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CRITICAL FUNCTIONAL FRAMEWORK AND MAXIMAL REGULARITY IN ACTION ON SYSTEMS OF INCOMPRESSIBLE FLOWS

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CRITICAL FUNCTIONAL FRAMEWORK AND MAXIMAL REGULARITY IN ACTION ON SYSTEMS OF INCOMPRESSIBLE FLOWS

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***Abstract.* —** This memoir is devoted to endpoint maximal regularity in Besov spaces for the evolutionary Stokes system in bounded or exterior domains of $\mathbb{R}^n$. We strive for *time independent* a priori estimates with L_1 time integrability.

In the whole space case, endpoint maximal regularity estimates are well known and have proved to be spectacularly powerful to investigate the well-posedness issue of PDEs related to fluid mechanics. They have been extended recently by the authors to the half-space setting [**15**]. The present work deals with the bounded and exterior domain cases. Although in both situations the Stokes system may be localized and reduced up to low order terms to the half-space and whole space cases, the exterior domain case is more involved owing to a bad control on the low frequencies of the solution (no Poincaré inequality is available whatsoever). In order to glean some global-in-time integrability, we adapt to the Besov space setting the approach introduced by P. Maremonti and V.A. Solonnikov in [**39**]. The price to pay is that we end up with estimates in intersections of Besov spaces, rather than in a single Besov space.

As a first application of our work, we solve locally for large data or globally for small data, the (slightly) inhomogeneous incompressible Navier-Stokes equations in critical Besov spaces, in an exterior domain. After observing that the L_1 time integrability allows to determine globally the streamlines of the flow, the whole system is recast in the Lagrangian coordinates setting. This, in particular, enables us to consider discontinuous densities, as in [**17**], [**19**].

The second application concerns a low Mach number system that has been studied recently in the whole space setting by the first author and X. Liao [**14**].

***Résumé* (Régularité critique, régularité maximale et application à la mécanique des fluides incompressibles)**

Ce mémoire traite de la régularité maximale limite dans les espaces de Besov pour le système de Stokes non stationnaire en domaine borné ou extérieur. Nous avons en vue des estimations avec intégrabilité globale en temps.

Les inégalités de régularité maximale limite sont bien connues dans l'espace entier et ont joué un rôle spectaculaire dans l'étude du problème de Cauchy associé à diverses EDPs de la mécanique des fluides. Ces inégalités ont été adaptées récemment par les auteurs au cas du demi-espace [**15**]. Nous considérons ici des domaines bornés ou extérieurs. Bien que dans les deux situations le système de Stokes puisse être localisé et l'étude, ramenée à celle de l'espace entier ou du demi-espace, le cas du domaine extérieur est plus compliqué car on ne dispose pas de contrôle *a priori* sur les basses fréquences via une inégalité de Poincaré par exemple. Afin d'exhiber une certaine forme d'intégrabilité globale en temps, nous adaptons au cadre Besov le travail de P. Maremonti et V.A. Solonnikov [**39**]. Nous obtenons ainsi le type d'inégalités voulu, mais dans l'intersection d'espaces de Besov.

Comme première application de ces nouvelles inégalités, nous résolvons localement (données initiales grandes) ou globalement (données initiales petites) les équations de Navier-Stokes incompressibles faiblement non homogènes en domaine extérieur dans des espaces de Besov critiques. La propriété d'intégrabilité L^1 en temps à valeurs Lipschitz pour le champ de vitesses solution assure l'équivalence entre les formulations lagrangiennes et eulériennes du système. Passer en coordonnées lagrangiennes permet de considérer des données initiales avec densité discontinue, comme observé récemment dans [**17**], [**19**].

Comme deuxième application, nous résolvons un système limite qui apparaît dans le régime à faible nombre de Mach et a été étudié récemment par le premier auteur et X. Liao [**14**].

CONTENTS

CHAPTER 1

INTRODUCTION

Description of motion of Newtonian fluids is based on the physical and thermodynamical laws governing the conservation of momentum, energy and mass. We expect in general information concerning these quantities to be enough to find out the velocity field at each point of the fluid region and, at least, on some time interval $[0,T]$ if the initial time is $t=0$. This is the Eulerian description of the fluid.

Another fundamental physical information is the knowledge of the *streamlines* or *particle paths* corresponding to the evolution of infinitesimal particles or fluid parcels. It is given by the following Ordinary Differential Equation:

$$\frac{dX}{dt} = v(t,X), \qquad X|_{t=0} = y. \tag{1.1}$$

Here y is the initial position of a particle of the fluid and $X(t,y)$ denotes the position of that particle at time t under the action of the velocity field v. Knowing X thus gives the evolution of an infinitesimal fluid parcel *labelled by its initial position* y as it moves through space and time. This is the Lagrangian description of the fluid under consideration. Equation (1.1) gives the relationship between the two descriptions of flows, that is the Eulerian and Lagrangian ones. The Eulerian coordinates system (t,x) uses the position x of the material at time t, while the Lagrangian coordinates system (t,y) uses the initial position y of a point of the medium. The change of coordinates is governed by the following identity which is the integrated counterpart of (1.1):

$$x = X(t,y) \quad \text{with} \quad X(t,y) = y + \int_0^t v(\tau, X(\tau,y))\, d\tau. \tag{1.2}$$

From the mathematical viewpoint, the basic question is whether those two descriptions are indeed equivalent: what are the conditions ensuring that one may go from one system of coordinates to the other without any loss of information on the flow ? From the basic theory of Ordinary Differential Equations, we know that, roughly, the minimal assumption is that

$$\nabla v \in L_1(0,T;L_{\infty,\mathrm{loc}}(\Omega)), \tag{1.3}$$

where Ω is the fluid domain. This in particular ensures (1.2) to have a unique solution X that is continuous in time and Lipschitz in space (see [**4**], [**23**], [**45**] for more general results concerning the flow and transport equations).

In the present work, we would like to find a functional framework – the largest one if possible – ensuring the velocity field to satisfy (1.3) and the system we are looking at, to be well-posed. We have in mind models describing the evolution of incompressible flows with nonconstant densities and, more specifically, mixtures of incompressible homogeneous fluids. Such models possess some invariance with respect to appropriate time and space dilations and it has been observed in many situations that the optimal functional framework – the so-called critical one – for studying the corresponding governing equations should have the same invariance (see the introduction of Chapter 5 for more explanations).

Resuming to the study of mixtures of incompressible flows, the basic question is whether the following initial configuration:

$$\rho_0 = 1+\sigma\chi_A, \tag{1.4}$$

where σ denotes some constant and χ_A, the characteristic function of some subset A of Ω, is stable through the time evolution. According to the Lagrangian description introduced above, we expect the density to be transported by the velocity field and thus to read

$$\rho(t) = 1+\sigma\chi_{A(t)} \quad \text{with} \quad A(t) := X(t,A). \tag{1.5}$$

Note that if (1.3) is fulfilled then the flow X is Lipschitz and thus $A(t)$ remains Lipschitz during the evolution, if it is Lipschitz initially.

To make it more concrete, consider the following *inhomogeneous incompressible Navier-Stokes system*:

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0 & \text{in} \quad (0,T)\times\Omega, \\ \rho(u_t+u\cdot\nabla u) - \nu\Delta u + \nabla P = 0 & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{at} \quad (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0, \qquad \rho|_{t=0} = \rho_0 & \text{on} \quad \Omega. \end{cases} \tag{1.6}$$

Here $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $P = P(t,x) \in \mathbb{R}$ stand for the density, velocity field and pressure of the fluid, respectively. For simplicity, the given positive viscosity coefficient ν is assumed to be constant.

From the viewpoint of hydrodynamics, the first equation is the mass conservation, the second one is the momentum conservation, and the third equation is the incompressibility constraint. Given some initial data ρ_0, u_0, we here want to determine $(\rho, u, \nabla P)$ in the case where the fluid domain Ω is a smooth bounded or exterior domain of $\mathbb{R}^n$.

As we plan to investigate the well-posedness issue of (1.6) for possibly discontinuous initial densities such as in (1.4), the classical strong solution theory developed in e.g. [**13, 34**] is too restrictive because ρ_0 has to be (at least) uniformly continuous therein. In order to understand what should be the relevant functional framework and tools for our analysis, a crucial point is to investigate the linearization of the momentum equation, namely the following evolutionary Stokes system with homogeneous Dirichlet boundary conditions:

$$\left\{\begin{array}{lll} u_t - \nu\Delta u + \nabla P = f & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} u = g & \text{in} & (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{at} & (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{on} & \Omega. \end{array}\right. \tag{1.7}$$

We need a functional framework so that if we plug the obtained solutions in the momentum equation of (1.6) then we get (1.7) with source terms f allowing to recover the regularity we started from. This is exactly in the same spirit as Schauder or L_p estimates for the Laplace operator: they ensure that if $f \in C^\alpha$ (resp. $f \in L_p$) then the solution to $\Delta u = f$, with, say, homogeneous Dirichlet boundary data satisfies $\nabla^2 u \in C^\alpha$ (resp. $\nabla^2 u \in L_p$), see [**29**].

In the setting of (1.7) with $g \equiv 0$ and $u_0 \equiv 0$, we expect u_t, $\nabla^2 u$ and ∇P to have the same regularity as f. In the standard cases (Ω stands for the whole space $\mathbb{R}^n$, the half-space $\mathbb{R}^n_+$, or a bounded or exterior domain of $\mathbb{R}^n$), that so-called *maximal regularity property* has been proved in a number of functional spaces. For instance, if $f \in L_q(0,T;L_p(\Omega))$ for some $1 < p,q < \infty$ and $g \equiv 0$ then there exists a constant C *independent of* T so that any solution to (1.7) satisfies (see e.g. [**28, 35, 40**]):

$$\|u_t, \nu\nabla^2 u, \nabla P\|_{L_q(0,T;L_p(\Omega))} \leq C\big(\|u_0\|_{\dot{B}^{2-\frac{2}{q}}_{p,q}(\Omega)} + \|f\|_{L_q(0,T;L_p(\Omega))}\big). \tag{1.8}$$

Inequalities (1.8) are based on Calderon-Zygmund theory for singular integrals (see [**24**], [**52**]) and related to the analyticity properties of the semi-group of the Stokes operator [**3**]. Therefore, unsurprisingly, they fail in the endpoint cases where one of the exponents p or q is 1 or ∞.

Compared to those results, the natural regularity required on the velocity for (1.2) to be uniquely solvable, is rather exotic: we need an L_1 in time bound. As we aim at working in the critical regularity framework, we cannot afford any loss in the estimates resulting from incautious use of Hölder inequality: this precisely means that we need an extension of (1.8) to the case $p=1$, with a gain of two full spatial derivatives with respect to the data. In other words, we look for a Banach space X with the property that any smooth enough solution $(u,\nabla P)$ to (1.7) satisfies (if $g\equiv 0$ for simplicity):

$$\|u_t,\nu\nabla^2 u,\nabla P\|_{L_1(0,T;X)}\leq C\big(\|u_0\|_X+\|f\|_{L_1(0,T;X)}\big)\quad\text{for all}\ \ T \tag{1.9}$$

with a constant C *independent of* T.

On the one hand, Inequality (1.9) fails whenever X is a reflexive Banach space. On the other hand, it is known to be true in the whole space setting if X is a homogeneous Besov space *with third index* 1, namely $\dot B^s_{p,1}(\mathbb{R}^n)$ (see e.g. [**6**]). The proof relies on a very simple argument based on Fourier analysis, which is recalled at the end of the proof of Theorem 4.1.1. In our recent work [**15**], we extended that inequality to the half-space $\mathbb{R}^n_+$ (assuming that $g\equiv 0$). There, we had to restrict ourselves to values of s close to 0 (namely $-1+1/p<s<1/p$), a limitation corresponding to the case where functions of $\dot B^s_{p,1}(\mathbb{R}^n_+)$ do not have a trace at the boundary and may thus be extended by 0 over $\mathbb{R}^n$, with no loss of regularity. One of the difficulties that we had to face is that, in contrast with the $\mathbb{R}^n$ situation, the half-space case is not amenable to the heat equation by projection, and cannot be reduced either to the $\mathbb{R}^n$ case by a suitable symmetric/antisymmetric extension. A great deal of the analysis was related to the use of the Fourier transform *with respect to tangential variables.*

In the present paper, we want to extend Inequality (1.9) with $X=\dot B^s_{p,1}(\Omega)$ to smooth bounded or exterior domains (the second case being wilder from the point of view of mathematical analysis). In passing, we also treat the case

$$g\not\equiv 0$$

which is of interest for some applications that we have in mind, and that is also needed in some intermediate steps of our proof. The general strategy is the same as in our recent paper dedicated to the heat equation [**20**] but, owing to the divergence constraint, the proof is much more involved and requires first a very careful analysis of the Poisson equation in domains and recent results of ours for the (generalized) divergence equation [**16**]. The basic idea consists in localizing the equation by means of a resolution of unity. We then have to deal with 'interior terms', the support of which do not intersect $\partial\Omega$, and 'boundary terms' with support intersecting $\partial\Omega$. After extension by zero, the interior terms may be handled according to the maximal regularity estimates for $\mathbb{R}^n$. As for boundary terms, we perform a change of variable in order to reduce the study to that of (1.7) in the half-space $\mathbb{R}^n_+$. Putting together all the local estimates does not quite yield the desired inequality, namely [(1)]

$$(1.10)\qquad \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\big).$$

In fact we get it either *up to a low order term involving* u or with a *time-dependent* constant C. In the bounded domain case, one may take advantage of the exponential decay of the Stokes semi-group so as to remove the time dependency. In exterior domains, according to the work by H. Iwashita [**32**], only algebraic decay – the so-called L_p-L_q estimates – is available. It turns out that adapting the work by P. Maremonti and V. Solonnikov in [**39**] allows to bound the $L_1(0,T;\dot B^s_{q,1}(K))$ norm of u for any compact subset K surrounding the boundary of Ω, *provided that* $1<q<n/2$ *and* s *is close enough to* 0. Hence we miss the two-dimensional case.

As an example, let us now state the result we get if $g\equiv 0$ (the general statement being given in Theorem 4.3.3):

Theorem. — Let Ω be a smooth exterior domain [(2)] of $\mathbb{R}^n$ with $n\geq 3$. Let $1<q\leq p<\infty$ with $q<n/2$, and let $s\in(-1+1/p,1/p)$. Assume that $u_0\in \dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega)$ with $\operatorname{div} u_0=0$ and $u_0\cdot\vec n|_{\partial\Omega}=0$ (here $\vec n$ is the outer unit normal vector at $\partial\Omega$) and that $f\in L_1(0,T;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega))$. Then System (1.7) with $g\equiv 0$ has a unique solution $(u,\nabla P)$ with

$$u\in\mathcal{C}(\mathbb{R}_+;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega))\quad\text{and}\quad \partial_t u,\nabla^2 u,\nabla P\in L_1(\mathbb{R}_+;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega)).$$

In addition, we have for all positive T:

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega))} &+\|(u_t,\nu\nabla^2 u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\ &\leq C\big(\|u_0\|_{\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega))}\big),\end{aligned}$$

where the above constant C is *independent of* T *and* ν.

1. Assuming just here that $g\equiv 0$ for simplicity.
2. That is the complement of a smooth simply connected bounded subset.

Such time independent estimates are of interest not only for the Stokes semigroup theory but also in a number of applications related to fluid mechanics. Having a time-independent constant in (1.9) is crucial for proving the global existence of strong solutions for systems related to the incompressible Navier-Stokes equations. In effect, the fact that two full derivatives may be gained with respect to the source term allows to consider not only the Stokes operator but also perturbations of it.

Generalizing the above theorem to the case $g \not\equiv 0$ will enable us to establish new well-posedness results for the inhomogeneous Navier-Stokes equations (1.6) in some critical functional framework related to the scaling of the equations (see Chapter 5 for more details). In the slightly inhomogeneous case, that is if ρ_0 is close enough to some positive constant, we shall prove global well-posedness for small initial velocity, and local well-posedness for large velocity. We shall see that choosing $s=-1+n/p$ in the above statement (which corresponds to the critical regularity framework mentioned above) ensures the velocity field to be L_1-in-time with values in the set of uniformly C^1 functions. Hence, it admits a unique C^1 flow for all time, and the system satisfied by u may thus be reformulated equivalently in Lagrangian variables as explained at the beginning of the introduction, exactly as in our recent work in [**17**] dedicated to the whole space setting (3). Looking at the system in Lagrangian coordinates will enable us to handle discontinuous initial densities and to justify (1.5) for the solutions to (1.6) whenever ρ_0 satisfies (1.4) with ∂A uniformly C^1, and we will get for free that $\partial A(t)$ remains C^1 during the time evolution. In other words, the inhomogeneous incompressible Navier-Stokes equations in a domain may be used for describing a free boundary problem for two incompressible homogeneous fluids with different densities separated by some interface. This is in sharp contrast with the standard approach where the free boundary is seen as an additional unknown (see e.g. [**49**]). Last but not least, our approach based on Lagrangian coordinates will enable us to recast our problem in terms of a suitable contracting mapping, and we will thus get uniqueness of the solution and Lipschitz dependence with respect to the data, with no additional regularity assumption whatsoever.

Another interesting application is given by the initial state $\rho_0=1+Z$ in the case where the nonnegative function Z is bounded and $\operatorname{Supp} Z$ is a connected

3. Note that the divergence free property is lost when performing the change of coordinates explained in (1.2). This, in itself, is a good motivation for considering nonzero divergence in (1.7).

set (a model for the description of pollution in a homogeneous liquid). The uniqueness of solutions implies that, during the time evolution, the polluted area cannot be split into several components because the support of $\rho(\cdot,t)-1$ remains connected. Besides, the bounds we have on the velocity field provide us with some control on the growth of the diameter and on the speed of propagation of the polluted area.

Let us mention that the local-in-time well-posedness for large jumps of the density has been proved in [**19**], by another approach that is not compatible with the critical functional setting. Another interesting development toward this issue has been done recently in [**38**].

As a second application, we solve globally in critical spaces the limit low Mach number system we get in the large entropy variations case for a heat conducting and viscous perfect gas. This is a nonlinear coupling between a heat equation for the temperature and a Stokes-like equation for the velocity that has been investigated recently in [**14**] in the whole space (see also [**2**] and the references therein). Being of parabolic type, there is no need to recast the system in Lagrangian coordinates. So we concentrate on the study of the original equations in the Eulerian coordinates in the case where the fluid domain is exterior (the bounded case being easier). The main difficulty encountered is that the incompressibility condition is violated by the structure of the model. Nevertheless, the generalization of Theorem 1 to nonzero divergence constraints obtained in Theorem 4.3.3 turns out to be appropriate to solve the system. In passing, we have to establish new maximal regularity results (in the spirit of Theorem 1), for the heat equation with Neumann boundary conditions, involving higher order norms. This will be done by combining the methods of the present Chapter 4 and of [**20**].

We end this introduction with a short description of the content of the memoir. In Chapter 2, we introduce most definitions and tools that will be needed in the paper. Besov spaces (and basic properties) are presented, first on $\mathbb{R}^n$, and next, on domains. In passing, we recall some results of ours concerning the divergence equation, and finally present changes of coordinates that will be useful in the analysis of the Stokes equation, and of the inhomogeneous Navier-Stokes equations. The next chapter is dedicated to the study of the Poisson equation with Neumann or Dirichlet boundary conditions. We mainly aim at proving estimates in homogeneous Besov spaces, in the low regularity framework. Those estimates will be one of the keys to the proof of maximal regularity estimates for the evolutionary Stokes system (Chapter 4). In the last two chapters, we give applications of those estimates : Chapter 5 is devoted to the the global well-posedness issue for (1.6) in a critical framework, thus generalizing our recent result in [**17**] and Chapter 6 is concerned with the proof of a similar result for a low Mach number limit system.

CHAPTER 2

TOOLS AND SPACES

In this chapter, we present basic definitions and tools that will be needed throughout the memoir. We first introduce the Littlewood-Paley decomposition (a dyadic decomposition with respect to the Fourier variable) and homogeneous Besov spaces over $\mathbb{R}^n$, then state several classical and fundamental properties : density results, embedding, product estimates, and so on. In the second part of this chapter, we extend the definition of Besov spaces and some of their properties to general domains of $\mathbb{R}^n$. In the third section, we recall some results for the divergence equation, after our recent study in [**16**]. The last part of this chapter is devoted to presenting different types of change of coordinates that will be used a number of times in this paper to transform a PDE problem at the boundary of a domain into a problem in the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$. In passing, we introduce the Lagrangian coordinates needed in Chapter 5, and derive related algebraic relations.

2.1. Besov spaces on $\mathbb{R}^n$

2.1.1. Definition and classical properties. — Throughout we fix a smooth nonincreasing function $\chi : \mathbb{R}_+ \to [0,1]$ supported in $[0,1)$ and such that $\chi \equiv 1$ on $[0,1/2)$, and set

$$\varphi(\xi) := \chi\big(|\xi|/2\big) - \chi\big(|\xi|\big).$$

Note that φ is valued in $[0,1]$, supported in $\{1/2 \leq |\xi| \leq 2\}$ and that

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1 \quad \text{for all} \quad \xi \neq 0. \tag{2.1}$$

Then we introduce the homogeneous Littlewood-Paley spectral truncation operators $(\dot{\Delta}_k)_{k\in\mathbb{Z}}$ over $\mathbb{R}^n$ by setting

$$\dot{\Delta}_k u := \varphi(2^{-k}D)u := \mathcal{F}^{-1}\big(\varphi(2^{-k}\cdot)\mathcal{F}u\big).$$

Above $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^n$. We also define the low frequency cut-off

$$\dot{S}_k := \chi(2^{-k}D). \tag{2.2}$$

For $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$, we define the following homogeneous Besov semi-norms on $\mathbb{R}^n$ as follows:

$$\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} := \big\|2^{sk}\|\dot{\Delta}_k u\|_{L_p(\mathbb{R}^n)}\big\|_{\ell_r(\mathbb{Z})}$$

and nonhomogeneous Besov norms:

$$\|u\|_{B^s_{p,r}(\mathbb{R}^n)} := \big\|2^{sk}\|\dot{\Delta}_k u\|_{L_p(\mathbb{R}^n)}\big\|_{\ell_r(\mathbb{N})} + \|\dot{S}_0 u\|_{L_p(\mathbb{R}^n)}.$$

It is obvious that Besov semi-norms vanish on the set of polynomials. To upgrade them to norms, we need a further control on low frequencies. To this end, we shall adopt the following definition borrowed from [**6**]:

$$\dot{B}^s_{p,r}(\mathbb{R}^n) = \Big\{u \in \mathcal{S}'_h(\mathbb{R}^n) \,:\, \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} < \infty\Big\},$$

where $\mathcal{S}'_h(\mathbb{R}^n)$ stands for the set of tempered distributions u over $\mathbb{R}^n$ such that for all smooth compactly supported function θ over $\mathbb{R}^n$, we have

$$\lim_{\lambda\to+\infty} \theta(\lambda D)u = 0 \quad \text{in} \quad L_\infty(\mathbb{R}^n). \tag{2.3}$$

That condition is obviously satisfied whenever $\theta(D)u \in L_p(\mathbb{R}^n)$ for some $p < \infty$ and $\theta \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\theta(0) \neq 0$. Note also that any distribution in $\mathcal{S}'_h(\mathbb{R}^n)$ tends weakly to 0 at infinity. In particular, $\mathcal{S}'_h(\mathbb{R}^n)$ contains no nonzero polynomial and if $u \in \mathcal{S}'_h(\mathbb{R}^n)$ then one may write

$$u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u \quad \text{in} \quad \mathcal{S}'_h(\mathbb{R}^n). \tag{2.4}$$

Conversely, if (2.4) is satisfied and $\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} < \infty$ for some index s such that $s < n/p$ (or $s \leq n/p$ if $r = 1$) then u is in $\dot{B}^s_{p,r}(\mathbb{R}^n)$.

The following fundamental properties are proved in e.g. [**6**], [**10**], [**11**]:

Proposition 2.1.1. — *Basic properties.*

1. *Completeness: the space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ is complete whenever*

$$s \leq n/p \ \ \textit{if} \ \ r = 1, \quad \textit{or} \quad s < n/p \ \ \textit{if} \ \ r > 1. \tag{2.5}$$

2. *Density: the set* $\mathcal{S}_0(\mathbb{R}^n)$ *of Schwartz functions with Fourier transform supported away from the origin is dense in* $\dot B^s_{p,r}(\mathbb{R}^n)$ *if and only if* p *and* r *are finite.*
3. *Action of derivatives: for any* $k \in \{1,\dots,n\}$*, the derivative operator* ∂_k *maps* $\dot B^s_{p,r}(\mathbb{R}^n)$ *in* $\dot B^{s-1}_{p,r}(\mathbb{R}^n)$*. Besides, we have for some constant* $C \geq 1$ *independent of* u*:*
$$C^{-1}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} \leq \|\nabla u\|_{\dot B^{s-1}_{p,r}(\mathbb{R}^n)} \leq C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$
4. *Embedding: if* $p_1 \leq p_2$ *and* $r_1 \leq r_2$ *then* $\dot B^{s+n/p_1}_{p_1,r_1}(\mathbb{R}^n)$ *is continuously embedded in* $\dot B^{s+n/p_2}_{p_2,r_2}(\mathbb{R}^n)$*.*
5. *Comparison with Lebesgue spaces:*
   - *for any* $1 \leq p \leq \infty$*, we have for some universal constant* c*,*
   $$c\|u\|_{\dot B^0_{p,\infty}(\mathbb{R}^n)} \leq \|u\|_{L_p(\mathbb{R}^n)} \leq \|u\|_{\dot B^0_{p,1}(\mathbb{R}^n)};$$
   - *for any* $1 < p < \infty$*, we have*
   $$\dot B^0_{p,\min(p,2)}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \hookrightarrow \dot B^0_{p,\max(p,2)}(\mathbb{R}^n);$$
   - *if* $1 \leq p < \infty$ *and* $0 < s < n/p$ *then*
   $$\dot B^s_{p,p}(\mathbb{R}^n) \hookrightarrow L_{p^*}(\mathbb{R}^n) \quad \textit{with} \quad \frac{n}{s}\Big(\frac{1}{p} - \frac{1}{p^*}\Big) = 1.$$
6. *Scaling properties: There exists a constant* C *depending only on* s *such that for all* $\lambda > 0$ *we have*
$$C^{-1}\lambda^{s-n/p}\|u\|_{\dot B^s_{p,r}} \leq \|u(\lambda\cdot)\|_{\dot B^s_{p,r}} \leq C\lambda^{s-n/p}\|u\|_{\dot B^s_{p,r}}.$$
7. *Duality: for all* $(s,p,r) \in \mathbb{R}\times[1,\infty]^2$*, we have*
$$\Big|\int_{\mathbb{R}^n} uv\,dx\Big| \leq C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)},$$
and the space $\dot B^{-s}_{p',r'}(\mathbb{R}^n)$ *coincides with the set of* $u \in \mathcal{S}'_h(\mathbb{R}^n)$ *such that*
$$\sup_v \Big|\int_{\mathbb{R}^n} uv\,dx\Big| < \infty \tag{2.6}$$
where the supremum is taken over functions v *in* $\mathcal{S}(\mathbb{R}^n)\cap\dot B^{-s}_{p',r'}(\mathbb{R}^n)$ *with* $\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)} \leq 1$*. The left-hand side of* (2.6) *is equivalent to* $\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}$*.*
8. *Fatou property: under Condition* (2.5)*, there exists a constant* C *such that for any bounded sequence* $(u_j)_{j\in\mathbb{N}}$ *of* $\dot B^s_{p,r}(\mathbb{R}^n)$ *converging to some* u *in* $\mathcal{S}'(\mathbb{R}^n)$*, we have*
$$\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} \leq C\liminf_{j\to+\infty}\|u_j\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

In some parts of the paper, we shall also use the more classical *nonhomogeneous* Besov spaces $B^s_{p,r}(\mathbb{R}^n)$ that are defined by

$$B^s_{p,r}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,:\, \|u\|_{B^s_{p,r}(\mathbb{R}^n)} < \infty \right\}\cdot$$

Those spaces have the above properties with no restriction (2.5). Furthermore both the set $\mathcal{C}^\infty_c(\mathbb{R}^n)$ of smooth functions with compact support, and the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ are dense in $B^s_{p,q}(\mathbb{R}^n)$ whenever p and r are finite.

2.1.2. Product laws. — We shall make an extensive use of the following inequalities, sometimes named *tame estimates* because of their linear dependence with respect to the highest norm.

PROPOSITION 2.1.2. — *Let $b^s_{p,r}$ denote $\dot B^s_{p,r}(\mathbb{R}^n)$ or $B^s_{p,r}(\mathbb{R}^n)$. Then the following estimates hold true*[(1)]*:*

– *For any $s > 0$,*

$$\|uv\|_{b^s_{p,r}} \lesssim \|u\|_{L_\infty}\|v\|_{b^s_{p,r}} + \|v\|_{L_\infty}\|u\|_{b^s_{p,r}}.$$

– *For any $s > 0$ and $t > 0$,*

$$\|uv\|_{b^s_{p,r}} \lesssim \|u\|_{L_\infty}\|v\|_{b^s_{p,r}} + \|v\|_{b^{-t}_{\infty,r}}\|u\|_{b^{s+t}_{p,\infty}}.$$

– *For any $t > 0$ and $s > -n/p'$,*

$$\|uv\|_{b^s_{p,r}} \lesssim \|u\|_{L_\infty}\|v\|_{b^s_{p,r}} + \|u\|_{b^{n/p'}_{p',\infty}}\|v\|_{b^s_{p,r}} + \|v\|_{b^{-t}_{\infty,r}}\|u\|_{b^{s+t}_{p,\infty}}.$$

Proof. — The proof is based on continuity results for the paraproduct and on Bony's decomposition that has been introduced in [**9**]:

$$uv = T_u v + R(u,v) + T_v u.$$

Above, T and R stand for the paraproduct and remainder operators, respectively, that may be defined in the homogeneous case by

$$T_u v := \sum_{j\in\mathbb{Z}} \dot S_{j-1}u\,\dot\Delta_j v \quad\text{and}\quad R(u,v) = \sum_{j\in\mathbb{Z}}\sum_{|i|\le1} \dot\Delta_j u\,\dot\Delta_{j+i}v,$$

and in the nonhomogeneous case by

$$T_u v = \sum_{j\ge1} \dot S_{j-1}u\,\Delta_j v \quad\text{and}\quad R(u,v) = \sum_{j\ge-1}\sum_{|i|\le1} \Delta_j u\,\Delta_{j+i}v,$$

with $\Delta_k = \dot\Delta_k$ if $k\ge0$, $\Delta_{-1} = \dot S_0$ and $\Delta_k = 0$ if $k \le -2$. Recall that $\dot S_k$ has been defined in (2.2).

1. From now on, we agree that $A \lesssim B$ means that $A \le CB$ for some harmless positive constant C.

So, in order to prove the above estimates, it suffices to use the classical properties of continuity for R and T, namely in the cases we are interested in:

$$\|T_u v\|_{b^s_{p,r}} \lesssim \|u\|_{L_\infty}\|v\|_{b^s_{p,r}} \quad\text{and}\quad \|T_u v\|_{b^s_{p,r}} \lesssim \|u\|_{b^{-t}_{\infty,r}}\|v\|_{b^{s+t}_{p,\infty}} \quad\text{if}\quad t>0,$$
$$\|R(u,v)\|_{b^s_{p,r}} \lesssim \|u\|_{L_\infty}\|v\|_{b^s_{p,r}} \quad\text{and}\quad \|R(u,v)\|_{b^s_{p,r}} \lesssim \|u\|_{b^{-t}_{\infty,r}}\|v\|_{b^{s+t}_{p,\infty}} \quad\text{if}\quad s>0,$$
$$\|R(u,v)\|_{b^s_{p,r}} \lesssim \|u\|_{b^{n/p'}_{p',\infty}}\|v\|_{b^s_{p,r}} \quad\text{if}\quad s>-n/p'.$$

As an example, let us establish the second inequality for $T_u v$ in the homogeneous case. The reader may refer to [**6**], [**48**], [**51**] for the proof of the other inequalities. Owing to the support properties of the function φ entering in the definition of $\dot\Delta_j$, we may write for all $j\in\mathbb{Z}$ and some large enough integer N:

$$\dot\Delta_j(T_u v) = \sum_{|k-j|\le N} \dot\Delta_j(\dot S_{k-1}u\dot\Delta_k v).$$

Hence

$$2^{js}\|\dot\Delta_j(T_u v)\|_{L_p(\mathbb{R}^n)} \le C\sum_{|k-j|\le N} 2^{(j-k)s}2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)}2^{k(s+t)}\|\dot\Delta_k v\|_{L_p(\mathbb{R}^n)},$$

and one may thus assert that

$$\|T_u v\|_{\dot B^s_{p,r}} \le C\Big\|2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)}\,2^{k(s+t)}\|\dot\Delta_k v\|_{L_p(\mathbb{R}^n)}\Big\|_{\ell_r(\mathbb{Z})}.$$

We may further write

$$2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)} \le \sum_{k'\le k-2} 2^{(k'-k)t}\,2^{-k't}\|\dot\Delta_{k'}u\|_{L_\infty(\mathbb{R}^n)}.$$

Because $t>0$, taking the $\ell_r(\mathbb{Z})$ norm of both sides and using convolution inequalities for series completes the proof. □

As a smooth compactly supported function belongs to any space $\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ with $1\le p\le\infty$, and to any Besov space $B^\sigma_{p,1}(\mathbb{R}^n)$, we deduce from the previous proposition and embedding the following localization properties [(2)]:

Corollary 2.1.1. — *Let θ be in $\mathcal{C}^\infty_c(\mathbb{R}^n)$. Then $u\mapsto\theta u$ is a continuous mapping of $b^s_{p,r}(\mathbb{R}^n)$*

- *for any $s\in\mathbb{R}$ and $1\le p,r\le\infty$, if $b^s_{p,r}(\mathbb{R}^n)=B^s_{p,r}(\mathbb{R}^n)$;*
- *for any $s\in\mathbb{R}$ and $1\le p,r\le\infty$ satisfying $-n/p'<s<n/p$ ($-n/p<s\le n/p$ if $r=1$ and $-n/p'\le s<n/p$ if $r=\infty$) if $b^s_{p,r}(\mathbb{R}^n)=\dot B^s_{p,r}(\mathbb{R}^n)$.*

2. In the nonhomogeneous case with very negative s, we need to resort to other continuity results for R than those that have been recalled above.

The above proposition will also enable us to compare $B^s_{p,r}(\mathbb{R}^n)$ and $\dot B^s_{p,r}(\mathbb{R}^n)$ for compactly supported functions:

Proposition 2.1.3. — *Let* $1 \le p, r \le \infty$ *and* $s > -n/p'$ *(or* $s \ge -n/p'$ *if* $r = \infty$*). Then for any* compactly supported *distribution* u *we have*

$$u \in B^s_{p,r}(\mathbb{R}^n) \iff u \in \dot B^s_{p,r}(\mathbb{R}^n)$$

and there exists a constant $C = C(s, p, r, n, K)$ *(with* $K = \operatorname{Supp} u$*) such that*

$$C^{-1}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} \le \|u\|_{B^s_{p,r}(\mathbb{R}^n)} \le C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

Proof. — Let us first treat the case $s > 0$. Then the embedding $B^s_{p,r}(\mathbb{R}^n) \hookrightarrow \dot B^s_{p,r}(\mathbb{R}^n)$ is clear [3]. Conversely, assume that u belongs to $\dot B^s_{p,r}(\mathbb{R}^n)$. In order to get $u \in B^s_{p,r}(\mathbb{R}^n)$, it suffices to establish that $u \in L_p(K)$. This is in fact obvious as one may write that

$$u = \dot S_0 u + (\mathrm{Id} - \dot S_0)u.$$

The first term belongs to $L_\infty(\mathbb{R}^n)$ (as u is in $\mathcal{S}'_h(\mathbb{R}^n)$) hence to $L_p(K)$. Next, because its Fourier transform is supported away from the origin and $s > 0$, the second term belongs to $L_p(\mathbb{R}^n)$. We claim that there exists some constant C depending only on p, K and s, such that

$$\|u\|_{L_p(K)} \le C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

Let us write that $u = \dot S_j u + (\mathrm{Id} - \dot S_j)u$ for some $j \in \mathbb{Z}$ to be chosen hereafter. We have

$$\begin{aligned}
\|u\|_{L_p(K)} &\le \|\dot S_j u\|_{L_p(K)} + \|(\mathrm{Id} - \dot S_j)u\|_{L_p(\mathbb{R}^n)} \\
&\le |K|^{\frac1p}\|\dot S_j u\|_{L_\infty(\mathbb{R}^n)} + C2^{-js}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.
\end{aligned}$$

Using Bernstein's inequalities and, again, that $\operatorname{Supp} u \subset K$, we thus get

$$\begin{aligned}
\|u\|_{L_p(K)} &\le C|K|^{\frac1p}2^{jn}\|u\|_{L_1(\mathbb{R}^n)} + C2^{-js}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} \\
&\le C|K|2^{jn}\|u\|_{L_p(K)} + C2^{-js}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.
\end{aligned}$$

So choosing j so that $2^{-n} < 2C|K|2^{jn} \le 1$, we discover that

$$\|u\|_{L_p(K)} \le C|K|^{\frac sn}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

3. Without any support assumption, it is obvious that if s is positive then we have $\|.\|_{\dot B^s_{p,r}(\mathbb{R}^n)} \lesssim \|.\|_{B^s_{p,r}(\mathbb{R}^n)}$, and that the opposite inequality holds true if s is negative.

Let us now focus on the case $s < 0$. It is clear that any (not necessarily compactly supported) distribution in $\dot{B}^s_{p,r}(\mathbb{R}^n)$ belongs to $B^s_{p,r}(\mathbb{R}^n)$, too. Conversely, consider some distribution $u \in B^s_{p,r}(\mathbb{R}^n)$ with compact support and fix a cut-off function $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with value 1 on $\operatorname{Supp} u$. We decompose u into

$$u = \eta \dot{S}_0 u + \eta(\mathrm{Id} - \dot{S}_0)u. \tag{2.7}$$

Note that $\mathrm{Id} - \dot{S}_0$ maps $B^s_{p,r}(\mathbb{R}^n)$ in $\dot{B}^s_{p,r}(\mathbb{R}^n)$, as there are no low frequencies. As $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, Corollary 2.1.1 implies that the last term in (2.7) belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ and that for some constant $C = C(s, p, n, \eta)$,

$$\|\eta(\mathrm{Id} - \dot{S}_0)u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} \leq C\|u\|_{B^s_{p,r}(\mathbb{R}^n)}. \tag{2.8}$$

Next, because $\dot{S}_0 u$ is a $\mathcal{C}^\infty$ bounded function, $\eta\dot{S}_0 u$ is in $L_1(\mathbb{R}^n)$. Hence, by embedding, it is also in $\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)$, and we may thus write

$$\|\eta\dot{S}_0 u\|_{\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)} \leq C\|\eta\dot{S}_0 u\|_{L_1(\mathbb{R}^n)} \leq C\|u\|_{B^s_{p,r}(\mathbb{R}^n)}.$$

Of course $\eta\dot{S}_0 u$ also belongs to $L_p(\mathbb{R}^n)$ hence to all the intermediate Besov spaces (with obvious estimates) between $\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n)$, and in particular to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if $-n/p' < s < 0$.

The limit case $s = 0$ follows by interpolation. □

In some applications we have in mind, we need not specify in which Besov space the two terms of the product belong. Typically, given u in some Banach space X, and some function ϕ, we just need to know that ϕu belongs to the same space X. This motivates the following definition of a *multiplier space*.

Definition 2.1.1. — *Let X be a Banach space. We designate by $\mathcal{M}(X)$ (multiplier space for X) the set of those tempered distributions ϕ so that ϕu is in X for all $u \in X$.*

The space $\mathcal{M}(X)$ is naturally endowed with a structure of Banach space if equipped with the following norm:

$$\|\phi\|_{\mathcal{M}(X)} = \sup_{\|u\|_X = 1} \|\phi u\|_X.$$

Even for very classical spaces (e.g. Sobolev spaces), describing the corresponding multiplier space in terms of standard functional spaces is hopeless (see e.g.[**41**]). From the first item of Proposition 2.1.2, one may assert that $\mathcal{M}(b^s_{p,r}(\mathbb{R}^n))$ contains $L_\infty(\mathbb{R}^n) \cap b^s_{p,r}(\mathbb{R}^n)$ if $s > 0$ and $1 \leq p, r \leq \infty$, but this is far from being optimal.

A direct application of Lemma 2.2.1 below ensures that if A is a subset of $\mathbb{R}^n$ with uniformly C^1 boundary then for all $1 \le q \le \infty$,

$$1_A \in \mathcal{M}(\dot{B}^s_{p,q}(\mathbb{R}^n)) \text{ if } s \in \big(-1+1/p, 1/p\big) \ \text{ with } 1 < p < \infty. \tag{2.9}$$

The following lemma will be useful when transforming a problem on the boundary to a problem on the half-space and also to justify the equivalence between the Eulerian and Lagrangian formulation of the systems of PDEs that we shall study in the last chapter.

Lemma 2.1.1. — *Let $Z : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$ with $-n/p' < s < n/p$. The linear map $u \mapsto u \circ Z$ is continuous on $\dot{B}^s_{p,r}(\mathbb{R}^n)$ whenever:*

- *either $0 < s < 1$ and $J_{Z^{-1}}$, DZ are bounded,*
- *or $-1 < s < 0$, J_Z, DZ^{-1} are bounded and $J_{Z^{-1}} \in \mathcal{M}(\dot{B}^{-s}_{p',r'}(\mathbb{R}^n))$.*

Above, we agree that $J_Z^{\pm 1} := |\det DZ^{\pm 1}|$.

Proof. — Let us first consider the case $s \in (0,1)$ and p, r finite (the limit cases being left to the reader). Using the characterization of homogeneous Besov semi-norms by means of finite differences (see e.g. [**6, 52**]), one may write up to an irrelevant constant:

$$\|u \circ Z\|^r_{\dot{B}^s_{p,r}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|u(Z(y)) - u(Z(x))|^p}{|y-x|^{n+sp}}\, dy\Big)^{\frac{r}{p}} dx.$$

So performing the change of variable $x' = Z(x)$ and $y' = Z(y)$, we see that

$$\|u \circ Z\|^r_{\dot{B}^s_{p,r}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|u(y') - u(x')|^p}{|Z^{-1}(y') - Z^{-1}(x')|^{n+sp}} J_{Z^{-1}}(y')\, dy'\Big)^{\frac{r}{p}} J_{Z^{-1}}(x)\, dx',$$

whence

$$\|u \circ Z\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} \le \|J_{Z^{-1}}\|^{\frac{1}{p}+\frac{1}{r}}_{L_\infty(\mathbb{R}^n)} \|DZ\|^{s+\frac{n}{p}}_{L_\infty(\mathbb{R}^n)} \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)}.$$

Let us emphasize that the condition that $s < n/p$ ensures in addition that u belongs to some Lebesgue space $L_{p^*}(\mathbb{R}^n)$ with $p^* < \infty$. Hence $u \circ Z \in L_{p^*}(\mathbb{R}^n)$, too, and one may thus conclude that $u \circ Z \in \dot{B}^s_{p,r}(\mathbb{R}^n)$.

The result for $s \in (-1,0)$ may be achieved by duality: we have

$$\|u \circ Z\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} = \sup_{\|v\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} v(z) u\big(Z(z)\big)\, dz.$$

Now, setting $x = Z(z)$ yields

$$\begin{aligned}\int_{\mathbb{R}^n} v(z)u(Z(z))\,dz &= \int_{\mathbb{R}^n} u(x)v(Z^{-1}(x))J_{Z^{-1}(x)}\,dx\\ &\leq \|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|J_{Z^{-1}}\,v\circ Z^{-1}\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}\\ &\lesssim \|J_Z\|^{\frac{1}{p'}+\frac{1}{r'}}_{L_\infty(\mathbb{R}^n)}\|DZ^{-1}\|^{-s+\frac{n}{p'}}_{L_\infty(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n))}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\end{aligned}$$

the last inequality being a consequence of the first part of the proof, of the definition of multiplier spaces and of $\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)} \leq 1$.

In order to show that $u\circ Z \in \mathcal{S}'_h(\mathbb{R}^n)$, one may use the fact that

$$\|\dot S_j(u\circ Z)\|_{L_\infty(\mathbb{R}^n)} = \sup_{\|v\|_{L_1(\mathbb{R}^n)}=1}\int_{\mathbb{R}^n} u\circ Z\,\dot S_j v\,dx$$

and follow the above computations. We still get

$$\begin{aligned}&\int_{\mathbb{R}^n} u\circ Z\,\dot S_j v\,dx\\ &\quad\leq \|J_Z\|^{\frac{1}{p'}+\frac{1}{r'}}_{L_\infty(\mathbb{R}^n)}\|DZ^{-1}\|^{-s+\frac{n}{p'}}_{L_\infty(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n))}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|\dot S_j v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}.\end{aligned}$$

By using Bernstein inequality and the fact that v is in $L_1(\mathbb{R}^n)$, it is not difficult to conclude that $\|\dot S_j(u\circ Z)\|_{L_\infty(\mathbb{R}^n)} \to 0$ when j goes to $-\infty$. This completes the proof. □

Remark 2.1.1. — The above lemma extends to $s = 0$ by interpolation. It also may be generalized to higher order regularities if making stronger assumptions on Z. For instance, if assuming that $1 < s < 2$ then the map $u \mapsto u\circ Z$ is continuous on $\dot B^s_{p,r}(\mathbb{R}^n)$ whenever $J_{Z^{-1}}$ and DZ are bounded, and

$$DZ \in \mathcal{M}(\dot B^{s-1}_{p,r}(\mathbb{R}^n)).$$

Likewise, if $-2 < s < -1$ then $u \to u\circ Z$ is continuous on $\dot B^s_{p,r}(\mathbb{R}^n)$ whenever J_Z and DZ^{-1} are bounded, and

$$J_{Z^{-1}} \in \mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n)) \quad\text{and}\quad DZ^{-1} \in \mathcal{M}(\dot B^{-s-1}_{p,r}(\mathbb{R}^n)).$$

Proof. — If $1 < s < 2$ then we look for a bound of $D(u\circ Z)$ in $\dot B^{s-1}_{p,r}(\mathbb{R}^n)$. Using the chain rule $D(u\circ Z) = (Du\circ Z)\cdot DZ$, the definition of multiplier spaces and the previous lemma, we may write

$$\begin{aligned}\|D(u\circ Z)\|_{\dot B^{s-1}_{p,r}(\mathbb{R}^n)} &\lesssim \|DZ\|_{\mathcal{M}(\dot B^{s-1}_{p,r}(\mathbb{R}^n))}\|Du\circ Z\|_{B^{s-1}_{p,r}(\mathbb{R}^n)}\\ &\lesssim \|DZ\|_{\mathcal{M}(\dot B^{s-1}_{p,r}(\mathbb{R}^n))}\|J_{Z^{-1}}\|^{\frac{1}{p}+\frac{1}{r}}_{L_\infty(\mathbb{R}^n)}\|DZ\|^{s-1+\frac{n}{p}}_{L_\infty(\mathbb{R}^n)}\|Du\|_{B^{s-1}_{p,r}(\mathbb{R}^n)}.\end{aligned}$$

As for the case $-2 < s < -1$, we argue by duality:

$$\int_{\mathbb{R}^n} v(z)u(Z(z))\,dz = \int_{\mathbb{R}^n} u(x)v(Z^{-1}(x))J_{Z^{-1}}(x)\,dx,$$
$$\leq \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)}\|v\circ Z^{-1}\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot{B}^{-s}_{p',r'}(\mathbb{R}^n))}.$$

As $1 < -s < 2$, applying the result for positive indices of regularity to $v\circ Z^{-1}$ enables us to conclude. □

2.2. Besov spaces on domains

We aim at extending the definition of homogeneous Besov spaces to general domains. We proceed by restriction as follows [(4)]:

Definition 2.2.1. — *For $s\in\mathbb{R}$ and $1\leq p,q\leq\infty$, we define the homogeneous Besov space $\dot{B}^s_{p,q}(\Omega)$ on Ω to be the restriction (in the distributional sense) of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ to Ω, that is*

$$\phi\in\dot{B}^s_{p,q}(\Omega) \iff \phi=\psi|_\Omega \quad \text{for some} \quad \psi\in\dot{B}^s_{p,q}(\mathbb{R}^n).$$

We then set

$$\|\phi\|_{\dot{B}^s_{p,q}(\Omega)} := \inf_{\psi|_\Omega=\phi}\|\psi\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.$$

As in the $\mathbb{R}^n$ case, the Besov spaces defined above are Banach spaces with the Fatou property whenever Condition (2.5) is satisfied. Moreover, interpolation and embedding properties may be deduced from those that have been stated in Proposition 2.1.1.

Owing to the definition by restriction, we expect the product estimates to be the same as in the whole space setting. For example, for $s>0$, it seems reasonable to have

$$\|uv\|_{\dot{B}^s_{p,q}(\Omega)} \lesssim \|u\|_{L_\infty(\Omega)}\|v\|_{\dot{B}^s_{p,q}(\Omega)} + \|v\|_{L_\infty(\Omega)}\|u\|_{\dot{B}^s_{p,q}(\Omega)}. \tag{2.10}$$

However the situation is not so simple because if we consider some extensions $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^n$ of u and v then $\bar{u}\bar{v}$ is an extension of uv over $\mathbb{R}^n$ but it is not clear that the restriction to Ω of $X(\mathbb{R}^n)\cap Y(\mathbb{R}^n)$ coincides with $X(\Omega)\cap Y(\Omega)$.

As regards (2.10), it may be fully justified for $0<s<1/p$ by using extensions by zero if Ω is uniformly C^1 (see Corollary 2.2.1 below). For larger values of s and if the domain is sufficiently smooth then there exists an explicit bounded extension operator $E: B^s_{p,r}(\Omega)\to B^s_{p,r}(\mathbb{R}^n)$ which is also bounded from $L_\infty(\Omega)\to L_\infty(\mathbb{R}^n)$, see [**1**],[**52**].

4. Nonhomogeneous Besov spaces on domains may be defined by the same token.

In most situations, the following result will be sufficient for our purposes:

Proposition 2.2.1. — *Let $b^s_{p,r}(\Omega)$ denote $\dot B^s_{p,r}(\Omega)$ or $B^s_{p,r}(\Omega)$, and Ω be a domain of $\mathbb{R}^n$. Then for any $p\in[1,\infty]$, s such that $-n/p'<s<n/p$ (or $-n/p'<s\leq n/p$ if $r=1$, or $-n/p'\leq s<n/p$ if $r=\infty$), the following inequality holds true:*

$$\|uv\|_{b^s_{p,r}(\Omega)}\leq C\|u\|_{b^{n/q}_{q,1}(\Omega)}\|v\|_{b^s_{p,r}(\Omega)}\quad\textit{with}\ \ q=\min(p,p').$$

Proof. — Let us assume for instance that $p\leq 2$ (so that $p'\geq p$). Let us consider some extensions $\widetilde u$ and $\widetilde v$ of u and v in $b^{n/p}_{p,1}(\mathbb{R}^n)$ and $b^s_{p,r}(\mathbb{R}^n)$, respectively. Then applying the last item of Proposition 2.1.2 to $\widetilde u$ and $\widetilde v$ with $t=n/p-s$, and noticing that our assumption on p guarantees that

$$b^{n/p}_{p,1}(\mathbb{R}^n)\hookrightarrow b^{n/q}_{q,\infty}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)\quad\text{for}\ \ q=p,p',$$

we get $uv=(\widetilde u\widetilde v)|_\Omega$ and

$$\|\widetilde u\,\widetilde v\|_{b^s_{p,r}(\mathbb{R}^n)}\leq C\|\widetilde u\|_{b^{n/p}_{p,1}(\mathbb{R}^n)}\|\widetilde v\|_{b^s_{p,r}(\mathbb{R}^n)}.$$

As this inequality holds true with the same constant for *any* extensions of u and v, we get the result. □

We shall often use the following compact embedding (see [**48**]).

