatrix}, \quad \iota \in \{-1, 1\},$$

with

$$U = \big(u_1, u_2, u_3, v_1, v_2, v_3\big) \in \mathbb{C}^{2(d+2)}, \quad U' = \big(u_1', u_2', u_3', v_1', v_2', v_3'\big) \in \mathbb{C}^{2(d+2)}.$$

For the rest, in particular t, x, A_1, L_0 and θ_0, the notations are borrowed from Section 5.2.

The spectral decomposition is

$$A(\xi) + A_0/i = \sum_{j=1}^{5} \lambda_j(\xi)\Pi_j(\xi),$$

with eigenvalues

$$\lambda_1(\xi) := \sqrt{\alpha_0^2 \omega_0^2 + |\xi|^2} = -\lambda_4(\xi), \quad \lambda_2(\xi) := \sqrt{\omega_0^2 + \theta_0^2|\xi|^2} = -\lambda_3(\xi), \quad \lambda_5(\xi) \equiv 0.$$

The verification of Assumption 2.1 goes exactly as in Sections 5.2.1. In the upcoming Sections, we check that Assumptions 2.2 and 2.8 are satisfied, then verify that there holds $\operatorname{sgn}\boldsymbol{\Gamma} = \operatorname{sgn}\iota$. Stability ensues if $\iota = -1$, and instability if $\iota = 1$.

5.3.1. Verification of Assumption 2.2: WKB expansion. —

We select a fundamental phase $\beta = (\omega, k) \in \mathbb{R}^{1+d}$ as follows:

– The phase β belongs to the slow positive Klein-Gordon branch on the variety:

$$\omega = \sqrt{\omega_0^2 + \theta_0^2|k|^2}.$$

– The only harmonics of β on the slow Klein-Gordon branches are -1 and 1.

– No harmonics of β belong to the fast branches on the variety.
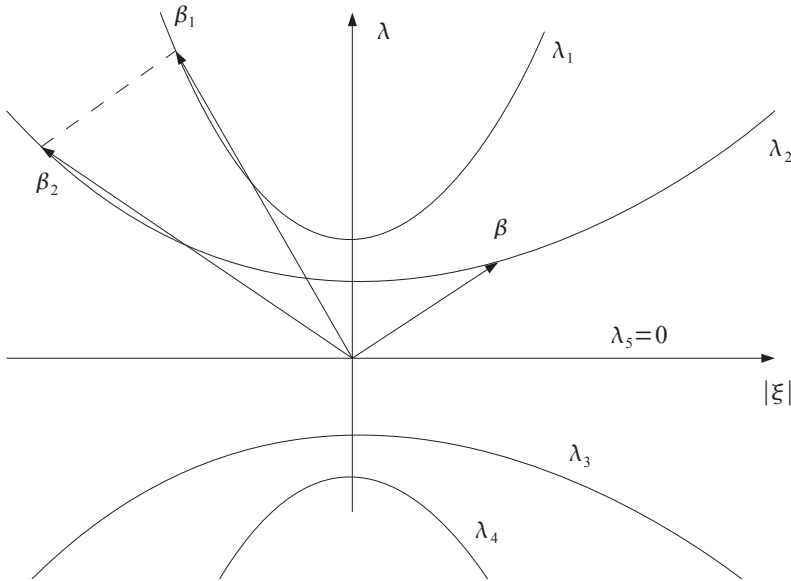
FIGURE 5. Coupled Klein-Gordon with different masses.

– There are no auto-resonances.

For more details, and comments on these conditions, see Section 5.2.2. In addition, we restrict the range of k as we assume

$$(5.45) \qquad |k|^2 < \frac{1}{\theta_0^2}(\alpha_0^2 - 1)\omega_0^2.$$

It is easy to verify that condition (5.45) implies that there are no $(1,5)$ resonances, and no $(5,4)$ resonances.

Denoting $\mathbf{\Pi}(p\beta)$ the orthogonal projector onto $\ker(-ip\omega + A(ipk) + A_0)$, we find the identities

$$\mathbf{\Pi}(0)U = (u_1, 0, 0, v_1, 0, 0), \quad \mathbf{\Pi}(\beta)U = (U, \vec{e}_1)\vec{e}_1, \quad \mathbf{\Pi}(-\beta)U = (U, \vec{e}_{-1})\vec{e}_{-1},$$

for any $U = (u_1, u_2, u_3, v_1, v_2, v_3)$, where

$$(5.46) \qquad \vec{e}_1 := \big(0_{\mathbb{C}^{d+2}}, -\frac{\theta_0 k}{\omega}, 1, \frac{i\omega_0}{\omega}\big), \quad \vec{e}_{-1} = (\vec{e}_1)^*.$$

Then, as in Section (5.2.2), there holds

$$\mathbf{\Pi}(0)B \equiv 0, \quad B\big(\mathbf{\Pi}(0), \cdot\big) \equiv 0, \quad B\big(\cdot, \mathbf{\Pi}(0)\big) \equiv 0,$$

and condition (6.48) is satisfied. By Proposition 6.21, Assumption 2.2 is then satisfied.

5.3.2. Verification of Assumption 2.8: **resonances and transparency.** — By the form of the characteristic variety, and choice of β (in particular, condition (5.45)), resonant pairs are

$$\mathfrak{R} = \big\{(1,2), \quad (2,5), \quad (3,4), \quad (5,3)\big\}.$$

A $(1,2)$ resonant frequency is pictured on Figure 5.

(i) Boundedness of $\mathfrak{R}$. The asymptotic branches on the variety are obviously distinct (Figure 5). Lemma 6.27 then implies boundedness of $\mathfrak{R}$.

(ii) Partial transparency. From the definition of B in (5.44) and $\vec{e}_1$ in (5.46), we see that

$$B(\vec{e}_1)U = \frac{1}{\sqrt{2}}\Big(0_{\mathbb{C}^d}, \frac{i\omega_0}{\omega}(u_3 + v_3), 0, 0_{\mathbb{C}^d}, -\iota u_2, 0\Big),$$

$$B(\vec{e}_{-1})U = \frac{1}{\sqrt{2}}\Big(0_{\mathbb{C}^d}, \frac{-i\omega_0}{\omega}(u_3 + v_3), 0, 0_{\mathbb{C}^d}, -\iota u_2, 0\Big).$$

An element in the image of $\Pi_5(\xi)$, the orthogonal projector onto the kernel of the matrix $A(\xi) + A_0/i$, has the form

$$U_5(\xi) = \Big(u_1, 0, \frac{-i\xi \cdot u_1}{\alpha_0 \omega_0}, v_1, 0, \frac{-i\theta_0 \xi \cdot v_1}{\omega_0}\Big), \qquad (u_1, v_1) \in \mathbb{C}^{2d}.$$

In particular, for all $U \in \mathbb{C}^{2(d+2)}$, $B(\vec{e}_{\pm 1})U$ belongs to the orthogonal of the range of $\Pi_5(\xi)$, so that

$$(5.47) \qquad\qquad\qquad \Pi_5(\cdot)B(\vec{e}_{\pm 1}) \equiv 0.$$

The other projectors are

$$\Pi_j(\xi)U = \frac{1}{\sqrt{2}}\big(U, \Omega_j(\xi)\big)\,\Omega_j(\xi),$$

where

$$\Omega_j(\xi) := \frac{1}{\sqrt{2}}\Big(-\frac{\xi}{\lambda_j}, 1, \frac{i\alpha_0\omega_0}{\lambda_j}, 0_{\mathbb{C}^{d+2}}\Big),\; j = 1, 4;$$

and

$$\Omega_{j'}(\xi) := \Big(0_{\mathbb{C}^{d+2}}, -\frac{\theta_0 \xi}{\lambda_{j'}}, 1, \frac{i\omega_0}{\lambda_{j'}}\Big),\; j' = 2, 3.$$

From there, we compute

$$(5.48) \qquad\qquad \Pi_2(\xi)B(\vec{e}_{\pm 1})\Pi_5(\xi') \equiv 0, \quad \Pi_3(\xi)B(\vec{e}_{\pm 1})\Pi_5(\xi') \equiv 0.$$

Together with (5.47) this implies that resonances $(2,5)$ and $(5,3)$ are transparent. Besides,

$$\Big(\Omega_1(\xi'),\, B(\vec{e}_1)\Omega_2(\xi)\Big) = \frac{-\omega_0^2}{2\omega\lambda_2(\xi)}, \qquad \Big(\Omega_2(\xi'),\, B(\vec{e}_{-1})\Omega_1(\xi)\Big) = \frac{-\iota}{2}.$$

In particular, the $(1,2)$ resonance is non-transparent: $(1,2) \in \mathfrak{R}_0$. Similarly, the $(3,4)$ resonance belongs to $\mathfrak{R}_0$.

The partial transparency is automatically satisfied because the sets in (2.18) and (2.19) are all empty, a direct consequence of the form of $\mathfrak{R}_0$ (in which no index appears more than once).

(iii) Rank-one coefficients: the eigenvalues λ_j, for $j \in \{1, 2, 3, 4\}$, are simple eigenvalues, implying that the interaction coefficients have rank at most one.

5.3.3. Stability index. — Computing as in Section 5.2.4 and using (5.48), we find

$$\Gamma_{12}(\xi) = \iota \frac{\omega_0^2}{4\omega\lambda_2(\xi)}, \qquad \Gamma_{34}(\xi) = \iota \frac{\omega_0^2}{4\omega\lambda_2(\xi + k)},$$

implying $\operatorname{sgn} \mathbf{\Gamma} = \operatorname{sgn} \iota$.

CHAPTER 6

APPENDIX

6.1. Symbols and operators

Given $m \in \mathbb{R}$, we denote S^m the set of matrix-valued symbols $a \in C^{\bar{s}}(\mathbb{R}_x^d; C^\infty(\mathbb{R}_\xi^d))$, such that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq \bar{s}$, for all $\beta \in \mathbb{N}^d$, for some $C_{\alpha\beta} > 0$, for all (x, ξ),

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}, \qquad \langle \xi \rangle := (1 + |\xi|^2)^{1/2}.$$

That is, we consider symbols with a finite, but large, spatial regularity $\bar{s}$, in connection with the finite Sobolev regularity s_a of the approximate solution u_a, postulated in Assumption 2.2.

We call S^m the space of classical symbols of order m. Given $a \in S^m$, the associated family of pseudo-differential operators in semi-classical quantization is denoted $\mathrm{op}_\varepsilon(a)$ and formally defined by their action on functions or distributions u in the variable x:

$$(6.1) \qquad \mathrm{op}_\varepsilon(a)u := \int e^{ix\cdot\xi} a(x, \varepsilon\xi)\hat{u}(\xi)\, d\xi, \qquad \varepsilon > 0.$$

The semi-classical Sobolev norms $\|\cdot\|_{\varepsilon,s}$ are defined by

$$(6.2) \qquad \|u\|_{\varepsilon,s}^2 := \int (1 + |\varepsilon\xi|^2)^s |\hat{u}(\xi)|^2\, d\xi.$$

6.1.1. Estimates for Fourier multipliers. — *Fourier multipliers are pseudo-differential symbols (by extension, the associated operators) which do not depend on x. Examples of Fourier multipliers in the text are given by the eigenprojectors Π_j and the eigenvalues λ_j. The interaction coefficients $\mathcal{B}$, b_{ij}, the normal form Q, and the interaction matrix M all depend on (x, ξ), but as tensor products $M(x, \xi) = M_1(x)M_2(\xi)$ they are handled just like Fourier multipliers.*

Given a Fourier multiplier $a \in S^0$, the associated operators $\mathrm{op}_\varepsilon(a)$ map H^s to H^s, for all s, and for all $u \in H^s$,

$$(6.3) \qquad \|\mathrm{op}(a)u\|_{\varepsilon,s} \lesssim |a|_{L^\infty} \|u\|_{\varepsilon,s}, \qquad a = a(\xi).$$

Also, denoting $|\cdot|_{\mathcal{F}L^1}$ the L^1 norm of the Fourier transform: $|u|_{\mathcal{F}L^1} := |\hat{u}|_{L^1}$, there holds, by Young's convolution inequality, the bound

$$(6.4) \qquad \left|a(x)\mathrm{op}_\varepsilon(b)u\right|_{\mathcal{F}L^1} \leq |b|_{L^\infty}|a|_{\mathcal{F}L^1}|u|_{\mathcal{F}L^1}, \qquad b = b(\xi) \in L^\infty.$$

Pointwise estimates follow from Hausdorff-Young and Cauchy-Schwartz:

$$(6.5) \qquad |\mathrm{op}_\varepsilon(a)u|_{L^\infty} \leq C|a|_{L^2}|u|_{L^2}, \qquad a = a(\xi).$$

Given a Fourier multiplier $a \in S^1$, and $f \in H^{s+d/2+\eta}$ for some $\eta > 0$, there holds for all $u \in H^s$,

$$(6.6) \qquad \left\|\left[\mathrm{op}_\varepsilon(a), f\right]u\right\|_{\varepsilon,s} \lesssim \varepsilon\,|\nabla a|_{L^\infty}\|f\|_{H^{s+d/2+\eta}}\|u\|_{\varepsilon,s}, \qquad a = a(\xi),$$

and

$$(6.7) \qquad \left|\mathcal{F}\left(\left[\mathrm{op}_\varepsilon(a), f\right]u\right)\right|_{L^1} \lesssim \varepsilon\,|\nabla a|_{L^\infty}|\widehat{\partial_x f}|_{L^1}|\hat{u}|_{L^1}, \qquad a = a(\xi).$$

Similarly, for the Fourier multiplier $\Lambda^s := \mathrm{op}_\varepsilon(\langle\cdot\rangle^s)$, given $f \in H^{s+d/2+\eta}$ for some $\eta > 0$, there holds for all $u \in H^{s-1}$:

$$(6.8) \qquad \left|\left[\Lambda^s, f\right]u\right|_{L^2} \lesssim \varepsilon\|f\|_{H^{s+d/2+\eta}}\|u\|_{\varepsilon,s-1}.$$

6.1.2. Estimates for pseudo-differential operators. — *Genuine pseudo-differential operators arise in the proof via S, the flow of M. Essentially, S is the exponential $\exp(tM)$, hence cannot be written as a function of x times a Fourier multiplier.*

We first introduce para-differential symbols, which are regularized pseudo-differential symbols. Then we give an action result (Proposition 6.5) and a composition result (Proposition 6.6), before giving a comparison result (Proposition 6.8).

