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FUNCTIONAL CALCULUS FOR FIRST ORDER SYSTEMS OF DIRAC TYPE AND BOUNDARY VALUE PROBLEMS

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FUNCTIONAL CALCULUS FOR FIRST ORDER SYSTEMS OF DIRAC TYPE AND BOUNDARY VALUE PROBLEMS

Pascal Auscher, Sebastian Stahlhut

***Abstract.* —** This memoir contains two articles.

1) In *A priori estimates for boundary value elliptic problems via first order systems*, we prove a number of *a priori* estimates for weak solutions of elliptic equations or systems with vertically independent coefficients in the upper-half space. These estimates are designed towards applications to boundary value problems of Dirichlet and Neumann type in various topologies. We work in classes of solutions which include the energy solutions. For those solutions, we use a description using the first order systems satisfied by their conormal gradients and the theory of Hardy spaces associated with such systems but the method also allows us to design solutions which are not necessarily energy solutions. We obtain precise comparisons between square functions, non-tangential maximal functions and norms of boundary trace. The main thesis is that the range of exponents for such results is related to when those Hardy spaces (which could be abstract spaces) are identified to concrete spaces of tempered distributions. We consider some adapted non-tangential sharp functions and prove comparisons with square functions. We obtain boundedness results for layer potentials, boundary behavior, in particular strong limits, which is new, and jump relations. One application is an extrapolation for solvability "à la Šneĭberg". Another one is stability of solvability in perturbing the coefficients in L^∞ without further assumptions. We stress that our results do not require De Giorgi-Nash assumptions, and we improve the available ones when we do so.

2) In *L^p-L^q theory for holomorphic functions of perturbed first order Dirac operators*, the aim is to prove L^p-L^q off-diagonal estimates and L^p-L^q boundedness for operators in the functional calculus of certain perturbed first order differential operators of Dirac type for with $p \leq q$ in a certain range of exponents. We describe the L^p-L^q off-diagonal estimates and the L^p-L^q boundedness in terms of the decay properties of the related holomorphic functions and give a necessary condition for L^p-L^q boundedness. Applications to Hardy-Littlewood-Sobolev estimates for fractional operators will be given.

Résumé **(Calcul fonctionnel des systèmes du premier ordre de type Dirac et problèmes aux limites)**

Ce mémoire comporte deux articles.

1) Dans *Estimations a priori pour les problèmes aux limites elliptiques via des systèmes du premier ordre*, on démontre un certain nombre d'estimations *a priori* pour les solutions faibles d'équations ou systèmes elliptiques sur le demi-espace dont les coefficients sont indépendents de la variable verticale. Ces estimations s'appliquent aux problèmes de Dirichlet ou de von Neumann dans des topologies variées. Nous considérons des classes de solutions comprenant les solutions d'énergie. Pour ces solutions, on utilise une approche par réduction à un système du premier ordre vérifié par le gradient conormal et une théorie des espaces de Hardy associés à ces systèmes. La méthode permet aussi de construire d'autres types de solutions. On obtient des comparaisons précises entre les normes de certaines fonctions d'aire, de certaines fonctions non-tangentielles maximales et de la trace des solutions du système. La thèse du mémoire est que l'ensemble des exposants pour lesquels on obtient ces comparaisons est relié à celui pour lesquels les espaces de Hardy associés (qui pourraient n'être que des espaces abstraits) sont identifiés à des espaces concrets dans les distributions tempérées. On compare aussi les normes des fonctions maximales dièses non-tangentielles à celle des fonctions d'aire. On obtient en particulier des résultats de continuité pour les opérateurs de simple et double couche généralisés, leur comportement au bord dans des topologies fortes, ce qui est nouveau, et les relations de saut. Une des applications est un résultat d'extrapolation locale « à la Šneĭberg » pour la résolubilité de nos équations elliptiques. Une autre est la stabilité de la résolubilité par perturbations dans L^∞ des coefficients. On observe que nos résultats n'utilisent pas la régularité locale des solutions (i.e., les conditions de DeGiorgi-Nash) et, lorsque nous la supposons, nous améliorons les résultats existants.

2) Dans *Théorie L^p-L^q pour le calcul holomorphe d'opérateurs de Dirac perturbés*, notre objectif est de démontrer des estimations de continuité hors diagonale L^p-L^q pour des opérateurs dans le calcul fonctionnel de certains opérateurs différentiels de type Dirac avec $p \leq q$ dans un certain intervalle d'exposants. Nous donnons des conditions suffisantes et des conditions nécessaires pour obtenir de telles estimations. Une application à des inégalités de type Hardy-Littlewood-Sobolev pour les puissances fractionnaires est donnée.

CONTENTS

PART ONE

PART TWO

PART I

A PRIORI ESTIMATES FOR BOUNDARY VALUE ELLIPTIC PROBLEMS VIA FIRST ORDER SYSTEMS

Pascal Auscher & Sebastian Stahlhut

1. INTRODUCTION

Our main goal in this work is to provide *a priori* estimates for boundary value problems for t-independent systems in the upper half space. We will apply this to perturbation theory for solvability. Of course, this topic has been much studied, but our methods and results are original in this context. We obtain new estimates and also design solutions in many different classes.

A remarkable feature is that we do not require any kind of existence or uniqueness to build such solutions. In fact, the point of the reduction of second order PDEs to first order systems is that for such systems the aim is to understand the initial value problem, and solving the PDE means inverting a boundary operator to create the initial data for the first order system. The initial value problem looks easier. However, the system has now a big null space and this creates other types of difficulties as we shall see.

If $E(\Omega)$ is a normed space of $\mathbb{C}$-valued functions on a set Ω and F a normed space, then $E(\Omega; F)$ is the space of F-valued functions with $\||f|_F\|_{E(\Omega)} < \infty$. More often, we forget about the underlying F if the context is clear.

We denote points in $\mathbb{R}^{1+n}$ by boldface letters $\boldsymbol{x}, \boldsymbol{y}, \ldots$ and in coordinates in $\mathbb{R} \times \mathbb{R}^n$ by (t, x), etc. We set $\mathbb{R}^{1+n}_+ = (0, \infty) \times \mathbb{R}^n$. Consider the system of m equations given by

$$\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i \big(A_{i,j}^{\alpha,\beta}(x) \partial_j u^\beta(\boldsymbol{x}) \big) = 0, \qquad \alpha = 1, \ldots, m \tag{1}$$

in $\mathbb{R}^{1+n}_+$, where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$ if $i = 1, \ldots, n$. For short, we write

$$Lu = -\operatorname{div} A \nabla u = 0$$

to mean (1), where we always assume that the matrix

$$A(x) = (A_{i,j}^{\alpha,\beta}(x))_{i,j=0,\ldots,n}^{\alpha,\beta=1,\ldots,m} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{m(1+n)})), \tag{2}$$

is bounded and measurable, independent of t, and satisfies the strict accretivity condition on a subspace $\mathcal{H}$ of $L^2(\mathbb{R}^n; \mathbb{C}^{m(1+n)})$, that is, for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \operatorname{Re}(A(x) f(x) \cdot \overline{f(x)})\, dx \geq \lambda \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 dx, \quad \forall f \in \mathcal{H}. \tag{3}$$

The subspace $\mathcal{H}$ is formed of those functions $(f_j^\alpha)_{j=0,\ldots,n}^{\alpha=1,\ldots,m}$ such that $(f_j^\alpha)_{j=1,\ldots,n}$ is curl-free in $\mathbb{R}^n$ for all α. The system (1) is always considered in the sense of distributions with weak solutions, that is $H^1_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m) = W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$ solutions.

It was proved in [**8**] that weak solutions of $Lu = 0$ in the classes

$$\mathcal{E}_0 = \{u \in \mathcal{D}'; \|\widetilde{N}_*(\nabla u)\|_2 < \infty\}$$

or

$$\mathcal{E}_{-1} = \{u \in \mathcal{D}'; \|S(t\,\nabla u)\|_2 < \infty\}$$

(where $\widetilde{N}_*(f)$ and $S(f)$ stand for a non-tangential maximal function and square function: definitions will be given later) have certain semigroup representation in their conormal gradient

$$\nabla_A u(t,x) := \begin{bmatrix} \partial_{\nu_A} u(t,x) \\ \nabla_x u(t,x) \end{bmatrix}.$$

More precisely, one has

$$\nabla_A u(t,\,.\,) = S(t)(\nabla_A u|_{t=0}) \tag{4}$$

for a certain semigroup $S(t)$ acting on the subspace $\mathcal{H}$ of L^2 in the first case and in the corresponding subspace in $\dot{H}^{-1}$, where $\dot{H}^s$ is the homogeneous Sobolev space of order s, in the second case. Actually, the second representation was only explicitly derived in subsequent works (see [**20, 83**]) provided one defines the conormal gradient at the boundary in this subspace of $\dot{H}^{-1}$. In [**83**], the semigroup representation was extended to weak solutions in the intermediate classes defined by

$$\mathcal{E}_s = \big\{u \in \mathcal{D}'; \|S(t^{-s}\nabla u)\|_2 < \infty\big\}$$

for $-1 < s < 0$ and the semigroup representation holds in $\dot{H}^s$. In particular, for $s = -1/2$, the class of weak solutions in $\mathcal{E}_{-1/2}$ is exactly the class of energy solutions used in [**20, 23**] (other classes were defined in [**73**] and used in [**55**]). And the boundary value problems associated to L can always be solved in the energy class. However, we shall not use this solvability property nor any other one until chapter 14.

Here, we intend to study the following problems:

PROBLEM 1: For which $p \in (0,\infty)$ do we have

$$\|\widetilde{N}_*(\nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{X_p} \sim \|S(t\partial_t \nabla u)\|_p \tag{5}$$

for solutions of $Lu = 0$ such that $u \in \mathcal{E} = \bigcup_{-1\le s\le 0} \mathcal{E}_s$?

PROBLEM 2: For which $p \in (0,\infty)$, do we have

$$\|S(t\,\nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{\dot{W}^{-1,p}} \tag{6}$$

for solutions of $Lu = 0$ such that $u \in \mathcal{E} = \bigcup_{-1\le s\le 0} \mathcal{E}_s$?

Here, $\dot{W}^{-1,p}$ is the usual homogeneous Sobolev space of order -1 on L^p: an estimate for partial derivatives in $\dot{W}^{-1,p}$ amounts to a usual L^p estimate. Moreover, do we have an analog when $p = \infty$, in which case we look for a Carleson measure estimate of $|t\,\nabla u|^2$ to the left and BMO^{-1} to the right, and a weighted Carleson measure estimate of $|t\,\nabla u|^2$ to the left and Hölder spaces $\dot{\Lambda}^{\alpha-1}$ to the right?

Let us comment on problem 1: here for the problem to make sense, we take $X_p = L^p$ if $p > 1$ and $X_p = H^p$, the Hardy space, for $p \leq 1$ and soon discover the constraint $p > \frac{n}{n+1}$. The equivalence between non-tangential maximal estimates and X_p norms is known in the following case: the inequality $\gtrsim$ is a very general fact proved for all weak solutions and $1 < p < \infty$ in [**72**] and when $\frac{n}{n+1} < p \leq 1$ in [**61**], and their arguments carry over to our situation. The inequality $\lesssim$ was proved in [**55**] for $1 < p < 2+\varepsilon$ and in [**23**] for $1-\varepsilon < p \leq 1$ (and also $1 < p < 2$ by interpolation) assuming some interior regularity of solutions (the De Giorgi-Nash condition) of $Lu = 0$. To our knowledge, *a priori* comparability with the square function $S(t\partial_t \nabla u)$ has not been studied so far, but this is a key feature of our analysis, roughly because the square function norms in (5) define spaces that interpolate while it is not clear for the spaces corresponding to non-tangential maximal norms in (5). The range of p in problem 1 allows one to formulate Neumann and regularity problems with L^p/H^p data, originally introduced in [**72**], in a meaningful way. By this, we mean that the conormal derivative and the tangential gradient at the boundary are in the natural spaces for those problems to have a chance to be solved with such solutions. Outside this range of p, there will be no solutions in our classes.

Let us turn to problem 2: that such comparability holds for a range of p containing $[2, \infty]$ and beyond under the De Giorgi-Nash condition on L was already used in [**23**]. We provide here the proof. The inequalities obtained in [**55**] contain extra terms and are less precise. The comparability in problem 2 allows one to formulate the Dirichlet problem with L^p data and even BMO or $\dot{\Lambda}^\alpha$ data and also a Neumann problem with $\dot{W}^{-1,p}$ or BMO^{-1} or $\dot{\Lambda}^{\alpha-1}$ data. Note that we are talking about square functions without mentioning non-tangential maximal estimates on the solutions u which are usually smaller in L^p sense. A beautiful result in [**54**] is the converse inequality for solutions of real elliptic equations.

We shall study comparability with appropriate non-tangential sharp functions, namely study when does

$$\|\widetilde{N}_*(u - u_0)\|_p \lesssim \|S(t\,\nabla u)\|_p$$

hold. The advantage of this inequality compared to the one with the non-tangential maximal function (which will be studied as well) is that we may allow $p = \infty$, in which case the right hand side should be replaced by the Carleson measure estimate of $|t\,\nabla u|^2$, and beyond using adapted versions for Hölder estimates.

The boundary spaces obtained in problem 1 for L and in problem 2 for L^* are usually in duality. This was used in [**23**] to give new lights, with sharper results, on the duality principles for elliptic boundary value problems studied first in [**72**] and then [**45**], [**55**], and to apply this to extrapolation.

Our main results are the following (here in dimension $1 + n \geq 2$).

THEOREM 1.1. — *The range of p in problem 1 for solutions $u \in \mathcal{E}$ of*

$$Lu = 0$$

is an interval I_L contained in $(\frac{n}{n+1}, \infty)$ and containing $(\frac{2n}{n+2} - \varepsilon, 2 + \varepsilon')$ for some $\varepsilon, \varepsilon' > 0$. Moreover, if $n = 1$ then $I_L = (\frac{1}{2}, \infty)$, if L has constant coefficients then $I_L = (\frac{n}{n+1}, \infty)$ and if $n \geq 2$ and $L_\parallel^*$ has the De Giorgi condition then $I_L = (1-\varepsilon, 2+\varepsilon')$ where ε is related to the regularity exponent in the De Giorgi condition.

Here, $L_\parallel$ is the tangential part of operator in L, obtained by deleting in L any term with a $\partial_t = \partial_0$ derivative in it. As L has t-independent coefficients, $L_\parallel$ is seen as an operator on $\mathbb{R}^n$ and the De Giorgi condition for $L_\parallel^*$ is about the regularity of weak solutions of $L_\parallel^* u = 0$ in $\mathbb{R}^n$. For example, this holds when $L_\parallel$ is a scalar real operator, but also when $1 + n = 2$ (this is due to Morrey). In that case, the other coefficients of L are arbitrary.

THEOREM 1.2. — *The range of exponents in problem 2 for solutions $u \in \mathcal{E}$ of*

$$L^* u = 0$$

is "dual" to the one in theorem 1.1. That is, for $p \in I_L$, we obtain (6) for p' if $p > 1$, the modification for BMO *if $p = 1$ and the modification for $\dot{\Lambda}^\alpha$ with $\alpha = n(\frac{1}{p} - 1)$ if $p < 1$.*

Although we can not define the objects in the context of this introduction, the main thesis of this work is as follows : *The exponents p in the first theorem are the exponents for which the Hardy space $\mathbb{H}^p_{DB}$ for the first order operator DB associated to L (as discovered in [**11**]) is identified to $\mathbb{H}^p_D$. The semigroup $S(t)$ mentioned above coincides with $e^{-t|DB|}$ seen as some kind of Poisson semigroup or Cauchy extension depending on the point of view. Hence, a large part of this work is devoted to say when $\mathbb{H}^p_{DB}$ and $\mathbb{H}^p_D$ are the same.*

A word on the *a priori* class $\mathcal{E}$ is in order: in fact, we want to work with a class for which the semigroup representation for the conormal gradient (4) is valid and this is the only reason for restricting to this class of solutions at this time. To make a parallel (and this case corresponds to the $L = -\Delta$ here), this is like proving such estimates for an harmonic function assuming it is the Poisson integral of an L^2 function: such estimates are in the fundamental work of Fefferman-Stein [**48**]. Removing this *a priori* information uses specific arguments on harmonic functions (also found in [**48**]). Removing that $u \in \mathcal{E}$ *a priori* will also require specific arguments. This will be the purpose of a forthcoming work by the first author with M. Mourgoglou [**24**]. It will be proved semigroup representation: every solution of L with $\|\widetilde{N}_*(\nabla u)\|_p < \infty$ in the range of p for theorem 1.1 has the semigroup representation (4) in an appropriate functional setting; and every solution of L^* with $\|S(t\,\nabla u)\|_p$ or even weighted Carleson control in the "dual" range of theorem 1.2 and a weak control at infinity has the semigroup representation in an appropriate functional setting.

We remark that the results obtained here impact on the boundary layer potentials. A. Rosén [**81**] proposed an abstract definition of boundary layer potentials $\mathcal{D}_t$ and $\mathcal{S}_t$ which turned out to coincide with the ones constructed in [**3**] for real equations of their perturbations via the fundamental solutions. These abstract definitions use the first order semi-group $S(t)$ mentioned above, instantly proving the L^2 boundedness of $\mathcal{D}_t$ and $\nabla \mathcal{S}_t$, which was a question raised by S. Hofmann [**62**]. Thus in the interval of p and its dual arising in the two theorems above, we obtain boundedness, jump relations, non-tangential maximal estimates and square functions estimates. In particular, we obtain strong limits as $t \to 0$, which is new for $p \neq 2$, the case $p = 2$ following from a combination of [**8**] and [**81**]. It goes without saying that these results are obtained without any kernel information nor fundamental solution: this is far beyond Calderón-Zygmund theory and subsumes the results in [**61**].

In the context of theorem 1.2, we also prove

$$\|\widetilde{N}_*(u - u_0)\|_{p'} \lesssim \|S(t\,\nabla u)\|_{p'}$$

in the same range and with modification for $p = 1$ and below. Our non-tangential sharp functions above can be seen as a part of non-tangential sharp functions adapted to the first order operators BD for which we have the equivalence

$$\|\widetilde{N}_\sharp(\phi(tBD)h)\|_{p'} \sim \|S(t\,\nabla u)\|_{p'}$$

for an appropriate h, where $\widetilde{N}_\sharp(\phi(tBD)h) = \widetilde{N}_*(\phi(tBD)h - h)$. Modified sharp functions, where averages are replaced by the action of more general operators, were introduced by Martell [**77**] and then used by [**47**] in developing their BMO theory associated with operators. Some versions were also used by [**59**] and [**60**] in the context of second order operators under divergence form on $\mathbb{R}^n$. All these versions used ϕ such that $\phi(tBD)$ have enough decay in some pointwise or averaged sense. Here we have to consider the Poisson type semigroup $e^{-t|BD|}$ to get back to solutions of L. The difficulty lies in the fact that these operators have small decay and we overcome this using the depth of Hardy space theory because these operators are bounded there while they may not be bounded on L^p.

Let us turn to boundary value problems for solutions of $Lu = 0$ or $L^*u = 0$ and formulate four such problems:

1) $(D)^{L^*}_Y = (R)^{L^*}_{Y^{-1}}$: $L^*u = 0$, $u|_{t=0} \in Y$, $t\,\nabla u \in \mathcal{T}$.
2) $(R)^L_X$: $Lu = 0$, $\nabla_x u|_{t=0} \in X$, $\widetilde{N}_*(\nabla u) \in \mathcal{N}$.
3) $(N)^{L^*}_{Y^{-1}}$: $L^*u = 0$, $\partial_{\nu_{A^*}} u|_{t=0} \in \dot{Y}^{-1}$, $t\,\nabla u \in \mathcal{T}$.
4) $(N)^L_X$: $Lu = 0$, $\partial_{\nu_A} u|_{t=0} \in X$, $\widetilde{N}_*(\nabla u) \in \mathcal{N}$.

Here X is a space X_p with $p \in I_L$, Y is the dual space of such an X (we are ignoring whether functions are scalar or vector-valued; context is imposing it) and $\dot{Y}^{-1} = \operatorname{div}_x(Y^n)$ with the quotient topology. Then $\mathcal{N} = L^p$ for $p \in I_L$ and $\mathcal{T}$ is a tent space $T_2^{p'}$ if $p \geq 1$ and a weighted Carleson measure space $T^\infty_{2,n(\frac{1}{p}-1)}$ if $p < 1$. In each case, we want to solve, possibly uniquely, with control from the data. For example, for $(D)^{L^*}_Y$ we want $\|t\,\nabla u\|_{\mathcal{T}} \lesssim \|u|_{t=0}\|_Y$, etc. The behavior at the boundary is continuity (strong or weak-$*$) at $t = 0$; non-tangential convergence can occur in some cases but is not part of the convergence at the boundary.

As in [**72, 23**], we say that a boundary value problem is solvable for the energy class if the energy solution corresponding to the data (assumed in the proper trace space as well) satisfies the required control by the data. For energy solutions, we have semi-group representation or, equivalently, boundary layer representation. By solvability, we mean existence of a solution for *any* boundary datum, with control. Precise definitions will be recalled in chapter 14 where we describe a method to construct solutions and show the following extrapolation theorem.

Theorem 1.3. — *Consider any of the four boundary value problems with a given space of boundary data in the list above. If it is solvable for the energy class then it is solvable in nearby spaces of boundary data.*

For example, if $X = X_p$, one can take X_q for q in a neighborhood of p. For Neumann and regularity problems, this seems to be new for $p \leq 1$ in this generality. See [**69**] for the case of the Laplacian on Lipschitz domains when $p = 1$. Also for $p = 1$ and duality, we get extrapolation for BMO solvability of the Dirichlet problem. We note that we only get solvability in the conclusion. In the case of real equations as in [**44**] where such an extrapolation of proved, harmonic measure techniques naturally lead to solvability for the energy class after perturbation.

We shall also prove a stability result for each boundary value problem with respect to perturbations in L^∞ with t-independent coefficients of the operator L. Such results when $p \leq 1$ are known assuming invertibility of the single layer potential with De Giorgi-Nash conditions in [**55**], which is not the case here.

Before we end this introduction, let us mention that most of the work to prove theorems 1.1 and 1.2 has not much to do with the elliptic system given by L and their solutions (except under De Giorgi-Nash conditions). In fact, this is mainly a consequence of inequalities for Hardy spaces associated to first order systems DB or BD on the boundary and the operators D can be much more general than the one arising from the boundary value problems. These type of operators were introduced in the topic by McIntosh and led to one proof of the L^2 boundeness of the Cauchy integral from the solution of the Kato square root problem in one dimension although the original article [**39**] does not present it this way (see also [**71**], [**21**]). An extended higher dimensional setting was introduced in [**29**], and further studied in [**64, 65, 63, 27**], where D is a differential first order operator with constant coefficients ha ving some coercivity and B is the operator of pointwise multiplication by an accretive matrix function. But the relation between elliptic systems (1) and boundary operators of the form DB was only established recently in [**11**], paving the way to the representations in [**8**] mentioned above. The Hardy space theory we need is the one associated to operators with Gaffney-Davies type estimates developed in [**22, 59**] and followers. We just mention that our operators are non injective, hence it makes the theory a little more delicate.

For the first part of the memoir, we shall review the needed material. Then we turn to the proof of estimates which will imply theorems 1.1 and 1.2 when specializing to solutions of $Lu = 0$. A large part of the end of the article is to study the case of operators with De Giorgi-Nash conditions. The application of our theory to perturbations for solvability of the boundary value problems is given in the last section.

We shall not attempt to treat intermediate situations for the boundary value problems, that is assuming some fractional order of regularity for the data. This has been recently done in [**32**] with data in Besov spaces for elliptic equations $Lu = 0$ assuming De Giorgi type conditions for L and L^*.