Proposition 2.2.2. — *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Then for any $s\in\mathbb{R}$, $(p,q)\in[1,+\infty]^2$ and $\varepsilon>0$, the space $B^s_{p,q}(\Omega)$ is compactly embedded in $B^{s-\varepsilon}_{p,q}(\Omega)$.*

In addition, any bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $B^s_{p,q}(\Omega)$ converges weakly star (up to an omitted extraction) to some u in $B^s_{p,q}(\Omega)$ and we have

$$\|u\|_{B^s_{p,q}(\Omega)}\leq C\liminf\|u_n\|_{B^s_{p,q}(\Omega)}\quad\textit{and}\quad u_n\longrightarrow u\ \ \textit{in any}\ \ B^{s-\varepsilon}_{p,q}(\Omega).$$

As already pointed out in the previous section, interpolation properties are a very useful feature of Besov spaces. We refer to [**7, 52**] for the proof of the following statement.

Proposition 2.2.3. — *Let $b^s_{p,q}$ denote $B^s_{p,q}(\Omega)$ or $\dot B^s_{p,q}(\Omega)$; $s\in\mathbb{R}$, $p\in(1,\infty)$ and $q\in[1,\infty]$. The real interpolation of Besov spaces gives if $s_1\neq s_2$:*

$$\big(b^{s_1}_{p,q_1}(\Omega),b^{s_2}_{p,q_2}(\Omega)\big)_{\theta,q}=b^s_{p,q}(\Omega)$$

with $s:=\theta s_2+(1-\theta)s_1$ and $\frac1p:=\frac{\theta}{p_2}+\frac{1-\theta}{p_1}\cdot$. Moreover, if $s_1\neq s_2$, $t_1\neq t_2$ and if $T:b^{s_1}_{p_1,q_1}(\Omega)+b^{s_2}_{p_2,q_2}(\Omega)\to b^{t_1}_{k_1,l_1}(\Omega)+b^{t_2}_{k_2,l_2}(\Omega)$ is a linear map, bounded from

$b^{s_1}_{p_1,q_1}(\Omega)$ to $b^{t_1}_{k_1,l_1}(\Omega)$ and from $b^{s_2}_{p_2,q_2}(\Omega)$ to $b^{t_2}_{k_2,l_2}(\Omega)$ then for any $\theta\in(0,1)$, the map T is also bounded from $b^s_{p,q}(\Omega)$ to $b^t_{k,q}(\Omega)$ with

$$s=\theta s_2+(1-\theta)s_1,\quad t=\theta t_2+(1-\theta)t_1,\quad \frac{1}{p}=\frac{\theta}{p_2}+\frac{1-\theta}{p_1},\quad \frac{1}{k}=\frac{\theta}{k_2}+\frac{1-\theta}{k_1}\cdot$$

An important question is whether one is allowed to extend functions in domains by 0, without changing their regularity. In the flat case, the following statement (see [**15**]) gives the answer.

Lemma 2.2.1. — *For $\varepsilon>0$, denote $\Phi_\varepsilon(u):x\mapsto\eta_\varepsilon(x_n)u(x)$ with*

$$\eta_\varepsilon(t):=\begin{cases}0 & \textit{for}\quad t<\varepsilon,\\ \frac{1}{\varepsilon}t-1 & \textit{for}\quad \varepsilon\le t\le 2\varepsilon,\\ 1 & \textit{for}\quad t>2\varepsilon.\end{cases}$$

Then for all $1\le p<\infty$, $1\le q\le\infty$ and $-1+1/p<s<1/p$ the operator Φ_ε maps $\dot B^s_{p,q}(\mathbb{R}^n)$ in $\dot B^s_{p,q}(\mathbb{R}^n)$ uniformly with respect to ε. Moreover, if q is finite then for all $u\in\dot B^s_{p,q}(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to0}\Phi_\varepsilon(u)=\Phi_0(u):=1_{\mathbb{R}^n_+}u\quad \textit{in}\quad \dot B^s_{p,q}(\mathbb{R}^n).$$

As a corollary, we readily get that $1_{\mathbb{R}^n_+}$ is in $\mathcal{M}(\dot B^s_{p,q}(\mathbb{R}^n))$ whenever (s,p,q) are as above. More generally, as already pointed out, Lemma 2.2.1 implies that 1_A is in $\mathcal{M}(\dot B^s_{p,q}(\mathbb{R}^n))$ if A is any uniformly C^1 domain of $\mathbb{R}^n$. Indeed, the C^1 regularity allows to transform locally the boundary to that of the half-space case (see Lemma A.7 in [**17**] for more details).

Now, if we consider some uniformly C^1 domain Ω and $u\in\dot B^s_{p,q}(\Omega)$ and some arbitrary extension $\widetilde u\in\dot B^s_{p,q}(\mathbb{R}^n)$ of u, then we deduce that $\widetilde u1_\Omega$ is still in $\dot B^s_{p,q}(\mathbb{R}^n)$ (with the expected control of the norm). In other words, we proved the following result [5]:

Corollary 2.2.1. — *For any uniformly C^1 domain Ω, $(p,q)\in[1,\infty]^2$ and $s\in(-1+1/p,1/p)$, the extension by 0 operator is continuous from $\dot B^s_{p,q}(\Omega)$ to $\dot B^s_{p,q}(\mathbb{R}^n)$.*

Remark 2.2.1. — Combining the above corollary with Proposition 2.1.3, we deduce that

$$B^s_{p,q}(\Omega)=\dot B^s_{p,q}(\Omega)\quad\text{if}\quad -1+1/p<s<1/p\quad\text{and } \Omega \text{ is a } C^1 \text{ bounded domain.}$$

5. The similar result for nonhomogeneous spaces is classical, see [**52**].

The Besov spaces can be naturally defined on sub-manifolds using their atlas. Indeed, as pointed out in the proof of Lemma 2.1.1 in the $\mathbb{R}^n$ case, for positive exponents, the Besov semi-norms may be expressed in terms of finite differences. This leads to the following definition of Besov spaces on manifolds:

DEFINITION 2.2.2. — *Let S be a C^1 m-dimensional submanifold and $s \in (0,1)$. Then the nonhomogeneous Besov space $B^s_{p,p}(S)$ is the set of $L_p(S)$ functions so that*

$$\|u\|_{B^s_{p,p}(S)} := \|u\|_{L_p(S)} + \|u\|_{\dot B^s_{p,p}(S)} < \infty \tag{2.11}$$

where $\|.\|_{\dot B^s_{p,q}(S)}$ stands for the following homogeneous semi-norm:

$$\|u\|_{\dot B^s_{p,p}(S)} := \left(\int_S\int_S \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,dx\,dy\right)^{1/p}. \tag{2.12}$$

The above double integral may be used to define the *homogeneous* Besov space $\dot B^s_{p,p}(S)$ on S if in addition $s < m/p$ (see the remark below). The spaces $\dot B^s_{p,p}(S)$ with $\max(-1, -m/p') < s < 0$ may be defined by duality: we set

$$\dot B^s_{p,p}(S) := (\dot B^{-s}_{p',p'}(S))^*.$$

The remaining spaces $\dot B^s_{p,q}(S)$ for

$$1 < p < \infty, \quad 1 \le q \le \infty \quad \text{and} \quad \max(-1, -m/p') < s < \min(1, m/p)$$

may be defined by interpolation according to the following relation:

$$\big(\dot B^{s_1}_{p,q_1}(S), \dot B^{s_2}_{p,q_2}(S)\big)_{\theta,q} = \dot B^s_{p,q}(S). \tag{2.13}$$

We just have to fix some $\max(-1, -m/p') < s_1 < s_2 < \min(1, m/p)$ and take $\theta \in (0,1)$ such that $s = \theta s_2 + (1-\theta)s_1$. Note that the space $\dot B^s_{p,q}(S)$ with $0 < s < 1$ may be equivalently defined by finite differences as in the proof of Lemma 2.1.1.

REMARK 2.2.2. — For $S = \mathbb{R}^n$ and $0 < s < \min(1, n/p)$, Definitions 2.2.1 and 2.2.2 give the same functional space. Indeed, knowing that $\dot B^s_{p,p}(\mathbb{R}^n)$ embeds in $L_m(\mathbb{R}^n)$ for some finite m, the decay to 0 at infinity is controlled (see (2.3)), and $\overline{C^\infty_c(\mathbb{R}^n)}^{\dot B^s_{p,p}(\mathbb{R}^n)}$ thus defines a Banach space which coincides with $\dot B^s_{p,p}(\mathbb{R}^n)$. For an arbitrary domain Ω one may thus define the homogeneous Besov space $\dot B^s_{p,p}(\Omega)$ (being a Banach space) by means of the following norm:

$$\|u\|_{\dot B^s_{p,p}(\Omega)} = \|u\|_{\dot B^s_{p,p}(\Omega)} + \|u\|_{L_m(\Omega)} \quad \text{with} \quad \frac{1}{p} - \frac{1}{m} = \frac{s}{n}. \tag{2.14}$$

More generally, if $0 < s < n/p$ and $k = [s]$ then one may define $\dot{B}^s_{p,p}$ as the subset of $L_m(\Omega)$ functions u (with m as above) with $\|\nabla^k u\|_{\dot{B}^{s-k}_{p,p}} < \infty$.

General spaces $\dot{B}^s_{p,q}(\Omega)$ with $0 < s < n/p$ and $1 \leq q \leq \infty$ may be defined by interpolation.

From Lemma 2.2.1 and localization property, i.e.

$$\text{if } \ \eta \in \mathcal{C}^\infty_c(\overline{\Omega}) \text{ and } u \in B^s_{p,q}(\Omega) \text{ (or } \dot{B}^s_{p,q}(\Omega)) \text{ then } \ \eta u \in B^s_{p,q}(\Omega) \text{ (or } \dot{B}^s_{p,q}(\Omega)),$$

one can get the following important corollary (more details may be found in [**53**], page 210).

Corollary 2.2.2. — *Let Ω be a uniformly C^1 domain of $\mathbb{R}^n$. For any $(p,q) \in [1,\infty)^2$ and $s \in (-1+1/p, 1/p)$, we have*

$$B^s_{p,q}(\Omega) = \overline{\{f \in B^s_{p,q}(\mathbb{R}^n) : \operatorname{Supp} f \subset \Omega\}}^{\|\cdot\|_{B^s_{p,q}(\Omega)}}$$
$$\text{and } \ \dot{B}^s_{p,q}(\Omega) = \overline{\{f \in \dot{B}^s_{p,q}(\mathbb{R}^n) : \operatorname{Supp} f \subset \Omega\}}^{\|\cdot\|_{\dot{B}^s_{p,q}(\Omega)}}$$

where the nonhomogeneous and homogeneous norms are defined in (2.11) *and* (2.12)*, respectively.*

In the case $q = \infty$, density holds true for the weak $*$ topology only.

Proof. — The nonhomogeneous case in standard (see e.g. [**52**]). The homogeneous case is a consequence of Corollary 2.2.1 and of the above remark. □

We shall use repeatedly the following trace theorem (see e.g. [**52**]).

Proposition 2.2.4. — *Let Ω be a sufficiently smooth simply connected domain. Suppose that $1 < p < \infty$ and $s > 1/p$. The trace map from Ω to $\partial\Omega$ extends to a continuous operator from $B^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,q}(\partial\Omega)$.*

We also need the following lemma proved in the appendix of [**15**], concerning the harmonic extension from the hyperplane $\partial\mathbb{R}^n_+$ to the half-space $\mathbb{R}^n_+$:

Lemma 2.2.2. — *Let $s > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Then there exists a constant C such that for all $h \in \dot{B}^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)$, we have*

$$\big\|\mathcal{F}^{-1}_{x'}[e^{-|\xi'|x_n}\mathcal{F}_{x'}[h]]\big\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)} \leq C\big\|h\big\|_{\dot{B}^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)}, \tag{2.15}$$

where $\mathcal{F}_{x'}$ stands for the Fourier transform with respect to $x' := (x_1, \ldots, x_{n-1})$ and ξ' denotes the corresponding Fourier variable.

Consequently, we get the following extension lemma in the nonflat situation.

LEMMA 2.2.3. — *Let Ω be a smooth domain with compact boundary. Then for $s \in (0, \frac{n}{p})$, $p \in (1,\infty)$ and $q \in [1,\infty]$ there is a continuous extension operator from $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ to $B^s_{p,q}(\Omega)$.*

Proof. — Let ψ be in $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$. Observe that the condition over s implies the space $B^{s-1/p}_{p,q}(\partial\Omega)$ to be stable under multiplication by compactly supported functions.

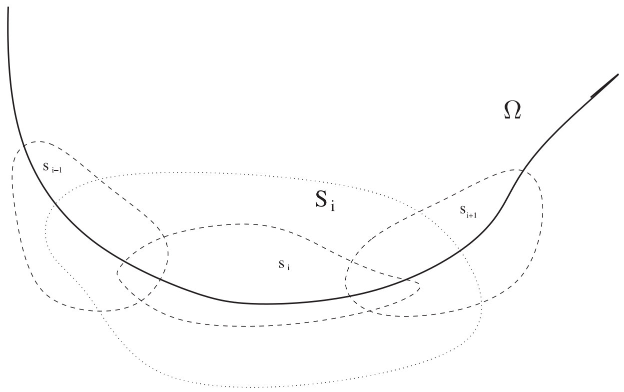

FIGURE 2.2.1. Covering of $\partial\Omega$

By compactness of $\partial\Omega$, one may find two coverings $(s_i)_{1\le i\le N}$ and $(S_i)_{1\le i\le N}$ of $\partial\Omega$ by open sets of $\mathbb{R}^n$ with $s_i \subset S_i$. Then, we fix N maps $Z_i : s_i \to \mathbb{R}^n$ with $Z_i : s_i \cap \Omega \to \mathbb{R}^n_+$ and $Z_i : s_i \cap \partial\Omega \to \partial\mathbb{R}^n_+$ (see the beginning of Section 2.4).

Let $(\eta_i)_{1\le i\le N}$ be a partition of unity associated to the covering $s_i \cap \partial\Omega$, with $\operatorname{Supp}\eta_i \subset s_i$. Further introduce a family $(\bar\eta_i)_{1\le i\le N}$ of smooth functions with $\bar\eta_i|_{s_i} \equiv 1$ and $\operatorname{Supp}\bar\eta_i \subset S_i$. Denoting $Z^*(\phi) := Z \circ \phi^{-1}$, we have according to Lemma 2.1.1,

$$Z_i^*(\eta_i\psi) \in \dot B^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+). \tag{2.16}$$

Then, thanks to Lemma 2.2.2, we may find some extension $\Psi_i \in \dot B^s_{p,q}(\mathbb{R}^n_+)$ of $Z_i^*(\eta_i\psi)$ such that

$$\|\Psi_i\|_{\dot B^s_{p,q}(\mathbb{R}^n_+)} \le c\|Z_i^*(\eta_i\psi)\|_{\dot B^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)} \tag{2.17}$$

and $\bar\eta_i(Z_i^{-1})^*(\Psi_i) \in \dot B^s_{p,q}(\Omega)$ and $\bar\eta_i(Z_i^{-1})^*(\Psi_i)|_{\partial\Omega} = \bar\eta_i\eta_i\psi$. Obviously $\Psi := \sum_i \bar\eta_i(Z_i^{-1})^*(\Psi_i)$ is an extension such that

$$\|\Psi\|_{B^s_{p,q}(\Omega)} \le c\|\psi\|_{B^{s-1/p}_{p,q}(\partial\Omega)} \text{ and } \Psi|_{\partial\Omega} = \psi. \tag{2.18}$$

This completes the proof of the lemma. □

Lemma 2.2.4. — *Let Ω be a C^1 domain with compact boundary. Let $\vec{n}$ be the outer unit normal vector at $\partial\Omega$. Then for any $1 < p < \infty$, $1 \le q \le \infty$ and $s \in (-1+1/p, 1/p)$, the normal trace operator $F \mapsto (F\cdot\vec{n})|_{\partial\Omega}$ acting on smooth divergence-free vector fields extends continuously from $B^s_{p,q}(\Omega)$ to $B^{s-1/p}_{p,q}(\partial\Omega)$.*

Proof. — In the smooth case, using the properties of duality of Besov spaces, one may write

$$\|F\cdot\vec{n}|_{\partial\Omega}\|_{B^{s-1/p}_{p,q}(\partial\Omega)} \le C\sup\left\{\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma : \phi\in B^{-s+1/p}_{p',q'}(\partial\Omega)\ \text{ and }\ \|\phi\|_{B^{-s+1/p}_{p',q'}(\partial\Omega)}\le 1\right\}\cdot$$

Because $\operatorname{div} F = 0$, we have

$$\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma = \int_\Omega F\cdot\nabla(E\phi)\,dx,$$

where $E\phi$ is the extension of ϕ in $B^{-s+1}_{p',q'}(\Omega)$ given by Lemma 2.2.3 – the assumptions guarantee that $-s+1>0$. Now, $\nabla(E\phi)\in B^{-s}_{p',q'}(\Omega)$ with $-s\in(-1+1/p', 1/p')$. Hence, thanks to Corollary 2.2.1, both functions $\nabla(E\phi)$ and F can be extended by zero outside Ω. We thus get (by using the duality properties for Besov spaces on $\mathbb{R}^n$):

$$\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma \le C\|\nabla E\phi\|_{B^{-s}_{p',q'}(\Omega)}\|F\|_{B^s_{p,q}(\Omega)} \le C\|\phi\|_{B^{-s+1/p}_{p',q'}(\partial\Omega)}\|F\|_{B^s_{p,q}(\Omega)}.$$

This completes the proof of the lemma. □

2.3. The divergence equation

Our analysis requires an accurate description of low regularity Besov spaces on domains, an issue that strongly depends on the problem we aim at considering. As an example, if we look at the Poisson equation $\Delta u = \operatorname{div} k$ with rough vector-field k (say just $L_p(\Omega)$) then the class of k for which the meaning of solution makes sense is larger if prescribing Neumann boundary conditions rather than Dirichlet conditions.

The present work requires our looking at the divergence operator in low regularity as a distribution acting on test functions *up to the boundary.* To this end, we adopt the following definition that is borrowed from our recent work [**16**]:

Definition 2.3.1. — *Let Ω be a domain of $\mathbb{R}^n$ with a compact Lipschitz boundary. If k is a distribution over Ω and ζ a distribution over $\partial\Omega$ then we designate by $\mathcal{DIV}\,[k;\zeta]$ the linear functional defined on the set $\mathcal{C}_c^\infty(\overline{\Omega})$ of smooth functions with compact support in $\overline{\Omega}$, by*

$$\mathcal{DIV}\,[k;\zeta](\varphi) := -\int_\Omega k\cdot\nabla\varphi\,dx + \int_{\partial\Omega}\zeta\,\varphi\,d\sigma.$$

For $1<p<\infty$, $-1+1/p<s<1/p$, $1\le q\le\infty$, the notation $\mathcal{B}^{s-1}_{p,q}(\Omega)$ designates the set of all functionals $\mathcal{DIV}\,[k;\zeta]$ such that [6]

$$k\in B^s_{p,q}(\Omega)\quad\text{and}\quad \zeta\in B^{s-\frac1p}_{p,q}(\partial\Omega)\qquad\text{with}\quad \int_{\partial\Omega}\zeta\,d\sigma=0. \tag{2.19}$$

The space $\mathcal{B}^{s-1}_{p,q}(\Omega)$ is endowed with the following norm:

$$\|\,\mathcal{DIV}\,[k;\zeta]\|_{\mathcal{B}^{s-1}_{p,q}(\Omega)} = \inf\Big(\|\widetilde{k}\|_{B^s_{p,q}(\Omega)} + \|\widetilde{\zeta}\|_{B^{s-\frac1p}_{p,q}(\partial\Omega)}\Big), \tag{2.20}$$

where the infimum is taken over all the couples $(\widetilde{k},\widetilde{\zeta})$ satisfying (2.19) and such that $\mathcal{DIV}\,[\widetilde{k};\widetilde{\zeta}] = \mathcal{DIV}\,[k;\zeta]$.

Analogously, for the same range of exponents, we define the homogeneous space $\dot{\mathcal{B}}^{s-1}_{p,q}(\Omega)$ for $k\in\dot{B}^s_{p,q}(\Omega)$ and $\zeta\in\dot{B}^{s-\frac1p}_{p,q}(\partial\Omega)$ endowed with the norm

$$\|\,\mathcal{DIV}\,[k;\zeta]\|_{\dot{\mathcal{B}}^{s-1}_{p,q}(\Omega)} = \inf\Big(\|\widetilde{k}\|_{\dot{B}^s_{p,q}(\Omega)} + \|\widetilde{\zeta}\|_{\dot{B}^{s-\frac1p}_{p,q}(\partial\Omega)}\Big), \tag{2.21}$$

where the infimum is taken over all the couples $(\widetilde{k},\widetilde{\zeta})$ satisfying (2.19) and such that $\mathcal{DIV}\,[\widetilde{k};\widetilde{\zeta}] = \mathcal{DIV}\,[k;\zeta]$.

For k and ζ as above, it is clear that if the vector-field v satisfies

$$\mathcal{DIV}\,[v;0] = \mathcal{DIV}\,[k;\zeta], \tag{2.22}$$

then it is a solution to the following system [7]:

$$\begin{cases} \operatorname{div} v = \operatorname{div} k & \text{in}\quad \Omega,\\ (k-v)\cdot\vec{n} = \zeta & \text{on}\quad \partial\Omega.\end{cases}$$

Rewriting the system in terms of the functional $\mathcal{DIV}\,[k;\zeta]$ enables us to incorporate the boundary condition either in the interior part or in the boundary part of the data (see [**16, 18, 19**] for more detailed explanations).

6. We make the convention that $\int_{\partial\Omega}\zeta\,d\sigma$ designates the distribution bracket $\langle\zeta,1\rangle$. That 1 is a test function comes from the fact that $\partial\Omega$ is compact.

7. That the boundary condition makes sense stems from Lemma 2.2.4.

In the present paper, the following result will be used a number of times:

THEOREM 2.3.1. — *Let Ω be a bounded or exterior C^2 domain. There exists a linear operator $\mathcal{B}_\Omega$ which is bounded from $\mathcal{B}^{s-1}_{p,q}(\Omega)$ to $B^s_{p,q}(\Omega;\mathbb{R}^n)$ whenever $1<p<\infty$, $1\leq q\leq\infty$ and $-1+1/p<s<1/p$, and such that for any $F=\mathcal{DIV}\,[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$, the vector-field $v:=\mathcal{B}_\Omega(F)$ fulfills* (2.22).

Furthermore if $k\in B^{m+s}_{p,q}(\Omega)$ with $m=1,2$ and $\zeta=(k\cdot\vec{n})|_{\partial\Omega}$ then v belongs to $B^{m+s}_{p,q}(\Omega)$, vanishes at the boundary and satisfies

$$\|v\|_{B^{m+s}_{p,q}(\Omega)}\leq C\|\operatorname{div}k\|_{B^{m-1+s}_{p,q}(\Omega)}. \tag{2.23}$$

Finally, if k is time dependent with k_t and $\operatorname{div}k$ in $L_1(0,T;B^s_{p,q}(\Omega))$, and $((k\cdot\vec{n})|_{\partial\Omega})_t\in L_1(0,T;B^{s-1/p}_{p,q}(\partial\Omega))$ then we have

$$\|v_t\|_{L_1(0,T;B^s_{p,q}(\Omega))}\leq C\Big(\|k_t\|_{L_1(0,T;B^s_{p,q}(\Omega)}+\|((k\cdot\vec{n})|_{\partial\Omega})_t\|_{L_1(0,T;B^{s-1/p}_{p,q}(\partial\Omega))}\Big).$$

Proof. — We just sketch the proof in the case where Ω is bounded and star-shaped with respect to some ball, the reader being referred to [**16**], [**18**], [**19**] for more details. Then the following Bogovskiĭ formula provides us with an example of operator $\mathcal{B}_\Omega$ fulfilling (2.23):

$$\mathcal{B}_\Omega(F)(x):=\int_\Omega f(y)\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}\,dr\,dy, \tag{2.24}$$

where ω stands for a smooth function with average 1 and support in a ball with respect to which Ω is star-shaped.

In [**16**], in order to achieve distributions F with lower regularity (e.g. $F=\mathcal{DIV}\,[k;\zeta]$), we performed a formal integration by parts in (2.24) so as to decompose the outer integral into an interior integral and a boundary integral. More precisely, we introduced the following operators [8] I_Ω and J_Ω:

(2.25)

$$I_\Omega(k)(x)=-\int_\Omega k(y)\cdot\nabla_y\Big[\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}dr\Big]dy$$

$$J_\Omega(\zeta)(x)=\int_{\partial\Omega}\zeta(y)\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}\,dr\,d\sigma_y.$$

Those two operators enabled us to extend Bogovskiĭ formula to the rough case. In effect, in the smooth case where $F=\operatorname{div}k$, it is obvious that

$$v=\mathcal{B}_\Omega(\operatorname{div}k)=I_\Omega(k)+J_\Omega(\zeta)\quad\text{with}\quad\zeta:=(k\cdot\vec{n})|_{\partial\Omega}. \tag{2.26}$$

8. These singular integrals have to be understood in the *principal value* meaning.

Under the assumptions of the theorem, it has been established in [**16**] that

$$v := I_\Omega(k) + J_\Omega(\zeta)$$

is indeed a solution to (2.22), and that v belongs to $B^s_{p,q}(\Omega;\mathbb{R}^n)$. More precisely, it has been shown that $I_\Omega : B^s_{p,q}(\Omega;\mathbb{R}^n) \to B^s_{p,q}(\Omega;\mathbb{R}^n)$ and J_Ω : $B^{s-1/p}_{p,q}(\partial\Omega;\mathbb{R}) \to B^s_{p,q}(\Omega;\mathbb{R}^n)$.

Finally, in the case where k is time-dependent, differentiating the above relation with respect to time yields

$$v_t = I_\Omega(k_t) + J_\Omega(\zeta_t).$$

Therefore, taking advantage of the continuity results for I_Ω and J_Ω, and integrating with respect to time gives the end of the statement. □

2.4. Change of coordinates

Investigating the Laplace and Stokes equations in general domains will rely on a localization of our problem. Obtaining local estimates at the interior and at the boundary of the domain will be two key steps. The analysis at the interior is amenable to equations in the whole space while boundary terms may be seen as the solution to model equations in the half-space after a suitable change of coordinates (so as to straighten the boundary). This section is devoted to introducing changes of coordinates so as to transform problems at the (nonflat) boundary of some C^r open set Ω ($r \geq 2$) to a problem at the boundary of $\mathbb{R}^n_+$.

Let us first present the general setting. By definition, having $\partial\Omega$ of class C^r means that for any point x_0 of $\partial\Omega$ there exists some small enough $\lambda > 0$ and a one-to-one C^r mapping

$$Z : B(x_0,\lambda) \longrightarrow \mathbb{R}^n, \quad x \longmapsto z,$$

such that

(i) Z is a C^r diffeomorphism from $B(x_0,\lambda)$ to $Z(B(x_0,\lambda))$;
(ii) $Z(x_0) = 0$ and $D_xZ(x_0) = \mathrm{Id}$;
(iii) $Z(\Omega \cap B(x_0,\lambda)) \subset \mathbb{R}^n_+$;
(iv) $Z(\partial\Omega \cap B(x_0,\lambda)) = \partial\mathbb{R}^n_+ \cap Z(B(x_0,\lambda))$.

If we denote $D_xZ = \mathrm{Id} + \mathcal{A}$ and assume that $\partial\Omega$ is uniformly C^r then there exist constants C_ℓ depending only on Ω and on $\ell \in \{1,\dots,r-1\}$ such that

$$\|D^\ell \mathcal{A}\|_{L_\infty(B(x_0,\lambda))} \leq C_\ell, \qquad (2.27)$$

a property which implies (by the mean value formula) that

$$\|\mathcal{A}\|_{L_\infty(B(x_0,\varepsilon))} \le C_1\varepsilon \quad \text{if} \quad 0<\varepsilon<\lambda, \tag{2.28}$$

hence by interpolation between the spaces $L_q(B(0,\varepsilon))$ and $W^{r-1}_q(B(0,\varepsilon))$,

$$\|\mathcal{A}\|_{B^{\frac{n}{q}}_{q,1}(B(x_0,\varepsilon))} \le C\varepsilon \quad \text{for all} \quad 1\le q<\infty \quad \text{such that} \quad n/q<r-1. \tag{2.29}$$

Let us introduce some examples of maps Z. We would like to consider a neighborhood of a point $x_0\in\partial\Omega$. After a rigid motion we may assume that $x_0=0$ and $T_{x_0}\partial\Omega=\partial\mathbb{R}^n_+$ and that in addition $(B(0,\epsilon)\cap\Omega)\cap\mathbb{R}^n_+\neq\varnothing$ for any $0<\epsilon<\lambda$.

First example: the basic change of coordinates.

The above assumptions ensure that the *interior* unit normal vector of $\partial\Omega$ at $x_0=0$ is $\vec{e}_n:=(0,\dots,0,1)$. Then one may set

$$Z(x',x_n):=(x',x_n-\phi(x')), \tag{2.30}$$

where the graph of the function ϕ coincides with the boundary $\partial\Omega$ in some neighborhood of $x_0=0$, hence satisfies $\phi(0)=0$ and $D_{x'}\phi(0)=0$. As $\partial\Omega\in C^r$, so do ϕ and Z. In addition, (2.27) and (2.28) are satisfied.

Second example: a normal preserving change of coordinates.

We would like the value of the normal derivative at the boundary to be invariant under the change of coordinates. Hence we define Z so that for small enough t and x', we have (with ϕ as above)

$$Z\big((x',\phi(x'))+t\vec{n}\big)=(x',t), \tag{2.31}$$

where $\vec{n}$ stands for the *interior* unit normal vector at the boundary.

Differentiating the above equality with respect to t, we see that $\partial_{\vec{n}}Z$ coincides with $\vec{e}_n$. Hence in particular, for any differentiable function q,

$$\partial_{\vec{n}}q(x)=\partial_{z_n}q(z)|_{z=Z(x)} \quad \text{for} \quad x \quad \text{in a neighborhood of } x_0. \tag{2.32}$$

Third example: a measure preserving change of coordinates.

This last example, borrowed from e.g. [**44**], [**47**] and [**50**], is more involved. We start with some bounded simply connected open set $S_1\subset\mathbb{R}^n_+$, star-shaped with respect to some point y inside Ω and such that ∂S_1 is a neighborhood of the point $x_0=0$ in $\partial\Omega$ (see the figure next page). We also fix another bounded simply connected open set $S_0\subset\mathbb{R}^n_+$ such that

1. $S_0\cap\partial\mathbb{R}^n_+$ is a neighborhood of 0 in $\partial\mathbb{R}^n_+$,
2. S_0 is star-shaped with respect to y,
3. $|S_0|=|S_1|$.

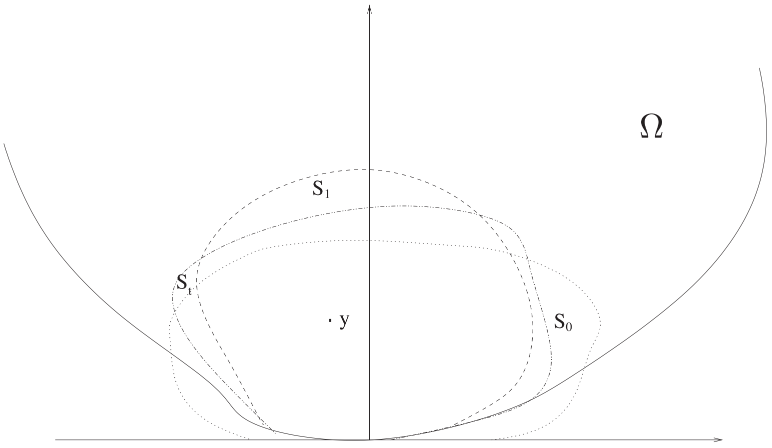

FIGURE 2.4.1. Construction of S_0

We aim at constructing a *measure preserving* change of coordinates Z satisfying the requirements enumerated at the beginning of Section 2.4, and so that $Z(S_1) = S_0$.

To achieve it, we first construct intermediate sets S_t^* between S_0 and S_1 in terms of y and $t \in (0,1)$ as follows:

$$S_t^* := \{x \in \mathbb{R}^n : x = y + s\omega,\ \omega \in \mathbb{S}^{n-1},\ s \in [0, \bar{s}_t(y,\omega)]\},$$

where $\bar{s}_t(y,\omega) = (1-t)\bar{s}_0(y,\omega) + t\bar{s}_1(y,\omega)$ and $\bar{s}_i$ are given by the relation

$$y + \bar{s}_i(y,\omega)\omega \in \partial S_i \qquad \text{for } i = 0, 1.$$

In general, S_t^* need not have the same measure as S_1. Hence we define S_t to be the image of S_t^* by some suitable dilation centered at point 0. Having constructed such a family S_t, we notice that V_t, the normal speed of deformation of ∂S_t at time t, satisfies the compatibility condition

$$\int_{\partial S_t} V_t \, d\sigma = 0.$$

To show this relation it is enough to note that since the area of S_t is preserved,

$$0 = \frac{d}{dt}\int_{S_t} dx = \int_{\partial S_t} V_t \, d\sigma.$$

Hence one may solve the following system:

$$\begin{array}{rcl} \Delta P_t = 0 & \text{in} & S_t, \\ \frac{\partial P_t}{\partial \vec{n}} = V_t & \text{at} & \partial S_t. \end{array} \tag{2.33}$$

In order to define the map Z we solve the differential equation

$$\frac{dz_x}{dt}(t) = \nabla P(z_x(t), t) \quad \text{with} \quad z_x(0) = x'.$$

Then for $x = (x', t)$, we set

$$Z(x) := x + \int_0^t \nabla P(z_x(s), s)\, ds.$$

The construction guarantees that $Z(0) = 0$ and Z is measure preserving since $\operatorname{div} \nabla P = 0$ (Liouville's theorem). In addition we are able to control the regularity of P_t (see e.g. [**26**], Th. 15): if $\partial S_t \in B^{1+s-\frac{1}{p}}_{p,q}$, then $V_t \in B^{s-\frac{1}{p}}_{p,q}$, so the solvability of (2.33) gives $\nabla P \in B^s_{p,q}$, hence eventually $Z \in B^s_{p,q}$.

To complete this section, let us explicit the effect of the above changes of coordinates on the differential operators that we shall consider throughout this paper.

We consider a general C^r-diffeomorphism $Z : \Omega \to \bar{\Omega}$. Let

$$H : \Omega \longrightarrow \mathbb{R}^n$$

denote some vector-field defined on Ω. Then we define the vector field

$$\overline{H} : \bar{\Omega} \longrightarrow \mathbb{R}^n$$

by $\overline{H} := Z^*(H) := H \circ Z^{-1}$. Similarly, for any function $f : \Omega \to \mathbb{R}$, we define $\bar{f} : \bar{\Omega} \to \mathbb{R}$ by $\bar{f} := Z^*(f) := f \circ Z^{-1}$. We thus have

$$\overline{H}(z) = H(x) \quad \text{and} \quad \bar{f}(z) = f(x) \quad \text{with} \quad z = Z(x).$$

From the chain rule, we get [9]

$$\begin{aligned} \operatorname{div}_x H(x) = D_z \overline{H}(z) : D_x Z(x) = \nabla_x Z(x) : \nabla_z \overline{H}(z) \\ \text{and} \quad \nabla_x f(x) = \nabla_x Z(x) \cdot \nabla_z \bar{f}(z). \end{aligned} \tag{2.34}$$

Therefore, setting $\bar{\mathcal{B}}(z) = \mathcal{B}(x) = D_x Z(x)$, we get

$$\Delta_x f(x) = \operatorname{div}_x (\nabla_x f)(x) = D_z\big({}^T\bar{\mathcal{B}}(z) \cdot \nabla_z \bar{f}(z)\big) : \bar{\mathcal{B}}(z).$$

We thus deduce, with the summation convention over repeated indices, that

$$\begin{aligned} \overline{\Delta_x f} &= \partial_{z_i}(\bar{\mathcal{B}}_{i,j}\bar{\mathcal{B}}_{k,j}\partial_{z_k}\bar{f}) - (\partial_{z_i}\bar{\mathcal{B}}_{i,j})\,\bar{\mathcal{B}}_{k,j}\partial_{z_k}\bar{f}, \\ &= \partial_{z_i}(\bar{\mathcal{B}}_{i,j}\bar{\mathcal{B}}_{k,j}\partial_{z_k}\bar{f}) - \partial_{z_k}(\partial_{z_i}\bar{\mathcal{B}}_{i,j}\,\bar{\mathcal{B}}_{k,j}\bar{f}) + \bar{f}\partial_{z_k}(\bar{\mathcal{B}}_{k,j}\partial_{z_i}\bar{\mathcal{B}}_{i,j}). \end{aligned}$$

9. In all the paper, we agree that $D_x Z$ stands for the $n \times n$ matrix with entries $\partial_{x_j} Z^i$ and that $\nabla_x Z$ stands for the matrix with entries $\partial_{x_i} Z^j$. Furthermore, for M and N two $n \times n$ matrices, we set $M : N = \operatorname{Tr} MN$.

Setting $\mathcal{B} = \mathrm{Id} + \mathcal{A}$, that formula also reads

$$\overline{\Delta_x f} = \Delta_z \bar{f} + \operatorname{div}_z \Big((\bar{\mathcal{B}}^T \bar{\mathcal{B}}) \nabla_z \bar{f} - \bar{f}\, \bar{\mathcal{B}} \operatorname{div}_z \bar{\mathcal{A}} \Big) + \bar{f} \operatorname{div}_z \big(\bar{\mathcal{B}} \operatorname{div}^T \bar{\mathcal{A}} \big) \tag{2.35}$$

with the convention that $(\operatorname{div} \bar{\mathcal{A}})^j := \sum_i \partial_i \bar{\mathcal{A}}_{ij}$.

In the case where Z is measure preserving, formula (2.34) for the divergence operator may be alternately written

$$\operatorname{div}_x H = \operatorname{div}_z \big(\bar{\mathcal{B}}\, \overline{H} \big). \tag{2.36}$$

This is the consequence of the following series of computation which holds true for any test function ϕ and uses the fact that $\det \mathcal{B} \equiv 1$:

$$\begin{aligned} \int \phi \operatorname{div}_x H \, dx &= -\int D_x \phi \cdot H \, dx \\ &= -\int D_z \bar{\phi}(z) \cdot \bar{\mathcal{B}}(z) \cdot \overline{H}(z) \, dz \\ &= \int \bar{\phi}(z) \operatorname{div}_z (\bar{\mathcal{B}}(z) \cdot \overline{H}(z)) \, dz \\ &= \int \phi(x) (\operatorname{div}_z (\bar{\mathcal{B}} \cdot \overline{H}))(Z(x)) \, dx. \end{aligned}$$

Hence we have

$$\Delta_x f = \operatorname{div}_z \big(\bar{\mathcal{B}}^T \bar{\mathcal{B}} \nabla_z \bar{f} \big). \tag{2.37}$$

For general diffeomorphism Z, Equality (2.36) extends as follows:

$$\operatorname{div}_x H(x) = \bar{J}_Z \operatorname{div}_z \big(\bar{J}_Z \bar{\mathcal{B}} \overline{H} \big),$$

with $\bar{J}_Z$ being the Jacobian of the change of coordinates.

This allows to write $\Delta_x \bar{f}$ in another way :

$$\Delta_x f = \bar{J}_Z \operatorname{div}_z \big(\bar{J}_Z \bar{\mathcal{B}}^T \bar{\mathcal{B}} \nabla_z \bar{f} \big).$$

Having different equivalent formulae for the divergence and Laplacian operator after change of variable turns out to be crucial in our study of the Stokes system and of the inhomogeneous incompressible Navier-Stokes equation.

CHAPTER 3

THE POISSON EQUATION

Here we prove auxiliary results for the Laplace operator supplemented with Dirichlet or Neumann boundary conditions. First we consider the equation in the whole space, then in different types of domains : the half-space, bounded or exterior domains, with either Dirichlet or Neumann boundary conditions. Even though some of those results belong to the mathematical folklore, our approach based on a new definition of very weak solutions for the Neumann problem (see in particular Section 3.3) sheds new light on this issue.

3.1. The whole space case

In this short section, we establish various existence results and estimates for the following Poisson equation in the whole space:

$$\Delta b = f \ \text{ in } \ \mathbb{R}^n, \qquad b \to 0 \quad \text{at} \quad \infty. \tag{3.1}$$

Our first statement concerns the case where the r.h.s. has low enough regularity in the scale of homogeneous Besov spaces.

Lemma 3.1.1. — *Let f be in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ with $\sigma \in \mathbb{R}$ and let $1 \leq p,q \leq \infty$ satisfy*

$$\sigma + 2 < \frac{n}{p} \quad \Big(\text{or} \ \ \sigma + 2 \leq \frac{n}{p} \ \ \text{if} \ \ q = 1\Big)\cdot \tag{3.2}$$

Then (3.1) admits a unique solution $b \in \dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n)$ and we have

$$\|b\|_{\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Proof. — For $f \in \mathcal{S}_0(\mathbb{R}^n)$ the solution to (3.1) in Fourier variables is given by

$$\widehat{b}(\xi) = -|\xi|^{-2}\widehat{f}(\xi),$$

and we get the desired inequality.

The general case follows by completion since $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{B}^\sigma_{p,q}(\mathbb{R}^n)$ (if both p and q are finite) and our assumption on σ ensures $\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n)$ to be a Banach space. If either p or q is ∞ then density of $\mathcal{S}_0(\mathbb{R}^n)$ holds true for the weak $*$ topology, which suffices to complete the proof. □

If it is known that the solution to (3.1) is compactly supported then estimates in Besov spaces with arbitrarily large regularity index are available:

Corollary 3.1.1. — *Assume that $1 \le p, q \le \infty$ and that $\sigma > -\min(1, n/p')$. If b is a compactly supported function with $f := \Delta b$ in $B^\sigma_{p,q}(\mathbb{R}^n)$ (or, equivalently, in $\dot{B}^\sigma_{p,q}(\mathbb{R}^n)$) then b is in $B^{\sigma+2}_{p,q}(\mathbb{R}^n)$ and the following inequality is true:*

$$\|b\|_{B^{\sigma+2}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{B^\sigma_{p,q}(\mathbb{R}^n)}. \tag{3.3}$$

Proof. — As $(-\Delta)^{-1}\nabla^2$ is an homogeneous Fourier multiplier, we readily have

$$\|\nabla^2 b\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)}.$$

In order to complete the proof of the corollary, it thus suffices to bound b in $L_p(\mathbb{R}^n)$. To this end, fix some $\sigma' < \sigma$ so that $\sigma' + 1 < n/p$ and $\sigma' > -\min(1, n/p')$. By embedding, we have $f \in B^{\sigma'}_{p,q}(\mathbb{R}^n)$. Furthermore, because f is compactly supported, Proposition 2.1.3 ensures that f is also in $\dot{B}^{\sigma'}_{p,q}(\mathbb{R}^n)$, and thus $\nabla f \in \dot{B}^{\sigma'-1}_{p,q}(\mathbb{R}^n)$. Now, ∇b tends to 0 at infinity (it is compactly supported), and satisfies

$$\Delta(\nabla b) = \nabla f \quad \text{in} \quad \mathbb{R}^n.$$

Hence Lemma 3.1.1 (recall that $\sigma' + 1 < n/p$) ensures that $\nabla b \in \dot{B}^{\sigma'+1}_{p,q}(\mathbb{R}^n)$ and that

$$\|\nabla b\|_{\dot{B}^{\sigma'+1}_{p,q}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{\dot{B}^{\sigma'-1}_{p,q}(\mathbb{R}^n)}.$$

Again, the compact support property allows to replace the homogeneous norm of b by the corresponding nonhomogeneous one and we thus have by embedding

$$\|\nabla b\|_{L_p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{\dot{B}^{\sigma'-1}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^\sigma_{p,q}(\mathbb{R}^n)}.$$

At that stage, one may take advantage of Poincaré inequality to bound $\|b\|_{L_p(\mathbb{R}^n)}$ by the above r.h.s. This completes the proof of the corollary. □

We shall also need the following result.

Lemma 3.1.2. — *Let $p \in [1, \infty)$, $q \in [1, \infty]$ and $\sigma \in \mathbb{R}$ such that $(\sigma - 2, p, q)$ satisfies* (3.2). *Then the operator*

$$\mathcal{S}_0(\mathbb{R}^n) \longrightarrow \mathcal{S}_0(\mathbb{R}^n), \quad f \longmapsto -(-\Delta)^{-1}\nabla \operatorname{div} f$$

admits a unique continuous extension $\Phi : \dot{B}^\sigma_{p,q}(\mathbb{R}^n) \to \dot{B}^\sigma_{p,q}(\mathbb{R}^n)$. *Furthermore, there exists a distribution* b *such that* $\nabla b = \Phi(f)$ (*and thus* $\Delta b = \operatorname{div} f$ *in* $\mathbb{R}^n$).

Proof. — In Fourier variables, we have

$$\mathcal{F}(\nabla b)(\xi) = \mathcal{F}^{-1}\Big(\frac{\xi\,(\xi\cdot\mathcal{F}f(\xi))}{|\xi|^2}\Big).$$

As 0 order Fourier multipliers are continuous on homogeneous Besov spaces (under Lemma's assumptions), and as $\mathcal{S}_0(\mathbb{R}^n)$ is *dense* in the *Banach space* $\dot{B}^\sigma_{p,q}(\mathbb{R}^n)$, we get the desired extension. The existence of b just follows from the fact that, by construction, $\operatorname{curl}\Phi(f) \equiv 0$. □

3.2. The homogeneous Neumann problem in bounded domains

Although we eventually aim at investigating the Laplace equation in exterior (unbounded) domains, proving first results for the bounded domain case is needed. In the present paragraph, we focus on the following so-called homogeneous Neumann problem:

$$\begin{cases} \Delta u = f & in \quad D, \\ \partial_{\vec{n}} u = 0 & on \quad \partial D, \end{cases} \qquad \int_D u\,dx = 0, \tag{3.4}$$

in a smooth bounded domain D of $\mathbb{R}^n$ with $n \geq 2$.