Given $\phi_0 \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \phi_0 \leq 1$, and real numbers $0 < A < B < 2A$ such that

$$\phi_0 \equiv 1 \text{ for } |\xi| \leq A, \text{ and } \phi_0 \equiv 0 \text{ for } |\xi| \geq B.$$

We let

$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi), \qquad \text{for } j \geq 1,$$

so that for $j \geq 1$, ϕ_j has support included in the annulus $\mathcal{C}_j := \{A2^{j-1} \leq |\xi| \leq B2^j\}$, and is constant equal to one in the annulus $\tilde{\mathcal{C}}_j := \{B2^{j-1} \leq |\xi| \leq A2^j\}$.

The function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined for $N \geq 2$ by

$$\psi(\eta,\xi) = \sum_{k\geq0}\phi_0(2^{-k+N}\eta)\phi_k(\xi).$$

is called a Bony admissible cut-off [**1**]. It satisfies

$$\psi(\eta,\xi) \equiv \begin{cases} 1, & |\eta| \leq 2^{-N}\langle\xi\rangle, \\ 0, & |\eta| \geq 2^{1-N}\langle\xi\rangle. \end{cases}$$

DEFINITION 6.1. — Given a Bony admissible cut-off ψ and $a \in \Gamma_k^m$, we call para-differential symbol associated with a the symbol

$$a^\psi(x,\xi) := \mathcal{F}^{-1}\Big(\psi(\cdot,\xi)\hat{a}(\cdot,\xi)\Big)(x) = \Big((\mathcal{F}^{-1}\psi(\cdot,\xi)) \star a(\cdot,\xi)\Big)(x),$$

where convolution takes place in the spatial variable x. The smooth function $\mathcal{F}^{-1}\psi(\cdot,\xi)(x)$ is the inverse Fourier transform of ψ in its first variable η. The pseudo-differential operator $\mathrm{op}^\psi(a) = \mathrm{op}(a^\psi)$ is said to be the para-differential operator associated with a in classical quantization.

We define the para-differential operator (precisely, the ε-dependent family of operators) associated with a in semi-classical quantization by

$$(6.9) \qquad \mathrm{op}_\varepsilon^\psi(a) := h_\varepsilon^{-1}\mathrm{op}^\psi(\tilde{a})h_\varepsilon, \qquad \text{with } \tilde{a}(x,\xi) := a(\varepsilon x,\xi),$$

where

$$(h_\varepsilon u)(x) := \varepsilon^{d/2}u(\varepsilon x), \qquad \|h_\varepsilon u\|_{H^s} = \|u\|_{\varepsilon,s}.$$

REMARK 6.2. — Note that the maps $a \to \tilde{a}$ and $a \to a^\psi$ do not commute:

$$\mathrm{op}_\varepsilon^\psi(a) = h_\varepsilon^{-1}\mathrm{op}\big((\tilde{a})^\psi\big)h_\varepsilon \neq h_\varepsilon^{-1}\mathrm{op}(\widetilde{a^\psi})h_\varepsilon = \mathrm{op}_\varepsilon(a^\psi),$$

so that the para-differential operator associated with a in semi-classical quantization is *not* $\mathrm{op}_\varepsilon(a^\psi)$. The classical symbol of $\mathrm{op}_\varepsilon^\psi(a)$ is

$$(x,\xi) \to \mathbf{a}(x,\xi) = \Big(\mathcal{F}^{-1}\psi \star \tilde{a}\Big)\Big(\frac{x}{\varepsilon}, \varepsilon\xi\Big) = \int \mathcal{F}^{-1}\psi(y,\varepsilon\xi)a(x-\varepsilon y,\varepsilon\xi)\,dy,$$

in the sense that $\mathrm{op}_\varepsilon^\psi(a) \equiv \mathrm{op}_1(\mathbf{a})$.

REMARK 6.3. — An admissible cut-off ψ satisfies the bound

$$\|\partial_\xi^\beta \mathcal{F}^{-1}(\partial_\eta^\alpha \psi)(\cdot,\xi)\|_{L^1(\mathbb{R}^d)} \leq C_\beta\langle\xi\rangle^{-|\beta|-|\alpha|}, \qquad \alpha,\beta \in \mathbb{N}^d, \ C_{\alpha\beta} > 0.$$

REMARK 6.4. — By property $\psi(0,\xi) \equiv 1$ of the Bony admissible cut-off, pseudo- and para-differential operators agree for Fourier multipliers:

$$\mathrm{op}_\varepsilon^\psi(a) \equiv \mathrm{op}_\varepsilon(a), \qquad a = a(\xi).$$

For tensor products $a(x,\xi) = a_1(x)a_2(\xi)$, there holds $(a_1 a_2)^\psi \equiv a_1^\psi a_2$, so that

$$\mathrm{op}_\varepsilon^\psi(a) = \mathrm{op}_\varepsilon^\psi(a_1)\mathrm{op}_\varepsilon(a_2) \equiv \mathrm{op}_\varepsilon^\psi(a_1)\mathrm{op}_\varepsilon^\psi(a_2), \qquad a_1 = a_1(x), \ a_2 = a_2(\xi).$$

PROPOSITION 6.5. — *Given $m \in \mathbb{R}$, $a \in S^m$, there holds for $s \in \mathbb{R}$, all $u \in H^{s+m}$ the bound*

$$\|\mathrm{op}_\varepsilon^\psi(a)u\|_{\varepsilon,s} \lesssim \mathbf{M}_{0,d}^m(a)\|u\|_{\varepsilon,s+m},$$

where

$$(6.10) \qquad \mathbf{M}_{k,k'}^m(a) = \sup_{\substack{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d \\ |\alpha|\leq k, |\beta|\leq k'}} \langle\xi\rangle^{-(m-|\beta|)}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|.$$

Proof. — See for instance Theorem 4.3.5 of [**35**].   □

PROPOSITION 6.6. — *For all $m_1, m_2 \in \mathbb{R}$, all $r \in \mathbb{N}^*$, with $r \leq \bar{s}$, given $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, there holds*

$$\mathrm{op}_\varepsilon^\psi(a_1)\mathrm{op}_\varepsilon^\psi(a_2) = \mathrm{op}_\varepsilon^\psi\big(a_1 \sharp_\varepsilon a_2\big) + \varepsilon^r R_r^\psi(a_1, a_2),$$

with the notation

$$a_1 \sharp_\varepsilon a_2 = \sum_{|\alpha|<r} \varepsilon^{|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_1 \partial_x^\alpha a_2,$$

and the bound, for $d^* = 2d + r + 1$, for all $s \in \mathbb{R}$, all $u \in H^{s+m_1+m_2-r}$,

$$\big\|R_r^\psi(a_1, a_2)u\big\|_{\varepsilon,s} \lesssim \Big(\mathbf{M}_{0,d^*}^{m_1}(a_1)\mathbf{M}_{r,d}^{m_2}(a_2) + \mathbf{M}_{r,d^*}^{m_1}(a_1)\mathbf{M}_{0,d}^{m_2}(a_2)\Big)\|u\|_{\varepsilon,s+m_1+m_2-r}.$$

Proof. — See for instance Theorem 6.1.4 of [35], or Proposition B.21 of [36]. □

The need for a different action result, one which involves a smaller number of ξ-derivatives, was evoked in Section 1.3 in the introduction.

PROPOSITION 6.7. — *Given $a \in S^0$, there holds for $s \geq 0$, $r \in \mathbb{N}$ with $r \leq s$, $u \in H^s$ the bound*

(6.11)
$$\|\mathrm{op}_\varepsilon^\psi(a)u\|_{\varepsilon,s} \lesssim \sum_{\substack{0 \leq |\alpha| \leq d+1 \\ 0 \leq |\beta| \leq r-1}} \varepsilon^\beta \sup_{\xi \in \mathbb{R}^d} |\partial_x^{\alpha+\beta} a(\cdot, \xi)|_{L^1(\mathbb{R}_x^d)} \|u\|_{\varepsilon,s} + \varepsilon^r \mathbf{M}_{r,d}^0(a)\|u\|_{\varepsilon,s-r},$$

with norms $\mathbf{M}_{k,k'}^m$ defined in (6.10).

Proof. — Theorem 18.8.1 in Hörmander's treatise [20] asserts the bound, for $b \in S^0$ and $u \in L^2$:

$$|\mathrm{op}_1(b)u|_{L^2} \lesssim \sum_{|\alpha| \leq d+1} \sup_{\xi \in \mathbb{R}^d} |\partial_x^\alpha b(\cdot, \xi)|_{L^1(\mathbb{R}_x^d)} |u|_{L^2}.$$

If $s = 0$, it suffices to apply the above to $b = \mathbf{a}$, with notation drawn from Remark 6.2. In the case $s > 0$, we introduce a commutator:

$$\|\mathrm{op}_\varepsilon^\psi(a)u\|_{\varepsilon,s} \leq |\mathrm{op}_\varepsilon^\psi(a)\Lambda^s u|_{L^2} + \big|\big[\Lambda^s, \mathrm{op}_\varepsilon^\psi(a)\big]u\big|_{L^2}.$$

For the first term in the above upper bound, we use the result for $s = 0$. For the commutator, we use Remark 6.4 and Proposition 6.6:

$$\big|\big[\Lambda^s, \mathrm{op}_\varepsilon^\psi(a)\big]u\big|_{L^2} = \big|\big[\mathrm{op}_\varepsilon^\psi(\langle\cdot\rangle^s), \mathrm{op}_\varepsilon^\psi(a)\big]u\big|_{L^2} \leq |\mathrm{op}_\varepsilon^\psi(\langle\cdot\rangle^s \sharp_\varepsilon a)u|_{L^2} + \varepsilon^r |R_r^\psi(\langle\cdot\rangle^s, a)u|_{L^2}.$$

There holds

$$\mathrm{op}_\varepsilon^\psi\big(\partial_\xi^\alpha \langle\cdot\rangle^s \partial_x^\alpha a\big)u \equiv \mathrm{op}_\varepsilon^\psi\big(\langle\cdot\rangle^{-(s-|\alpha|)} \partial_\xi^\alpha \langle\cdot\rangle^s \partial_x^\alpha a\big)\Lambda^{s-|\alpha|}u,$$

so that, by (6.11) for $s = 0$:

$$|\mathrm{op}_\varepsilon^\psi(\partial_\xi^\alpha \langle\cdot\rangle^s \partial_x^\alpha a)u|_{L^2} \lesssim \sum_{|\beta| \leq d+1} \sup_{\xi \in \mathbb{R}^d} |\partial_x^{\alpha+\beta} a(\cdot, \xi)|_{L^1(\mathbb{R}_x^d)} \|u\|_{\varepsilon,s-|\alpha|}.$$

Besides, by Proposition 6.6, $|R_r^\psi(\langle\cdot\rangle^s, a)u|_{L^2} \lesssim \mathbf{M}_{r,d}^0(a)\|u\|_{\varepsilon,s-r}$. □

We finally give two para-linearization estimates:

PROPOSITION 6.8. — *For all $r \in \mathbb{N}^*$, $s \geq r$, given $a \in H^s$, there holds for all $u \in L^\infty$,*

$$(6.12) \qquad \left\| \left(a - \mathrm{op}_\varepsilon^\psi(a) \right) u \right\|_{\varepsilon,s} \lesssim \| (\varepsilon \partial_x)^r a \|_{\varepsilon,s-r} |u|_{L^\infty},$$

and for all $u \in L^2$,

$$(6.13) \qquad \left\| \left(a - \mathrm{op}_\varepsilon^\psi(a) \right) u \right\|_{\varepsilon,s} \lesssim |(\varepsilon \partial_x)^s a|_{L^\infty} |u|_{L^2}.$$

Proof. — For (6.12), see Proposition 5.2.2 in [**35**]; for (6.13) see Theorem 5.2.8 in [**35**]. □

6.1.3. Product laws in weighted Sobolev spaces. — *The need for product laws arises from the semilinear nature of the equations* (1.1).