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2. SETUP

2.1. Boundary function spaces

We work on the upper-half space $\mathbb{R}^{1+n}_+$ and its boundary identified to $\mathbb{R}^n$. We consider a variety of function spaces defined on the boundary $\mathbb{R}^n$ and valued in $\mathbb{C}^N$ for some integer N. Function or distribution spaces $X(\mathbb{R}^n;\mathbb{C}^N)$ will often be written X if this is not confusing. For example, $L^q := L^q(\mathbb{R}^n;\mathbb{C}^N)$ is the standard Lebesgue space. For $0 < q \le 1$, H^q denotes the Hardy space in its real version. It will be sometimes convenient to set $H^q = L^q$ even when $q > 1$.

The dual of H^q for a duality extending the L^2 sesquilinear pairing when $q > 1$ is thus $H^{q'}$ and is the space $\dot\Lambda^{n(\frac{1}{q}-1)}$ when $q \le 1$. Here, $\dot\Lambda^0$ denotes BMO for convenience; for $0 < s < 1$, $\dot\Lambda^s$ is the Hölder space of those continuous functions with $|f(x)-f(y)| \le C|x-y|^s$ (equipped with a semi-norm); for $s \ge 1$, we say $f \in \dot\Lambda^s$ if the distributional partial derivatives of f belong to $\dot\Lambda^{s-1}$.

For $q > 1$, $W^{1,q}$ is the standard Sobolev space of order 1 on L^q and $\dot W^{1,q}$ denotes its homogeneous version: the space of Schwartz distributions with $\|\nabla f\|_q < \infty$ or, equivalently, the closure of $W^{1,q}$ for $\|\nabla f\|_q$. It becomes a Banach space when moding out the constants. For $\frac{n}{n+1} < q \le 1$, we also set $\dot H^{1,q}$, the space of Schwartz distributions with $\nabla f \in H^q$ (componentwise). Again, we sometimes use the notation $\dot H^{1,q} = \dot W^{1,q}$ also when $q > 1$ for convenience.

The dual of $\dot W^{1,q}$ is $\dot W^{-1,q'} := \operatorname{div}(L^{q'})^n$ with quotient topology. The dual of $\dot H^{1,q}$, $q \le 1$, is $\dot\Lambda^{s-1} := \operatorname{div}(\dot\Lambda^s)^n$ when $s = n(\frac{1}{q}-1) \in [0,1)$, equipped with the quotient topology.

We shall also use the homogeneous Sobolev spaces $\dot{\mathcal{H}}^s$ for $s \in \mathbb{R}$. We mention that for $s \ge 0$, they can be realized within L^2_{loc} and equipped with a semi-norm. For $s < 0$, the homogeneous Sobolev spaces embed in the Schwartz distributions.

2.2. Bisectorial operators

The space of continuous linear operators between normed vector spaces E, F is denoted by $\mathcal{L}(E,F)$ or $\mathcal{L}(E)$ if $E = F$. For an unbounded linear operator $\mathcal{A}$, its domain is denoted by $\mathsf{D}(\mathcal{A})$, its null space $\mathsf{N}(\mathcal{A})$ and its range $\mathsf{R}(\mathcal{A})$. The spectrum is denoted by $\sigma(\mathcal{A})$.

An unbounded linear operator $\mathcal{A}$ on a Banach space $\mathcal{X}$ is called *bisectorial* of angle $\omega \in [0, \pi/2)$ if it is closed, its spectrum is contained in the closure of

$$S_\omega := S_{\omega+} \cup S_{\omega-},$$

where $S_{\omega+} := \{z \in \mathbb{C}; |\arg z| < \omega\}$ and $S_{\omega-} := -S_{\omega+}$, and one has the resolvent estimate

$$\|(I+\lambda\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C_\mu, \quad \forall\lambda \notin S_\mu,\ \forall\mu > \omega. \tag{7}$$

Assuming $\mathcal{X}$ is reflexive, this implies that the domain is dense and also the fact that the null space and the closure of the range split. More precisely, we say that the operator $\mathcal{A}$ *kernel/range decomposes* if $\mathcal{X} = \mathsf{N}(\mathcal{A}) \oplus \overline{\mathsf{R}(\mathcal{A})}$ ($\oplus$ means that the sum is topological). Bisectoriality in a reflexive space is stable under taking adjoints.

For any bisectorial operator in a reflexive Banach space, one can define a calculus of bounded operators by the Cauchy integral formula,

$$\begin{aligned} &\psi(\mathcal{A}) := \frac{1}{2\pi i}\int_{\partial S_\nu} \psi(\lambda)(I - \frac{1}{\lambda}\mathcal{A})^{-1}\frac{d\lambda}{\lambda}, \\ &\qquad \psi \in \Psi(S_\mu) := \{\phi \in H^\infty(S_\mu) : \phi(z) = O\big(\inf(|z|, |z^{-1}|)^\alpha\big), \alpha > 0\}, \end{aligned} \tag{8}$$

with $\mu > \nu > \omega$ and where $H^\infty(S_\mu)$ is the space of bounded holomorphic functions in S_μ. If one can show the estimate $\|\psi(\mathcal{A})\| \lesssim C_\mu\|\psi\|_\infty$ for all $\psi \in \Psi(S_\mu)$ and all μ with $\omega < \mu < \frac{\pi}{2}$, then this allows to extend the calculus on $\overline{\mathsf{R}(\mathcal{A})}$ to all $\psi \in H^\infty(S_\mu)$ and all μ with $\omega < \mu < \frac{\pi}{2}$ in a consistent way for different values of μ. In that case, $\mathcal{A}$ is said to have an H^∞-*calculus* of angle ω on $\overline{\mathsf{R}(\mathcal{A})}$, and $b(\mathcal{A})$ is defined by a limiting procedure for any $b \in H^\infty(S_\mu)$. For those b which are also defined at 0, one extends the H^∞-calculus to $\mathcal{X}$ by setting $b(\mathcal{A}) = b(0)I$ on $\mathsf{N}(\mathcal{A})$. For later use, we shall say that a holomorphic function on S_μ is non-degenerate if it is non identically 0 on each connected component of S_μ.

2.3. The first order operator D

We assume that D is a first order differential operator on $\mathbb{R}^n$ acting on Schwartz distributions valued in $\mathbb{C}^N$, whose symbol satisfies the conditions (D0), (D1) and (D2) in [**63**]. Later, we shall assume that D is self-adjoint on L^2 but for what follows in this section, this is not necessary by observing that the three conditions can be shown to be stable under taking the adjoint symbol and operator. For completeness, we recall the three conditions here although what we will be using are the consequences below.

First D has the form

$$D = -i\sum_{j=1}^n \widehat{D}_j\partial_j, \quad \widehat{D}_j \in \mathcal{L}(\mathbb{C}^N). \tag{D0}$$

It can also be viewed as the Fourier multiplier operator with symbol

$$\widehat{D}(\xi) = \sum_{j=1}^n \widehat{D}_j\xi_j.$$

The symbol is required to satisfy the following properties:

(D1) $$\kappa|\xi|\cdot|e| \le |\widehat{D}(\xi)e|, \quad \forall\xi\in\mathbb{R}^n,\ \forall e\in \mathsf{R}(\widehat{D}(\xi)),$$

where $\mathsf{R}(\widehat{D}(\xi))$ stands for the range of $\widehat{D}(\xi)$, and

(D2) $$\sigma(\widehat{D}(\xi)) \subseteq \overline{S_\omega},$$

where $\kappa > 0$ and $\omega \in [0, \pi/2)$ are some constants.

For $1 < q < \infty$, this induces the unbounded operator D_q on each L^q with domain

$$\mathsf{D}_q(D) := \mathsf{D}_{L^q}(D) = \{u \in L^q; Du \in L^q\}$$

and $D_q = D$ on $\mathsf{D}_q(D)$. We keep using the notation D instead of D_q for simplicity. The following properties have been shown in [**65**], except for the last one shown in [**27**].

1) D is a bisectorial operator with H^∞-calculus in L^q.
2) $L^q = \mathsf{N}_q(D) \oplus \overline{\mathsf{R}_q(D)}$, the closure being in the L^q topology.
3) $\mathsf{N}_q(D)$ and $\overline{\mathsf{R}_q(D)}$, $1 < q < \infty$, are complex interpolation families.
4) D has the coercivity condition
$$\|\nabla u\|_q \lesssim \|Du\|_q \quad \text{for all } u \in \mathsf{D}_q(D) \cap \overline{\mathsf{R}_q(D)} \subset W^{1,q}.$$
Here, we use the notation ∇u for $\nabla \otimes u$.
5) $\mathsf{D}_q(D)$, $1 < q < \infty$, is a complex interpolation family.

The results in [**65**] are obtained by applying the Mikhlin multiplier theorem to the resolvent and also to the projection from L^q on $\overline{\mathsf{R}_q(D)}$ along $\mathsf{N}_q(D)$ by checking the symbol is C^∞ away from 0 and has the appropriate estimates for all its partial derivatives. This projection, which we denote by $\mathbb{P}$, will play an important role (it does not depend on q) and we have

$$\mathbb{P}(L^q) = \overline{\mathsf{R}_q(D)}.$$

This theorem can be shown to apply to the operators $b(D)$ of the bounded holomorphic functional calculus. Moreover, 4) is a consequence of the L^q boundedness of $\nabla D^{-1}\mathbb{P}$, which again follows from Mikhlin multiplier theorem. Even if this is not done this way in [**27**], one can show the property 5) using the Mikhlin multiplier theorem as in [**65**].

By standard singular integral theory, all operators to which the (C^∞ case of the) Mikhlin multiplier theorem applies extend boundedly to the Hardy spaces H^q, $0 < q \le 1$. In particular, $\mathbb{P}$ is a bounded projection on H^q so $\mathbb{P}(H^q)$ is a closed complemented subspace of H^q.

Set $X_p = L^p$ when $1 < p < \infty$, $X_p = H^p$ when $p \le 1$ and also $X_\infty = \mathrm{BMO}$ the space of bounded mean oscillations functions.

We mention the following consequence: For $0 < q < \infty$, each $\mathbb{P}(X_q)$ contains $\mathbb{P}(\mathcal{D}_0)$ as a dense subspace where $\mathcal{D}_0$ is the space of C^∞ functions with compact support and all vanishing moments. Note that a Fourier transform argument shows $\mathbb{P}(\mathcal{D}_0) \subset \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space. Similarly, the same statement holds if $\mathcal{D}_0$ is replaced by the subspace $\mathcal{S}_0$ of $\mathcal{S}$ of those functions with compactly supported Fourier transform away from the origin.

As said, all this applies to the adjoint of D (we shall assume D self-adjoint subsequently). Hence, the resolvent of D is bounded on X_p^*, the dual space to X_p with the estimate (7), and $\mathbb{P}$ is a bounded projection on X_p^*. In particular, $\mathbb{P}(X_p^*)$ is complemented in X_p^*. Also, using that the X_p, $0 < p \leq \infty$, spaces form a complex interpolation scale, the same holds for the spaces $\mathbb{P}(X_p)$, $0 < p \leq \infty$.

2.4. The operators DB and BD

We let D as defined above and we assume from now on that D is self-adjoint on L^2. We consider an operator B of multiplication by a matrix $B(x) \in \mathcal{L}(\mathbb{C}^N)$. We assume that as a function, $B \in L^\infty$ and note $\|B\|_\infty$ its norm. Thus as a multiplication operator, B is bounded on all L^q spaces with norm equal to $\|B\|_\infty$ when $1 < q < \infty$. We also assume that B is strictly accretive in $\overline{\mathsf{R}_2(D)}$, that is for some $\kappa > 0$,

$$\mathrm{Re}\langle u, Bu\rangle \geq \kappa \|u\|_2^2, \quad \forall u \in \overline{\mathsf{R}_2(D)}. \tag{9}$$

In this case, let

$$\omega := \sup_{u \in \overline{\mathsf{R}_2(D)}, u \neq 0} |\arg(\langle u, Bu\rangle)| < \frac{\pi}{2} \tag{10}$$

denote the *angle of accretivity* of B on $\overline{\mathsf{R}_2(D)}$. Note that B may not be invertible on L^2. Still for X a subspace of L^2, we set $B^{-1}X = \{u \in L^2 ; Bu \in X\}$. Note that B^* is also strictly accretive on $\overline{\mathsf{R}_2(D)}$ with the same lower bound and angle of accretivity.

Proposition 2.1. — *With the above assumptions, we have the following facts.*

(i) *The operator DB, with domain $B^{-1}\mathsf{D}_2(D)$, is bisectorial with angle ω, i.e. $\sigma(DB) \subseteq \overline{S_\omega}$ and there are resolvent bounds $\|(\lambda I - DB)^{-1}\| \lesssim 1/\operatorname{dist}(\lambda, S_\mu)$ when $\lambda \notin S_\mu$, $\omega < \mu < \pi/2$.*

(ii) *The operator DB has range $\mathsf{R}_2(DB) = \mathsf{R}_2(D)$ and null space $\mathsf{N}_2(DB) = B^{-1}\mathsf{N}_2(D)$ such that topologically (but not necessarily orthogonally) one has*
$$L^2 = \overline{\mathsf{R}_2(DB)} \oplus \mathsf{N}_2(DB).$$

(iii) *The restriction of DB to $\overline{\mathsf{R}_2(DB)}$ is a closed, injective operator with dense range in $\overline{\mathsf{R}_2(D)}$. Moreover, the same statements on spectrum and resolvents as in* (i) *hold.*

(iv) *Statements similar to* (i), (ii) *and* (iii) *hold for BD with $\mathsf{D}_2(BD) = \mathsf{D}_2(D)$, defined as the adjoint of DB^* or equivalently by*
$$BD = B(DB)B^{-1}$$

on $\overline{\mathsf{R}_2(BD)} \cap \mathsf{D}_2(D)$ *with* $\mathsf{R}_2(BD) := B\mathsf{R}_2(D)$*, and* $BD = 0$ *on the null space* $\mathsf{N}_2(BD) := \mathsf{N}_2(D)$*.*

For a proof, see [**2**]. Note that the accretivity of B is only needed on $\mathsf{R}_2(D)$. The fact that D is self-adjoint is used in this statement. In fact, for a self-adjoint operator D on a separable Hilbert space instead of L^2 and a bounded operator B which is accretive on $\overline{\mathsf{R}_2(D)}$, the statement above is valid.

We come back to the concrete D and B above. We isolate this result as it will play a special role throughout.

Proposition 2.2. — *Consider the orthogonal projection* $\mathbb{P}$ *from* L^2 *onto* $\overline{\mathsf{R}_2(D)}$*. Then* $\mathbb{P}$ *is an isomorphism between* $\overline{\mathsf{R}_2(BD)}$ *and* $\overline{\mathsf{R}_2(D)}$*.*

Proof. — Using $\mathsf{N}_2(BD) = \mathsf{N}_2(D)$, we have the splittings

$$L^2 = \overline{\mathsf{R}_2(BD)} \oplus \mathsf{N}_2(D) = \overline{\mathsf{R}_2(D)} \oplus \mathsf{N}_2(D).$$

It is then a classical fact from operator theory that $\mathbb{P} : \overline{\mathsf{R}_2(BD)} \to \overline{\mathsf{R}_2(D)}$ is invertible with inverse being $\mathbb{P}_{BD} : \overline{\mathsf{R}_2(D)} \to \overline{\mathsf{R}_2(BD)}$, where $\mathbb{P}_{BD}$ is the projection onto $\overline{\mathsf{R}_2(BD)}$ along $\mathsf{N}(D)$ associated to the first splitting. Indeed, if $h \in \overline{\mathsf{R}_2(D)}$, then $h - \mathbb{P}_{BD}h \in \mathsf{N}_2(D)$, thus $\mathbb{P}(h - \mathbb{P}_{BD}h) = 0$. It follows that

$$h = \mathbb{P}h = (\mathbb{P} \circ \mathbb{P}_{BD})h.$$

Similarly, we obtain $h = (\mathbb{P}_{BD} \circ \mathbb{P})h$ for $h \in \overline{\mathsf{R}_2(BD)}$. □

We also state the following decay estimates. See [**10**].

Lemma 2.3 (L^2 off-diagonal decay). — *Let* $T = BD$ *or* DB*. For every integer* N *there exists* $C_N > 0$ *such that*

$$\|1_E\,(I + itT)^{-1}u\|_2 \le C_N \langle \operatorname{dist}(E, F)/|t|\rangle^{-N} \|u\|_2 \tag{11}$$

for all $t \neq 0$*, whenever* $E, F \subseteq \mathbb{R}^n$ *are closed sets,* $u \in L^2$ *is such that* $\operatorname{supp} u \subseteq F$*. We have set*

$$\langle x \rangle := 1 + |x| \quad \textit{and} \quad \operatorname{dist}(E, F) := \inf\{|x - y|\,;\, x \in E, y \in F\}.$$

Remark 2.4. — Any operator satisfying such estimates with $N > \frac{n}{2}$ has an extension from L^∞ into L^2_{loc}.

3. HOLOMORPHIC FUNCTIONAL CALCULUS

3.1. L^2 results

We begin with recalling the following result due to [**29**]. A direct proof is in [**10**].

Proposition 3.1. — *If $T = DB$ or $T = BD$, then one has the equivalence*

$$\int_0^\infty \|tT(I+t^2T^2)^{-1}u\|_2^2 \,\frac{dt}{t} \sim \|u\|_2^2, \quad \text{for all } \ u \in \overline{\mathsf{R}_2(T)}. \tag{12}$$

Note that if $u \in \mathsf{N}_2(T)$ then $tT(I+t^2T^2)^{-1}u = 0$. Thus by the kernel/range decomposition, we have the inequality $\lesssim$ for all $u \in L^2$.

The next result summarizes the needed consequences of this quadratic estimate. This statement, contrarily to the previous one, is abstract and applies to $T = BD$ or DB on L^2.

Proposition 3.2. — *Let T be an ω-bisectorial operator on a separable Hilbert space $\mathcal{H}$ with $0 \le \omega < \pi/2$. Assume that the quadratic estimate*

$$\int_0^\infty \|tT(I+t^2T^2)^{-1}u\|^2 \,\frac{dt}{t} \sim \|u\|^2 \ \text{ holds for all } \ u \in \overline{\mathsf{R}(T)}. \tag{13}$$

Then, the following statements hold.

- *T has an H^∞-calculus on $\overline{\mathsf{R}(T)}$, which can be extended to $\mathcal{H}$ by setting $b(T) = b(0)I$ on $\mathsf{N}(T)$ whenever b is also defined at 0.*
- *For any $\omega < \mu < \pi/2$ and any non-degenerate $\psi \in \Psi(S_\mu)$, the comparison*

$$\int_0^\infty \|\psi(tT)u\|^2 \,\frac{dt}{t} \sim \|u\|^2 \text{ holds for all } \ u \in \overline{\mathsf{R}(T)}. \tag{14}$$

- *$\overline{\mathsf{R}(T)}$ splits topologically into two spectral subspaces*

$$\overline{\mathsf{R}(T)} = \mathcal{H}_T^+ \oplus \mathcal{H}_T^- \tag{15}$$

  with $\mathcal{H}_T^\pm = \chi^\pm(T)(\overline{\mathsf{R}(T)})$ and $\chi^\pm(T)$ are projections with $\chi^\pm(z) = 1$ if $\pm \mathrm{Re}\, z > 0$ and $\chi^\pm(z) = 0$ if $\pm \mathrm{Re}\, z < 0$.
- *The operator $\mathrm{sgn}(T) = \chi^+(T) - \chi^-(T)$ is a bounded involution on $\overline{\mathsf{R}(T)}$.*
- *The operator $|T| = \mathrm{sgn}(T)T = \sqrt{T^2}$ with $\mathsf{D}(|T|) = \mathsf{D}(T)$ is an ω-sectorial operator with H^∞-calculus on $\mathcal{H}$ and $-|T|$ is the infinitesimal generator of a bounded analytic semigroup of operators $(e^{-z|T|})_{z \in S_{\frac{\pi}{2}-\omega+}}$ on $\mathcal{H}$.*

- *For $h \in \mathsf{D}(T)$, $h \in \mathcal{H}_T^{\pm}$ if and only if $|T|h = \pm Th$. As a consequence $e^{\mp zT}$ are well-defined operators on $\mathcal{H}_T^{\pm}$ respectively, and $e^{-zT}\chi^+(T)$ and $e^{+zT}\chi^-(T)$ are well-defined operators on $\mathcal{H}$ for $z \in S_{\frac{\pi}{2}-\omega+}$.*

Finally, all these properties hold for the adjoint T^* of T.

This result is for later use.

Proposition 3.3. — *If $b \in H^\infty(S_\mu)$ and b is defined at 0, then, for all $h \in L^2$,*

$$\mathbb{P}b(BD)\mathbb{P}h = \mathbb{P}b(BD)h.$$

If $\psi \in \Psi(S_\mu)$, then for all $h \in L^2$, $\psi(BD)\mathbb{P}h = \psi(BD)h$.

Proof. — Remark that $h - \mathbb{P}h \in \mathsf{N}_2(D) = \mathsf{N}_2(BD)$. Thus,

$$b(BD)(h - \mathbb{P}h) = b(0)(h - \mathbb{P}h).$$

Hence $\mathbb{P}b(BD)(h - \mathbb{P}h) = 0$. If $b = \psi$ then $\psi(BD)$ annihilates the null space of BD, hence $\psi(BD)(h - \mathbb{P}h) = 0$ (This is consistent with the fact that one can set $\psi(0) = 0$ by continuity).

□

3.2. L^p results

There has been a series of works [**64, 65, 1, 63, 27**] concerning extension to L^p of the L^2 theory. We summarize here the results described in [**27**].

Let D and B be as before and $1 < q < \infty$. Then we have a meaning of D and B as operators on L^q, thus of BD and DB as unbounded operators with natural domains $\mathsf{D}_q(D)$ and $B^{-1}\mathsf{D}_q(D)$ respectively. Introduce the set of coercivity of B (it also depends on D) as

$$\mathcal{I}(BD) = \{q \in (1,\infty)\,;\ \|Bu\|_q \gtrsim \|u\|_q \text{ for all } u \in \mathsf{R}_q(D)\}.$$

By density, we may replace $\mathsf{R}_q(D)$ by its closure. The following observation will be frequently used.

Lemma 3.4. — *If $q \in \mathcal{I}(BD)$ then*

$$B|_{\overline{\mathsf{R}_q(D)}} : \overline{\mathsf{R}_q(D)} \longrightarrow \overline{\mathsf{R}_q(BD)}$$

is an isomorphism and $\mathsf{R}_q(BD) = B\mathsf{R}_q(D)$. Moreover, $\mathsf{N}_q(BD) = \mathsf{N}_q(D)$.

Proof. — See proposition 2.1, (2) and (3), in [**27**]. □

Remark 3.5. — It is shown in [**63, 27**] that the set of coercivity of B is open. As it contains $q = 2$, let $\mathcal{I}_2$ be the connected component of $\mathcal{I}(BD) \cap \mathcal{I}(B^*D)$ that contains 2. Remark that if $B(x)$ is invertible in L^∞ then B is invertible in $\mathcal{L}(L^q)$ for all $1 < q < \infty$ and $\mathcal{I}_2 = (1,\infty)$. Otherwise, we do not even know if the set of coercivity of B is connected.

For an interval $I \subset (1,\infty)$, its dual interval is $I' = \{p'; p \in I\}$ where p' is the conjugate exponent to p. The following result is taken from [**27**] with a cosmetic modification in the statement.

Theorem 3.6. — *There exists an open interval*

$$I(BD) = (p_-(BD), p_+(BD)),$$

maximal in $\mathcal{I}_2$, containing 2, with the following dichotomy: bisectoriality of BD with angle ω, H^∞-calculus with angle ω in L^p, and kernel/range decomposition hold for BD in L^p if $p \in I(BD)$ and all fail if $p = p_\pm(BD)$ and $p \in \mathcal{I}_2$. The same property holds for DB with $I(DB) = I(BD)$. The same property holds for $B^*D = (DB)^*$ and $DB^* = (BD)^*$ in the dual interval $I(DB^*) = I(B^*D) = (I(BD))'$. Thus we have the relations,

$$p_\pm(BD) = p_\pm(DB), \qquad p_\pm(BD) = p_\mp(B^*D)'. \tag{16}$$

If $p_\pm(BD)$ is an endpoint of $\mathcal{I}_2$, then we do not know what happens for $p = p_\pm(BD)$ from this theory.