As we shall mostly work with low regularity solutions, the above system has to be understood as follows:

$$\forall \varphi \in \mathcal{C}^\infty_c(\bar{D}),\ -\int_D \nabla u\cdot\nabla\varphi\,dx = \int_D f\varphi\,dx \quad\text{and}\quad \int_D u\,dx = 0. \tag{3.5}$$

The rest of the paragraph is devoted to proving the following

Proposition 3.2.1. — *Let* D *be a bounded* $C^{2,1/p}$ *domain*[(1)] *of* $\mathbb{R}^n$ *with* $n \geq 2$. *Let* $1 < p < \infty$, $-1 + 1/p < \sigma < 1/p$ *and* $1 \leq q \leq \infty$. *Let* $f \in B^\sigma_{p,q}(D)$ *such that* $\int_D f\,dx = 0$. *Then* (3.5) *admits a unique solution* u *in* $B^{2+\sigma}_{p,q}(D)$ *and the following estimate is valid:*

$$\|u\|_{B^{2+\sigma}_{p,q}(D)} \leq C\|f\|_{B^\sigma_{p,q}(D)}. \tag{3.6}$$

Proof. — If the domain is C^∞ then this result is a particular case of e.g. Theorem 13 in [**26**] devoted to general elliptic equations. Here we write out the details for the Laplace operator supplemented with homogeneous Neumann

1. We do not claim our regularity assumption to be optimal.

boundary conditions, both for the reader convenience and because it sheds light on the key points of our approach *which is not based on any explicit representation of the solution.*

We claim that any smooth function u with $\partial_{\vec{n}} u|_{\partial D} \equiv 0$ satisfies

$$\|u\|_{B^{2+\sigma}_{p,q}(D)} \leq C(\|f\|_{B^{\sigma}_{p,q}(D)} + \|u\|_{B^{1+\sigma}_{p,q}(D)}) \quad \text{with} \quad f := \Delta u. \tag{3.7}$$

Indeed, introduce a partition of unity $\{\eta^0, \eta^1, \dots, \eta^k\}$ of D such that

- η^0 is compactly supported in the interior of D;
- the support of each η^ℓ with $1 \leq \ell \leq k$ has diameter of order λ and nonempty intersection with ∂D;
- The following bounds hold true:

$$\|\partial_\alpha \eta^\ell\|_{L_\infty(\mathbb{R}^n)} \leq C_\alpha \lambda^{-|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}^n \quad \text{and} \quad 1 \leq \ell \leq k. \tag{3.8}$$

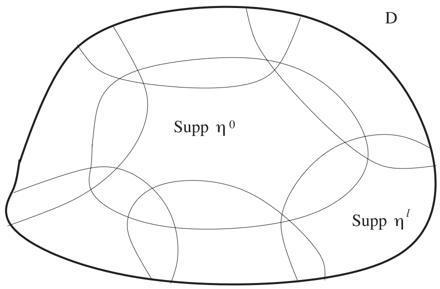

FIGURE 3.2.1. Partition of unity $(\eta^\ell)_{0\leq \ell \leq k}$ of D

Let $U^\ell := \eta^\ell u$ and $f^\ell := \eta^\ell f$. Note that as the functions η^ℓ are smooth and compactly supported, Corollary 2.1.1 and our definition of Besov spaces by restriction guarantee that the functions U^ℓ (resp. f^ℓ) are in $B^{\sigma+2}_{p,q}(D)$ (resp. $B^{\sigma}_{p,q}(D)$). Now, the equation for U^ℓ reads

$$\Delta U^\ell = 2\,\mathrm{div}\,(u\nabla \eta^\ell) - u\Delta \eta^\ell + f^\ell \quad \text{in } D. \tag{3.9}$$

For $\ell = 0$, the above equation also holds in $\mathbb{R}^n$. Hence, because U^0 is compactly supported, using Corollary 3.1.1 readily gives

$$\|U^0\|_{B^{\sigma+2}_{p,q}(\mathbb{R}^n)} \lesssim \|u\nabla\eta^0\|_{B^{\sigma+1}_{p,q}(\mathbb{R}^n)} + \|u\Delta\eta^0\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)} + \|f^0\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Then taking advantage of Proposition 2.1.2, we easily get

$$\|U^0\|_{B^{\sigma+2}_{p,q}(\mathbb{R}^n)} \lesssim \|u\|_{B^{\sigma+1}_{p,q}(D)} + \|f\|_{B^{\sigma}_{p,q}(D)}.$$

In order to treat the boundary terms $U^1, \dots, U^k$, introduce local coordinates so as to transform (3.9) into a problem over the half-space. We choose a change of coordinates Z^ℓ that preserves the normal vector at the boundary in order to have homogeneous Neumann boundary condition (see Subsection 2.4, second example). Denoting $\overline{g} := Z_\ell^*(g) = g \circ Z_\ell^{-1}$, we get

$$(3.10)\quad \begin{aligned} &\Delta_z \overline{U^\ell} = (\Delta_z - \Delta_x)\overline{U^\ell} + 2\,\overline{\operatorname{div}_x(u\nabla_x \eta^\ell)} - \overline{u\Delta_x\eta^\ell} + \overline{f^\ell} && \text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{U^\ell}|_{z_n=0} = 0 && \text{on} \quad \partial\mathbb{R}^n_+. \end{aligned}$$

Hence, using (2.35), the above system rewrites

$$(3.11)\quad \begin{aligned} &\Delta_z \overline{U^\ell} = F^\ell && \text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{U^\ell}|_{z_n=0} = 0 && \text{on} \quad \partial\mathbb{R}^n_+, \end{aligned}$$

with, denoting $\mathcal{A}^\ell := DZ_\ell \circ Z_\ell^{-1} - \operatorname{Id}$ and $\mathcal{B}^\ell := \mathcal{A}^\ell + {}^T\!\mathcal{A}^\ell + \mathcal{A}^\ell\,{}^T\!\mathcal{A}^\ell$,

$$\begin{aligned} F^\ell = -\operatorname{div}_z\Big(\mathcal{B}^\ell\nabla_z\overline{U^\ell} - \overline{U^\ell}(\operatorname{Id}+\mathcal{A}^\ell)\cdot\operatorname{div}_z\mathcal{A}^\ell\Big)\\ - \operatorname{div}_z\big((\operatorname{Id}+\mathcal{A}^\ell)\cdot\operatorname{div}_z\mathcal{A}^\ell\big)\overline{U^\ell} - \overline{u\Delta_x\eta^\ell} + 2\,\overline{\operatorname{div}_x(u\nabla_x\eta^\ell)} + \overline{f^\ell}. \end{aligned}$$

Let $\widetilde{U}^\ell$ and $\widetilde{F}^\ell$ be the symmetric extensions of $\overline{U^\ell}$ and F^ℓ. We have

$$\Delta_z \widetilde{U}^\ell = \widetilde{F}^\ell \quad \text{in} \quad \mathbb{R}^n$$

and, as $1/p > \sigma > -1 + 1/p$, Remark 2.2.1 gives $\widetilde{F}^\ell \in \dot{B}^\sigma_{p,q}(\mathbb{R}^n)$ and

$$\|\widetilde{F}^\ell\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)} \lesssim \|F^\ell\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)}.$$

We also have to keep in mind that, by construction, $\widetilde{U}^\ell$ is compactly supported. Hence, taking advantage of Corollary 3.1.1 and of Lemma 2.1.1, we end up with

$$(3.12)\quad \|\overline{U^\ell}\|_{B^{2+\sigma}_{p,q}(\mathbb{R}^n_+)} \le C\|F^\ell\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

We claim that

$$(3.13)\quad \begin{aligned} \|F^\ell\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)} \lesssim \lambda\|\overline{U^\ell}\|_{B^{2+\sigma}_{p,q}(\mathbb{R}^n_+)} + \|\overline{U^\ell}\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n_+)}\\ + \|\overline{f^\ell}\|_{B^\sigma_{p,q}(\mathbb{R}^n_+)} + \lambda^{-1}\|\overline{u}\|_{B^{1+\sigma}_{p,q}(B(0,\lambda))}. \end{aligned}$$

Indeed, to bound the first term of F^ℓ, it suffices to interpolate between $\dot{W}^{-1}_p(B(0,\lambda))$ and $\dot{W}^1_p(B(0,\lambda))$. First, owing to (2.27) Leibniz' rule, we have for $f \in W^3_{p;0}(B(0,\lambda))$ [(2)],

$$(3.14)\quad \|\operatorname{div}_z(\mathcal{B}^\ell\nabla_z f)\|_{\dot{W}^1_p(B(0,\lambda))} \le C\lambda\|f\|_{\dot{W}^3_p(B(0,\lambda))} + C\|\nabla_z f\|_{W^1_p(B(0,\lambda))},$$

2. $W^3_{p;0}$ stands for W^3_p with zero trace at the boundary

but the compactness of the support allows us to take advantage of the Poincaré inequality, so (3.14) reduces to

$$\|\operatorname{div}_z(\mathcal{B}^\ell\nabla_z f)\|_{\dot W^1_p(B(0,\lambda))} \lesssim \lambda\|f\|_{\dot W^3_p(B(0,\lambda))}. \tag{3.15}$$

Similarly, we find that for $f\in W^1_{p;0}(B(0,\lambda))$, we have

$$\|\operatorname{div}_z(\mathcal{B}^\ell\nabla_z f)\|_{\dot W^{-1}_p(B(0,\lambda))} \lesssim \|\mathcal{B}^\ell\nabla_z f\|_{L_p(B(0,\lambda))} \lesssim \lambda\|\nabla_z f\|_{L_p(B(0,\lambda))}. \tag{3.16}$$

Interpolating between (3.15) and (3.16) we eventually get

$$\|\operatorname{div}_z(\mathcal{B}^\ell\nabla_z\overline{U^\ell})\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \leq C\lambda\|\overline{U^\ell}\|_{\dot B^{2+\sigma}_{p,q}(\mathbb{R}^n_+)}. \tag{3.17}$$

Bounding the other terms in F^ℓ goes along the same lines. For instance:

$$\|\operatorname{div}_z((\operatorname{div}\mathcal{A}^\ell)\,\overline{U^\ell})\|_{\dot W^1_p(\mathbb{R}^n_+)} \leq C\|\overline{U^\ell}\|_{\dot W^2_p(\mathbb{R}^n_+)},$$
$$\|\operatorname{div}_z((\operatorname{div}\mathcal{A}^\ell)\,\overline{U^\ell})\|_{\dot W^{-1}_p(\mathbb{R}^n_+)} \leq C\|\overline{U^\ell}\|_{L_p(\mathbb{R}^n_+)},$$

hence

$$\|\operatorname{div}_z((\operatorname{div}\mathcal{A}^\ell)\,\overline{U^\ell})\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+))} \leq C\|\overline{U^\ell}\|_{\dot B^{1+\sigma}_{p,q}(\mathbb{R}^n_+)},$$

and

$$\|\overline{\operatorname{div}_x(u\nabla_x\eta^\ell)}\|_{\dot W^1_p(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|u\|_{\dot W^2_p(B(0,\lambda))},$$
$$\|\overline{\operatorname{div}_x(u\nabla_x\eta^l)}\|_{\dot W^{-1}_p(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|\overline{u}\|_{L_p(B(0,\lambda))},$$

whence

$$\|\overline{\operatorname{div}_x(u\nabla_x\eta^\ell)}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|u\|_{\dot B^{\sigma+1}_{p,q}(\mathbb{R}^n_+)}.$$

Assuming that λ is small enough, one may absorb the first term in the r.h.s. of (3.13) by the l.h.s. of (3.12). Hence, using that, by virtue of the composition lemma 2.1.1 and of remark 2.1.1, we may write

$$\|u\|_{B^{2+\sigma}_{p,q}(D)} \leq \sum_\ell\|\eta^\ell u\|_{B^{2+\sigma}_{p,q}(D)} \lesssim \sum_\ell\|\overline{U^\ell}\|_{B^{2+\sigma}_{p,q}(\mathbb{R}^n_+)} \lesssim \sum_\ell\|F^\ell\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)},$$

we get the desired estimate (3.7).

Next, we claim that

$$\|u\|_{B^{1+\sigma}_{p,q}(D)} \leq C\|\Delta u\|_{B^\sigma_{p,q}(D)} \tag{3.18}$$

for all $u\in B^{2+\sigma}_{p,q}(D)$ such that $\displaystyle\int_D u(x)\,dx=0$ and $\partial_{\vec n}u|_{\partial D}=0$.

The proof is based on compactness arguments : if (3.18) were not true, then there would exist a sequence $(u^k)_{k\in\mathbb{N}}$ of $B^{2+\sigma}_{p,q}(D)$ functions such that

$$1 = \|u^k\|_{B^{1+\sigma}_{p,q}(D)} > k\|\Delta u^k\|_{B^{\sigma}_{p,q}(D)}. \tag{3.19}$$

According to (3.7) we have for all k:

$$\|u^k\|_{B^{2+\sigma}_{p,q}(D)} \leq C(\|u^k\|_{B^{1+\sigma}_{p,q}(D)} + \|\Delta u^k\|_{B^{\sigma}_{p,q}(D)}) \leq C. \tag{3.20}$$

So the compactness properties of Besov spaces (see Proposition 2.2.2) imply that there exists a subsequence $\{u^{k_n}\}$ and $u^* \in B^{2+\sigma}_{p,q}(D)$ such that

$$u^{k_n} \to u^* \qquad \text{in } B^{1+\sigma}_{p,q}(D) \tag{3.21}$$

and

$$u^{k_n} \rightharpoonup u^* \qquad \text{in } B^{2+\sigma}_{p,q}(D). \tag{3.22}$$

Relations (3.19) and (3.21) imply that

$$\|u^*\|_{B^{1+\sigma}_{p,q}(D)} = 1 \quad \text{and} \quad \|\Delta u^*\|_{B^{\sigma}_{p,q}(D)} = 0. \tag{3.23}$$

In other words, u^* fulfills:

$$\begin{cases} \Delta u^* = 0 & \text{in} \quad D, \\ \partial_{\vec{n}} u^* = 0 & \text{on} \quad \partial D, \end{cases} \qquad \int_D u^*\,dx = 0. \tag{3.24}$$

If $B^{2+\sigma}_{p,q}(D) \hookrightarrow H^1(D)$ (which is always the case if $p \geq 2$) then one can immediately conclude that $u^* \equiv 0$, which stands in contradiction with (3.23). Hence estimate (3.18) holds true, and approximating the data f by smooth data allows to fully justify the proof of existence of a $B^{\sigma+2}_{p,q}(D)$ solution to (3.4) in that case (see the details below).

Now, if $B^{2+\sigma}_{p,q}(D)$ is not embedded in $H^1(D)$ then and one may argue by duality as follows: take any ψ in $B^{-\sigma}_{p',q'}(D)$ *with average* 0 and solve according to the case we have just completed the following Neumann problem:

$$\begin{cases} \Delta \phi = \psi & \text{in} \quad D, \\ \partial_{\vec{n}} \phi = 0 & \text{on} \quad \partial D, \end{cases} \qquad \int_D \phi\,dx = 0.$$

Then we get a solution ϕ is in $B^{2-\sigma}_{p',q'}(D)$. As both u^* and ϕ satisfy homogeneous Neumann boundary conditions, we get the following chain of equalities:

$$\int_D u^*\psi\,dx = -\int_D \nabla u^* \cdot \nabla\phi\,dx = \int_D \Delta u^*\,\phi\,dx = 0.$$

Therefore $u^* \equiv 0$, thus contradicting (3.23).

To end the proof of Proposition 3.2.1, we ought to prove the existence of solutions. To this end, let us write f as the limit in $B^{\sigma}_{p,q}(D)$ of a sequence of smooth functions $f_j \in \mathcal{C}^{\infty}(\overline{D})$ with $\displaystyle\int_D f_j\,dx = 0$. Then we know (see [**36**]) that there exists a solution u_j in $C^{2,\frac{1}{p}}$ of

$$\begin{cases} \Delta u_j = f_j & \text{in} \quad D, \\ \partial_{\vec{n}} u_j = 0 & \text{on} \quad \partial D, \end{cases} \qquad \int_D u_j\,dx = 0, \tag{3.25}$$

whenever the domain is $C^{2,\frac{1}{p}}$. Of course u_j is also in $B^{\sigma+2}_{p,q}$ for $\sigma < 1/p$ as the domain D is bounded. Therefore, resuming to (3.6), one may write

$$\|u_j - u_k\|_{B^{2+\sigma}_{p,q}(D)} \leq C\|f_j - f_k\|_{B^{\sigma}_{p,q}(D)} \quad \text{for all} \quad (j,k) \in \mathbb{N}^2,$$

which ensures that $(u_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $B^{2+\sigma}_{p,q}(D)$. One can thus conclude that there exists $u \in B^{2+\sigma}_{p,q}(D)$ satisfying (3.5) and (3.6). □

Remark 3.2.1. — One may extend Proposition 3.2.1 to more general σ. For higher regularity this requires extra compatibility conditions on f. We omit this interesting issue (see e.g. [**26**]) since it is very technical and not needed for the analysis of the Stokes system we want to perform here. The case of more negative σ will be treated below in Lemma 3.3.3.

Remark 3.2.2. — In [**46**], an alternative approach, based on the analysis of regularity of the weak solutions to (3.4), is proposed. It turns out to be more efficient in the case of critical regularity of the boundary.

Let us state an important consequence of Proposition 3.2.1.

Corollary 3.2.1. — *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Consider a compactly supported function f in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ for some real number σ. For f to belong to $\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)$, it suffices that*

– *either $\sigma > -1/p'$ and*

$$\int_{\mathbb{R}^n} f\,dx = 0; \tag{3.26}$$

– *or $\sigma > 1 - n/p'$.*

Furthermore, there exists a constant C such that if $\operatorname{Supp} f \subset B(0,\lambda)$ *then*

$$\|f\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)} \leq C\lambda\|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Proof. — Performing a suitable dilation of the space variable and using the scaling properties of homogeneous Besov semi-norms (see Proposition 2.1.1) reduces the study to $\lambda = 1$.

Now, if $\sigma > 1 - n/p'$ then the result is an easy corollary of Proposition 2.1.3, because as f is compactly supported, one may write

$$\|f\|_{\dot B^{\sigma-1}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^{\sigma-1}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Next, let us consider the case $-1/p' < \sigma < \min(-1 + n/p, 0)$ and $p = q$ under Assumption (3.26). Then Proposition 2.1.1 guarantees that

$$\|f\|_{\dot B^{\sigma-1}_{p,p}(\mathbb{R}^n)} = \sup_{\|\phi\|_{\dot B^{1-\sigma}_{p',p'}(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} f\,\phi\,dx. \tag{3.27}$$

Arguing by density, it suffices to consider functions ϕ in $\mathcal{C}^\infty_c(\mathbb{R}^n)$. Furthermore, as (3.26) is satisfied, Proposition 3.2.1 (here we need $\sigma > -1/p'$) ensures that there exists some function c in $B^{\sigma+2}_{p,p}(B(0,1))$ so that

$$\begin{cases} \Delta c = f & \text{in} \quad B(0,1), \\ \partial_{\vec n} c = 0 & \text{on} \quad \partial B(0,1), \end{cases} \qquad \int_{B(0,1)} c\,dx = 0, \tag{3.28}$$

and, in addition,

$$\|c\|_{B^{\sigma+2}_{p,p}(B(0,1))} \le C\|f\|_{B^{\sigma}_{p,p}(B(0,1))}.$$

Next, we define

$$\widetilde{\nabla c}(x) = \begin{cases} \nabla c(x) & \text{for} \quad x \in B(0,1), \\ 0 & \text{for} \quad x \in \mathbb{R}^n \setminus B(0,1). \end{cases} \tag{3.29}$$

Remark that by construction, we have, owing to the homogeneous Neumann boundary condition over c and the support properties of f,

$$-\int_{\mathbb{R}^n} \widetilde{\nabla c} \cdot \nabla\phi\,dx = \int_{\mathbb{R}^n} f\phi\,dx \quad \text{for all} \quad \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n). \tag{3.30}$$

Therefore,

$$\Big|\int_{\mathbb{R}^n} f\phi\,dx\Big| \le \|\widetilde{\nabla c}\|_{L_{p^*}(\mathbb{R}^n)} \|\nabla\phi\|_{L_{(p^*)'}(\mathbb{R}^n)}. \tag{3.31}$$

To bound the right-hand side, it suffices to use the embedding result stated in Proposition 2.1.1 and its dual version. We get

$$\|\widetilde{\nabla c}\|_{L_{p^*}(\mathbb{R}^n)} \lesssim \|\widetilde{\nabla c}\|_{\dot B^{\sigma+1}_{p,p}(\mathbb{R}^n)} \quad \text{for} \ \tfrac{n}{\sigma+1}\Big(\tfrac{1}{p} - \tfrac{1}{p^*}\Big) = 1 \quad \text{as} \ 0 < \sigma+1 < \tfrac{n}{p},$$

$$\|\nabla\phi\|_{L_{(p^*)'}(\mathbb{R}^n)} \lesssim \|\nabla\phi\|_{\dot B^{-\sigma}_{p',p'}(\mathbb{R}^n)} \ \text{for} \ \tfrac{n}{-\sigma}\Big(\tfrac{1}{p'} - \tfrac{1}{(p^*)'}\Big) = 1 \ \text{as} \ 0 < -\sigma < \tfrac{n}{p'}.$$

Now, (3.31) and the above inequalities imply that

$$\Big|\int_{\mathbb{R}^n} f\phi\,dx\Big| \leq C\|\widetilde{\nabla c}\|_{\dot{B}^{1+\sigma}_{p,p}(\mathbb{R}^n)}\|\nabla\phi\|_{\dot{B}^{-\sigma}_{p',p'}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{\sigma}_{p,p}(\mathbb{R}^n)}\|\phi\|_{\dot{B}^{1-\sigma}_{p',p'}(\mathbb{R}^n)}.$$

So (3.27) yields the desired inequality for $-1/p' < \sigma < \min(-1+n/p, 0)$ and $p = q$. The remaining cases follow by interpolation. □

As a consequence, we get the following improvement of Corollary 3.1.1 in the case where the source term of (3.1) has average 0:

Lemma 3.2.1. — *Let $f \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ for some $p \in (1,\infty)$, $\sigma \in (-1+1/p, 1/p)$ and $q \in [1,\infty]$. Assume in addition that f fulfills (3.26) and is supported in $\overline{B(0,\lambda)}$.*

Then the Poisson equation (3.1) has a unique solution b in $L_m(\mathbb{R}^n)$ for some finite[3] *m. Furthermore ∇b and $\nabla^2 b$ are in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and satisfy*

$$\|\nabla^2 b\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} + \lambda^{-1}\|\nabla b\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Proof. — Uniqueness stems from Liouville theorem, so let us concentrate on the proof of existence. Being compactly supported, the function f is also in $B^{\sigma}_{p,q}(\mathbb{R}^n)$ and, more generally, in all spaces $B^{\sigma'}_{r,q}(\mathbb{R}^n)$ with $1 \leq r \leq p$ and $\sigma' \leq \sigma$.

If $\sigma \geq 0$ then applying Corollary 3.2.1 thus ensures that $f \in \dot{B}^{\sigma'-1}_{r,q}(\mathbb{R}^n)$ if

$$1 < r \leq p \quad \text{and} \quad -1+1/r < \sigma' \leq \sigma. \tag{3.33}$$

Therefore, whenever r and σ' are taken so that $\sigma'+1 < n/r$, Lemma 3.1.1 provides a solution $b \in \dot{B}^{\sigma'+1}_{r,q}(\mathbb{R}^n)$, and combining with Sobolev embedding guarantees that $b \in L_m(\mathbb{R}^n)$ for all $n/(n-1) < m < n/\min(0, -1-\sigma+n/p)$.

If $\sigma < 0$, then the condition $-1+1/r < \sigma$ is no longer satisfied for $r \to 1$. Hence (3.33) implies that $r > 1/(\sigma+1)$ and we thus eventually get $b \in L_m(\mathbb{R}^n)$ only for $n/((n-1)(1+\sigma)) < m < n/\min(0, -1-\sigma+n/p)$.

In order to prove that in addition ∇b and $\nabla^2 b$ are in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$, consider a sequence $f_j \to f$ with $f_j \in \mathcal{S}_0(\mathbb{R}^n)$ for all $j \in \mathbb{Z}$, and define $b_j := -(-\Delta)^{-1} f_j$. We already know that $b_j \to b$ in $L_m(\mathbb{R}^n)$. Furthermore, because all functions

3. In fact, b belongs to all spaces $L_m(\mathbb{R}^n)$ with

$$\frac{n}{n-1}\,\frac{1}{1+\min(0,\sigma)} < m < \frac{n}{\max(0, -1-\sigma+n/p)}. \tag{3.32}$$

f_j are in $\mathcal{S}_0(\mathbb{R}^n)$, it is obvious that for all $(j,k) \in \mathbb{Z}^2$,

$$\|\nabla b_j - \nabla b_k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f_j - f_k\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)},$$
$$\|\nabla^2 b_j - \nabla^2 b_k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f_j - f_k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Therefore $(\nabla b_j)_{j\in\mathbb{Z}}$ and $(\nabla^2 b_j)_{j\in\mathbb{Z}}$ are Cauchy sequences in the complete space $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and we can thus assert that ∇b and $\nabla^2 b$ are in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ with

$$\|\nabla b\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)} \quad \text{and} \quad \|\nabla^2 b\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

The fact that $\|f\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)}$ may be replaced with $\lambda\|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}$ stems from Corollary 3.2.1. □

3.3. The half-space case

Let us first concentrate on the homogeneous Dirichlet problem:

$$\begin{cases} \Delta u = h & \text{in} \quad \mathbb{R}^n_+, \\ u|_{x_n=0} = 0 & \text{on} \quad \partial\mathbb{R}^n_+. \end{cases} \tag{3.34}$$

The first equation has to be understood in the distributional sense in $\mathbb{R}^n_+$, and the second equation means that we require the trace at $\partial\mathbb{R}^n_+$ to our solution to be defined and equal to 0. For smooth enough solutions, this is equivalent to the fact that the antisymmetric extension $\widetilde{u}$ of u on $\mathbb{R}^n$ satisfies the Poisson equation (3.1) with r.h.s. $\widetilde{h}$ (the antisymmetric extension of h on $\mathbb{R}^n$).

Lemma 3.3.1. — *Let h be in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ with $p \in (1,\infty)$, $q \in [1,\infty]$ and $\sigma \in (-1+1/p, 1/p)$.*

1. *If in addition (σ,p,q) fulfills (3.2) then (3.34) has a unique solution $u \in \dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n_+)$, and we have*

$$\|u\|_{\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n_+)} \leq C\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

2. *If in addition h is compactly supported then (3.34) has a unique solution $u \in L_m(\mathbb{R}^n_+)$ for some finite m, and we have*

$$\|\nabla^2 u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

3. *If in addition*

$$\sigma + 1 < \frac{n}{p} \quad \left(\textit{or} \quad \sigma + 1 \leq \frac{n}{p} \quad \textit{if} \quad q = 1\right) \tag{3.35}$$

then (3.34) *has a unique solution satisfying* $\nabla u \in \dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)$, *and we have*

$$\|\nabla u\|_{\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)} \leq C\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}. \tag{3.36}$$

4. *If* $\nabla h \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *and* $u = (G\star\widetilde{h})|_{\partial\mathbb{R}^n_+}$ *where* G *stands for the fundamental solution of* Δ *in* $\mathbb{R}^n$ *and* $\widetilde{h}$ *is the antisymmetric extension of* h, *then we have* $\nabla^3 u \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *and*
$$\|\nabla^3 u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|\nabla h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$
5. *If* $h = \operatorname{div} k$ *for some* $k \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *then* (3.34) *has a unique solution* u *with* $\nabla u \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *and we have*
$$\|\nabla u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Proof. — To prove the existence part of the first item, introduce the antisymmetric extension $\widetilde{h}$ of h, and solve (3.1) with right-hand side $\widetilde{h}$. As $\sigma \in (-1+1/p, 1/p)$, Remark 2.2.1 ensures that $\widetilde{h}$ is in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and

$$\|\widetilde{h}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

As in addition (3.2) is fulfilled, Lemma 3.1.1 provides us with a solution $\widetilde{u}$ in $\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n)$ to (3.1), satisfying

$$\|\widetilde{u}\|_{\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n)} \leq C\|\widetilde{h}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}. \tag{3.37}$$

Uniqueness for the Poisson equation in $\mathbb{R}^n$ ensures $\widetilde{u}$ to be antisymmetric. Hence its well defined trace at $\partial\mathbb{R}^n_+$ vanishes, and $u := \widetilde{u}|_{\mathbb{R}^n_+}$ satisfies our claim.

For proving the second item, we note that the antisymmetric extension $\widetilde{h}$ to h satisfies (3.26) and is compactly supported. Hence there exists a unique solution $\widetilde{u} \in L_m(\mathbb{R}^n)$ to (3.1) (see Lemma 3.2.1), and

$$\|\nabla^2\widetilde{u}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|\widetilde{h}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Setting $u := \widetilde{u}|_{\mathbb{R}^n_+}$ yields the desired result as the components of $\nabla^2\widetilde{u}$ coincide either with the symmetric or with the antisymmetric extension of ∇u on $\mathbb{R}^n$.

To prove the third item, approximate h by a sequence $(h_j)_{h\in\mathbb{N}}$ of smooth compactly supported functions. Because (3.35) is fulfilled, arguing as in the previous item gives a sequence $(u_j)_{j\in\mathbb{N}}$ of solutions with $\nabla u_j \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$, to (3.34) with r.h.s. h_j. Besides, we have

$$\|\nabla(u_j - u_k)\|_{\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)} \leq C\|h_j - h_k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

As Condition (3.35) ensures $\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)$ to be complete, we deduce that $(\nabla u_j)_{j\in\mathbb{N}}$ converges to some ∇u in $\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)$ with u fulfilling (3.36) and the first line of (3.34) in the distributional meaning. Combining the localization properties of Besov spaces with the trace theorem (see Prop. 2.2.4), we discover that the trace of $\varphi\nabla u$ at $\partial\mathbb{R}^n_+$ is well-defined for all smooth compactly supported function φ on $\mathbb{R}^n_+$, and that in addition

$$\|\varphi\nabla(u_j-u)|_{x_n=0}\|_{B^{\sigma+1-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\leq C\|\nabla(u_j-u)\|_{\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)}\leq C\|h_j-h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

As $u_j|_{x_n=0}\equiv 0$ implies $\nabla_{x'}u_j|_{x_n=0}\equiv 0$, we conclude that $\nabla_{x'}u|_{x_n=0}\equiv 0$ and thus $u|_{x_n=0}$ is independent of x'. Subtracting a constant as the case may be, one may thus ensure the second line of (3.34).

To prove the fourth item, we start with the remark that (still denoting with tilde the antisymmetric extensions):

$$\widetilde{\nabla_{x'}u}=(G\star\widetilde{\nabla_{x'}h})=G\star\nabla_{x'}\widetilde{h}.$$

Hence arguing as for the second item yields $\nabla^2\nabla_{x'}u\in\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ and

$$\|\nabla^2\nabla_{x'}u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}\leq C\|\nabla_{x'}h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Let us notice that only the term $\partial^3_{x_n}u$ of $\nabla^3 u$ has not been estimated yet. Now we have $\Delta u=h$ in $\mathbb{R}^n_+$, and thus

$$\partial^3_{x_n}u=\partial_{x_n}h-\Delta_{x'}\partial_{x_n}u\quad\text{in}\quad\mathbb{R}^n_+.$$

This completes the proof of the fourth item.

To prove the last item, we consider the antisymmetric/symmetric extension $\check{k}$ of k over $\mathbb{R}^n$, that is defined for all $x_n<0$ and $x'\in\mathbb{R}^{n-1}$ by

$$\check{k}(x',x^n)=(\check{k}',\check{k}^n)(x',x_n):=(-k',k^n)(x',-x^n).$$

Remark 2.2.1 ensures that $\check{k}\in\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ and

$$\|\check{k}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}\leq\|k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Furthermore, $\operatorname{div}\check{k}$ coincides with the antisymmetric extension of $\operatorname{div}k$ and Lemma 3.1.2 thus provides us with a function $\widetilde{u}$ such that $\nabla\widetilde{u}\in\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$,

$$\Delta\widetilde{u}=\operatorname{div}\check{k}\quad\text{in}\quad\mathbb{R}^n,$$

and

$$\|\nabla\widetilde{u}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}\lesssim\|\operatorname{div}\check{k}\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)}.$$

Recall that uniqueness to the Poisson equation in $\mathbb{R}^n$ holds up to an harmonic polynomial. As we restrict our attention to functions with gradient in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ (which implies decay at infinity in the distributional sense), we deduce that the constructed solution $\widetilde{u}$ is antisymmetric up to some constant. Subtracting that constant as the case may be, we conclude that the restriction u of $\widetilde{u}$ to $\mathbb{R}^n_+$ satisfies (3.34) and the desired inequality.

For completeness, let us say a few words on the proof of uniqueness. It suffices to establish that if u satisfies (3.34) with r.h.s. 0 and $\nabla u \in L_m(\mathbb{R}^n_+)$, (or more generally in $\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)$) then $\nabla u \equiv 0$. Let $(\chi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers compactly supported in $\mathbb{R}^n_+$. Then $\chi_\varepsilon \star \nabla u \to \nabla u$ in $L_m(\mathbb{R}^n_+)$ for ε going to 0, and all functions $\chi_\varepsilon \star \nabla u$ are smooth on $\overline{\mathbb{R}^n_+}$ and harmonic on $\mathbb{R}^n_+$. As they vanish on $\{x_n = 0\}$, the maximum principle ensures that $\chi_\varepsilon \star u \equiv 0$ in $\mathbb{R}^n_+$. Hence $\nabla u \equiv 0$ in $\mathbb{R}^n_+$ and thus $u \equiv 0$ as well, given the homogeneous Dirichlet boundary condition. □

Remark 3.3.1. — Let us comment the condition $u \to 0$ at $x \to \infty$. Having regularity controlled by Condition (3.35) indeed ensures that the gradient of the solution vanishes at the infinity. It is sufficient to control the distributional meaning of the solutions, and first of all the uniqueness of them. Of course, restricting our attention to the stronger condition (3.2) we are guaranteed that the solution truly goes to zero at infinity.

Corollary 3.3.1. — *Let $h \in \dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)$ (with $1 < p < \infty$, $1 \leq q \leq \infty$ and $-1 + 1/p < \sigma < 1/p$) be such that $\operatorname{Supp} h \subset \overline{B(0,\lambda) \cap \mathbb{R}^n_+}$. Then the Dirichlet problem* (3.34) *has a unique solution u belonging to $L_m(\mathbb{R}^n_+)$ for some finite m and such that $\nabla u \in \dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)$. Furthermore, we have*

$$\|\nabla u\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)} \leq C\lambda \|h\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

Proof. — Scaling arguments reduces the study to the case $\lambda = 1$. Let $\widetilde{h}$ be the antisymmetric extension of h on $\mathbb{R}^n$. Clearly, it is supported in $B(0,1)$, and Proposition 2.1.3 guarantees that $\widetilde{h} \in B^\sigma_{p,q}(\mathbb{R}^n)$ and that

$$\|\widetilde{h}\|_{B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

As the compatibility condition $\int_{B(0,1)} \widetilde{h}\, dx = 0$ is satisfied (a consequence of antisymmetric extension), we know from e.g. [**42**] that there exists some function $k \in B^{1+\sigma}_{p,q}(B(0,1))$ such that $\operatorname{div} k = \widetilde{h}$ in $B(0,1)$, $k|_{\partial B(0,1)} = 0$ and

$$\|k\|_{B^{1+\sigma}_{p,q}(B(0,1))} \leq C\|\widetilde{h}\|_{B^\sigma_{p,q}(B(0,1))} \leq C\|h\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

Note also that if denote by k_0 the extension of k by 0 on $\mathbb{R}^n$ then ∇k_0 is just the extension of ∇k by 0. Hence we have $k_0 \in B^{1+\sigma}_{p,q}(\mathbb{R}^n)$ and thus

$$\|k_0\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

As k is compactly supported, applying once again Proposition 2.1.3 leads to the following series of inequalities:

$$\|k\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \leq \|k_0\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|k_0\|_{B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|k_0\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

Finally, applying the last part of Lemma 3.3.1, we get a solution $u \in L_m(\mathbb{R}^n_+)$ for some finite m, to the problem

$$\begin{cases} \Delta u = h = \operatorname{div} k & \text{in} \quad \mathbb{R}^n_+, \\ u|_{x_n=0} = 0 & \text{on} \quad \partial\mathbb{R}^n_+ \end{cases}$$

with, in addition,

$$\|\nabla u\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \leq C\|k\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

This completes the proof of the lemma. □

We now turn to the study of the *Neumann problem* for the Poisson equation in the half-space, namely

$$\text{(3.38)} \qquad \begin{cases} \Delta u = h & \text{in} \quad \mathbb{R}^n_+, \\ \partial_{x_n} u|_{x_n=0} = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases} \qquad u \to 0 \ \text{ as } \ |x| \to \infty.$$

The solution has to be understood in the weak sense (see (3.5)).

Let us first establish an existence result in the 'smooth' case.

LEMMA 3.3.2. — *Let $p \in (1,\infty)$, $q \in [1,\infty]$ and $\sigma \in (-1+1/p, 1/p)$. Let h be in $\dot B^\sigma_{p,q}(\mathbb{R}^n_+)$ with*

$$\int_{\mathbb{R}^n_+} h(x)\,dx = 0 \quad \textit{and} \quad \operatorname{Supp} h \subset \overline{B(0,\lambda) \cap \mathbb{R}^n_+}.$$

Then (3.38) *has a unique solution $u \in L_m(\mathbb{R}^n_+)$ for some finite m (and even for all m satisfying* (3.32)*) with ∇u and $\nabla^2 u$ in $\dot B^\sigma_{p,q}(\mathbb{R}^n_+)$. Furthermore,*

$$\|\nabla^2 u\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} + \lambda^{-1}\|\nabla u\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \leq C\|h\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

Proof. — As h is in $\dot B^\sigma_{p,q}(\mathbb{R}^n_+)$ with $-1+1/p < \sigma < 1/p$, the symmetric extension h_{sym} of h belongs to $\dot B^\sigma_{p,q}(\mathbb{R}^n)$ and satisfies

$$\|h_{sym}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}, \quad \int_{\mathbb{R}^n} h_{sym}(x)\,dx = 0 \ \text{ and } \ \operatorname{Supp} h_{sym} \subset \overline{B(0,\lambda)}.$$

Therefore, according to Lemma 3.2.1, the problem

$$\Delta u_{sym} = h_{sym} \quad \text{in} \ \ \mathbb{R}^n, \qquad u_{sym} \to 0 \ \text{ at } \ \infty$$

has a unique solution u_{sym} in $L_m(\mathbb{R}^n)$ and we have

$$\|\nabla^2 u_{sym}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} + \lambda^{-1}\|\nabla u_{sym}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|h_{sym}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}.$$

This ensures that ∇u_{sym} is at least locally in $B^{\sigma+1}_{p,q}(\mathbb{R}^n)$ and thus has a trace at $\partial\mathbb{R}^n_+$. Owing to the symmetry of h_{sym} with respect to the hyperplane $x_n = 0$, the solution u_{sym} is symmetric, too. Hence the homogeneous Neumann boundary condition on $\{x_n = 0\}$ is satisfied. So setting $u := u_{sym}|_{\mathbb{R}^n_+}$ provides the desired solution for (3.38). □

Let us now investigate the case where the source term h in (3.38) has so low regularity that the meaning of $\partial_{x_n} u$ at the boundary cannot be understood in the classical way. The relevant framework will be taken from Definition 2.3.1: we want to solve (3.38) with source term $h = \mathcal{DIV}\,[k;\zeta]$ in $\dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)$, that is to find some distribution u so that

$$\mathcal{DIV}\,[\nabla u; 0] = \mathcal{DIV}\,[k;\zeta],$$

or in other words

$$(3.39)\quad -\int_{\mathbb{R}^n_+} \nabla u\cdot\nabla\varphi\,dx = -\int_{\mathbb{R}^n_+} k\cdot\nabla\varphi\,dx + \int_{\partial\mathbb{R}^n_+} \zeta\varphi\,d\varsigma \quad \text{for all } \varphi\in\mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}).$$

Note that as $\partial\mathbb{R}^n_+$ is noncompact, the compatibility condition over ζ in Definition 2.3.1 is somehow hidden in the definition of the space $\dot{B}^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)$.

Lemma 3.3.3. — *Let $h = \mathcal{DIV}\,[k;\zeta] \in \dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)$ with $-1+1/p<\sigma<1/p$. Assume either that (σ,p,q) satisfies* (3.35) *or that both k and ζ are compactly supported* (*in which case ζ must have average* 0).

Then equation (3.39) *admits a unique solution $u\in L_m(\mathbb{R}^n_+)$ with*[4] *m defined by $n/m = -1-\sigma+n/p$, and $\nabla u\in\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)$. Moreover*

$$(3.40)\qquad \|\nabla u\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)} \le C\|\,\mathcal{DIV}\,[k;\zeta]\|_{\dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)}.$$

In the compact support case, we have $u\in L_m(\mathbb{R}^n_+)$ for all m satisfying (3.32).

Proof. — Uniqueness may be proved thanks to a regularizing argument, exactly as in Lemma 3.3.1.

In order to prove the existence of a solution satisfying the required properties, we shall first construct some function H going to 0 at infinity, so that

$$-\int_{\mathbb{R}^n_+}\nabla H\cdot\nabla\varphi\,dx = \int_{\partial\mathbb{R}^n_+}\zeta\varphi\,d\varsigma \quad \text{for all } \varphi\in\mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}).$$

4. We have to use the Lorentz space $L_{m,q}(\mathbb{R}^n_+)$ if $q>m$, and $\mathcal{C}_0(\mathbb{R}^n_+)$ if $\sigma=-1+n/p$ and $q=1$.

In other words, we want to solve

$$\text{(3.41)}\qquad \begin{cases} \Delta H = 0 & \text{in} \quad \mathbb{R}^n_+, \\ -\partial_{x_n} H|_{x_n=0} = \zeta & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases} \qquad H \to 0 \ \text{ at } \ \infty.$$

If $\zeta \in \mathcal{S}_0(\partial\mathbb{R}^n_+)$ then using the Fourier transform with respect to tangential variables x' yields

$$-|\xi'|^2 \mathcal{F}_{x'} H + \partial^2_{x_n x_n} \mathcal{F}_{x'} H = 0, \qquad \mathcal{F}_{x'} H \to 0 \ \text{ for } \ x_n \to +\infty,$$

the solution of which is given by the explicit formula

$$H = \mathcal{F}^{-1}_{x'} \Big[\frac{1}{|\xi'|} e^{-|\xi'| x_n} \mathcal{F}_{x'} \zeta \Big].$$

From Lemma 2.2.2, we thus infer that

$$\text{(3.42)}\qquad \|\nabla H\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C \|\zeta\|_{\dot{B}^{\sigma - \frac{1}{p}}_{p,q}(\partial\mathbb{R}^n_+)}.$$

Now, if $(\sigma - 1, p, q)$ fulfills (3.2) then $\mathcal{S}_0(\partial\mathbb{R}^n_+)$ is dense in $\dot{B}^{\sigma-\frac{1}{p}}_{p,q}(\partial\mathbb{R}^n_+)$, and the space $\dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)$ is complete. Hence we get $H \in \dot{B}^{\sigma+1}_{p,q}(\mathbb{R}^n_+)$ together with Inequality (3.42) for general ζ in $\dot{B}^{\sigma-\frac{1}{p}}_{p,q}(\partial\mathbb{R}^n_+)$.

The case where $\sigma \geq n/p - 1$ and ζ is compactly supported reduces to the previous one as we have $\zeta \in \dot{B}^{\sigma^* - \frac{1}{p^*}}_{p^*,q}(\partial\mathbb{R}^n_+)$ for some $\sigma^* \leq \sigma$ and $-1 + 1/p^* < \sigma^* < 1/p^*$ with $\sigma^* < n/p^* - 1$. So, as in Lemma 3.2.1, the previous construction combined with embedding provides a solution $H \in L_m(\mathbb{R}^n_+)$ for all m given by (3.32), still satisfying (3.42).

Next, let us construct some distribution w satisfying

$$\text{(3.43)}\qquad \int_{\mathbb{R}^n_+} \nabla w \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n_+} k \cdot \nabla \varphi \, dx \quad \text{for all} \quad \varphi \in \mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}).$$

To this end, consider the symmetric/antisymmetric extension k_{div} of k over $\mathbb{R}^n$, namely the function k_{div} defined by

$$k_{\mathrm{div}} = k \ \text{ on } \ \mathbb{R}^n_+ \quad \text{and} \quad (k'_{\mathrm{div}}, k^n_{\mathrm{div}})(x', x^n) := (k', -k^n)(x', -x^n) \quad \text{for} \ \ x^n < 0.$$

Because $-1 + 1/p < \sigma < 1/p$, we have $k_{\mathrm{div}} \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and

$$\|k_{\mathrm{div}}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C \|k_{\mathrm{div}}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Now, let us solve, according to Lemma 3.1.1 (if $(\sigma - 1, p, q)$ fulfills (3.2)), or to Lemma 3.2.1 (if k is compactly supported) the following equation:

$$\text{(3.44)}\qquad \Delta w_{\mathrm{sym}} = \operatorname{div} k_{\mathrm{div}} \quad \text{in} \ \ \mathbb{R}^n.$$

We get a distribution $w_{\rm sym}$ belonging to some Lebesgue space $L_m(\mathbb{R}^n)$ with finite m (or to $\mathcal{C}_0(\mathbb{R}^n)$) and satisfying

$$\|\nabla w_{\rm sym}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|k_{\rm div}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}. \tag{3.45}$$

The restriction w of $w_{\rm sym}$ to $\mathbb{R}^n_+$ is also in $L_m(\mathbb{R}^n_+)$ and satisfies

$$\|\nabla w\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)} \leq C\|k\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

In addition, because the distribution $\operatorname{div} k_{\rm div}$ is symmetric with respect to the hyperplane $\{x_n = 0\}$, so does the function $w_{\rm sym}$. Therefore (3.44) implies (3.43), and setting $u := w + H$ completes the proof of the lemma. □

3.4. The Neumann problem in bounded or exterior domains

This section is devoted to solving the nonhomogeneous Neumann problem

$$\begin{cases} \Delta P = 0 & \text{in} \quad \Omega, \\ \partial_{\vec n} P = b & \text{on} \quad \partial\Omega \end{cases} \tag{3.46}$$

in an exterior domain (that is in the complement of some simply connected compact subset of $\mathbb{R}^n$) or in a bounded domain of $\mathbb{R}^n$ (that need not be simply connected), for rough boundary data.

In the exterior domain case, we supplement the equation with

$$P \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.47}$$

and, in the bounded domain case, with

$$\int_\Omega P\,dx = 0. \tag{3.48}$$

The main result of this section reads:

Theorem 3.4.1. — *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$, or a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Let b be in $B^{\sigma-\frac1p}_{p,q}(\partial\Omega)$ for some $p \in (1,\infty)$, $q \in [1,\infty]$ and $-1 + 1/p < \sigma < 1/p$, and satisfy the following compatibility condition in the distributional meaning:*

$$\int_{\partial\Omega} b\,d\sigma = 0. \tag{3.49}$$

Then System (3.46) supplemented with (3.47) or (3.48) has a unique solution P such that:

– *Bounded domain case: $P \in B^{\sigma+1}_{p,q}(\Omega)$ and*

$$\|P\|_{B^{1+\sigma}_{p,q}(\Omega)} \leq C\|b\|_{B^{\sigma-\frac1p}_{p,q}(\partial\Omega)}. \tag{3.50}$$

– *Exterior domain case:* $P \in L_m(\Omega)$ *for all* m *satisfying* (3.32), $\nabla P \in \dot{B}^\sigma_{p,q}(\Omega)$ *and* $P \in B^\sigma_{p,q}(K)$ *for any compact subset* K *of* $\mathbb{R}^n$ *such that* $\mathrm{dist}(\partial\Omega, \Omega \setminus K) > 0$ *(see Fig. 3.4). In addition, we have*

$$\|P\|_{L_m(\Omega)} + \|\nabla P\|_{\dot{B}^\sigma_{p,q}(\Omega)} \leq C\|b\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)}, \tag{3.51}$$

$$\|P\|_{B^\sigma_{p,q}(K)} \leq C_K\|b\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)}. \tag{3.52}$$

Proof. — We focus on exterior domains, just indicating when needed what has to be changed in the easier bounded domain case.

Let us first prove *a priori* estimates for smooth enough (up to the boundary) harmonic functions in Ω satisfying either (3.47) or (3.48). We shall first establish that for all m satisfying (3.32), we have

$$\|P\|_{L_m(\Omega)} + \|\nabla P\|_{\dot{B}^\sigma_{p,q}(\Omega)} \leq C(\|\partial_{\vec{n}}P|_{\partial\Omega}\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)} + \|P\|_{B^\sigma_{p,q}(K)}) \tag{3.53}$$

and next that

$$\|P\|_{B^\sigma_{p,q}(K)} \leq C\|\partial_{\vec{n}}P|_{\partial\Omega}\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)}. \tag{3.54}$$

To show (3.53), we localize the system by means of a partition of unity $\{\eta^\ell\}_{0\leq\ell\leq L}$ of Ω such that:

1. $\eta^0 \equiv 0$ in a neighborhood of $\mathbb{R}^n \setminus \Omega$ and $\eta^0 \equiv 1$ on $\Omega \setminus K$;
2. η^ℓ with $1 \leq \ell \leq L$ is supported in some open set Ω^ℓ of size λ that intersects $\partial\Omega$, and such that $\{\Omega^\ell\}_{1\leq\ell\leq L}$ is a covering of $\partial\Omega$;
3. $\|\nabla\eta^\ell\|_{L_\infty} \leq C\lambda^{-1}$ if $1 \leq \ell \leq L$;
4. $\sum_{\ell=0}^{L} \eta^\ell \equiv 1$ on Ω.

If Ω is bounded then (1) has to be replaced with:

(1′) η^0 is supported in a compact subset K of Ω that does not intersect $\partial\Omega$.

Let $P^\ell := \eta^\ell P$ and $b^\ell := \eta^\ell b$ with $b := \partial_{\vec{n}}P|_{\partial\Omega}$. It is clear that P^ℓ fulfills

$$\begin{cases} \Delta P^\ell = 2\,\mathrm{div}\,(P\nabla\eta^\ell) - P\Delta\eta^\ell & \text{in } \Omega, \\ \partial_{\vec{n}}P^\ell = b^\ell & \text{on } \partial\Omega. \end{cases} \tag{3.55}$$

For $\ell = 0$ one can recast the problem in the whole space since $\mathrm{Supp}\,\eta^0 \cap \partial\Omega = \varnothing$. To estimate P^0 in terms of b and P, decompose ∇P^0 into $\nabla P^0_1 + \nabla P^0_2$, where

$$\nabla P^0_1 = -2(-\Delta)^{-1}\nabla\,\mathrm{div}\,(P\nabla\eta^0) \quad \text{and} \quad \Delta P^0_2 = -P\Delta\eta^0 \quad \text{in } \mathbb{R}^n.$$

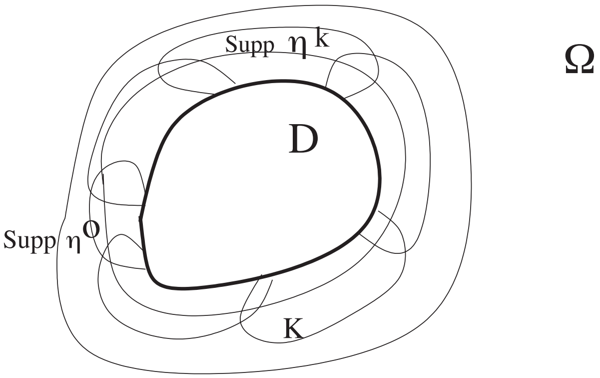

FIGURE 3.4.1. The subset K and the partition of unity $(\eta^k)_{0\le k\le L}$ of Ω

Because $\operatorname{Supp}\nabla\eta^0\subset K$ and $\dot B^\sigma_{p,q}(\mathbb{R}^n)$ is stable by multiplication by $\mathcal{C}^\infty_c(\mathbb{R}^n)$ functions, Corollary 2.1.1 and Proposition 2.1.3 ensure that

$$\|P\nabla\eta^0\|_{\dot B^\sigma_{p,q}(\Omega)}\lesssim\|\widetilde P\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}\quad\text{for any extension }\ \widetilde P\ \text{ of }\ P|_K\ \text{ on }\ \mathbb{R}^n,$$

and thus

$$\|P\nabla\eta^0\|_{\dot B^\sigma_{p,q}(\Omega)}\le C\|P\|_{B^\sigma_{p,q}(K)}.$$

As $P\nabla\eta^0$ is compactly supported away from $\partial\Omega$, it may be seen as a function of $B^{\sigma'}_{r,q}(\mathbb{R}^n)$ for all $\sigma'\le\sigma$ and $1\le r\le p$. Proposition 2.1.3 thus yields for all $1\le r\le p$ and $-n/r'<\sigma'\le\sigma$:

$$\|P\nabla\eta^0\|_{\dot B^{\sigma'}_{r,q}(\mathbb{R}^n)}\le C\|P\|_{B^\sigma_{p,q}(K)}.$$

Then applying Lemma 3.1.2 yields $\nabla P^0_1\in\dot B^{\sigma'}_{r,q}(\mathbb{R}^n)$ and thus also, by embedding, $P^0_1\in L_m(\mathbb{R}^n)$ for all $n/(n-1)<m<n/\max(0,-1-\sigma+n/p)$.

To estimate P^0_2, we aim at taking advantage of Lemma 3.2.1. This is possible because the following compatibility condition

$$\int_{\mathbb{R}^n}P\Delta\eta^0\,dx=0$$

is satisfied since P is harmonic and goes to zero at infinity.

Now, arguing as above we get $P\Delta\eta^0\in\dot B^\sigma_{p,q}(\mathbb{R}^n)$ and

$$\|P\Delta\eta^0\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}\le C\|P\|_{B^\sigma_{p,q}(K)}.$$

Hence Lemma 3.1.2 yields $P^2_0\in L_m(\mathbb{R}^n)$ for all m satisfying (3.32) as well as $\nabla P^2_0\in\dot B^\sigma_{p,q}(\mathbb{R}^n)$, and we have the following inequality

$$\|P^2_0\|_{L_m(\mathbb{R}^n)}+\|\nabla P^2_0\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}\le C\|P\|_{B^\sigma_{p,q}(K)}.$$

Putting together the inequalities for P_1^0 and P_0^2, one may thus conclude that

$$\|P^0\|_{L_m(\mathbb{R}^n)} + \|\nabla P^0\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)} \leq C\|P\|_{B^\sigma_{p,q}(K)}. \tag{3.56}$$

Let us now consider boundary terms (viz. $\ell \in \{1, \dots, L\}$), keeping in mind that P^ℓ is compactly supported in K, and thus belongs to $B^{1+\sigma}_{p,q}(\Omega)$.

We want to perform a change of variables in order to transform (3.46) into a Neumann problem in the half-space. To keep track of the information at the boundary, we use the *normal preserving* change of coordinates $z = Z^\ell(x)$ (see Chapter 2 and [**43**] for details), and get:

$$\begin{cases} \Delta_z \overline{P^\ell} = 2\,\overline{\mathrm{div}_x\,(P\nabla_x \eta^\ell)} - \overline{P\Delta_x \eta^\ell} + (\Delta_z - \Delta_x)\overline{P^\ell} & \text{in} \quad \mathbb{R}^n_+, \\ \partial_{z_n}\overline{P^\ell}|_{z_n=0} = \overline{b^\ell} & \text{on} \quad \partial\mathbb{R}^n_+. \end{cases} \tag{3.57}$$

Setting $\mathcal{B}^\ell := DZ_\ell \circ Z_\ell^{-1}$ and $\mathcal{A}^\ell := \mathcal{B}^\ell - \mathrm{Id}$, and taking into account that $\mathrm{div}_x = {}^T\mathcal{B}^\ell : \nabla_z$ the above system recasts in

$$\begin{cases} \Delta_z \overline{P^\ell} = \mathrm{div}_z\, k^\ell + g^\ell & \text{in} \quad \mathbb{R}^n_+, \\ \partial_{z_n}\overline{P^\ell}|_{z_n=0} = \overline{b^\ell} & \text{on} \quad \partial\mathbb{R}^n_+ \end{cases} \tag{3.58}$$

with

$$\begin{aligned} k^\ell &:= (\mathrm{Id} - \mathcal{B}^{\ell T}\mathcal{B}^\ell)\nabla_z \overline{P^\ell} + \overline{P^\ell}\mathcal{B}^\ell\,\mathrm{div}\,\mathcal{A}^\ell + 2\,{}^T\mathcal{B}^\ell\,\overline{P\nabla_x \eta^\ell}, \\ g^\ell &:= -\overline{P^\ell}\mathcal{B}^\ell\,\mathrm{div}\,{}^T\mathcal{A}^\ell - \overline{P\Delta_x \eta^\ell}. \end{aligned}$$

Note that by construction of $\overline{P^\ell}$, we have [(5)]

$$-\int_{\partial\mathbb{R}^n_+} \overline{b}^\ell\,dx' = \int_{\mathbb{R}^n_+} g^\ell\,dx' - \int_{\partial\mathbb{R}^n_+} k^\ell_n\,dx'. \tag{3.59}$$

That compatibility condition will be important in the sequel.