Given $u, v \in H^s \cap L^\infty$, with $s \geq 0$, there holds

$$(6.14) \qquad \|uv\|_{\varepsilon,s} \leq C \big(|u|_{L^\infty} \|v\|_{\varepsilon,s} + |v|_{L^\infty} \|u\|_{\varepsilon,s} \big),$$

where $C > 0$ does not depend on u, v. Estimate (6.14), used in connection with the Sobolev embedding $H^s \hookrightarrow L^\infty$, for $s > d/2$, gives a product law in $\| \cdot \|_{\varepsilon,s}$. A problem with (6.14) is that the Sobolev embedding has a large norm $\sim \varepsilon^{-d/2}$ when H^s is equipped with $\| \cdot \|_{\varepsilon,s}$.

A way around this difficulty is given by the following estimate:

$$(6.15) \qquad \|uv\|_{\varepsilon,s} \leq C \big(|u|_{L^\infty} \|v\|_{\varepsilon,s} + |(\varepsilon \partial_x)^s u|_{L^\infty} |v|_{L^2} \big),$$

where $C > 0$ does not depend on u, v. Estimate (6.15) is a direct consequence of Proposition 6.5 and estimate (6.13) in Proposition 6.8.

6.2. An integral representation formula

We adapt to the present context an integral representation formula introduced in [**46**]. Consider the initial value problem

$$(6.16) \qquad \partial_t u + \frac{1}{\sqrt{\varepsilon}} \mathrm{op}_\varepsilon^\psi(M) u = f, \qquad u(0) \in H^s,$$

where $f \in L^\infty([0, T_\star |\ln \varepsilon|], H^s(\mathbb{R}^d))$, for some $T_\star > 0$. For the matrix-valued, time-dependent family of symbols $M = M(\varepsilon, t, x, \xi)$, we assume the following:

ASSUMPTION 6.9. — *The family of symbols $M(\varepsilon, t, x, \xi) \in S^0$ is constant in x outside the ball $\{|x| \leq x_\star\}$, for some $x_\star > 0$ which does not depend on (ε, t, ξ). Besides, there holds the uniform bounds*

$$\langle \xi \rangle^{|\beta|} \big| \partial_x^\alpha \partial_\xi^\beta M(\varepsilon, t, x, \xi) \big| \leq C_{\alpha\beta} < \infty, \qquad \alpha, \beta \in \mathbb{N}^d,$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$, $t \in [0, T_\star |\ln \varepsilon|]$, and $(x, \xi) \in \mathbb{R}^{2d}$.

Above and below, multi-indices α for x-derivatives are restricted to $|\alpha| \leq \bar{s}$, where $\bar{s}$ is the spatial regularity index introduced in Appendix 6.1.

We define the *flow* S_0 of $\varepsilon^{-1/2} M$ as the solution to the following system of linear ordinary differential equations, $0 \leq \tau \leq t \leq T_0 |\ln \varepsilon|$:

$$(6.17) \qquad \partial_t S_0(\tau;t) + \frac{1}{\sqrt{\varepsilon}} M S_0(\tau;t) = 0, \qquad S_0(\tau;\tau) = \mathrm{Id}.$$

For S_0 we assume an exponential growth in time:

ASSUMPTION 6.10. — *There holds for some $\gamma^+ > 0$, for all α,*

$$(6.18) \qquad |\partial_x^\alpha S_0(\tau,t)| \lesssim |\ln \varepsilon|^* \exp\left((t-\tau)\gamma^+\right),$$

uniformly in $0 \leq \tau \leq t \leq T_\star |\ln \varepsilon|$, where $|\ln \varepsilon|^* = |\ln \varepsilon|^{N_*}$ for some $N_* > 0$ independent of $(\varepsilon, \tau, t, x, \xi)$.

In (6.18), we do not assume that x-derivatives of S_0 lead to losses in powers of ε, in spite of the $1/\sqrt{\varepsilon}$ prefactor in front of M in (6.17). This assumption is tailored for our application to M defined by (3.26), in which x-dependent terms have a $\sqrt{\varepsilon}$ prefactor.

We introduce correctors $\{S_q\}_{1 \leq q \leq q_0}$, defined as the solutions of the triangular system of linear ordinary differential equations

$$(6.19) \qquad \begin{cases} \partial_t S_q + \dfrac{1}{\sqrt{\varepsilon}} M S_q + \displaystyle\sum_{1 \leq |\alpha| \leq [(q+1)/2]} \dfrac{(-i)^{|\alpha|}}{|\alpha|!} \partial_\xi^\alpha M \partial_x^\alpha S_{q+1-2|\alpha|} = 0, \\[3mm] \hspace{9cm} S_q(\tau;\tau) = 0. \end{cases}$$

LEMMA 6.11. — *Under Assumptions 6.9 and 6.10, there holds, for all $q \in [0, q_0]$, all α, β, the bounds, for $0 \leq \tau \leq t \leq T_\star |\ln \varepsilon|$:*

$$\langle \xi \rangle^{|\beta|} |\partial_x^\alpha \partial_\xi^\beta S_q(\tau;t)| \lesssim \varepsilon^{-|\beta|/2} |\ln \varepsilon|^* \exp((t-\tau)\gamma^+).$$

Proof. — By (6.17) and (6.19), there holds for $q \geq 1$

$$(6.20) \quad S_q(\tau;t) = \sum_{1 \leq |\alpha| \leq [(q+1)/2]} \frac{(-i)^{|\alpha|}}{|\alpha|!} \int_\tau^t S_0(t';t) \partial_\xi^\alpha M(t') \partial_x^\alpha S_{q+1-2|\alpha|}(\tau;t') \, dt'.$$

From there, we see immediately that the bound $|\partial_x^\alpha S_q| \lesssim \exp((t-\tau)\gamma^+)$, which holds true for $q = 0$ by Assumption 6.10, propagates from q to $q+1$. The case $|\beta| > 0$ is proved similarly by induction. The loss of half a power of ε with each ξ-derivative comes of course from the prefactor $1/\sqrt{\varepsilon}$ in front of M in (6.17). $\qquad \square$

LEMMA 6.12. — *Under Assumptions 6.9 and 6.10, there holds, for all $q \in [0, q_0]$, all $u \in L^2$, the bound*

$$(6.21) \qquad \|\mathrm{op}_\varepsilon^\psi(S_q(\tau;t)) u\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^* e^{(t-\tau)\gamma^+} \|u\|_{\varepsilon,s}, \qquad 0 \leq \tau \leq t \leq T_\star |\ln \varepsilon|.$$

Proof. — We start with $q = 0$. Let $\varphi_\star \in C_c^\infty(\mathbb{R}_x^d)$, with $0 \leq \varphi_\star \leq 1$, and such that $\varphi_\star \equiv 1$ on $\{|x| \leq x_\star\}$, where $x_\star$ is introduced in Assumption 6.9. Then, the symbols $(1 - \varphi_\star)M$, and consequently $(1 - \varphi_\star)S_0$ are independent of x. We can thus apply estimate (6.6) for Fourier multipliers to bound the action of $(1 - \varphi_\star)S_0$:

$$\|\mathrm{op}_\varepsilon((1 - \varphi_\star)S_0)u\|_{\varepsilon,s} \leq \|u\|_{\varepsilon,s},$$

since $|(1 - \varphi_\star)S_0| \leq 1$. For $\varphi_\star S_0$, we use Proposition 6.7:

$$\|\mathrm{op}_\varepsilon^\psi(\varphi_\star S_0)u\|_{\varepsilon,s} \lesssim \sum_{\substack{|\alpha| \leq d+1 \\ |\beta| \leq r-1}} \varepsilon^\beta \sup_{\xi \in \mathbb{R}^d} |\partial_x^{\alpha+\beta}(\varphi_\star S_0)(\cdot,\xi)|_{L^1(\mathbb{R}_x^d)} \|u\|_{\varepsilon,s}$$

$$+ \varepsilon^r \mathbf{M}_{r,d^*}^0(\varphi_\star S_0)\|u\|_{\varepsilon,s-r}.$$

For the L^1 norms, we use compactness of the support of $\varphi_\star$ and Assumption 6.10:

$$|\partial_x^\alpha(\varphi_\star S_0)(\cdot,\xi)|_{L^1(\mathbb{R}_x^d)} \lesssim \sup_{|\alpha'| \leq |\alpha|} |\partial_x^{\alpha'} S_0|_{L^\infty} \lesssim |\ln \varepsilon|^* \exp\left((t-\tau)\gamma^+\right).$$

We bound the remainder with Lemma 6.11:

$$\varepsilon^r \mathbf{M}_{r,d_*}^0(\varphi_\star S_0) \lesssim \varepsilon^{r-d_*/2}|\ln \varepsilon|^* e^{(t-\tau)\gamma^+}.$$

If r is large enough, then $r - d_*/2 = r/2 - d - 1/2 > 0$, and we conclude that (6.21) holds for $q = 0$.

From (6.20), we see that the correctors S_q, for $q \geq 1$, vanish identically outside the ball $\{|x| \leq x_\star\}$. Thus it suffices to consider $\varphi_\star S_q$, and we use Proposition 6.7 again. $\square$

We let

$$S := \sum_{0 \leq q \leq q_0} \varepsilon^{q/2} S_q.$$

The following Lemma expresses the fact that $\mathrm{op}_\varepsilon^\psi(S)$ is an approximate solution operator:

LEMMA 6.13. — *Under Assumptions* 6.9 *and* 6.10, *there holds*

$$(6.22) \qquad\qquad \mathrm{op}_\varepsilon^\psi(\partial_t S) = \mathrm{op}_\varepsilon^\psi(M)\mathrm{op}_\varepsilon^\psi(S) + \rho_S,$$

where for $0 \leq \tau \leq t \leq T_\star|\ln \varepsilon|$, *for all* $s \in \mathbb{R}$ *and* $u \in H^s$:

$$(6.23) \qquad \|\rho_S(\tau;t)u\|_{\varepsilon,s} \lesssim \varepsilon^{q_0/4-(d+2)}|\ln \varepsilon|^* \exp\left((t-\tau)\gamma^+\right)\|u\|_{\varepsilon,s-1}.$$

Proof. — By definition of S and (6.19),

$$(6.24) \qquad\qquad\qquad -\partial_t \mathrm{op}_\varepsilon^\psi(S) = \mathrm{I} + \mathrm{II},$$

with the notation

$$\mathrm{I} := \sum_{0 \le q \le q_0} \varepsilon^{(q-1)/2} \mathrm{op}_\varepsilon^\psi(MS_q),$$

$$\mathrm{II} := \sum_{\substack{0 \le q \le q_0 \\ 1 \le |\alpha| \le [(q+1)/2]}} \varepsilon^{q/2} \frac{(-i)^{|\alpha|}}{|\alpha|!} \mathrm{op}_\varepsilon^\psi \left(\partial_\xi^\alpha M \partial_x^\alpha S_{q+1-2|\alpha|} \right),$$

where by convention a sum over the empty set is zero. By Proposition 6.6,

$$\mathrm{op}_\varepsilon^\psi(MS_q) = \mathrm{op}_\varepsilon^\psi(M)\mathrm{op}_\varepsilon^\psi(S_q) - \sum_{1 \le |\alpha| \le [(q_0-q+1)/2]} \varepsilon^{|\alpha|} \frac{(-i)^{|\alpha|}}{|\alpha|!} \mathrm{op}_\varepsilon^\psi \left(\partial_\xi^\alpha M \partial_x^\alpha S_q \right)$$
$$- \varepsilon^{1+[(q_0-q+1)/2]} R_{1+[(q_0-q+1)/2]}^\psi(M, S_q),$$

so that

$$\mathrm{I} = \varepsilon^{-1/2}\mathrm{op}_\varepsilon^\psi(M)\mathrm{op}_\varepsilon^\psi(S) - \sum_{\substack{0 \le q \le q_0-1 \\ 1 \le |\alpha| \le [(q_0-q+1)/2]}} \varepsilon^{(q-1)/2+|\alpha|} \frac{(-i)^{|\alpha|}}{|\alpha|!} \mathrm{op}_\varepsilon^\psi \left(\partial_\xi^\alpha M \partial_x^\alpha S_q \right) - \rho_S,$$

with

$$\rho_S := - \sum_{0 \le q \le q_0} \varepsilon^{(1+q)/2+[(q_0-q+1)/2]} R_{1+[(q_0-q+1)/2]}^\psi(M, S_q).$$