We remark that the calculi in L^p are consistent for all $p \in I(BD)$. For example, if $T_p = BD$ with domain $\mathsf{D}_p(D)$ then $(I+iT_p)^{-1}u = (I+iT_q)^{-1}u$ whenever $u \in L^p \cap L^q$ and $p, q \in I(BD)$. Thus, we do not distinguish them from now on.

Corollary 3.7. — *If $q \in I(BD) = I(DB)$, then $\mathsf{R}_q(DB) = \mathsf{R}_q(D)$.*

The inclusion $\mathsf{R}_q(DB) \subset \mathsf{R}_q(D)$ is always true. The converse is not clear when $q \notin I(BD)$, so we shall use this equality only for q in this range.

Proof. — The above theorem and corollary 2.3 in [**27**] give us the assumptions of proposition 2.1, (4) in [**27**], of which $\mathsf{R}_q(DB) = \mathsf{R}_q(D)$ is a consequence. □

Proposition 3.8. — *Consider the orthogonal projection $\mathbb{P}$ from L^2 onto $\overline{\mathsf{R}_2(D)}$. For $p \in I(BD)$, $\mathbb{P}$ extends to an isomorphism between $\overline{\mathsf{R}_p(BD)}$ and $\overline{\mathsf{R}_p(D)}$ with $\|h\|_p \sim \|\mathbb{P}h\|_p$ for all $h \in \overline{\mathsf{R}_p(BD)}$.*

Proof. — Using $\mathsf{N}_p(BD) = \mathsf{N}_p(D)$ from lemma 3.4, and the kernel/range decomposition for D and for BD if $p \in I(BD)$,

$$L^p = \overline{\mathsf{R}_p(BD)} \oplus \mathsf{N}_p(D) = \overline{\mathsf{R}_p(D)} \oplus \mathsf{N}_p(D).$$

The projection onto $\overline{\mathsf{R}_p(D)}$ along $\mathsf{N}_p(D)$ is the extension of $\mathbb{P}$ to L^p. The projection from L^p onto $\overline{\mathsf{R}_p(BD)}$ along $\mathsf{N}_p(D)$ is the extension $\mathbb{P}_{BD}$ defined on L^2 in the proof of proposition 2.2. Using the same notation for the extensions, it follows that $\mathbb{P} : \overline{\mathsf{R}_p(BD)} \to \overline{\mathsf{R}_p(D)}$ and $\mathbb{P}_{BD} : \overline{\mathsf{R}_p(D)} \to \overline{\mathsf{R}_p(BD)}$ are inverses of each other. □

Corollary 3.9. — *For all $p \in I(BD)$, the conclusions of proposition 3.2 hold for $T = BD$ and DB on L^p in place of $\mathcal{H}$ with the exception of (14) which reads*

$$\Big\| \Big(\int_0^\infty |\psi(tT)u|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \sim \|u\|_p \text{ holds for all } u \in \overline{\mathsf{R}_p(T)} \tag{17}$$

for any $\omega < \mu < \pi/2$ and any non-degenerate $\psi \in \Psi(S_\mu)$. Furthermore, one has $\lesssim$ in general for all $u \in L^p$.

The last part of the corollary follows from extension of an abstract theorem of Le Merdy [**75**, corollary 2.3], saying that for an injective sectorial operator T on L^p, the H^∞-calculus on L^p is equivalent to the square function estimate (17). This uses the notion of R-sectoriality which we have not defined here but follows from the H^∞-calculus. The extension to injective bisectorial operators is straightforward with the notion of R-bisectoriality. If T is not injective but one has the kernel/range decomposition, then its restriction to $\overline{\mathsf{R}_p(T)}$ is injective, and the proof of Le Merdy's theorem extends easily also in this case. In our situation, for $p \in I(BD)$, $T = BD$ or DB may not be injective on L^p but its restriction to $\overline{\mathsf{R}_p(T)}$ is injective as one has the kernel/range decomposition. One can apply Le Merdy's extended theorem to T on $\overline{\mathsf{R}_p(T)}$ and obtain H^∞-calculus on $\overline{\mathsf{R}_p(T)}$ (which, for this particular T, is equivalent to the R-bisectoriality on L^p, see [**63, 27**]), and then extend it to all of L^p as described before.

Note also that by interpolation between lemma 2.3 and the boundedness on L^p of the resolvent for $p \in I(BD)$, one has

Lemma 3.10 (L^p off-diagonal decay). — *Let $T = BD$ or DB and $p \in I(BD)$. For every integer N there exists $C_N > 0$ such that*

$$\|1_E\,(I + itT)^{-1} 1_F u\|_p \le C_N \langle \mathrm{dist}(E,F)/|t| \rangle^{-N} \|u\|_p \tag{18}$$

for all $t \neq 0$, whenever $E, F \subseteq \mathbb{R}^n$ are closed sets, $u \in L^p$ is such that $\operatorname{supp} u \subseteq F$.

Actually, it is observed in [**63**] that the proof for $p = 2$ (lemma 2.3) goes through, which gives another argument.

3.3. The one dimensional case

Proposition 3.11. — *Assume D and B are as above and $n = 1$. Assume that $\widehat{D}(\xi)$ is invertible for all $\xi \neq 0$. Then $p_-(DB) = 1$ and $p_+(DB) = \infty$. In particular, DB and BD have bounded holomorphic functional calculi on L^p spaces for $1 < p < \infty$.*

Proof. — We fix $1 < p < \infty$. By theorem 3.6, it suffices to show that the kernel/range decomposition holds on L^p for BD.

First, as $\widehat{D}(\xi)$ is invertible for all $\xi \neq 0$, (D0) implies that for all $u \in L^p(\mathbb{R}^n; \mathbb{C}^N)$,

$$Du = -i\widehat{D}_1 u'$$

with $\widehat{D}_1$ being an invertible matrix on $\mathbb{C}^N$. Thus, we have that $N_p(D) = \{0\}$ and $\overline{\mathsf{R}_p(D)} = L^p$, the closure being taken in L^p. As a consequence, if B is accretive on $\overline{\mathsf{R}_2(D)} = L^2$, it is invertible in L^∞ by Lebesgue differentiation theorem, and one has $\mathcal{I}_2 = (1, \infty)$. By lemma 3.4, we have, since $p \in \mathcal{I}_2$, $\mathsf{N}_p(BD) = \mathsf{N}_p(D) = \{0\}$ and

$$\overline{\mathsf{R}_p(BD)} = \overline{B\mathsf{R}_p(D)} = B\overline{\mathsf{R}_p(D)} = L^p.$$

Thus the kernel/range decomposition holds trivially. □

Remark 3.12. — If one does not assume $\widehat{D}(\xi)$ invertible for all $\xi \neq 0$, it is not clear whether one has the kernel/range decomposition, even assuming B invertible on L^∞. Assume B invertible on L^∞. By the results in [**63**] (see [**27**], lemma 5.2, for the explicit statement), BD is (R-)bisectorial on $\overline{\mathsf{R}_p(BD)}$ when $p \in (1, \infty) \cap (\frac{2}{3}, \infty) = (1, \infty)$. It is trivially bisectorial on $\mathsf{N}_p(BD)$. The only thing missing might be the kernel/range decomposition.

3.4. Constant coefficients

We come back to arbitrary dimensions. A simple example is when B is a constant and strictly accretive matrix on $\overline{\mathsf{R}_2(D)}$ with D being still self-adjoint. Then it follows from [**63**, proposition A.8] that the interval of coercivity is all $(1, \infty)$.

Now BD is another first order differential operator which satisfies (D0), (D1) and (D2) of section 2.3 with ω being the angle of accretivity of B. Thus the conclusion is that BD is a bisectorial operator with H^∞-calculus in L^q for all $q \in (1, \infty)$.

Therefore the theory above tells that $p_-(BD) = 1$ and $p_+(BD) = \infty$.

3.5. L^p-L^q estimates

We summarize here estimates that we will use later. Proofs can be found in Part II. They concern only the exponents in the interval

$$I(BD) = I(DB) = (p_-, p_+).$$

First, we introduce subclasses of $H^\infty(S_\mu)$. For $\sigma, \tau \geq 0$, let

$$\Psi_\sigma^\tau(S_\mu) = \{\psi \in H^\infty(S_\mu) : \psi(z) = O\big(\inf(|z|^\sigma, |z|^{-\tau})\big)\},$$

with convention that $|z|^0 = 1$. For $\sigma, \tau > 0$, $\Psi_\sigma^\tau(S_\mu) \subset \Psi(S_\mu)$. For $\sigma = 0$, we have no vanishing at 0, for $\tau = 0$, no decay at ∞, and $\Psi_0^0(S_\mu) = H^\infty(S_\mu)$.

Proposition 3.13. — *Let $T = BD$ or DB. Let $p, q \in I(T)$ with $p \leq q$. Let $\psi \in \Psi_\sigma^\tau(S_\mu)$ with $\sigma > 0, \tau > \frac{n}{p} - \frac{n}{q}$ and $g \in H^\infty(S_\mu)$. Then for all $t > 0$, closed sets $E, F \subset \mathbb{R}^n$ and $u \in L^p$ with support in F:*

$$\|1_E g(T)\psi(tT)1_F u\|_q \lesssim \|g\|_\infty t^{\frac{n}{q}-\frac{n}{p}} \langle \mathrm{dist}(E,F)/t\rangle^{-\sigma c}\|u\|_p. \tag{19}$$

If, furthermore, $g(z) = \varphi(rz)$ with $|\varphi(z)| \le \inf(|z|^M, 1)$ for some $M > 0$, then for all $t \ge r > 0$, closed sets $E, F \subset \mathbb{R}^n$ and $u \in L^p$ with support in F

$$\|1_E\varphi(rT)\psi(tT)1_F u\|_q \lesssim t^{\frac{n}{q}-\frac{n}{p}} \langle \mathrm{dist}(E,F)/r\rangle^{-Mc}\|u\|_p. \tag{20}$$

Here, c is any positive number smaller than $1-(\frac{1}{p}-\frac{1}{q})(\frac{1}{p_-}-\frac{1}{p_+})^{-1}$ and can be taken equal to 1 when $p = q$. The implicit constants are independent of t, E, F, r and u.

Besides the precise values, it is important to notice that the exponent expressing the decay grows linearly with the order of decay of ψ at 0 in the first estimate and with the order of decay of φ at 0 in the second one. Notice that the first estimate contains in particular global L^p-L^q estimates

$$\|\psi(tT)u\|_q \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|u\|_p \tag{21}$$

for all ψ as above. Such an estimate is not true for the resolvent if $p < q$ unless T has a trivial null space. See Part II for more.

Here is an extension of remark 2.4.

Corollary 3.14. — *If $\tau > 0$, $\sigma > \frac{n}{p}$, $2 < p < p_+$ and $\psi \in \Psi_\sigma^\tau(S_\mu)$, then $\psi(tT)$ has a bounded extension from L^∞ to L^p_{loc}.*

Proof. — We take $h \in L^\infty$ and B a ball of radius t. Write $h = \sum h_j$ where $h_0 = h1_{2B}$ and $h_j = h1_{2^{j+1}B\setminus 2^jB}$. Then

$$\|\psi(tT)h_j\|_{L^p(B)} \lesssim 2^{-j\sigma}\|h_j\|_p \lesssim 2^{-j(\sigma-\frac{n}{p})}\|h\|_\infty.$$

It remains to sum. □

4. HARDY SPACES

The theory of Hardy spaces associated to operators allows us to introduce a scale of abstract spaces. One goal will be to identify ranges of p for which they agree with subspaces of L^p or H^p.

4.1. Tent spaces: notation and some review

For $0 < q < \infty$, T_2^q is the tent space of [**40**]. This is the space of $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ functions F such that

$$\|F\|_{T_2^q} = \|SF\|_q < \infty$$

with for all $x \in \mathbb{R}^n$,

$$(SF)(x) := \Big(\iint_{t>0, |x-y|<at} |F(t,y)|^2 \, \frac{dtdy}{t^{n+1}} \Big)^{1/2}, \tag{22}$$

where $a > 0$ is a fixed number. Two different values a give equivalent T_2^q norms.

For $q = \infty$, $T_2^\infty(\mathbb{R}^{1+n}_+)$ is defined via Carleson measures by $\|F\|_{T_2^\infty} < \infty$, where $\|F\|_{T_2^\infty}$ is the smallest positive constant C in

$$\iint_{T_{x,r}} |F(t,y)|^2 \, \frac{dtdy}{t} \le C^2 |B(x,r)|$$

for all open balls $B(x,r)$ in $\mathbb{R}^n$ and $T_{x,r} = (0,r) \times B(x,r)$. For $0 < \alpha < \infty$, $T^\infty_{2,\alpha}(\mathbb{R}^{1+n}_+)$ is defined by $\|F\|_{T^\infty_{2,\alpha}} < \infty$ where $\|F\|_{T^\infty_{2,\alpha}}$ is the smallest positive constant C in

$$\iint_{T_{x,r}} |F(t,y)|^2 \, \frac{dtdy}{t} \le C^2 |B(x,r)|^{1+\frac{2\alpha}{n}}$$

for all open balls $B(x,r)$ in $\mathbb{R}^n$. For convenience, we set $T^\infty_{2,0} = T^\infty_2$.

For $1 \le q < \infty$ and p the conjugate exponent to q, T_2^p is the dual of T_2^q for the duality

$$(F,G) := \iint_{\mathbb{R}^{1+n}_+} F(t,y)\overline{G(t,y)} \, \frac{dtdy}{t}.$$

For $0 < q \le 1$ and $\alpha = n(\frac{1}{q} - 1)$, $T^\infty_{2,\alpha}$ is the dual of T_2^q for the same duality form. Although not done explicitly there, it suffices to adapt the proof of [**40**, theorem 1].

4.2. General theory

We summarize here the theory pionnered in [**22, 59**] for operators T satisfying L^2 off-diagonal estimates of any polynomial order (11) and further developed in [**67, 47, 60, 66, 57, 46, 4**], etc. Here, there is an issue about homogeneity of the operator and notice that both DB and BD are of order 1. We stick to this homogeneity. The needed assumptions on T for what follows is bisectoriality on L^2 with H^∞-calculus on $\overline{\mathsf{R}_2(T)}$ and L^2 off-diagonal estimates (11). Let $\omega \in [0, \pi/2)$ be the angle of the H^∞-calculus. In what follows, μ is an arbitrary real number with $\omega < \mu < \pi/2$.

For $\psi \in H^\infty(S_\mu)$, let

$$\mathbb{Q}_{\psi,T} f = (\psi(tT)f)_{t>0}, \quad f \in L^2$$

and

$$\mathbb{S}_{\psi,T} F = \int_0^\infty \psi(tT)F(t,.)\,\frac{dt}{t}, \quad F \in T^2_2.$$

The second definition is provided one can make sense of the integral. Precisely, for $\psi \in \Psi(S_\mu)$ (the class is defined in (8)), the operators

$$\mathbb{Q}_{\psi,T} : L^2 \longrightarrow T^2_2 \quad \text{and} \quad \mathbb{S}_{\psi,T} : T^2_2 \longrightarrow L^2$$

are bounded as follows from the square function estimates (14) for T and its adjoint T^*. Indeed, $\mathbb{S}_{\psi,T}$ is the adjoint to $\mathbb{Q}_{\psi^*,T^*}$ where $\psi^*(z) = \overline{\psi(\bar z)}$.

Recall that for $\sigma, \tau \geq 0$,

$$\Psi^\tau_\sigma(S_\mu) = \{\psi \in H^\infty(S_\mu) : \psi(z) = O\big(\inf(|z|^\sigma, |z|^{-\tau})\big)\}.$$

So

$$\Psi(S_\mu) = \bigcup_{\sigma>0,\tau>0} \Psi^\tau_\sigma(S_\mu).$$

For $0 < \gamma$, let

$$\Psi^\gamma(S_\mu) = \bigcup_{\sigma>0,\tau>\gamma} \Psi^\tau_\sigma(S_\mu), \quad \Psi_\gamma(S_\mu) = \bigcup_{\sigma>\gamma,\tau>0} \Psi^\tau_\sigma(S_\mu).$$

Set $\gamma(p) = |\frac{n}{p} - \frac{n}{2}|$ for $0 < p \leq \infty$. If $p \leq 1$ and $\alpha = n(\frac{1}{p} - 1)$, then $\gamma(p) = \frac{n}{2} + \alpha$.

Consider the table

exponents =	$\mathcal{T} =$	$\Psi_{\mathcal{T}}(S_\mu) =$	$\Psi^{\mathcal{T}}(S_\mu) =$
$0 < p \leq 2$	T^p_2	$\Psi^{\gamma(p)}(S_\mu)$	$\Psi_{\gamma(p)}(S_\mu)$
$2 \leq p \leq \infty$	T^p_2	$\Psi_{\gamma(p)}(S_\mu)$	$\Psi^{\gamma(p)}(S_\mu)$
$0 \leq \alpha = n\big(\frac{1}{p} - 1\big) < \infty$	$T^\infty_{2,\alpha}$	$\Psi_{\gamma(p)}(S_\mu)$	$\Psi^{\gamma(p)}(S_\mu)$

Note that $\Psi^{\gamma(2)}(S_\mu) = \Psi_{\gamma(2)}(S_\mu) = \Psi(S_\mu)$ so the next result is consistent with the L^2 theory.

Proposition 4.1. — *For any space $\mathcal{T}$ in the table, $\psi \in \Psi_{\mathcal{T}}(S_\mu), \varphi \in \Psi^{\mathcal{T}}(S_\mu)$ and $b \in H^\infty(S_\mu)$, then $\mathbb{Q}_{\psi,T} b(T) \mathbb{S}_{\varphi,T}$ initially defined on T^2_2, extends to a bounded operator on $\mathcal{T}$ by density if $\mathcal{T} = T^p_2$ and by duality if $\mathcal{T} = T^\infty_{2,\alpha}$.*

Proof. — One can extract these classes in the range $1 < p < \infty$ from [**66**] and in the other ranges from [**60**] (replacing $\frac{n}{4}$ adapted to second order operators to $\frac{n}{2}$ here). Actually, there is a possible interpolation method to reobtain directly the results in [**66**] without recoursing to UMD technology, once one knows the results for $0 < p \le 1$. See [**86**]. $\square$

We also recall the Calderón reproducing formula in this context (See [**22**, remark 2.1]). As the proof is not given there, we sketch one possible argument.

Proposition 4.2. — *For any $\sigma_1, \tau_1 \ge 0$ and non-degenerate $\psi \in \Psi^{\tau_1}_{\sigma_1}(S_\mu)$ and any $\sigma, \tau > 0$, there exists $\varphi \in \Psi^\tau_\sigma(S_\mu)$ such that*

$$\int_0^\infty \varphi(tz)\psi(tz)\,\frac{dt}{t} = 1, \quad \forall z \in S_\mu. \tag{23}$$

As a consequence,

$$\mathbb{S}_{\varphi,T}\mathbb{Q}_{\psi,T} f = f, \quad \forall f \in \overline{R_2(T)}. \tag{24}$$

Proof. — Assume $\psi \in \Psi^{\tau_1}_{\sigma_1}(S_\mu)$ with $\sigma_1, \tau_1 \ge 0$ and ψ is non-degenerate. Let

$$\theta(z) = e^{-[z]-[z]^{-1}}$$

with $[z] = z$ if $\operatorname{Re} z > 0$ and $[z] = -z$ if $\operatorname{Re} z < 0$. Clearly $\theta \in \cap_{\sigma>0,\tau>0}\Psi^\tau_\sigma(S_\mu)$ and so does

$$\varphi(z) = \begin{cases} c_+\overline{\psi(\bar{z})}\theta(z) & \text{for } z \in S_{\mu+}, \\ c_-\overline{\psi(\bar{z})}\theta(-z) & \text{for } z \in S_{\mu-}. \end{cases}$$

The constants $c_\pm$ are chosen such that $\int_0^\infty \psi(\pm t)\varphi(\pm t)\,\frac{dt}{t} = 1$ (note that the integrals are positive numbers because ψ is non-degenerate, hence $|\psi(\pm t)| > 0$ almost everywhere, so that there is such a choice for $c_\pm$). Next, (23) follows by analytic continuation. $\square$

Remark 4.3. — The function ψ can be taken without any decay at 0 and ∞: it is enough that the product $\psi\varphi$ has both decay.

Let $\mathcal{T}$ be any of the spaces in the table above and $\psi \in \Psi(S_\mu)$. Set

$$\mathbb{H}^{\mathcal{T}}_{\mathbb{Q}_{\psi,T}} = \{f \in \overline{R_2(T)}; \mathbb{Q}_{\psi,T} f \in \mathcal{T}\}$$

equipped with the (quasi-)norm $\|f\|_{\mathbb{H}^{\mathcal{T}}_{\mathbb{Q}_{\psi,T}}} = \|\mathbb{Q}_{\psi,T} f\|_{\mathcal{T}}$ and

$$\mathbb{H}^{\mathcal{T}}_{\mathbb{S}_{\psi,T}} = \{\mathbb{S}_{\psi,T} F; F \in \mathcal{T} \cap T^2_2\}$$

equipped with the (quasi-)norm $\|f\|_{\mathbb{H}^{\mathcal{T}}_{\mathbb{S}_{\psi,T}}} = \inf\{\|F\|_{\mathcal{T}}; f = \mathbb{S}_{\psi,T}F,\ F \in \mathcal{T} \cap T^2_2\}$. We do not need to introduce completions at this point.

Corollary 4.4. — *For any $\mathcal{T}$ in the above table, non-degenerate $\psi \in \Psi_{\mathcal{T}}(S_\mu)$ and $\varphi \in \Psi^{\mathcal{T}}(S_\mu)$, we have*

$$\mathbb{H}^{\mathcal{T}}_{\mathbb{Q}_{\psi,T}} = \mathbb{H}^{\mathcal{T}}_{\mathbb{S}_{\varphi,T}}$$

with equivalent (quasi-)norms.

▷ We set $\mathbb{H}^{\mathcal{T}}_T$ this space and call it the pre-Hardy space associated to $(T, \mathcal{T})$. For any $b \in H^\infty(S_\mu)$, this space is preserved by $b(T)$ and $b(T)$ is bounded on it.

▷ For $\mathcal{T} = T^p_2$, we simply set $\mathbb{H}^p_T = \mathbb{H}^{T^p_2}_T$ and

▷ for $\alpha > 0$, we set $\mathbb{L}^\alpha_T = \mathbb{H}^{T^\infty_{2,\alpha}}_T$.

Of course, the pre-Hardy space associated to $(T, \mathcal{T})$ is not complete as defined. The issue of finding a completion within a classical space is not an easy one.

We shall say that ψ is *allowable* for $\mathbb{H}^{\mathcal{T}}_T$ if we have the equality $\mathbb{H}^{\mathcal{T}}_{\mathbb{Q}_{\psi,T}} = \mathbb{H}^{\mathcal{T}}_T$ with equivalent (quasi-)norms. The set of allowable ψ contains the non-degenerate functions in $\Psi_{\mathcal{T}}(S_\mu)$ but could be larger in some cases.

As the H^∞-calculus extends to $\mathbb{H}^{\mathcal{T}}_T$, the operators $e^{-s|T|}$ extend to bounded operators on $\mathbb{H}^{\mathcal{T}}_T$ with uniform bound in $s > 0$ and have the semigroup property. A question is the continuity on $s \geq 0$, which as is well-known reduces to continuity at $s = 0$. In the reflexive Banach space case, this can be solved by abstract methods for bisectorial operators (see below). However, this excludes the quasi-Banach case we are also interested in. The following result seems new in the theory (this is not an abstract one as it uses the fact that we work with operators defined on L^2 and measure theory) and includes the reflexive range $p > 1$ as well.