We plan to bound $\|\nabla\overline{P}^\ell\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}$ according to Lemma 3.3.3 (compactly supported case). This requires our writing g^ℓ and the boundary condition $\overline{b}^\ell$ in terms of the generalized divergence operator $\mathcal{DIV}$. To achieve it, let us consider the following problem

$$\begin{cases} \mathrm{div}\, L^\ell = E_{\mathrm{ant}} g^\ell & \text{in} \quad B(0,\lambda), \\ L^\ell = 0 & \text{on} \quad \partial B(0,\lambda), \end{cases} \tag{3.60}$$

where E_{ant} denotes the antisymmetric extension operator.

5. The minus sign is due to the downward orientation of the exterior normal on $\partial\mathbb{R}^n_+$.

The fact that $-1+1/p<\sigma<1/p$ ensures that $E_{\rm ant}g^\ell\in\dot B^\sigma_{p,q}(\mathbb{R}^n)$ and that

$$\|E_{\rm ant}g^\ell\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}\leq C\|g^\ell\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

In addition, by construction, we have

$$\operatorname{Supp} E_{\rm ant}g^\ell\subset\overline{B(0,\lambda)}\quad\text{and}\quad\int_{B(0,\lambda)}E_{\rm ant}g^\ell\,dx=0.$$

Therefore Theorem 2.3.1 ensures that (3.60) has a solution L^ℓ such that

$$\|L^\ell\|_{B^{1+\sigma}_{p,q}(B(0,\lambda))}\lesssim\|E_{\rm ant}g^\ell\|_{B^\sigma_{p,q}(B(0,\lambda))}\lesssim\|g^\ell\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

As, by construction,

$$L^\ell\in B^{1+\sigma}_{p,q}(B(0,\lambda))\quad\text{and}\quad L^\ell=0\ \text{ on }\ \partial B(0,\lambda),$$

the function

$$\widetilde{L^\ell}:=\begin{cases}L^\ell & \text{in}\quad B(0,\lambda),\\ 0 & \quad\text{elsewhere}\end{cases}$$

belongs to $B^{\sigma+1}_{p,q}(\mathbb{R}^n)$ and satisfies

$$\|\widetilde{L^\ell}\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n)}\lesssim\|g^\ell\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.\tag{3.61}$$

So, setting $H^\ell:=k^\ell+\widetilde{L^\ell}$, we have

$$\operatorname{div}H^\ell=\operatorname{div}k^\ell+g^\ell\quad\text{in}\quad\mathbb{R}^n_+$$

and solving (3.1) thus recasts in

$$\mathcal{DIV}\,[\nabla\overline{P^\ell};0]=\mathcal{DIV}\,[H^\ell;-H^\ell_n+\overline{b^\ell}].$$

In order to apply Lemma 3.3.3, it suffices to establish that

$$H^\ell\in\dot B^\sigma_{p,q}(\mathbb{R}^n_+)\quad\text{and}\quad(H^\ell_n-\overline{b^\ell})\in\dot B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+).$$

Because H^ℓ is compactly supported, the first condition is equivalent to $H^\ell\in B^\sigma_{p,q}(\mathbb{R}^n_+)$. Likewise, since $\sigma-1/p>-1-(n-1)/p'$ (which is equivalent to $\sigma>-n/p'$), the second condition is equivalent to $(H^\ell_n-\overline{b^\ell})\in B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)$. This latter property will come up as a consequence of the trace theorem (as regards H^ℓ_n) and of the stability of nonhomogeneous spaces by right composition (see Lemma 2.1.1) and localization, as regards $\overline{b^\ell}$.

As a final consequence, because (3.59) ensures that

$$\int_{\partial\mathbb{R}^n_+}(\overline{b^\ell}-H^\ell_n)\,dx'=0,\tag{3.62}$$

we will get thanks to Lemma 3.3.3 that

$$\|\nabla\overline{P^\ell}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}\lesssim\|H^\ell\|_{B^\sigma_{p,q}(\mathbb{R}^n_+)}+\|(\overline{b^\ell}-k^\ell_n-\widetilde{L}^\ell_n)|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}.\tag{3.63}$$

The important and nice fact is that our normal preserving change of coordinates gives $\mathcal{B}^{\ell T}\mathcal{B}^{\ell}\nabla_z\overline{P^\ell}\cdot\vec{e}_n|_{\partial\mathbb{R}^n_+}=\overline{b^\ell}$, and thus the highest order term of k_n^ℓ vanishes at the boundary. Therefore we have

$$\|k_n^\ell|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\lesssim\|\mathcal{B}^\ell(\operatorname{div}\mathcal{A}^\ell)\overline{P^\ell}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}+\|{}^T\mathcal{B}^\ell\overline{P\nabla_x\eta^\ell}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}.$$

Of course, any space $B^{\varepsilon}_{p,q}(\partial\mathbb{R}^n_+)$ with $\varepsilon>0$ embeds in $B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)$, so that the trace theorem implies that

$$\|k_n^\ell|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\lesssim\|\mathcal{B}^\ell(\operatorname{div}\mathcal{A}^\ell)\overline{P^\ell}\|_{B^{\varepsilon+1/p}_{p,q}(\mathbb{R}^n_+)}+\|{}^T\mathcal{B}^\ell\overline{P\nabla_x\eta^\ell}\|_{B^{\varepsilon+1/p}_{p,q}(\mathbb{R}^n_+)}.$$

As nonhomogeneous Besov spaces are stable by multiplication by compactly supported smooth functions, we thus deduce that

$$\|k_n^\ell|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\lesssim\|\overline{P^\ell}\|_{B^{\varepsilon+1/p}_{p,q}(\mathbb{R}^n_+)}+\lambda^{-1}\|P\|_{B^{\varepsilon+1/p}_{p,q}(K)}.$$

Note that one may take some $\varepsilon>0$ such that $\sigma<\varepsilon+1/p<\sigma+1$. So arguing by interpolation, we conclude that for any small enough α, we have

$$\|(k_n^\ell-\overline{b^\ell})|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\leq\alpha\|\nabla\overline{P^\ell}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}+C_\alpha\|\overline{P^\ell}\|_{B^{\sigma}_{p,q}(\mathbb{R}^n_+)}+C\lambda^{-1}\|P\|_{B^{\epsilon+1/p}_{p,q}(K)}+\|\overline{b^\ell}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}.\tag{3.64}$$

Next, we see that the trace theorem, (3.61) and the definition of g^ℓ imply

$$\|(\widetilde{L}_n^\ell)|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\lesssim\|(\operatorname{div}{}^TA^\ell)\overline{P^\ell\nabla_x\eta^\ell}\|_{B^{\sigma}_{p,q}(\mathbb{R}^n_+)}+\|\overline{P\Delta_x\eta^\ell}\|_{B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

So we get

$$\|(\widetilde{L}_n^\ell)|_{z_n=0}\|_{B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\lesssim\lambda^{-1}\|\overline{P^\ell}\|_{B^{\sigma}_{p,q}(\mathbb{R}^n_+)}+\lambda^{-2}\|P\|_{B^{\sigma}_{p,q}(K)}.$$

Let us now estimate H^ℓ. Most of the terms entering in its definition have already been bounded above. The only definitely new term is $(\mathrm{Id}-\mathcal{B}^{\ell T}\mathcal{B}^\ell)\nabla_z\overline{P^\ell}$. Now, from product estimates, we get (see Proposition 2.1.2)

$$\|(\mathrm{Id}-\mathcal{B}^{\ell T}\mathcal{B}^\ell)\nabla_z\overline{P^\ell}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}\lesssim\|\mathrm{Id}-\mathcal{B}^{\ell\,T}\mathcal{B}^\ell\|_{\dot B^{n/p'}_{p',1}(\mathbb{R}^n_+)}\|\nabla_z\overline{P^\ell}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

According to (2.29), we have

$$\|\mathrm{Id}-\mathcal{B}^{\ell\,T}\mathcal{B}^\ell\|_{\dot B^{n/p'}_{p',1}(\mathbb{R}^n_+)}\leq C\lambda.$$

Hence one may conclude that

$$\|H^\ell\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}\lesssim\lambda\|\nabla_z\overline{P^\ell}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}+\lambda^{-2}\|P\|_{B^{\sigma}_{p,q}(K)}+\lambda^{-1}\|\overline{P^\ell}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Plugging all the previous estimates in (3.63) (take $\alpha=\lambda$ in (3.64)), we get

$$\|\nabla_z\overline{P^\ell}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}\lesssim\lambda\|\nabla_z\overline{P^\ell}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}+\lambda^{-2}\|P\|_{B^\sigma_{p,q}(K)}$$
$$+\|\overline{b^\ell}\|_{B^{\sigma-\frac1p}_{p,q}(\partial\mathbb{R}^n_+)}+C_\lambda\|\overline{P^\ell}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

Of course, the first term of the r.h.s may be absorbed by the l.h.s if taking λ small enough. Hence one may write (see Lemma 2.1.1)

$$\|\nabla P\|_{\dot B^\sigma_{p,q}(\Omega)}\lesssim\sum_\ell\|\nabla P^\ell\|_{\dot B^\sigma_{p,q}(\Omega)}\lesssim\|\nabla P^0\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}+\sum_{\ell\geq1}\|\nabla\overline{P^\ell}\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n_+)},$$

and a similar inequality for the norm of P in $L_m(\Omega)$.

So from (3.56) and the above inequalities, we get for all m satisfying (3.32)

$$\|P\|_{L_m(\Omega)}+\|\nabla P\|_{\dot B^\sigma_{p,q}(\Omega)}\leq C_\lambda\Big(\|P\|_{B^\sigma_{p,q}(K)}+\|b\|_{B^{\sigma-\frac1p}_{p,q}(\partial\Omega)}\Big).\tag{3.65}$$

Note that C_λ blows up as $\lambda\to0$, but remains finite for all $\lambda>0$, since the sum is finite.

Next we want to prove (3.54). We claim that

$$\|P\|_{B^\sigma_{p,q}(K)}\leq C\|\partial_{\vec n}P\|_{B^{\sigma-1/p}_{p,q}(\partial\Omega)}.\tag{3.66}$$

Let us first consider the (easier) bounded domain case, taking $K=\Omega$ with no loss of generality. We argue by contradiction and assume that (3.66) fails. Then there exists a sequence $(P_k)_{k\in\mathbb{N}}$ of harmonic functions in Ω with average 0, and such that

$$1=\|P_k\|_{B^\sigma_{p,q}(\Omega)}>k\|\partial_{\vec n}P_k\|_{B^{\sigma-\frac1p}_{p,q}(\partial\Omega)}.\tag{3.67}$$

By (3.53), this implies that $(P_k)_{k\in\mathbb{N}}$ is bounded in $B^{1+\sigma}_{p,q}(\Omega)$. Since this latter space is compactly embedded in $L_p(\Omega)$, we deduce that there exists a function $P^*\in B^{1+\sigma}_{p,q}(\Omega)$, and some subsequence $(P_{k_n})_{n\in\mathbb{N}}$ so that

$$P_{k_n}\to P^*\text{ in }L_p(\Omega).\tag{3.68}$$

Note that we also have, owing to (3.67),

$$\partial_{\vec n}P_{k_n}|_{\partial\Omega}\to0\text{ in }B^{\sigma-\frac1p}_{p,q}(\partial\Omega).$$

So finally, $P^*\in B^{1+\sigma}_{p,q}(\Omega)$ must fulfill the system

$$\begin{cases}\Delta P^*=0 & \text{in} & \Omega,\\ \partial_{\vec n}P^*=0 & \text{on} & \partial\Omega,\end{cases}\qquad\int_\Omega P^*\,dx=0,$$

the only solution of which is $P^* \equiv 0$, as already pointed out in (3.24) (note that the computations therein just require P^* to be in $B^{1+\sigma}_{p,q}(\Omega)$). Now, the strong convergence given by (3.68) implies that $\|P^*\|_{L_p(\Omega)} = 1$, a contradiction. Hence (3.50) has been proved if Ω is a bounded domain.

Let us now assume that Ω is an exterior domain. As before, we argue by contradiction and suppose that there exists a sequence $(P_k)_{k\in\mathbb{N}}$ of harmonic functions going to 0 at infinity, with gradient in $\dot B^\sigma_{p,q}(\Omega)$ and such that

$$1 = \|P_k\|_{B^\sigma_{p,q}(K)} > k\|\partial_{\vec n} P_k\|_{B^{\sigma-\frac1p}_{p,q}(\partial\Omega)}.$$

By (3.53), we thus get

1. $(P_k)_{k\in\mathbb{N}}$ bounded in $L_m(\Omega)$ for all m satisfying (3.32),
2. $(\nabla P_k)_{k\in\mathbb{N}}$ is bounded in $\dot B^\sigma_{p,q}(\Omega)$,
3. $(P_k)_{k\in\mathbb{N}}$ is bounded in $B^{\sigma+1}_{p,q}(K)$.

Since the embedding of $B^{\sigma+1}_{p,q}(K)$ in $B^\sigma_{p,q}(K)$ is compact and as the Besov spaces have the Fatou property, we get some subsequence $(P_{k_n})_{n\in\mathbb{N}}$ and a function $P^* : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &P_{k_n} \to P^* \ \text{ in } \ B^\sigma_{p,q}(K),\\ &P_{k_n} \rightharpoonup P^* \ \text{ in } \ L_m(\Omega) \ \text{ and } \ \nabla P_{k_n} \rightharpoonup \nabla P^* \ \text{ in } \ \dot B^\sigma_{p,q}(\Omega). \end{aligned} \tag{3.69}$$

By construction, $\Delta P_{k_n} = 0$ for all $n \in \mathbb{N}$ and $\partial_{\vec n} P_{k_n}|_{\partial\Omega} \to 0$ in $B^{\sigma-\frac1p}_{p,q}(\partial\Omega)$. Therefore

$$\begin{cases} \Delta P^* = 0 & \text{in} \quad \Omega,\\ \partial_{\vec n} P^* = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.70}$$

We claim that

$$\nabla P^* \in L_2(\Omega) \quad \text{and} \quad P^* \to 0 \ \text{ at } \ \infty. \tag{3.71}$$

If (3.71) holds true then the standard energy argument implies that $P^* \equiv 0$ which contradicts the strong convergence in $B^\sigma_{p,q}(K)$, and thus completes the proof of (3.66).

Of course $P^* \to 0$ is ensured by $P^* \in L_m(\Omega)$. So, as a first step toward (3.71), let us establish that

$$\nabla P^* \in B^{\sigma+1}_{p,q}(\Omega). \tag{3.72}$$

To this end, using again the cut-off function η^0 (see Fig. 3.4), we see that on the one hand

$$\begin{aligned} \Delta((1-\eta^0)P^*) = -2\nabla\eta^0\cdot\nabla P^* - P^*\Delta\eta^0 \quad \text{in} \quad K, \\ \partial_{\vec{n}}(1-\eta^0)P^* = 0 \quad \text{on} \quad \partial K \end{aligned} \tag{3.73}$$

and that on the other hand,

$$\Delta(\eta^0 P^*) = 2\nabla\eta^0\cdot\nabla P^* + P^*\Delta\eta^0 \quad \text{in} \ \ \mathbb{R}^n. \tag{3.74}$$

Note that $P^*\in B^{\sigma+1}_{p,q}(K)$ ensures that the r.h.s. of (3.73) is in $B^{\sigma}_{p,q}(K)$. Hence Proposition 3.2.1 guarantees that $(1-\eta^0)P^*\in B^{\sigma+2}_{p,q}(K)$. Likewise, the r.h.s. of (3.74) is in $B^{\sigma}_{p,q}(\mathbb{R}^n)$ and supported in K. Hence Lemma 3.2.1 gives us $\nabla(\eta^0 P^*), \nabla^2(\eta^0 P^*)\in \dot{B}^{\sigma}_{p,q}(\Omega)$. Of course, being compactly supported the r.h.s. also belong to $B^{\sigma'}_{p,q}(K)$ with $-1/p'<\sigma'\le\sigma$ and thus we also have $\nabla(\eta^0P^*)$ and $\nabla^2(\eta^0 P^*)$ in $\dot{B}^{\sigma'}_{p,q}(\Omega)$ for some negative σ', which eventually implies (3.72). We can thus assume from now on with no loss of generality that $0<\sigma<1/p$, which allows to conclude that (3.71) holds true if $p\ge 2$ or, more generally, if $p<2$ and $\sigma+1>n/p-n/2$ (by embedding in $B^{1+\sigma+n/2-n/p}_{2,q}(\Omega)$).

If $\sigma+1\le n/p-n/2$ then we resume to the above argument starting with the regularity $\dot{B}^{1+\sigma+n/2-n/p}_{2,q}(\Omega)$ instead of $\dot{B}^{\sigma}_{p,q}(\Omega)$, and we end up with $\nabla P^*\in B^{2+\sigma+n/2-n/p}_{2,q}(\Omega)$. This yields (3.71) if $\sigma+2>n/p-n/2$. It is now clear that it is always possible to achieve $\nabla P^*\in L_2(\Omega)$ within a finite number of steps.

Of course, the proof of uniqueness of a solution to (3.46) reduces to the study of (3.70) and thus works exactly the same.

Let us finally say a few words about the proof of existence. Here only $n\ge 3$ is needed. If the boundary is smooth then we may use the L_2 approach. Take a sequence of smooth functions $b_k\in C^\infty(\partial\Omega)$ such that $b_k\to b$ in $B^{\sigma-\frac{1}{p}}_{p,1}(\partial\Omega)$. For each b_k we are able to construct a smooth solution such that $\nabla P_k\in L_2(\Omega)$ (via the Lax-Milgram theorem). In particular, if $n\ge3$ then, owing to Sobolev embedding, $P_k\in L_{\frac{2n}{n-2}}(\Omega)$ so that $P_k\to 0$ at infinity. Furthermore, P_k satisfies (3.65). Then, passing to the limit we get the existence of our solution in the desired class of regularity, and (3.51) and (3.52) are fulfilled. □

Remark 3.4.1. — As regards the proof of existence in an exterior domain of $\mathbb{R}^2$, the simple argument just below does not guarantee that $P_k\to 0$ at ∞, although we expect the compatibility condition (3.49) to ensure decay to 0 (see e.g. [**5**]). As the restriction $n\ge 3$ will appear elsewhere when investigating the evolutionary Stokes system in exterior domains, we here prefer to omit a more detailed study of the two-dimensional case.

3.5. Helmholtz projection

Function spaces with the divergence-free property naturally arise in the mathematical theory of incompressible flows. Those spaces may be obtained as the image of a suitable continuous projection operator on a space of *vector valued* functions over the domain Ω. Such an operator $\mathcal{P} : X \to X$ is often called *Helmholtz or Leray projector* and has the property that for any $f \in X$,

$$\operatorname{div} \mathcal{P}f = 0 \quad \text{in} \quad \Omega, \qquad \text{and} \qquad \mathcal{P}f \cdot \vec{n} = 0 \quad \text{at} \quad \partial\Omega.$$

Formally, $\mathcal{P}$ may be defined by $\mathcal{P}f := f - \nabla P$ where ∇P is a solution to

$$\begin{cases} \Delta P = \operatorname{div} f & \text{in} \quad \Omega, \\ (\nabla P - f) \cdot \vec{n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.75}$$

The proposition below gives a suitable functional framework for solving (3.75).

Proposition 3.5.1. — *Let Ω be either an exterior domain of $\mathbb{R}^n$ (with $n \geq 3$), or a bounded domain, the whole space or the half-space with $n \geq 2$. Assume that f is in $\dot{B}^{\sigma}_{p,q}(\Omega)$ for some $p \in (1, \infty)$, $q \in [1, \infty]$ and $\sigma \in (-1 + 1/p, 1/p)$. Then* (3.75) *has a solution ∇P in $\dot{B}^{\sigma}_{p,q}(\Omega)$ with*

$$\|\nabla P\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} \leq C\|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)}. \tag{3.76}$$

Proof. — In the whole space case, the solution is provided by Lemma 3.1.2. If Ω is not $\mathbb{R}^n$ then we fix some $\varepsilon > 0$ and consider an extension $\widetilde{f}$ of f on the whole space $\mathbb{R}^n$ such that

$$\|\widetilde{f}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq \|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon.$$

Lemma 3.1.2 yields some $\nabla \widetilde{P}$ in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ satisfying

$$\Delta \widetilde{P} = \operatorname{div} \widetilde{f} \quad \text{in} \quad \mathbb{R}^n,$$

and

$$\|\nabla \widetilde{P}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|\widetilde{f}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

By construction, $\operatorname{div}(\widetilde{f} - \nabla \widetilde{P}) = 0$ in $\mathbb{R}^n$ and $\widetilde{f} - \nabla \widetilde{P}$ is in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$. Hence $(\widetilde{f} - \nabla \widetilde{P}) \cdot \vec{n}$ has a trace at $\partial\Omega$ (this is Lemma 2.2.4) and

$$\|(\widetilde{f} - \nabla \widetilde{P}) \cdot \vec{n}\|_{\dot{B}^{\sigma - \frac{1}{p}}_{p,q}(\partial\Omega)} \leq C\|\widetilde{f} - \nabla \widetilde{P}\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} \leq C(\|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon).$$

Therefore, setting $P := \widetilde{P} + P_{\text{new}}$, we see that P_{new} has to satisfy (3.46) with boundary data $b := (f - \nabla\widetilde{P}) \cdot \vec{n}$. Note that because $\operatorname{div}(f - \nabla\widetilde{P}) = 0$ in Ω, the compatibility condition for b holds. Then defining P_{new} according to Theorem 3.4.1, we find that $\nabla P_{\text{new}} \in \dot{B}^{\sigma}_{p,q}(\Omega)$ and that

$$\|\nabla P_{\text{new}}\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} \leq C\big(\|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon\big).$$

Finally, the half-space case may be easily deduced from Lemma 3.3.3. □

Corollary 3.5.1. — *Under the above assumptions, there exists a continuous Helmholtz projector* $\mathcal{P} : \dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n) \to \dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n)$.

Proof. — Let $\mathcal{P}f := f - \nabla P$ with P given by Proposition 3.5.1. Then we have $\operatorname{div}\mathcal{P}f = 0$ and $\mathcal{P}f \cdot \vec{n} = 0$ at the boundary, and also

$$\|\nabla P\|_{\dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n)}.$$

This completes the proof of the corollary. □

CHAPTER 4

THE EVOLUTIONARY STOKES SYSTEM

This section is devoted to endpoint maximal regularity estimates for the evolutionary Stokes system. First we concentrate on the whole and half-space cases, then we consider the problem in exterior or bounded domains. In the last section, we adapt Maremonti and Solonnikov's trick in [**39**] so as to establish a low order bound for the velocity on a compact set. This will enable us to discard the time dependency in the estimates if choosing the data in appropriate intersections of Besov spaces.

4.1. The whole space case

Here we investigate the following evolutionary Stokes system:

$$\begin{cases} u_t - \nu\Delta u + \nabla P = f & \text{in} \quad (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u = g & \text{in} \quad (0,T)\times\mathbb{R}^n, \\ u|_{t=0} = u_0 & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{4.1}$$

The main result of this part reads:

Theorem 4.1.1. — *Let $p \in (1,\infty)$ and $-1+1/p < s < 1/p$. Let $f \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$, $g \in \mathcal{C}([0,T];\dot B^{s-1}_{p,1}(\mathbb{R}^n))$ with $\nabla g \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot B^s_{p,1}(\mathbb{R}^n)$. Assume in addition that for some $\lambda > 0$, we have*

$$g_t = \operatorname{div} B + A, \ \textit{with} \ \operatorname{Supp} A(t,\cdot) \subset \overline{B(0,\lambda)} \ \textit{and} \ \int_{\mathbb{R}^n} A(t,x)\,dx = 0, \tag{4.2}$$

where $A, B \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$. Finally, suppose that the compatibility condition $\operatorname{div} u_0 = g|_{t=0}$ on $\mathbb{R}^n$ is satisfied.

Then System (4.1) *has a unique solution* $(u, \nabla P)$ *with*

$$u \in \mathcal{C}([0,T); \dot{B}^s_{p,1}(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T; \dot{B}^s_{p,1}(\mathbb{R}^n))$$

and the following estimate is valid:

$$\begin{aligned}(4.3)\quad & \|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \\ & \leq C(\|f, \nu\nabla g, B\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \lambda\|A\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}),\end{aligned}$$

where C *is an absolute constant with no dependence on* ν, T *and* λ.

Proof. — Applying the divergence operator to the first equation of (4.1) and remembering the constraint $g_t = \operatorname{div} B + A$, we see that the pressure is determined by

$$\Delta P = \operatorname{div} f + \nu\Delta g - \operatorname{div} B - A \qquad \text{in} \quad (0,T) \times \mathbb{R}^n.$$

We thus set $\nabla P := \nabla P_0 + \nabla P_1$ with

$$(4.4)\qquad \Delta P_1 = \operatorname{div}(f + \nu\nabla g - B) \qquad \text{in} \quad (0,T) \times \mathbb{R}^n,$$

and

$$\Delta P_0 = -A \qquad \text{in} \quad (0,T) \times \mathbb{R}^n.$$

Determining ∇P_1 may be done according to Lemma 3.1.2 (treating t as a parameter). We get ∇P_1 in $L_1(0,T; \dot{B}^s_{p,1}(\mathbb{R}^n))$ satisfying

$$(4.5)\qquad \|\nabla P_1\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \leq C\|f, \nu\nabla g, B\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))}.$$

Constructing ∇P_0 stems from Lemma 3.2.1 which yields

$$(4.6)\qquad \|\nabla P_0\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \leq C\lambda\|A\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))}.$$

Hence (4.5) and (4.6) give

$$(4.7)\quad \|\nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \lesssim \|f, \nu\nabla g, B\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \lambda\|A\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))}.$$

Now that ∇P has been constructed, we look at u as the solution to the following heat equation:

$$\begin{cases} u_t - \nu\Delta u = f - \nabla P & \text{in} \quad (0,T) \times \mathbb{R}^n, \\ u|_{t=0} = u_0 & \text{on} \qquad\quad \mathbb{R}^n. \end{cases}$$

The endpoint maximal property for the heat equation (see e.g. [**6**], Chap. 2) ensures the existence of u in the desired functional space, together with

$$\begin{aligned}(4.8)\quad & \|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_t, \nu\nabla^2 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \\ & \qquad \lesssim \|f, \nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}.\end{aligned}$$

Let us sketch the proof of (4.8) for the reader convenience. It just follows from the fact that there exist two constants c and C such that for all $j \in \mathbb{Z}$ and $\alpha \in \mathbb{R}^+$ one has (see e.g. [**6**])

$$\|e^{\alpha\Delta}\dot{\Delta}_j h\|_{L_p(\mathbb{R}^n)} \leq Ce^{-c\alpha 2^{2j}}\|\dot{\Delta}_j h\|_{L_p(\mathbb{R}^n)}.$$

Now, as u satisfies

$$\dot{\Delta}_j u(t) = e^{\nu t\Delta}\dot{\Delta}_j u_0 + \int_0^t e^{\nu(t-\tau)\Delta}\dot{\Delta}_j(f - \nabla P)\, d\tau,$$

we readily get

$$\|\dot{\Delta}_j u(t)\|_{L_p(\mathbb{R}^n)} \leq C\Big(e^{-c\nu t 2^{2j}}\|\dot{\Delta}_j u_0\|_{L_p(\mathbb{R}^n)} + \int_0^t e^{-c\nu(t-\tau)2^{2j}}\|\dot{\Delta}_j(f - \nabla P)\|_{L_p(\mathbb{R}^n)}\, d\tau\Big),$$

whence

$$\|\dot{\Delta}_j u\|_{L_\infty(0,T;L_p(\mathbb{R}^n))} + \nu 2^{2j}\|\dot{\Delta}_j u\|_{L_1(0,T;L_p(\mathbb{R}^n))} \lesssim \|\dot{\Delta}_j u_0\|_{L_p(\mathbb{R}^n)} + \|\dot{\Delta}_j(f - \nabla P)\|_{L_1(0,T;L_p(\mathbb{R}^n))}.$$

Multiplying the inequality by 2^{js} and summing up over j yields (4.8). Remembering (4.7) implies the sought inequality (4.3).

To complete the proof of the theorem, one has to check whether the constraint $\operatorname{div} u = g$ is fulfilled on $[0,T)\times\mathbb{R}^n$. Applying the divergence operator to the equation for u and using the definition of ∇P and the assumption on $g|_{t=0}$, we see that

$$\partial_t(\operatorname{div} u - g) - \nu\Delta(\operatorname{div} u - g) = 0, \quad (\operatorname{div} u - g)|_{t=0} = 0.$$

As uniqueness holds true in $\mathcal{C}([0,T);\mathcal{S}'(\mathbb{R}^n))$, we have $\operatorname{div} u - g \equiv 0$ on $[0,T)\times\mathbb{R}^n$ and one may thus conclude that $(u, \nabla P)$ satisfies System (4.1). □

Remark 4.1.1. — By the same token, one may prove that for general $1 < p < \infty$, $-1 + 1/p < s < 1/p$ and $1 \leq q \leq \infty$, if $u_0 \in \dot{B}^{s+2-\frac{2}{q}}_{p,q}(\mathbb{R}^n)$, f and ∇g are in $L_q(0,T;\dot{B}^s_{p,q}(\mathbb{R}^n))$, $g \in \mathcal{C}([0,T);\dot{B}^{s+1-\frac{2}{q}}_{p,q}(\mathbb{R}^n))$ with in addition (4.2) for some A and B in $L_q(0,T;\dot{B}^s_{p,q}(\mathbb{R}^n))$ then $u \in L_\infty(0,T;\dot{B}^{s+2-\frac{2}{q}}_{p,q}(\mathbb{R}^n))$ and $(\partial_t u, \nabla^2 u, \nabla P) \in L_q(0,T;\dot{B}^s_{p,q}(\mathbb{R}^n))$ with an estimate similar to (4.3).

4.2. The Stokes system in the half-space

The purpose of this part is to extend Theorem 4.1.1 to the half-space setting $\mathbb{R}^n_+$. We thus consider

$$\begin{cases} u_t - \nu\Delta u + \nabla P = f & \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} u = g & \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ u|_{x_n=0} = 0 & \text{on} \quad (0,T)\times\partial\mathbb{R}^n_+,\\ u|_{t=0} = u_0 & \text{on} \quad \mathbb{R}^n_+. \end{cases} \tag{4.9}$$

This section is devoted to the proof of the following statement:

Theorem 4.2.1. — *Let $p\in(1,\infty)$ and $s\in(-1+1/p,1/p)$. Assume that $f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ and that $u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0=0$ in $\mathbb{R}^n_+$ and $u_0\cdot\vec e_n|_{\partial\mathbb{R}^n_+}\equiv 0$. Further assume that $g=\operatorname{div} R$ for some $R\in\mathcal{C}(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$, that $\nabla g\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ and that $g(0)=0$. Finally, we suppose that*

$$g_t = A + \mathcal{DIV}\,[B,b] \tag{4.10}$$

for some $b\in L_1(0,T;\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ and $A,B\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ with $\operatorname{Supp} A(t,\cdot)\subset\overline{B(0,\lambda)\cap\mathbb{R}^n_+}$ for some $\lambda>0$.

Then System (4.9) has a unique solution $(u,\nabla P)$ with

$$u\in\mathcal{C}_b([0,T);\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad u_t,\nabla^2u,\nabla P\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$$

and the following estimate is valid:

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &+ \|u_t,\nu\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\le C\Big(\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} + \|f,\nu\nabla g,B\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\qquad + \lambda\|A\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|b\|_{L_1(0,T;\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\Big),\end{aligned}$$

where C is an absolute constant with no dependence on ν, T and λ.

Remark 4.2.1. — The case where one prescribes the trace of u at the boundary to be equal to some nonzero given h reduces to the homogeneous situation, if assuming that h admits some extension $\widetilde h$ over $(0,T)\times\mathbb{R}^n_+$ so that $\widetilde h_t-\nu\Delta\widetilde h$ (resp. $\operatorname{div}\widetilde h$) satisfies the same assumptions as f (resp. g).

The proof of Theorem 4.2.1 is based essentially on the results of [**15**] concerning the case $g\equiv 0$ and on our recent work in [**16**] so as to handle the nonhomogeneous divergence constraint. Recall the statement for $g\equiv 0$:

THEOREM 4.2.2. — *If $g \equiv 0$ and u_0, f fulfill the assumptions of Theorem 4.2.1 then System (4.9) has a unique solution $(u, \nabla P)$ satisfying*

$$u \in \mathcal{C}_b([0,T); \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad u_t, \nabla^2 u, \nabla P \in L_1(0,T; \dot{B}^s_{p,1}(\mathbb{R}^n_+)).$$

Besides, the following estimate is valid:

$$\|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq C(\|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}), \tag{4.11}$$

where C is an absolute constant with no dependence on ν and T.

Proof of Theorem 4.2.1. — As a first step, let us reduce our study to the case $u_0 \equiv 0$ and $f \equiv 0$. To this end, solve System (4.9) with $g \equiv 0$, according to Theorem 4.2.2. We get a solution $(u_1, \nabla P_1)$ satisfying (4.11). Now, setting

$$u = u_{\text{new}} + u_1 \quad \text{and} \quad \nabla P = \nabla P_{\text{new}} + \nabla P_1, \tag{4.12}$$

we see that $(u_{\text{new}}, \nabla P_{\text{new}})$ satisfies System (4.9) with $f \equiv 0$ and $u_0 \equiv 0$ and the same g (since $\operatorname{div} u_1 = 0$). Additionally one may extend the system on the whole time line, setting $u_{\text{new}} = \nabla P_{\text{new}} = 0$ as well as $f_{\text{new}} = g_{\text{new}} = 0$ for $t < 0$. Using the fact that $u_0 \equiv 0$, we eventually get (dropping the index *new* for simplicity),

$$\begin{cases} u_t - \nu\Delta u + \nabla P = 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}^n_+, \\ \operatorname{div} u = g & \text{in} \quad \mathbb{R} \times \mathbb{R}^n_+, \\ u|_{x_n=0} = 0 & \text{on} \quad \mathbb{R} \times \partial\mathbb{R}^n_+. \end{cases} \tag{4.13}$$

Let us emphasize that, owing to $g(0) = 0$, (4.10) is now satisfied on $\mathbb{R} \times \mathbb{R}^n_+$ (if extending A and B by 0 for negative t of course). □

In what follows we thus concentrate on the proof of the following lemma:

LEMMA 4.2.1. — *Let $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Let $g = \operatorname{div} R$ with $R \in \mathcal{C}(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ satisfying in addition $\nabla g \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $g_t = A + \mathcal{DIV}\,[B, b]$ with $A, B \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, $b \in L_1(\mathbb{R}; \dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ and, for some $\lambda > 0$,*

$$\operatorname{Supp} A(\cdot, t) \subset \overline{B(0,\lambda) \cap \mathbb{R}^n_+} \quad a.e.\ t \in \mathbb{R}.$$

Then System (4.13) has a unique solution $(u, \nabla P)$ with $u \in \mathcal{C}_b(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, $u_t, \nabla^2 u, \nabla P \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n))$. Furthermore, there exists some constant C so *that*

$$\begin{aligned}&\|u\|_{L_\infty(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|u_t,\nu\nabla^2u,\nabla P\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\leq C\big(\|b\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}+\|\nu\nabla g,B\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\big).\end{aligned}$$

Proof. — Performing the rescaling

$$(u_{\rm new},P_{\rm new},g_{\rm new})(t,x)=(\nu u_{\rm old},P_{\rm old},\nu g_{\rm old})(\nu^{-1}t,x),\tag{4.14}$$

we see that $(u_{\rm new},P_{\rm new})$ satisfies System (4.13) with $\nu=1$. Hence one may assume with no loss of generality that $\nu=1$. As a preliminary step, we want to discard the source term g. To this end, we define for all $t\in\mathbb{R}$ the function $w(t)$ to be the solution of

$$\begin{cases}\Delta w(t)=g(t)=\operatorname{div}R(t) & \text{in}\quad \mathbb{R}^n_+,\\ w(t)|_{x_n=0}=0 & \text{on}\quad \partial\mathbb{R}^n_+.\end{cases}\tag{4.15}$$

Applying the last part of Lemma 3.3.1 provides us with a solution w satisfying $\nabla w\in\mathcal{C}(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))$ and for all $t\in\mathbb{R}$,

$$\|\nabla w(t)\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\lesssim\|R(t)\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.\tag{4.16}$$

As by construction w is the restriction to the half-space of $G\star\widetilde g$ (where G is the fundamental solution of Δ and $\widetilde g$, the antisymmetric extension of g), applying the fourth item of Lemma 3.3.1 yields the following additional estimate:

$$\|\nabla^3 w\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|\nabla g\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}.\tag{4.17}$$

Differentiating (4.15) with respect to the time variable and using the assumption on g_t, we also discover that

$$\begin{cases}\Delta w_t=g_t=A+\operatorname{div}B & \text{in}\quad \mathbb{R}^n_+,\\ w_t|_{x_n=0}=0 & \text{on}\quad \partial\mathbb{R}^n_+.\end{cases}\tag{4.18}$$

Hence using the last part of Lemma 3.3.1 and Corollary 3.3.1 to handle the parts of w_t coming from B and A, respectively, we end up with

$$\|\nabla w_t\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq C\big(\|B\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\big).\tag{4.19}$$

Then we look for a solution $(u,\nabla P)$ to (4.13) with $\nu=1$ in the following form:

$$u=u_{\rm new}+\nabla w,\qquad \nabla P=\nabla P_{\rm new}-\nabla w_t+\nabla g.$$

Dropping the index *new*, we thus get the following system [1]:

$$\left\{\begin{array}{lll} u_t - \Delta u + \nabla P = 0 & \text{in} & \mathbb{R}\times\mathbb{R}^n_+, \\ \operatorname{div} u = 0 & \text{in} & \mathbb{R}\times\mathbb{R}^n_+, \\ u|_{x_n=0} = -\nabla w|_{x_n=0} & \text{on} & \mathbb{R}\times\partial\mathbb{R}^n_+. \end{array}\right. \tag{4.20}$$

Let (ξ_0,ξ') denote the Fourier variables for the Fourier transform $\mathcal{F}_{t,x'}$ with respect to t and x'. We claim that the pressure P obeys the formula

$$\widehat{P}(\xi_0,\xi',x_n) := \mathcal{F}_{t,x'}P(\xi_0,\xi',x_n) = \widehat{P}_b(\xi_0,\xi')e^{-|\xi'|x_n}, \tag{4.21}$$

with

$$\widehat{P}_b(\xi_0,\xi') := -\Big(\frac{i\xi_0}{|\xi'|} + r + |\xi'|\Big)\widehat{\partial_{x_n} w}|_{x_n=0} \quad \text{and} \quad r^2 := i\xi_0 + |\xi'|^2. \tag{4.22}$$

Indeed, it is only a matter of looking at (4.20) as the following heat equation:

$$u_t - \Delta u = -\nabla P, \qquad u|_{x_n=0} = -\nabla w|_{x_n=0}.$$

Then taking the Fourier transform with respect to time and tangential directions, and remembering that our functional framework requires u to tend to 0 at infinity, we obtain from the standard theory of linear ordinary differential equations,

$$\begin{aligned}\widehat{u}(\xi_0,\xi',x_n) &= \widehat{u}(\xi_0,\xi',0)e^{-rx_n} \\ &\quad + \frac{1}{2r}\int_0^\infty [e^{-r|x_n-s_n|} - e^{-r(x_n+s_n)}]\begin{pmatrix} -i\xi' \\ |\xi'| \end{pmatrix}\widehat{P}_b(\xi_0,\xi')e^{-|\xi'|s_n}\,ds_n.\end{aligned}$$

So differentiating the n-th component with respect to x_n and letting x_n go to 0 gives (see [**15**] for more details):

$$\partial_{x_n}\widehat{u}_n(\xi_0,\xi',x_n)|_{x_n=0} = r\widehat{\partial_{x_n} w}|_{x_n=0} + \Big(\int_0^\infty |\xi'|e^{-(r+|\xi'|)s_n}\,ds_n\Big)\widehat{P}_b(\xi_0,\xi').$$

We know that the tangential parts of the boundary data is zero and that $\operatorname{div} u = 0$, thus $\partial_{x_n}u_n|_{x_n=0} = 0$. Therefore, the above equality implies that

$$\frac{|\xi'|}{r+|\xi'|}\widehat{P}_b(\xi_0,\xi') = -r\widehat{\partial_{x_n} w}|_{x_n=0}.$$

This yields formula (4.22).

Now, in order to determine the pressure, we proceed as follows:

1. we construct an extension G_1 of $\mathcal{F}^{-1}_{t,x'}[|\xi'|\widehat{\partial_{x_n} w}|_{x_n=0}]$ on $\mathbb{R}\times\mathbb{R}^n_+$ such that $\nabla G_1 \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;

1. Note that only $\partial_{x_n} w$ may be nonzero at the boundary.

2. we construct an extension G_2 of $\mathcal{F}^{-1}_{t,x'}[r\widehat{\partial_{x_n} w}|_{x_n=0}]$ on $\mathbb{R}\times\mathbb{R}^n_+$ such that $\nabla G_2 \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;
3. we construct an extension V of $\mathcal{F}^{-1}_{t,x'}[i\xi_0/|\xi'|\widehat{\partial_{x_n} w}|_{x_n=0}]$ on $\mathbb{R}\times\mathbb{R}^n_+$ such that $\nabla V \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;
4. keeping in mind that P has to be harmonic, we write
$$P = P_{\text{new}} - G_1 - G_2 - V$$
where P_{new} is a solution to
$$\Delta P_{\text{new}} = \Delta(G_1+G_2+V) \quad \text{in } \mathbb{R}^n_+, \qquad P_{\text{new}}|_{x_n=0} = 0 \quad \text{on } \partial\mathbb{R}^n_+,$$
and establish that $\nabla P_{\text{new}} \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ with a suitable estimate.

First step: construction of G_1. — We just have to set $G_1 := |D'|\partial_{x_n} w$ where the pseudo-differential operator $|D'|$ is defined by
$$|D'|z := \mathcal{F}^{-1}_{x'}(|\xi'|\widehat{z}).$$
Indeed, we have $\nabla^3 w \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$, hence
$$\nabla|D'|\partial_{x_n} w \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+)).$$

Second step: construction of G_2. — It suffices to set $G_2 := -\partial_{x_n} y$ with y the solution to
$$\left\{\begin{array}{lll} y_t - \Delta y = 0 & \text{in} & \mathbb{R}\times\mathbb{R}^n_+, \\ y|_{x_n=0} = (\partial_{x_n} w)|_{x_n=0} & \text{on} & \mathbb{R}\times\partial\mathbb{R}^n_+, \end{array}\right. \qquad y \to 0 \ \text{ at } \ \infty. \tag{4.23}$$
Indeed, we observe that $\mathcal{F}_{t,x'} y = e^{-rx_n}\widehat{\partial_{x_n} w}|_{x_n=0}$, whence
$$\partial_{x_n} y = -\mathcal{F}^{-1}_{t,x'}[re^{-rx_n}\widehat{\partial_{x_n} w}|_{x_n=0}],$$
and $\partial_{x_n} y$ is thus an extension of $-\mathcal{F}^{-1}_{t,x'}(r\widehat{\partial_{x_n} w}|_{x_n=0})$ to $\mathbb{R}\times\mathbb{R}^n_+$.

In order to solve (4.23), we decompose y into $y = z + \partial_{x_n} w$ with z the solution to
$$\left\{\begin{array}{lll} z_t - \Delta z = \Delta(\partial_{x_n} w) - (\partial_{x_n} w)_t & \text{in} & \mathbb{R}\times\mathbb{R}^n_+, \\ z|_{x_n=0} = 0 & \text{on} & \mathbb{R}\times\partial\mathbb{R}^n_+, \end{array}\right. \qquad z \to 0 \ \text{ at } \ \infty.$$
Note that the right-hand side is in $L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ by construction of w and satisfies, owing to (4.17) and (4.19),
$$\|\Delta(\partial_{x_n} w) - (\partial_{x_n} w)_t\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq C\|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{4.24}$$
Thus, as a consequence of Prop. 6 in [**15**] or Th. 6.2. in [**20**], we get that $z_t, \nabla^2 z \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and are bounded by the right-hand side of

(4.24). Hence $\nabla^2 y$ too, and one can conclude that $\nabla G_2 \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and satisfies

$$\|\nabla G_2\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \le C\|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.$$

Third step: construction of V. — In order to extend the term coming from

$$\mathcal{F}^{-1}_{t,x'}\Big(\frac{i\xi_0}{|\xi'|}\widehat{\partial_{x_n} w}|_{x_n=0}\Big) = |D'|^{-1}\partial_{x_n} w_t|_{x_n=0} \quad \text{with} \quad w_t \ \text{ fulfilling (4.18)},$$

it is natural to set

$$Z' := \mathcal{F}^{-1}_{x'}\Big(e^{-|\xi'|x_n}\frac{i\xi'}{|\xi|'}\mathcal{F}_{x'}(\partial_{x_n} w_t|_{x_n=0})\Big),$$
$$Z^n := -\mathcal{F}^{-1}_{x'}\big(e^{-|\xi'|x_n}\mathcal{F}_{x'}(\partial_{x_n} w_t|_{x_n=0})\big).$$

We observe that the vector field (Z', Z^n) is the gradient of some potential V which is the sought extension. As we plan to use Lemma 2.2.2, the problem thus reduces to bounding[2] $\partial_{x_n} w_t|_{x_n=0}$ in $L_1(\mathbb{R}; \dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$. To this end, we first have to find some vector field $H \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and scalar function $h \in L_1(\mathbb{R}; \dot{B}^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+))$ such that

$$A + \mathcal{DIV}\,[B; b] = \mathcal{DIV}\,[H; h]. \tag{4.25}$$

So consider the antisymmetric extension $E_{\rm ant}A$ of A on the whole space. By construction, we are guaranteed that

$$\int_{\mathbb{R}^n} E_{\rm ant}A\,dx = 0,$$

and thus Proposition 3.2.1 enables us to solve

$$\begin{cases} \Delta a = E_{\rm ant}A & \text{in} \quad B(0,\lambda),\\ \partial_{\vec{n}} a = 0 & \text{on} \quad \partial B(0,\lambda), \end{cases} \qquad \int_{B(0,\lambda)} a\,dx = 0 \tag{4.26}$$

and provides us with the following bound in nonhomogeneous Besov space:

$$\|a\|_{B^{s+2}_{p,1}(B(0,\lambda))} \le C\|A\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{4.27}$$

The above inequality combined with the scaling argument of Corollary 3.2.1 yields for some constant C independent of λ:

$$\|\nabla a\|_{\dot{B}^s_{p,1}(B(0,\lambda))} \le C\lambda\|A\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

2. Note that $\partial_{x_n} w_t = \partial_t(\partial_{x_n} w)$ is well defined at the boundary as ∇w is, by construction, in $L_2(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$, and $s + 1 > 1/p$.

Next, consider the extension $\widetilde{\nabla a}$ of ∇a by 0, on $\mathbb{R}^n$. In light of Corollary 2.2.1, $\widetilde{\nabla a}$ is in $L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n))$ and satisfies

$$\|\widetilde{\nabla a}\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(B(0,\lambda)))} \le C\lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{4.28}$$

Additionally, by (3.5),

$$-\int_{B(0,\lambda)} \nabla a\cdot\nabla\varphi\,dx = \int_{B(0,\lambda)} E_{\mathrm{ant}}A\,\varphi\,dx \quad \text{for all } \ \varphi\in\mathcal{C}^\infty(\overline{B(0,\lambda)}).$$

Hence, we gather that

$$\operatorname{div}\widetilde{\nabla a} = E_{\mathrm{ant}}A \quad \text{in the space } \ L_1(\mathbb{R};\dot{B}^{s-1}_{p,1}(\mathbb{R}^n)).$$

Now, setting $H := B + \widetilde{\nabla a}|_{\mathbb{R}^n_+}$, the above arguments enable us to conclude that (4.25) is fulfilled with [3] $h := b - \partial_{x_n}a|_{x_n=0}$ and H satisfying

$$\|H\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \le C\big(\lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|B\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\big). \tag{4.29}$$

Let us emphasize that thanks to (4.26) the trace of $\partial_{x_n}a$ on $\partial\mathbb{R}^n_+\cap B(0,\lambda)$ is well defined in $L_1(\mathbb{R};B^{s+1-1/p}_{p,1}(\partial\mathbb{R}^n_+\cap B(0,\lambda)))$. Hence h makes sense in $L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$. Besides, we have

$$\|h\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))} \le C\big(\lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|b\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\big). \tag{4.30}$$

We are now ready to bound $\partial_{x_n}w_t|_{x_n=0}$ in $L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$. We start with the definition of w in (4.15) which gives for all $t\in\mathbb{R}$ and $\varphi\in\mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+})$,

$$-\int_{\mathbb{R}^n_+}\nabla w\cdot\nabla\varphi\,dx - \int_{\partial\mathbb{R}^n_+}\partial_{x_n}w\,\varphi\,d\sigma = \int_{\mathbb{R}^n_+} g\,\varphi\,dx.$$

Therefore, differentiating with respect to time, and using (4.10) and (4.25),

$$-\int_{\mathbb{R}^n_+}\nabla w_t\cdot\nabla\varphi\,dx - \int_{\partial\mathbb{R}^n_+}\partial_{x_n}w_t\,\varphi\,d\sigma = -\int_{\mathbb{R}^n_+} H\cdot\nabla\varphi\,dx + \int_{\partial\mathbb{R}^n_+} h\,\varphi\,d\sigma. \tag{4.31}$$

This clearly implies that $\operatorname{div}(\nabla w_t - H) = 0$ in $\mathbb{R}^n_+$ (take φ supported in $\mathbb{R}^n_+$) and thus, according to Lemma 2.2.4, the distribution $(\nabla w_t - H)\cdot\vec{e}_n$ has a trace on $\partial\mathbb{R}^n_+$ which belongs to $L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ and satisfies

$$\|\big((\nabla w_t - H)\cdot\vec{e}_n\big)|_{x_n=0}\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))} \le C\|\nabla w_t - H\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{4.32}$$

Furthermore, (4.31) guarantees that

$$\int_{\partial\mathbb{R}^n_+}\big((\nabla w_t - H)\cdot\vec{e}_n\big)\varphi\,d\sigma = \int_{\mathbb{R}^n_+}(H-\nabla w_t)\cdot\nabla\varphi\,dx = \int_{\partial\mathbb{R}^n_+}(h+\partial_{x_n}w_t)\varphi\,d\sigma,$$

3. The minus sign is due to the orientation of the exterior normal unit vector at $\partial\mathbb{R}^n_+$.

and thus

$$\partial_{x_n} w_t|_{x_n=0} = \big((\nabla w_t - H)\cdot \vec{e}_n\big)|_{x_n=0} - h. \tag{4.33}$$

Now, bounding the r.h.s. of (4.32) according to (4.19) and (4.29), and using also (4.30), we conclude that

$$\begin{aligned}\|\nabla V\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} &\lesssim \|B\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\ &\quad + \lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|b\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}.\end{aligned} \tag{4.34}$$

Last step: construction of the pressure and velocity. — Recall that the pressure defined in (4.21) has to fulfill the system

$$\begin{cases} \Delta P = 0 & \text{in} \quad \mathbb{R}^n_+,\\ P|_{x_n=0} = P_b & \text{on} \quad \partial\mathbb{R}^n_+.\end{cases}$$

Now, setting $EP_b := -G_1 - G_2 - V$, the previous steps ensure that $\nabla EP_b \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $EP_b|_{x_n=0} = P_b$. In addition, (4.24) and (4.34) yield

$$\|\nabla EP_b\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \lesssim \|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|b\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}.$$

Hence decomposing P into $P = P_{\text{new}} + EP_b$ and dropping the index *new* as usual, we see that it suffices to consider the system

$$\begin{cases} \Delta P = -\operatorname{div}\nabla EP_b & \text{in} \quad \mathbb{R}^n_+,\\ P|_{x_n=0} = 0 & \text{on} \quad \partial\mathbb{R}^n_+.\end{cases}$$

Because $\nabla EP_b \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$, the last part of Lemma 3.3.1 readily gives a unique solution P with $\nabla P \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$.