Changing variables in the double sum, we find

$$\mathrm{I} = \varepsilon^{-1/2}\mathrm{op}_\varepsilon^\psi(M)\mathrm{op}_\varepsilon^\psi(S) - \sum_{\substack{1 \le q' \le q_0 \\ 1 \le |\alpha| \le [(q'+1)/2]}} \varepsilon^{q'/2} \frac{(-i)^{|\alpha|}}{|\alpha|!} \mathrm{op}_\varepsilon^\psi \left(\partial_\xi^\alpha M \partial_x^\alpha S_{q'+1-2|\alpha|} \right) - \rho_S,$$

hence

$$(6.25) \qquad\qquad \mathrm{I} + \mathrm{II} = \varepsilon^{-1/2}\mathrm{op}_\varepsilon^\psi(M)\mathrm{op}_\varepsilon^\psi(S) - \rho_S.$$

Identities (6.24) and (6.25) prove (6.22). From Proposition 6.6 and Assumption 6.9, we deduce, for $\tilde{q} := 1 + [(q_0 - q + 1)/2]$:

$$\|R_{\tilde{q}}^\psi(M, S_q)u\|_{\varepsilon,s} \lesssim \mathbf{M}_{\tilde{q},2d+1+\tilde{q}}^0(S_q)\|u\|_{\varepsilon,s-\tilde{q}},$$

and with Lemma 6.11 this implies

$$\|R_{\tilde{q}}^\psi(M, S_q)u\|_{\varepsilon,s} \lesssim \varepsilon^{-(d+2)-(q_0-q)/4} \exp\left((t-\tau)\gamma^+ \right) \|u\|_{\varepsilon,s-\tilde{q}},$$

whence (6.23) follows.   □

The following theorem gives an integral representation formula for a solution to (6.16) in terms in $\mathrm{op}_\varepsilon^\psi(S)$:

THEOREM 6.14. — *Under Assumptions* 6.9 *and* 6.10, *the initial value problem* (6.16) *has a unique solution* $u \in C^0([0, T_\star|\ln \varepsilon|, H^s(\mathbb{R}^d))$, *which satisfies the representation*

$$(6.26) \quad u = \mathrm{op}_\varepsilon^\psi(S(0;t))u(0) + \int_0^t \mathrm{op}_\varepsilon^\psi(S(t';t))(\mathrm{Id} + \varepsilon^\zeta R_1)\big(f + \varepsilon^\zeta R_2(\cdot)u(0)\big)(t')\, dt',$$

for some $\zeta > 0$, where for all $v \in L^\infty H^s$, all $w \in H^s$,

$$(6.27) \quad \|(R_1 v)(t)\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^* \sup_{0 \le t' \le T_\star |\ln \varepsilon|} \|v(t')\|_{\varepsilon,s}, \quad \|R_2(t)w\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^* \|w\|_{\varepsilon,s},$$

uniformly in $0 \le t \le T_\star |\ln \varepsilon|$.

The parameter ζ is defined in (6.29) below. There we see that the larger $T_\star$ and γ^+, the larger q_0 needs to be. In other words, given an observation time $T_\star$, the spatial regularity that we need for M is a function of $|M|_{L^\infty}$. Indeed, spatial regularity is connected with q_0, since the proof is essentially a Taylor expansion at order q_0, and the growth rate γ^+ is connected with the L^∞ norm of M : think of M as being independent of t; then $S = \exp(-tM/\sqrt{\varepsilon})$, and γ^+ indeed appears as the sup of M.

Proof. — Let $g \in L^\infty([0, T_\star |\ln \varepsilon|], H^s)$. By Lemma 6.13, the map

$$u := \mathrm{op}_\varepsilon^\psi(S(0;t))u(0) + \int_0^t \mathrm{op}_\varepsilon^\psi(S(t';t))g(t')\,dt'$$

solves (6.16) if and only if there holds for all $0 \le t \le T_1 |\ln \varepsilon|$,

$$(6.28) \qquad\qquad (\mathrm{Id} + r)g(t) = f(t) - \rho_S(0;t)u(0),$$

where r is the linear integral operator

$$r : \qquad v \in L^\infty H^s \to \left(t \to \int_0^t \rho_S(\tau;t)v(\tau)\,d\tau\right) \in L^\infty H^s,$$

and ρ_S is the remainder in Lemma 6.13. We now choose the index q_0 that appears in the definition of S such that

$$(6.29) \qquad\qquad \zeta := \frac{q_0}{4} - (d+2) - T_1 \gamma^+ > 0.$$

Then, by estimate (6.23), for all $0 \le t \le T_\star |\ln \varepsilon|$, the operator r maps $L^\infty(0, T_\star |\ln \varepsilon|, H^s)$ to itself, with the bound

$$(6.30) \qquad\qquad \sup_{0 \le t \le T(\varepsilon)} \|(rv)(t)\|_{\varepsilon,s} \lesssim \varepsilon^\zeta \sup_{0 \le t \le T(\varepsilon)} \|v\|_{\varepsilon,s}$$

In particular, for ε small enough the operator $\mathrm{Id}+r$ is invertible, with inverse $(\mathrm{Id}+r)^{-1}$ bounded as an operator from $L^\infty([0, T_\star |\ln \varepsilon|], H^s_\varepsilon)$ to itself, uniformly in ε. As a consequence, we can solve (6.28) in $L^\infty([0, T_\star |\ln \varepsilon|], H^s_\varepsilon)$, and obtain the representation formula (6.26), with

$$\varepsilon^\zeta R_1 := (\mathrm{Id} + r)^{-1} - \mathrm{Id}, \qquad \varepsilon^\zeta R_2(t) := -\rho_S(0;t).$$

Bound (6.27) follows from (6.30) and (6.23). Since $M \in S^0$, $\mathrm{op}_\varepsilon^\psi(M)$ is linear bounded from H^s to H^s, hence uniqueness is a consequence of the Cauchy-Lipschitz theorem.
$\square$

6.3. Bounds for the symbolic flow

We give here a proof of Proposition 3.9 page 39, which we reproduce below:

PROPOSITION 6.15. — *For all $T > 0$, all $0 \leq \tau \leq t \leq T|\ln\varepsilon|$, $\alpha \in \mathbb{N}^d$, the solution S_0 to*

$$(6.31) \qquad \partial_t S_0 + \frac{1}{\sqrt{\varepsilon}} M S_0 = 0, \qquad S_0(\tau, \tau) = \mathrm{Id},$$

with M defined in (3.26), satisfies the bound

$$|\partial_x^\alpha S_0(\tau, t)| \lesssim |\ln\varepsilon|^* \exp(t\gamma^+),$$

where the growth rate γ^+ is defined in (3.30): $\gamma^+ = |a|_{L^\infty} \left| \max_{\xi \in \mathcal{R}_{12}^h} \Re e \left(\Gamma(\xi)^{1/2} \right) \right|$.

In a first step (Section 6.3.1), we approximate M by its value $M(\varepsilon, 0, x, \xi)$ at $t = 0$. Then the solution to (6.31) is a matrix exponential. The general case follows by a simple perturbation argument (Section 6.3.2).

6.3.1. The autonomous case. — The matrix $M(\varepsilon, 0, x, \xi)$ is block-diagonal:

$$M(\varepsilon, 0, x, \xi) = \begin{pmatrix} i\mu_1 & -\sqrt{\varepsilon}\, b_{12} & 0 & \cdots & 0 \\ -\sqrt{\varepsilon}\, b_{21} & i\mu_2 & 0 & \cdots & 0 \\ 0 & 0 & i\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & i\lambda_J \end{pmatrix}$$

denoting $\mu_1 = \lambda_1(\cdot + k) - \omega$, $\mu_2 = \lambda_2$, as in (3.15), and

$$b_{12}(x, \xi) := \varphi_1(x) a(x) \chi_0(\xi) b_{12}^+(\xi), \qquad b_{21}(x, \xi) = \varphi_1(x) a(x)^* \chi_0(\xi) b_{21}^-(\xi).$$

Above, b_{12}^+ and $b_{21}^- \in \mathbb{C}^{N \times N}$ are the interaction coefficients associated with resonance $(1, 2)$, as in (3.28), a is the initial amplitude (1.2), and φ_1 and χ_0 are spatial and frequency cut-offs, respectively, as defined on page 36.

We prove in this Section the bound:

$$(6.32) \qquad \left| \exp\left(-\varepsilon^{-1/2} t M(\varepsilon, 0, x, \xi) \right) \right| \lesssim |\ln\varepsilon|^* \exp(t\gamma^+).$$

By reality of λ_j, and the fact that the truncated interaction coefficients vanish identically outside $\mathrm{supp}\,\varphi_1 \times \mathrm{supp}\,\chi_0 \subset \mathrm{supp}\,\varphi_1 \times \mathcal{R}_{12}^h$, it suffices to prove the bound

$$(6.33) \quad |\exp(-\varepsilon^{-1/2} t M_0)| \lesssim \exp(t\gamma^+), \qquad (x, \xi) \in \mathrm{supp}\,\varphi_1 \times \mathcal{R}_{12}^h, \quad 0 \leq t \leq T|\ln\varepsilon|,$$

where M_0 is the top left block of $M(\varepsilon, 0, x, \xi)$:

$$M_0 := \begin{pmatrix} i\mu_1 & -\sqrt{\varepsilon} b_{12} \\ -\sqrt{\varepsilon} b_{21} & i\mu_2 \end{pmatrix} \in \mathbb{C}^{2N \times 2N}.$$

The top left block of M_0 being a multiple of the identity, we can compute the characteristic polynomial of M_0 by blocks:

$$\det(x\mathrm{Id} - M_0) = \det\big((x - i\mu_1)(x - i\mu_2)\mathrm{Id} - \varepsilon b_{21}b_{12}\big), \qquad x \in \mathbb{C}.$$

The characteristic polynomial of M_0 has a zero at x if and only if $(x - i\mu_1)(x - i\mu_2)$ is an eigenvalue of $\varepsilon b_{21}b_{12}$. The ranks of b_{12} and b_{21} are at most one (Assumption 2.6(iii)), implying that the rank of $b_{21}b_{12}$ is at most one. Then the only possible nonzero eigenvalue for $b_{21}b_{12}$ is its trace $\mathrm{tr}(b_{12}b_{21})$, and the spectrum of M_0 appears as

$$(6.34) \qquad i\mu_1, \; i\mu_2, \; \mu_{\pm} := \frac{i}{2}\big(\mu_1 + \mu_2\big) \pm \frac{1}{2}\Big(4\varepsilon\mathrm{tr}b_{12}b_{21} - (\mu_1 - \mu_2)^2\Big)^{1/2},$$

with respective algebraic multiplicities $N - 1$, $N - 1$, one and one.

LEMMA 6.16 (Near the resonant set). — *If $|\mu_1(\xi) - \mu_2(\xi)| \leq \sqrt{\varepsilon}|\ln\varepsilon|^2$, then there holds the bound*

$$|\exp(-\varepsilon^{-1/2}tM_0)| \lesssim |\ln\varepsilon|^* \exp(t\gamma^+),$$

for $(x, \xi) \in supp\,\varphi_1 \times \mathcal{R}_{12}^h$, and $0 \leq t \leq T|\ln\varepsilon|$.

Proof. — Substracting $i\mu_1\mathrm{Id}$, we equivalently bound the exponential of

$$\varepsilon^{-1/2}\tilde{M}_0 = \begin{pmatrix} 0 & -b_{12} \\ -b_{21} & i\varepsilon^{-1/2}(\mu_2 - \mu_1) \end{pmatrix}.$$

In the space-frequency domain under consideration, the entries of $\varepsilon^{-1/2}\tilde{M}_0$ are $O(|\ln\varepsilon|^2)$. The eigenvalues of $\varepsilon^{-1/2}\tilde{M}_0$ are 0, $i\varepsilon^{-1/2}(\mu_2 - \mu_1)$, $\varepsilon^{-1/2}(\mu_{\pm} - i\mu_1)$, as given in (6.34). In particular, in the relation of unitary similarity to an upper triangular matrix (Schur decomposition): $\varepsilon^{-1/2}\tilde{M}_0 = Q^*UQ$, the real parts of the diagonal entries of the upper triangular matrix U are equal to 0 or $\Re e\, \varepsilon^{-1/2}\mu_{\pm}$. Since the norm of $\varepsilon^{-1/2}\tilde{M}_0$ controls the norm of U, and since entries of U are controlled by the norm of U, the entries of U above the diagonal are $O(|\ln\varepsilon|^2)$. Let the $2N \times 2N$ diagonal matrix

$$Q_\varepsilon = \mathrm{diag}\,\big(1, |\ln\varepsilon|^4, |\ln\varepsilon|^8, \ldots, |\ln\varepsilon|^{4(2N-1)}\big).$$

Then, the matrix

$$\tilde{U} = Q_\varepsilon U Q_\varepsilon^{-1}$$

is upper triangular, with the same diagonal entries as U, and with entries above the diagonal which are $O(|\ln\varepsilon|^{-2})$. There holds moreover

$$\big|\exp(-\varepsilon^{-1/2}t\tilde{M}_0)\big| = \big|\exp(tU)\big| \lesssim |\ln\varepsilon|^* \big|\exp(t\tilde{U})\big|.$$