Proposition 4.5. — *For all $0 < p < \infty$ and $h \in \mathbb{H}^p_T$, we have the strong limit*

$$\lim_{s \to 0} \|e^{-s|T|}h - h\|_{\mathbb{H}^p_T} = 0.$$

Proof. — We choose

$$\psi(z) = [z]^N e^{-[z]},$$

with $N > \frac{n+1}{2}$ and $N > |\frac{n}{p} - \frac{n}{2}|$. Set $\Gamma(x)$ the cone of (t, y) with $0 \leq |x - y| < t$, and for $0 < \delta \leq R < \infty$, $\Gamma_\delta(x)$ its truncation for $t \leq \delta$, $\Gamma^R(x)$ its truncation for $t \geq R$ and $\Gamma^R_\delta(x) = \Gamma^R(x) \setminus \Gamma_\delta(x)$. Set

$$\Sigma h = S(\psi(tT)h) \quad \text{and} \quad \Sigma h(x) = \Big(\iint_{\Gamma(x)} |\psi(tT)h(y)|^2 \, \frac{dtdy}{t^{n+1}} \Big)^{1/2}$$

so that $\|\Sigma h\|_{L^p} \sim \|h\|_{\mathbb{H}^p_T}$ as ψ is allowable for $\mathbb{H}^p_T$. Let $\Sigma^R h(x)$, $\Sigma_\delta h(x)$ and $\Sigma^R_\delta h(x)$ be defined as $\Sigma h(x)$ with integral on $\Gamma^R(x)$, $\Gamma_\delta(x)$ and $\Gamma^R_\delta(x)$ respectively. Remark that by the choice of ψ, we have

$$(\Sigma h)^2(x) = \iint_{\Gamma(x)} t^{2N-n-1} |\cdot |T|^N e^{-t|T|} h(y)|^2 \, dtdy.$$

It easy to see that $\Sigma(e^{-s|T|}h)(x) \leq Sh(x)$ for all $s > 0$ by using $2N - n - 1 > 0$ and observing that the translated cone $\Gamma(x) + (s, 0)$ is contained in $\Gamma(x)$. Thus we have

$$\Sigma(e^{-s|T|}h - h)(x) \leq 2\Sigma h(x),$$

so that by the Lebesgue convergence theorem, it suffices to show that $\Sigma(e^{-s|T|}h - h)(x)$ converges to 0 almost everywhere. Using the same idea, we have

$$\begin{aligned}\Sigma(e^{-s|T|}h - h)(x) &\leq \Sigma_\delta(e^{-s|T|}h - h)(x) + \Sigma_\delta^R(e^{-s|T|}h - h)(x) \\ &\qquad + \Sigma^R(e^{-s|T|}h - h)(x) \\ &\leq 2\Sigma_{\delta+s}h(x) + \Sigma_\delta^R(e^{-s|T|}h - h)(x) + 2\Sigma^R(h)(x).\end{aligned}$$

Pick $x \in \mathbb{R}^n$ so that $\Sigma h(x) < \infty$ and let $\varepsilon > 0$. Then pick R large and δ small so that $\Sigma^R h(x) < \varepsilon$ and $\Sigma_{2\delta} h(x) < \varepsilon$. Hence, for $s < \delta$, we have

$$\Sigma(e^{-s|T|}h - h)(x) \leq 4\varepsilon + \Sigma_\delta^R(e^{-s|T|}h - h)(x).$$

Now, a rough estimate using the L^2 boundedness of $\psi(tT)$ yields

$$\Sigma_\delta^R(e^{-s|T|}h - h)^2(x) \leq \int_\delta^R \frac{dt}{t^{n+1}} \|e^{-s|T|}h - h\|_2^2$$

and, as $h \in \overline{\mathsf{R}_2(T)}$ and the semigroup is continuous on L^2, the proof is complete. □

For later use, we also have behavior at ∞.

Proposition 4.6. — *For all $0 < p < \infty$ and $h \in \mathbb{H}_T^p$, we have the strong limit*

$$\lim_{s\to\infty} \|e^{-s|T|}h\|_{\mathbb{H}_T^p} = 0.$$

Proof. — With the same square function Σ as above, we have $\Sigma(e^{-s|T|}h) \leq \Sigma h \in L^p$ and $\Sigma(e^{-s|T|}h) \to 0$ almost everywhere when $s \to \infty$. We conclude from the Lebesgue dominated convergence. □

Let us turn to some duality statements.

Proposition 4.7. — *Let $\mathcal{T} = T_2^p$, $0 < p < \infty$ and $\mathcal{T}^*$ be its dual space. Let ψ be allowable for $\mathbb{H}_T^{\mathcal{T}}$ and $\mathbb{H}_{T^*}^{\mathcal{T}^*}$, (for example, $\psi \in \Psi_{\gamma(p)}(S_\mu) \cap \Psi^{\gamma(p)}(S_\mu)$). For any $G \in \mathcal{T}^*$, then*

$$J(G) : f \longmapsto (\mathbb{Q}_{\psi,T} f, G) \in (\mathbb{H}_T^{\mathcal{T}})^*.$$

Conversely, to any $\ell \in (\mathbb{H}_T^{\mathcal{T}})^*$, there corresponds a $G \in \mathcal{T}^*$, such that $\ell(f) = J(G)(f)$ for any $f \in \mathbb{H}_T^{\mathcal{T}}$.

Proof. — The proof is quite standard. That $J(G) \in (\mathbb{H}^{\mathcal{T}}_{T})^*$ follows using that ψ is allowable for $\mathbb{H}^{\mathcal{T}}_{T}$, hence $\|\mathbb{Q}_{\psi,T}f\|_{\mathcal{T}} \sim \|f\|_{\mathbb{H}^{\mathcal{T}}_{T}}$ and the duality of tent spaces. Conversely, let φ associated to ψ as in proposition 4.2. Let $\ell \in (\mathbb{H}^{\mathcal{T}}_{T})^*$, then $\ell \circ \mathbb{S}_{\varphi,T}$ is defined on $\mathcal{T} \cap T^2_2$ and $|\ell \circ \mathbb{S}_{\varphi,T}(F)| \lesssim \|F\|_{\mathcal{T}}$. By density in $\mathcal{T}$ and duality, there exists $G \in \mathcal{T}^*$ such that $|\ell \circ \mathbb{S}_{\varphi,T}(F)| = (F, G)$ for all $F \in \mathcal{T} \cap T^2_2$. Inserting $F = \mathbb{Q}_{\psi,T}f$, we obtain the $(\mathbb{Q}_{\psi,T}f, G) = \ell \circ \mathbb{S}_{\varphi,T}(\mathbb{Q}_{\psi,T}f) = \ell(f)$. □

It will be easier to work within $\mathbb{H}^2_T = \overline{R_2(T)}$. This is why we systematically use pre-Hardy spaces.

Proposition 4.8. — *Let $\mathcal{T} = T^p_2$, $0 < p < \infty$ and $\mathcal{T}^*$ be its dual space. Denote by $\langle .\,,.\rangle$ the L^2 sesquilinear inner product. Then for any $f \in \mathbb{H}^{\mathcal{T}}_{T}$, $g \in \mathbb{H}^{\mathcal{T}^*}_{T^*}$*

$$|\langle f, g\rangle| \lesssim \|f\|_{\mathbb{H}^{\mathcal{T}}_{T}} \cdot \|g\|_{\mathbb{H}^{\mathcal{T}^*}_{T^*}}.$$

More generally, for any $f \in \overline{R_2(T)}, g \in \overline{R_2(T^*)}$ and any $\psi, \varphi \in \Psi(S_\mu)$ for which the Calderón reproducing formula (23) *holds, one has*

$$|\langle f, g\rangle| \leq \|\mathbb{Q}_{\psi,T}f\|_{\mathcal{T}} \cdot \|\mathbb{Q}_{\varphi^*,T^*}g\|_{\mathcal{T}^*}.$$

Next, for any $g \in \mathbb{H}^{\mathcal{T}^*}_{T^*}$,

$$\|g\|_{\mathbb{H}^{\mathcal{T}^*}_{T^*}} \sim \sup\{|\langle f, g\rangle|; f \in \mathcal{T}, \|f\|_{\mathbb{H}^{\mathcal{T}}_{T}} = 1\}.$$

When $1 < p < \infty$, we can revert the roles of $\mathcal{T}$ and $\mathcal{T}^*$, that is, $\langle .\,,.\rangle$ is a duality for the pair of spaces $(\mathbb{H}^p_T, \mathbb{H}^{p'}_{T^*})$.

We mention as a corollary the usual principle that upper bounds in square functions for allowable ψ imply lower bounds for all φ with the dual operator.

Proposition 4.9. — *Let $\mathcal{T} = T^p_2$, $1 < p < \infty$ so that $\mathcal{T}^* = T^{p'}_2$. Assume that $\langle .\,,.\rangle$ is a duality for the pair of normed spaces (X, Y) with $X \subset \overline{R_2(T)}$ and $Y \subset \overline{R_2(T^*)}$ and that for any allowable $\psi \in \Psi(S_\mu)$ for $\mathbb{H}^{\mathcal{T}}_{T}$, we have*

$$\|\mathbb{Q}_{\psi,T}f\|_{\mathcal{T}} \lesssim \|f\|_X$$

for all $f \in \overline{R_2(T)}$. Then for any non-degenerate $\varphi \in \Psi(S_\mu)$, we have

$$\|g\|_Y \leq \|\mathbb{Q}_{\varphi^*,T^*}g\|_{\mathcal{T}^*}$$

for all $g \in \overline{R_2(T^*)}$.

Proof. — This a consequence of the previous result with the fact that given a non-degenerate φ, one can find ψ in any class $\Psi^\tau_\sigma(S_\mu)$, thus one allowable ψ for $\mathbb{H}^{\mathcal{T}}_{T}$, for which the Calderón reproducing formula (23) holds. □

For $0 < p \leq 1$, we can take advantage of the notion of molecules. We follow [**60**]. For a cube (or a ball) $Q \subset \mathbb{R}^n$ denote the dyadic annuli by $S_i(Q)$, which is defined for $i = 1, 2, 3, ...$ by

$$S_i(Q) := 2^i Q \backslash 2^{i-1} Q$$

and $S_0(Q) := Q$. Here λQ is the cube with same center as Q and sidelength $\lambda \ell(Q)$. Let $0 < p \leq 1$, $\epsilon > 0$ and $M \in \mathbb{N}$. We say that a function $m \in L^2$ is a $(\mathbb{H}^p_T, \epsilon, M)$-molecule if there exists a cube $Q \subset \mathbb{R}^n$ and a function $b \in \mathsf{D}_2(T^M)$ such that $T^M b = m$ and

$$\|\left(\ell(Q)\, T\right)^{-k} m\|_{L^2(S_i(Q))} \leq \left(2^i \ell(Q)\right)^{\frac{n}{2} - \frac{n}{p}} 2^{-i\epsilon} \tag{25}$$

for $i = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, M$. Remark that $m \in \mathsf{R}_2(T)$ and also that $m \in L^p$ with $\|m\|_p \lesssim 1$ independently of Q.

Definition 4.10. — Let $0 < p \leq 1$, $\epsilon > 0$ and $M \in \mathbb{N}$. For $f \in \overline{\mathsf{R}_2(T)}$, $f = \sum_j \lambda_j m_j$ is a molecular $(\mathbb{H}^p_T, \epsilon, M)$-representation of f if each m_j is an $(\mathbb{H}^p_T, \epsilon, M)$-molecule, $(\lambda_j) \in \ell^p$ and the series converges in L^2. We define

$$\mathbb{H}^p_{T,\mathrm{mol},M} := \left\{ f \in \overline{\mathsf{R}_2(T)};\, f \text{ has a molecular } (\mathbb{H}^p_T, \epsilon, M)\text{-representation } \right\}$$

with the quasi-norm (it is a norm only when $p = 1$)

$$\|f\|_{\mathbb{H}^p_{T,\mathrm{mol},M}} := \inf \left\{ \|(\lambda_j)\|_{\ell^p} \right\},$$

taken over all molecular $(\mathbb{H}^p_T, \epsilon, M)$-representations $f = \sum_{j=0}^{\infty} \lambda_j m_j$, where $\|(\lambda_j)\|_{\ell^p} := \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}}$.

Remark 4.11. — Note the continuous inclusion $\mathbb{H}^p_{T,\mathrm{mol},M_1} \subset \mathbb{H}^p_{T,\mathrm{mol},M_2}$ if $M_2 \geq M_1$. In particular, $\mathbb{H}^p_{T,\mathrm{mol},M} \subset \mathbb{H}^p_{T,\mathrm{mol},1}$.

Proposition 4.12. — *Let $0 < p \leq 1$, $M \in \mathbb{N}$ with $M > \frac{n}{p} - \frac{n}{2}$. Then*

$$\mathbb{H}^p_{T,\mathrm{mol},M} = \mathbb{H}^p_T$$

with equivalence of quasi-norms.

Proof. — Adapt [**59, 60**]. ☐

Remark 4.13. — It would also make sense to consider the atomic versions but at this level of generality, we do not know whether $\mathbb{H}^p_T$ has an atomic decomposition.

The following corollary is a useful consequence.

Corollary 4.14. — *For $0 < p < 2$, then $\mathbb{H}^p_T \subset L^p$ with $\|f\|_p \lesssim \|f\|_{\mathbb{H}^p_T}$.*

Proof. — For $0 < p \leq 1$, this is a consequence of the fact that any $(\mathbb{H}^p_T, \epsilon, M)$-molecule satisfies $\|m\|_p \lesssim 1$ and of the previous proposition. For $1 \leq p \leq 2$, we proceed by interpolation as follows. Fix one $\varphi \in \Psi_{\frac{n}{2}}(S_\mu)$ and consider the map $\mathbb{S}_{\varphi,T}$. By proposition 4.1, it is bounded from T^2_2 to L^2 and from $T^1_2 \cap T^2_2$ to $\mathbb{H}^1_T$ with $\|\mathbb{S}_{\varphi,T} F\|_{\mathbb{H}^1_T} \lesssim \|F\|_{T^1_2}$ so that it maps $T^2_1 \cap T^2_2$ to L^1 with $\|\mathbb{S}_{\varphi,T} F\|_1 \lesssim \|F\|_{T^2_1}$. By interpolation, the bounded extension on T^1_2 is bounded from T^p_2 into L^p. It is a standard duality argument to show that this extension agrees with $\mathbb{S}_{\varphi,T}$ on $T^p_2 \cap T^2_2$. ☐

We finish with non-tangential maximal estimates. Recall the Kenig-Pipher functional

$$\widetilde{N}_*(g)(x) := \sup_{t>0} \Big(\iint_{W(t,x)} |g|^2 \Big)^{1/2}, \qquad x \in \mathbb{R}^n, \tag{26}$$

with $W(t,x) := (c_0^{-1}t, c_0 t) \times B(x, c_1 t)$, for some fixed constants $c_0 > 1$, $c_1 > 0$.

LEMMA 4.15. — *For all $0 < p \le 1$, one has the estimate*

$$\|\widetilde{N}_*(e^{-t|T|}h)\|_p \lesssim \|h\|_{\mathbb{H}^p_T}, \quad \forall h \in \overline{\mathsf{R}_2(T)}. \tag{27}$$

Furthermore, it also holds for $1 < p < 2$ if it holds at $p = 2$.

Proof. — For $0 < p \le 1$, this comes from $\|\widetilde{N}_*(e^{-t|T|}m)\|_p \lesssim 1$ for any $(\mathbb{H}^p_T, \epsilon, M)$-molecule with $M \in \mathbb{N}$ for M large enough depending on n, p. (Adapt the proofs in [**59**], [**67**] or [**46**]. See also [**86**] for an explicit argument. It is likely that one can prove that the lower bound $M > \frac{n}{p} - \frac{n}{2}$ works but we don't need such a precision.) This implies the inequality for any $f \in \mathbb{H}^p_{T,\mathrm{mol},M} = \mathbb{H}^p_T$.

We use interpolation for $1 < p < 2$ as follows. Fix $\varphi \in \Psi_{\frac{n}{2}}(S_\mu)$ and $\psi \in \Psi^{\frac{n}{2}}(S_\mu)$ such that $\mathbb{S}_{\varphi,T}\mathbb{Q}_{\psi,T} = I$ on $\overline{\mathsf{R}_2(T)}$. From the assumption, the sublinear operator

$$V : F \longmapsto \widetilde{N}_*(e^{-t|T|}\mathbb{S}_{\varphi,T}F)$$

is bounded from T^2_2 into L^2 with $\|VF\|_2 \lesssim \|F\|_{T^2_2}$ and we just proved it is bounded from $T^2_2 \cap T^1_2$ into L^1 with $\|VF\|_1 \lesssim \|F\|_{T^1_1}$. Using real interpolation (see [**40**], corrected in [**33**]), and density (of $T^1_2 \cap T^2_2$ into $T^p_2 \cap T^2_2$ for the T^p_2 topology) this implies that V maps $T^2_2 \cap T^p_2$ into L^p with $\|VF\|_p \lesssim \|F\|_{T^p_2}$. Applying this to $F = \mathbb{Q}_{\psi,T}h$ when $h \in \overline{\mathsf{R}_2(T)}$, this implies (27). □

4.3. Spaces associated to D

We specialize the general theory to the situation where $T = D$. Because D is self-adjoint, we can also consider the $(\mathbb{H}^p_D, M)$-atoms, which are those $(\mathbb{H}^p_D, \epsilon, M)$-molecules associated to a cube Q and supported in Q. The atomic space

$$\mathbb{H}^p_{D,\mathrm{ato},M}$$

is defined similarly to the molecular one and one has $\mathbb{H}^p_{D,\mathrm{ato},M} = \mathbb{H}^p_D$ when $0 < p \le 1$ and $M > \frac{n}{p} - \frac{n}{2}$. The proof is explicitly done in [**19**] for $p = 1$ and applies *in extenso* to $p < 1$. See also [**57**] for the case of second order operators and $p \le 1$.

We also remark that

$$\mathbb{H}^p_{D,\mathrm{ato},1} = \mathbb{H}^p_{D,\mathrm{mol},1}$$

with equivalence of norms (this argument is due to A. McIntosh).

The inclusion $\subset$ is obvious. In the opposite direction, if m is an $(\mathbb{H}^p_D, \epsilon, 1)$-molecule, then one can write $m = Db$ with estimate (25) and $M = 1$. Then we write $b = \sum \chi_i b$ with (χ_i) a smooth partition of unity associated to the annular set $S_i(Q)$: they satisfy

$$0 \le \chi_i \le 1, \quad \|\nabla \chi_i\|_\infty \lesssim (2^i \ell(Q))^{-1}$$

and χ_i supported on $S_i(2Q)$. Then, it is easy to show that $a_i = 2^{i\epsilon} D(\chi_i b)$ is up to a dimensional constant an $(\mathbb{H}^p_D, 1)$-atom and that the series $m = \sum 2^{-i\epsilon} a_i$ converges in L^2.

THEOREM 4.16. — *Let $\frac{n}{n+1} < p \le 1$. Then*

$$\mathbb{H}^p_D = \mathbb{H}^p_{D,\mathrm{mol},1} = \mathbb{H}^p_{D,\mathrm{ato},1} = H^p \cap \overline{\mathsf{R}_2(D)} = H^p \cap \mathbb{P}(L^2) = \mathbb{P}(H^p \cap L^2) \tag{28}$$

with

$$\|f\|_{\mathbb{H}^p_D} \sim \|f\|_{\mathbb{H}^p_{D,\mathrm{mol},1}} \sim \|f\|_{\mathbb{H}^p_{D,\mathrm{ato},1}} \sim \|f\|_{H^p}, \quad \forall f \in \overline{\mathsf{R}_2(D)}.$$

Let $1 < p < \infty$. Then

$$\mathbb{H}^p_D = \overline{\mathsf{R}_p(D)} \cap \overline{\mathsf{R}_2(D)} = L^p \cap \overline{\mathsf{R}_2(D)} = L^p \cap \mathbb{P}(L^2) = \mathbb{P}(L^p \cap L^2) \tag{29}$$

with

$$\|f\|_{\mathbb{H}^p_D} \sim \|f\|_{L^p}, \quad \forall f \in \overline{\mathsf{R}_2(D)}.$$

Let $p = \infty$. Then

$$\mathbb{BMO}_D = \mathrm{BMO} \cap \overline{\mathsf{R}_2(D)} = \mathrm{BMO} \cap \mathbb{P}(L^2) = \mathbb{P}(\mathrm{BMO} \cap L^2) \tag{30}$$

with

$$\|f\|_{\mathbb{BMO}_D} \sim \|f\|_{\mathrm{BMO}}, \quad \forall f \in \overline{\mathsf{R}_2(D)}.$$

Let $0 \le \alpha < 1$. Then

$$\mathbb{L}^\alpha_D = \dot{\Lambda}^\alpha \cap \overline{\mathsf{R}_2(D)} = \dot{\Lambda}^\alpha \cap \mathbb{P}(L^2) = \mathbb{P}(\dot{\Lambda}^\alpha \cap L^2) \tag{31}$$

with

$$\|f\|_{\mathbb{L}^\alpha_D} \sim \|f\|_{\dot{\Lambda}^\alpha}, \quad \forall f \in \overline{\mathsf{R}_2(D)}.$$

Proof. — Let us assume first $p \le 1$. As $\overline{\mathsf{R}_2(D)} = \mathbb{P}(L^2)$ the fourth equality is a trivial. The inclusion $\supset$ of the fifth equality comes from the fact that $\mathbb{P}$ is bounded on H^p, and for the converse, if $h \in H^p \cap \mathbb{P}(L^2)$ then $h = \mathbb{P}h \in \mathbb{P}(H^p \cap L^2)$. By general theory and the discussion above,

$$\mathbb{H}^p_D = \mathbb{H}^p_{D,\mathrm{mol},n} \subset \mathbb{H}^p_{D,\mathrm{mol},1} = \mathbb{H}^p_{D,\mathrm{ato},1}.$$

Now a $(\mathbb{H}^p_D, 1)$-atom $a = Db$ belongs to $\mathsf{R}_2(D)$ and also to H^p as $p > \frac{n}{n+1}$ and $\int a = \int Db = 0$. As convergence of atomic decompositions is in L^2, so also in tempered distributions, it follows that

$$\mathbb{H}^p_{D,\mathrm{ato},1} \subset \overline{\mathsf{R}_2(D)} \cap H^p.$$

It remains to show $\mathbb{P}(H^p \cap L^2) \subset \mathbb{H}^p_D$. Let $L = D^2\mathbb{P} - \Delta(I - \mathbb{P})$ where Δ is the ordinary negative self-adjoint Laplacian on L^2. Clearly L is self-adjoint on L^2, positive, it has a homogeneous of order 2 symbol, C^∞ away from 0, with $\widehat{L}(\xi) \sim |\xi|^2$ (in the sense of self-adjoint matrices). One can estimate the kernel of the convolution operator $t^2 L e^{-t^2 L}$ and find pointwise decay in $t^{-n}(1 + \frac{|x|}{t})^{-n-2}$. Similarly for all its partial derivatives with $-n-3$ replacing $-n-2$. By standard theory for the Hardy space as in [**40**], for $h \in \mathbb{P}(H^p \cap L^2)$,

$$F(t,.) = t^2 L e^{-t^2 L} h \in T_2^p \cap T_2^2,$$

thus for any φ such that $\mathbb{H}_D^p = \mathbb{H}_{\mathbb{S}_{\varphi,D}}^{T_2^p}$ we have $\mathbb{S}_{\varphi,D} F \in \mathbb{H}_D^p$. Now $L\mathbb{P} = D^2\mathbb{P} = D^2$. Thus, as $h = \mathbb{P}h$,

$$F(t,.) = t^2 D^2 e^{-t^2 D^2} h = \psi(tD)h$$

with $\psi(z) = z^2 e^{-z^2}$. If one chooses $\varphi \in \Psi_{\gamma(p)}(S_\mu)$ such that (23) holds then

$$\mathbb{S}_{\varphi,D} F = \mathbb{S}_{\varphi,D}\mathbb{Q}_{\psi,D} h = h$$

so that $h \in \mathbb{H}_D^p$.