Collecting all the steps and changes of unknown functions, we conclude that

$$\|\nabla P\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq C\|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.$$

Once the pressure term has been estimated, one just has to define the velocity u to be the solution to the following heat equation:

$$\begin{cases} u_t - \Delta u = -\nabla P & \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+,\\ u|_{x_n=0} = 0 & \text{on} \quad \partial\mathbb{R}^n_+.\end{cases}$$

Solving this equation in our functional framework has been done in [**15**], Prop. 6. We get a velocity u with the required property. Lemma 4.2.1 is proved. Subsequently, the proof of Theorem 4.2.1 is complete, too. □

Remark 4.2.2. — A direct approach, based on the explicit solution formula as in [**21**] is possible if one assumes that $\partial_t(\mathcal{H}(R|_{x_n=0})) \in L_1(\mathbb{R}_+;\dot{B}^s_{p,1})$, where $\mathcal{H}(R|_{x_n=0})$ stands for the harmonic extension of $R|_{x_n=0}$, as in (3.41).

REMARK 4.2.3. — In the case where g is compactly supported, one may provide a *shorter* proof based on our work in [**16**]. Indeed, in this case, it is possible to remove directly the divergence part of u by means of the (generalized) Bogovskiĭ formula, resorting to the $\mathcal{DIV}$ functional introduced in Chapter 2. Here, we treated the general case where g is not compactly supported because it shows how the approach of [**15**] has to be adapted so as to handle nonzero divergence condition. Besides, it is needed to investigate systems for incompressible fluids in Lagrangian coordinates (see Chapter 5).

REMARK 4.2.4. — We did not deliver here any sketch of the proof of Theorem 4.2.2 from [**15**]. As in the proof of Theorem 4.2.1, constructing the pressure term from formula (4.21) is the key idea. Finding out suitable extension operators is the main difficulty.

4.3. The exterior domain case

This section is devoted to solving the evolutionary Stokes system

$$\begin{cases} u_t - \nu\Delta u + \nabla P = f & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = g & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{on} \quad \Omega, \end{cases} \tag{4.35}$$

in an *exterior* or *bounded* domain Ω.

Extending the results of the previous section to this new situation is our main objective here. We shall focus on the unbounded case which is more tricky and just indicate at the end of this section what has to be changed for bounded domains.

4.3.1. Proof of time-dependent estimates. — As a preliminary step, we shall establish the following time-dependent estimates for (4.35):

THEOREM 4.3.1. — *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$. Let $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Let $u_0 \in \dot{B}^s_{p,1}(\Omega)$, $f \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$ and $g \in \mathcal{C}([0,T);\dot{B}^{s-1}_{p,1}(\Omega))$ with $g(0) = \operatorname{div} u_0$. Assume in addition that $\nabla g \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$ and that $g = \operatorname{div} R$ for some vector field R with the following properties*[4]*:*

4. Note that the last two properties recast in $g_t = \mathcal{DIV}\,[R_t;\varrho]$. That $\frac{d}{dt}((R\cdot\vec{n})|_{\partial\Omega})$ is defined in the sense of distributions is due to the fact that $R|_K$ is in $L_1(0,T;B^{1+s}_{p,1}(K))$, hence its trace on ∂K (and thus on $\partial\Omega$) is well defined (see Proposition 2.2.4).

1. $R \in L_1(0,T;L_m(\Omega))$ *for some* $m \in (1,\infty)$ *and* $R|_K \in L_1(0,T;B^{1+s}_{p,1}(K))$ *where* K *stands for some bounded subset of* Ω *surrounding* $\partial\Omega$ *and such that* $\mathrm{dist}(\partial\Omega, \Omega\setminus K) > 0$ *(see Figure* 3.4*);*
2. $\displaystyle\int_{\partial\Omega} R\cdot\vec{n}\,d\varsigma = 0$*;*
3. $R_t \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$*;*
4. $\varrho := \frac{d}{dt}((R\cdot\vec{n})|_{\partial\Omega}) \in L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))$.

Then System (4.35) *has a unique solution* $(u,\nabla P)$ *such that*

$$u \in \mathcal{C}([0,T];\dot{B}^s_{p,1}(\Omega)), \qquad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T;\dot{B}^s_{p,1}(\Omega)), \tag{4.36}$$

and the following estimate is valid:

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))} &+ \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\\ &\leq Ce^{CT\nu}\big(\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}\\ &\qquad + \|f,\nu\nabla g, R_t\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} + \|\varrho\|_{L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))}\\ &\qquad\quad + \nu\|R\|_{L_1(0,T;L_m(\Omega))} + \nu\|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))}\big),\end{aligned} \tag{4.37}$$

where the constant C *depends only on* K, Ω, s, m *and* p.

Additionally, there holds

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))} &+ \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\\ &\leq C\big(\|u_0\|_{\dot{B}^s_{p,1}(\Omega)} + \|f,\nu\nabla g, R_t\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\\ &\qquad + \nu\|R\|_{L_1(0,T;L_m(\Omega))} + \|\varrho\|_{L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))}\\ &\qquad\quad + \nu\|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))} + \nu\|u|_K\|_{L_1(0,T;B^s_{p,1}(K))}\big),\end{aligned} \tag{4.38}$$

where C *depends only on* K, Ω, s, m *and* p.

Proof. — Let us first say a few words about the existence issue. The first step of the proof (see below) will guarantee that one may restrict to $g \equiv 0$ and source term f with $\operatorname{div} f = 0$ in Ω and $f\cdot\vec{n}|_{\partial\Omega} = 0$. Even in this case, the task is more complex than for standard parabolic systems because we have to keep the pressure term under control. To this end, one may use a suitable approximation in L_2-type spaces or the results of Giga-Sohr [**28**] and Maremonti-Solonikov [**40**], to obtain the solvability for smooth data together with (4.37). This latter inequality enables us to pass to the limit so as to get a solution satisfying (4.36), as it ensures that the sequence corresponding to smooth data is a *Cauchy sequence* in the space defined in (4.36).

The rest of the proof is devoted to establishing estimates (4.37) and (4.38). We may suppose that we are given a smooth enough solution. As usual, performing the change of variables (4.14) reduces the study to the case $\nu = 1$. So we shall make this assumption in all that follows.

First step: removing g. — In order to remove the inhomogeneity from the r.h.s. of $(4.35)_2$, we shall construct a solution to

$$\operatorname{div} v = g \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega, \tag{4.39}$$

such that $v = v^1 + v^2$ with $v^1 \in L_1(0,T; L_m(\Omega))$ and v^2 with support in $(0,T)\times K$ and in the space $L_1(0,T; B^{s+2}_{p,1}(\Omega))$. We want v to satisfy in addition $v_t, \nabla^2 v \in L_1(0,T; \dot B^s_{p,1}(\Omega))$ and also, owing to our assumption on g_t,

$$\mathcal{DIV}\,[v_t, 0] = \mathcal{DIV}\,[R_t, \varrho]. \tag{4.40}$$

In view of our results in [**16**], if g *has a compact support* then the most natural approach is to construct v by means of the (generalized) Bogovskiĭ formula associated to the domain Ω: we set

$$v = \mathcal{B}_\Omega(g) = I_\Omega(R) + J_\Omega\big((R\cdot\vec n)|_{\partial\Omega}\big),$$

where the operators I_Ω and J_Ω have been defined in the proof of Theorem 2.3.1. Then, owing to the properties of these two operators (see [**16**]), we may write that

$$-\int_\Omega v\cdot\nabla\varphi\,dx = -\int_\Omega R\cdot\nabla\varphi + \int_{\partial\Omega}(R\cdot\vec n)\varphi\,d\sigma = \int_\Omega \varphi\, g\,dx \quad \text{for all} \quad \varphi\in\mathcal{C}^\infty_c(\overline{\Omega}).$$

Differentiating with respect to time we thus get

$$-\int_\Omega v_t\cdot\nabla\varphi\,dx = -\int_\Omega R_t\cdot\nabla\varphi + \int_{\partial\Omega}\varrho\varphi\,d\sigma \quad \text{for all} \quad \varphi\in\mathcal{C}^\infty_c(\overline{\Omega}),$$

which is exactly what we wanted.

That approach works whenever g has a compact support for it suffices to solve (4.39) in a bounded subdomain of Ω. The result is given by Theorem 2.3.1, and as nonhomogeneous and homogeneous Besov norms coincide for compactly supported functions, we are done.

In the applications we have in mind however (see Chapters 5 and 6), g need not have a compact support. The rest of the section is devoted to the study of that more involved case.

Lemma 4.3.1. — *There exists a vector field $v = v^1 + v^2$ with v^1 supported in $(0,T)\times K$, fulfilling* (4.39) *and* (4.40), *supported in $\overline{\Omega}$ and such that*

$$\|v^1\|_{L_1(0,T;L_m(\Omega))} + \|v^2\|_{L_1(0,T;B^{s+2}_{p,1}(K))} + \|\nabla^2 v\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}$$
$$\leq C\big(\|\operatorname{div} R\|_{L_1(0,T;\dot B^{s+1}_{p,1}(\Omega))} + \|R\|_{L_1(0,T;L_m(\Omega))} + \|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))}\big),$$
$$\|v_t\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C\big(\|R_t\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} + \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}\big).$$

Proof. — We aim at reducing the study to the compact support case. As the time variable does not play any role here, we just omit it in the following computations.

Let $\eta^0 : \mathbb{R}^n \to [0,1]$ be a smooth cut-off function such that $\eta^0 \equiv 0$ on a neighborhood of $\mathbb{R}^n\backslash\Omega$ and $\eta^0 \equiv 1$ on a neighborhood of $\Omega\backslash K$ (see Figure 3.4). Let us consider the following problem:

$$\Delta G = \operatorname{div}(\eta^0 R) \quad \text{in} \quad \mathbb{R}^n. \tag{4.41}$$

We define ∇G by the formula

$$\nabla G := -(-\Delta)^{-1}\nabla\operatorname{div}(\eta^0 R),$$

so that we also have

$$\nabla G_t = -(-\Delta)^{-1}\nabla\operatorname{div}(\eta^0 R_t).$$

Because $(-\Delta)^{-1}\nabla\operatorname{div}$ is an homogeneous multiplier of degree 0, we gather from [**6**] and standard results on singular integrals that

$$\|\nabla^3 G\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \lesssim \|\nabla\operatorname{div}(\eta^0 R)\|_{\dot B^s_{p,1}(\mathbb{R}^n)},$$
$$\|\nabla G\|_{L_m(\mathbb{R}^n)} \lesssim \|\eta^0 R\|_{L_m(\mathbb{R}^n)} \quad \text{and} \quad \|\nabla G_t\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \lesssim \|\eta^0 R_t\|_{\dot B^s_{p,1}(\mathbb{R}^n)}.$$

Therefore, using the decomposition

$$\operatorname{div}(\eta^0 R) = \eta^0 g + R\nabla\eta^0 \quad \text{on} \quad \Omega,$$

we get

$$\begin{cases} \|\nabla^3 G\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \lesssim \|g\|_{\dot B^{1+s}_{p,1}(\Omega)} + \|R|_K\|_{B^{1+s}_{p,1}(K)}, \\ \|\nabla G\|_{L_m(\mathbb{R}^n)} \lesssim \|R\|_{L_m(\Omega)}, \qquad \|\nabla G_t\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \lesssim \|R_t\|_{\dot B^s_{p,1}(\Omega)}. \end{cases} \tag{4.42}$$

Then we look for v in the following form

$$v = \eta^0\nabla G + w_1 \tag{4.43}$$

with $w_1|_{\partial\Omega} = 0$ and

$$\operatorname{div} w_1 = \operatorname{div}\big((1-\eta^0)R\big) + \operatorname{div}\big((1-\eta^0)\nabla G\big). \tag{4.44}$$

Decomposing $(\eta^0-1)\nabla G$ into [5]

$$(\eta^0-1)\nabla G=(\eta^0-1)\dot S_0\nabla G+(\eta^0-1)(\mathrm{Id}\,-\dot S_0)\nabla G,$$

and using product estimates in $\mathbb{R}^n$ (recall that η^0-1 is compactly supported), and Bernstein inequality, we get

$$\|(\eta^0-1)\nabla G\|_{\dot B^{s+2}_{p,1}(\mathbb{R}^n)}\lesssim\|\nabla G\|_{L_m(\mathbb{R}^n)}+\|\nabla^3 G\|_{\dot B^{s}_{p,1}(\mathbb{R}^n)}.$$

Hence, combining with (4.42),

$$\|(\eta^0-1)\nabla G\|_{\dot B^{s+2}_{p,1}(\mathbb{R}^n)}\lesssim\|g\|_{\dot B^{1+s}_{p,1}(\Omega)}+\|R|_K\|_{B^{1+s}_{p,1}(K)}+\|R\|_{L_m(\Omega)}. \tag{4.45}$$

In order to reduce (4.44) to solving some divergence equation on the bounded set K we have to prove that the average over K of the r.h.s. of (4.44) vanishes. In fact, owing to the support properties of $1-\eta^0$ and to Condition (2) in Theorem 4.3.1, we can write

$$\begin{aligned}\int_K \mathrm{div}\,\big((1-\eta^0)(R+\nabla G)\big)dx&=\int_{\partial\Omega}\big(\vec n\cdot(R+\nabla G)\big)d\sigma\\ &=\int_{\partial\Omega}\vec n\cdot\nabla G\,d\sigma=-\int_{\mathbb{R}^n\setminus\Omega}\mathrm{div}\,(\eta^0R)\,dx=0.\end{aligned}$$

Hence we may solve (4.44) via the Bogovskiĭ formula in the set K according to Theorem 2.3.1: setting

$$\begin{aligned}w_1&=\mathcal{B}_K\big[\,\mathrm{div}\,\big((1-\eta^0)(R+\nabla G)\big)\big]\\ &=\mathcal{DIV}_K\big[(1-\eta^0)(R+\nabla G),(1-\eta^0)(\varrho+(\vec n\cdot\nabla G)|_{\partial K})\big],\end{aligned} \tag{4.46}$$

we readily get, by virtue of continuity results for $\mathcal{B}_K$,

$$\begin{aligned}\|w_1\|_{B^{s+2}_{p,1}(K)}&\lesssim\|(1-\eta^0)(\,\mathrm{div}\,R+\Delta G)\|_{B^{s+1}_{p,1}(K)}+\|\nabla\eta^0\cdot(R+\nabla G)\|_{B^{1+s}_{p,1}(K)}\\ &\lesssim\|g\|_{B^{s+1}_{p,1}(K)}+\|R|_K\|_{B^{1+s}_{p,1}(K)}.\end{aligned} \tag{4.47}$$

Therefore by (4.42), (4.43), (4.45) and (4.47), we conclude that

$$\|v\|_{\dot B^{s+2}_{p,1}(\Omega)}\lesssim\|g\|_{\dot B^{s+1}_{p,1}(\Omega)}+\|R\|_{L_m(\Omega)}+\|R|_K\|_{B^{s+1}_{p,1}(K)}. \tag{4.48}$$

Let us now concentrate on the proof of estimates for v_t. We have

$$v_t=\nabla G_t+(\eta^0-1)\nabla G_t+w_{1,t}.$$

Differentiating (4.46) yields

$$w_{1,t}=I_K[(1-\eta^0)(R_t+\nabla G_t)]+J_K[(1-\eta^0)(\varrho+\partial_{\vec n}G_t|_{\partial K})].$$

5. The low frequency cut-off $\dot S_0$ has been defined in (2.2).

That $\partial_{\vec{n}} G_t$ has a trace at the boundary ∂K is a consequence of Lemma 2.2.4 as, by construction, $\operatorname{div}(\nabla G_t) = 0$ in $\mathbb{R}^n \setminus \Omega$. Furthermore, there holds

$$\|\partial_{\vec{n}} G_t|_{\partial\Omega}\|_{\dot{B}^{s-1/p}_{p,1}(\partial\Omega)} \lesssim \|\nabla G_t\|_{\dot{B}^s_{p,1}(\mathbb{R}^n\setminus\Omega)} \lesssim \|R_t\|_{\dot{B}^s_{p,1}(\Omega)} \quad \text{for a.e.} \quad t > 0.$$

Thus, we obtain

$$\|w_{1,t}\|_{\dot{B}^s_{p,1}(\Omega)} \lesssim \|R_t\|_{\dot{B}^s_{p,1}(\Omega)} + \|\varrho\|_{B^{s-1/p}_{p,1}(\partial\Omega)}. \tag{4.49}$$

Putting (4.42) and (4.49) together, one may conclude that

$$\|v_t\|_{\dot{B}^s_{p,1}(\Omega)} \lesssim \|R_t\|_{\dot{B}^s_{p,1}(\Omega)} + \|\varrho\|_{B^{s-1/p}_{p,1}(\partial\Omega)}. \tag{4.50}$$

Integrating (4.48) with respect to time completes the proof of the lemma. □

The construction of v given by Lemma 4.3.1 reduces the proof of Theorem 4.3.1 to the case $g \equiv 0$. Indeed, if we set

$$u_{\text{new}} = u_{\text{old}} - v \quad \text{and} \quad f_{\text{new}} = f_{\text{old}} - v_t + \Delta v, \tag{4.51}$$

then $u = u_{\text{new}}$ has to satisfy

$$\begin{cases} u_t - \Delta u + \nabla P = f_{\text{new}} & \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T) \times \Omega, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{on} \quad \Omega. \end{cases} \tag{4.52}$$

According to (4.48) and (4.50), the regularity of the new function f is preserved and

$$\begin{aligned} \|f_{\text{new}}\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} &\lesssim \|f_{\text{old}}, \nabla g, R_t\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \\ &+ \|\varrho\|_{L_1(0,T;B^{s-\frac{1}{p}}_{p,1}(\partial\Omega))} + \|R\|_{L_1(0,T;L_m(\Omega))} + \|R\|_{L_1(0,T;B^{1+s}_{p,1}(K))}. \end{aligned} \tag{4.53}$$

REMARK 4.3.1. — As already pointed out, in the case of a bounded domain Ω, one may directly apply Theorem 2.3.1 so as to remove g. Then making the change of unknown (4.51), we eventually get (4.52) with f_{new} satisfying

$$\|f_{\text{new}}\|_{L_1(0,T;B^s_{p,1}(\Omega))} \lesssim \|f_{\text{old}}, \nabla g, R_t\|_{L_1(0,T;B^s_{p,1}(\Omega))} + \|\varrho\|_{L_1(0,T;B^{s-\frac{1}{p}}_{p,1}(\partial\Omega))}.$$

REMARK 4.3.2. — Note that in this first step, the time variable is just treated as a parameter. Hence reducing the study to the divergence free case may be done in any Besov space $\dot{B}^s_{p,r}(\Omega)$ with $1 \le r \le \infty$, $1 < p < \infty$ and $-1 + 1/p < s < 1/p$.

Second step: an estimate for the pressure. — At this point, one may remove the potential part of f and its normal component at the boundary. Indeed, Proposition 3.5.1 enables us to solve the following problem:

$$\begin{cases} \Delta Q = \operatorname{div} f & \text{in} \quad \Omega, \\ (\nabla Q - f)\cdot\vec{n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then one may change f to $f - \nabla Q$, putting ∇Q in the pressure.

So we may assume from now on that

$$\operatorname{div} f = 0 \quad \text{in} \ \ \Omega \quad \text{and} \quad f\cdot\vec{n} = 0 \ \ \text{on} \ \ \partial\Omega. \tag{4.54}$$

Since the Stokes system is not quite of parabolic type, an extra information on the pressure is needed so as to adapt the purely parabolic techniques of e.g. [**33**], [**35**]. One of the difficulties is that the basic energy estimate does not supply any reasonable bound for the pressure. The estimates that we shall obtain below will enable us to control lower order terms which will appear as a consequence of the localization procedure, and to 'close the estimates' for small enough times.

In order to get this extra information, we take the divergence of $(4.35)_1$. Under assumption (4.54), we obtain:

$$\begin{cases} \Delta P = 0 & \text{in} \quad \Omega, \\ \partial_{\vec{n}} P = \Delta u\cdot\vec{n} & \text{on} \quad \partial\Omega, \end{cases} \qquad P \to 0 \ \ \text{at} \ \ \infty. \tag{4.55}$$

The boundary condition is just taken directly from the equation $(4.35)_1$. Note that, as $\operatorname{div} f = 0$ we have $\operatorname{div}\Delta u = 0$ with $\Delta u \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$, hence the boundary condition makes sense according to Lemma 2.2.4.

Lemma 4.3.2. — *Let $-1+1/p < s < 1/p$ and u be a divergence free vector field over Ω with $u\cdot\vec{n} = 0$ at $\partial\Omega$ and*

$$\Delta u \in L_1(0,T;\dot{B}^s_{p,1}(\Omega)) \cap L_\infty(0,T;\dot{B}^{-2+s}_{p,1}(\Omega)).$$

Then there exists a unique distributional solution P to (4.55) such that

$$\|P\|_{L_1(0,T;B^{s-2a}_{p,1}(K))} + \|\nabla P\|_{L_1(0,T;\dot{B}^{s-2a}_{p,1}(\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(K))}\|\Delta u\|^a_{L_1(0,T;B^{-2+s}_{p,1}(K))}, \tag{4.56}$$

$$\|P\|_{L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))} + \|\nabla P\|_{L_{1+\kappa}(0,T;\dot{B}^{s-2a}_{p,1}(\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(K))}\|\Delta u\|^a_{L_\infty(0,T;B^{-2+s}_{p,1}(K))}, \tag{4.57}$$

where the constant C is independent of T and $-1+1/p < s-2a < 1/p$ with $1+\kappa = 1/(1-a)$.

Proof. — Of course we have

$$\Delta u \in L_1(0,T;\dot B^s_{p,1}(K)) \cap L_\infty(0,T;\dot B^{-2+s}_{p,1}(K)),$$

and this implies, by interpolation, that $\Delta u \in L_{1+\kappa}(0,T;\dot B^{s-2a}_{p,1}(K))$ with κ defined as in the statement. If $s-2a > -1+1/p$ then, owing to the compactness of K, we have $\Delta u \in L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))$, and by the trace Lemma (see Lemma 2.2.4) combined with the fact that $\operatorname{div}\Delta u = 0$ in K, we are guaranteed that $\Delta u\cdot\vec n|_{\partial\Omega}$ is defined in $L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))$ and that

$$\|\Delta u\cdot\vec n\|_{L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \le C\|\Delta u\|_{L_1(0,T;B^{s-2a}_{p,1}(K))}.$$

Combining with the interpolation inequality, we thus get

$$\|\Delta u\cdot\vec n\|_{L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \le C\|\Delta u\|^{1-a}_{L_1(0,T;\dot B^{s}_{p,1}(K))}\|\Delta u\|^{a}_{L_1(0,T;\dot B^{-2+s}_{p,1}(K))}$$

where C is independent of T. Now, applying Theorem 3.4.1 with $b := \Delta u\cdot\vec n$ gives (4.56).

Let us now turn to the proof of (4.57). Starting from the fact that

$$\|\Delta u\cdot\vec n\|_{L_{1+\kappa}(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \le C\|\Delta u\|_{L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))},$$

and using once again the equivalence between homogeneous and nonhomogeneous norms in our context, we get by means of an elementary interpolation argument:

$$\|\Delta u\cdot\vec n\|_{L_{1+\kappa}(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \le C\|\Delta u\|^{1-a}_{L_1(0,T;\dot B^{s}_{p,1}(\Omega))}\|\Delta u\|^{a}_{L_\infty(0,T;\dot B^{-2+s}_{p,1}(\Omega))},$$

where C is independent of T.

Therefore, applying Theorem 3.4.1 with $b := \Delta u\cdot\vec n$ gives (4.57). □

REMARK 4.3.3. — For Ω a bounded domain of $\mathbb R^n$ $(n\ge 2)$ and $K=\bar\Omega$, the proof works the same. Under the above conditions, we get

$$\|P\|_{L_1(0,T;B^{s+1-2a}_{p,1}(\Omega))} \le C\|\Delta u\|^{1-a}_{L_1(0,T;B^{s}_{p,1}(\Omega))}\|\Delta u\|^{a}_{L_1(0,T;B^{-2+s}_{p,1}(\Omega))},$$

$$\|P\|_{L_{1+\kappa}(0,T;\dot B^{s+1-2a}_{p,1}(\Omega))} \le C\|\Delta u\|^{1-a}_{L_1(0,T;B^{s}_{p,1}(\Omega))}\|\Delta u\|^{a}_{L_\infty(0,T;B^{-2+s}_{p,1}(\Omega))}.$$

Now we can tackle the proof of estimates in Theorem 4.3.1 under the assumption $g\equiv 0$, $\operatorname{div} f\equiv 0$ and $f\cdot\vec n|_{\partial\Omega}\equiv 0$. Throughout we fix some covering $(\Omega^\ell)_{0\le\ell\le L}$ of Ω such that $\operatorname{Supp}\eta^0\subset\Omega^0$ with $\Omega^0\cap\partial\Omega=\varnothing$ (here η^0 is the function introduced in the first step of the proof just after (4.42)), and $(\Omega^\ell)_{1\le\ell\le L}$ constitutes a covering of $\partial\Omega$ with $\Omega^\ell\subset\Omega$, $\Omega^\ell\cap\partial\Omega\neq\varnothing$, $\Omega\cap\Omega^\ell$ star-shaped with respect to some ball, and $\operatorname{diam}(\Omega^\ell)\approx\lambda$ (see Figure 3.4). Then we consider a subordinate partition of unity $(\eta^\ell)_{1\le\ell\le L}$ such that:

1. $\sum_{0\leq\ell\leq L} \eta^\ell = 1 \quad \text{on} \quad \Omega;$
2. $\|\nabla^k\eta^\ell\|_{L_\infty(\mathbb{R}^n)} \leq C_k\lambda^{-k}$ for $k\in\mathbb{N}$ and $1\leq\ell\leq L$;
3. $\operatorname{Supp}\eta^\ell\subset\Omega^\ell$.

We also introduce a smooth function $\widetilde{\eta}^0$ supported in K and with value 1 on $\operatorname{Supp}\nabla\eta^0$, and smooth functions $\widetilde{\eta}^1,\dots,\widetilde{\eta}^L$ with compact support in Ω^ℓ and such that $\widetilde{\eta}^\ell\equiv 1$ on $\operatorname{Supp}\eta^\ell$. Obviously those functions can be defined on the whole space.

Note that, for $\ell\in\{1,\dots,L\}$, the bounds for the derivatives of η^ℓ together with the fact that $\left|\operatorname{Supp}\nabla\eta^\ell\right|\approx\lambda^n$ and interpolation implies that for $k=1,2$ and any $q\in[1,\infty]$, we have

$$\|\nabla^k\eta^\ell\|_{\dot B^{n/q}_{q,1}(\mathbb{R}^n)}\lesssim\lambda^{-k}. \tag{4.58}$$

The same holds for the functions $\widetilde{\eta}^\ell$.

Throughout, we set $U^\ell := u\eta^\ell$ and $P^\ell := P\eta^\ell$. We first prove *an interior estimate* (that is an estimate for (U^0,P^0)) then *boundary estimates*, which will eventually lead to the desired estimates (4.37) and (4.38).

Third step: the interior estimate. — The couple (U^0,P^0) satisfies:

$$\begin{cases} U^0_t-\Delta U^0+\nabla P^0=f^0+\eta^0 f & \text{in} \quad (0,T)\times\mathbb{R}^n,\\ \operatorname{div}U^0=g^0 & \text{in} \quad (0,T)\times\mathbb{R}^n,\\ U^0|_{t=0}=u_0\eta^0 & \text{on} \quad \mathbb{R}^n,\end{cases} \tag{4.59}$$

with

$$f^0:=-2\nabla\eta^0\cdot\nabla u+u\Delta\eta^0+P\nabla\eta^0 \quad\text{and}\quad g^0:=u\cdot\nabla\eta^0.$$

The localization procedure destroys the divergence-free assumption. Hence we have to check whether the r.h.s. of $(4.59)_2$ is of the form that we considered in Theorem 4.1.1. Let us observe that

$$g^0_t=u_t\cdot\nabla\eta^0=(u_t-f)\cdot\nabla\eta^0+f\cdot\nabla\eta^0=(\Delta u-\nabla P)\cdot\nabla\eta^0+f\cdot\nabla\eta^0.$$

Hence, denoting $\mathbb{D}u:={}^t\nabla u+\nabla u$, one may write

$$g^0_t=\operatorname{div}\left[\mathbb{D}u\cdot\nabla\eta^0-P\nabla\eta^0\right]-\mathbb{D}u:\nabla^2\eta^0+P\Delta\eta^0+f\cdot\nabla\eta^0.$$

To match the assumptions of Theorem 4.1.1, we thus set

$$B^0:=\mathbb{D}u\cdot\nabla\eta^0-P\nabla\eta^0 \qquad\text{and}\qquad A^0:=-\mathbb{D}u:\nabla^2\eta^0+P\Delta\eta^0+f\cdot\nabla\eta^0.$$

Next we notice that by virtue of the Stokes formula,

$$\int_{\mathbb{R}^n}A^0\,dx=\int_\Omega\operatorname{div}(U^0_t-B^0)\,dx=\int_{\partial\Omega}(U^0_t-B^0)\cdot\vec{n}\,d\sigma=0,$$

hence Theorem 4.1.1 yields:

$$\begin{aligned}\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ \leq C(\|\eta^0 f,f^0,\nabla g^0,B^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ +\|A^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\eta^0u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}).\end{aligned}$$

Let us emphasize that as A^0, B^0, f^0 and g^0 are compactly supported, we may replace the homogeneous norms by nonhomogeneous ones. As a consequence, because the function $\nabla\eta^0$ is in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\widetilde\eta^0\equiv 1$ on $\operatorname{Supp}\nabla\eta^0$, Corollary 2.1.1 ensures that

$$\begin{aligned}\|\nabla g^0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|A^0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|B^0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\\ \lesssim\|\widetilde\eta^0u\|_{B^{s+1}_{p,1}(\mathbb{R}^n)}+\|\widetilde\eta^0P\|_{B^s_{p,1}(\mathbb{R}^n)}+\|\widetilde\eta^0f\|_{B^s_{p,1}(\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ \lesssim\|\widetilde\eta^0f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\widetilde\eta^0u\|_{L_1(0,T;B^{s+1}_{p,1}(\mathbb{R}^n))}+\|\widetilde\eta^0P\|_{L_1(0,T;B^s_{p,1}(\mathbb{R}^n))}.\end{aligned}\tag{4.60}$$

By interpolation, u belongs to $L_2(0,T;\dot B^{1+s}_{p,1}(\Omega))$ and

$$\|u\|_{L_2(0,T;\dot B^{1+s}_{p,1}(\Omega))}\leq C\|u\|^{\frac12}_{L_1(0,T;\dot B^{2+s}_{p,1}(\Omega))}\|u\|^{\frac12}_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}.$$

Additionally, as the definition of Besov spaces by restriction ensures that

$$\|\widetilde\eta^0P\|_{B^s_{p,1}(\mathbb{R}^n)}\leq C\|P\|_{B^s_{p,1}(K)},$$

combining Inequality (4.57) and Hölder inequality gives

$$\|\widetilde\eta^0P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\leq CT^a\|u\|^{1-a}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^a_{L_\infty(0,T;B^s_{p,1}(K))}$$

whenever $-1+1/p<s-2a<1/p$.

In this way, we may conclude that

$$\begin{aligned}\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ \lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ +(T^{1/2}+T^a)\|u\|_{L_1(0,T;\dot B^{2+s}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}.\end{aligned}\tag{4.61}$$

If we want to prove (4.38) then we rather estimate the terms from (4.60) as follows:

$$(4.62)\quad \|f^0,\nabla g^0,B^0,A^0\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C\Big(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|^{1/2}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^{1/2}_{L_1(0,T;B^s_{p,1}(K))} + \|P\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|u_0\|_{\dot B^s_{p,1}(\Omega)}\Big)$$

and we use the following estimate for the pressure (a consequence of (4.56) and of an interpolation inequality involving the two terms in the l.h.s. of (4.56)):

$$\|P\|_{L_1(0,T;\dot B^s_{p,1}(K))}\leq C\|u\|^{1-a}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^{a}_{L_1(0,T;B^s_{p,1}(K))}.$$

Hence, using Young's inequality, we find the following interior inequality for all $\varepsilon>0$:

$$(4.63)\quad \|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)} +\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\varepsilon\|u\|_{L_1(0,T;B^{2+s}_{p,1}(K))}+c(\varepsilon)\|u\|_{L_1(0,T;B^s_{p,1}(K))}.$$

Fourth step: the boundary estimate. — We now consider $\ell\in\{1,\dots,L\}$ so that $\operatorname{Supp}\eta^\ell\cap\partial\Omega\neq\varnothing$. The localization leads to the following problem:

$$(4.64)\quad \begin{cases} U^\ell_t-\Delta U^\ell+\nabla P^\ell=f^\ell & \text{in}\quad (0,T)\times\Omega,\\ \operatorname{div}U^\ell=g^\ell & \text{in}\quad (0,T)\times\Omega,\\ U^\ell=0 & \text{on}\quad (0,T)\times\partial\Omega,\\ U^\ell_t|_{t=0}=u_0\eta^\ell & \text{on}\quad \Omega,\end{cases}$$

with

$$f^\ell:=-2\nabla\eta^\ell\cdot\nabla u+u\Delta\eta^\ell+P\nabla\eta^\ell+\eta^\ell f\quad\text{and}\quad g^\ell:=u\cdot\nabla\eta^\ell.$$

As a first step for proving the boundary estimate, we want to reduce the problem to the case $\operatorname{div}U^\ell\equiv0$. For that, we shall resort once again to the (generalized) Bogovskiĭ formula.

Since $g^\ell=\nabla\eta^\ell\cdot u$, it is compactly supported (in Ω^ℓ for instance). In addition, we notice that

$$(4.65)\quad \int_{\Omega^\ell}g^\ell\,dx=\int_{\partial\Omega}\eta^\ell u\cdot\vec n\,d\sigma-\int_\Omega\eta^\ell\operatorname{div}u\,dx=0\quad\text{and}\quad g^\ell=0\ \text{ on }\ \partial\Omega^\ell.$$

Therefore, setting

$$(4.66)\qquad v^\ell:=\mathcal{B}_{\Omega^\ell}(g^\ell),$$

where $\mathcal{B}_{\Omega^\ell}$ stands for the Bogovskiĭ operator defined in (2.24), we get a vector field $v^\ell \in L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))$ such that $\operatorname{Supp} v^\ell(t,\cdot) \subset \overline{\Omega^\ell}$,

$$\|v^\ell\|_{L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))} \lesssim \|g^\ell\|_{L_1(0,T;B^{1+s}_{p,1}(\Omega))} \quad \text{and} \quad \operatorname{div} v^\ell = g^\ell \ \text{ in } \ \Omega.$$

Then, using the stability of Besov spaces by multiplication by smooth compactly supported functions, we conclude that

$$\|v^\ell\|_{L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))} \leq C_\lambda \|\widetilde{\eta}^\ell u\|_{L_1(0,T;B^{1+s}_{p,1}(\Omega))}, \tag{4.67}$$

where the constant C_ℓ depends only on (s,p), on Ω_ℓ and on Ω.

Next, differentiating (4.66) with respect to time yields

$$v^\ell_t = \mathcal{B}_{\Omega^\ell}(A^\ell) + I_{\Omega^\ell}(B^\ell) + J_{\Omega^\ell}(B^\ell_n) \tag{4.68}$$

with I_{Ω^ℓ} and J_{Ω^ℓ} defined in (2.25),

$$A^\ell := -\mathbb{D}u : D^2\eta^\ell + P\Delta\eta^\ell + f\cdot\nabla\eta^\ell \quad \text{and} \quad B^\ell := \mathbb{D}u\cdot\nabla\eta^\ell - P\nabla\eta^\ell.$$

Using again Corollary 2.1.1, we easily find that

$$\|A^\ell\|_{B^s_{p,1}(\Omega)} \leq C_\lambda\Big(\|\widetilde{\eta}^\ell f\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{s+1}_{p,1}(\Omega)}\Big) \tag{4.69}$$

and that

$$\|B^\ell\|_{B^s_{p,1}(\Omega)} \leq C_\lambda\Big(\|\widetilde{\eta}^\ell P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{s+1}_{p,1}(\Omega)}\Big). \tag{4.70}$$

Besides, $u \in L_1(0,T;\dot{B}^{s+2}_{p,1}(\Omega))$ and $P \in L_1(0,T;\dot{B}^{s+1}_{p,1}(K))$. Hence the product laws in Besov spaces (Corollary 2.1.1) ensure that B^ℓ belongs to $L_1(0,T;B^{s+1}_{p,1}(\Omega))$. Therefore $B^\ell_n := B^\ell\cdot\vec{n}$ has a trace at the boundary and Relation (4.68) is thus valid.

Now, differentiating (4.65) with respect to time implies that

$$\int_{\Omega^\ell} (A^\ell + \operatorname{div} B^\ell)\,dx = 0.$$

As A^ℓ and B^ℓ are compactly supported in Ω^ℓ, we deduce that the compatibility condition for A^ℓ and B^ℓ is satisfied on Ω^ℓ. By virtue of (4.69), we thus have

$$\begin{aligned}\|\mathcal{B}_{\Omega^\ell}(A^\ell)\|_{L_1(0,T;B^{s+1}_{p,1}(\Omega))} \leq C_\ell\Big(&\|(\widetilde{\eta}^\ell f,\widetilde{\eta}^\ell P)\|_{L_1(0,T;B^s_{p,1}(\Omega))}\\ &+ \|\widetilde{\eta}^\ell u\|_{L_1(0,T;B^{s+1}_{p,1}(\Omega))}\Big).\end{aligned} \tag{4.71}$$

For bounding $I_{\Omega^\ell}(B^\ell)$, it is only a matter of using the results of [**16**] and (4.70). We get

$$\|I_{\Omega^\ell}(B^\ell)\|_{B^s_{p,1}(\Omega)} \leq C_l\Big(\|\widetilde{\eta}^\ell P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{s+1}_{p,1}(\Omega)}\Big). \tag{4.72}$$

As usual, owing to the compact support of $\mathcal{B}_{\Omega^\ell}(A^\ell)$ and of $I_{\Omega^\ell}(B^\ell)$, the nonhomogeneous norms may be replaced with homogeneous ones in the left-hand side of (4.71) and (4.72).

For bounding B^ℓ_n in $B^{s-\frac{1}{p}}_{p,1}(\partial\Omega^\ell)$, we use that $B^\ell \in L_1(0,T;B^{s+1}_{p,1}(\Omega^\ell))$. Hence applying Proposition 2.2.4 yields for any ε in $(0,s+1-1/p]$,

$$\|B^\ell_n\|_{B^\varepsilon_{p,1}(\partial\Omega^\ell)} \leq C_\lambda \|B^\ell\|_{B^{\varepsilon+1/p}_{p,1}(\Omega^\ell)}.$$

If in addition $\varepsilon + 1/p < n/p$, then, owing to Proposition 2.2.1 and (4.58),

$$\|B^\ell\|_{B^{\varepsilon+1/p}_{p,1}(\Omega^\ell)} \leq C_\lambda \big(\|\widetilde{\eta}^\ell P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\big),$$

whence

$$\|B^\ell_n\|_{B^{s-1/p}_{p,1}(\partial\Omega^\ell)} \leq C\|B^\ell_n\|_{B^\varepsilon_{p,1}(\partial\Omega^\ell)} \leq C_\lambda \big(\|\widetilde{\eta}^\ell P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\big).$$

Then using the results from [**16**], one may conclude that

$$\|J_{\Omega^\ell}(B^\ell_n)\|_{B^s_{p,1}(\Omega^\ell)} \leq C_\lambda \big(\|\widetilde{\eta}^\ell P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^\ell u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\big).$$

So finally, putting together the above inequalities and bearing in mind (4.67), we get

$$(4.73)\quad \|v^\ell_t, \nabla^2 v^\ell\|_{L_1(0,T;B^s_{p,1}(\Omega))}$$
$$\leq C_\lambda \Big(\|\widetilde{\eta}^\ell f\|_{L_1(0,T;B^s_{p,1}(\Omega))} + \|\widetilde{\eta}^\ell P\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))} + \|\widetilde{\eta}^\ell u\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\Big).$$

Next, making use of v^ℓ we modify System (4.64) into

$$\begin{cases} U^\ell_t - \Delta U^\ell + \nabla P^\ell = f^\ell_{\mathrm{new}} & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} U^\ell = 0 & \text{in} \quad (0,T)\times\Omega, \\ U^\ell = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ U^\ell|_{t=0} = u_0\eta^\ell - \mathcal{B}_{\Omega^\ell}\operatorname{div}(u_0\eta^\ell) & \text{on} \quad \Omega, \end{cases}$$

with

$$U^\ell_{\mathrm{new}} := U^\ell_{\mathrm{old}} - v^\ell \quad \text{and} \quad f^\ell_{\mathrm{new}} := f^\ell_{\mathrm{old}} - v^\ell_t + \Delta v^\ell.$$

Note that (4.73) implies that

$$(4.74)\quad \|f^\ell_{\mathrm{new}}\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}$$
$$\leq \|\widetilde{\eta}^\ell f^\ell_{\mathrm{old}}\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}$$
$$+ C_\lambda \big(\|\widetilde{\eta}^\ell P\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))} + \|\widetilde{\eta}^\ell u\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\big)$$

and it is also clear that

$$\|U^\ell|_{t=0}\|_{\dot B^s_{p,1}(\Omega)} \leq C_\lambda \|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{4.75}$$

Let us now recast (4.35) on $\mathbb{R}^n_+$ according to the *volume preserving* change of coordinates introduced in Chapter 2. Let

$$V^\ell := Z^*_\ell U^\ell := U^\ell \circ Z_\ell^{-1} \quad \text{and} \quad Q^\ell := Z^*_\ell P^\ell.$$

The system satisfied by (V^ℓ, Q^ℓ) reads

$$\begin{cases} V^\ell_t - \Delta_z V^\ell + \nabla_z Q^\ell = F^\ell & \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div}_z V^\ell = G^\ell & \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ V^\ell|_{z_n=0} = 0 & \text{on} \quad (0,T)\times\partial\mathbb{R}^n_+,\\ V^\ell|_{t=0} = Z^*_\ell(U^\ell|_{t=0}) & \text{on} \quad \partial\mathbb{R}^n_+,\end{cases}$$

with

$$F^\ell := Z^*_\ell f^\ell + (\Delta_x - \Delta_z)V^\ell - (\nabla_x - \nabla_z)Q^\ell \quad \text{and} \quad G^\ell := (\operatorname{div}_z - \operatorname{div}_x)V^\ell.$$

Let us stress that, according to Chapter 2, we have

$$G^\ell = -{}^T\!\mathcal{A}^\ell : \nabla_z V^\ell = -\operatorname{div}_z(\mathcal{A}^\ell V^\ell) \quad \text{with} \quad \mathcal{A}^\ell(z) := D_x Z_\ell(x) - \mathrm{Id}\,.$$

Hence $G^\ell|_{t=0} = 0$ and

$$G^\ell_t = -\operatorname{div}_z(\mathcal{A}^\ell V^\ell_t).$$

According to Theorem 4.2.1, we thus get

$$\begin{aligned}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &+ \|V^\ell_t, \nabla^2 V^\ell, \nabla Q^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\lesssim \|Z^*_\ell f^\ell, (\Delta_x - \Delta_z)V^\ell, (\nabla_z - \nabla_x)Q^\ell, \nabla G^\ell, \mathcal{A}^\ell V^\ell_t\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\qquad + \|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.\end{aligned}$$

The first and last terms in the right-hand side may be dealt with thanks to Lemma 2.1.1: we have

$$\begin{aligned}\|Z^*_\ell f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\lesssim \|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))},\\ \|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} &\lesssim \|U^\ell|_{t=0}\|_{\dot B^s_{p,1}(\Omega)}.\end{aligned}$$

So the definitely new terms are $(\nabla_x - \nabla_z)Q^\ell$, $\mathcal{A}^\ell V^\ell_t$, $(\Delta_x - \Delta_z)V^\ell$ and ∇G_ℓ. First, we notice that, denoting $\mathcal{B}^\ell = \mathcal{A}^\ell + \mathrm{Id} = D_x Z_\ell$, we have

$$(\nabla_x - \nabla_z)Q^\ell = {}^T\!\mathcal{B}^\ell \nabla_z Q^\ell = {}^T\!\widetilde{\mathcal{B}}^\ell \nabla_z Q^\ell \quad \text{with} \quad \widetilde{\mathcal{B}}^\ell := \mathcal{B}^\ell\, Z^*_\ell \widetilde{\eta}^\ell.$$

Hence Proposition 2.2.1 ensures that

$$\|(\nabla_x - \nabla_z)Q^\ell\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} \leq C\|{}^T\!\mathcal{B}^\ell\, Z^*_\ell \widetilde{\eta}^\ell\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\cap \dot B^{\frac{n}{p'}}_{p',1}(\mathbb{R}^n_+)} \|\nabla_z Q^\ell\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

Together with Lemma 2.1.1 and Inequality (2.29), this implies that

$$\|(\nabla_x - \nabla_z)Q^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \leq C\lambda\|\nabla Q^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

From similar arguments, we get

$$\|\mathcal{A}^\ell V^\ell_t\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \leq C\lambda\|V^\ell_t\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

Bounding $(\Delta_x - \Delta_z)V^\ell$ is more involved. It relies on the formula

$$(\Delta_x - \Delta_z)V^\ell = \operatorname{div}_z\big(\widetilde{\mathcal{A}}^\ell \cdot (\mathrm{Id} + {}^T\widetilde{\mathcal{A}}^\ell) \cdot \nabla_z V^\ell\big) + \operatorname{div}_z\big({}^T\widetilde{\mathcal{A}}^\ell \cdot \nabla_z V^\ell\big) \tag{4.76}$$

with $\widetilde{\mathcal{A}}^\ell = Z^*_\ell \eta^\ell \mathcal{A}^\ell$. Using the fact that

$$\operatorname{div}_z\big({}^T\widetilde{\mathcal{A}}^\ell \cdot \nabla_z V^\ell\big) = (\nabla^T\widetilde{\mathcal{A}}^\ell) \cdot \nabla_z V^\ell + {}^T\widetilde{\mathcal{A}}^\ell \cdot \nabla_z \operatorname{div}_z V^\ell,$$

we may write, by virtue of Proposition 2.2.1,

$$\begin{aligned}\|\operatorname{div}_z({}^T\widetilde{\mathcal{A}}^\ell \cdot \nabla_z V^\ell)\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \lesssim \|\nabla_z\widetilde{\mathcal{A}}^\ell\|_{\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)}\|\nabla_z V^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}&\\ +\|\widetilde{\mathcal{A}}^\ell\|_{\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)}\|\nabla^2_z V^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}&.\end{aligned}$$

The first term in the right-hand side of (4.76) obeys a similar inequality. Hence, using Inequality (2.29), one may conclude that

$$\|(\Delta_x - \Delta_z)V^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \lesssim \lambda\|\nabla^2_z V^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|\nabla_z V^\ell\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{4.77}$$

The last term, $\nabla_z G^\ell$, may be treated in the same way. Hence, putting together the above inequalities, we finally get

$$\begin{aligned}&\|V^\ell\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|V^\ell_t, \nabla^2 V^\ell, \nabla Q^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\ &\quad\lesssim \|U^\ell|_{t=0}\|_{\dot{B}^s_{p,1}(\Omega)} + \|f^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\\ &\qquad + \lambda\|V^\ell_t, \nabla^2 V^\ell, \nabla Q^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla V^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

By interpolation, we have

$$\|\nabla V^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq \|\nabla^2 V^\ell\|^{1/2}_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\|V^\ell\|^{1/2}_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.$$

Now, using Young's inequality to handle the term with ∇B^ℓ, taking λ so small as the term

$$\lambda\|V^\ell_t, \nabla^2 V^\ell, \nabla Q^\ell\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}$$

to be absorbed by the l.h.s., and using (4.74) and (4.75), we end up with

$$\begin{aligned}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|V^\ell_t,\nabla^2V^\ell,\nabla Q^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\lesssim\|u_0^\ell\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\&\quad+\lambda^{-1}\big(\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\widetilde\eta^\ell P\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))}\\&\qquad+\|\widetilde\eta^\ell u\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\big).\end{aligned}$$

In order to handle the last two terms, there are two ways of proceeding depending on whether we want a time dependent constant or not. Throughout, we fix some $a\in(0,1/2)$ given by Lemma 4.3.2 and choose ε so that $s+1-2a=\varepsilon+1/p$. The first possibility is to write that, by interpolation and Hölder's inequality,

$$\|\widetilde\eta^\ell u\|_{L_1(0,T;\dot B^{s+2-2a}_{p,1}(\Omega))}\quad\leq\quad T^a\|\widetilde\eta^\ell u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(\Omega))\cap L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

and that, according to (4.57), we have

$$\|\widetilde\eta^\ell P\|_{L_1(0,T;\dot B^{\varepsilon+1/p}_{p,1}(\Omega)\cap\dot B^s_{p,1}(\Omega))}\lesssim T^a\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}.$$

This yields

$$\begin{aligned}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|V^\ell_t,\nabla^2V^\ell,\nabla Q^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\lesssim\|u_0^\ell\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\&\quad+\lambda^{-1}T^a\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}\\&\qquad+\lambda^{-1}T\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}\tag{4.78}$$

The second possibility is to write that

$$\|\widetilde\eta^\ell u\|_{L_1(0,T;\dot B^{s+2-2a}_{p,1}(\Omega))}\quad\leq\quad\|\widetilde\eta^\ell u\|^{1-a}_{L_1(0,T;\dot B^{s+2}_{p,1}(\Omega))}\|\widetilde\eta^\ell u\|^a_{L_1(0,T;\dot B^s_{p,1}(\Omega))},$$

and to bound the pressure term according to (4.56). We eventually get

$$\begin{aligned}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|V^\ell_t,\nabla^2V^\ell,\nabla Q^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\lesssim\|u_0^\ell\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\lambda^{-1}\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\quad+\lambda^{-1}\|u\|^{1-a}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^a_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}\tag{4.79}$$

Fifth step: global a priori estimates. In view of Lemma 2.1.1, we may write

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}&\leq\sum_\ell\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\\&\lesssim\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\sum_{\ell\geq1}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\end{aligned}$$

and similar inequalities for the other terms of the l.h.s of (4.78). Of course, Proposition 2.1.2 ensures that

$$\|u_0^\ell\|_{\dot B^s_{p,1}(\Omega)} \lesssim \|u_0\|_{\dot B^s_{p,1}(\Omega)} \quad \text{and} \quad \|\widetilde{\eta}^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \lesssim \|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}.$$

So using also (4.61) and the fact that $L \approx \lambda^{-n}$, and bearing in mind (4.53), we get

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} &+ \|(u_t,\nabla^2 u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\lesssim \lambda^{-n}\Big(\|u_0\|_{B^s_{p,1}(\mathbb{R}^n)} + \|(f,\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\qquad + \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))} + \|R\|_{L_1(0,T;L_m(\Omega)\cap L_1(0,T;B^{1+s}_{p,1}(K))}\Big)\\ &\qquad\quad + \lambda^{-n-1}(T^a+T)\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}.\end{aligned}$$

Hence

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1})} &+ \|u_t,\nabla^2 u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1})}\\ &\le C\Big(\|u_0\|_{B^s_{p,1}(\mathbb{R}^n)} + \|(f,\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\qquad + \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))} + \|R\|_{L_1(0,T;L_m(\Omega)\cap B^{1+s}_{p,1}(K))}\Big)\end{aligned}$$

for a very short time T depending only on λ.