Above $|\cdot|$ denotes the sup norm of the entries of a matrix in $\mathbb{C}^{2N \times 2N}$. Let $\|\cdot\|$ denote the canonical Hermitian norm in $\mathbb{C}^{2N}$, and also the associated norm in $\mathbb{C}^{2N \times 2N}$. Let $z(t) = \exp(t\tilde{U})z(0)$. Then z solves $z' = \tilde{U}z$, and

$$\partial_t\|z\|^2 = \langle\partial_t z, z\rangle + \langle z, \partial_t z\rangle = 2\langle\Re e\,\tilde{U}z, z\rangle, \quad 2\Re e\,\tilde{U} := \tilde{U} + \tilde{U}^*.$$

so that

$$\frac{1}{2}\partial_t \|z\|^2 \leq \|\Re e\, \tilde{U}\|\|z\|^2,$$

implying, by Gronwall's lemma, for some $C > 0$,

$$\left|\exp(t\tilde{U})\right| \leq C \exp\left(t\|\Re e\, \tilde{U}\|\right),$$

where $|\cdot|$ denotes the sup norm in $\mathbb{C}^{2N \times 2N}$. Given $x \in \mathbb{C}^{2N}$ with $\|x\| = 1$, there holds

$$\|\Re e\, \tilde{U}x\|^2 = \sum_{1 \leq i \leq 2N} (\Re e\, \tilde{U}x)_i^2 = \sum_{1 \leq i \leq 2N} \left((\Re e\, \tilde{U})_{i,i} x_i + O(|\ln \varepsilon|^{-2})\right)^2,$$

so that

$$\|\Re e\, \tilde{U}x\|^2 \leq \max_i (\Re e\, \tilde{U})_{i,i} \|x\|^2 + O(|\ln \varepsilon|^{-2}),$$

and this implies

$$\|\Re e\, \tilde{U}\| \leq \max_i (\Re e\, \tilde{U})_{i,i} + O(|\ln \varepsilon|^{-2}).$$

The diagonal entries of $\Re e\, \tilde{U}$ are the real parts of the diagonal entries of U, that is the real parts of the eigenvalues of $\varepsilon^{-1/2}\tilde{M}_0$. Here μ_1 and μ_2 contribute zero. Thus it suffices to bound from above $\Re e\, \varepsilon^{-1/2}\mu_\pm$. Based on the explicit formula

$$\Re e\, (a + ib)^{1/2} = \frac{1}{2}\left((a^2 + b^2)^{1/2} + a\right)^{1/2}, \qquad a, b \in \mathbb{R},$$

we observe that

$$\Re e\, \varepsilon^{-1/2}\mu_\pm \leq \Re e\, \mathrm{tr}\left(b_{12}b_{21}\right)^{1/2}.$$

Since $\mathrm{tr}\, b_{12}b_{21} = (\varphi_1(x)\chi_0(\xi))^2|a(x)|^2\Gamma(\xi)$, with Γ defined in (2.10), the result follows from the bound $\Re e\, (\mathrm{tr}\, b_{12}b_{21})^{1/2} \leq \gamma^+$. $\square$

LEMMA 6.17 (Away from the resonant set). — *If $|\mu_1(\xi) - \mu_2(\xi)| > \sqrt{\varepsilon}|\ln \varepsilon|^2$, then for $(x, \xi) \in supp\, \varphi_1 \times \mathcal{R}_{12}^h$, and $0 \leq t \leq T|\ln \varepsilon|$, there holds*

$$(6.35) \qquad\qquad |\exp(-\varepsilon^{-1/2}tM_0)| \lesssim 1.$$

Proof. — We use a non-stationary phase argument that is analogous to the normal form reduction of Section 3.1.2.2. Consider $\tilde{S}_0 := \begin{pmatrix} e^{-it\mu_1/\sqrt{\varepsilon}} & 0 \\ 0 & e^{-it\mu_2/\sqrt{\varepsilon}} \end{pmatrix} S_0$. A column $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$ of $\tilde{S}_0$ solves the system

$$y_1' + e^{-it(\mu_1-\mu_2)/\sqrt{\varepsilon}}b_{12}y_2 = 0, \quad y_2' + e^{-it(\mu_2-\mu_1)/\sqrt{\varepsilon}}b_{21}y_1 = 0.$$

Integrating in time and then integrating by parts, we find

$$y_1(t) = y_1(0) - \frac{\sqrt{\varepsilon}}{i(\mu_1 - \mu_2)}\left(e^{-it(\mu_1-\mu_2)/\sqrt{\varepsilon}}b_{12}y_2(t) - b_{12}y_2(0)\right)$$

$$- \frac{\sqrt{\varepsilon}}{i(\mu_2 - \mu_1)}\int_0^t b_{12}b_{21}y_1(t')\, dt'.$$

This implies the bound, for $z_j(t) := \max_{[0,t]} |y_j|$,

$$z_1(t) \leq |y_1(0)| + C|\ln \varepsilon|^{-2}z_2(t) + Ct|\ln \varepsilon|^{-2}z_1(t).$$

In a time interval $[0, T|\ln \varepsilon|]$, this gives $z_1(t) \lesssim |y_1(0)| + |\ln \varepsilon|^{-2} z_2(t)$. By symmetry, we find the same bound for z_2. We conclude that the symbolic flow is uniformly bounded in ε, t, x, ξ in the frequency domain under consideration. $\qquad \square$

Conclusion: Lemmas 6.16 and 6.17 imply bound (6.33), which in turns implies bound (6.32).

6.3.2. The general case. — Here we use a perturbative argument to show that the bound (6.32) for $\exp(tM(\varepsilon, 0, x, \xi)/\sqrt{\varepsilon})$ carries over to a bound for $\partial_x^\alpha S_0$. This will prove Proposition 6.15.

It suffices to bound $|\partial_x^\alpha S_0|$ on the compact set $(x, \xi) \in \operatorname{supp} \varphi_1 \times \mathcal{R}_{12}^h$, since M is diagonal with purely imaginary entries outside of it.

By assumption on g (Assumption 2.2), we may Taylor expand M at $t = 0$: there holds $M = M(0) + \varepsilon M_1$, with $M(0) = M(\varepsilon, 0, x, \xi)$, and $|M_1| \lesssim |\ln \varepsilon|^*$, uniformly in ε, t, x, ξ in the domain under consideration. The equation in $\partial_x^\alpha S_0$ is

$$(6.36) \qquad \partial_t \partial_x^\alpha S_0 + \varepsilon^{-1/2} M(0) \partial_x^\alpha S_0 = -\varepsilon^{1/2} M_1 \partial_x^\alpha S_0 - \varepsilon^{-1/2} [\partial_x^\alpha, M] S_0.$$

In the right-hand side of (6.36), the first term involves the unknown $\partial_x^\alpha S_0$ but is small. The second term is bounded in ε since $[\partial_x^\alpha, M] = O(\sqrt{\varepsilon})$, and involves only lower-order derivatives $\partial_x^{\alpha'} S_0$, with $|\alpha'| < |\alpha|$. From (6.36) and definition of S_0 (6.31), we deduce the implicit integral representations

$$(6.37) \qquad S_0(\tau; t) = \mathbf{S}(\tau; t) - \varepsilon^{1/2} \int_\tau^t \mathbf{S}(t'; t) M_1(t') S_0(\tau; t'),$$

where $\mathbf{S}(\tau; t) := \exp\big(-\varepsilon^{-1/2}(t - \tau) M(\varepsilon, 0, x, \xi)\big)$, and for $|\alpha| > 0$:

$$
\begin{aligned}
(6.38) \qquad \partial_x^\alpha S_0(\tau; t) = {}& -\varepsilon^{1/2} \int_\tau^t \mathbf{S}(t'; t) M_1(t') \partial_x^\alpha S_0(\tau; t') \\
& + \varepsilon^{-1/2} \int_\tau^t \mathbf{S}(t'; t) [\partial_x^\alpha, M(t')] S_0(\tau; t')) \, dt'.
\end{aligned}
$$

We factor out the exponential growth by consideration of

$$S_0^\flat(\tau; t) := \exp\big((t - \tau)\gamma^+\big) S_0(\tau; t),$$

and define similarly $\mathbf{S}^\flat(\tau; t) := \exp\big((t - \tau)\gamma^+\big) \mathbf{S}(\tau; t)$, so that, by bound 6.32, there holds $|\mathbf{S}^\flat(\tau; t)| \leq C|\ln \varepsilon|^*$. This gives for $t \leq T|\ln \varepsilon|$ the bound

$$|S_0^\flat(\tau; t)| \leq C|\ln \varepsilon|^* \Big(1 + \varepsilon^{1/2} \int_\tau^t |S_0^\flat(\tau; t')| \, dt'\Big),$$

from which we immediately deduce $|S_0^\flat(\tau; t)| \leq C(T_1)|\ln \varepsilon|^*$. Finally, assuming inductively the bound

$$|\partial_x^\alpha S_0(\tau; t)| \leq C_\alpha |\ln \varepsilon|^* \exp\big((t - \tau)\gamma^+\big),$$

for $|\alpha| < \alpha_0$ and some $C_\alpha > 0$, we deduce from (6.38) with $0 < |\alpha| = \alpha_0$ the bound

$$|\partial_x^\alpha S_0^\flat| \leq \varepsilon^{1/2} C |\ln \varepsilon|^* \int_\tau^t |\partial_x^\alpha S_0^\flat| \, dt' + C |\ln \varepsilon|^*,$$

implying $|\partial_x^\alpha S_0^\flat| \leq C |\ln \varepsilon|^*$, which concludes the proof.

6.4. A Gronwall Lemma

Consider the integral inequality in $y : [0, T |\ln \varepsilon|] \to \mathbb{R}_+$:

$$(6.39) \qquad y(t) \leq e^{t\gamma_0} y_0 + \delta_0 \int_0^t y(t') \, dt' + \varepsilon^{\eta_0} \int_0^t e^{(t-t')\gamma_0} y(t') \, dt',$$

where $\gamma_0 > 0$, $\delta_0 > 0$ and $\eta_0 > 0$. This type of inequality is typical of situations in which we perform mixed-type estimates: the second term in the right-hand side comes from an L^2 "energy" estimate (as in Section 3.1.5), while the third term comes from a semi-group estimate (as in Section 3.1.3).

LEMMA 6.18. — *If inequality* (6.39) *holds for all* $t \in [0, T |\ln \varepsilon|]$, *then under the condition*

$$(6.40) \qquad \eta_0 - T\delta_0 > 0,$$

there holds, for ε small enough, the bound

$$(6.41) \qquad y(t) \leq C(y_0, \delta_0) \exp\big(t \max(\gamma_0, \delta_0)\big), \qquad C(y_0, \delta_0) > 0.$$

The proof is elementary and based on three applications of the standard Gronwall's lemma.

Proof. — We let $z := y - \delta_0 \int_0^t y(t') \, dt'$, so that, by Gronwall's lemma,

$$y(t) \leq e^{t\delta_0} \bar{z}(t), \qquad \bar{z}(t) := \max_{[0,t]} z,$$

and

$$\bar{z}(t) \leq y_0 e^{t\gamma_0} + \varepsilon^{\eta_0} \int_0^t e^{(t-t')\gamma_0} e^{t'\delta_0} \bar{z}(t') \, dt'.$$

We now let $w(t) := e^{-t\gamma_0} \bar{z}(t)$, so that

$$w(t) \leq y_0 + \varepsilon^{\eta_0} \int_0^t e^{t'\delta_0} w(t') \, dt',$$

and, with Gronwall's lemma

$$w(t) \leq y_0 \exp\Big(\frac{\varepsilon^{\eta_0}}{\delta_0} e^{t\delta_0}\Big).$$

Under condition (6.40), for ε small enough and $t \leq T |\ln \varepsilon|$, this gives $w(t) \leq 2y_0$. Hence

$$y \leq 2y_0 e^{t\gamma_0} + \int_0^t \delta_0 y(t') \, dt',$$

from which we deduce (6.41) by another application of Gronwall's lemma. □

6.5. On regularity of the spectral decomposition

Coalescing eigenvalues of smooth matrices are typically *not* smooth; the canonical example being $\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$. In some cases, however, there is an ordering of the eigenvalues so that regularity is preserved, an example being $\begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$. A symmetric example is given by $\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$. The smoothness condition in Assumption 2.1 should be understood with this latter example in mind. Besides this smoothness condition, Assumption 2.1 states that the eigenvalues λ_j and eigenprojectors Π_j of the family $A_0/i + A(\xi)$, where $A(\xi) = \sum_j \xi_j A_j$, satisfy bounds (2.2) page 17, which we reproduce here: for all $\beta \in \mathbb{N}^d$, some $C_\beta > 0$:

$$(6.42) \qquad |\partial_\xi^\beta \lambda_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{(1-|\beta|)/2}, \quad |\partial_\xi^\beta \Pi_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{-|\beta|/2}.$$

Consider the family of matrices $A_\infty(\omega, x) = A(\omega) - ixA_0$, with $(\omega, x) \in \mathbb{S}^{d-1} \times \mathbb{R}$.