If $1 < p < \infty$, the third equality is trivial, the fourth and the inclusion $\mathbb{P}(L^p \cap L^2) \subset \mathbb{H}_D^p$ are obtained as above. By using truncation in t for T_2^p functions in a Calderón reproducing formula $\mathbb{S}_{\varphi,D}\mathbb{Q}_{\psi,D} = I$, we see easily that $\mathbb{H}_D^p \subset \overline{\textsf{R}_p(D)} \cap \overline{\textsf{R}_2(D)}$ and obviously $\overline{\textsf{R}_p(D)} \cap \overline{\textsf{R}_2(D)} \subset L^p \cap \overline{\textsf{R}_2(D)}$.

The proof for BMO type spaces is obtained by duality from $p = 1$, noticing that the duality form is the same for $\mathbb{H}_D^1, \mathbb{BMO}_D$ and H^1, BMO, and that $\mathbb{P} = \mathbb{P}^*$ is bounded on H^1 and BMO.

The proof for $\dot{\Lambda}^\alpha$ type space is also obtained by duality from the case $p < 1$. We omit further details. □

4.4. General facts about comparison of $\mathbb{H}_{DB}^p$ and $\mathbb{H}_D^p$

Of course, by definition $\mathbb{H}_{DB}^2 = \mathbb{H}_D^2$, thus we look at other values of p.

Proposition 4.17. — *For $\frac{n}{n+1} < p < 2$, we have $\mathbb{H}_{DB}^p \subset \mathbb{H}_D^p$ with continuous inclusion. More precisely, the inequality*

$$\|h\|_{H^p} \lesssim \|\mathbb{Q}_{\psi,DB} h\|_{T_2^p}, \quad \forall h \in \overline{\textsf{R}_2(D)},$$

holds when $\psi \in \Psi^{\gamma(p)}(S_\mu)$, where $H^p = L^p$ when $p > 1$.

Proof. — Indeed, if $p \le 1$ it is clear that an $(\mathbb{H}_{DB}^p, \epsilon, M)$-molecule $a = (DB)^M b$ writes $a = D(B(DB)^{M-1}b)$, hence is an $(\mathbb{H}_D^p, \epsilon, 1)$-molecule. We conclude using theorem 4.16. For $1 < p < 2$, we use the interpolation argument of lemma 4.15. □

Proposition 4.18. — *For $2 < p < \infty$, we have $\mathbb{H}_D^p \subset \mathbb{H}_{DB}^p$ with continuous inclusion. More precisely, the inequality*

$$\|\mathbb{Q}_{\psi,DB} h\|_{T_2^p} \lesssim \|h\|_p, \quad \forall h \in \overline{\textsf{R}_2(D)},$$

holds when $\psi \in \Psi_{\gamma(p)}(S_\mu)$.

Proof. — It suffices to prove this for $\psi \in \Psi_{\frac{n}{2}}(S_\mu)$. The method of lemma 5.16 below proves in particular for such ψ that

$$\|\mathbb{Q}_{\psi,DB}h\|_{T^p_2} \lesssim \|h\|_p, \quad \forall h \in L^p \cap L^2.$$

If $h \in \mathbb{H}^p_D$, then $h = \mathbb{P}h$ and $\|\mathbb{P}h\|_p \sim \|h\|_{\mathbb{H}^p_D}$ by theorem 4.16 and as $\mathbb{Q}_{\psi,DB}h = \mathbb{Q}_{\psi,DB}\mathbb{P}h$, we obtain

$$\|\mathbb{Q}_{\psi,DB}\mathbb{P}h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_p \sim \|h\|_{\mathbb{H}^p_D}. \qquad \square$$

We are interested in the equality $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$.

THEOREM 4.19. — *Let $\frac{n}{n+1} < p < \infty$. Assume that $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalent norms. Then for any $b \in H^\infty(S_\mu)$, $b(DB)$ is bounded on $\mathbb{H}^p_D$. Thus,*

$$\|b(DB)h\|_{H^p} \lesssim \|b\|_\infty \cdot \|h\|_{H^p}$$

for any $h \in \mathbb{H}^p_D$ where $H^p = L^p$ if $p > 1$. Furthermore, $(e^{-t|DB|})_{t>0}$ is a strongly continuous semigroup on $\mathbb{H}^p_D$.

Proof. — From corollary 4.4, we know that $b(DB)$ is bounded on $\mathbb{H}^p_{DB}$ for any $0 < p < \infty$. The strong continuity of the semigroup on $\mathbb{H}^p_{DB}$ is also shown in proposition 4.5. The same properties hold for any equivalent topology. $\square$

We turn to dual statements. The result which will guide our discussion is the following one. Recall that $\mathbb{P}$ is the orthogonal projection from L^2 onto $\overline{R_2(D)}$.

THEOREM 4.20. — *Let $\frac{n}{n+1} < q < \infty$. Assume that $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalent norms. Then if $q > 1$ and $p = q'$,*

$$\mathbb{P} : \mathbb{H}^p_{BD} \longrightarrow \mathbb{H}^p_D$$

is an isomorphism and if $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$,

$$\mathbb{P} : \mathbb{L}^\alpha_{BD} \longrightarrow \mathbb{L}^\alpha_D$$

is an isomorphism. In the range $q > 1$ and $p = q'$, the converse holds: if $\mathbb{P} : \mathbb{H}^p_{BD} \to \mathbb{H}^p_D$ is an isomorphism then $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalent norms.

Proof. — This is in fact a simple functional analytic statement. Let us prove the direct part. We have $\frac{n}{n+1} < q < \infty$ and $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$, with equivalence of norms. We want to show the isomorphism property of $\mathbb{P}$. We know that

$$\mathbb{P} : \overline{R_2(BD)} = \mathbb{H}^2_{BD} \longrightarrow \overline{R_2(D)} = \mathbb{H}^2_D$$

is isomorphic, thus bijective. It suffices to prove the norm comparison. Assume first $1 < q$. Set $p = q'$. Let $g \in \overline{R_2(BD)}$. Then using proposition 4.8 for $T = DB^*$ and also for D, one has

$$\begin{aligned}\|g\|_{\mathbb{H}^p_{BD}} &\sim \sup\{|\langle g, f\rangle|\,;\, \|f\|_{\mathbb{H}^q_{DB^*}} \le 1\} \\ &\sim \sup\{|\langle g, f\rangle|\,;\, \|f\|_{\mathbb{H}^q_D} \le 1\} \\ &= \sup\{|\langle \mathbb{P}g, f\rangle|\,;\, \|f\|_{\mathbb{H}^q_D} \le 1\} \sim \|\mathbb{P}g\|_{\mathbb{H}^p_D}.\end{aligned} \tag{32}$$

Next, if we assume $q \le 1$, then we work with $\alpha = n(\frac{1}{q} - 1)$ and $\mathbb{L}^\alpha_{BD}$, and exactly the same argument applies.

For the converse in the case $q > 1$, it suffices to reverse the role of the spaces. Let $f \in \overline{R_2(D)}$. Then,

$$\begin{aligned}\|f\|_{\mathbb{H}^q_{DB^*}} &\sim \sup\{|\langle g, f\rangle|\,;\, \|g\|_{\mathbb{H}^p_{BD}} \le 1\} \\ &= \sup\{|\langle \mathbb{P}g, f\rangle|\,;\, \|g\|_{\mathbb{H}^p_{BD}} \le 1\} \\ &\sim \sup\{|\langle h, f\rangle|\,;\, \|h\|_{\mathbb{H}^p_D} \le 1\} \sim \|f\|_{\mathbb{H}^q_D}.\end{aligned}$$

□

Corollary 4.21. — *Let $\frac{n}{n+1} < q < \infty$. Assume that $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalent norms. Let $b \in H^\infty(S_\mu)$. If $q > 1$ and $p = q'$, then*

$$\|\mathbb{P}b(BD)\|_p \lesssim \|\mathbb{P}h\|_p, \quad \forall h \in \mathbb{H}^2_{BD}.$$

If $q \le 1$, then for $\alpha = n(\frac{1}{q} - 1)$,

$$\|\mathbb{P}b(BD)h\|_{\dot\Lambda^\alpha} \lesssim \|\mathbb{P}h\|_{\dot\Lambda^\alpha}, \quad \forall h \in \mathbb{H}^2_{BD}.$$

Proof. — This is just using the similarity induced by $\mathbb{P}$ from the previous theorem and the H^∞-calculus on $\mathbb{H}^p_{BD}$ or $\mathbb{L}^\alpha_{BD}$ from corollary 4.4. □

Another version is also useful.

Corollary 4.22. — *Let $\frac{n}{n+1} < q < \infty$. Assume that*

$$\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$$

with equivalent norms. Then if $q > 1$ and $p = q'$, for any $b \in H^\infty(S_\mu)$ which is defined at 0, $\mathbb{P}b(BD)$ is bounded on $\mathbb{H}^p_D$ with

$$\|\mathbb{P}b(BD)h\|_p \lesssim \|h\|_p$$

for all $h \in \mathbb{H}^p_D$. Also $(\mathbb{P}e^{-t|BD|})_{t>0}$ is a strongly continuous semigroup on $\mathbb{H}^p_D$. If $q \le 1$, then for $\alpha = n(\frac{1}{q} - 1)$, $\mathbb{P}b(BD)$ is bounded on $\mathbb{L}^\alpha_D$ with

$$\|\mathbb{P}b(BD)h\|_{\mathbb{L}^\alpha_D} \lesssim \|h\|_{\mathbb{L}^\alpha_D}$$

for all $h \in \mathbb{L}^\alpha_D$. Furthermore, $(\mathbb{P}e^{-t|BD|})_{t>0}$ is a weakly-$*$ continuous semigroup on $\mathbb{L}^\alpha_D$.

Proof. — Let us begin with the case $q > 1$. Let $h \in \mathbb{H}^p_D$. From the previous theorem, there exists a unique $h' \in \mathbb{H}^p_{BD}$ such that $h = \mathbb{P}h'$. By proposition 3.3, since b is defined at 0,

$$\mathbb{P}b(BD)h = \mathbb{P}b(BD)\mathbb{P}h' = \mathbb{P}b(BD)h'.$$

Thus, by the previous corollary,

$$\|\mathbb{P}b(BD)h\|_p = \|\mathbb{P}b(BD)h'\|_p \lesssim \|\mathbb{P}h'\|_p.$$

The proof when $q \leq 1$ is similar and we skip it. □

REMARK 4.23. — The assumption $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalent norms can be weakened in theorem 4.20 by the following one. This will be useful when $q \leq 2$. It suffices to assume that $\mathbb{H}^q_{DB^*} \subset \mathbb{H}^q_D$, that $\|h\|_{\mathbb{H}^q_D} \sim \|h\|_{\mathbb{H}^q_{DB^*}}$ for all $h \in \mathbb{H}^q_{DB^*}$ and that the inclusion is dense for the $\mathbb{H}^q_D$ topology. Indeed, when $q > 1$, (32) rewrites

$$\begin{aligned}
\|g\|_{\mathbb{H}^p_{BD}} &\sim \sup\{|\langle g, f\rangle|\,;\, f \in \mathbb{H}^q_{DB^*}, \|f\|_{\mathbb{H}^q_{DB^*}} \leq 1\}\\
&\sim \sup\{|\langle g, f\rangle|\,;\, f \in \mathbb{H}^q_{DB^*}, \|f\|_{\mathbb{H}^q_D} \leq 1\}\\
&\sim \sup\{|\langle g, f\rangle|\,;\, f \in \mathbb{H}^q_{D}, \|f\|_{\mathbb{H}^q_D} \leq 1\}\\
&= \sup\{|\langle \mathbb{P}g, f\rangle|\,;\, f \in \mathbb{H}^q_{D}, \|f\|_{\mathbb{H}^q_D} \leq 1\} \sim \|\mathbb{P}g\|_{\mathbb{H}^p_D}.
\end{aligned}$$

The second line comes from the equivalence of norms, and the third from the density. The same reasoning holds when $q \leq 1$. The same weaker assumption can be taken in corollary 4.21 as well.

4.5. The spectral subspaces

The pre-Hardy spaces split in two spectral subspaces. This will become useful when relating this to boundary value problems as these spectral subspaces will identify to trace spaces for elliptic systems.

Because $T = DB$ or BD is bisectorial with H^∞-calculus on L^2, we have two spectral subspaces of $H^2_T = \overline{\mathsf{R}_2(T)}$, called $H^{2,\pm}_T$ as defined in proposition 3.2 by

$$H^{2,\pm}_T = \chi^\pm(T)(H^2_T).$$

This can be extended to the pre-Hardy spaces $\mathbb{H}^{\mathcal{T}}_T$ by setting

$$\mathbb{H}^{\mathcal{T},\pm}_T := \chi^\pm(T)(\mathbb{H}^{\mathcal{T}}_T) = \mathbb{H}^{\mathcal{T}}_T \cap H^{2,\pm}_T.$$

This leads to the spaces $\mathbb{H}^{p,\pm}_T$ for $0 < p \leq \infty$ and $\mathbb{L}^{\alpha,\pm}_T$ for $\alpha \geq 0$.

We have the following properties.

1) $\mathbb{H}^{\mathcal{T}}_T = \mathbb{H}^{\mathcal{T},+}_T \oplus \mathbb{H}^{\mathcal{T},-}_T$ where the sum is topological for the topology of $\mathbb{H}^{\mathcal{T}}_T$.

2) $(e^{\mp tT}\chi^\pm(T))_{t>0}$ are semigroups on $\mathbb{H}^{\mathcal{T},\pm}_T$, which coincide with $(e^{-t|T|})_{t>0}$. Thus, they are strongly continuous if $\mathcal{T} = T^p_2$ and weakly-$*$ continuous if $\mathcal{T} = T^\infty_{2,\alpha}$.

5. PRE-HARDY SPACES IDENTIFICATION

The range of q for which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ will be our goal, together with the determination the classes of allowable ψ, as we will need something more precise than what the general theory predicts. This is the most important chapter of this article and we give full details.

Recall that

$$I(BD) = I(DB) = (p_-(BD), p_+(BD)).$$

We sometimes set

$$p_- = p_-(BD) = p_-(DB) \quad \text{and} \quad p_+ = p_+(BD) = p_+(DB)$$

to simplify notation. The situation for the operator DB is simple to state and meets our needs for applications to elliptic PDEs. We use the notation $q_* = \frac{nq}{n+q}$ for the lower Sobolev exponent of q and $q^* = \frac{nq}{n-q}$ for the Sobolev exponent of q when $q < n$. At this level of generality, we have the following range for comparison of Hardy spaces. Later (chapter 13) we obtain a much bigger range under some De Giorgi assumptions when DB arises from a second order equation or system.

Theorem 5.1. — *For $(p_-(DB))_* < p < p_+(DB)$, we have $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalent norms. More precisely, the comparison*

$$\|\mathbb{Q}_{\psi,DB}h\|_{T^p_2} \sim \|h\|_{H^p}, \quad \forall h \in \overline{\mathsf{R}_2(DB)} = \overline{\mathsf{R}_2(D)},$$

holds when $\psi \in \Psi^{\gamma(p)}(S_\mu)$ if $(p_-)_* < p < 2$ and $\psi \in \Psi(S_\mu)$ if $2 \le p < p_+$. In particular, we have the square function estimates

$$\|S(tDBe^{-t|DB|}h)\|_p \sim \|S(t\partial_t e^{-t|DB|}h)\|_p \sim \|h\|_{H^p}, \quad \forall h \in \overline{\mathsf{R}_2(D)}.$$

Remark 5.2. — In the case of constant B as in section 3.4, or under the assumption of proposition 3.11 if $n = 1$, we have $p_-(DB) = 1$ and $p_+(DB) = \infty$, hence the interval is the largest possible one $(\frac{n}{n+1}, \infty)$.

The situation for BD is a little more complicated, since we want $\psi(z) = O(z)$ for applications and also since the functions of BD do not give all the information we need for the elliptic PDEs. We state this in three different results.

Theorem 5.3. — *For $p_-(DB) < p < (p_+(DB))^*$, we have $\mathbb{P} : \mathbb{H}^p_{BD} \to \mathbb{H}^p_D$ is an isomorphism. More precisely, the comparison*

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^p_2} \sim \|\mathbb{P}h\|_{\mathbb{H}^p_D}, \quad \forall h \in \overline{\mathsf{R}_2(BD)},$$

holds when $\psi \in \Psi^{\gamma(p)}(S_\mu)$ *if* $p_- < p < 2$, $\psi \in \Psi(S_\mu)$ *if* $2 < p < p_+$, *and* $\psi \in \Psi_{\frac{n}{p_+}-\frac{n}{p}}(S_\mu)$ *if* $p_+ \leq p < (p_+)^*$. *Moreover, if* $p_+ > n$, *then for* $0 < \alpha < 1 - \frac{n}{p_+}$, *we have that* $\mathbb{P} : \mathbb{L}^\alpha_{BD} \to \mathbb{L}^\alpha_D$ *is an isomorphism with*

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^\infty_{2,\alpha}} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{\textsf{R}_2(BD)},$$

when $\psi \in \Psi_{\alpha+\frac{n}{p_+}}(S_\mu)$.

Corollary 5.4. — *For* $p \in I(BD)$,

$$\mathbb{H}^p_{BD} = \overline{\textsf{R}_2(BD)} \cap \overline{\textsf{R}_p(BD)} = \overline{\textsf{R}_2(BD)} \cap L^p.$$

Proof. — Remark that for $p \in I(BD)$, we have $\|\mathbb{P}h\|_p \sim \|h\|_p$ for all $h \in \overline{\textsf{R}_2(BD)}$ by proposition 3.8. So $\mathbb{H}^p_{BD}$ is the set of $h \in \overline{\textsf{R}_2(BD)}$ for which $\|h\|_p < \infty$, which is $\overline{\textsf{R}_2(BD)} \cap \overline{\textsf{R}_p(BD)} = \overline{\textsf{R}_2(BD)} \cap L^p$. □

Remark 5.5. — If, furthermore, B is invertible in L^∞, then $\mathbb{H}^p_{BD} = \overline{\textsf{R}_2(BD)} \cap L^p$ also for $\max(1, (p_-)_*) \leq p < p_-$. See [**86**]. In particular, $\mathbb{H}^p_{BD}$ is also a subspace of L^p but this is not so useful in practice.

Remark 5.6. — In each theorem, the classes of ψ for the upper bounds are what is expected from the general theory when $p < 2$ and a little better when $p > 2$. In particular, the classes for the upper bounds of $\|\mathbb{Q}_{\varphi,T}h\|_{\mathcal{T}}$ obtained for $p > 2$ will require a specific statement (corollary 5.18 and proposition 5.19). Note that all these classes allow the behavior $\psi(z) = O(z)$ at 0. This will be important for applications to elliptic equations. However, it could be that we want to use square functions with some $\psi(z) = O(z)$ at 0 for p beyond the exponent $(p_+(BD))^*$. Indeed, the value of $p_+(BD)$ is usually close to 2 while one needs to consider $p = \infty$. This is the object of the next result where the failure of good vanishing order at 0 is compensated by being approximable to higher order at 0 on each component of S_μ.

We introduce specific classes in $H^\infty(S_\mu)$. We let $\mathcal{R}^k(S_\mu)$, $k = 1, 2$, be the subclasses of $H^\infty(S_\mu)$ of those ϕ of the form

$$\phi(z) = \sum_{m=1}^{M} c_m(1 + imz)^{-k} \tag{33}$$

for some integer $M \geq 1$ and $c_m \in \mathbb{C}$. Next, for $\sigma > 0$, we define $\mathcal{R}^k_\sigma(S_\mu)$ as the subset of those $\psi \in H^\infty(S_\mu)$ for which there exist $\phi_\pm \in \mathcal{R}^k(S_\mu)$ with

$$|\psi(z) - \phi_\pm(z)| = O(|z|^\sigma), \quad \forall z \in S_{\mu\pm}. \tag{34}$$

We mean here that we may use different approximations of ψ in each sector $S_{\mu+}$ and $S_{\mu-}$. The main example is for us $\psi(z) = [z]e^{-[z]}$. For $z \in S_{\mu+}$, $\psi(z) = ze^{-z}$, so this is the restriction of an analytic function on $\mathbb{C}$ and for any given $\sigma > 1$, it is easy (by solving a finite dimensional linear Vandermonde system) to find $\phi_+ \in \mathcal{R}^1(S_\mu)$

such that $|\psi(z) - \phi_+(z)| = O(|z|^\sigma)$ for $z \in S_{\mu+}$. The same thing can be done in $S_{\mu-}$ but ϕ_- must be different.

THEOREM 5.7. — *Assume that for some q with $\frac{n}{n+1} < q < p_+(DB^*)$ we have*

$$\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$$

with equivalent norms. Let $\psi \in \mathcal{R}^1_\sigma(S_\mu) \cap \Psi^\tau_1(S_\mu)$ with $\sigma > \gamma(q)$ and $\tau > 0$ if $q < 2$, and $\psi \in \Psi^\tau(S_\mu)$ with $\tau > \gamma(q)$ if $q > 2$.

If $q > 1$ and $p = q'$, we have

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^p_2} \sim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in \overline{R_2(BD)},$$

and if $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$,

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^\infty_{2,\alpha}} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{R_2(BD)}.$$

In particular, if $q > 1$, we have the square function estimates

$$\|S(tBDe^{-t|BD|}h)\|_p \sim \|S(t\partial_t e^{-t|BD|}h)\|_p \sim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in \overline{R_2(BD)},$$

and, if $q \leq 1$, the weighted Carleson measure estimates

$$\|tBDe^{-t|BD|}h\|_{T^\infty_{2,\alpha}} \sim \|t\partial_t e^{-t|BD|}h\|_{T^\infty_{2,\alpha}} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{R_2(BD)}.$$

When $(p_-(DB^*))_* < q < p_+(DB^*)$, theorem 5.1 already takes care of the conclusions without the condition $\mathcal{R}^1_\sigma(S_\mu)$. We state it this way for later use. In fact, for the boundary value problems later, we also need tent space estimates for $tDe^{-t|BD|}h$. When B^{-1} exists in L^∞, in particular, this covers the case of second order equations, these results are enough for our needs. But for systems with $B^{-1} \in L^\infty$ not granted, one still has to work a little bit. This result covers both situations.

THEOREM 5.8. — *Assume that for some q with $\frac{n}{n+1} < q < p_+(DB^*)$ we have*

$$\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$$

with equivalent norms. Let $\phi \in \mathcal{R}^2_\sigma(S_\mu) \cap \Psi^\tau_0(S_\mu)$ with $\sigma > \gamma(q), \tau > 1$ if $q < 2$ and $\phi \in \Psi^\tau_0(S_\mu)$ with $\tau > 1 + \gamma(q)$ if $q > 2$.

If $q > 1$ and $p = q'$, we have

$$\|tD\phi(tBD)h\|_{T^p_2} \sim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in \overline{R_2(BD)},$$

and, if $q \leq 1$, and $\alpha = n(\frac{1}{q} - 1)$,

$$\|tD\phi(tBD)h\|_{T^\infty_{2,\alpha}} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{R_2(BD)}.$$

In particular, if $q > 1$, we have the square function estimate

$$\|S(tDe^{-t|BD|}h)\|_p \sim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in \overline{R_2(BD)},$$

and, if $q \leq 1$, the weighted Carleson measure estimate

$$\|tDe^{-t|BD|}h\|_{T^\infty_{2,\alpha}} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{R_2(BD)}.$$

Compare the conclusions of the last two theorems: in one case we have $t|BD|e^{-t|BD|}$ and $tBDe^{-t|BD|}$; in the other we have $tDe^{-t|BD|}$. So we have cancelled B. Other conditions on ϕ suffice for this theorem to hold. We shall stop here the search on such conditions.