Repeating the argument over the interval $[T,2T]$ and so on, we get (4.37) with the constant Ce^{CT} for some suitably large constant C.

Removing the time-dependency is just a matter of starting from (4.63) and (4.79) instead of (4.61) and (4.78). After a few computations and use of (4.53), we get for some constant C depending on λ,

$$\begin{aligned}&\|u_t,\nabla^2 u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\quad\le C\Big(\|u_0\|_{\dot B^s_{p,1}(\Omega)} + \|f,\nabla g,R_t\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}\\ &\qquad + \|R\|_{L_1(0,T;L_m(\Omega))\cap L_1(0,T;B^{1+s}_{p,1}(K))}\\ &\qquad\quad + \|u\|^{1/2}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^{1/2}_{L_1(0,T;\dot B^s_{p,1}(K))}\\ &\qquad\qquad + \|u\|^{1-a}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^{a}_{L_1(0,T;\dot B^s_{p,1}(K))} + \|u\|_{L_1(0,T;B^s_{p,1}(K))}\Big).\end{aligned}$$

Using Young's inequality, it is easy to absorb the second line, up to an error term which may be bounded by the last term. □

REMARK 4.3.4. — Let us emphasize that the term $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ may be replaced by any lower norm taken over a compact set K. In particular s can be put to zero.

REMARK 4.3.5. — In the case where the domain Ω is bounded, the proof is very similar : we still have to introduce some resolution of unity $(\eta^\ell)_{0\leq\ell\leq L}$ where, now, η^0 is supported in the interior of Ω hence has compact support. Step one (removing g) is directly based on our work in [**16**]. The main difference is in step 2 because Proposition 3.5.1 holds true in bounded domains for *any* $n \geq 2$. Hence, Theorem 4.3.1 is valid for $n \geq 2$, with K replaced by Ω.

4.3.2. A low order bound for the velocity on a compact set. — This section is to establish *time-independent* bounds in $L_1(0,T;\dot B^s_{p,1}(K))$ for the velocity satisfying System (4.35). Let us emphasize that this lower order term does not appear when removing the divergence part of the velocity (first step of the proof of Theorem 4.3.1). Therefore, we shall concentrate on the case of a divergence free solution, namely System (4.52).

The main result of the section is the following one.

LEMMA 4.3.3. — *Assume that* $n \geq 3$ *and that* $1 < p < n/2$. *There exists some* $s_p > 0$ *(depending only on* p *and* n*) such that for all* $s \in (-s_p, s_p)$ *sufficiently smooth solutions to* (4.52) *fulfill*

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))} \leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} + \|u_0\|_{\dot B^s_{p,1}(\Omega)}),$$

where C *is* independent of T.

Proof. — The proof consists in extending to the Besov setting the Lebesgue type estimates proved in [**39**].

To start with, we take advantage of the linearity of the system and of the uniqueness of solutions provided by (4.37) so as to decompose u into $u_1 + u_2$ with u_1 the solution of the system with zero initial data and source term f, and u_2 the solution of the system with no source term and initial data u_0. In other words $u = u_1 + u_2$ with

$$\begin{cases} u_{1,t} - \Delta u_1 + \nabla P_1 = f, & u_{2,t} - \Delta u_2 + \nabla P_2 = 0 \quad \text{in} \quad (0,T)\times\Omega,\\ \operatorname{div} u_1 = 0, & \operatorname{div} u_2 = 0 \quad \text{in} \quad (0,T)\times\Omega,\\ u_1|_{x\in\partial\Omega} = 0, & u_2|_{x\in\partial\Omega} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega,\\ u_1|_{t=0} = 0, & u_2|_{t=0} = u_0 \quad \text{on} \quad \Omega. \end{cases}$$

Let us first focus on u_1. Arguing by duality, one may write that

$$\|u_1(t)\|_{\dot{B}^s_{p,1}(K)} \leq C \sup \int_K u_1(t,x) \cdot \psi(x)\, dx,$$

where the supremum is taken over those $\psi \in \dot{B}^{-s}_{p',\infty}(K;\mathbb{R}^n)$ such that $\|\psi\|_{\dot{B}^{-s}_{p',\infty}(K)} = 1$. Recall that, by virtue of Corollary 2.2.1, such functions may be extended by 0 over $\mathbb{R}^n$. So we may assume that the supremum is taken over those functions ψ satisfying

$$\psi \in \dot{B}^{-s}_{p',\infty}(\mathbb{R}^n;\mathbb{R}^n) \quad \text{with norm } 1 \text{ and } \operatorname{Supp}\psi \subset K. \tag{4.80}$$

In what follows, it will be important to restrict our attention to functions ψ which are divergence free and satisfy $\psi \cdot \vec{n}|_{\partial\Omega} = 0$. Note that according to Corollary 3.5.1, the Helmholtz projector $\mathcal{P}$ is a selfmap over $\dot{B}^{-s}_{p',\infty}(\Omega)$ and that, since u_1 is divergence free, we may write

$$\int_K u_1(t,x) \cdot \psi(x)\, dx = \int_\Omega u_1(t,x) \cdot \mathcal{P}\psi(x)\, dx.$$

Set $\eta_0 := \mathcal{P}\psi$ and consider the solution η to the problem:

$$\begin{cases} \eta_t - \Delta\eta + \nabla Q = 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\eta = 0 & \text{in } (0,T)\times\Omega, \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \eta|_{t=0} = \eta_0 & \text{on } \Omega. \end{cases} \tag{4.81}$$

Testing the equation for u_1 by $\eta(t-\cdot)$ we discover that

$$\int_\Omega u_1(t,x)\cdot\eta_0(x)\, dx = \int_0^t \int_\Omega f(\tau,x)\cdot\eta(t-\tau,x)\, dx\, d\tau. \tag{4.82}$$

As we plan to adapt the approach of [**39**] to Besov spaces, we first have to extend to that setting the classical so-called $L_a - L_b$ estimates of the Stokes semigroup. Recall that in exterior domains we have in any dimension $n \geq 2$ (see e.g. [**32, 40**]):

$$\|\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{for} \quad 1 < b \leq a < \infty, \tag{4.83}$$

$$\|\nabla\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})-\frac{1}{2}} \quad \text{for} \quad 1 < b \leq a \leq n. \tag{4.84}$$

We claim that for all $1 < p \leq q < \infty$, there exists some (small) positive $s_{p,q}$ depending only on n, p, q such that

$$\|\eta(t)\|_{\dot{B}^s_{q,q}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^s_{p,p}(\Omega)} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } s \in (0, s_{p,q}). \tag{4.85}$$

Indeed, first we use the classical embedding theorem (see e.g. [**48**], p. 31) [6]:

$$L_a(\mathbb{R}^n) \hookrightarrow \dot{B}^{-s_0}_{\bar{a},\bar{a}}(\mathbb{R}^n) \quad \text{and} \quad \dot{B}^{s_0}_{\bar{b},\bar{b}}(\mathbb{R}^n) \hookrightarrow L_b(\mathbb{R}^n)$$

for

$$\frac{1}{a} - \frac{1}{\bar{a}} = \frac{s_0}{n} = \frac{1}{\bar{b}} - \frac{1}{b} \tag{4.86}$$

and $1 < a < \bar{a} < \infty$, $1 < \bar{b} < b < \infty$ to write that, according to (4.83),

$$\|\eta(t)\|_{\dot{B}^{-s_0}_{\bar{a},\bar{a}}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^{s_0}_{\bar{b},\bar{b}}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.87}$$

By the same token, (4.84) implies that

$$\|\eta(t)\|_{\dot{B}^{1-s_1}_{\bar{a}_1,\bar{a}_1}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^{s_1}_{\bar{b}_1,\bar{b}_1}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b_1}-\frac{1}{a_1})-\frac{1}{2}}, \tag{4.88}$$

provided

$$\frac{1}{a_1} - \frac{1}{\bar{a}_1} = \frac{s_1}{n} = \frac{1}{\bar{b}_1} - \frac{1}{b_1} \tag{4.89}$$

under the restriction $1 < b_1 \leq a_1 \leq n$, $1 < \bar{b}_1 < b_1$ and $\bar{a}_1 > a_1$.

Now, let us recall the interpolation property:

$$(\dot{B}^{s_1}_{a,a}(\Omega), \dot{B}^{s_2}_{b,b}(\Omega))_{\theta,q} = \dot{B}^{s}_{q,q}(\Omega)$$

with

$$\theta s_2 + (1-\theta)s_1 = s, \qquad \frac{1}{q} = \frac{\theta}{b} + \frac{1-\theta}{a}\cdot$$

Let us fix some small enough positive s. Then we see that combining Inequalities (4.87) and (4.88) with the above interpolation property yields (4.85) with decay exponent σ given by

$$\sigma := \frac{n}{2}\Big(\frac{1}{b} - \frac{1}{a}\Big)\theta + \frac{n}{2}\Big(\frac{1}{b_1} - \frac{1}{a_1}\Big)(1-\theta) + \frac{1}{2}(1-\theta),$$

provided one may find some $\theta \in (0,1)$, a, $\bar{a}$, b, $\bar{b}$, a_1, $\bar{a}_1$, b_1, $\bar{b}_1$, s_0 and s_1 so that (4.86) and (4.89) are satisfied together with,

$$\theta(-s_0) + (1-\theta)(1-s_1) = \theta s_0 + (1-\theta)s_1 = s,$$
$$\frac{1}{q} = \frac{\theta}{\bar{a}} + \frac{1-\theta}{\bar{a}_1}, \qquad \frac{1}{p} = \frac{\theta}{\bar{b}} + \frac{1-\theta}{\bar{b}_1}\cdot$$

Thus we see that θ must satisfy

$$s = \frac{1}{2}(1-\theta).$$

6. Which naturally extends to general domains, owing to our definition of spaces by restriction.

Finally, we get

$$\begin{aligned}\sigma &= \frac{n}{2}\Big(\frac{1}{\bar{\bar{b}}}-\frac{1}{\bar{a}}-\frac{2s_0}{n}\Big)\theta+\frac{n}{2}\Big(\frac{1}{\bar{\bar{b}}_1}-\frac{1}{\bar{a}_1}-\frac{2s_1}{n}\Big)(1-\theta)+\frac{1}{2}(1-\theta)\\ &= \frac{n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big)-s+\frac{1}{2}(1-\theta)=\frac{n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big),\end{aligned}$$

and we are done.

Let us emphasize that if s is close to zero then θ is close to 1. Hence $a, b, \bar{a}, \bar{b}$ may be chosen very close to p, q. Therefore Inequality (4.85) is valid *for all* $1 < p \le q < \infty$.

In order to extend (4.85) to negative indices s and $1 < q < \infty$, we consider the dual problem

$$\begin{cases} \zeta_t - \Delta\zeta + \nabla Q' = 0 & \text{in} \quad (0,T)\times\Omega,\\ \operatorname{div}\zeta = 0 & \text{in} \quad (0,T)\times\Omega,\\ \zeta = 0 & \text{on} \quad (0,T)\times\partial\Omega,\\ \zeta|_{t=0} = \zeta_0 & \text{on} \quad \Omega, \end{cases} \tag{4.90}$$

where $\zeta_0 \in \dot{B}^{-s}_{b',q'}(\Omega)$, is divergence free and satisfies $\zeta_0 \cdot \vec{n} = 0$ at the boundary.

Testing (4.90) by $\eta(t-\cdot)$ yields

$$\int_\Omega \eta(t,x)\cdot\zeta_0(x)\,dx = \int_\Omega \eta_0(x)\cdot\zeta(t,x)\,dx. \tag{4.91}$$

Let us observe that

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\sup_{\zeta_0}\int_\Omega \eta(t,x)\cdot\zeta_0(x)\,dx,$$

where the supremum is taken over all $\zeta_0 \in \dot{B}^{-s}_{a',q'}(\Omega)$ such that $\operatorname{div}\zeta_0 = 0$, $\zeta\cdot\vec{n}|_{\partial\Omega} = 0$ and $\|\zeta_0\|_{\dot{B}^{-s}_{a',q'}(\Omega)} = 1$. Thus by virtue of (4.91), we get:

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\sup_{\zeta_0}\int_\Omega \eta_0(x)\cdot\zeta(t,x)\,dx \le \|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)}\sup_{\zeta_0}\|\zeta(t)\|_{\dot{B}^{-s}_{b',q'}(\Omega)}.$$

Because $-s$ is positive we can apply (4.85) (if s is close enough to 0) and get

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{a'}-\frac{1}{b'})}\sup_{\zeta_0}\|\zeta_0\|_{\dot{B}^{-s}_{a',q'}(\Omega)}.$$

Since $\frac{1}{a'}-\frac{1}{b'} = \frac{1}{b}-\frac{1}{a}$, we conclude that

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.92}$$

In order to get the remaining case $s = 0$, one may argue by interpolation between (4.85) and (4.92). One can thus conclude that for all $1 < b \le a < \infty$, $q \in [1,\infty]$ and s close enough to 0, we have

$$\|\eta(t)\|_{\dot B^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot B^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.93}$$

Now we return to the initial problem of bounding u_1. Starting from (4.82) and using duality, one may write

$$\Big|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\Big| \lesssim \int_0^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta(t-\tau)\|_{\dot B^{-s}_{p',\infty}(\Omega)}\,d\tau.$$

Hence splitting the interval $(0,t)$ into $(0,\max(0,t-1))$ and $(\max(0,t-1),t)$ and applying (4.93) yields for small enough ε,

$$\begin{aligned}\Big|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\Big| \\ \lesssim \int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}\,d\tau \\ + \int_0^{\max(0,t-1)} \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)}(t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)}\,d\tau.\end{aligned}$$

Recall that $\eta_0 = \mathcal{P}\psi$ with ψ satisfying (4.80). Using the properties of continuity of $\mathcal{P}$, we can thus write for $1 < a \le p'$,

$$\|\eta_0\|_{\dot B^{-s}_{a,\infty}(\Omega)} \le C\|\psi\|_{\dot B^{-s}_{a,\infty}(\Omega)}.$$

Now, as ψ is supported in K, one has

$$\|\psi\|_{\dot B^{-s}_{a,\infty}(\Omega)} \le C|K|^{\frac{1}{p}+\frac{1}{a}-1}\|\psi\|_{\dot B^{-s}_{p',\infty}(\Omega)}.$$

This may be proved by introducing a suitable smooth cut-off function with value 1 over K, taking advantage of Proposition 2.1.2. A scaling argument yields the dependency of the norm of the embedding with respect to $|K|$. Hence we have for some constant C depending on K,

$$\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}} \le C\|\psi\|_{\dot B^{-s}_{p',\infty}(\Omega)}.$$

So, keeping in mind (4.82) and the fact that the supremum is taken over all the functions ψ satisfying (4.80), we deduce that

$$\begin{aligned}&\|u_1(t)\|_{\dot B^s_{p,1}(K)} \\ &\le C\Big(\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,ds + \int_0^{\max(0,t-1)} (t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,ds\Big).\end{aligned}$$

Therefore,

$$\int_1^T \|u_1\|_{\dot B^s_{p,1}(K)}dt \le C\Big(1+\int_1^T s^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\,ds\Big)\int_0^T \|f(t)\|_{\dot B^s_{p,1}(\Omega)}\,dt. \tag{4.94}$$

For the time interval $[0,1]$, we merely have

$$\int_0^1 \|u_1(t)\|_{\dot B^s_{p,1}(K)}\,dt \le C\int_0^1 \|f(t)\|_{\dot B^s_{p,1}(\Omega)}\,dt. \tag{4.95}$$

Now, provided that one may find some $\varepsilon > 0$ such that

$$\frac{n}{2}\Big(\frac{1}{p}-\epsilon\Big) > 1,$$

a condition which is equivalent to $p < n/2$, the constant in (4.94) may be chosen independent of T, and we conclude that

$$\int_0^T \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \le C\int_0^T \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt.$$

Bounding u_2 is rather straightforward. We first write that

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \le C\|u_0\|_{\dot B^s_{p,1}(\Omega)}$$

and

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \le C|K|^{\frac{1}{p}-\epsilon}\|u_2(t)\|_{\dot B^s_{\frac{1}{\epsilon},1}(K)} \le C|K|^{\frac{1}{p}-\epsilon}\|u_0\|_{\dot B^s_{p,1}(\Omega)}t^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}.$$

Then decomposing the integral on $[0,\min(1,T)]$ into an integral on $[0,1]$ and on $[1,\min(1,T)]$, we easily get

$$\int_0^T \|u_2(t)\|_{\dot B^s_{p,1}(K)}\,dt \lesssim \Big(1+\int_{\min(1,T)}^T t^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\,dt\Big)\|u_0\|_{\dot B^s_{p,1}(\Omega)} \lesssim \|u_0\|_{\dot B^s_{p,1}(\Omega)}.$$

Putting this together with (4.94) and (4.95) completes the proof. □

We end this section with a few remarks concerning the case where Ω is a bounded domain of $\mathbb{R}^n$ with $n \ge 2$. Then it is standard (a consequence of e.g. [**27**]) that the solution η to (4.81) satisfies for some $c > 0$,

$$\|\eta(t)\|_{L_p(\Omega)} \le Ce^{-ct}\|\eta_0\|_{L_p(\Omega)},$$

and it is also true that, denoting by A the Stokes operator,

$$\|A\eta(t)\|_{L_p(\Omega)} \le Ce^{-ct}\|A\eta_0\|_{L_p(\Omega)}.$$

By interpolation, we thus have for any $1 < p < \infty$ and $-1+1/p < s < 1/p$,

$$\|\eta(t)\|_{\dot B^s_{p,1}(\Omega)} \le Ce^{-ct}\|\eta_0\|_{\dot B^s_{p,1}(\Omega)}.$$

Therefore we may write

$$\Big|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\Big| \leq C\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}\int_0^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}e^{-c(t-\tau)}\,d\tau,$$

thus giving

$$\|u_1\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}.$$

Similarly, we have

$$\|u_2\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C\|u_0\|_{\dot B^s_{p,1}(\Omega)}.$$

So we end up with the following statement:

Lemma 4.3.4. — *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n\geq 2$. Then for all $1<p<\infty$ and $-1+1/p<s<1/p$, there exists a constant C such that for all $T>0$, sufficiently smooth solutions to (4.52) fulfill*

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u_0\|_{\dot B^s_{p,1}(\Omega)}).$$

4.3.3. The final result. — If Ω is a bounded domain then putting together Remark 4.3.5 with $K=\Omega$, and Lemma 4.3.4 yields the following statement:

Theorem 4.3.2. — *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n\geq 2$. Let $1<p<\infty$ and $-1+1/p<s<1/p$. Let $u_0\in\dot B^s_{p,1}(\Omega)$, $f\in L_1(0,T;\dot B^s_{p,1}(\Omega))$, $g\in\mathcal{C}([0,T];\dot B^{s-1}_{p,1}(\Omega))$ with*

$$g(0)=\operatorname{div}u_0,\quad \nabla g\in L_1(0,T;\dot B^s_{p,1}(\Omega))\quad\text{and}\quad g=\operatorname{div}R$$

with R satisfying all the conditions of Theorem 4.3.1.

Then there exists a unique solution $(u,\nabla P)$ to System (4.35) such that

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} &+\|(u_t,\nu u,\nu\nabla^2u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|(f,\nu\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\qquad+\|R\|_{L_1(0,T;B^{s+1}_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}\big),\end{aligned}$$

where C is independent of T and ν.

Let us finally give the statement if Ω is an exterior domain of $\mathbb{R}^n$ with $n\geq 3$.

Theorem 4.3.3. — *Let $1<q\leq p<\infty$ with $q<n/2$. Let $-1+1/p<s<1/p$ and s' close enough to 0. Assume that*

$$u_0\in\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega),\qquad f\in L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)),$$

$g\in\mathcal{C}([0,T];\dot B^{s-1}_{p,1}\cap\dot B^{s'-1}_{q,1}(\Omega))$ *with* $g(0)\equiv\operatorname{div}u_0$, $\nabla g\in L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))$

and $g = \operatorname{div} R$ *with* R *satisfying the conditions of Theorem* 4.3.1 *with respect to* (s,p) *and* (s',q).

Then there exists a unique solution $(u,\nabla P)$ *to System* (4.35) *such that*

$$\begin{aligned}(4.96)\quad &\|u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}+\|(u_t,\nu\nabla^2u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\\&\qquad\qquad+\nu\|u|_K\|_{L_1(0,T;B^{s'}_{q,1}(K))}\\&\leq C\big(\|u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)}+\|(f,\nu\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\\&\quad+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}\cap B^{s'-\frac1q}_{q,1}(\partial\Omega))}+\nu\|R\|_{L_1(0,T;L_m(\Omega)\cap B^{1+s'}_{q,1}(K)\cap B^{1+s}_{p,1}(K))}\big),\end{aligned}$$

where C *in* (4.96) *is independent of* T *and* ν.

Proof. — Granted with Theorem 4.3.1 and Inequality (4.38), it is enough to show that

$$\|u|_K\|_{L_1(0,T;B^s_{p,1}(K)\cap B^{s'}_{q,1}(K))}$$

may be bounded by the right-hand side of (4.96). Of course, in the case $p<n/2$, it readily stems from Lemma 4.3.3 (combined with interpolation if s is not close enough to 0).

Now, if $p\geq n/2$ then we use the continuous embedding

$$\dot B^{s'+2}_{q,1}(\Omega)\hookrightarrow\dot B^s_{q^*,1}(\Omega)\quad\text{with}\quad\frac{1}{q^*}=\frac1q-\frac2n+\frac{s-s'}{n}\cdot$$

Then combining with interpolation and Lemma 4.3.3 allows to absorb the term $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ whenever $p<q^*$. This completes the proof of Theorem 4.3.3 for those values of s,s' if $1<p<q^*$.

If $p\geq q^*$ then we know by interpolation that data fulfill the assumptions of the theorem we want to prove for any $q\leq p_1<q^*$. Therefore, we may use the embedding of $B^{s+2}_{p_1,1}(\Omega)$ in $B^s_{q_1^*,1}$ with $1/q_1^*=1/q_1-2/n$ to treat the case where $p<q_1^*$. It is now clear that any finite value of p may be reached after a finite number of iterations. This completes the proof of Theorem 4.3.3. □

CHAPTER 5

INHOMOGENEOUS NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

This chapter presents a first important application of the results that we established so far for the Stokes system. We analyze here the Navier-Stokes equations modeling flows of incompressible *and* inhomogeneous fluids. In this context, the density is constant along the stream lines. We shall see that the L_1-integrability in time property for the velocity field that has been proved in the previous chapter enables us to recast the whole system of equations in the Lagrangian coordinates. This will allow us to construct unique strong solutions for quite general initial data : as regards the initial density, piecewise constant initial configurations may be considered for instance. Let us emphasize that according to several recent works [**19**], [**31**], [**38**] it is even possible to build strong unique solutions assuming that the initial density is only bounded and bounded away from zero. However, the velocity has to be smooth enough therein. Here, following our recent work in [**17**] devoted to the case where the fluid domain is the whole space $\mathbb{R}^n$, we assume the initial velocity to have *critical regularity*, which requires to slightly enhance the regularity assumptions on the density. Nevertheless we shall see that initial densities as in (1.4) may be considered.

This chapter unfolds as follows. In the first section, we present the inhomogeneous Navier-Stokes equations in Eulerian and Lagrangian coordinates. The next section is devoted to the study of a suitable linearization of those equations. This will eventually enable us to prove the local (resp. global) existence of strong solutions for large (resp. small) data in the rest of the chapter.

5.1. Lagrangian stream lines setting

In the Eulerian coordinates, the inhomogeneous incompressible Navier-Stokes equations read

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0, & \text{in} \quad (0,T)\times\Omega, \\ \rho(u_t + u\cdot\nabla u) - \mu\Delta u + \nabla P = 0 & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

Above, $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $P = P(t,x) \in \mathbb{R}$ stand for the density, velocity field and pressure of the fluid, respectively. The viscosity coefficient μ is positive and constant. For simplicity, we assume that there is no external force. We aim at constructing solutions $(\rho, u, \nabla P)$ so that ∇u is in $L_1(\mathbb{R}_+; L_\infty(\Omega))$. This will imply that the velocity field u has a unique measure preserving flow X, defined on $\mathbb{R}_+\times\Omega$. It will be thus possible to recast System (5.1) in Lagrangian coordinates, and to prove uniqueness under rather mild assumptions on the density (in particular small jumps are admitted). Using so-called *critical spaces*, that is, in our context, functional spaces with norm invariant for all $\ell > 0$ by the following transform

$$(\rho, u, \nabla P)(t,x) \longmapsto (\rho, \ell u, \ell^3\nabla P)(\ell^2 t, \ell x) \tag{5.2}$$

has become a classical approach nowadays, in the case where $\Omega = \mathbb{R}^n$ (see [**13**]). For more general domains, the above rescaling is no longer relevant as it changes the domain. However it still gives us a hint on the minimal local regularity that has to be assumed for the data, so as to prove the well-posedness of the equations. In the Besov spaces scale for instance, this suggests our taking u_0 in $\dot{B}^{n/p-1}_{p,1}(\Omega)$, which is in fact the only possibility ensuring the constructed velocity u to be in $L_1(\mathbb{R}_+; L_\infty(\Omega))$. Supplementary 'out of scaling' conditions are needed to control the decay of u, if Ω is an exterior domain.

Let us give more details on the 'Lagrangian approach'. The change from the Eulerian coordinates (t,x) to the Lagrangian coordinates (t,y) is defined by setting $x = X(t,y)$ with X the solution to the following (integrated) Ordinary Differential Equation:

$$X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau.$$

Because $u = 0$ at the boundary, this transform preserves the domain of the fluid : we have $X(t, \Omega) = \Omega$. Then we set

$$\bar\rho(t, y) := \rho(t, X(t, y)), \quad \overline{P}(t, y) := P(t, X(t, y)), \quad \bar u(t, y) := u(t, X(t, y)).$$

Given the definition of X and according to the chain rule, it is obvious that

$$\partial_t\bar\rho(t, y) = (\partial_t\rho + u\cdot\nabla_x\rho)(t, X(t, y)) \quad \text{and} \quad \partial_t\bar u(t, y) = (\partial_t u + u\cdot\nabla_x u)(t, X(t, y)).$$

Besides, denoting by $Y(t,\cdot)$ the inverse diffeomorphism of $X(t,\cdot)$, we may write

$$\nabla_x P(t, x) = {}^T\overline{B}(t, y)\cdot\nabla_y\overline{P}(t, y) \tag{5.3}$$

with $x := X(t, y)$ and $\overline{B}(t, y) := D_xY(t, x)$.

The fact that X is measure preserving implies that for any smooth enough vector field H one has (see (2.36) and (2.37))

$$\operatorname{div}_x H = {}^T\overline{B} : \nabla_y\bar u = \operatorname{div}_y(\overline{B}\,\overline{H}), \tag{5.4}$$

$$\Delta_x H^i = \operatorname{div}_x\nabla_x H^i = \operatorname{div}_y(\overline{B}^T\overline{B}\nabla_y\overline{H}^i). \tag{5.5}$$

So finally, we see that, at least formally, $(\rho, u, \nabla_x P)$ satisfies (5.1) if and only if $\bar\rho \equiv \rho_0$ and $(\bar u, \nabla_y\overline{P})$ satisfies

$$\begin{cases} \rho_0\bar u_t - \mu\operatorname{div}_y(\overline{B}^T\overline{B}\nabla_y u) + {}^T\overline{B}\nabla_y\overline{P} = 0, \\ \operatorname{div}_y(\overline{B}\bar u) = 0 \end{cases} \tag{5.6}$$

with

$$\overline{B}(t, y) = D_xY(t, x) = (D_yX(t, y))^{-1} \quad \text{and} \quad X(t, y) = y + \int_0^t \bar u(\tau, y)\,d\tau.$$

By adapting the arguments that have been used in [**17**] to the domain setting, one may show that systems (5.1) and (5.6) are equivalent in the functional framework that we shall use. Hence we shall focus on solving (5.6) rather than (5.1), in the rest of this chapter.

5.2. The linearized equations

We are concerned with the following linearization of System (5.6):

$$\begin{cases} \rho_t + v\cdot\nabla\rho = 0 & \text{in} \quad (0, T)\times\Omega, \\ q(u_t + v\cdot\nabla u) - \mu\Delta u + \nabla P = 0 & \text{in} \quad (0, T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0, T)\times\Omega, \\ u = 0 & \text{at} \quad (0, T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{at} \quad \Omega. \end{cases} \tag{5.7}$$

In the above system the vector-field v and the positive function q are given. We assume in addition that $\operatorname{div} v = 0$ and that the trace of v is zero at the boundary.

Introducing Lagrangian coordinates with respect to the vector field v, that is setting $y := Y_v(t,x)$ with $Y_v(t,\cdot) := (X_v(t,\cdot))^{-1}$ and X_v defined by

$$X_v(t,y) = y + \int_0^t v(\tau, X_v(\tau,y))\,d\tau, \tag{5.8}$$

and

$$\overline{B}_v(t,y) := D_x Y_v(t,x) = (D_y X_v(t,y))^{-1}, \quad \bar\rho(t,y) := \rho(t, X_v(t,y)),$$
$$\overline{P}(t,y) := P(t, X_v(t,y)) \;\text{ and }\; \bar u(t,y) := u(t, X_v(t,y)),$$

we see that, under suitable regularity assumptions, $(\rho, u, \nabla_x P)$ satisfies (5.7) if and only if $\bar\rho(t,y) = \rho_0(y)$ and $(\bar u, \nabla_y \overline{P})$ satisfies

$$\begin{cases} \rho_0 \bar u_t - \mu \operatorname{div}_y (\overline{B}_v{}^T \overline{B}_v \nabla_y \bar u) + {}^T\overline{B}_v \nabla_y \overline{P} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div}_y (\overline{B}_v \bar u) = 0 & \text{in} \quad (0,T)\times\Omega, \\ \bar u = 0 & \text{at} \quad (0,T)\times\partial\Omega, \\ \bar u|_{t=0} = u_0 & \text{at} \quad \Omega. \end{cases} \tag{5.9}$$

Here we aim at proving existence results for (5.7) in the critical functional framework in which (5.1) and (5.6) are going to be solved.

Before giving the main statement, let us introduce a few notation. First, we denote by X_T^p the set of $(u, \nabla P)$ so that $u|_{\partial\Omega} = 0$,

$$u \in \mathcal{C}([0,T]; \dot B^{n/p-1}_{p,1}(\Omega)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T; \dot B^{n/p-1}_{p,1}(\Omega)),$$

and set

$$\|(u,\nabla P)\|_{X_T^p} := \|u\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))} + \|u_t, \mu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}. \tag{5.10}$$

Proposition 5.2.1. — *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ $(n \geq 3)$ or bounded domain $(n \geq 2)$. Let $\overline{T} > 0$ and $p \in (n-1, 2n)$. Assume that $v \in \mathcal{C}([0,\overline{T}]; \dot B^{n/p-1}_{p,1}(\Omega))$ with $\operatorname{div} v = 0$, $\nabla v \in L_1(0,\overline{T}; \dot B^{n/p}_{p,1}(\Omega))$ and $v|_{\partial\Omega} = 0$. There exists a constant $c = c(n,p,\Omega)$ so that if*

$$\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))} + \|\nabla \bar v\|_{L_1(0,\overline{T};\dot B^{n/p}_{p,1}(\Omega))} \leq c, \tag{5.11}$$

then for any divergence-free data $u_0 \in \dot B^{n/p-1}_{p,1}(\Omega)$ with $(u_0 \cdot \vec n)|_{\partial\Omega} = 0$ System (5.9) has a unique solution $(\bar u, \nabla\overline{P})$ on $[0,\overline{T}]$, belonging to $X^p_{\overline{T}}$ and so *that for some constant* $C = C(n,p,\Omega)$ *we have*

$$\|(\bar{u},\nabla\overline{P})\|_{X^p_{\overline{T}}} \leq Ce^{C\overline{T}}\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}. \tag{5.12}$$

Proof. — We focus on the exterior domain case. Using (4.14) enables us to restrict ourselves to the case $\mu = 1$. So we make this assumption throughout. The proof is based on Theorem 4.3.1 and on the Banach fixed point theorem after observing that System (5.9) recasts in

$$\begin{cases} \bar{u}_t - \Delta_y\bar{u} + \nabla_y\overline{P} = f_v(\bar{u},\nabla\overline{P}) & \text{in} \quad (0,T)\times\Omega,\\ \operatorname{div}_y\bar{u} = g_v(\bar{u}) & \text{in} \quad (0,T)\times\Omega,\\ \bar{u} = 0 & \text{at} \quad (0,T)\times\partial\Omega,\\ \bar{u}|_{t=0} = u_0 & \text{at} \quad \Omega, \end{cases} \tag{5.13}$$

with [1]

$$\begin{aligned} f_v(\bar{u},\nabla\overline{P}) &:= (1-\rho_0)\partial_t\bar{u} + \operatorname{div}\left((\overline{B}_v{}^T\overline{B}_v - \operatorname{Id})\nabla_y\bar{u}\right) + (\operatorname{Id} - {}^T\overline{B}_v)\nabla_y\overline{P},\\ g_v(\bar{u}) &:= (\operatorname{Id} - {}^T\overline{B}_v) : \nabla_y\bar{u} = \operatorname{div} R_v(\bar{u}) \quad \text{with} \quad R_v(\bar{u}) := (\operatorname{Id} - \overline{B}_v)\bar{u}. \end{aligned}$$

Hence to show existence for (5.13), it suffices to find a fixed point for the map

$$\Phi : (\bar{w},\nabla\overline{Q}) \to (\bar{u},\nabla\overline{P}) \tag{5.14}$$

with $(\bar{u},\nabla\overline{P})$ the solution to

$$\begin{cases} \bar{u}_t - \Delta\bar{u} + \nabla\overline{P} = f_v(\bar{w},\nabla\overline{Q}) & \text{in} \quad (0,T)\times\Omega,\\ \operatorname{div}\bar{u} = g_v(\bar{w}) & \text{in} \quad (0,T)\times\Omega,\\ \bar{u} = 0 & \text{at} \quad (0,T)\times\partial\Omega,\\ \bar{u}|_{t=0} = u_0 & \text{at} \quad \Omega. \end{cases} \tag{5.15}$$

Let us decompose $\Phi(\bar{w},\nabla\overline{Q})$ into

$$\Phi(\bar{w},\nabla\overline{Q}) = (u_L,\nabla P_L) + \Psi(\bar{w},\nabla\overline{Q}),$$

where $(u_L,\nabla P_L)$ stands for the free solution to the Stokes system with initial data u_0, namely

$$\begin{cases} \partial_t u_L - \mu\Delta u_L + \nabla P_L = 0 & \text{in} \quad (0,T)\times\Omega,\\ \operatorname{div} u_L = 0 & \text{in} \quad (0,T)\times\Omega,\\ u_L = 0 & \text{at} \quad (0,T)\times\partial\Omega,\\ u_L|_{t=0} = u_0 & \text{at} \quad \Omega. \end{cases} \tag{5.16}$$

1. That $g_v(\bar{u})$ may be written in two different ways is a consequence of (5.4) because $\operatorname{div} v = 0$; this is of course fundamental.

Theorem 4.3.1 guarantees that $(u_L, \nabla P_L) \in X^p_T$ for all $T \in \mathbb{R}_+$, and that [2]

$$\|(u_L, \nabla P_L)\|_{X^p_T} \leq Ce^{CT\mu}\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}. \tag{5.17}$$

Hence in order to establish that the map Φ fulfills the conditions of Banach fixed point theorem, it is only a matter of finding a condition under which the linear map Ψ is a self-map on X^p_T, with norm smaller than, say, 1/2. Now, we notice that $R_v(\bar{w})$ vanishes at the boundary and one may thus apply Theorem 4.3.1 to bound $\Psi(\bar{w}, \nabla\overline{Q})$ in X^p_T. Taking $s = n/p - 1$ and $m = 2n$ leads to

$$\begin{aligned}\|\Psi(\bar{w}, \nabla\overline{Q})\|_{X^p_T} \leq Ce^{CT}\Big(&\|R_v(\bar{w})\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\\ &+ \|(f_v(\bar{w}, \nabla\overline{Q}), \nabla g_v(\bar{w}), (R_v(\bar{w}))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\Big),\end{aligned} \tag{5.18}$$

where K is any bounded subset of Ω with $d(\Omega \setminus K, \partial\Omega) > 0$.

In the following computations, we agree that $(s, r) = (n/p - 1, p)$. First, from the expression of f_v and the definition of multiplier spaces, we readily have

$$\begin{aligned}&\|f_v(\bar{w}, \nabla\overline{Q})\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}\\ &\quad\lesssim \|1-\rho_0\|_{\mathcal{M}(\dot{B}^s_{r,1}(\Omega))}\|\bar{w}_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}\\ &\qquad+ \|\overline{B}_v{}^T\overline{B}_v - \mathrm{Id}\,\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla^2\bar{w}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}\\ &\qquad+ \|\nabla(\overline{B}_v{}^T\overline{B}_v)\|_{L_\infty(0,T;\dot{B}^s_{r,1}(\Omega))}\|\nabla\bar{w}\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\\ &\qquad+ \|\mathrm{Id} - \overline{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla\overline{Q}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}.\end{aligned} \tag{5.19}$$

Next, we have

$$\begin{aligned}&\|\nabla g_v(\bar{w})\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}\\ &\quad\lesssim \|\nabla\overline{B}_v\|_{L_\infty(0,T;\dot{B}^s_{r,1})}\|\nabla\bar{w}\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\\ &\qquad+ \|\mathrm{Id} - \overline{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla^2\bar{w}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))},\end{aligned} \tag{5.20}$$

$$\begin{aligned}&\|(R_v(\bar{w}))_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}\\ &\quad\lesssim \|\bar{w}\|_{L_\infty(0,T;\dot{B}^s_{r,1}(\Omega))}\|(\overline{B}_v)_t\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\\ &\qquad+ \|\mathrm{Id} - \overline{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\bar{w}_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}.\end{aligned} \tag{5.21}$$

2. Of course $\operatorname{div} u_0 \equiv 0$ implies that $\nabla P_L \equiv 0$.

In order to go further in the computations, we have to use Lemma 5.5.1 below which implies that all the above multiplier norms are controlled by the norm in $\dot{B}^{n/p}_{p,1}$. Therefore,

$$
\begin{aligned}
(5.22)\quad & \|f_v(\bar w,\nabla\overline{Q})\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&\quad +\|\nabla g_v(\bar w)\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+\|(R_v(\bar w))_t\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&\quad\quad \lesssim \big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1})}\big)\|(\bar w,\nabla\overline{Q})\|_{X^p_T}.
\end{aligned}
$$

We also have (use that $B^{n/p}_{p,1}(K)$ is an algebra, Proposition 2.1.3, and interpolation):

$$
\begin{aligned}
\|R_v(\bar w)\|_{L_1(0,T;B^{n/p}_{p,1}(K))} &\leq \|\mathrm{Id}-\overline{B}_v\|_{L_\infty(0,T;B^{n/p}_{p,1}(K))}\|\bar w\|_{L_1(0,T;B^{n/p}_{p,1}(K))}\\
&\lesssim T^{1/2}\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\nabla\bar w\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\qquad\qquad +\|\bar w\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\big),
\end{aligned}
$$

and, because $\dot B^{n/p-1/2}_{p,1}(\Omega)$ embeds in $L_{2n}(\Omega)$,

$$
\begin{aligned}
\|R_v(\bar w)\|_{L_1(0,T;L_{2n}(\Omega))} &\leq \|\mathrm{Id}-\overline{B}_v\|_{L_\infty(0,T;L_\infty(\Omega))}\|\bar w\|_{L_1(0,T;L_{2n}(\Omega))}\\
&\lesssim \|\nabla\bar v\|_{L_1(0,T;L_\infty(\Omega))}\|\bar w\|_{L_1(0,T;\dot B^{n/p-1/2}_{p,1}(\Omega))}\\
&\lesssim T^{3/4}\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\nabla\bar w\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\qquad\qquad +\|\bar w\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\big).
\end{aligned}
$$

So plugging all the previous estimates in (5.18), we conclude that

$$
\begin{aligned}
&\|\Psi(\bar w,\nabla\overline{Q})\|_{X^p_T}\\
&\quad\leq Ce^{CT}\big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big)\|(\bar w,\nabla\overline{Q})\|_{X^p_T}.
\end{aligned}
$$

Therefore, if we take $\eta>0$ so that $e^{C\eta}\leq 2$ and assume that

$$8C\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}\leq 1,$$

then we have

$$(5.23)\qquad \|\Psi(\bar w,\nabla\overline{Q})\|_{X^p_T}\leq\frac12\|(\bar w,\nabla\overline{Q})\|_{X^p_T},$$

whenever

$$(5.24)\qquad 8C\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1})}\leq 1\quad\text{and}\quad T\leq\eta.$$

In fact, if (5.24) is satisfied for $T=\overline{T}$, then one can get rid of the condition that $\overline{T}\leq\eta$: it suffices to split the interval $[0,\overline{T}]$ into subintervals $[T_i,T_{i+1}]$

$(i=0,\dots,k-1)$ of size at most η, and to use the norm (with obvious notation, see (5.10))

$$\|(\bar{w},\nabla\overline{Q})\|_{\widetilde{X}^p_{\overline{T}}} := \sum_{i=0}^{k-1}\|(\bar{w},\nabla\overline{Q})\|_{X^p_{T_i,T_{i+1}}}.$$

Now the above argument leading to (5.23) applies on every subinterval $[T_i,T_{i+1}]$ and we end up with

$$\|\Psi(\bar{w},\nabla\overline{Q})\|_{\widetilde{X}^p_{\overline{T}}} \le \frac{1}{2}\|(\bar{w},\nabla\overline{Q})\|_{\widetilde{X}^p_{\overline{T}}}.$$

Then applying the fixed point theorem in $X^p_{\overline{T}}$ endowed with the norm $\|.\|_{\widetilde{X}^p_{\overline{T}}}$ ensures the existence of a solution $(\bar{u},\nabla\overline{P})$ in $X^p_{\overline{T}}$ for (5.9). Note that by construction we have,

$$\|(\bar{u},\nabla\overline{P})\|_{\widetilde{X}^p_{\overline{T}}} \le 2\|(u_L,\nabla P_L)\|_{\widetilde{X}^p_{\overline{T}}},$$

which yields Inequality (5.12). □

Remark 5.2.1. — It is possible to extend the above proposition to other regularity exponents. However, owing to the properties of the multipliers spaces involved in (5.19), (5.20) and (5.21) we have to assume regularity in intersection of Besov spaces and the computations become quite cumbersome.

The above proposition will enable us to establish the local-in-time existence for (5.1). At the same time, it is not suitable for proving a global statement as *it does not* provide any bound on the gradient of the constructed velocity field in $L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))$. In order to overcome this, we shall establish a second existence result for the linear system (5.7), based on Theorem 4.3.3 so as to discard the time dependency in the estimates.

We shall work in the subspace $X^{p,q}_T$ of couples $(u,\nabla P)$ of X^p_T (see the definition in (5.10)) satisfying the additional property that

$$u\in\mathcal{C}([0,T];\dot{B}^0_{q,1}(\Omega)) \quad\text{and}\quad \partial_t u,\nabla^2 u,\nabla P\in L_1(0,T;\dot{B}^0_{q,1}(\Omega)),$$

and we shall set

$$\|(u,\nabla P)\|_{X^{p,q}_T} = \|(u,\nabla P)\|_{X^p_T} + \|(u,\nabla P)\|_{X^q_T}. \tag{5.25}$$

We agree that $X^{p,q}$ corresponds to the above definition with $T=+\infty$.

Proposition 5.2.2. — *Let $1<q\le p<2n$ with $q<n/2$ and $p>n-1$. Let v be a divergence-free vector field in $\mathcal{C}(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))$ with $v|_{\partial\Omega}=0$. There exists a constant c so that if*

$$\|\nabla v\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}\cap\dot{B}^1_{q,1}(\Omega))} \le c \tag{5.26}$$

and

$$\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))} \le c \tag{5.27}$$

then System (5.9) *with divergence free initial velocity* $u_0 \in \dot B^{n/p-1}_{p,1}(\Omega)\cap \dot B^0_{q,1}(\Omega)$ *satisfying* $u_0\cdot\vec n|_{\partial\Omega}=0$ *has a unique global solution* $(\bar u,\nabla\overline{P})$ *in* $X^{p,q}$, *and we have for some constant* $C=C(n,p,q)$,

$$\|(\bar u,\nabla\overline{P})\|_{X^{p,q}} \le C\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega)}.$$

Proof. — The proof is similar to that of the previous proposition except that it is now based on Theorem 4.3.3 to have *time independent* estimates. We readily get for any $m\in(1,\infty)$

$$\begin{aligned}\|\Psi(\bar w,\nabla\overline{Q})\|_{X^{p,q}} \le C\Big(&\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K)\cap B^{n/p}_{p,1}(K))}\\ &+\|(f_v(\bar w,\nabla\overline{Q}),\nabla g_v(\bar w),(R_v(\bar w))_t)\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\Big),\end{aligned} \tag{5.28}$$

where K is a bounded subset of Ω with $d(\Omega\setminus K,\partial\Omega)>0$ (see Fig. 3.4).

In order to go further in the computations, we use Lemma 5.5.1 that implies that all the multiplier norms in (5.19), (5.20) and (5.21) with $(s,r)=(n/p-1,p)$ or $(s,r)=(0,q)$ are controlled by the norm in $\dot B^{n/p}_{p,1}$. We get

$$\begin{aligned}&\|f_v(\bar w,\nabla\overline{Q}),\nabla g_v(\bar w),(R_v(\bar w))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\quad\lesssim \|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\|\bar w_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\qquad+\|\overline{B}_v-\mathrm{Id}\,,\,\overline{B}_v{}^T\overline{B}_v-\mathrm{Id}\,\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla^2\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\qquad+\|\mathrm{Id}\,-\overline{B}_v\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla Q\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\qquad+\|\nabla\overline{B}_v,\nabla(\overline{B}_v{}^T\overline{B}_v)\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ &\qquad+\|(\overline{B}_v)_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\bar w\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}.\end{aligned}$$

Using also Inequalities (5.49) to (5.54) below, we readily get

$$\begin{aligned}&\|f_v(\bar w,\nabla\overline{Q}),\nabla g_v(\bar w),(R_v(\bar w))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\quad\le C\Big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\|\bar w_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap \dot B^0_{q,1}(\Omega))}\\ &\qquad+\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|(\bar w,\nabla\overline{Q})\|_{X^{p,q}}\\ &\qquad\quad+\|\nabla^2 v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\Big).\end{aligned} \tag{5.29}$$

Let us observe that, owing to $1 < q < n/2$, the Besov space $\dot B^2_{q,1}(\Omega)$ embeds in the Lebesgue space $L_m(\Omega)$ with $m = \frac{qn}{n-2q}\cdot$ So we shall take this value of m in (5.28). We get

$$\begin{aligned}\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;L_m(\Omega))} &\leq \|\mathrm{Id}\, - B_v\|_{L_\infty(\mathbb{R}_+;L_\infty(\Omega))}\|\bar w\|_{L_1(\mathbb{R}_+;L_m(\Omega))}\\ &\leq \|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}.\end{aligned}\tag{5.30}$$

Let us now bound $R_v(\bar w)$ in $L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))$. We have

$$\begin{aligned}\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))} &\lesssim \|\mathrm{Id}\, - B_v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\\ &\lesssim \|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.\end{aligned}$$

Because $B^2_{q,1}(K)$ embeds in $L^m(K)$, it is not difficult to prove (just use the corresponding inequality in $\mathbb{R}^n$ and some suitable extension operator) that

$$\|\bar w\|_{B^{n/p}_{p,1}(K)} \lesssim \|\nabla\bar w\|_{B^{n/p}_{p,1}(K)} + \|\bar w\|_{B^2_{q,1}(K)}.\tag{5.31}$$

Therefore

$$\begin{aligned}&\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\\ &\qquad\lesssim \|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(\Omega))}\big(\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))} + \|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big).\end{aligned}\tag{5.32}$$

Let us finally bound $R_v(\bar w)$ in $L_1(\mathbb{R}_+;B^1_{q,1}(K))$. We use the fact that, because K is bounded and $q \leq m$,

$$\begin{aligned}\|R_v(\bar w)\|_{B^1_{q,1}(K)} &\lesssim \|R_v(\bar w)\|_{L_q(K)} + \|\nabla(R_v(\bar w))\|_{B^0_{q,1}(K)}\\ &\lesssim \|R_v(\bar w)\|_{L_m(\Omega)} + \|\nabla(R_v(\bar w))\|_{B^0_{q,1}(K)}.\end{aligned}\tag{5.33}$$

The first term in the r.h.s. may be handled according to (5.30). We decompose the second one into

$$\nabla(R_v(\bar w)) = (\mathrm{Id}\, - B_v)\nabla\bar w - \nabla B_v\,\bar w,$$

and use (5.48). Combining with the results of Section 5.5, we end up with

$$\begin{aligned}&\|\nabla(R_v(\bar w))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\\ &\qquad\lesssim \|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla\bar w\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\\ &\qquad\qquad + \|\nabla^2\bar v\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.\end{aligned}$$

Again, one may use (5.31), and the fact that

$$\|\nabla\bar w\|_{B^0_{q,1}(K)} \lesssim \|\bar w\|_{L_m(K)} + \|\bar w\|_{\dot B^2_{q,1}(K)} \lesssim \|\bar w\|_{\dot B^2_{q,1}(\Omega)},\tag{5.34}$$

owing to the boundedness of K. So finally

$$\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^1_{q,1}(K))} \lesssim \|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}\big(\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}+\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big). \tag{5.35}$$

Plugging Inequalities (5.29) to (5.35) in (5.28), we end up with

$$\|\Psi(\bar w,\nabla\overline{Q})\|_{X^{p,q}} \le C\Big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}+\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}\cap\dot B^1_{q,1}(\Omega))}\Big)\|(\bar w,\nabla\overline{Q})\|_{X^{p,q}}.$$

Therefore assuming that c is small enough in (5.26) and (5.27), we conclude that

$$\|\Psi(\bar w,\nabla\overline{Q})\|_{X^{p,q}}\le\frac{1}{2}\|(\bar w,\nabla\overline{Q})\|_{X^{p,q}}.$$

Applying the fixed point theorem in the Banach space $X^{p,q}$ completes the proof of Proposition 5.2.2. □

5.3. Local-in-time existence

This section is devoted to proving local-in-time existence for System (5.1) with slightly nonhomogeneous density and arbitrarily large initial velocity field. Here is the main statement:

THEOREM 5.3.1. — *Let $p\in(n-1,2n)$ and $u_0\in\dot B^{n/p-1}_{p,1}(\Omega)$ with $\operatorname{div}u_0=0$ and $u_0\cdot\vec n|_{\partial\Omega}=0$. Assume that $\rho_0\in\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))$ and that, for a small enough constant c,*

$$\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}\le c. \tag{5.36}$$

There exists $T>0$ such that System (5.6) has a unique solution $(\bar u,\nabla\overline{P})$ in the space X^p_T defined in (5.10), with

$$\|(\bar u,\nabla\overline{P})\|_{X^p_T}\le C\|u_0\|_{\dot B^{n/p-1}_{p,1}}.$$

Proof. — We consider the map

$$\mathcal{T}:(\bar v,\nabla\overline{Q})\longmapsto(\bar u,\nabla\overline{P}),$$

where $(\bar u,\nabla\overline{P})$ is the solution to (5.9) with $\bar v$ defining $\overline{B}_v$ constructed in Proposition 5.2.1.