LEMMA 6.19. — *If for all $\omega \in \mathbb{S}^{d-1}$, the family A_∞ has smooth eigenvalues and eigenprojectors in a neighborhood of $(\omega, 0)$ in $\mathbb{S}^{d-1} \times \mathbb{R}$, then bounds (6.42) hold.*

Proof. — By the assumed smoothness of λ_j and Π_j, we only need to prove bounds (6.42) for large $|\xi|$. There holds

$$A_0/i + A(\xi) = |\xi| A_\infty(\omega, x), \quad \text{with } \omega = \frac{\xi}{|\xi|}, \ x = \frac{1}{|\xi|}.$$

Thus, denoting λ_j^∞ and Π_j^∞ the eigenvalues and eigenprojectors of A_∞, there holds the correspondence

$$(6.43) \qquad \lambda_j(\xi) = |\xi| \lambda_j^\infty(\omega, x), \quad \Pi_j(\xi) = \Pi_j^\infty(\omega, x).$$

By assumption, the eigenvalues of A_∞ have Taylor expansions at all orders in x at $(\omega, 0)$, with coefficients that are smooth in ω :

$$(6.44) \qquad \lambda_j^\infty(\omega, x) = \lambda_{j0}^\infty(\omega) + x\lambda_{j1}^\infty(\omega) + \cdots + x^m \tilde{\lambda}_{jm}^\infty(\omega, x),$$

and smooth remainders $\tilde{\lambda}_{jm}^\infty$. Via (6.43), these translate into (6.42) for λ_j. Similarly, the Taylor expansions of the Π_j^∞ around at $(\omega, 0)$ translate into large-frequency bounds for $\partial_\xi^\beta \Pi_j$. □

In one space dimension, Rellich's theorem [**39, 40**] ensures that analytic family of symmetric matrices have analytic eigenvalues and eigenvectors, so that the assumption of Lemma 6.19 is always satisfied. In dimension greater than one, eigenvalues are Lipschitz (a consequence of the characteristic polynomial being hyperbolic; see

Brohnstein [**4**], or Kurdyka and Paunescu [**28**]), but eigenvectors may fail to be even continuous, as shown by Example 6.1 in [**28**]:

$$\begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix},$$

for which the eigenvectors are (x_1, x_2) and $(x_2, -x_1)$.

We conclude this paragraph by noting that under a smoothness condition at infinity, we have an asymptotic expansion for the eigenvalues. This will be useful in Appendix 6.7, where we discuss existence of resonances at infinity.

LEMMA 6.20. — *Under the assumption of Lemma* 6.19, *the eigenvalues* λ_j *have asymptotic expansions*

$$(6.45) \qquad\qquad \lambda_j(\xi) = c_j(\omega)|\xi| + O\left(\frac{1}{|\xi|}\right).$$

Proof. — Indeed, since A_j are real symmetric and A_0 skew-symmetric, the transpose matrix of $A_\infty(\omega, x)$ is equal to $A_\infty(\omega, -x)$. But then the determinant of a matrix is equal to the determinant of its transpose, so that $A_\infty(\omega, x)$ and $A_\infty(\omega, -x)$ have the same eigenvalues. That is, there is an ordering of the eigenvalues of $A_\infty(\omega, \cdot)$ so that the eigenvalues are even in x. In their Taylor expansions (6.44) at $x = 0$, only even powers of x appear. Via the correspondence (6.43), this means in particular only odd powers of $|\xi|^{-1}$ in the asymptotic expansion of λ_j, implying (6.45). $\qquad\square$

6.6. On existence of WKB approximate solutions

We give here sufficient conditions for Assumption 2.2, stating that the family of systems (1.1) admits WKB approximate solutions, to hold true.

We first remark on conditions (2.5), reproduced here:

$$(6.46) \qquad \left(A_0 + \sum_{1 \le j \le d} A_j \partial_{x_j}\right)\left(e^{\pm i(k \cdot x - \omega t)/\varepsilon} \vec{e}_{\pm 1}\right) = 0, \quad \vec{e}_{-1} = (\vec{e}_1)^*.$$

In (6.46) we state that (i) the matrix $A_0/i + A(k) = A_0/i + \sum_j k_j A_j$ has real eigenvalue ω, (ii) the matrix $A_0/i - A(k)$ has eigenvalue $-\omega$, and (iii) these eigenvalues are associated with respective eigenvectors $\vec{e}_1$ and $\vec{e}_{-1}$ that satisfy the component-by-component conjugation relation $\vec{e}_{-1} = (\vec{e}_1)^*$.

Points (ii) and (iii) are consequences of point (i), and the structure of the differential operator. Indeed, from the equality $(A_0/i + A(k))\vec{e}_1 = \omega \vec{e}_1 \in \mathbb{C}^N$, applying component-by-component complex conjugation we find

$$(-A_0/i + A(k))(\vec{e}_1)^* = \omega(\vec{e}_1)^*,$$

which by linearity of $A(\cdot)$, translates into points (ii) and (iii).

A characteristic phase $\beta = (\omega, k) \in \mathbb{R}^{1+d}$ is given, satisfying (6.46), such that $k \neq 0$. For some $j_0 \in [1, J]$, there holds $\omega = \lambda_{j_0}(k)$. (The eigenvalues λ_j are introduced in Assumption 2.1). We assume that

$$(6.47) \qquad \det\big(p\omega + A_0/i + A(pk)\big) = 0 \iff p \in \{-1, 0, 1\},$$

meaning in particular that higher harmonics of the fundamental phase are not characteristic. We now show that Assumption 2.2 holds under the *weak transparency* assumption

$$(6.48) \qquad \mathbf{\Pi}(p\beta) \sum_{p_1 + p_2 = p} B\big(\mathbf{\Pi}(p_1\beta)u, \mathbf{\Pi}(p_2\beta)v\big) \equiv 0, \qquad p \in \{-1, 0, 1\},$$

for all $u, v \in \mathbb{C}^N$, where $\mathbf{\Pi}(\omega', k')$ denotes the orthogonal projector onto the kernel of the skew-hermitian matrix $\omega' + A_0 + A(ik')$. If β, for $p \in \{-1, 0, 1\}$, is a simple point on the characteristic variety, meaning that the branch $\xi \to \lambda_{j_0}(\xi)$ in the spectrum of the family $\big(A_0/i + A(\xi)\big)_{\xi \in \mathbb{R}^d}$ does not change multiplicity at k, then $\mathbf{\Pi}(\beta) = \Pi_{j_0}(k)$. We do not want to exclude the case of coalescing eigenvalues, however, especially at the phase $p\beta = (0, 0)$, since it is frequently met in applications (see for instance the examples given in Section 2.3 of [47]).

PROPOSITION 6.21. — *Given (ω, k) satisfying (6.46), conditions (6.47)* [1] *and (6.48) imply Assumption 2.2.*

Proof. — The goal is to construct u_a satisfying (2.3), in the form (2.4), such that the polarization conditions (2.5)-(2.6) and the bounds (2.7) hold. From (2.4), we see in particular that we are considering highly-oscillating data:

$$(6.49) \quad u_a(\varepsilon, 0, x) = e^{-ik \cdot x/\varepsilon} g(0, x)^* \vec{e}_{-1} + e^{ik \cdot x/\varepsilon} g(0, x) \vec{e}_1 + \sqrt{\varepsilon} v_a(\varepsilon, 0, x), \quad k \neq 0.$$

We introduce notation borrowed from [25]:

$$L(\beta\partial_\theta) := -\omega\partial_\theta + A_0 + A(k\partial_\theta) = -\omega\partial_\theta + A_0 + \sum_{1 \leq j \leq d} k_j A_j \partial_\theta,$$

and

$$L_1(\partial_t, \partial_x) := \partial_t + A(\partial_x),$$

and look for an approximate solution u_a in the form of a *profile*:

$$u_a(\varepsilon, t, x) = \big[\mathbf{u}_a(\varepsilon, t, x, \theta)\big]_{\theta = (k \cdot x - \omega t)/\varepsilon}$$

where $\mathbf{u}_a(\varepsilon, t, x, \theta)$ is 2π-periodic in the fast variable θ. Then, for u_a to satisfy (2.3) it suffices that its representation $\mathbf{u}_a$ satisfies

$$(6.50) \qquad \frac{1}{\varepsilon} L(\beta\partial_\theta)\mathbf{u}_a + L_1(\partial_t, \partial_x)\mathbf{u}_a = \frac{1}{\sqrt{\varepsilon}} B(\mathbf{u}_a, \mathbf{u}_a) + \varepsilon^{K_a} \mathbf{r}_a^\varepsilon,$$

---

1. Condition (6.47) is introduced here only as a matter of notational simplification; without any additional difficulty we could allow for a larger set of characteristic harmonics.

for some remainder $\mathbf{r}_a^\varepsilon$ with a trace r_a^ε satisfying bound (2.7). We look for a solution to (6.50) in the form of a WKB expansion:

$$(6.51) \qquad \mathbf{u}_a = \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{u}_1 + \varepsilon\mathbf{u}_2 + \cdots + \varepsilon^{K_a}\mathbf{u}_{2K_a},$$

where each $\mathbf{u}_k$ is a profile and in particular can be expanded in Fourier series in θ. We denote $u_{k,p} = u_{k,p}(t,x)$ the p-th Fourier coefficient in θ of the k-th profile $\mathbf{u}_k$, and *assume* that the coefficients $u_{0,-1}$ and $u_{0,1}$ of $\mathbf{u}_0$ satisfy the polarization conditions (2.5)-(2.6).

From (6.50) we derive a cascade of WKB equations, which are sufficient conditions for (6.51) to solve (6.50). The first, comprising terms of order $O(1/\varepsilon)$, is

$$(6.52) \qquad L(\beta\partial_\theta)\mathbf{u}_0 = 0.$$

Decomposing in Fourier series, we find that (6.52) is equivalent to

$$(6.53) \qquad L(ip\beta)u_{0,p} = 0, \qquad p \in \mathbb{Z}.$$

Under condition (6.47), equation (6.53) is equivalent to

$$(6.54) \qquad u_{0,p} = 0,\ p \notin \{-1,0,1\}, \qquad \mathbf{\Pi}(p\beta)u_{0,p} = u_{0,p}, \quad p \in \{-1,0,1\},$$

where we recall that notation $\mathbf{\Pi}$ was introduced just below (6.48). In agreement with (6.49), we let $u_{0,0} \equiv 0$. Then, conditions $\mathbf{\Pi}(\pm\beta)u_{0,\pm1} = u_{0,\pm1}$ are implied by conditions (2.5)-(2.6).

Thus (6.53) is satisfied, and we move on to the equation at order $O(1/\sqrt{\varepsilon})$:

$$L(p\partial_\theta)\mathbf{u}_1 = B(\mathbf{u}_0, \mathbf{u}_0),$$

or, at the level of the Fourier coefficients:

$$(6.55) \qquad L(p\beta)u_{1,p} = \sum_{p_1+p_2=p} B(u_{0,p_1}, u_{0,p_2}).$$

For $|p| > 2$, the above right-hand side is identically zero, and since by (6.47) for such p the matrix $L(ip\beta)$ is invertible, we solve (6.55) by $u_{1,p} = 0$. For $p \in \{-1,1\}$, the right-hand side of (6.55) is identically zero as well, since $u_{0,0} \equiv 0$, so that (6.55) reduces to the polarization conditions

$$(6.56) \qquad u_{1,p} = \mathbf{\Pi}(p\beta)u_{1,p}, \qquad p \in \{-1,1\}.$$

For $p \in \{-2,0,2\}$, projecting with $\mathbf{\Pi}(p\beta)$ onto the kernel of matrix $L(ip\beta)$, we find

$$(6.57) \qquad \mathbf{\Pi}(p\beta) \sum_{p_1+p_2=p} B(u_{0,p_1}, u_{0,p_2}) = 0.$$

By the polarization (6.54) and the assumed transparency (6.48), identity (6.57) holds. The matrix $L(ip\beta)$ being skew-hermitian, there holds $\mathbb{C}^N = \ker L(ip\beta) \oplus \operatorname{ran} L(ip\beta)$, and we can define a partial inverse $L(ip\beta)^{(-1)}$ by

$$L(ip\beta)^{(-1)}(x + y) = z, \quad x \in \ker L(ip\beta), \quad y \in \operatorname{ran} L(ip\beta), \quad y = L(ip\beta)z,$$

so that

$$L(ip\beta)^{(-1)}L(ip\beta) = \text{Id} - \mathbf{\Pi}(p\beta).$$

If $p \in \{-1, 0, 1\}$, then $L(ip\beta)^{(-1)} = L(ip\beta)^{-1}$, the actual matrix inverse. Multiplying (6.55) to the left by $L(ip\beta)^{(-1)}$, we then find

$$(6.58) \qquad (1 - \mathbf{\Pi}(p\beta))u_{1,p} = L(ip\beta)^{(-1)} \sum_{p_1 + p_2 = p} B(u_{0,p_1}, u_{0,p_2}), \qquad p \in \{-2, 0, 2\}.$$

At this stage, equation (6.55) is solved. The $\mathbf{\Pi}(pk)u_{1,p}$, for $|p| \le 1$, are still undetermined.