5.1. Proof of theorem 5.1

5.1.1. Upper bounds. — We begin with upper bounds separating the cases $p > 2$ and $p < 2$. The case $p = 2$ is, of course, contained in proposition 3.2.

PROPOSITION 5.9. — *For $T = DB$ or BD, $2 < p < p_+(DB)$ and $\psi \in \Psi(S_\mu)$, it holds*

$$\|\mathbb{Q}_{\psi,T} h\|_{T_2^p} \lesssim \|h\|_p, \quad \forall h \in \overline{\mathcal{R}_2(T)}.$$

Proof. — It is well known (see [**87**]) that for $p > 2$

$$\|\mathbb{Q}_{\psi,T} h\|_{T_2^p} \lesssim \Big\| \Big(\int_0^\infty |\psi(tT)h|^2 \,\frac{dt}{t} \Big)^{1/2} \Big\|_p.$$

Then we use (17). □

REMARK 5.10. — Observe that the inequality $\|\mathbb{Q}_{\psi,T} h\|_{T_2^p} \lesssim \|h\|_p$ holds for $h \in L^p \cap L^2$ for p in the above range. Indeed, $h = h_N + h_R$ where h_N is in the null part of T and h_R in the closure of the range of T. We have $\mathbb{Q}_{\psi,T} h_N = 0$ and the inequality applies to h_R. As $h_R = \mathbb{P}_T h$ and the projection is bounded on L^p by the kernel/range decomposition, $\|h_R\|_p \lesssim \|h\|_p$.

Now the main estimate is the following:

THEOREM 5.11. — *For $(p_-(DB))_* < p < 2$ and $\psi \in \Psi^{\gamma(p)}(S_\mu)$, it holds*

$$\|\mathbb{Q}_{\psi,DB} h\|_{T_2^p} \lesssim \|h\|_p, \quad \forall h \in \overline{\mathcal{R}_2(D)}. \tag{35}$$

The proof is quite long and will be divided in two cases:

$$(p_-(DB))_* > 1 \quad \text{and} \quad (p_-(DB))_* \le 1.$$

In the first case, we go via weak type estimates and extend an argument of [**63**] to square functions. In the second case, we use atomic theory.

We remark that, thanks to the equivalence of norms, it is enough to show the inequality for $\psi \in \Psi_\sigma^\tau(S_\mu)$ for σ, τ as large as one needs. We shall do this and we will not try to track their precise values.

To treat the first case and, in fact, exponents $1 < p < 2$, we show the following extrapolation lemma. It is convenient to use the notation

$$S_{\psi,DB} h = S(\mathbb{Q}_{\psi,DB} h),$$

where S is the square function defined in (22) with $a = 1$ so that

$$\|S_{\psi,DB} h\|_p \sim \|\mathbb{Q}_{\psi,DB} h\|_{T_2^p}.$$

Recall also that the homogeneous Sobolev space $\dot{W}^{1,q}$ is the closure of the inhomogeneous Sobolev space $W^{1,q}$ for the semi-norm $\|u\|_{\dot{W}^{1,q}} = \|\nabla u\|_q < \infty$.

The following is implicit in [**63**].

Lemma 5.12. — *Let $1 < q < \infty$. Then $h \in \overline{\mathsf{R}_2(D)} \cap \overline{\mathsf{R}_q(D)}$ if and only if*

$$h = Du$$

for some $u \in \dot{W}^{1,2} \cap \dot{W}^{1,q}$ with $\|h\|_q \sim \|\nabla u\|_q$ and $\|h\|_2 \sim \|\nabla u\|_2$.

Proof. — Let $h \in \overline{\mathsf{R}_2(D)} \cap \overline{\mathsf{R}_q(D)}$. Let

$$h_k = \Big(I + \frac{i}{k}D\Big)^{-1} h - (I + ikD)^{-1}h = Du_k, \quad k \geq 1,$$

where $u_k = i(k - \frac{1}{k})(I + ikD)^{-1}(I + \frac{i}{k}D)^{-1}h$. We have $u_k \in \mathsf{D}_2(D) \cap \mathsf{D}_q(D)$, $u_k \in \overline{\mathsf{R}_2(D)} \cap \overline{\mathsf{R}_q(D)}$ as resolvents preserve the closure of the range. Also $h_k \in \mathsf{R}_2(D) \cap \mathsf{R}_q(D)$ and h_k converges to h is both L^2 and L^q topologies (see section 2.3). Using the coercivity property of D, we have

$$\|\nabla(u_k - u_\ell)\|_q \lesssim \|D(u_k - u_\ell)\|_q = \|h_k - h_\ell\|_q$$

and similarly in L^2. Taking limits of the Cauchy sequences, we obtain $u \in \dot{W}^{1,2} \cap \dot{W}^{1,q}$ with the required property. Conversely, let $u \in \dot{W}^{1,2} \cap \dot{W}^{1,q}$ such that $\|Du\|_q \sim \|\nabla u\|_q$ and $\|Du\|_2 \sim \|\nabla u\|_2$. Then, one can find $u_k \in W^{1,2} \cap W^{1,q}$ such that ∇u_k converges to ∇u in both L^2 and L^q topologies. Thus, $Du_k \in \mathsf{R}_2(D) \cap \mathsf{R}_q(D)$ converges to Du in both L^2 and L^q topologies, so that $Du \in \overline{\mathsf{R}_2(D)} \cap \overline{\mathsf{R}_q(D)}$. □

Armed with this lemma, the inequality (35) is equivalent to

$$\|S_{\psi,DB}Du\|_q \lesssim \|u\|_{\dot{W}^{1,q}}, \quad \forall u \in \dot{W}^{1,2} \cap \dot{W}^{1,q}.$$

Lemma 5.13. — *Let $p_-(DB) < q < 2$. Fix $\psi \in \Psi_\sigma^\tau(S_\mu)$ with $\sigma, \tau \gg 1$ as needed. If*

$$\|S_{\psi,DB}Du\|_q \lesssim \|u\|_{\dot{W}^{1,q}}, \quad \forall u \in \dot{W}^{1,2} \cap \dot{W}^{1,q}.$$

then for $\max(1, q_*) < p < q$, *one has*

$$\|S_{\psi,DB}Du\|_p \lesssim \|u\|_{\dot{W}^{1,p}}, \quad \forall u \in \dot{W}^{1,2} \cap \dot{W}^{1,p}.$$

Let us conclude (35) from this. By lemma 5.12, if $u \in \dot{W}^{1,2}$, then

$$h = Du \in \overline{\mathsf{R}_2(D)},$$

so that the inequality holds for $q = 2$ by H^∞-calculus. Then one can iterate lemma 5.13 at most a finite number of times to obtain the inequality when $\max(1, (p_-(DB))_*) < p < 2$. Applying lemma 5.12 yields the inequality (35) for all such p.

Proof of lemma 5.13. — It is enough to show the weak type estimate

$$\|S_{\psi,DB}Du\|_{p,\infty} \lesssim \|u\|_{\dot{W}^{1,p}} \tag{36}$$

for $u \in \dot{W}^{1,2} \cap \dot{W}^{1,p}$. Indeed, one can use N. Badr's theorem [**30**], which says that the homogeneous Sobolev spaces have the real interpolation property, and interpolate with the inequality at $p = q$ for the sublinear operator $u \mapsto S_{\psi,DB}Du$.

To prove (36), we use the Calderón-Zygmund decomposition of Sobolev functions in [**7**], extended straight forwardly to C^N-valued functions.

Fix $\lambda > 0$ and $u \in \dot{W}^{1,2} \cap \dot{W}^{1,p}$. Choose a collection of cubes (Q_j), (vector-valued) functions g and b_j such that $u = g + \sum_j b_j$ and the following properties hold:

$$\| \nabla g\|_{L^\infty} \leq C\lambda, \tag{37}$$

$$b_j \in W_0^{1,p}\left(Q_j, \mathbb{C}^N\right) \text{ and } \int_{Q_j} |\nabla b_j|^p \leq C\lambda^p |Q_j|, \tag{38}$$

$$\sum_j |Q_j| \leq C\lambda^{-p} \int_{\mathbb{R}^n} |\nabla u|^p, \tag{39}$$

$$\sum_j 1_{Q_j} \leq C', \tag{40}$$

where C and C' depend only on dimension and p. Remark that (38), Sobolev-Poincaré inequality with a real r such that $p \leq r \leq p^*$, and in particular $r = q$, gives us

$$\|b_j\|_r \lesssim |Q_j|^{\frac{1}{r}-\frac{1}{p}+\frac{1}{n}} \| \nabla b_j\|_p \lesssim \lambda |Q_j|^{\frac{1}{r}+\frac{1}{n}}. \tag{41}$$

Also, we note that the bounded overlap (40) implies that

$$\| \sum_j \nabla b_j\|_p + \|\nabla g\|_p \lesssim \|\nabla u\|_p,$$

hence for all $r \geq p$,

$$\lambda^{-r} \|\nabla g\|_r^r \lesssim \lambda^{-p} \|u\|_{\dot{W}^{1,p}}^p. \tag{42}$$

In particular, this holds for $r = 2$ so that we also have the qualitative bound $\| \sum_j \nabla b_j\|_2 + \|\nabla g\|_2 < \infty$ and the decomposition is also in $\dot{W}^{1,2}$ (It follows from the construction that $b_j \in W^{1,2}$ for each j).

Introduce for some integer $M > 1$, chosen large enough in the course of the argument,

$$\varphi(z) := \sum_{m=0}^{M} \binom{M}{m} (-1)^m (1 + imz)^{-1} \in H^\infty(S_\mu)$$

as in [**63**, section 4]. This function satisfies $|\varphi(z)| \lesssim \inf(|z|^M, 1)$. We decompose

$$u = g + \tilde{g} + b,$$

where $\tilde{g} := \sum_j (I - \varphi(\ell_j BD))b_j$ and $b = \sum_j \varphi(\ell_j BD)b_j$ with $\ell_j := \ell(Q_j)$. As usual, the set $\{S_{\psi,DB}Du > 3\lambda\}$ is contained in the union of

$$A_1 = \{S_{\psi,DB}Dg > \lambda\}, \quad A_2 = \{S_{\psi,DB}D\tilde{g} > \lambda\}, \quad A_3 = \{S_{\psi,DB}Db > \lambda\}.$$

For A_1 we use the hypothesis and (42), and

$$|A_1| \lesssim \lambda^{-q}\|\nabla g\|_q^q \lesssim \lambda^{-p}\|u\|_{\dot{W}^{1,p}}^p.$$

For A_2, we use also the hypothesis but in the form (35) to get

$$|A_2| \lesssim \lambda^{-q}\|D\tilde{g}\|_q^q \lesssim \lambda^{-p}\|u\|_{\dot{W}^{1,p}}^p.$$

For the last inequality, notice that $\|D\tilde{g}\|_q \lesssim \|BD\tilde{g}\|_q$ as $q \in I(BD)$ and

$$BD\tilde{g} = \sum_{m=1}^{M} c_m \sum_j (I + im\ell_j BD)^{-1} BDb_j,$$

so that the inequality is shown in [**63**, section 4.1].

The main new part compared to [**63**] is the treatment of the set A_3, for which we follow, in part, [**15**]. As usual, since $|\cup 4Q_j|$ is under control from (39), it is enough to control the measure of $\widetilde{A}_3 = \{S_{\psi,DB}Db > \lambda\} \cap F$ where $F = \mathbb{R}^n \setminus \bigcup 4Q_j$. We use then the L^2 Markov inequality

$$|\widetilde{A}_3| \leq \lambda^{-2}\int_F |S_{\psi,DB}Db|^2 = \lambda^{-2}\iint |\psi(tDB)Db(y)|^2 \frac{|B(y,t)\cap F|}{t^n}\frac{dydt}{t}.$$

We decompose $\psi(tDB)Db(y) = f_{\mathrm{loc}}(t,y) + f_{\mathrm{glob}}(t,y)$ with

$$f_{\mathrm{loc}}(t,y) = \sum_j 1_{2Q_j}(y)\psi(tDB)D\varphi(\ell_j BD)b_j(y)$$

and

$$f_{\mathrm{glob}}(t,y) = \sum_j 1_{(2Q_j)^c}(y)\psi(tDB)D\varphi(\ell_j BD)b_j(y).$$

Let us call I_{loc} and I_{glob} the integrals obtained. We begin with the estimate of I_{loc}. If $y \in 2Q_j$ and $t \leq 2\ell_j$ then $B(y,t) \subset 4Q_j$, hence $B(y,t) \cap F = \varnothing$. Thus, in I_{loc} we may replace f_{loc} by

$$\tilde{f}_{\mathrm{loc}}(t,y) = \sum_j 1_{2Q_j}(y)1_{(2\ell_j,\infty)}(t)\psi(tDB)D\varphi(\ell_j BD)b_j(y).$$

At this point we dualize against H with $\iint |H(t,y)|^2 \frac{dydt}{t} = 1$, so that using Fubini's theorem and Cauchy-Schwarz inequality

$$I_{\mathrm{loc}}^{1/2} \lesssim \iint \tilde{f}_{\mathrm{loc}}(t,y)\overline{H(t,y)}\frac{dydt}{t} \leq \sum_j I_j |Q_j|^{1/2} \inf_{x\in Q_j} \boldsymbol{M}_2\widetilde{H}(x),$$

where

$$I_j^2 := \int_{2\ell_j}^{\infty}\int_{\mathbb{R}^n} |\psi(tDB)D\varphi(\ell_j BD)b_j(y)|^2 \frac{dydt}{t},$$

$$\widetilde{H}(y)^2 := \int_0^\infty |H(t,y)|^2 \,\frac{dt}{t}, \quad \boldsymbol{M}_2\widetilde{H} := (\boldsymbol{M}|\widetilde{H}|^2)^{1/2}$$

and $\boldsymbol{M}$ is the Hardy-Littlewood maximal operator. We have

$$\psi(tDB)D\varphi(\ell_j BD)b_j = D\psi(tBD)\varphi(\ell_j BD)b_j$$

(remark that $b_j \in L^2$, so we can use functional calculus and the commutation holds) and using the accretivity of B and (19)

$$\|D\psi(tBD)\varphi(\ell_j BD)b_j\|_2 \lesssim \|BD\psi(tBD)\varphi(\ell_j BD)b_j\|_2 \lesssim t^{-1}t^{\frac{n}{2}-\frac{n}{q}}\|b_j\|_q.$$

It follows easily using (38) that

$$I_j \lesssim \lambda|Q_j|^{1/2}.$$

It is classical from Kolmogorov's inequality and the weak type (1,1) of $\boldsymbol{M}$ that

$$\sum_j |Q_j| \inf_{x\in Q_j} \boldsymbol{M}_2\widetilde{H}(x) \lesssim \|\widetilde{H}\|_2^{1/2}|\cup Q_j|^{1/2} = |\cup Q_j|^{1/2}. \tag{43}$$

Altogether, we conclude using (39) that

$$\lambda^{-2}I_{\mathrm{loc}} \lesssim |\cup Q_j| \lesssim \lambda^{-p}\|u\|^p_{\dot{W}^{1,p}}.$$

We next turn to I_{glob}. Using the same dualization argument, we have for some H as above,

$$I_{\mathrm{glob}}^{1/2} \lesssim \sum_{j,r\geq 1} I_{j,r}2^{rn/2}|Q_j|^{1/2} \inf_{x\in Q_j} \boldsymbol{M}_2\widetilde{H}(x),$$

where

$$I_{j,r}^2 := \int_0^\infty \int_{S_r(2Q_j)} |\psi(tDB)D\varphi(\ell_j BD)b_j(y)|^2 \,\frac{dydt}{t},$$

and we use the notation $S_r(Q)$ introduced for molecules. Since the integrals are localized we cannot use the same argument as before by using the accretivity of B on the range. Nevertheless, we prove a local version in the following lemma, which will be used many times later on.

Lemma 5.14 (Local coercivity inequality). — *For any $u \in L^2_{\mathrm{loc}}$ with $Du \in L^2_{\mathrm{loc}}$, any ball $B(x,r)$ in $\mathbb{R}^n$ and $c > 1$,*

$$\int_{B(x,r)} |Du|^2 \lesssim \int_{B(x,cr)} |BDu|^2 + r^{-2}\int_{B(x,cr)} |u|^2, \tag{44}$$

with the implicit constant depending only on the ellipticity constants of B, dimension, N and c.

We postpone the proof of the lemma. As

$$\psi(tDB)D\varphi(\ell_j BD)b_j = D\psi(tBD)\varphi(\ell_j BD)b_j,$$

we can apply it to $u_j = \psi(tBD)\varphi(\ell_j BD)b_j$, which leads to bound $I_{j,r}^2$ by two integrals with slightly larger regions $\widetilde{S}_r(2Q_j)$ of the same type as $S_r(2Q_j)$ and with integrands $|BDu_j|^2$ and $|(2^{-r}\ell_j)^{-1}u_j|^2$ respectively. We then truncate both integrals at ℓ_j.

For $t \leq \ell_j$, using the $L^q - L^2$ off-diagonal estimate (19) (which requires τ large enough),

$$\int_{\widetilde{S}_r(2Q_j)} |BDu_j(y)|^2\, dy \lesssim t^{-2} t^{\frac{2n}{2}-\frac{2n}{q}} \langle 2^r \ell_j/t\rangle^{-2\sigma c} \|b_j\|_q^2$$

which, using (41), leads to

$$\int_0^{\ell_j} \int_{\widetilde{S}_r(2Q_j)} |BDu_j(y)|^2\, \frac{dydt}{t} \lesssim \ell_j^{-2+\frac{2n}{2}-\frac{2n}{q}} 2^{-2r\sigma c} \|b_j\|_q^2 \lesssim 2^{-2r\sigma c} \lambda^2 |Q_j|.$$

The argument for $(2^{-r}\ell_j)^{-1}u_j$ replacing BDu_j is the same if $q < 2$ and leads to a similar estimate with $1+\sigma c$ in place of σc. If $q = 2$, we may use an $L^s - L^2$ estimate for some $s < 2$ instead and (41) for $\|b_j\|_s$.

When $t \geq \ell_j$, we deduce from (20) (provided τ is large enough)

$$\int_{\widetilde{S}_r(2Q_j)} |BDu_j(y)|^2\, dy \lesssim t^{-2} t^{\frac{2n}{2}-\frac{2n}{q}} \langle 2^r \ell_j/\ell_j\rangle^{-2Mc} \|b_j\|_q^2$$

and then

$$\int_{\ell_j}^{\infty} \int_{\widetilde{S}_r(2Q_j)} |BDu_j(y)|^2\, \frac{dydt}{t} \lesssim 2^{-2rMc} \lambda^2 |Q_j|.$$

The argument for $(2^{-r}\ell_j)^{-1}u_j$ replacing BDu_j is the same if $q < 2$ and leads to a similar estimate with $1+Mc$ in place of Mc. If $q = 2$, we may use an $L^s - L^2$ estimate for some $s < 2$ instead and (41) for $\|b_j\|_s$.

In total, we obtain an estimate

$$I_{j,r} \lesssim \sum_{j,r\geq 1} 2^{-rK} \lambda |Q_j|^{1/2},$$

where K can be arbitrary large (upon choosing σ, M large) so that using (43)

$$I_{\text{glob}}^{1/2} \lesssim \sum_{j,r\geq 1} \lambda 2^{r(\frac{n}{2}-K)} |Q_j| \inf_{x\in Q_j} \boldsymbol{M}_2 \widetilde{H}(x) \lesssim \lambda |\cup Q_j|^{1/2}$$

and the desired conclusion follows. □

Proof of lemma 5.14. — For this inequality, we let χ be a scalar-valued cut-off function with $\chi = 1$ on $B(x,r)$, supported in $B(x,cr)$ and with $\|\nabla\chi\|_\infty \lesssim r^{-1}$. As $\chi u \in \mathsf{D}(D)$ and using that the commutator between χ and D is the pointwise multiplication by a matrix with bound controlled by $|\nabla\chi|$,

$$\int_{B(x,r)} |Du|^2 \leq \int_{\mathbb{R}^n} |\chi Du|^2 \lesssim \int_{\mathbb{R}^n} |D(\chi u)|^2 + \int_{\mathbb{R}^n} |\nabla\chi|^2 |u|^2 .$$

Since B is accretive on $\mathsf{R}_2(D)$, we have $\int_{\mathbb{R}^n} |D(\chi u)|^2 \lesssim \int_{\mathbb{R}^n} |BD(\chi u)|^2$. Now, we use again the commutation between χ and D together with $\|B\|_\infty$. This proves (44). □

To continue the proof of theorem 5.11, we have to consider the case $p \leq 1$, which occurs only when $(p_-)_* < 1$. In this case, it is enough to consider a $(\mathbb{H}^p_D, 1)$-atom

$a = Db$ with a, b supported in a cube Q and show that $\|S_{\psi,DB}a\|_p \lesssim 1$ uniformly for some $\psi \in \Psi_\sigma^\tau(S_\mu)$ with σ, τ as large as one needs.

As usual the local term is handled by the L^2 bound

$$\|S_{\psi,DB}a\|_{L^p(4Q)} \leq |4Q|^{\frac{1}{p}-\frac{1}{2}}\|S_{\psi,DB}a\|_{L^2(4Q)} \lesssim |4Q|^{\frac{1}{p}-\frac{1}{2}}\|a\|_2 \lesssim 1.$$

Next, for the non-local term, we remark that if $x \notin 4Q$ and $t \in (0,\infty)$, then

$$\langle \mathrm{dist}(B(x,2t),Q)/t\rangle \geq C\langle \mathrm{dist}(x,Q)/t\rangle.$$

Using $\psi(tDB)a = \psi(tDB)Db = D\psi(tBD)b$, the local coercivity inequality (44) and L^q-L^2 off-diagonal estimates (19) (provided τ is large enough), we have

$$\begin{aligned}\|\psi(tDB)a\|_{L^2(B(x,t))} &\lesssim \|BD\psi(tBD)b\|_{L^2(B(x,2t))} + t^{-1}\|\psi(tBD)b\|_{L^2(B(x,2t))}\\ &\lesssim t^{-1}t^{\frac{n}{2}-\frac{n}{q}}\langle \mathrm{dist}(x,Q)/t\rangle^{-K}\|b\|_q,\end{aligned}$$

where K can taken as large as one wants upon taking σ large, and one chooses q with $p_- < q < p^*$ and $q \leq 2$, which is possible as $(p_-)_* < p \leq 1$. Thus, for $x \notin 4Q$

$$S_{\psi,DB}a(x) \lesssim (d(x,Q))^{-1-\frac{n}{q}}\|b\|_q.$$

As $q < p^*$, it follows that $1 + \frac{n}{q} > \frac{n}{p}$, so one can integrate the pth power and get

$$\|S_{\psi,DB}a\|_{L^p((4Q)^c)} \lesssim \ell(Q)^{-1-\frac{n}{q}+\frac{n}{p}}\|b\|_q \lesssim 1,$$

where the last inequality is merely Hölder's inequality and $\|b\|_2 \lesssim \ell(Q)^{1+\frac{n}{2}-\frac{n}{p}}$.

We have obtained all the upper bounds in theorem 5.1. We complete the proof by proving the lower bounds.