We claim that $\mathcal{T}$ is a contraction in some suitable closed ball of X_T^p with sufficiently small T. As we aim at considering *large* initial velocity u_0 with critical regularity however, we take a ball centered at the solution $(u_L, \nabla P_L)$ to the homogeneous Stokes system (5.16) (that satisfies (5.17)). Then we focus on the discrepancy to $(u_L, \nabla P_L)$, namely

$$(\widetilde{u}, \nabla \widetilde{P}) := (\bar{u} - u_L, \nabla(\overline{P} - P_L))$$

and

$$(\widetilde{v}, \nabla \widetilde{Q}) := (\bar{v} - u_L, \nabla(\overline{Q} - P_L)).$$

The couple $(\widetilde{u}, \nabla \widetilde{P})$ satisfies the following modification of (5.9) (if $\mu = 1$ for simplicity):

$$\text{(5.37)} \quad \begin{cases} \partial_t \widetilde{u} - \Delta \widetilde{u} + \nabla \widetilde{P} = f_v(\widetilde{u}, \nabla \widetilde{P}) + f_v(u_L, \nabla P_L) & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} \widetilde{u} = g_v(\widetilde{u}) + g_v(u_L) & \text{in} \quad \Omega \times (0,T), \\ \widetilde{u}|_{\partial\Omega} = 0 & \text{at} \quad \partial\Omega \times (0,T), \\ \widetilde{u}|_{t=0} = 0 & \text{at} \quad \Omega. \end{cases}$$

Thanks to Proposition 5.2.1 we are ensured that solutions to (5.37) exist at least on a short time interval $[0, T]$, so far as

$$\text{(5.38)} \qquad \int_0^T \|\nabla \bar{v}\|_{\dot{B}^{n/p}_{p,1}(\Omega)} \, dt \le c.$$

We claim that there exists $R > 0$ and $T > 0$ (depending only on u_L) so that the map

$$\text{(5.39)} \qquad \widetilde{\mathcal{T}} : (\widetilde{v}, \nabla \widetilde{Q}) \longmapsto (\widetilde{u}, \nabla \widetilde{P})$$

is a contraction on $\overline{B}_{X_T^p}(0, R)$. Indeed, applying Theorem 4.3.1 yields

$$\begin{aligned} &\|(\widetilde{u}, \nabla \widetilde{P})\|_{X_T^p} \\ &\quad \le Ce^{CT} \Big(\|(f_v(\widetilde{u}, \nabla \widetilde{P}), \nabla g_v(\widetilde{u}), (R_v(\widetilde{u}))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))} \\ &\qquad + \|(f_v(u_L, \nabla P_L), \nabla g_v(u_L), (R_v(u_L))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))} \\ &\qquad\quad + \|R_v(\widetilde{u})\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))} + \|R_v(u_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))} \Big). \end{aligned}$$

Thus, arguing as in the proof of Proposition 5.2.1, we get

$$
\begin{aligned}
(5.40)\quad &\|(\widetilde{u},\nabla\widetilde{P})\|_{X_T^p}\\
&\leq Ce^{CT}\Big[\Big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\Big)\\
&\qquad\times\Big(\|(\widetilde{v},\nabla\widetilde{Q})\|_{X_T^p}+\|\partial_t u_L,\nabla^2u_L,\nabla P_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\Big)\\
&\qquad+(T^{1/2}+T^{3/4})\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\qquad\times\Big(\|\widetilde{u},u_L\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla\widetilde{u},\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\Big)\\
&\qquad+\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla v\|_{L_2(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\Big].
\end{aligned}
$$

This may be obtained by using (5.19), (5.20) and so on for $f_v(u_L,\nabla P_L)$, $g_v(u_L)$ and $R_v(u_L)$. The only difference lies in the use of (5.21) : we now write that

$$
\begin{aligned}
\|(R_v(u_L))_t\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\lesssim{}&\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla v\|_{L_2(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&+\|\partial_t u_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}.
\end{aligned}
$$

Bounding the first term in that way is important as we need to show that it is small when T goes to 0, *even if the initial velocity is large.*

Now, if we assume that $e^{CT}\leq 2$ and that c in (5.36) and (5.38) is small enough, Inequality (5.40) and interpolation imply that

$$
\begin{aligned}
&\|(\widetilde{u},\nabla\widetilde{P})\|_{X_T^p}\\
&\quad\leq\frac14\Big(\|(\widetilde{v},\nabla\widetilde{Q})\|_{X_T^p}+\|\partial_t u_L,\nabla^2u_L,\nabla P_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&\qquad+T^{1/2}\big(\|u_L\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big)\Big)\\
&\qquad\quad+C\|u_L\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\|\nabla v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\qquad\qquad+C\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|v\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}.
\end{aligned}
$$

It is now clear that if one takes $R=2cC\|u_0\|_{\dot B^{n/p-1}_{p,1}(\Omega)}$ and T fulfilling in addition

$$
\begin{aligned}
&\|\partial_t u_L,\nabla^2u_L,\nabla P_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+T^{1/2}\big(\|u_L\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&\quad+\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big)+\|u_0\|_{\dot B^{n/p-1}_{p,1}(\Omega)}\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\qquad\leq\eta\|u_0\|_{\dot B^{n/p-1}_{p,1}(\Omega)}
\end{aligned}
$$

for a small enough $\eta > 0$, then the above inequality implies that, whenever $\|(\widetilde{v}, \nabla\widetilde{Q})\|_{X_T^p} \le R$ we have $\|(\widetilde{u}, \nabla\widetilde{P})\|_{X_T^p} \le R$, too.

In order to prove that the map $\mathcal{T}$ is a contraction if T is sufficiently small, let us consider two data $(v_1, \nabla Q_1)$ and $(v_2, \nabla Q_2)$ in $\overline{B}_{X_T^p}(u_L, R)$, and set $(u_i, \nabla P_i) = \mathcal{T}(v_i, \nabla Q_i)$, $i = 1, 2$. We also use the notation $f_i := f_{v_i}$, $g_i := g_{v_i}$ and $R_i := R_{v_i}$ for $i = 1, 2$. Then we may look at

$$(\delta u, \nabla\delta P) := (u_2 - u_1, \nabla(P_2 - P_1))$$

as the solution to the following evolutionary Stokes system on $[0, T] \times \Omega$:

$$\begin{aligned}\partial_t\delta u - \Delta\delta u + \nabla\delta P &= f_2(\widetilde{u}_2, \nabla\widetilde{P}_2) - f_1(\widetilde{u}_1, \nabla\widetilde{P}_1) + f_2(u_L, \nabla P_L) - f_1(u_L, \nabla P_L),\\ \operatorname{div}\delta u &= g_2(\widetilde{u}_2) - g_1(\widetilde{u}_1) + g_2(u_L) - g_1(u_L)\\ &= \operatorname{div}\big(R_2(\widetilde{u}_2) - R_1(\widetilde{u}_1) + R_2(u_L) - R_1(u_L)\big).\end{aligned}$$

Therefore, applying Theorem 4.3.1 implies under the small time condition of the previous step

$$\begin{aligned}(5.41)\qquad &\|(\delta u, \nabla\delta P)\|_{X_T^p}\\ &\le C\Big(\|f_2(\widetilde{u}_2, \nabla\widetilde{P}_2) - f_1(\widetilde{u}_1, \nabla\widetilde{P}_1)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\quad + \|f_2(u_L, \nabla P_L) - f_1(u_L, \nabla P_L)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\quad + \|g_2(\widetilde{u}_2) - g_1(\widetilde{u}_1)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\ &\quad + \|g_2(u_L) - g_1(u_L)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\ &\quad + \|(R_2(\widetilde{u}_2) - R_1(\widetilde{u}_1))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\quad + \|(R_2(\widetilde{u}_L) - R_1(\widetilde{u}_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\quad + \|R_2(\widetilde{u}_2) - R_1(\widetilde{u}_1)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\\ &\quad + \|R_2(\widetilde{u}_L) - R_1(\widetilde{u}_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\Big).\end{aligned}$$

Keeping in mind that both v_1 and v_2 satisfy (5.38), we may bound the right-hand side as follows:

• *Term* $f_2(\widetilde{u}_2,\nabla\widetilde{P}_2)-f_1(\widetilde{u}_1,\nabla\widetilde{P}_1)$. — We rewrite this term as (with $B_i:=\overline{B}_{v_i}$ for $i=1,2$):

$$\begin{aligned}&f_2(\widetilde{u}_2,\nabla\widetilde{P}_2)-f_1(\widetilde{u}_1,\nabla\widetilde{P}_1)\\&\qquad=(1-\rho_0)\partial_t\delta u+\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla\widetilde{u}_2)\big)\\&\qquad\qquad+\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id}\,)\nabla\delta u\big)+{}^T(B_1-B_2)\nabla\widetilde{P}_2+(\mathrm{Id}\,-{}^TB_1)\nabla\delta P.\end{aligned}$$

Hence, using Lemma 5.5.1 and (5.51), (5.52), (5.53),

$$\begin{aligned}&\|f_2(\widetilde{u}_2,\nabla\widetilde{P}_2)-f_1(\widetilde{u}_1,\nabla\widetilde{P}_1)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\&\quad\lesssim\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\partial_t\delta u\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\&\qquad+\big(\|\nabla\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\nabla\widetilde{P}_2\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\&\qquad\quad+\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\nabla\delta P\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big).\end{aligned}$$

Above we used Inequalities (5.56) and (5.57).

• *Term* $f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)$. — Because

$$f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)=\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_L)\big)+{}^T(B_1-B_2)\nabla P_L,$$

we readily get

$$\begin{aligned}&\|f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\&\quad\lesssim\big(\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\nabla P_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.\end{aligned}$$

• *Term* $\|g_2(\widetilde{u}_2)-g_1(\widetilde{u}_1)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}$. — We write

$$g_2(\widetilde{u}_2)-g_1(\widetilde{u}_1)=(\mathrm{Id}\,-{}^T\overline{B}_1):\nabla\delta u+{}^T(B_1-B_2):\nabla\widetilde{u}_2.$$

Now, because $\dot{B}^{n/p}_{p,1}(\Omega)$ is an algebra, we readily get, by virtue of (5.51),

$$\begin{aligned}\|g_2(\widetilde{u}_2)-g_1(\widetilde{u}_1)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\lesssim&\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\&+\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla\delta u\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.\end{aligned}$$

• *Term* $\|g_2(u_L)-g_1(u_L)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}$. — Because

$$g_2(u_L)-g_1(u_L)={}^T(B_1-B_2):\nabla u_L,$$

we get

$$\|g_2(u_L)-g_1(u_L)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\lesssim\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.$$

• *Term* $\|(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$. — We use the fact that

$$(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t=-(B_1)_t\delta u+(\mathrm{Id}-B_1)\partial_t\delta u+(B_1-B_2)_t\widetilde{u}_2+(B_1-B_2)\partial_t\widetilde{u}_2.$$

Hence using (5.51), (5.54), (5.56) and (5.59),

$$\begin{aligned}
&\|(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\
&\quad\lesssim\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\delta u\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_t\delta u\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\\
&\qquad+\big(\|\widetilde{u}_2\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_t\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.
\end{aligned}$$

• *Term* $\|(R_2(\widetilde{u}_L)-R_1(\widetilde{u}_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$. — We just have to write that

$$(R_2(u_L)-R_1(u_L))_t=(B_1-B_2)_tu_L+(B_1-B_2)\partial_tu_L.$$

If we proceed as for bounding the previous term then we get the term $\|u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$ in the r.h.s. that does not need to be small for T going to 0. Hence, we proceed slightly differently: we apply (5.59) with $s=n/p-1$ in order to bound the term $(B_1-B_2)_t$. We eventually get

$$\begin{aligned}
&\|(R_2(u_L)-R_1(u_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\
&\quad\lesssim\|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla\delta v\|_{L_2(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\
&\qquad+\|\partial_tu_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.
\end{aligned}$$

• *Term* $\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;L_{2n}(\Omega)\cap\dot{B}^{n/p}_{p,1}(K))}$. — We start with the following expansion:

$$R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)=(\mathrm{Id}-B_1)\delta u+(B_1-B_2)\widetilde{u}_2.$$

Then applying Hölder inequality, embedding and interpolation inequality as in the proof of Proposition 5.2.1,

$$\begin{aligned}
&\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;L_{2n}(\Omega))}\\
&\quad\lesssim\|\nabla v_1\|_{L_1(0,T;L_\infty(\Omega))}\|\delta u\|_{L_1(0,T;L_{2n}(\Omega))}\\
&\qquad\qquad+\|\nabla\delta v\|_{L_1(0,T;L_\infty(\Omega))}\|\widetilde{u}_2\|_{L_1(0,T;L_{2n}(\Omega))}\\
&\quad\lesssim T^{3/4}\Big(\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\delta u,\nabla\delta P)\|_{X^p_T}\\
&\qquad\qquad\qquad+\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\widetilde{u}_2,\nabla\widetilde{P}_2)\|_{X^p_T}\Big).
\end{aligned}$$

For bounding the norm in $L_1(0,T;\dot{B}^{n/p}_{p,1}(K))$, we just use the corresponding norm on the larger set Ω and write that

$$\begin{aligned}\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}&\\ \lesssim \|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\delta u\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}&\\ +\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}&\\ \lesssim T^{1/2}\big(\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\delta u,\nabla\delta P)\|_{X^p_T}&\\ +\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\widetilde{u}_2,\nabla\widetilde{P}_2)\|_{X^p_T}\big).&\end{aligned}$$

- *Term* $\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap\dot{B}^{n/p}_{p,1}(\Omega))}$. — We have

$$R_{v_2}(u_L)-R_{v_1}(u_L)=(B_1-B_2)u_L.$$

Hence arguing as in the previous item, we get

$$\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;L_{2n}(\Omega))}\lesssim T^{3/4}\|\nabla\delta v\|_{L_1(0,T;L_\infty(\Omega))}\|(u_L,\nabla P_L)\|_{X^p_T}$$

and

$$\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\lesssim T^{1/2}\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(u_L,\nabla P_L)\|_{X^p_T}.$$

Putting all the above inequalities in (5.41), using the definition of the norm of X^p_T and the fact that T is small eventually yields

$$\begin{aligned}&\|(\delta u,\nabla\delta P)\|_{X^p_T}\\ &\quad\lesssim\big(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big)\|(\delta u,\nabla\delta P)\|_{X^p_T}\\ &\qquad+\big(T^{1/2}\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}+\|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\ &\qquad\quad+\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\partial_t u_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\qquad\quad+\|\nabla P_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|(\widetilde{u}_2,\nabla\widetilde{P}_2)\|_{X^p_T}\big)\|(\delta v,\nabla\delta Q)\|_{X^p_T}.\end{aligned}$$

Now, according to (5.36) and (5.38), the first term of the r.h.s. may be absorbed by the l.h.s. As the factor of $\|(\delta v,\nabla\delta Q)\|_{X^p_T}$ becomes less than R as T tends to 0, we conclude that the map $\mathcal{T}$ is indeed a contraction on $\overline{B}_{X^p_T}(u_L,R)$, if T and R have been chosen small enough. This completes the proof of existence.

Proving uniqueness or the continuity of the flow map stems from similar arguments. The details are left to the reader. Theorem 5.3.1 is proved. □

Remark 5.3.1. — Let us emphasize that the smallness of $\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}$ is completely independent of the largeness of the velocity data. In effect, it is only needed because $1-\rho_0$ appears as a factor of u_t. As pointed out in [**17**], this allows to consider discontinuous initial densities of the type $\rho_0 = c_1 1_{A_0} + c_2 1_{A_0^c}$ with A_0 any uniformly C^1 domain, provided $|c_2 - c_1|$ is small enough.

Remark 5.3.2. — Theorem 5.3.1 also holds for the original system (5.1) in Eulerian coordinates. In the functional framework we used, the two formulations turn out to be equivalent whenever the velocity satisfies (5.11) (see the Appendix of [**17**] for more details).

5.4. Global in time existence

This section is devoted to proving the main result of this chapter for the inhomogeneous incompressible Navier-Stokes equations (5.1), which is the following global in time existence statement.

Theorem 5.4.1. — *Assume that* $1 < q \leq p < 2n$ *with* $q < n/2$ *and* $p > n-1$. *Let* u_0 *be in* $\dot{B}^{n/p-1}_{p,1} \cap \dot{B}^0_{q,1}(\Omega)$ *with* $\operatorname{div} u_0 = 0$, $u_0 \cdot \vec{n} = 0$ *at the boundary. There exists a small positive constant c such that if*

$$\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1} \cap \dot{B}^0_{q,1}(\Omega))} \leq c \quad \text{and} \quad \|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1} \cap \dot{B}^0_{q,1}(\Omega)} \leq c\mu, \tag{5.42}$$

then System (5.1) *has a unique global solution* $(\rho, u, \nabla P)$ *satisfying*

$$\rho \in \mathcal{C}([0,T]; \mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1} \cap \dot{B}^0_{q,1}(\Omega)))$$

and $(u, \nabla P) \in X^{p,q}$ *(see the definition in* (5.25)*). Besides, there exists some constant* C *so that*

$$\|(u, \nabla P)\|_{X^{p,q}} \leq C \|u_0\|_{\dot{B}^0_{q,1} \cap \dot{B}^{n/p-1}_{p,1}(\Omega)}.$$

Proof. — The idea is to apply the global maximal regularity estimate for solutions to the Stokes system in exterior domains (namely Theorem 4.3.3) so as to get a global-in-time existence result for small data. As usual, we restrict to $\mu = 1$. We have already established, under smallness conditions (5.26) and (5.27) the existence of the solution map

$$\mathcal{T} : (\bar{v}, \nabla \overline{Q}) \longrightarrow (\bar{u}, \nabla \overline{P}) \tag{5.43}$$

to System (5.9), from the subset of $X^{p,q}$ with $\bar{v}$ satisfying (5.26), to $X^{p,q}$. Hence, in order to complete the proof of the theorem, it is only a matter of exhibiting some positive R so small as (5.26) to be satisfied, so that $\mathcal{T}$ maps the closed ball $\overline{B}_{X^{p,q}}(0,R)$ into itself, and is contractive. In light of Proposition 5.2.2, we have

$$\|(\bar u,\nabla\overline{P})\|_{X^{p,q}}\leq C\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}.$$

Hence one may take $R=C\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}$ if $\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}$ is small enough (in order that $\|(\bar v,\nabla\overline{Q})\|_{X^{p,q}}\leq C\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}$ implies (5.26)).

Let us now go to the proof of contractivity. Using the same notations as in the proof of Theorem 5.3.1, we have

$$\begin{cases}\partial_t\delta u-\Delta\delta u+\nabla\delta P=f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1),\\ \operatorname{div}\delta u=g_2(u_2)-g_1(u_1)=\operatorname{div}\big(R_2(u_2)-R_1(u_1)\big).\end{cases}$$

Applying Theorem 4.3.3 with $m:=\frac{nq}{n-2q}$, we thus get

$$\begin{aligned}(5.44)\quad \|(\delta u,\nabla\delta P)\|_{X^{p,q}}&\lesssim\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K)\cap B^{n/p}_{p,1}(K))}\\ &+\|f_2(\cdots)-f_1(\cdots),g_2(u_2)-g_1(u_1),(R_2(u_2)-R_1(u_1))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}.\end{aligned}$$

Following the computations of the proof of Theorem 5.3.1, we readily get

$$\begin{aligned}&\|f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1),g_2(u_2)-g_1(u_1),(R_2(u_2)-R_1(u_1))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}\\ &\lesssim\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}\\ &\qquad+\big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|\nabla P_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}\\ &\qquad+\|u_2\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}+\|\partial_t u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ &+\|\nabla v_1,\nabla v_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|\nabla\delta P\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))}\big).\end{aligned}$$

Next, let us go to the proof of estimates in $L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))$ for $f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)$. Again, we use the decomposition

$$\begin{aligned}&f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)\\ &\quad=(1-\rho_0)\partial_t\delta u+\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)\\ &\qquad\qquad+\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id})\nabla\delta u\big)\\ &\qquad\qquad\quad+{}^T(B_1-B_2)\nabla P_2+(\mathrm{Id}-{}^TB_1)\nabla\delta P.\end{aligned}$$

We further write that

$$\begin{aligned}\|\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)\|_{\dot B^0_{q,1}(\Omega)}\\ \leq \|({B_2}^TB_2-{B_1}^TB_1)\otimes\nabla^2u_2\|_{\dot B^0_{q,1}(\Omega)}\\ +\|\nabla({B_2}^TB_2-{B_1}^TB_1)\otimes\nabla u_2\|_{\dot B^0_{q,1}(\Omega)}.\end{aligned}$$

Hence, using Lemma 5.5.1 and flow estimates in Section 5.5, we get

$$\begin{aligned}\|\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\ \lesssim \|\nabla^2u_2\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ +\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ \times\big(\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\nabla^2v_1\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big).\end{aligned}\tag{5.45}$$

Similarly, using that

$$\begin{aligned}\|\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id}\,)\nabla\delta u\big)\|_{\dot B^0_{q,1}(\Omega)}\\ \leq\|\nabla({B_1}^TB_1)\otimes\nabla\delta u\|_{\dot B^0_{q,1}(\Omega)}+\|({B_1}^TB_1-\mathrm{Id}\,)\nabla^2\delta u\|_{\dot B^0_{q,1}(\Omega)},\end{aligned}$$

we get

$$\begin{aligned}\|\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id}\,)\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\ \lesssim\|\nabla^2v_1\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ +\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla^2\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}.\end{aligned}\tag{5.46}$$

Bounding the last two terms ${}^T(B_1-B_2)\nabla P_2$ and $(\mathrm{Id}\,-{}^TB_1)\nabla\delta P$ also follows from Lemma 5.5.1 and estimates for the flow. As it is totally similar to the above terms, we do not provide more details. We eventually get

$$\begin{aligned}\|f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\ \lesssim\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}(\Omega))}\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\ +\big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}+\|\nabla P_2\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\ +\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla\delta P\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\ +\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}.\end{aligned}$$

Bounding $g_2(u_2)-g_1(u_1)$ is the same. As for $(R_2(u_2)-R(u_1))_t$, we write

$$(R_2(u_2)-R(u_1))_t = -(B_1)_t\delta u + (\mathrm{Id}-B_1)\partial_t\delta u + (B_1-B_2)_t u_2 + (B_1-B_2)\partial_t u_2.$$

Given that the product maps $\dot B^{n/p}_{p,1}(\Omega)\times\dot B^0_{q,1}(\Omega)$ in $\dot B^0_{q,1}(\Omega)$, one may proceed exactly as in the proof of Theorem 5.3.1. Indeed, all the terms pertaining to B_1 or B_2 just have to be bounded in spaces $L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))$ or $L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))$. We end up with

$$\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))} \lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\delta u\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big) +\big(\|u_2\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\partial_t u_2\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}.$$

Let us now bound $R_2(u_2)-R_1(u_1)$ in $L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))$. Recall that

$$R_2(u_2)-R_1(u_1) = (\mathrm{Id}-B_1)\delta u + (B_1-B_2)u_2. \tag{5.47}$$

Hence, using that $B^{n/p}_{p,1}(K)$ is an algebra, one may write that

$$\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))} \lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\delta u\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))} + \|u_2\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.$$

Then using (5.31) enables us to get

$$\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))} \lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\big) + \big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|u_2\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.$$

Let us finally bound $R_2(u_2)-R_1(u_1)$ in $L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K))$. We shall use again that, according to (5.33),

$$\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K))} \lesssim \|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}+\|\nabla(R_2(u_2)-R_1(u_1))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}.$$

For the first term, using the decomposition (5.47) and the bounds for $\mathrm{Id}-B_1$ and B_1-B_2 in Section 5.5, we find out that

$$\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))} \lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;L_\infty(\Omega))}\|\delta u\|_{L_1(\mathbb{R}_+;L_m(\Omega))} + \|\nabla\delta v\|_{L_1(\mathbb{R}_+;L_\infty(\Omega))}\|u_2\|_{L_1(\mathbb{R}_+;L_m(\Omega))},$$

whence, using the embeddings $\dot B^{n/p}_{p,1}(\Omega)\hookrightarrow L_\infty(\Omega)$ and $\dot B^2_{q,1}(\Omega)\hookrightarrow L_m(\Omega)$,

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}&\\ \lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}&\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\\ &+\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|u_2\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}K))}.\end{aligned}$$

Finally, differentiating (5.47) with respect to y yields

$$\begin{aligned}&\nabla(R_2(u_2)-R_1(u_1))\\ &\qquad=-\nabla B_1\delta u+\nabla(B_1-B_2)\,u_2+(\mathrm{Id}\,-B_1)\nabla\delta u+(B_1-B_2)\,\nabla u_2.\end{aligned}$$

We have (see Section 5.5)

$$\begin{aligned}\|\nabla(B_1)\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}&\lesssim\|\nabla^2 v_1\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\delta u\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))},\\ \|\nabla(B_1-B_2)\,u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}&\lesssim\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))},\\ \|(\mathrm{Id}\,-B_1)\nabla\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))},\\ \|(B_1-B_2)\,\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}&\lesssim\|\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.\end{aligned}$$

Therefore, using (5.34),

$$\begin{aligned}&\|\nabla(R_2(u_2)-R_1(u_1))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\\ &\quad\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\\ &\qquad+\|\nabla^2 v_1\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}+\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big)\\ &\qquad\;+\big(\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big)\\ &\qquad\qquad\times\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega)\cap\dot B^{n/p}_{p,1}(\Omega))}.\end{aligned}$$

Plugging all the above estimates in (5.44), we conclude that

$$\begin{aligned}&\|(\delta u,\nabla\delta P)\|_{X^{p,q}}\\ &\quad\lesssim\|(u_2,\nabla Q_2)\|_{X^{p,q}}\|(\delta v,\nabla\delta Q)\|_{X^{p,q}}\\ &\qquad\quad+\big(\|(u_1,\nabla Q_1)\|_{X^{p,q}}+\|(u_2,\nabla Q_2)\|_{X^{p,q}}+\|(v_1,\nabla P_1)\|_{X^{p,q}}\\ &\qquad\qquad+\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}\big)\|(\delta u,\nabla\delta P)\|_{X^{p,q}}.\end{aligned}$$

It is now clear that if R and ρ_0 have been chosen so that

$$2\big(\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}+CR\big)<1,$$

then the above inequality entails that

$$\|(\delta u,\nabla\delta P)\|_{X^{p,q}}\leq\kappa\|(\delta v,\nabla\delta Q)\|_{X^{p,q}}$$

for some $\kappa < 1$ whenever $(v_1, \nabla Q_1)$ and $(v_2, \nabla Q_2)$ are in $\overline{B}_{X^{p,q}}(0,R)$. This completes the proof of the global existence. The proof of uniqueness is similar to that of contractivity. The details are left to the reader. □

5.5. Estimates of nonlinearities

In this section we establish a few estimates for nonlinear terms in Besov spaces. First, let us give some insight on the structure of the multiplier spaces $\mathcal{M}(\dot{B}^s_{p,1})$.

LEMMA 5.5.1. — *The following inequality holds true:*

$$\|u\|_{\mathcal{M}(\dot{B}^s_{p,1}(\Omega))} + \|u\|_{\mathcal{M}(\dot{B}^0_{q,1}(\Omega))} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}(\Omega)}$$

whenever $1 < p < \infty$, $-\min(n/p, n/p') < s \leq n/p$ *and* $1 < q < \infty$.

If in addition $\max(p,q) \leq n$ *then we also have*

$$\|u\|_{\mathcal{M}(\dot{B}^1_{q,1}(\Omega))} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}(\Omega)}.$$

Proof. — For the first item, it suffices to establish that the product maps $\dot{B}^s_{p,1}(\Omega) \times \dot{B}^{n/p}_{p,1}(\Omega)$ in $\dot{B}^s_{p,1}(\Omega)$, and $\dot{B}^0_{q,1}(\Omega) \times \dot{B}^{n/p}_{p,1}(\Omega)$ in $\dot{B}^0_{q,1}(\Omega)$. Our definition of Besov norms by restriction allows us to consider only the case $\Omega = \mathbb{R}^n$. Then the result is sort of classical. Note that the result for $\mathcal{M}(\dot{B}^s_{p,1}(\mathbb{R}^n))$ has already been proved in Proposition 2.2.1. As for $\mathcal{M}(\dot{B}^0_{q,1}(\mathbb{R}^n))$, one may use continuity results for the paraproduct, and functional embedding. Indeed, we have $uv = T_u v + R(u,v) + T_v u$, and

- T maps $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n) \times \dot{B}^0_{q,1}(\mathbb{R}^n)$ in $\dot{B}^0_{q,1}(\mathbb{R}^n)$ (as $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ embeds in $L_\infty(\mathbb{R}^n)$),
- R maps $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n) \times \dot{B}^0_{q,1}(\mathbb{R}^n)$ in $\dot{B}^0_{q,1}(\mathbb{R}^n)$ as $n/p + 0 > n\max(0, 1/q - 1/p')$,
- T maps $\dot{B}^0_{q,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ in $\dot{B}^0_{q,1}(\mathbb{R}^n)$ (use first that $\dot{B}^0_{q,1}(\mathbb{R}^n)$ is embedded in $\dot{B}^{n/p'-n/q}_{p',1}(\mathbb{R}^n)$ if $q \leq p'$).

In order to prove the last item, we use the fact that T and R map $L_\infty(\mathbb{R}^n) \times \dot{B}^1_{q,1}(\mathbb{R}^n)$ in $\dot{B}^1_{q,1}(\mathbb{R}^n)$, together with the embedding of $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ in $L_\infty(\mathbb{R}^n)$, and also that T maps $\dot{B}^1_{q,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ in $\dot{B}^1_{q,1}(\mathbb{R}^n)$, if $\max(p,q) \leq n$. □

Let us finally prove some useful 'flow estimates'. The important fact that we shall use repeatedly is that, as a consequence of the above lemma, the space $\dot{B}^{n/p}_{p,1}(\Omega)$ is a (quasi)-Banach algebra (and of course so does $B^{n/p}_{p,1}(K)$). Hence if $\int_0^t D\bar{v}\,d\tau$ is small enough, a condition that will be ensured by the smallness of the data, then one may just write

$$\overline{B}_v = \sum_{k\geq 0}\Big(-\int_0^t D\bar{v}\,d\tau\Big)^k. \tag{5.48}$$

Therefore, if $\|\nabla\bar{v}\|_{L_1(0,T;L_\infty(\Omega))} < 1$ then $\mathrm{Id} - \overline{B}_v \in L_\infty(0,T;L_\infty(\Omega))$ and

$$\|\mathrm{Id} - \overline{B}_v\|_{L_\infty(0,T;L_\infty(\Omega))} \leq \frac{\|\nabla\bar{v}\|_{L_1(0,T;L_\infty(\Omega))}}{1-\|\nabla\bar{v}\|_{L_1(0,T;L_\infty(\Omega))}}. \tag{5.49}$$

Likewise, since $\dot{B}^{n/p}_{p,1}(\Omega)$ is a quasi-Banach algebra, there exist two constants $c = c(n,p)$ and $C = c(n,p)$ such that if

$$\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))} < c, \tag{5.50}$$

then

$$\|\mathrm{Id} - \overline{B}_v\|_{L_\infty(0,T;\dot{B}^{n/p}_{p,1}(\Omega))} \leq C\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}. \tag{5.51}$$

Let us emphasize that we have

$$D_x v(t,x) = D_y\bar{v}(t,y)\,\overline{B}_v(t,y) = D_y\bar{v}(t,y)\sum_{k\geq 0}\Big(-\int_0^t D_y\overline{B}(\tau,y)\,d\tau\Big)^k.$$

Hence Condition (5.50) holds simultaneously for $\bar{v}$ or v (up to a harmless change of c), a fact that we used freely and repeatedly throughout this chapter.

Note also that, as

$$\mathrm{Id} - \overline{B}_v{}^T\overline{B}_v = (\mathrm{Id} - \overline{B}_v)({}^T\overline{B}_v - \mathrm{Id}\,) + (\mathrm{Id} - \overline{B}_v) + (\mathrm{Id} - {}^T\overline{B}_v),$$

we also have, under Condition (5.51),

$$\|\mathrm{Id} - \overline{B}_v{}^T\overline{B}_v\|_{L_\infty(0,T;\dot{B}^{n/p}_{p,1}(\Omega))} \leq C\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}. \tag{5.52}$$

Similarly, by taking the gradient of (5.48) and using

$$\nabla(\overline{B}_v{}^T\overline{B}_v) = \nabla\overline{B}_v{}^T\overline{B}_v + \overline{B}_v\nabla({}^T\overline{B}_v),$$

we find out that for $(s,r)\in\{(n/p-1,p),(0,q)\}$, we have

$$\begin{aligned}\|\nabla\overline{B}_v\|_{L_\infty(0,T;\dot{B}^{s}_{r,1}(\Omega))} + \|\nabla(\overline{B}_v{}^T\overline{B}_v)\|_{L_\infty(0,T;\dot{B}^{s}_{r,1}(\Omega))}&\\ \lesssim \|\nabla^2\bar{v}\|_{L_1(0,T;\dot{B}^{s}_{r,1}(\Omega))}&.\end{aligned} \tag{5.53}$$

Finally, by taking one time derivative of (5.48), we get

$$\|(\overline{B}_v)_t\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))} \leq C\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}. \tag{5.54}$$

In order to prove stability, we need some estimates on B_2-B_1 where $B_i := \overline{B}_{v_i}$ for $i=1,2$. The starting point is that, owing to (5.48), we may write if both v_1 and v_2 satisfy (5.50), the following identity [3]:

$$B_2-B_1 = \sum_{k\geq 1}\Big(\Big(-\int_0^t D\bar v_2\,d\tau\Big)^k-\Big(-\int_0^t D\bar v_1\,d\tau\Big)^k\Big)$$

$$(5.55)\qquad = -\Big(\int_0^t D\delta v\,d\tau\Big)\sum_{k\geq 1}\sum_{j=0}^{k-1}\Big(-\int_0^t D\bar v_2\,d\tau\Big)^j\Big(-\int_0^t D\bar v_1\,d\tau\Big)^{k-1-j}.$$

Therefore Lemma 5.5.1 guarantees that we have for all positive t,

$$(5.56)\qquad \|(B_2-B_1)(t)\|_{\dot B^s_{r,1}(\Omega)} \lesssim \|\nabla\delta v\|_{L_1(0,t;\dot B^s_{r,1}(\Omega))}$$

whenever $1<p<\infty$ and $-\min(n/p,n/p')<s\leq n/p$ if $r=p$, or with $s=0$ and $r\in(1,\infty)$.

Note that because

$$B_2{}^TB_2-B_1{}^TB_1=(B_2-B_1)^TB_2+B_1{}^T(B_2-B_1),$$

we also have, for the same couples (s,r) as in (5.56),

$$(5.57)\qquad \|(B_2{}^TB_2-B_1{}^TB_1)(t)\|_{\dot B^s_{r,1}(\Omega)} \lesssim \|\nabla\delta v\|_{L_1(0,t;\dot B^s_{r,1}(\Omega))}.$$

Next, we want to estimate $D(B_2-B_1)$. Differentiating (5.55), we get

$$D(B_2-B_1) = -\int_0^t D^2\delta v\,d\tau\sum_{k\geq 1}k\Big(-\int_0^t D\bar v_2\,d\tau\Big)^{k-1}$$
$$+\sum_{k\geq 2}k\Big(\int_0^t D^2 v_1\,d\tau\Big)\Big(\int_0^t D\delta v\,d\tau\Big)\sum_{j=0}^{k-2}\Big(-\int_0^t D\bar v_2\,d\tau\Big)^j\Big(-\int_0^t D\bar v_1\,d\tau\Big)^{k-2-j}.$$

Hence, still assuming (5.50) for v_1 and v_2, and using Lemma 5.5.1,

$$(5.58)\qquad \|\nabla(B_1-B_2)(t)\|_{\dot B^s_{r,1}(\Omega)} \lesssim \|\nabla^2\delta v\|_{L_1(0,t;\dot B^s_{r,1}(\Omega))}+\|\nabla^2 v_1\|_{L_1(0,t;\dot B^s_{r,1}(\Omega))}\|\nabla\delta v\|_{L_1(0,t;\dot B^{n/p}_{p,1}(\Omega))}.$$

3. Rigorously speaking (5.55) is not quite correct for the different matrices involved need not to commute. From the point of view of a priori estimates, everything happens as if they did, though.

Finally, taking one time derivative of (5.55) yields

$$D(B_1 - B_2) = -D\delta v \sum_{k\geq 1} k\Big(-\int_0^t D\bar{v}_2\,d\tau\Big)^{k-1}$$
$$+\sum_{k\geq 2} kD\bar{v}_1\Big(\int_0^t D\delta v\,d\tau\Big)\sum_{j=0}^{k-2}\Big(-\int_0^t D\bar{v}_2\,d\tau\Big)^j\Big(-\int_0^t D\bar{v}_1\,d\tau\Big)^{k-2-j}.$$

Hence,

$$\|\partial_t(B_1 - B_2)(t)\|_{\dot{B}^s_{r,1}(\Omega)} \tag{5.59}$$
$$\lesssim \|\nabla\delta v(t)\|_{\dot{B}^s_{r,1}(\Omega)}\|\nabla\bar{v}_2\|_{L_1(0,t;\dot{B}^{n/p}_{p,1}(\Omega))} + \|\nabla\bar{v}_1(t)\|_{\dot{B}^s_{r,1}(\Omega)}\|\nabla\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1}(\Omega))}.$$

CHAPTER 6

THE LOW MACH NUMBER SYSTEM

The last part of the present memoir is dedicated to the analysis of a limit system for the Navier-Stokes-Fourier equations that may be derived in the low Mach number asymptotics and has been studied recently in [**14**] in the whole space setting (see [**12**], [**25**] for older related results). This system is a non-linear coupling between a Stokes-like system and a heat-like equation. As a consequence of its derivation, the divergence of the velocity is determined by the heat flux and is thus nonzero if the fluid is heat-conductive. In contrast with the previous chapter, the full system is of parabolic type in the Eulerian coordinates framework. Hence it is not helpful to switch to Lagrangian coordinates to solve the system by means of the Banach fixed point theorem.

We here aim at extending the results of [**14**] to the case where the fluid domain is an exterior domain of $\mathbb{R}^n$ with $n \geq 3$. To simplify the presentation, we shall concentrate on the proof of global-in-time solutions with critical regularity. In passing, we will establish a new regularity result for the heat equation with Neumann boundary condition in exterior domains, which is of independent interest.

6.1. The system

We aim at investigating the following type of systems:

$$\begin{cases} \beta(\vartheta)(\partial_t\vartheta + u\cdot\nabla\vartheta) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = 0 & \text{in} \quad \mathbb{R}_+\times\Omega,\\ \rho(\vartheta)(\partial_t u + u\cdot\nabla u) - \operatorname{div}\tau + \nabla P = 0 & \text{in} \quad \mathbb{R}_+\times\Omega,\\ \operatorname{div} u = a(\vartheta)\operatorname{div}(\kappa(\vartheta)\nabla\vartheta) & \text{in} \quad \mathbb{R}_+\times\Omega, \end{cases}$$

with $\tau := \mu\mathbb{D}u + \lambda(\operatorname{div} u)\mathrm{Id}$, where $\mathbb{D}u$ stands for (twice) the deformation tensor of the fluid, that is $\mathbb{D}u = \nabla u + {}^T\nabla u$. We suppose that ρ (the density of the fluid) and also β, λ, μ, κ and a are given smooth functions of ϑ satisfying

$$\kappa > 0, \quad \rho > 0, \quad \beta > 0, \quad \mu > 0 \text{ and } \lambda + 2\mu > 0.$$

This type of system may be derived in the low Mach number asymptotics in the large entropy variations regime (see e.g. [**14, 37**] and the references therein). For simplicity, we here consider only perfect gases. Then we have for some reference positive constant pressure P_0,

$$\rho(\vartheta) = \frac{P_0}{R\vartheta}, \quad \beta(\vartheta) = C_p\rho(\vartheta) = \frac{\gamma}{\gamma-1}R\rho(\vartheta) \quad \text{and} \quad a(\vartheta) = \frac{\gamma-1}{\gamma P_0}$$

with $R > 0$ and $\gamma > 1$.

We here focus on small perturbations of some constant positive reference temperature, that we can normalize at 1. Setting $\theta := \vartheta - 1$, $\nabla Q := \nabla(P + \lambda \operatorname{div} u)$, and keeping the same notation for the functions β, ρ, κ and μ, expressed in terms of θ, we eventually get

$$\text{(6.1)} \quad \begin{cases} \beta(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0 & \text{in} \quad \mathbb{R}_+\times\Omega, \\ \rho(\theta)(\partial_t u + u\cdot\nabla u) - \operatorname{div}(\mu(\theta)\mathbb{D}u) + \nabla Q = 0 & \text{in} \quad \mathbb{R}_+\times\Omega, \\ \operatorname{div} u = a \operatorname{div}(\kappa(\theta)\nabla\theta) & \text{in} \quad \mathbb{R}_+\times\Omega. \end{cases}$$

We supplement System (6.1) with the boundary constraints

$$\text{(6.2)} \qquad u|_{\mathbb{R}_+\times\partial\Omega} = 0, \quad \partial_{\vec{n}}\theta|_{\mathbb{R}_+\times\partial\Omega} = 0 \qquad \text{at } \mathbb{R}_+\times\partial\Omega$$

and the initial data

$$\text{(6.3)} \qquad u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 \qquad \text{in } \Omega,$$

interrelated through the compatibility condition $\operatorname{div} u_0 = a \operatorname{div}(\kappa(\theta_0)\nabla\theta_0)$.

Scaling arguments similar to those of Chapter 5 suggest us to use a functional framework in which the temperature has one more derivative than the velocity. Besides, in order to have some control on the conductivity and viscosity coefficients (that may depend on the temperature), we need θ to be at least continuous. Keeping our maximal regularity results in mind, this eventually leads to consider the initial velocity u_0 in the space $\dot{B}^{n/p-1}_{p,1}$ and the initial (relative) temperature θ_0 in $\dot{B}^{n/p}_{p,1}$ with $n/p - 1$ close to 0.

Looking at the structure of the linearization of System (6.1), we see that we have to deal with the heat equation and the Stokes system *with some non-divergence free constraint*. Therefore, the full system is of (generalized) parabolic type and using the Eulerian coordinates will enable us to show the well-posedness by means of the Banach fixed point theorem (in contrast with

Chapter 5 where we had to switch to the Lagrangian coordinates). Note that the coupling between the temperature and velocity equations is rather harmless: once θ has been determined as a solution to a transport-diffusion type equation, the velocity may be controlled almost as if solving the homogeneous incompressible Navier-Stokes equation. The dependency of $\operatorname{div} u$ with respect to θ will turn out to be compatible with the statement of Theorem 4.3.3.

Before starting our investigation of System (6.1), we have to establish a new maximal regularity result concerning the heat equation in exterior domains, in the spirit of [**20**], but for *Neumann* boundary conditions. Besides, as we plan to handle initial temperatures in $\dot B^{n/p}_{p,1}$ with n/p close to 1, we will have to prove regularity estimates as well.

6.2. The heat equation with Neumann boundary conditions

The starting point is the following result in the whole space (see e.g. [**6**]).

THEOREM 6.2.1. — *Let $p\in[1,\infty]$ and $s\in\mathbb{R}$. Let $f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ and $u_0\in\dot B^s_{p,1}(\mathbb{R}^n)$. The system*

$$\begin{cases} u_t-\nu\Delta u=f & \text{in}\quad (0,T)\times\mathbb{R}^n,\\ u=u_0 & \text{on}\qquad\qquad \mathbb{R}^n\end{cases}$$

has a unique solution u in

$$\mathcal{C}([0,T);\dot B^s_{p,1}(\mathbb{R}^n))\quad\text{with}\quad \partial_t u,\nabla^2u\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$$

and the following inequality holds true:

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|u_t,\nu\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ \leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}).$$

6.2.1. The heat equation in the half-space. — The purpose of this paragraph is to extend Theorem 6.2.1 to the half-space setting : we now consider

$$\begin{cases} u_t-\nu\Delta u=f & \text{in}\quad (0,T)\times\mathbb{R}^n_+,\\ \partial_{x_n}u|_{x_n=0}=0 & \text{on}\quad (0,T)\times\partial\mathbb{R}^n_+,\\ u|_{t=0}=u_0 & \text{on}\quad \mathbb{R}^n_+.\end{cases}\tag{6.4}$$

As we are looking for solutions u such that

$$u\in\mathcal{C}([0,T);\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad u_t,\nabla^2u\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+)),\tag{6.5}$$

we expect ∇u to be in the *nonhomogeneous space* $L_2(0,T;B^s_{p,1}(\mathbb{R}^n_+))$ (by interpolation) and the trace of ∇u at $(0,T)\times\partial\mathbb{R}^n_+$ to be thus defined (see Proposition 2.2.4). Therefore, we just have to solve in the sense of distributions the first and third equations of (6.4) in the class of functions u satisfying (6.5) and having *null trace at* $(0,T)\times\partial\mathbb{R}^n_+$.

The rest of this paragraph is devoted to proving the following statement.

THEOREM 6.2.2. — *Let* $u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+)$ *and* $f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ *with* $p\in[1,\infty)$ *and* $s\in(-1+1/p,1/p)$. *Then* (6.4) *has a unique solution* u *satisfying* (6.5) *and the following inequality is valid:*

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|u_t,\nu\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}),\tag{6.6}$$

where C is an absolute constant with no dependence on ν and T.

If in addition $\nabla u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+)$ *and* $\nabla f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ *then* ∇u *satisfies* (6.5) *and we have*

$$\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla u_t,\nu\nabla^3u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq C(\|\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).\tag{6.7}$$

Proof. — Let $\widetilde u_0$ and $\widetilde f$ be the symmetric extensions over $\mathbb{R}^n$ of u_0 and f. Because $-1+1/p<s<1/p$, we have $\widetilde u_0\in\dot B^s_{p,1}(\mathbb{R}^n)$, $\widetilde f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ with (see Corollary 2.2.1)

$$\|\widetilde u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\approx\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\quad\text{and}\quad\|\widetilde f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\approx\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Let $\widetilde u$ be the solution given by Theorem 6.2.1. As this solution is unique in the corresponding functional framework, the symmetry properties of the data ensure that $\widetilde u$ is symmetric with respect to $\{x_n=0\}$, and thus vanishes there if it is smooth enough. Arguing by density, we gather that $\partial_{x_n}\widetilde u|_{x_n=0}=0$ under the regularity assumptions of the theorem.

Now, we observe that the restriction u of $\widetilde u$ to the half-space satisfies the first and last equations of (6.4), and that

- $\widetilde u_t$ coincides with the symmetric extension of u_t,
- $\nabla^2_{x'}\widetilde u$ coincides with the symmetric extension of $\nabla^2_{x'}u$,
- $\nabla_{x'}\partial_{x_n}\widetilde u$ coincides with the antisymmetric extension of $\nabla_{x'}\partial_{x_n}u$,
- $\partial^2_{x_n,x_n}\widetilde u=(\Delta-\Delta_{x'})\widetilde u$ hence coincides with $\widetilde u_t-\widetilde f-\Delta_{x'}\widetilde u$.

Hence we get

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|u_t, \nu\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \\ \leq \|\widetilde{u}\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \|\widetilde{u}_t, \nu\nabla^2 \widetilde{u}\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))},$$

which implies (6.6).

To prove higher regularity estimates, we first use the fact that $\nabla_{x'}\widetilde{u}$ coincides with the symmetric extension of $\nabla_{x'}u$ on $\mathbb{R}^n$, and satisfies the heat equation on $(0,T)\times\mathbb{R}^n$, supplemented with initial data $\nabla_{x'}\widetilde{u}_0$ on $\mathbb{R}^n$. Therefore applying Theorem 6.2.1 and arguing as in the proof of (6.6) (with $\nabla_{x'}u$ instead of u), we conclude that

$$\|\nabla_{x'}u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla_{x'}u_t, \nu\nabla^2\nabla_{x'} u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \\ \leq C(\|\nabla_{x'}f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla_{x'}u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).$$

As regards the vertical derivative, we have

$$\begin{cases} (\partial_{x_n}u)_t - \nu\Delta\partial_{x_n}u = \partial_{x_n}f & \text{in} \quad (0,T)\times\mathbb{R}^n_+, \\ \partial_{x_n}u|_{x_n=0} = 0 & \text{on} \quad (0,T)\times\partial\mathbb{R}^n_+, \\ \partial_{x_n}u|_{t=0} = \partial_{x_n}u_0 & \text{on} \quad \mathbb{R}^n_+. \end{cases}$$

The fact that $\nabla u \in L_2(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ (given by the first part of the statement and interpolation) implies that u decays to 0 at infinity. Hence applying the results for the heat equation *with Dirichlet boundary conditions* in [**15**] yields

$$\|\partial_{x_n}u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_{x_n}u_t, \nu\nabla^2\partial_{x_n} u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \\ \leq C(\|\partial_{x_n}f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_{x_n}u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).$$

This completes the proof of the theorem. □

6.2.2. The exterior domain case. — Here we extend Theorem 6.2.1 to the case where Ω is a smooth exterior domain.

THEOREM 6.2.3. — *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$. Let $1 < q \leq p < \infty$ with $q < n/2$. Let $-1+1/p < s < 1/p$ and $-1+1/q < s' < 1/q - 2/n$. Let*

$$u_0 \in \dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega) \quad \textit{and} \quad f \in L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)).$$

Then there exists a unique solution u to

$$\begin{cases} u_t - \nu\Delta u = f & \text{in} \quad (0,T)\times\Omega, \\ \partial_{\vec{n}} u = 0 & \text{at} \quad (0,T)\times\partial\Omega, \\ u = u_0 & \text{on} \quad \Omega \end{cases} \tag{6.8}$$

such that

$$u \in \mathcal{C}([0,T];\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)), \qquad u_t,\nabla^2 u \in L_1(0,T;\dot{B}^s_{p,1}\cap B^{s'}_{q,1}(\Omega))$$

and the following inequality is satisfied:

$$\begin{aligned} \|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} &+ \|u_t,\nu\nabla^2 u\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} \\ &\leq C\big(\|u_0\|_{\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)} + \|f\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))}\big), \end{aligned} \tag{6.9}$$

where the constant C is independent of T and ν.