The equations at order $O(1)$ are

$$L(\beta\partial_\theta)\mathbf{u}_2 + L_1(\partial_t, \partial_x)\mathbf{u}_0 = B(\mathbf{u}_0, \mathbf{u}_1) + B(\mathbf{u}_1, \mathbf{u}_0),$$

corresponding to equations

$$(6.59) \qquad L(ip\beta)u_{2,p} + L_1(\partial_t, \partial_x)u_{0,p} = \sum_{p_1 + p_2 = p} B(u_{0,p_1}, u_{1,p_2}) + B(u_{1,p_1}, u_{0,p_2})$$

for the Fourier coefficients.

For $|p| > 3$, the above right-hand side is identically zero, and we solve (6.59) by $u_{2,p} \equiv 0$. For $|p| = 2$, equation (6.59) reduces to

$$u_{2,p} = \sum_{|p_1 + p_2| = 2} L(ip\beta)^{-1} \Big(B(u_{0,p_1}, u_{1,p_2}) + B(u_{1,p_1}, u_{0,p_2}) \Big), \qquad |p| = 2.$$

In particular, the coefficients $u_{2,2}$ and $u_{2,-2}$ are determined as soon as $\mathbf{\Pi}(p\beta)u_{1,1}$ and $u_{1,-1}$ are determined. For $p = 0$, projecting (6.59) onto the kernel of $L(ip\beta)$, we find that necessarily

$$\sum_{p_1 + p_2 = 0} \mathbf{\Pi}(0) \Big(B(u_{0,p_1}, u_{1,p_2}) + B(u_{1,p_1}, u_{0,p_2}) \Big) = 0,$$

an identity which holds indeed by (6.48), (6.54) and (6.56). For $p = 0$, the other component is

$$(1 - \mathbf{\Pi}(0))u_{2,0} = \sum_{p_1 + p_2 = 0} L(0)^{(-1)} \Big(B(u_{0,p_1}, u_{1,p_2}) + B(u_{1,p_1}, u_{0,p_2}) \Big).$$

Finally, for $|p| = 1$, projecting (6.59), we find

$$\mathbf{\Pi}(p\beta)L_1(\partial_t, \partial_x)\mathbf{\Pi}(p\beta)u_{0,p} = \sum_{|p_1 + p_2| = 1} \mathbf{\Pi}(p\beta) \Big(B(u_{0,p_1}, u_{1,p_2}) + B(u_{1,p_1}, u_{0,p_2}) \Big).$$

Only $u_{1,\pm 2}$ and $u_{1,0}$ contribute to the above right-hand side. By transparency (6.48), we see that actually only $(1 - \mathbf{\Pi}(\pm 2\beta))u_{1,\pm 2}$ and $(1 - \mathbf{\Pi}(0))u_{1,0}$ contribute to the above right-hand side. With (6.58), we obtain

$$
\begin{aligned}
(6.60) \qquad \mathbf{\Pi}(\beta)L_1(\partial_t, \partial_x)\mathbf{\Pi}(\beta)u_{0,1} &= \mathbf{\Pi}(p\beta)B(u_{0,-1})\, L(2ik)^{-1}B(u_{0,1}, u_{0,1}) \\
&\quad + \mathbf{\Pi}(p\beta)B(u_{0,1})\, L(0)^{(-1)}\big(B(u_{0,1})\, u_{0,-1}\big),
\end{aligned}
$$

and a similar equation in $u_{0,-1}$. In (6.60), we used Notation 2.3: $B(u)\,v = B(u,v) + B(v,u)$.

The operator $\Pi(p\beta)L_1(\partial_t,\partial_x)\Pi(p\beta)$ is a transport operator at group velocity, or a family of transport operators, depending on whether $p\beta$ is a simple point on the characteristic variety or not. If β is a simple point on the variety, meaning that there is no change in multiplicity for the branch λ_{j_0} at β, then the operator $\Pi(\beta)L_1(\partial_t,\partial_x)\Pi(\beta)$ is the scalar transport operator

$$\Pi_{j_0}(k)L_1(\partial_t,\partial_x)\Pi_{j_0}(k) = \big(\partial_t + \big(\nabla_\xi\lambda_{j_0}\big)(k)\cdot\partial_x\big)\Pi_{j_0}(k).$$

If there is a change in multiplicity for λ_{j_0} at β, meaning that several eigenvalues coalesce at β, then $\mathbf{\Pi}(\beta)L_1(\partial_t,\partial_x)\mathbf{\Pi}(\beta)$ is a family of transport operators , with velocities given by the directional derivatives of the branches that intersect at β[2].

In both cases, simple or coalescing eigenvalues, equation (6.60) is hyperbolic with a cubic nonlinearity, and can be solved locally in time for smooth Sobolev data.

The other component of (6.59) for $|p| = 1$ is

$$(1 - \mathbf{\Pi}(p\beta))u_{2,p} = \sum_{|p_1+p_2|=1} L(ipk)^{(-1)}\Big(B(u_{0,p_1},u_{1,p_2}) + B(u_{1,p_1},u_{0,p_2})\Big).$$

Summing up, we see that at this stage:
- the leading term $\mathbf{u}_0 = u_{0,-1}e^{-i\theta} + u_{0,1}e^{i\theta}$ in (6.51) is completely determined, with amplitudes $u_{0,\pm 1}$ solving semilinear hyperbolic equations (6.60);
- the first corrector $\mathbf{u}_1 = \sum_{|p|\leq 2} e^{ip\theta}u_{1,p}$ is known, except for $\mathbf{\Pi}(p\beta)u_{1,p}$, $|p| \leq 1$, for which we have no information so far;
- the second corrector $\mathbf{u}_2 = \sum_{|p|\leq 3} e^{ip\theta}u_{2,p}$ is known, except for $u_{2,\pm 2}$, which will be determined by $\mathbf{\Pi}(\pm\beta)u_{1,\pm 1}$, and for $\mathbf{\Pi}(p\beta)u_{2,p}$, $|p| \leq 1$, for which we have no information so far.

We can go on with the expansion up to any order $2K_a$. The components $\mathbf{\Pi}(p\beta)u_{1,p}$, for $|p| \leq 1$, are determined by the equation at order $O(\sqrt{\varepsilon})$: they satisfy *linear* transport equations

$$\mathbf{\Pi}(p\beta)L_1(\partial_t,\partial_x)\mathbf{\Pi}(p\beta)u_{1,p} = \big(F_1(\mathbf{u}_0)\mathbf{u}_1\big)_p,$$

where a typical term in the source F_1 is

$$\mathbf{\Pi}(p\beta)B\Big(\mathbf{\Pi}(p\beta)u_{1,p}, L(0)^{(-1)}B(u_{0,1},u_{0,-1})\Big).$$

---

2. In the case of separated eigenvalues, these facts are proved in [**31, 15**]; in the case of coalescing eigenvalues, these facts are proved in [**29, 47**]. The article [**47**] contains unified proofs for both simple and coalescing cases, and also for higher-order operators, such as Schrödinger, that arise in three-scale approximations.

Similarly, the components $\mathbf{\Pi}(p\beta)u_{2,p}$ are determined by the equations at order $O(\varepsilon)$; more generally, the components $\mathbf{\Pi}(p\beta)u_{\ell,p}$ are determined by the equations at order $O(\varepsilon^{\ell/2})$, they satisfy linear tranport equations, with source terms which are polynomials in the Fourier coefficients of the lower-order profiles $\mathbf{u}_{\ell'}$, with $0 \leq \ell' < \ell$. In particular, these equations can be solved over any interval of existence for (6.60).

By construction, the correctors $\mathbf{v}_a$ and the remainder $\mathbf{r}_a^{\varepsilon}$ are trigonometric polynomials in θ, so that bounds (2.7) hold. This completes the proof of Proposition 6.21. $\square$

Condition (6.48), introduced by Joly, Métivier and Rauch [**25**], can be linked to transparency in the sense of Definition 2.5:

PROPOSITION 6.22. — *Assuming*
  – *condition* (6.47) *describing the set of harmonics of the fundamental phase;
  and denoting* j_0, j_1 *the indices such that* $\lambda_{j_0}(k) = -\lambda_{j_1}(-k) = \omega_0$;
  – *assuming that* $\vec{e}_1$ *and* $\vec{e}_{-1}$ *generate the kernels of* $i\omega + A_0 + A(k)$ *and* $-i\omega + A_0 + A(-ik)$ *respectively;*
  – *denoting* $\mathcal{J}$ *the set of indices* j *such that* $\lambda_j(0) = 0$, *and assuming that the resonances* (j_0, j) *and* (j, j_1) *are transparent, for all* $j \in \mathcal{J}$,
then weak transparency (6.48) *holds.*

Proof. — There holds $\Pi_{j_0}(k) = \mathbf{\Pi}(\beta)$, $\Pi_{j_1}(-k) = \mathbf{\Pi}(-\beta)$, and $\mathbf{\Pi}(0) = \sum_{j \in \mathcal{J}} \Pi_j(0)$, so that the weak transparency condition (6.48) is implied by conditions

$$
(6.61) \quad
\begin{cases}
\Pi_j(0)\Big(B(\vec{e}_1, \vec{e}_{-1}) + B(\vec{e}_{-1}, \vec{e}_1) \Big) = 0, \\[2mm]
\Pi_{j_0}(k)\Big(B(\vec{e}_1, \Pi_j(0)\,\cdot\,) + B(\Pi_j(0)\,\cdot\,, \vec{e}_1) \Big) = 0, \\[2mm]
\Pi_{j_1}(-k)\Big(B(\vec{e}_{-1}, \Pi_j(0)\,\cdot\,) + B(\Pi_j(0)\,\cdot\,, \vec{e}_{-1}) \Big) = 0.
\end{cases}
$$

With Notation 2.3, conditions (6.61) take the form [3]
$$
(6.62) \quad \Pi_j(0)B(\vec{e}_{-1})\Pi_{j_0}(k) = 0, \quad \Pi_{j_0}(k)B(\vec{e}_1)\Pi_j(0) = 0, \quad \Pi_{j_1}(-k)B(\vec{e}_{-1})\Pi_j(0) = 0.
$$

Since $j \in \mathcal{J}$, the (trivial) resonance

occurs at $\xi = 0$. Similarly, the trivial resonance

$$
(6.63) \qquad\qquad \omega = \lambda_j(0) - \lambda_{j_1}(-k) = 0 - \lambda_{j_1}(-k)
$$

occurs at $\xi = -k$. The first two conditions in (6.62) can then be seen as a (partial) transparency condition for the resonances (j_0, j), with $j \in \mathcal{J}$. By partial we mean here that under (6.62) the bounds of Definition 2.5 hold a priori only at $\xi = 0$. The

---

3. The lack of symmetry between $\vec{e}_1$ and $\vec{e}_{-1}$ in (6.62) is only apparent: it suffices indeed to reformulate the first condition in (6.62) as $\Pi_j(0)B(\vec{e}_1)\Pi_{j_1}(-k) = 0$, associated with resonance (6.63), to restore symmetry.

third condition in (6.62) is a (partial, i.e., only at $\xi = -k$) transparency condition for resonances (j, j_1), with $j \in \mathcal{J}$. Since by assumption the resonances (j_0, j) and (j, j_1) are transparent, condition (6.62) holds, implying (6.48). □

Propositions 6.21 and 6.22 lead to sufficient conditions for Assumption 2.8 to imply Assumption 2.2, as follows:

COROLLARY 6.23. — *Under* (6.46)-(6.47)*, if $\vec{e}_{\pm 1}$ generate the respective kernels of the matrices $\pm i\omega + A_0 + A(\pm ik)$, and if eigenvalues λ_j such that $\lambda_j(0) = 0$ are not involved in non-transparent resonances, meaning that for such j, for all j', there holds $(j, j') \notin \mathfrak{R}_0$ and $(j', j) \notin \mathfrak{R}_0$, then Assumption* 2.8 *implies Assumption* 2.2.

Proof. — This is an immediate consequence of Propositions 6.21 and 6.22. □

REMARK 6.24. — In different settings, weak transparency conditions such as (6.48) may not guarantee existence of WKB solutions. One possible obstruction is a lack of well-posedness of the limiting equations.

For instance, the regime considered by Joly, Métivier and Rauch in [**25**] is

$$(6.64) \qquad \partial_t u + \frac{1}{\varepsilon} A_0 u + \sum_{1 \le j \le d} A_j \partial_{x_j} u = \frac{1}{\varepsilon} B(u, u).$$

This is a more singular regime than (1.1), on which we further comment in Remark 6.26 below. In particular, in the context of (6.64) the profile equations in $\Pi_{j_0}(pk)u_{0,p}$ are typically *quasi-linear*, when (6.60) was semi-linear. For triangular source terms B, Joly, Métivier and Rauch give sufficient conditions, in the form of transparency conditions, for these quasi-linear profiles equations to be well-posed. This is Assumption 2.2 in [**25**], and it is strictly stronger than their weak transparency assumption (Assumption 2.1 in [**25**]) guaranteeing existence of a WKB cascade, and strictly weaker than the conditions that guarantee stability (Assumption 2.5 in [**25**]).