5.1.2. Lower bounds. — Those have already been obtained in proposition 4.17 for $\frac{n}{n+1} < p < 2$ and we remark that $(p_-)_* > \frac{n}{n+1}$. It remains to see them for $2 < p < p_+$. We have seen in proposition 4.8 that for all $h \in \overline{\mathsf{R}_2(D)}$ and $g \in \overline{\mathsf{R}_2(B^*D)}$ and any $\psi, \varphi \in \Psi(S_\mu)$ for which the Calderón reproducing formula (23) holds, one has

$$|\langle h, g\rangle| \leq \|\mathbb{Q}_{\psi,DB}h\|_{T_2^p} \cdot \|\mathbb{Q}_{\varphi^*,B^*D}g\|_{T_2^{p'}}.$$

Now, we have $\varphi^*(tB^*D)g = B^*\varphi^*(tDB^*)(B^*)^{-1}g$. Using that B^* is bounded, $p' \in I(DB^*) = I(BD)'$ since $p \in I(BD)$ and B^* is an isomorphism from $\overline{\mathsf{R}_{p'}(D)}$ onto $\overline{\mathsf{R}_{p'}(B^*D)}$,

$$\|\mathbb{Q}_{\varphi^*,B^*D}g\|_{T_2^{p'}} \lesssim \|\mathbb{Q}_{\varphi^*,DB^*}(B^*)^{-1}g\|_{T_2^{p'}} \lesssim \|(B^*)^{-1}g\|_{p'} \sim \|g\|_{p'}, \tag{45}$$

provided φ is allowable for $\mathbb{H}^{p'}_{DB^*}$ which is the case if we choose, as we may, $\varphi \in \Psi^{\gamma(p')}(S_\mu)$. Thus

$$|\langle h, g\rangle| \leq \|\mathbb{Q}_{\psi,DB}h\|_{T_2^p} \cdot \|g\|_{p'}.$$

Now, from [**27**], proposition 2.1, (5), $\overline{\mathsf{R}_p(D)}$ and $\overline{\mathsf{R}_{p'}(B^*D)}$ are dual spaces for the L^2 pairing: this and a density argument yield $\|h\|_p \lesssim \|\mathbb{Q}_{\psi,DB}h\|_{T_2^p}$. This completes the proof of theorem 5.1.

5.2. Proof of theorem 5.3

Before we move to the proof, let us explain the ranges of p and α. In theorem 5.1, the range for q for $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ is $(p_-(DB^*))_* < q < p_+(DB^*)$. But

$$p_+(DB^*)' = p_-(BD) \quad \text{and} \quad p_-(DB^*)' = p_+(BD),$$

so this is the range $(p_+(BD)')_* < q < p_-(BD)'$. If $p_+(BD) \leq n$, we have $(p_+(BD)')_* = (p_+(DB^*)^*)'$ (with $n^* = \infty$ by convention). If $p_+(BD) > n$, then we obtain the range $[0, \alpha(BD))$ with $\alpha(BD) = n(\frac{1}{(p'_+(BD))_*} - 1) = 1 - \frac{n}{p_+(BD)}$. In all, we obtain the ranges for p and α specified in the statement.

5.2.1. Lower bounds. — The lower bounds of the tent space norms $\|\mathbb{Q}_{\varphi,BD}h\|_{\mathcal{T}}$ by norms on $\mathbb{P}h$ is a modification of the arguments in proposition 4.9. For example, for $p = q'$ and $q > 1$, take $\psi, \varphi \in \Psi(S_\mu)$ for which the Calderón reproducing formula (23) holds. then

$$\begin{aligned}
\|\mathbb{P}g\|_p &= \sup\{|\langle \mathbb{P}g, f\rangle| \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\sim \sup\{|\langle g, f\rangle| \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\leq \sup\{\|\mathbb{Q}_{\psi,BD}f\|_{T^p_2}\|\mathbb{Q}_{\varphi^*,DB^*}g\|_{T^q_2} \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\lesssim \sup\{\|\mathbb{Q}_{\psi,BD}g\|_{T^p_2}\|f\|_q \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\lesssim \|\mathbb{Q}_{\psi,BD}g\|_{T^p_2}.
\end{aligned}$$

The fourth line holds provided we also choose φ allowable for $\mathbb{H}^q_{DB^*}$ while ψ can be arbitrary.

The same argument holds when $q \leq 1$, working in the Hölder spaces $\mathbb{L}^\alpha_{BD}$ and $\mathbb{L}^\alpha_D$ and corresponding tent space $T^\infty_{2,\alpha}$.

5.2.2. Upper bounds. — For $p_- < p < 2$, we have just seen the desired upper bound in (45) up to changing p' to p and B^* to B. Proposition 5.9 takes care of the case $2 < p < p_+$.

Next, we consider the case $p_+ \leq p < (p_+)^*$. We adapt an argument of [**15**] which works for both BD or DB. Let $\psi \in \Psi^\tau_\sigma(S_\mu)$ with $\sigma > 0$ and $\tau > 0$. Recall that $[z] = \operatorname{sgn}(z)z$. Consider for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$,

$$\psi_\alpha(z) = \frac{[z]^{\alpha-\sigma}}{(1+[z])^{\alpha-\sigma}}\ \psi(z).$$

Remark that

$$\frac{[z]^\alpha}{(1+[z])^\alpha} = (1+[z]^{-1})^{-\alpha}$$

and since $z \in S_\mu$ implies $[z], [z]^{-1}, 1+[z]^{-1} \in S_{\mu+}$, we have that

$$\sup_{z\in S_\mu}\left|\frac{[z]^\alpha}{(1+[z])^\alpha}\right| \leq e^{\mu|\operatorname{Im}\alpha|}.$$

It follows that $\psi_\alpha \in \Psi^{\tau}_{\operatorname{Re}\alpha}(S_\mu)$ with

$$|\psi_\alpha(z)| \le Ce^{\mu|\operatorname{Im}\alpha|}\inf(|z|^{\operatorname{Re}\alpha}, |z|^{-\tau}).$$

Clearly, the map $\alpha \mapsto \psi_\alpha$ is analytic from $\operatorname{Re}\alpha > 0$ to $\Psi(S_\mu)$ with $\psi = \psi_\sigma$.

For $T = DB$ of BD, set

$$Q_\alpha f = \mathbb{Q}_{\psi_\alpha, T} f = (\psi_\alpha(tT)f)_{t>0}, \quad f \in L^2.$$

Thus Q_α is an analytic family of bounded operators from L^2 to T^2_2 with

$$\|Q_\alpha f\|_{T^p_2} \lesssim e^{\mu|\operatorname{Im}\alpha|}\|f\|_2.$$

In the statements below, implicit or explicit constants C are allowed to depend on the real part of α but not on its imaginary part.

Lemma 5.15. — *For $\operatorname{Re}\alpha > 0$, Q_α maps $L^p \cap L^2$ to T^p_2 when $2 \le p < p_+$ with*

$$\|Q_\alpha f\|_{T^p_2} \lesssim e^{\mu|\operatorname{Im}\alpha|}\|f\|_p.$$

Proof. — This is a reformulation of proposition 5.9 together with the remark that follows it. We note that the control of the norm with $e^{\mu|\operatorname{Im}\alpha|}$ comes from examination of the proof of Le Merdy's theorem [**75**] to get (17). □

Lemma 5.16. — *For $\operatorname{Re}\alpha > \frac{n}{p_+(T)}$, Q_α maps $L^p \cap L^2$ to T^p_2 when $2 \le p \le \infty$ with*

$$\|Q_\alpha f\|_{T^p_2} \lesssim e^{\mu|\operatorname{Im}\alpha|}\|f\|_p.$$

Proof. — For fixed α it is enough to consider the case $p = \infty$ as one can then complex interpolate from [**40**] between T^2_2 and T^∞_2. We claim that for any $2 < q < p_+$, and any ball B_r of $\mathbb{R}^n$, with radius r, setting $\Omega = (0, r) \times B_r$,

$$\text{(46)} \quad \Big(\frac{1}{|B_r|}\iint_\Omega |\psi_\alpha(tT)f(x)|^2\,\frac{dtdx}{t}\Big)^{1/2} \le Ce^{\mu|\operatorname{Im}\alpha|}\sum_{j=1}^{\infty} 2^{-j\,(\operatorname{Re}\alpha - \frac{n}{q})}\Big(\fint_{2^jB_r}|f|^q\Big)^{1/q}.$$

Admitting this claim, the right hand side is dominated by the L^∞ norm of f by using $\operatorname{Re}\alpha > \frac{n}{p_+}$ and choosing $q < p_+$ appropriately. Then the supremum over all B_r of the left hand side is precisely the T^∞_2 norm of $Q_\alpha f$.

To prove the claim, we write

$$f = f_{\text{loc}} + f_{\text{glob}},$$

where $f_{\text{loc}} = f\,1_{4B_r}$. Then, using the $L^2 - T^2_2$ boundedness of Q_α,

$$\begin{aligned}\frac{1}{|B_r|}\iint_\Omega |\psi_\alpha(tT)f_{\text{loc}}(x)|^2\,\frac{dtdx}{t} &\le \frac{1}{|B_r|}\int_{\mathbb{R}^n}\Big(\int_0^\infty |\psi_\alpha(tT)f_{\text{loc}}(x)|^2\,\frac{dt}{t}\Big)\,dx \\ &\lesssim \frac{1}{|B_r|}\int_{\mathbb{R}^n}|f_{\text{loc}}|^2 \lesssim \fint_{4B_r}|f|^2.\end{aligned}$$

It is then enough to show

$$\Big(\fint_{B_r} |\psi_\alpha(tT)f_{\mathrm{glob}}|^2\Big)^{1/2} \le Ce^{\mu|\operatorname{Im}\alpha|}\,\frac{t^{\operatorname{Re}\alpha}}{r^{\operatorname{Re}\alpha}}\sum_{j=2}^{\infty} 2^{-j\,(\operatorname{Re}\alpha-\frac{n}{q})}\Big(\fint_{2^{j+1}B_r}|f|^q\Big)^{1/q}.$$

Indeed, plugging this estimate in the integral on the Carleson region Ω, we obtain the claim.

To this end, we set $f_j = f\,1_{S_j(B_r)}$, so that $f_{\mathrm{glob}} = \sum_{j\ge 3} f_j$ and by Minkowski's and Hölder's inequalities

$$\Big(\fint_{B_r} |\psi_\alpha(tT)f_{\mathrm{glob}}|^2\Big)^{1/2} \le \sum_{j\ge 3}\Big(\fint_{B_r}|\psi_\alpha(tT)f_j|^q\Big)^{1/q}.$$

Fix $j\ge 3$ and use (19) with $p=q$ to obtain

$$\Big(\fint_{B_r}|\psi_\alpha(tT)f_j|^q\Big)^{1/q} \le Ce^{\mu|\operatorname{Im}\alpha|}\,\frac{t^{\operatorname{Re}\alpha}}{r^{\operatorname{Re}\alpha}}\,2^{-j\,(\operatorname{Re}\alpha-\frac{n}{q})}\Big(\fint_{2^{j+1}B_r}|f|^q\Big)^{1/q}.$$

The claim is proved. □

Lemma 5.17. — *For $0<\operatorname{Re}\alpha\le \frac{n}{p_+}$, Q_α maps $L^p\cap L^2$ to T^p_2 when $2\le p<\frac{np_+}{n-p_+\operatorname{Re}\alpha}$ with*

$$\|Q_\alpha f\|_{T^p_2}\lesssim e^{\mu|\operatorname{Im}\alpha|}\|f\|_p.$$

Proof. — This is *verbatim* the interpolation argument in [**15**]. □

Corollary 5.18. — *For $p_+\le p$ and $\psi\in\Psi_{\frac{n}{p_+}-\frac{n}{p}}(S_\mu)$, then $\mathbb{Q}_{\psi,T}$ maps $L^p\cap L^2$ to T^p_2 with*

$$\|\mathbb{Q}_{\psi,T}f\|_{T^p_2}\lesssim\|f\|_p.$$

In particular, if $\psi\in\Psi_1(S_\mu)$, then $\mathbb{Q}_{\psi,T}$ maps $L^p\cap L^2$ to T^p_2 when $2\le p<(p_+(T))^*$.

Proof. — This is an easy consequence of the previous construction when σ is any number larger than $\frac{n}{p_+}-\frac{n}{p}$ to start with. We leave details to the reader. □

This corollary proves the part of theorem 5.3 that concerns upper bounds for $T=BD$ and $2<p<(p_+)^*$.

To finish the proof of theorem 5.3, it suffices to prove the following stronger result.

Proposition 5.19. — *If $p_+=p_+(BD)>n$, then for $0\le\alpha<1-\frac{n}{p_+}$,*

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^\infty_{2,\alpha}}\lesssim\|h\|_{\dot\Lambda^s},\quad \forall h\in\dot\Lambda^\alpha\cap L^2,$$

when $\psi\in\bigcup_{\sigma>\alpha+\frac{n}{p_+},\tau>0}\Psi^\tau_\sigma(S_\mu)$ and in particular for $\psi\in\Psi_1(S_\mu)$.

Proof. — We observe that (46) applies to ψ replacing $\operatorname{Re}\alpha$ by σ and in the right hand side h by $h-c$ where c is any constant. Indeed, constants are annihilated by BD, or more concretely $\psi(tBD)c=0$. The action of $\psi(tBD)$ on L^∞ is guaranteed by corollary 3.14 applied with q close to p_+ and $\sigma>\frac{n}{p_+}$. Thus the left hand side of (46) remains the same. Now, we choose c to be the mean value of f on B_r. When $h \in \dot\Lambda^\alpha$, a telescoping argument yields

$$\Big(\fint_{2^{j+1}B_r} |h-c|^q\Big)^{1/q} \lesssim \|h\|_{\dot\Lambda^\alpha} 2^{j\alpha} r^\alpha.$$

Thus the series in j converges as long as $\sigma - \frac{n}{q} + \alpha > 0$, which is possible since $\sigma > \alpha + \frac{n}{p_+}$ and choosing $q < p_+$ close to p_+, and we obtain the desired conclusion when $\alpha > 0$.

The same argument works for $h \in \mathrm{BMO} = \dot\Lambda^0$, and $2^{j\alpha}$ is replaced by $\ln(j+1)$. $\square$

5.3. Proof of theorem 5.7

5.3.1. Lower bounds. — The argument is the same as for theorem 5.3 in section 5.2.1.

5.3.2. Upper bounds. — We begin with the case $2 < q < p_+(DB^*)$, that is $p_-(BD) < p < 2$. Then $\psi \in \Psi^\tau(S_\mu)$ is allowable for $\mathbb{H}^p_{BD}$ when $\tau > \gamma(p)$, which is the case as $\gamma(q) = \gamma(p)$.

We turn to $q < 2$. We proceed with the following lemma.

Lemma 5.20. — *Let $\phi \in \mathcal{R}^k(S_\mu)$, $k = 1, 2$, with $\phi(0) = 0$. Then for all $2 < p < \infty$*

$$\|\mathbb{Q}_{\phi,BD}h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in L^2,$$

and for all $0 \le \alpha < 1$,

$$\|\mathbb{Q}_{\phi,BD}h\|_{T^\infty_{2,\alpha}} \lesssim \|\mathbb{P}h\|_{\dot\Lambda^\alpha}, \quad \forall h \in L^2.$$

Proof. — The proof is basically the same as for lemma 5.16. Let $h \in L^2$. Fix a ball B_r, with radius r and set $\Omega = (0, r) \times B_r$. Using that we have L^2 off-diagonal decay of any order $N \ge 1$ for the resolvent and its iterates, and $\phi(0) = 0$ so that we have a square function estimate with $\phi(tBD)$, we obtain as in (46)

$$\Big(\frac{1}{|B_r|}\iint_\Omega |\phi(tBD)h(x)|^2\,\frac{dtdx}{t}\Big)^{1/2} \lesssim \sum_{j=1}^\infty 2^{-j\,(N-\frac{n}{2})}\Big(\fint_{2^jB_r}|h|^2\Big)^{1/2}. \tag{47}$$

Taking $N > \frac{n}{2}$, this shows that $\|\mathbb{Q}_{\phi,BD}h\|_{T^\infty_2} \lesssim \|h\|_\infty$ for all $h \in L^\infty \cap L^2$. Interpolating with the $L^2 \to T^2_2$ estimate, we obtain the T^p_2 estimate for all $h \in L^2 \cap L^p$. Since $\phi(0) = 0$,

$$\phi(tBD)h = \phi(tBD)\mathbb{P}h$$

and replacing h by $\mathbb{P}h$, the T^p_2 estimate $\|\mathbb{Q}_{\phi,BD}h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_{L^p}$ holds for $2 < p < \infty$ and $h \in L^2$.

Now, letting $f = \mathbb{P}h - \fint_{B_r}\mathbb{P}h$, we have

$$\phi(tBD)h = \phi(tBD)f.$$

Here we used that $\phi(tBD)$ maps L^∞ to L^2_{loc} and annihilates constants. Applying (47) with f replacing h, and using that

$$\Big(\fint_{2^jB_r}|f|^2\Big)^{1/2}\lesssim 2^{j\alpha}r^\alpha\|\mathbb{P}h\|_{\dot\Lambda^\alpha}$$

if $\alpha>0$ and $\lesssim \ln(1+j)\|\mathbb{P}h\|_{\dot\Lambda^0}$ (with convention $\dot\Lambda^0=\mathrm{BMO}$ if $\alpha=0$) we obtain

$$\Big(\frac{1}{|B_r|}\iint_\Omega |\phi(tBD)h(x)|^2\,\frac{dtdx}{t}\Big)^{1/2}\lesssim\sum_{j=1}^\infty 2^{-j\,(N-1-\frac{n}{2})}r^\alpha\|\mathbb{P}h\|_{\dot\Lambda^\alpha}$$

and we are done provided we choose $N>1+\frac{n}{2}$. □

We turn to prove the upper bounds in theorem 5.7. As we assume $\mathbb{H}^q_{DB^*}=\mathbb{H}^q_D$, corollary 4.21 implies that for $h\in\overline{R_2(BD)}$,

$$\|\mathbb{P}\chi^\pm(BD)h\|_p\lesssim\|\mathbb{P}h\|_p$$

for $p=q'$ if $q>1$ or $\|\mathbb{P}\chi^\pm(BD)h\|_{\dot\Lambda^\alpha}\lesssim\|\mathbb{P}h\|_{\dot\Lambda^\alpha}$ for $\alpha=n(\frac1q-1)$ if $q\le1$.

Next, let $\psi\in\mathcal{R}^1_\sigma(S_\mu)\cap\Psi_1^\tau(S_\mu)$ with $\sigma>\gamma(q)$ and construct $\phi_\pm\in\mathcal{R}^1(S_\mu)$ such that

$$|\psi(z)-\phi_\pm(z)|=O(|z|^\sigma),\quad\forall z\in S_{\mu\pm}.$$

Remark that necessarily, $\phi_\pm(0)=0$. The key point is the following observation: the functions $\psi_\pm=(\psi-\phi_\pm)\chi^\pm\in\Psi_\sigma^{\inf(1,\tau)}(S_\mu)$ and, for $h\in\overline{R_2(BD)}$, using $h=\chi^+(BD)h+\chi^-(BD)h$, we have the decomposition

$$\psi(tBD)h=\psi_+(tBD)h+\psi_-(tBD)h+\phi_+(BD)(\chi^+(BD)h)+\phi_-(BD)(\chi^-(BD)h).$$

Now, the condition $\sigma>\gamma(q)$ implies that $\psi_\pm$ are allowable for $\mathbb{H}^{\mathcal{T}}_{BD}$ where $\mathcal{T}=T^p_2$ if $p=q'$ and for $\mathcal{T}=T^\infty_{2,\alpha}$ if $\alpha=n(\frac1q-1)$. In the case $q=p'$ we deduce from this and lemma 5.20

$$\|\mathbb{Q}_{\psi,BD}h\|_{T^p_2}\lesssim\|\mathbb{P}h\|_p+\|\mathbb{P}h\|_p+\|\mathbb{P}\chi^+(BD)h\|_p+\|\mathbb{P}\chi^-(BD)h\|_p\lesssim\|\mathbb{P}h\|_p.$$

The argument when $q\le1$ and $\alpha=n(\frac1q-1)$ is similar. This completes the proof of the upper bounds in theorem 5.7.

5.4. Proof of theorem 5.8

5.4.1. Lower bounds. — The lower bounds of the tent space norms $\|tD\varphi(tBD)h\|_{\mathcal{T}}$ by norms on $\mathbb{P}h$ is again a modification of the arguments in proposition 4.9. Take ψ,φ for which the Calderón reproducing formula (23) holds. Here we take $\varphi\in H^\infty(S_\mu)$ and $\psi(z)=z\widetilde\psi(z)$ where $\widetilde\psi$ is allowable for $H^q_{DB^*}$. We observe that for $g\in\overline{R_2(BD)}$ and $f\in\overline{R_2(D)}$,

$$\langle g,f\rangle=\int_0^\infty\langle\varphi(tBD)g,tDB^*\widetilde\psi^*(tDB^*)f\rangle\,\frac{dt}{t}=\int_0^\infty\langle tD\varphi(tBD)g,B^*\widetilde\psi^*(tDB^*)f\rangle\,\frac{dt}{t}$$

using the self-adjointness of D.

Now we may proceed as in the proof of theorem 5.7. For $p = q'$ and $q > 1$,

$$\begin{aligned}
\|\mathbb{P}g\|_p &\sim \sup\{|\langle \mathbb{P}g, f\rangle| \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&= \sup\{|\langle g, f\rangle| \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\leq \sup\{\|tD\varphi(tBD)g\|_{T^p_2} \|B^*\|_\infty \|\mathbb{Q}_{\widetilde{\psi}^*, DB^*} f\|_{T^q_2} \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\lesssim \sup\{\|tD\varphi(tBD)g\|_{T^p_2} \|f\|_q \,;\, \|f\|_{\mathbb{H}^q_D} \lesssim 1\} \\
&\lesssim \|tD\varphi(tBD)g\|_{T^p_2}.
\end{aligned}$$

The fourth line holds since we chose $\widetilde{\psi}$ allowable for $\mathbb{H}^q_{DB^*}$.

The same argument holds when $q \leq 1$, working in the Hölder space $\mathbb{L}^\alpha_{BD}$ and corresponding tent space $T^\infty_{2,\alpha}$.

5.4.2. Upper bounds. — We begin with the case

$$2 < q < p_+(DB^*),$$

that is $p_-(BD) < p < 2$ and $\phi \in \Psi^\tau_0(S_\mu)$. Now, for $h \in \overline{\textsf{R}_2(BD)}$,

$$tD\phi(tBD)h = tD\phi(tBD)BB^{-1}h = tDB\phi(tDB)(B^{-1}h).$$

As $B^{-1}h \in \overline{\textsf{R}_2(D)}$, $z\phi \in \Psi^{\tau-1}_1(S_\mu)$ with $\tau - 1 > \gamma(p)$, we can use theorem 5.1 and then the invertibility of $B : \overline{\textsf{R}_p(D)} \to \overline{\textsf{R}_p(BD)}$ to obtain

$$\|tD\phi(tBD)h\|_{T^p_2} = \|tDB\phi(tDB)(B^{-1}h)\|_{T^p_2} \lesssim \|B^{-1}h\|_p \lesssim \|h\|_p.$$

We turn to $q < 2$.

Lemma 5.21. — *Let $\phi \in \mathcal{R}^2(S_\mu)$. Then for all $2 < p < \infty$*

$$\|tD\phi(tBD)h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_{L^p}, \quad \forall h \in L^2,$$

and for all $0 \leq \alpha < 1$,

$$\|tD\phi(tBD)h\|_{T^\infty_{2,\alpha}} \lesssim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in L^2.$$

Proof. — It suffices to do it for $\phi(tBD) = (I + itBD)^{-2}$. The proof is roughly the same as for lemma 5.20 (playing with the projection and constants) as soon as we establish the following: Let $h \in L^2$. Fix a ball $B_r \subset \mathbb{R}^n$, with radius r, set $\Omega = (0, r) \times B_r$, then

$$\Big(\frac{1}{|B_r|} \iint_\Omega |tD\phi(tBD)h(x)|^2 \, \frac{dtdx}{t}\Big)^{1/2} \lesssim \sum_{j=1}^{\infty} 2^{-j\,(N - \frac{n}{2})} \Big(\fint_{2^j B_r} |h|^2\Big)^{1/2}. \tag{48}$$

Indeed, one can always write

$$tD\phi(tBD)h = tD\phi(tBD)(\mathbb{P}h) = tD\phi(tBD)(\mathbb{P}h - c)$$

for any constant c, noting that $tD\phi(tBD)(c) = tD(\phi(0)c) = 0$ and apply this inequality as needed. Again the proof of (48) follows by decomposing $h = h_0 + h_1 + \dots$ The terms h_j for $j \geq 1$ are localized in annuli away from the ball B_r. One can use lemma 5.14 to control integrals $\int_{B_r} |tD\phi(tBD)h_j|^2$ by the sum of $\int_{\widetilde{B}_r} |tBD\phi(tBD)h_j|^2$

and $\int_{\widetilde{B}_r} |\phi(tBD)h_j|^2$ with slightly larger balls $\widetilde{B}_r$. Now, one uses the L^2 off-diagonal decay of combinations and iterates of resolvents. It remains to look at the term with $h_0 = h1_{2B}$. One has

$$\int_{B_r} |tD\phi(tBD)h_0|^2 \leq \int_{\mathbb{R}^n} |tD\phi(tBD)h_0|^2 \lesssim \int_{\mathbb{R}^n} |tBD\phi(tBD)h_0|^2$$

using the accretivity of B on $R_2(D)$. We conclude by plugging this in the dt integral and using the square function bounds for $tBD\phi(tBD)$. □

Armed with this lemma, we begin as in the proof of the upper bounds for theorem 5.7 by observing that our assumption implies for $h \in \overline{R_2(BD)}$,

$$\|\mathbb{P}\chi^{\pm}(BD)h\|_p \lesssim \|\mathbb{P}h\|_p$$

for $p = q'$ if $q > 1$ or $\|\mathbb{P}\chi^{\pm}(BD)h\|_{\dot{\Lambda}^\alpha} \lesssim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}$ for $\alpha = n(\frac{1}{q} - 1)$ if $q \leq 1$.