If in addition $\nabla u_0 \in \dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)$ and $\nabla f \in L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))$, then u also satisfies

$$\nabla u \in \mathcal{C}([0,T];\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)), \quad \nabla u_t,\nabla^3 u \in L_1(0,T;\dot{B}^s_{p,1}\cap B^{s'}_{q,1}(\Omega))$$

and

$$\begin{aligned} \|\nabla u\|_{L_\infty(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} &+ \|\nabla u_t,\nu\nabla^3 u\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} \\ &\leq C\big(\|u_0,\nabla u_0\|_{\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)} + \|f,\nabla f\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))}\big). \end{aligned} \tag{6.10}$$

Proving this theorem relies on the following statement, and on lower order estimates (see Lemma 6.2.1 below) so as to remove the time dependency.

THEOREM 6.2.4. — *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n\geq 2$. Let $1<p<\infty$, $-1+1/p<s<1/p$, $f\in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$, and $u_0\in\dot{B}^s_{p,1}(\Omega)$. Then Equation* (6.8) *has a unique solution u such that*

$$u\in\mathcal{C}([0,T];\dot{B}^s_{p,1}(\Omega)), \qquad \partial_t u,\nabla^2 u\in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$$

and the following estimates are valid:

$$\begin{aligned} \|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))} &+ \|u_t,\nu\nabla^2 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \\ &\leq Ce^{CT\nu}\big(\|u_0\|_{\dot{B}^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\big), \end{aligned} \tag{6.11}$$

$$\begin{aligned} \|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))} &+ \|u_t,\nu\nabla^2 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \\ &\leq C_K\big(\|u_0\|_{\dot{B}^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} + \nu\|u|_K\|_{L_1(0,T;\dot{B}^s_{p,1}(K))}\big), \end{aligned} \tag{6.12}$$

where K stands for any compact subset of Ω such that $\operatorname{dist}(\partial\Omega,\Omega\setminus K)>0$.

If in addition $\nabla u_0 \in \dot{B}^s_{p,1}(\Omega)$ *and* $\nabla f \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$ *then we also have*

$$\|\nabla u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))}+\|\nabla u_t,\nu\nabla^3 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \leq Ce^{CT\nu}\big(\|u_0,\nabla u_0\|_{\dot{B}^s_{p,1}(\Omega)}+\|f,\nabla f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\big), \tag{6.13}$$

$$\|\nabla u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\Omega))}+\|\nabla u_t,\nu\nabla^3 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \leq C_K\big(\|u_0,\nabla u_0\|_{\dot{B}^s_{p,1}(\Omega)}+\|f,\nabla f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}+\nu\|u|_K\|_{L_1(0,T;\dot{B}^s_{p,1}(K))}\big). \tag{6.14}$$

Proof. — We suppose that we are given a smooth enough solution and focus on the proof of the estimates. We shall do it in three steps: first we prove interior estimates, next boundary estimates and finally global estimates after summation. By performing a suitable change of time variable and source term, one may reduce our study to the case $\nu=1$.

Throughout we fix some covering $(B(x^\ell,\lambda))_{1\leq\ell\leq L}$ of K by balls of radius λ and take some neighborhood $\Omega^0\subset\Omega$ of $\mathbb{R}^n\setminus K$ such that $d(\Omega^0,\partial\Omega)>0$. We assume in addition that the first M balls do not intersect $\partial\Omega$ while the last $L-M$ balls are centered at some point of $\partial\Omega$.

Let $\eta^0:\mathbb{R}^n\to[0,1]$ be a smooth function supported in Ω^0 and with value 1 on a neighborhood of $\Omega\setminus K$. Then we consider a subordinate partition of unity $(\eta^\ell)_{1\leq\ell\leq L}$ such that (see e.g. [**35, 43**]):

1. $\sum_{0\leq\ell\leq L}\eta^\ell=1$ on Ω;
2. $\|\nabla^k\eta^\ell\|_{L_\infty(\mathbb{R}^n)}\leq C_k\lambda^{-k}$ for $k\in\mathbb{N}$ and $1\leq\ell\leq L$;
3. $\operatorname{Supp}\eta^\ell\subset B(x^\ell,\lambda)$,
4. $\partial_{\vec n}\eta^\ell=0$ on $\partial\Omega$.

We also introduce another smooth function $\widetilde\eta^0$ supported in K and with value 1 on $\operatorname{Supp}\nabla\eta^0$ and smooth functions $\widetilde\eta^1,\dots,\widetilde\eta^L$ with support in $B(x^\ell,\lambda)$ and such that $\widetilde\eta^\ell\equiv1$ on $\operatorname{Supp}\eta^\ell$.

Note that for $\ell\in\{1,\dots,L\}$, the bounds for the derivatives of η^ℓ together with the fact that $\big|\operatorname{Supp}\nabla\eta^\ell\big|\approx\lambda^n$ implies that for any $k\in\mathbb{N}$ and $q\in[1,\infty]$,

$$\|\nabla^k\eta^\ell\|_{\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)}+\|\nabla^k\widetilde\eta^\ell\|_{\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)}\lesssim\lambda^{-k}.$$

First step: the interior estimate. — For $\ell\in\{0,\dots,M\}$, function $U^\ell:=u\eta^\ell$ satisfies

$$\begin{cases} U^\ell_t-\Delta U^\ell=f^\ell & \text{in} \quad (0,T)\times\mathbb{R}^n,\\ U^\ell|_{t=0}=u_0\eta^\ell & \text{on} \qquad \mathbb{R}^n\end{cases}$$

with

$$f^\ell:=\eta^\ell f-2\nabla\eta^\ell\cdot\nabla u-u\Delta\eta^\ell. \tag{6.15}$$

Applying Theorem 6.2.1 yields the estimates:

$$\begin{aligned}
&\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^\ell_t,\nabla^2U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\
&\qquad\lesssim \|\eta^\ell u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))},\\
&\|\nabla U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\nabla U^\ell_t,\nabla^3U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\
&\qquad\lesssim \|\nabla(\eta^\ell u_0)\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\nabla f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}.
\end{aligned}$$

Because the function $\nabla\eta^\ell$ is in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\widetilde{\eta}^\ell\equiv 1$ on $\operatorname{Supp}\nabla\eta^\ell$, we get according to the results of Chapter 2,

$$\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\le\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+C_\lambda\|\widetilde{\eta}^\ell u,\widetilde{\eta}^\ell\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}. \tag{6.16}$$

As may be proved by writing that $\nabla(\eta^\ell z)=z\nabla\eta^\ell+\eta^\ell\nabla z$, for any $z\in\dot B^s_{p,1}(\mathbb{R}^n)$ with $-n/p'<s\le n/p$, we have

$$\|\nabla(\eta^\ell z)\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\le C\big(\|\widetilde{\eta}^l z\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\widetilde{\eta}^\ell\nabla z\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\big). \tag{6.17}$$

Hence, we also have

$$\begin{aligned}\|\nabla f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\le C\|\eta^\ell f,\widetilde{\eta}^l\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}&\\
+C_\lambda\|\widetilde{\eta}^\ell u,\widetilde{\eta}^\ell\nabla u,\widetilde{\eta}^\ell\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(K))}&.\end{aligned} \tag{6.18}$$

Plugging (6.16) and (6.18) in the inequalities for U^ℓ and ∇U^ℓ, we end up with

$$\begin{aligned}&\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^\ell_t,\nabla^2U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\
&\le C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\big)+C_\lambda\|\widetilde{\eta}^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))},\end{aligned} \tag{6.19}$$

$$\begin{aligned}&\|\nabla U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\nabla U^\ell_t,\nabla^2\nabla U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\
&\qquad\le C\big(\|\widetilde{\eta}^\ell(u_0,\nabla u_0)\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\widetilde{\eta}^\ell(f,\nabla f)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\big)\\
&\qquad\qquad+C_\lambda\|\widetilde{\eta}^\ell(u,\nabla u,\nabla^2u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned} \tag{6.20}$$

Second step: the boundary estimate. — We now consider $\ell\in\{L+1,\dots,M\}$ so that $B(x^\ell,\lambda)$ is centered at a point of $\partial\Omega$. The localization leads to

$$\left\{\begin{array}{lll}U^\ell_t-\Delta U^\ell=f^\ell & \text{in} & (0,T)\times\Omega,\\ \partial_{\vec n}U^\ell=0 & \text{on} & (0,T)\times\partial\Omega,\\ U^\ell|_{t=0}=u_0\eta^\ell & \text{on} & \Omega,\end{array}\right. \tag{6.21}$$

with f^ℓ defined by (6.15), hence satisfying (6.16).

Let us now make a change of variables so as to recast (6.21) in the half-space. As $\partial\Omega$ is smooth and compact, if λ has been chosen small enough then for fixed ℓ we may find a map Z_ℓ so that (see Chapter 2):

(i) Z_ℓ is a $\mathcal{C}^\infty$ diffeomorphism from $B(x^\ell,\lambda)$ to $Z_\ell(B(x^\ell,\lambda))$;
(ii) $Z_\ell(x^\ell)=0$ and $D_xZ(x^\ell)=\mathrm{Id}$;
(iii) $Z_\ell(\Omega\cap B(x^\ell,\lambda))\subset\mathbb{R}^n_+$;
(iv) $Z_\ell(\partial\Omega\cap B(x^\ell,\lambda))=\partial\mathbb{R}^n_+\cap Z_\ell(B(x^\ell,\lambda))$;
(v) Z_ℓ is normal preserving.

Setting $\nabla_x Z_\ell(x)=\mathrm{Id}+\mathcal{A}_\ell(z)$ then one may assume in addition that there exist constants C_j depending only on Ω and on $j\in\mathbb{N}$ such that

$$\|D^j\mathcal{A}_\ell\|_{L_\infty(B(x^\ell,\lambda))}\leq C_j, \tag{6.22}$$

a property which implies (by the mean value formula) that

$$\|\mathcal{A}_\ell\|_{L_\infty(B(x^\ell,\lambda))}\leq C_1\lambda, \tag{6.23}$$

hence by interpolation between the spaces $L_q(B(x^\ell,\lambda))$ and $W^k_q(B(x^\ell,\lambda))$,

$$\|\mathcal{A}_\ell\|_{B^{\frac nq}_{q,1}(B(x^\ell,\lambda))}\leq C\lambda\quad\text{for all}\ \ 1\leq q<\infty. \tag{6.24}$$

Let $V^\ell:=Z_\ell^*U^\ell:=U^\ell\circ Z_\ell^{-1}$. The system satisfied by V^ℓ reads

$$\begin{cases} V^\ell_t-\Delta_zV^\ell=F^\ell & \text{in}\quad (0,T)\times\mathbb{R}^n_+,\\ \partial_{z_n}V^\ell|_{z_n=0}=0 & \text{on}\quad (0,T)\times\partial\mathbb{R}^n_+,\\ V^\ell|_{t=0}=Z^*_\ell(U^\ell|_{t=0}) & \text{on}\quad \partial\mathbb{R}^n_+,\end{cases} \tag{6.25}$$

with

$$F^\ell:=Z^*_\ell f^\ell+(\Delta_x-\Delta_z)V^\ell.$$

According to Theorem 6.2.2, we thus get

$$\begin{aligned}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|V^\ell_t,\nabla^2_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\lesssim\|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|Z^*_\ell f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\qquad\qquad+\|(\Delta_x-\Delta_z)V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

The first two terms in the right-hand side may be dealt with thanks to composition estimates:

$$\begin{aligned}\|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}&\lesssim\|U^\ell|_{t=0}\|_{\dot B^s_{p,1}(\Omega)},\\ \|Z^*_\ell f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&\lesssim\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}.\end{aligned}$$

Compared to the first step, the only definitely new term is $(\Delta_x-\Delta_z)V^\ell$, which has already been bounded in (4.77). We get

$$\|(\Delta_x-\Delta_z)V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \lesssim \lambda\|\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Using also (6.16), we end up for small enough λ with

$$\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^\ell_t,\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \leq C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)\\ +C_\lambda\|\widetilde\eta^\ell(u,\nabla_xu)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.$$

Recall the following interpolation inequality for any smooth domain D (see [**8**], Chap. 18):

$$\|\nabla W\|_{\dot B^s_{p,1}(D)}\lesssim\|\nabla^2W\|^{1/2}_{\dot B^s_{p,1}(D)}\|W\|^{1/2}_{\dot B^s_{p,1}(D)}+\|W\|_{\dot B^s_{p,1}(D)}. \tag{6.26}$$

Applying it to $G=\nabla_zV^\ell$ and $D=\mathbb{R}^n_+$ and using Young's inequality allows to reduce the above inequality to

$$\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^\ell_t,\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \leq C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)\\ +C_\lambda\|\widetilde\eta^\ell(u,\nabla_xu)\|_{L_1(0,T;\dot B^s_{p,1}(K))}. \tag{6.27}$$

To prove regularity estimates, we apply Inequality (6.7) to (6.25), and thus get

$$\|\nabla_zV^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|(\nabla_zV^\ell)_t,\nabla_z^3V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \lesssim\|\nabla_z(Z^*_\ell(U^\ell|_{t=0}))\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|\nabla_z(Z^*_\ell f^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ +\|\nabla_z(\Delta_x-\Delta_z)V^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}. \tag{6.28}$$

Using $\nabla_z(Z^*_\ell g)=Z^*_\ell\nabla_xg\cdot\nabla_zZ^{-1}_\ell$ for $g=U^\ell|_{t=0}$ and $g=f^\ell$, composition and product estimates together with (6.24) and (6.15) yields

$$\begin{aligned}\|\nabla_z(Z^*_\ell(U^\ell|_{t=0}))\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}&\leq C\|\eta^\ell\nabla_xu_0\|_{\dot B^s_{p,1}(\Omega)},\\ \|\nabla_z(Z^*_\ell f^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&\leq C_\lambda\|\widetilde\eta^\ell(u,\nabla_xu,\nabla_x^2u)\|_{L_1(0,T;\dot B^s_{p,1}(K)}.\end{aligned}$$

From (4.76), we also see that $\nabla_z(\Delta_x-\Delta_z)V^\ell$ is a linear combination of components of $\nabla_z^3V_\ell\otimes\mathcal{A}^\ell$, $\nabla_z^2V_\ell\otimes\nabla_z\mathcal{A}^\ell$ and $\nabla_zV_\ell\otimes\nabla_z^2\mathcal{A}^\ell$. Now, for λ small enough, Inequalities (6.22), (6.23) and (6.24) guarantee that

$$\begin{aligned}
\|\nabla_z^3 V_\ell \otimes \mathcal{A}^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C\lambda \|\nabla_z^3 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\\
\|\nabla_z^2 V_\ell \otimes \nabla_z\mathcal{A}^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C \|\nabla_z^2 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\\
\|\nabla_z V_\ell \otimes \nabla_z^2\mathcal{A}^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C \|\nabla_z V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}$$

Resuming to (6.28), taking λ small enough and using the interpolation inequality (6.26) to eliminate the term pertaining to $\nabla_z^2 V^\ell$, we conclude that

$$\begin{aligned}
\text{(6.29)}\quad & \|\nabla_z V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|(\nabla_z V^\ell)_t, \nabla_z^3 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\leq C\big(\|\widetilde\eta^\ell(u_0,\nabla_x u_0)\|_{\dot B^s_{p,1}(\Omega)} + \|\widetilde\eta^\ell(f,\nabla_x f)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\
&\quad + \|\nabla_z V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big) + C_\lambda \|\widetilde\eta^\ell(u,\nabla_x u,\nabla_x^2 u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.
\end{aligned}$$

Third step: global a priori estimates : low regularity. — To establish (6.9) and (6.11), we start from the observation that, according to (6.19) and (6.27),

$$\begin{aligned}
&\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\\
&\quad\leq \sum_\ell \|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\\
&\quad\lesssim \sum_{0\leq\ell\leq M} \|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \sum_{M<\ell\leq L} \|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\quad\lesssim \sum_{0\leq\ell\leq L}\Big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)} + \|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\
&\qquad\qquad + \|\widetilde\eta^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}\Big) + \sum_{M<\ell\leq L} \|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},
\end{aligned}$$

and similar inequalities for $\|\partial_t u, \nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}$.

As the space $\dot B^s_{p,1}(\Omega)$ has the localization property (because $-n/p' < s \leq n/p$), we may write

$$\begin{aligned}
\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)} &\lesssim \|u_0\|_{\dot B^s_{p,1}(\Omega)},\\
\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} &\lesssim \|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))},\\
\|\widetilde\eta^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} &\lesssim \|(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))},\\
\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\lesssim \|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.
\end{aligned}$$

Therefore

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|u_t,\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}\\ +\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.$$

Once again, using (6.26) enables us to eliminate the last term, and we thus end up with Inequality (6.12). Now, because

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\le T\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

the last term of (6.12) may be absorbed by the left-hand side if T is small enough. Repeating the argument for $[T,2T]$ and so on, leads to (6.11).

Fourth step: global a priori estimates : high regularity. — Owing to (6.20) and (6.29), we have

$$\|\nabla_xu\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\sum_{0\le\ell\le M}\|\nabla_xU^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ +\sum_{M<\ell\le L}\|\nabla_zV^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \lesssim\sum_{0\le\ell\le L}\Big(\|\widetilde\eta^\ell(u_0,\nabla_xu_0)\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^\ell(f,\nabla_xf)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ +\|\widetilde\eta^\ell(u,\nabla_xu,\nabla^2_xu)\|_{L_1(0,T;\dot B^s_{p,1}(K))}\Big)+\sum_{M<\ell\le L}\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},$$

and similar inequalities for $\|\nabla u_t,\nabla^3u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}$.

By using the fact that $\nabla_zV^\ell=Z^*_\ell\nabla_xU^\ell\cdot\nabla_zZ_\ell^{-1}$ and arguing as in the previous step, we get

$$\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla u_t,\nabla^3u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ \lesssim\|u_0,\nabla u_0\|_{\dot B^s_{p,1}(\Omega)}\\ +\|f,\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}\\ +\|\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.$$

The last term may be handled by interpolation, and eliminated, and we get (6.14). If we use the fact that

$$\|u,\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\le T\|u,\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

and add up to Inequality (6.12), then we get (6.13) on a small time interval $[0,T]$. Then repeating the argument leads to (6.13) on $\mathbb{R}_+$.

Proving the existence is a rather standard issue (see e.g. [**35**]). We may consider smooth approximations of data f and u_0, which will generate W^2_p approximate solutions (see e.g. [**35**]). Estimates (6.11), (6.13) may thus be derived not only for those approximate solutions but also for the differences of them. We readily get that the sequence of approximate solutions is indeed a Cauchy sequence in the required space (which is complete owing to the small value of s) and it is then easy to pass to the limit in (6.8). □

In order to complete the proof of Theorem 6.2.3, we have to bound the last term of (6.12) and (6.14), *independently of* T. This is the goal of the next lemma. We here adapt Lemma 4.3.3 to the heat equation.

LEMMA 6.2.1. — *Assume that* $n \geq 3$ *and that* $1 < p < n/2$. *Then for any* $s \in (-1+1/p, 1/p-2/n)$ *smooth solutions to* (6.8) *with* $\nu = 1$ *fulfill*

$$\|u\|_{L_1(0,T;\dot{B}^s_{p,1}(K))} \leq C\big(\|u_0\|_{\dot{B}^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\big),$$

where C *is* independent of T.

Proof. — We split the solution u to (6.8) in $u = u_1 + u_2$ with

$$\begin{cases} u_{1,t} - \Delta u_1 = f & \text{in } (0,T)\times\Omega,\\ \partial_{\vec{n}} u_1 = 0 & \text{on } (0,T)\times\partial\Omega,\\ u_1|_{t=0} = 0 & \text{on } \Omega; \end{cases} \qquad \begin{cases} u_{2,t} - \Delta u_2 = 0 & \text{in } (0,T)\times\Omega,\\ \partial_{\vec{n}} u_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ u_2|_{t=0} = u_0 & \text{on } \Omega. \end{cases}$$

Let us first focus on u_1. From Corollary 2.2.1 and duality properties of Besov spaces, we infer that

$$\|u_1(t)\|_{\dot{B}^s_{p,1}(K)} = \sup \int_K u_1(t,x)\eta_0(x)\,dx,$$

where the supremum is taken over all

$$\eta_0 \in \dot{B}^{-s}_{p',\infty}(\mathbb{R}^n) \quad \text{with} \quad \|\eta_0\|_{\dot{B}^{-s}_{p',\infty}(\mathbb{R}^n)} = 1 \quad \text{and} \quad \operatorname{Supp}\eta_0 \subset K. \tag{6.30}$$

Consider the solution η to the problem:

$$\begin{cases} \eta_t - \Delta\eta = 0 & \text{in} \quad (0,T)\times\Omega,\\ \partial_{\vec{n}}\eta = 0 & \text{on} \quad (0,T)\times\partial\Omega,\\ \eta|_{t=0} = \eta_0 & \text{on} \quad \Omega. \end{cases} \tag{6.31}$$

Testing the equation for u_1 by $\eta(t-\cdot)$ we discover that

$$\int_\Omega u_1(t,x)\eta_0(x)\,dx = \int_0^t \int_\Omega f(\tau,x)\eta(t-\tau,x)\,dx\,d\tau. \tag{6.32}$$

The general theory for the heat operator in exterior domains implies that

$$\|\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{for} \quad 1 < b \leq a < \infty. \tag{6.33}$$

This is a consequence of Gaussian estimates for the kernel pertaining to the heat equation with Neumann boundary condition. More precisely, in [**30**], Theorem 2 (see also [**22**]), it has been proved that for fairly general domains there exists some function $\theta \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ so that for all $t > 0$ and $x \in \Omega$, we have

$$\eta(t,x) = \int_\Omega \frac{1}{t^{n/2}}\theta\Big(\frac{x-y}{\sqrt{t}}\Big)\eta_0(y)\,dy.$$

Using standard convolution estimates obviously yields (6.33).

Next, we observe that smooth solutions to (6.31) satisfy $\partial_{\vec{n}}\Delta\eta|_{\partial\Omega} = \partial_{\vec{n}}\eta_t|_{\partial\Omega} = 0$. Hence Inequality (6.33) applies to $\Delta\eta$ and we eventually get

$$\|\nabla^2\eta(t)\|_{L_a(\Omega)} \le C\|\Delta\eta(t)\|_{L_a(\Omega)} \le C'\|\Delta\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{6.34}$$

Interpolating between (6.33) and (6.34) thus yields for $0 < s < 1/b$ and $1 \le r \le \infty$,

$$\|\eta(t)\|_{\dot{B}^s_{b,r}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{a,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{if } \ 1 < a \le b < \infty. \tag{6.35}$$

In order to extend (6.35) to negative indices s, we consider the dual problem:

$$\left\{\begin{array}{lll} \zeta_t - \Delta\zeta = 0 & \text{in} & (0,T)\times\Omega, \\ \partial_{\vec{n}}\zeta = 0 & \text{on} & (0,T)\times\partial\Omega, \\ \zeta|_{t=0} = \zeta_0 & \text{on} & \Omega, \end{array}\right. \qquad \zeta_0 \in B^{-s}_{b',r'}(\Omega). \tag{6.36}$$

Testing (6.36) by $\eta(t-\cdot)$ yields

$$\int_\Omega \eta(t,x)\zeta_0(x)\,dx = \int_\Omega \eta_0(x)\zeta(t,x)\,dx.$$

Thus we get:

$$\|\eta(t)\|_{\dot{B}^s_{b,r}(\Omega)} = \sup_{\zeta_0}\int_\Omega \eta_0(x)\zeta(t,x)\,dx \le \sup_{\zeta_0}\Big(\|\eta_0\|_{\dot{B}^s_{a,r}(\Omega)}\|\zeta(t)\|_{\dot{B}^{-s}_{a',r'}(\Omega)}\Big),$$

where the supremum is taken over all $\zeta_0 \in \dot{B}^{-s}_{b',r'}(\Omega)$ such that $\|\zeta_0\|_{\dot{B}^{-s}_{b',r'}(\Omega)} = 1$.

As $-s$ is positive, applying (6.35) to bound $\|\zeta(t)\|_{\dot{B}^{-s}_{a',r'}(\Omega)}$ yields

$$\|\eta(t)\|_{\dot{B}^s_{a,r}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{if } \ s > -1+1/a.$$

The remaining case $s = 0$ follows by interpolation. So finally for all $1 < b \le a < \infty$, $q \in [1,\infty]$ and $-1+1/a < s < 1/b$, we have

$$\|\eta(t)\|_{\dot{B}^s_{a,r}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{6.37}$$

Resuming to the problem of bounding u_1 and starting from (6.32), one may write

$$\Big|\int_\Omega u_1(t,x)\eta_0(x)\,dx\Big| \lesssim \int_0^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta(t-\tau)\|_{\dot B^{-s}_{p',\infty}(\Omega)}\,d\tau.$$

Hence applying (6.37) yields for any $\varepsilon\in(\max(0,s),1)$,

$$\begin{aligned}\Big|\int_\Omega u_1(t,x)\eta_0(x)\,dx\Big| \lesssim &\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}\,d\tau\\ &+\int_0^{\max(0,t-1)} \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)}(t-\tau)^{-\frac n2(\frac1p-\varepsilon)}\,d\tau.\end{aligned}$$

As η_0 is supported in K, one has for some constant C depending on K:

$$\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)} \le C\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}.$$

So, keeping in mind (6.32) and the fact that the supremum is taken over all the functions η_0 satisfying (6.30), we deduce that

$$\begin{aligned}\|u_1(t)\|_{\dot B^s_{p,1}(K)} \le C\Big(&\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,d\tau\\ &+\int_0^{\max(0,t-1)} (t-\tau)^{-\frac n2(\frac1p-\varepsilon)}\|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,d\tau\Big).\end{aligned}$$

Therefore,

$$\int_1^T \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \le C\Big(1+\int_1^T \tau^{-\frac n2(\frac1p-\varepsilon)}\,d\tau\Big)\int_0^T \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt. \tag{6.38}$$

On $[0,1]$, we merely have

$$\int_0^1 \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \le C\int_0^1 \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt. \tag{6.39}$$

Now, provided $\max(0,s)<\varepsilon<1/p-2/n$ (which requires $p<n/2$), the constant in (6.38) may be made independent of T and we conclude that

$$\int_0^T \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \le C\int_0^T \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt.$$

Let us finally bound u_2. We first write that

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \le C\|u_0\|_{\dot B^s_{p,1}(\Omega)} \tag{6.40}$$

and, if $-1+\varepsilon<s<1/p$,

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \le C_K\|u_2(t)\|_{\dot B^s_{\frac1\varepsilon,1}(K)} \le C_K\|u_0\|_{\dot B^s_{p,1}(\Omega)}t^{-\frac n2(\frac1p-\varepsilon)}.$$

Then decomposing the integral over $[0,T]$ into an integral over $[0,\min(1,T)]$ and $[\min(1,T),T]$, we easily get

$$\int_0^T \|u_2(t)\|_{\dot B^s_{p,1}(K)}\,dt \le C\Big(1+\int_{\min(1,T)}^T t^{-\frac n2(\frac1p-\varepsilon)}\,dt\Big)\|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{6.41}$$

The integrant in the r.h.s. of (6.41) is finite for $\frac n2\big(\frac1p-\varepsilon\big)>1$. Hence,

$$\int_0^T \|u_2(t)\|_{\dot B^s_{p,1}(K)}\,dt \le C\|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{6.42}$$

Combining (6.42), (6.38) and (6.39) completes the proof of the lemma. □

We are now ready to prove Theorem 6.2.3. According to (6.12), in order to get (6.9), it suffices to show that

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))}\lesssim\|u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))}.$$

Of course, $\|u\|_{L_1(0,T;\dot B^{s'}_{q,1}(K))}$ may be directly bounded from Lemma 6.2.1, and it is also the case of $\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}$ if $p<n/2$ and $s<1/p-2/n$.

If $p\ge n/2$, then we use the fact that

$$\dot B^{s'+2}_{q,1}(\Omega)\subset\dot B^s_{q^*,1}(\Omega)\qquad\text{with}\qquad \frac1{q^*}=\frac1q-\frac2n+\frac{s-s'}{n}.$$

Therefore, if $q<n/2\le p<q^*$ then one may combine interpolation and Lemma 6.2.1 so as to absorb $\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}$ by the left-hand side of (6.9), changing the constant C if necessary.

If $p\ge q^*$ then one may repeat the argument again until the all possible values of p in $(n/2,\infty)$ are exhausted. This completes the proof of (6.9).

In order to establish the regularity estimate (6.10), we add up (6.12) and (6.14) (pertaining to Besov spaces $\dot B^s_{p,1}(\Omega)$ and $\dot B^{s'}_{q,1}(K)$) and use the interpolation inequality (6.26) so as to eliminate the term $\|\nabla u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))}$. We eventually get

$$\begin{aligned}&\|u,\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}+\|u_t,\nabla u_t,\nabla^2u,\nabla^3u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\\&\lesssim\|u_0,\nabla u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)}+\|f,\nabla f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}+\|u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))}.\end{aligned}$$

The last term may be handled by means of Lemma 6.2.1, as in the proof of (6.9). This completes the proof of Theorem 6.2.3.

Remark 6.2.1. — Here we decided to concentrate on the exterior domain case. Similar results hold true for the solutions to (6.8) supplemented with the condition that $\int_\Omega u\,dx=0$ in any smooth bounded domain Ω of $\mathbb R^n$ with $n\ge2$

(instead of $n \geq 3$ for exterior domains). The first part of the analysis, namely the proof of Theorem 6.2.4, works the same, and Lemma 6.2.1 may be improved given that $L_p - L_q$ estimates may be replaced by exponential decay.

6.3. Solving a low Mach number system

We can now tackle the well-posedness issue of System (6.1) for data having critical regularity, and Ω an exterior domain of $\mathbb{R}^n$ with $n \geq 3$. For simplicity, we focus on the global-in-time existence for small perturbations of the trivial constant state $(\theta, u) = (0, 0)$ and consider data (θ_0, u_0) with bulk regularity in the critical spaces $\dot{B}^{n/p}_{p,1}(\Omega) \times \dot{B}^{n/p-1}_{p,1}(\Omega)$ with $p = n$ (see [**14**] for more general results in the case $\Omega = \mathbb{R}^n$).

First we need to introduce a few notation. For $s \in \mathbb{R}$ and $1 \leq p, q \leq +\infty$, we denote by $X^s_{p,q}$ the set of functions $z : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $z \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))$ and $\partial_t z, D^2 z \in L_1(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))$ endowed with the norm

$$\|z\|_{X^s_{p,q}} := \|z\|_{L_\infty(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))} + \|\partial_t z, \nabla^2 z\|_{L_1(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))}. \tag{6.43}$$

We shall keep the same notation for vector fields with components in $X^s_{p,q}$.

Next, we denote by $\widetilde{X}^s_{p,q}$ the subspace of functions $\theta \in X^{s-1}_{p,q}$ satisfying $\nabla\theta \in X^{s-1}_{p,q}$, and set

$$\|\theta\|_{\widetilde{X}^s_{p,q}} := \|\theta\|_{X^{s-1}_{p,q}} + \|\nabla\theta\|_{X^{s-1}_{p,q}}. \tag{6.44}$$

It will also be convenient to use the notation $\widetilde{B}^s_{p,1}(\Omega)$ to designate the space of those functions $\theta \in \dot{B}^{s-1}_{p,1}(\Omega)$ so that $\nabla\theta \in \dot{B}^{s-1}_{p,1}(\Omega)$, endowed with the norm

$$\|\theta\|_{\widetilde{B}^s_{p,1}(\Omega)} := \|\theta\|_{\dot{B}^{s-1}_{p,1}(\Omega)} + \|\nabla\theta\|_{\dot{B}^{s-1}_{p,1}(\Omega)}. \tag{6.45}$$

Here is our main global well-posedness result for System (6.1).

THEOREM 6.3.1. — *Assume that Ω is an exterior domain of $\mathbb{R}^n$ with $n \geq 3$. Let $\theta_0 \in \widetilde{B}^1_{q,1} \cap \dot{B}^1_{n,1}(\Omega)$ and $u_0 \in \dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)$ with $1 < q < n/2$. If the compatibility condition*

$$\operatorname{div} u_0 = a \operatorname{div}(\kappa(\theta_0)\nabla\theta_0) \tag{6.46}$$

is satisfied, and

$$\|\theta_0\|_{\widetilde{B}^1_{q,1} \cap \dot{B}^1_{n,1}(\Omega)} + \|u_0\|_{\dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)} \leq c \tag{6.47}$$

for sufficiently small c, then there exists a unique global-in-time solution $(\theta, u, \nabla Q)$ *to System* (6.1) *such that*

$$(\theta, u) \in \widetilde{X}^1_{n,q} \times X^0_{n,q} \quad \textit{and} \quad \nabla Q \in L_1(\mathbb{R}_+; \dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)). \tag{6.48}$$

Besides, for some constant $C = C(n, q, \Omega)$,

$$\begin{aligned}\|\nabla Q\|_{L_1(\mathbb{R}_+; \dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega))} + \|\theta\|_{\widetilde{X}^1_{n,q}} + \|u\|_{X^0_{n,q}} \\ \leq C\big(\|\theta_0\|_{\widetilde{B}^1_{q,1} \cap \dot{B}^1_{n,1}(\Omega)} + \|u_0\|_{\dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)}\big).\end{aligned} \tag{6.49}$$

Some comments are in order, concerning the data. First, as in [**14**], we should be able to consider large data $\theta_0 \in \widetilde{B}^1_{n,1}(\Omega)$ and $u_0 \in \dot{B}^0_{n,1}(\Omega)$ and get the local existence of unique strong solutions, provided that there is no vacuum initially (or equivalently that ϑ_0 is positive). Second, we did not assume that θ_0 is in $\dot{B}^0_{n,1}(\Omega)$ because it is guaranteed by the following embedding:

$$\dot{B}^0_{q,1} \cap \dot{B}^1_{n,1} \cap \dot{B}^1_{q,1}(\Omega) \hookrightarrow \dot{B}^0_{n,1}(\Omega). \tag{6.50}$$

Finally, we expect a similar statement to be true for more general gases where the function a smoothly depends on ϑ. However, this would require us to generalize our maximal regularity estimates to the Stokes system with a divergence constraint which reads $\operatorname{div} R + A$.

Proof of Theorem 6.3.1. — Proving the existence and uniqueness of a solution for (6.1) is based on the Banach fixed point theorem. As a preliminary step, we shall derive a priori estimates. This will enable us to find out the solution space and an appropriate smallness condition on the data ensuring a global-in-time control on the solutions. Next, we shall introduce a suitable map $\mathcal{T}$ the fixed points of which are global solutions to (6.1). Slight modifications of the estimates obtained in the preliminary step will enable us to justify that the hypotheses of the fixed point theorem are fulfilled. This will complete the proof of the global-in-time existence. Proving uniqueness is almost the same as proving that $\mathcal{T}$ is contractive on a suitably small ball, and is thus omitted.

Step 1. A priori estimates. — Let $\bar{\rho} := \rho(0)$, $\bar{\mu} := \mu(0)$, $\bar{\kappa} := \kappa(0)$ and $\bar{\beta} := \beta(0)$. Set $\widetilde{\rho} := \rho - \bar{\rho}$, $\widetilde{\mu} := \mu - \bar{\mu}$, $\widetilde{\kappa} := \kappa - \bar{\kappa}$, and $\widetilde{\beta} := \beta - \bar{\beta}$. Let us recast

System (6.1) as follows:

$$
(6.51)\quad \begin{cases}
\bar\beta\partial_t\theta - \operatorname{div}(\bar\kappa\nabla\theta) = \operatorname{div}(\widetilde\kappa(\theta)\nabla\theta) - \beta(\theta)u\cdot\nabla\theta - \widetilde\beta(\theta)\partial_t\theta,\\
\bar\rho\partial_t u - \operatorname{div}(\bar\mu\mathbb{D}u) + \nabla Q = \operatorname{div}(\widetilde\mu(\theta)\mathbb{D}u) - \rho(\theta)u\cdot\nabla u - \widetilde\rho(\theta)\partial_t u,\\
\operatorname{div} u = a\operatorname{div}(\kappa(\theta)\nabla\theta),\\
u|_{\partial\Omega} = 0,\\
\partial_{\vec n}\theta|_{\partial\Omega} = 0,\\
u|_{t=0} = u_0,\\
\theta|_{t=0} = \theta_0.
\end{cases}
$$

Before going further in the computations, let us point out that our results for the Stokes system in Section 4.3 hold for $\operatorname{div}\mathbb{D}u$ instead of Δu. Indeed, as a first step, we removed the compressibility (the right-hand side of $(6.51)_3$) to obtain a divergence-free vector field. Then $\operatorname{div}\mathbb{D}u = \Delta u$ for $\operatorname{div} u = 0$. One may alternately remark that for a general vector-field we have $\operatorname{div}(\mathbb{D}u) = \Delta u + \nabla\operatorname{div} u$, and incorporate the last term in the pressure. In any case, our bounds for (6.51) will follow from Theorem 4.3.3 for the Stokes system and Theorem 6.2.4 for the heat equation.

More precisely, on the one hand, applying Theorem 6.2.4 yields

$$
(6.52)\quad \|\theta\|_{\widetilde X^1_{n,q}} \lesssim \|\theta_0\|_{\widetilde B^1_{q,1}\cap\dot B^1_{n,1}(\Omega)} + \|\operatorname{div}(\widetilde\kappa(\theta)\nabla\theta) - \beta(\theta)u\cdot\nabla\theta - \widetilde\beta(\theta)\partial_t\theta\|_{L_1(\mathbb{R}_+;\widetilde B^1_{q,1}\cap\dot B^1_{n,1}(\Omega))}.
$$

On the other hand, applying Theorem 4.3.3 to the momentum equation (recall that $\kappa(\theta)\nabla\theta\cdot\vec n \equiv 0$ at $\partial\Omega$), we get for all $1 < m < \infty$,

$$
\begin{aligned}
(6.53)\quad & \|u\|_{X^0_{n,q}} + \|\nabla Q\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\lesssim \|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}\\
&\quad + \|\operatorname{div}(\widetilde\mu(\theta)\mathbb{D}u) - \widetilde\rho(\theta)\partial_t u - \rho(\theta)u\cdot\nabla u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad + \|\operatorname{div}(\kappa(\theta)\nabla\theta)\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}\cap\dot B^1_{n,1}(\Omega))}\\
&\quad + \|(\kappa(\theta)\nabla\theta)_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))} + \|\kappa(\theta)\nabla\theta\|_{L_1(\mathbb{R}_+;L_m(\Omega))}.
\end{aligned}
$$

Note that no bounds for the pressure are needed to close the estimates, as it does not appear in the right-hand sides of (6.52) and (6.53). This quantity can be controlled at the end of our analysis. Another observation is that the left-hand side of (6.52) allows to estimate the highest order term in θ in the right-hand side of (6.53). To close the a priori estimates, we need to get suitable bounds for the right-hand side of (6.52) and (6.53). As the full proof is quite repetitive, we just consider a few terms by way of example.

For instance, we have, keeping (6.50) in mind,

$$\begin{aligned}(6.54)\quad &\|\operatorname{div}(\widetilde{\kappa}(\theta)\nabla\theta)\|_{\widetilde{B}^1_{q,1}\cap\widetilde{B}^1_{n,1}(\Omega)}\\ &\lesssim \|\widetilde{\kappa}(\theta)\nabla\Delta\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}+\|\widetilde{\kappa}'(\theta)\nabla\theta\otimes\nabla^2\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &\quad+\|\widetilde{\kappa}''(\theta)|\nabla\theta|^3\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}+\|\widetilde{\kappa}(\theta)\Delta\theta+\widetilde{\kappa}'(\theta)|\nabla\theta|^2\|_{\dot{B}^0_{q,1}(\Omega)}.\end{aligned}$$

First let us note that

$$(6.55)\qquad \|\theta\|_{\mathcal{M}(\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}\lesssim\|\theta\|_{\dot{B}^1_{n,1}(\Omega)}\lesssim\|\theta,\nabla\theta\|_{\dot{B}^0_{n,1}(\Omega)}.$$

The first inequality stems from Lemma 5.5.1 and the second one from the corresponding inequality in $\mathbb{R}^n$ (use a standard extension operator after reducing the proof to the bounded domain case). We eventually get, after using composition estimates:

$$\begin{aligned}(6.56)\quad &\|\widetilde{\kappa}(\theta)\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}\\ &\qquad\lesssim\|\theta,\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}.\end{aligned}$$

The second term in (6.54) may be handled similarly. Using (6.55), we get

$$\begin{aligned}&\|\widetilde{\kappa}'(\theta)\nabla\theta\otimes\nabla^2\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}\\ &\qquad\lesssim\|\theta,\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^2\theta,\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}.\end{aligned}$$

By the same arguments, we see that the third term obeys

$$\||\nabla\theta|^2\|_{L_1(\mathbb{R}_+;\dot{B}^1_{n,1}(\Omega))}\lesssim\|\theta,\nabla\theta\|^3_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^2\theta,\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}.$$

The (lower order) last term in (6.54) may be handled along the same lines.

Next, bounding $\beta(\theta)u\cdot\nabla\theta$ in $L_1(\mathbb{R}_+;\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))$ follows from the fact that $\beta(\theta)\in L_\infty(\mathbb{R}_+;\mathcal{M}(\dot{B}^0_{n,1}))$ and $u\in L_2(\mathbb{R}_+;\mathcal{M}(\dot{B}^0_{n,1}))$. We also see that

$$\begin{aligned}&\|\widetilde{\beta}(\theta)\partial_t\theta\|_{\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega)}\\ &\quad\lesssim\|\widetilde{\beta}(\theta)\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &\qquad+\|\widetilde{\beta}'(\theta)\nabla\theta\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}+\widetilde{\beta}(\theta)\nabla\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &\quad\lesssim\|\theta\|_{\dot{B}^1_{n,1}(\Omega)}\big(\|\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &\qquad+\|\nabla\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\|\partial_t\theta\|_{\dot{B}^1_{n,1}(\Omega)}+\|\nabla\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\big).\end{aligned}$$

Finally, let us bound some terms in the right-hand side of (6.53). We observe for instance that

$$\|\operatorname{div}(\kappa(\theta)\nabla\theta)\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}\cap\dot B^1_{n,1}(\Omega))}\lesssim\|\nabla^2\theta\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}\cap\dot B^1_{n,1}(\Omega))}+\|\theta\|^2_{\widetilde X^1_{n,q}}$$

and that

$$\begin{aligned}
&\|(\kappa(\theta)\nabla\theta)_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad\le\|\kappa'(\theta)\theta_t\nabla\theta\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}+\kappa(\theta)\nabla\theta_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad\lesssim\|\theta_t\|_{L_1(\mathbb{R}_+;\mathcal{M}(\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\|\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}+\|\nabla\theta_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad\lesssim\|\theta\|_{\widetilde X^1_{n,q}}(1+\|\theta\|_{\widetilde X^1_{n,q}}).
\end{aligned}$$

Note also that taking $m=qn/(n-2q)$ and using the embedding $\dot B^2_{q,1}(\Omega)\hookrightarrow L_m(\Omega)$ enables us to write that

$$\|\kappa(\theta)\nabla\theta\|_{L_1(\mathbb{R}_+;L_m(\Omega))}\le C\|\theta\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}.$$

Moreover,

$$\begin{aligned}
&\|\operatorname{div}(\widetilde\mu(\theta)\mathbb{D}(u))\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad\lesssim\|\theta\|_{L_\infty(\mathbb{R}_+;\dot B^1_{n,1}(\Omega))}\|\nabla^2u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\qquad\quad+\|\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\|\nabla u\|_{L_1(\mathbb{R}_+;\mathcal{M}(\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\
&\quad\lesssim\|\theta\|_{\widetilde X^1_{n,q}}\|u\|_{X^0_{n,q}}.
\end{aligned}$$

Putting together all the above estimates, we end up with

$$\begin{aligned}
\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\le C\Big(&\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\widetilde B^1_{q,1}\cap\dot B^1_{n,1}(\Omega)}\\
&+\big(\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\big)^2+\big(\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\big)^4\Big).
\end{aligned}\tag{6.57}$$

Hence we deduce from an elementary bootstrap argument that (6.49) follows from (6.47) if c has been taken small enough.

Step 2. The proof of the existence for small data. — The proof of the existence will be an elementary consequence of the Banach fixed point theorem. Let us introduce the map

$$\mathcal{T}:E^0_{n,q}\longrightarrow E^0_{n,q}\tag{6.58}$$

with $E^0_{n,q}:=X^0_{n,q}\times\widetilde X^1_{n,q}$ and

$$\mathcal{T}(\check u,\check\theta)=(u,\theta)\tag{6.59}$$

such that (u,θ) is the solution to the following linear system in $(0,T)\times\Omega$:

$$
\text{(6.60)}\qquad\begin{cases}
\bar\beta\partial_t\theta-\operatorname{div}(\bar\kappa\nabla\theta)=\operatorname{div}(\widetilde\kappa(\check\theta)\nabla\check\theta)-\beta(\bar\theta)\check u\cdot\nabla\check\theta-\widetilde\beta(\check\theta)\partial_t\check\theta,\\
\bar\rho\partial_t u-\operatorname{div}(\bar\mu\mathbb{D}u)+\nabla Q\\
\qquad\qquad\qquad\qquad=\operatorname{div}(\widetilde\mu(\check\theta)\mathbb{D}\check u)-\widetilde\rho(\check\theta)\partial_t\check u-\rho(\check\theta)\check u\cdot\nabla\check u,\\
\operatorname{div}u=a\operatorname{div}(\kappa(\check\theta)\nabla\check\theta),\\
u|_{\partial\Omega}=0\quad\text{and}\quad\partial_{\vec n}\theta|_{\partial\Omega}=0,\\
u|_{t=0}=u_0,\quad\theta|_{t=0}=\theta_0.
\end{cases}
$$

The solvability of System (6.60) in the space $E^0_{n,q}$ follows from the fact that the left-hand sides are just heat equation with Neumann boundary conditions, and the Stokes system with Dirichlet boundary conditions. Hence one may directly apply Theorems 4.3.3 and 6.2.3 and follow the computations of the previous step. More precisely, (6.57) (with $(\check\theta,\check u)$ in the right-hand side) implies that $\mathcal{T}$ maps the closed ball $\overline{B}(0,R)$ of $E^0_{n,q}$ into itself if choosing

$$
\text{(6.61)}\qquad R=2C\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)
$$

and assuming that

$$
\begin{aligned}
4C^2\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)&\\
+16C^4\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)^3&\le 1.
\end{aligned}
$$

To complete the proof of global existence, it suffices to show that $\mathcal{T}$ is a contraction on $\overline{B}(0,R)$, viz. that for all $(\check u_1,\check\theta_1)$ and $(\check u_2,\check\theta_2)$ in $\overline{B}(0,R)$, we have

$$
\text{(6.62)}\qquad \|\mathcal{T}(\check u_1,\check\theta_1)-\mathcal{T}(\check u_2,\check\theta_2)\|_{E^0_{n,q}}\le\frac12\|(\check u_1-\check u_2,\check\theta_1-\check\theta_2)\|_{E^0_{n,q}}.
$$

In order to guarantee (6.61) we consider the following system being a subtraction of (6.60) for the first and second solution. Setting $\delta\theta:=\theta_1-\theta_2$,

$\delta\check\theta := \check\theta_1 - \check\theta_2$, and so on, we get

$$
(6.63)\qquad \begin{cases}
\bar\beta\partial_t\delta\theta - \operatorname{div}(\bar\kappa\nabla\delta\theta) = \operatorname{div}(\widetilde\kappa(\check\theta_1)\nabla\check\theta_1) - \beta(\check\theta_1)\check u_1\cdot\nabla\check\theta_1\\
\qquad -\widetilde\beta(\check\theta_1)\partial_t\check\theta_1 - \operatorname{div}(\widetilde\kappa(\check\theta_2)\nabla\check\theta_2) + \beta(\check\theta_2)\check u_2\cdot\nabla\check\theta_2 + \widetilde\beta(\check\theta_2)\partial_t\check\theta_2,\\
\bar\rho\partial_t\delta u - \operatorname{div}(\bar\mu\mathbb{D}\delta u) + \nabla\delta Q = -\widetilde\rho(\check\theta_1)\partial_t\check u_1 - \rho(\check\theta_1)\check u_1\cdot\nabla\check u_1\\
\qquad + \operatorname{div}(\widetilde\mu(\check\theta_1)\mathbb{D}\check u_1) + \widetilde\rho(\check\theta_2)\partial_t\check u_2 + \rho(\check\theta_2)\check u_2\cdot\nabla\check u_2 - \operatorname{div}(\widetilde\mu(\check\theta_2)\mathbb{D}\check u_2),\\
\operatorname{div}\delta u = a\operatorname{div}(\kappa(\check\theta_1)\nabla\check\theta_1 - \kappa(\check\theta_2)\nabla\check\theta_2),\\
\delta u|_{\partial\Omega} = 0 \quad\text{and}\quad \partial_{\vec n}\delta\theta|_{\partial\Omega} = 0,\\
\delta u|_{t=0} = 0, \quad \delta\theta|_{t=0} = 0.
\end{cases}
$$

We claim that our results for the linear systems (for the left-hand sides of (6.63)) combined with nonlinear estimates give, up to a change of C,

$$
(6.64)\qquad \|(\delta u,\delta\theta)\|_{E^0_{n,q}} \le CR\|(\delta\check u,\delta\check\theta)\|_{E^0_{n,q}}.
$$

Let us just show how to bound $\partial_t\big(\kappa(\check\theta_1)\nabla\check\theta_1 - \kappa(\check\theta_2)\nabla\check\theta_2\big)$. We write that

$$
\begin{aligned}
&\|\partial_t\big(\kappa(\check\theta_1)\nabla\check\theta_1 - \kappa(\check\theta_2)\nabla\check\theta_2\big)\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}\\
&\quad\lesssim \|\partial_t(\kappa(\check\theta_1)\nabla\delta\check\theta)\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)} + \|\partial_t(((\kappa(\check\theta_1) - \kappa(\check\theta_2))\nabla\check\theta_2)\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}\\
&\quad\lesssim \|\kappa(\check\theta_1)\nabla\delta\check\theta_t, \kappa'(\check\theta)\check\theta_t\nabla\delta\check\theta, (\kappa(\check\theta_1) - \kappa(\check\theta_2))\nabla\check\theta_{2,t}\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}\\
&\qquad\qquad + \|\kappa'(\check\theta_1)\delta\check\theta_t\nabla\check\theta_2, (\kappa'(\check\theta_1) - \kappa'(\check\theta_2))\check\theta_{2,t}\nabla\check\theta_2\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}.
\end{aligned}
$$

Performing a time integration and using again (6.55) several times, it is easy to conclude that

$$
\|\partial_t\big(\kappa(\check\theta_1)\nabla\check\theta_1 - \kappa(\check\theta_2)\nabla\check\theta_2\big)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))} \le CR\|\delta\check\theta\|_{\widetilde X^1_{n,q}}.
$$

Taking c small enough in (6.47), and keeping the definition of R as in (6.61), it is clear that one may ensure that $CR \le 1/2$. The contraction mapping theorem thus ensures the existence of a fixed point for the map $\mathcal{T}$, which defines a unique solution to the original problem (6.1). Theorem 6.3.1 is proved. □

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