Another example is given in [**49**]. There, the second author considered *quasi-linear* Euler-Maxwell systems in the scaling

$$(6.65) \qquad \partial_t u + \frac{1}{\varepsilon} A_0 + \frac{1}{\sqrt{\varepsilon}} \sum_{1 \le j \le d} A_j \partial_{x_j} u + \sum_{1 \le j \le d} \tilde{A}_j(u) \partial_{x_j} u = \frac{1}{\sqrt{\varepsilon}} B(u, u),$$

and proved stability of WKB solutions with leading amplitudes solving the Zakharov system describing Langmuir turbulence. In particular, for (6.65) just like for (6.64), and as opposed to (1.1), the well-posed character of the limiting equations is far from trivial. For the Zakharov system, local-in-time well-posedness in smooth Sobolev spaces was first proved by Schochet and Weinstein [**42**] and Ozawa and Tsutsumi [**37**].

REMARK 6.25. — For some physical systems, the weak transparency condition (6.48) is actually *not* satisfied. This was proved for Maxwell-Bloch by Joly, Métivier and Rauch [**25**], and by the second author for Euler-Maxwell [**48**]. WKB solutions can then sometimes be constructed for a restricted set of initial data.

Consider for instance the situation in which after spelling out one component of (6.48), we arrive at condition

$$(6.66) \qquad n_{0,0} v_{0,1} = 0,$$

where n and v are components of the solution u (with n representing for instance variation of density and v velocity, as in Euler-Maxwell). Here we are using notation introduced in the proof of Proposition 6.21, so that $n_{0,0}$ denotes the mean mode of the leading term in the variation of density. Equation (6.66) can be solved by $n_{0,0} = 0$, if compatible with the datum, for instance in the case of highly-oscillating initial data in velocity, with $O(\varepsilon)$ initial variations of density.

In such situations, the limiting system often involves both $v_{0,1}$ and $n_{1,0}$, meaning a coupling between leading order terms and corrector terms: the term $\varepsilon n_{1,0}$, which vanishes in the limit $\varepsilon \to 0$, in particular cannot be measured, has a measurable effect on the leading term $v_{0,1}$. This is akin to the *ghost effect* that was studied in depth by the Kyoto school [**43, 45**] for rarefied gas dynamics.

REMARK 6.26. — We finally comment on the link between the specific regime in Joly, Métivier and Rauch's article, as described in Remark 6.24 above, absence of transparency as described in Remark 6.25, and our supercritical regime (1.1).

From (1.1), imagine that the hyperbolic operator is block-diagonal, like for instance system (5.36) from Section 5.2. This is the case for the Maxwell-Bloch equations. Suppose then that in the coordinate system $u = (u_1, u_2)$ in which the hyperbolic operator is block-diagonal, the source B has the form

$$B(u,u) = \begin{pmatrix} B_1(u_1, u_2) \\ B_2(u_1, u_1) \end{pmatrix},$$

and assume that B_1 does not satisfy the weak transparency condition (6.48). Then, rescaling $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) := (u_1, u_2/\sqrt{\varepsilon})$, we find (6.64), with $(1/\varepsilon)B$ replaced by

$$\frac{1}{\varepsilon} \begin{pmatrix} 0 \\ B_2(\tilde{u}_1, \tilde{u}_1) \end{pmatrix} + \begin{pmatrix} B_1(\tilde{u}_1, \tilde{u}_2) \\ 0 \end{pmatrix}.$$

The regime is now more singular, but the source is more transparent, and has a triangular structure, as in [**25**].

6.7. On structure of the resonant set

LEMMA 6.27. — *Under the assumption of Lemma* 6.19, *the set $\mathfrak{R}$ comprising all resonant frequencies is bounded as soon as the asymptotic branches on the characteristic variety are distinct: $c_i \neq c_j$ for $i \neq j$, with notation borrowed from Lemma* 6.20.

Proof. — If the resonant set is unbounded, then some $\mathcal{R}_{ij}$ has an accumulation point at infinity. By Lemmas 6.19 and 6.20, this gives an equality

$$c_i|\xi + k| = \omega + c_j|\xi| + O\left(\frac{1}{|\xi|}\right),$$

along a sequence $|\xi| \to \infty$, implying $c_i = c_j$.   □

6.7.1. Euler-Maxwell. — We verify that the assumptions of Lemma 6.27 are satisfied by the Euler-Maxwell equations (EM) of Section 5.1.2. First we check that the assumption of Lemma 6.19 is satisfied. Here $A_\infty(\omega)$ is block-diagonal:

$$A_\infty(\omega) = \begin{pmatrix} 0 & \omega\times & 0 & 0 & 0 & 0 \\ -\omega\times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_e\omega & 0 & 0 \\ 0 & 0 & \theta_e\omega\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2\theta_i\omega \\ 0 & 0 & 0 & 0 & \theta_i\omega\cdot & 0 \end{pmatrix}.$$

In particular, the eigenvalues of $A_\infty(\omega)$ are $\{\pm|\omega|,\ \pm\theta_e|\omega|,\ \pm\alpha\theta_i|\omega|\}$. Given ω on the sphere, the eigenvalues are separated, since $\theta_i \ll \theta_e \ll 1$, and $\alpha < 1$. By Rouché's theorem, smoothness is preserved under small perturbations. In particular, the eigenvalues of $A_\infty(\omega) - ixA_0$ are smooth, with respect to ω and x, locally around $(\omega, 0) \in \mathbb{S}^{d-1} \times \mathbb{R}$. This verifies the assumption of Lemma 6.19.

   Next the asymptotic description of the eigenvalues of (EM) in (5.16), or a look at Figure 1, shows that the separation assumption at infinity is satisfied.

   Hence Lemma 6.27 applies: the resonant set for Euler-Maxwell is bounded.

6.7.2. Maxwell-Landau-Lifschitz. — We conclude this Appendix by giving an example in which the assumptions of Lemma 6.27 do *not* appear to be satisfied. The Maxwell-Landau-Lifschitz equations are

(6.67)
$$\begin{cases} \partial_t E - \nabla \times H = 0, \\ \partial_t H + \nabla \times E = M \times H, \\ \qquad \partial_t M = -M \times H. \end{cases}$$

For the linearized equations around the family of constant solutions

$$(E, H, M) = (0, \alpha M_0, M_0) \in \mathbb{R}^9,$$

with $\alpha \in \mathbb{R}$, coordinatizing $M_0 = (1, 0, 0)$, the characteristic variety has equation

$$\lambda^3\left(\lambda^6 - 2(2 + |\xi|^2)\lambda^4 + (|\xi|^2(6 + |\xi|^2) - 2\xi_1^2)\lambda^2 - |\xi|^2(2|\xi|^2 - \xi_1^2)\right) = 0.$$

The one-dimensional Maxwell-Landau-Lifshitz are transparent in a strong sense; this was shown by the first author in [**34**].
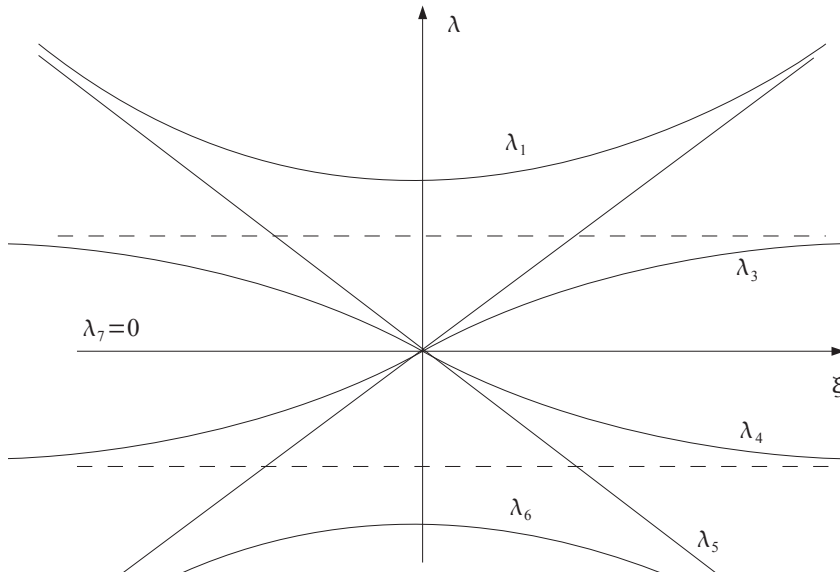
FIGURE 6. The characteristic variety for the Maxwell-Landau-Lifschitz equations.

In one space dimension, the variety is pictured on Figure 6. The asymptotic branches are not distinct, meaning that we cannot apply Lemma 6.27 in order to prove boundedness of the resonant set.

In three space dimensions, numerical calculations by the first author show existence of resonances for large values of $|\xi|$, meaning that Assumption 2.8(i) is probably not satisfied by (6.67). Moreover, these resonances are non-transparent, suggesting instability.

6.8. Notation index

$\lesssim$: inequality up to a constant, p. 30

$(\cdot,\cdot)$: scalar product in $\mathbb{C}^N$, p. 91

$|\cdot|$: sup norm in $\mathbb{C}^N, \mathbb{C}^{N\times N}$, p. 20

$\|\cdot\|_{\varepsilon,s}$: weighted Sobolev norm, p. 10

$\prec$: binary relation for cut-offs, p. 36

$\mathcal{A}$: real diagonal symbol, (3.7) p. 30

$\mathbf{A}$: real diagonal symbol, p. 37

$B(\cdot)$: linearized source term, p. 18

B_p : avatar of B, (3.9) p. 31

$\mathcal{B}^r$: avatar of B, (3.11) p. 32

$\check{\mathcal{B}}$: avatar of B, (3.18) p. 35

$b_{ij}^{\pm}$: interaction coefficients, p. 39 and 60

γ : maximal growth coefficient, p. 24

γ_{ij} : growth coefficient, p. 24

γ^- : lower growth rate, p. 41

γ^+ : upper growth rate, p. 39

$\boldsymbol{\Gamma}$: stability index, p. 19 and 24

Γ : trace of interaction coefficients, p. 19

g : leading WKB amplitude, p. 18

$\theta = (k \cdot x - \omega t)/\varepsilon$, p. 30

$|\ln \varepsilon|^*$: arbitrary power of $|\ln \varepsilon|$, p. 39

λ_j : eigenvalues, p. 17

μ_j : shifted eigenvalues (3.15), p. 33

$\boldsymbol{\mu}_\alpha$: shifted eigenvalues, p. 59

ξ_0 : frequency argmax, (3.20) p. 36

$\mathrm{op}_\varepsilon^\psi$: para-differential operator, p. 101

op_ε : pseudo-differential operator, p. 99

Π_j : eigenprojectors, p. 17

$\varphi_{...}$: spatial cut-offs, p. 36 and 64

$\chi_{...}$: frequency cut-offs, p. 36 and 57

$\mathfrak{R}$: resonant frequencies, p. 23

$\mathfrak{R}_0$: non-transparent resonant frequencies, p. 23

R_0 : uniform remainder, p. 33

$\mathcal{R}_{ij}$: (i,j) resonant frequencies, p. 18

$\mathcal{R}_{12}^h$: neighborhood of $(1,2)$ resonant frequencies, p. 32

$S_0(0;t)$: symbolic flow, p. 40 and 104

σ_{+p} : frequency-shifted symbol σ, p. 30

x_0 : spatial argmax, (3.19) p. 36

6.9. Parameter list

Temporal parameters:

  – T_0, the final observation time, is defined in (2.12) page 20.

  – T_1 is an observation time for mixed $\mathcal{F}L^1$-H^s estimates, defined in (3.44) page 44.

  – T_2 is an observation time for purely H^s estimates, introduced in (3.55) page 48.

  – $T_\star$ in Appendix 6.2 is arbitrary.

  – T in Appendix 6.3 is arbitrary.

Localization parameters:

  – $h > 0$ is a security distance from the resonance (Section 3.1.2.2 page 32).

  – $\rho > 0$ is the radius of an observation ball in the unstable case (Theorem 2.7 page 20).

  – $\delta_{\varphi_0} > 0$ controls the size of the support of the spatial truncation φ_0. It is introduced in the proof of Lemma 3.13 page 45. There holds $\varphi_0 \to 1$ as $\delta_{\varphi_0} \to 0$. The smaller δ_{φ_0}, the smaller the errors associated with (spatially) non-localized terms.

  – $x_\star$ in Appendix 6.2 corresponds to the radius of the support of φ_0 in the main proof.

Amplitude parameters:

  – K_a measures the consistency of the WKB approximation (defined in (2.3) page 18).

  – K measures the size of the initial perturbation in (1.2).

– K_0 is the amplification exponent in the main result, introduced in (2.12) page 20.

– K_0' and K_0'', introduced respectively in (2.24) page 25 and (2.26) page 26, are the amplification exponents in Theorems 2.11 and 2.12.

– η_1 controls the size of the $\mathcal{F}L^1$ norm of the solution for times $< T_1$. It can be made arbitrarily small (in statement of Lemma 3.13 page 44).

– η_2 controls the size of the L^∞ norm of the solution for times $< T_2$. It can be made arbitrarily small (in proof of Proposition 3.17 page 48).

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