Now let $\phi \in \mathcal{R}^2_\sigma(S_\mu) \cap \Psi^\tau_0(S_\mu)$ with $\tau > 1$. Pick $\phi_\pm \in \mathcal{R}^2(S_\mu)$ such that

$$|\phi(z) - \phi_\pm(z)| = O(|z|^\sigma), \quad \forall z \in S_{\mu\pm}.$$

The key point is the following observation: the functions $\psi_\pm(z) := z\tilde{\psi}_\pm(z)$ with

$$\tilde{\psi}_\pm(z) := (\phi - \phi_\pm)(z)\chi^\pm(z)$$

satisfy $\tilde{\psi}_\pm \in \Psi^\tau_\sigma(S_\mu)$ and $\psi_\pm \in \Psi^{\tau-1}_{\sigma+1}(S_\mu)$. Hence, for $h \in \overline{R_2(BD)}$, using

$$h = \chi^+(BD)h + \chi^-(BD)h = h^+ + h^-,$$

we have the decomposition

$$tD\phi(tBD)h = tD\widetilde{\psi}_+(tBD)h + tD\widetilde{\psi}_-(tBD)h + tD\phi_+(BD)h^+ + tD\phi_-(BD)h^-.$$

In the case $q = p'$ we deduce from lemma 5.21

$$\|tD\phi_+(BD)h^+\|_{T^p_2} \lesssim \|\mathbb{P}h^+\|_p \lesssim \|\mathbb{P}h\|_p$$

and similarly for the term with h^-. Now using the local coercivity assumption (44), up to opening the cones in the definition of the square function, we have

$$\|tD\widetilde{\psi}_\pm(tBD)h\|_{T^p_2} \lesssim \|\mathbb{Q}_{\psi_\pm,BD}h\|_{T^p_2} + \|\mathbb{Q}_{\widetilde{\psi}_\pm,BD}h\|_{T^p_2}.$$

But $\psi_\pm$ and $\tilde{\psi}_\pm$ are allowable for $\mathbb{H}^p_{BD}$ as we assumed $\sigma > \gamma(q) = \gamma(p)$, thus

$$\|tD\widetilde{\psi}_\pm(tBD)h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_p.$$

The argument when $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$ is similar. This completes the proof of the upper bounds in theorem 5.8.

6. COMPLETIONS

As said, completions of the pre-Hardy spaces may lead to abstract spaces. The results above will give us favorable situations in appropriate ranges. This is in spirit with the results in [**60**] obtained for second order operators in divergence form.

Strictly speaking, we could proceed this article without including such completions except in chapter 14. This can be skipped in a first reading.

Here $T = DB$ or BD on L^2 but the theory could be more generally defined.

For $0 < p < \infty$, define H^p_T to be the completion of $\mathbb{H}^p_T$ with respect to $\|\mathbb{Q}_{\psi,T}h\|_{T^p_2}$ for any allowable ψ. For $p < 1$, this is a quasi-Banach space.

For $p = \infty$, we have two options. One is $h^\infty_T = \dot\lambda^0_T$ be the completion $\mathbb{H}^\infty_T$ with respect to $\|\mathbb{Q}_{\psi,T}h\|_{T^\infty_2}$ for any allowable ψ. We do not see any crucial use of it but we mention it for completeness. The other one is $H^\infty_T = \dot\Lambda^0_T$ be the dual space of $H^1_{T^*}$. For $\alpha > 0$, let $\dot\lambda^\alpha_T$ be the completion of $\mathbb{L}^\alpha_T$ with respect to any of the allowable norms $\|\mathbb{Q}_{\psi,T}h\|_{T^\infty_{2,\alpha}}$. Alternately, let $\dot\Lambda^\alpha_T$ be the dual space of $H^p_{T^*}$ with $\alpha = n(\frac{1}{p} - 1)$.

The following properties hold:

1) For $1 < p < \infty$, H^p_T and $H^{p'}_{T^*}$ are dual spaces for a duality extending the L^2 sesquilinear inner product. In particular, H^p_T is reflexive.

2) $\dot\lambda^\alpha_T$ is a closed subspace of $\dot\Lambda^\alpha_T$ when $\alpha \geq 0$.

3) On each H^p_T, $1 < p < \infty$, there is a unique bisectorial operator $U = U_{H^p_T}$ with H^∞-calculus such that for all $b \in H^\infty(S_\mu)$, $b(U)h = b(T)h$ for all $h \in \mathbb{H}^p_T$. In particular there is a continuous, bounded and analytic semigroup $(e^{-t|U|})_{t>0}$ which extends the semigroup $(e^{-t|T|})_{t>0}$ on $\overline{\mathsf{R}_2(T)}$. Moreover, U is injective. Finally, $(U_{H^p_T})^* = U_{H^{p'}_{T^*}}$.

4) If $p \leq 1$, the H^∞-calculus originally defined on $\overline{\mathsf{R}_2(T)}$ extends to H^p_T. In particular, we have bounded extension of the operators $e^{-t|T|}$, $t \geq 0$. They form a semigroup and we have shown the strong continuity at 0 on a dense subspace in proposition 4.5. Thus strong continuity at 0 remains on the completion. Similarly, we can define the spectral spaces $H^{p,\pm}_T$ as the completion of $\mathbb{H}^{p,\pm}_T$ (within H^p_T) or, equivalently, as the image of the extension to H^p_T of $\chi^\pm(T)$. Similarly, by taking adjoints (in the duality extending the L^2 sesquilinear inner product), we can extend the H^∞-calculus originally defined on $\overline{\mathsf{R}_2(T)}$ to $\dot\Lambda^\alpha_T$ when $\alpha \geq 0$ and then the semigroup is weakly-$*$ continuous. Moreover, $\dot\Lambda^{\alpha,\pm}_T$ is the dual space to $H^{p,\pm}_{T^*}$.

5) The spaces H^p_T can be defined in such a way they form a complex interpolation family for $0 < p \le \infty$.

See [**22**] for 1) and 5). Assertions 2) and 4) are easy. We give a proof of 3) together with the construction.

Proof of 3). — Fix $1 < p < \infty$. Define $H^{p,\pm}_T$ as the completion of $\mathbb{H}^{p,\pm}_T$ (within H^p_T). Clearly, the splitting of the pre-Hardy spectral subspaces passes to completion. Also $(e^{\mp tT}\chi^\pm(T))_{t>0}$ extends to an analytic semigroup on $H^{p,\pm}_T$ in the open sector $S_{(\pi/2-\omega)+}$. As $H^{p,\pm}_T$ is a Banach space, this semigroup has a generator $-U_\pm$ which is ω-sectorial and densely defined (see [**80**]). On $H^p_T = H^{p,+}_T \oplus H^{p,-}_T$, define

$$Uh = U_+h^+ - U_-h^-, \quad \mathsf{D}(U) = \{h \in H^p_T\,;\, h^\pm \in \mathsf{D}(U_\pm)\}.$$

Then, U is clearly ω-bisectorial and densely defined on H^p_T. As $e^{\mp zU_\pm}$ coincides with $e^{\mp zT}\chi^\pm(T)$ on $\mathbb{H}^{p,\pm}_T$ when $z \in S_{(\pi/2-\omega)+}$, and $\chi^\pm(T)$ is the identity on $\mathbb{H}^{p,\pm}_T$, the resolvents $(I+isU)^{-1}$ and $(I+isT)^{-1}$ coincide on both $\mathbb{H}^{p,\pm}_T$, thus on their direct sum $\mathbb{H}^p_T$, when $s \in S_\nu$ where $0 \le \nu < \pi/2-\omega$. As a consequence, $\psi(T)$ and $\psi(U)$ coincide on $\mathbb{H}^p_T$ for any $\psi \in \Psi(S_\mu)$ by the Cauchy formula. As $\psi(T)$ has a bounded extension to H^p_T with norm controlled by $\|\psi\|_\infty$, this implies that U has a H^∞-calculus on H^p_T and that $b(T)$ and $b(U)$ coincide on $\mathbb{H}^p_T$ for any $b \in H^\infty(S_\mu)$.

The uniqueness of $-U$ follows from that of $-U_\pm$ as generators of semigroups.

The operator $|U|$ may now be defined as

$$|U| = \operatorname{sgn}(U)U,$$

or alternately as $|U|h = U_+h^+ + U_-h^-$ with $\mathsf{D}(|U|) = \{h \in H^p_T\,;\, h^\pm \in \mathsf{D}(U_\pm\} = \mathsf{D}(U)$. The semigroup generated by $-|U|$ thus coincides with the one generated by $-|T|$ on $\mathbb{H}^p_T$.

The injectivity is a little trickier. We have seen in proposition 4.6 that for any $h \in \mathbb{H}^p_T$, $\lim_{s\to\infty} \|e^{-s|T|}h\|_{\mathbb{H}^p_T} = 0$. By density, we have $\lim_{s\to\infty} \|e^{-s|U|}h\|_{H^p_T} = 0$ for any $h \in H^p_T$. If $h \in \mathsf{N}(U)$, then $h \in \mathsf{N}(|U|)$ and thus $e^{-s|U|}h = h$ for all $s > 0$. Taking the limit at ∞ yields $h = 0$.

Finally, calling $U = U_{H^p_T}$ and using the duality between H^p_T and $H^{p'}_{T^*}$, it it is easy to conclude that $(U_{H^p_T})^* = U_{H^{p'}_{T^*}}$. □

Remark 6.1. — Except for the last duality formula, the proof works for H^1_T, which is a Banach space, as reflexivity is not used.

Let us come back to our concrete situation.

Proposition 6.2. — *Let $\frac{n}{n+1} < p < \infty$. If $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms, then they have same completions $H^p_{DB} = H^p_D$ with equivalence of norms. In particular, H^p_{DB} is a complemented subspace of H^p where $H^p = L^p$ if $p > 1$. Moreover, the extended semigroup of $(e^{-t|DB|})_{t>0}$ is strongly continuous in H^p_D.*

Proof. — That $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms implies they have same completion is an exercise in functional analysis. We have seen in theorem 4.16 that $\mathbb{H}^p_D = \mathbb{P}(H^p \cap L^2)$. As $H^p \cap L^2$ is dense in H^p and $\mathbb{P}$ has a bounded extension to H^p, we have $H^p_D = \mathbb{P}(H^p)$, hence $H^p_{DB} = H^p_D$ is a complemented subspace of H^p. We have seen that the semi-group is strongly continuous on H^p_{DB}. This passes to H^p_D. $\square$

The following result is in spirit of [**60**] and [**19**].

Proposition 6.3. — *Let $\frac{n}{n+1} < p < \infty$. If $H^p_{DB} = H^p_D$ with equivalence of norms, then $H^p_{DB} \cap L^2 = \mathbb{H}^p_{DB}$ and $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms.*

Proof. — If $p = 2$, there is nothing to prove. In the other case, it suffices to show the first set equality as the second one, with the equivalence of norms, follows from it. If $p > 2$, then

$$\mathbb{H}^p_{DB} \subset H^p_{DB} \cap L^2 = H^p_D \cap L^2 = \mathbb{H}^p_D \subset \mathbb{H}^p_{DB}$$

using theorem 4.16 and proposition 4.18.

Assume now that $p < 2$. It is enough to show $H^p_{DB} \cap L^2 \subset \mathbb{H}^p_{DB}$ as the other inclusion is by construction. Let $h \in H^p_{DB} \cap L^2$. Take an allowable ψ for $\mathbb{H}^p_{DB}$. We have to show that $\mathbb{Q}_{\psi,DB}h \in T^p_2$. By definition, there exists $h_k \in \mathbb{H}^p_{DB}$ such that h_k converges to h in H^p_{DB}. Thus, $(\mathbb{Q}_{\psi,DB}h_k)$ is a Cauchy sequence in T^p_2 and has a limit H. Also, by the assumption, (h_k) converges to h for the H^p topology. It remains to show that $H = \mathbb{Q}_{\psi,DB}h$, for example in the sense of distributions in $\mathbb{R}^{1+n}_+$. Let $F \in C^\infty_0(\mathbb{R}^{1+n}_+)$, then we can write

$$(H - \mathbb{Q}_{\psi,DB}h, F) = (H - \mathbb{Q}_{\psi,DB}h_k, F) + \langle h_k - h, \mathbb{S}_{\psi^*,B^*D}F\rangle,$$

the computation being justified by the H^2_{DB} theory. The first term of the right hand side converges to 0, since $F \in (T^p_2)^*$ as easily checked. For the second term, we remark that it equals $\langle h_k - h, \mathbb{P}\mathbb{S}_{\psi^*,B^*D}F\rangle$ and we claim that $\mathbb{P}\mathbb{S}_{\psi^*,B^*D}F \in (H^p)^*$. Thus convergence to 0 follows and finishes the argument.

To prove the claim, let $[a, b] \times \mathbb{R}$ contain the support of F. Then

$$\mathbb{P}\mathbb{S}_{\psi^*,B^*D}F = \int_a^b \mathbb{P}\psi^*(tB^*D)F(t,.)\,\frac{dt}{t} = \int_a^b \mathbb{P}\psi^*(tB^*D)\mathbb{P}F(t,.)\,\frac{dt}{t}.$$

Remark that for each t, $F(t,.) \in (H^p)^* \cap L^2$ with uniform bound for $t \in [a, b]$. Thus $\mathbb{P}F(t,.) \in \mathbb{P}((H^p)^* \cap L^2) = \mathbb{H}^{p'}_D$ or $\mathbb{L}^\alpha_D$ depending on the value of p. We now verify the assumption of remark 4.23 : From $p < 2$, we know $\mathbb{H}^p_{DB} \subset \mathbb{H}^p_D$ (proposition 4.17). Next, from $H^p_{DB} = H^p_D$ with equivalence of norms, we see that $\|h\|_{\mathbb{H}^p_D} \sim \|h\|_{\mathbb{H}^p_{DB}}$ for all $h \in \mathbb{H}^p_{DB}$. Finally, the density of $\mathbb{H}^p_{DB}$ in H^p_{DB} guarantees that the above set inclusion is dense for the $\mathbb{H}^p_D$ topology. Thus, the conclusion of corollary 4.21 applies: $\mathbb{P}\psi^*(tB^*D)$ bounded on $\mathbb{H}^{p'}_D$ or $\mathbb{L}^\alpha_D$ uniformly in t. This implies $\mathbb{P}\mathbb{S}_{\psi^*,B^*D}F \in (H^p)^*$ as desired. $\square$

Proposition 6.4. — *The set of exponents $q \in (\frac{n}{n+1}, \infty)$ for which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms is equal to the set of those q for which $H^q_{DB} = H^q_D$ with equivalence of norms. Moreover, it is an interval which contains $((p_-(DB))_*, p_+(DB))$.*

Proof. — The first statement follows from the previous two propositions. We know from H^∞-calculus in L^2 that the identity map $I : H^2_D = \overline{\mathsf{R}_2(D)} \to H^2_{DB}$ is an isomorphism. Let q be such that $H^q_{DB} = H^q_D$ with equivalence of norms. It means that I is an isomorphism from H^q_D onto H^q_{DB}. As H^p_D and H^p_{DB} are complex interpolation families for $0 < p < \infty$, this shows that the set of q for which $H^q_{DB} = H^q_D$ is an interval which contains 2. That it contains $((p_-(DB))_*, p_+(DB))$ has been proved in theorem 5.1 for the pre-Hardy spaces, hence for their completions. $\square$

Proposition 6.5. — *The interval of exponents $q \in (\frac{n}{n+1}, \infty)$ for which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms is open.*

Proof. — We begin with openness about an exponent $q < 2$. Take $\psi \in \Psi^{\frac{n}{2}+1}(S_\mu)$ and $\varphi \in \Psi_{\frac{n}{2}+1}(S_\mu)$ for which the Calderón formula (24) holds. We have the bounded maps $\mathbb{Q}_{\psi,DB} : \mathbb{H}^p_{DB} \to T^p_2 \cap T^2_2$ and $\mathbb{S}_{\varphi,DB} : T^p_2 \cap T^2_2 \to \mathbb{H}^p_D$ for all $p \in (\frac{n}{n+1}, 2]$ by proposition 4.1 and proposition 4.17. The composition is the identity map. Consider bounded extensions $H^p_{DB} \to T^p_2$ and $T^p_2 \to H^p_D$ that are consistent for this range of p. The composition is assumed to be the identity at $p = q$. By the result in [**84, 69**], it remains invertible for p in a neighborhood of q. It readily follows that H^p_{DB} and H^p_D are isomorphic for those p. Since we already have the inclusion $\mathbb{H}^p_{DB} \subset \mathbb{H}^p_D$, it is easy to conclude the isomorphism is the identity.

In the case $p > 2$, we know from proposition 4.18 that $\mathbb{H}^p_D \subset \mathbb{H}^p_{DB}$. So we revert the roles of DB and D and consider $\mathbb{Q}_{\psi,D}$ and $\mathbb{S}_{\varphi,D}$ for appropriate ψ, φ. We skip details. $\square$

7. OPENNESS

We would like to prove that for any p such that $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms, then the same holds for small L^∞ perturbations of B. We do not know this in the abstract. However, we can do this in the range found in theorem 5.1.

Proposition 7.1. — *Fix $p \in ((p_-(DB))_*, p_+(DB))$. Then for any B' with $\|B - B'\|_\infty$ small enough (depending on p),*

$$\mathbb{H}^p_{DB'} = \mathbb{H}^p_D$$

with equivalence of norms. Furthermore, for any $b \in H^\infty(S_\mu)$ with $\omega_B < \mu < \pi/2$, we have

$$\|b(DB) - b(DB')\|_{\mathcal{L}(\mathbb{H}^p_D)} \lesssim \|b\|_\infty \|B - B'\|_\infty. \tag{49}$$

Proof. — We shall use analyticity: let $B_z = B - zM$ for $M(x)$ normalized with L^∞ norm 1 and $z \in \mathbb{C}$ so that $B_0 = B$. We shall show the conclusion for $B' = B_z$ with z in a small enough disk. First there is $r > 0$ such that for $|z| < r$, B_z is accretive on $\overline{R_2(D)}$ with constant half the one for B and bounded with L^∞ bound twice that of B. Using a Neumann series expansion, for $\lambda \notin S_\mu$, where $\mu > \omega_B$ (the accretivity angle of B)

$$\begin{aligned}(\lambda - DB_z)^{-1} &= \sum_{k=0}^{\infty} z^k \big((\lambda - DB)^{-1} DM\big)^k (\lambda - DB)^{-1} \\ &= \sum_{k=0}^{\infty} z^k \big((\lambda - DB)^{-1} DBB^{-1} M\big)^k (\lambda - DB)^{-1}.\end{aligned}$$

Thus if $|z| < \varepsilon_2$, the series converges in $\mathcal{L}(L^2)$ and this shows that DB_z is ω_B bisectorial on L^2 for all $|z| < \varepsilon_2$. As B_z has the same form as B, it follows that DB_z has H^∞-calculus on bisectors S_μ with uniform bounds with respect to $|z| < \varepsilon_2$. Furthermore $z \mapsto b(DB_z)$ is an analytic $\mathcal{L}(L^2)$-valued function for any $b \in H^\infty(S_\mu)$. This is shown in [**29**, section 6] together with (49) in $\mathcal{L}(L^2)$.

Now, the same Neumann series shows that DB_z is also bisectorial on L^p if $p_-(DB) < p < p_+(DB)$ and $|z| < \varepsilon_p$ small enough. Thus such operators also have H^∞-calculus on L^p by the theory recalled in section 3.2. Furthermore, analyticity of $z \mapsto b(DB_z)$ in $\mathcal{L}(\mathbb{H}^p_D)$ on $|z| < \varepsilon_p$ for any $b \in H^\infty(S_\mu)$ can be proved as follows. First, for any $\lambda \notin S_\mu$, the Neumann series, show that $z \mapsto (\lambda - DB_z)^{-1}$ is analytic in $\mathcal{L}(L^p)$ on $|z| < \varepsilon_p$. Next, for $\psi \in \Psi(S_\mu)$, using the Cauchy formula, one has analyticity of $z \mapsto \psi(DB_z)$ in $\mathcal{L}(L^p)$ on $|z| < \varepsilon_p$. Finally, b can be approximated for the topology of the uniform convergence on compact subsets of S_μ by a sequence (ψ_k) with $\psi_k \in \Psi(S_\mu)$ for each k. This implies strong convergence of $\psi_k(DB_z)$ to $b(DB_z)$ in $\mathcal{L}(\mathbb{H}^p_D)$ and examination shows it is uniform on compact subsets of $|z| < \varepsilon_p$. Analyticity follows and also (49) in $\mathcal{L}(\mathbb{H}^p_D)$.

We next turn to values $(p_-(DB))_* < p \le p_-(DB)$. For those, the method of proof of theorem 5.1 (in particular lemma 5.13) shows that for a suitable ε_p (which can be taken equal to ε_q for some $q \in (p_-(DB), p^*))$ and a suitable allowable ψ, $\|\mathbb{Q}_{\psi, DB_z} h\|_{T^p_2} \lesssim \|h\|_{\mathbb{H}^p_D}$ when $h \in \overline{R_2(D)}$ uniformly on compact subsets of $|z| < \varepsilon_p$. Hence, $\mathbb{H}^p_{DB_z} = \mathbb{H}^p_D$ with equivalence of norms, uniformly on compact subsets of $|z| < \varepsilon_p$. This implies that $b(DB_z)$ are uniformly bounded operators on $\mathbb{H}^p_D$ when $|z| < \varepsilon_p$.

If $1 < p$, this gives analyticity as follows: for $h \in \mathbb{H}^p_D, g \in \mathbb{H}^{p'}_D$, the map $z \mapsto \langle b(DB_z)h, g\rangle$ is uniformly bounded, and analytic because of the L^2 case. Then (49) follows from Cauchy estimates.

If $p \le 1$, it is likely that the abstract results developed in Kalton [**68**] apply. We follow a different route taking advantage of the atomic-molecular theory.

Let us admit the following lemma for the moment.

Lemma 7.2. — *Let $(p_-(DB))_* < p \le 1$ and $b \in H^\infty(S_\mu)$. For some $\varepsilon > 0$ depending only on p and n, then for all $(\mathbb{H}^p_D, 1)$-atoms a, with associated cube Q and all $j \ge 0$,*

$$\|b(DB)a\|_{L^2(S_j(Q))} \lesssim \|b\|_\infty \left(2^j \ell(Q)\right)^{\frac{n}{2} - \frac{n}{p}} 2^{-j\varepsilon}$$

and moreover $\int b(DB)a = 0$. In all, $b(DB)a$ is a classical H^p molecule.

Now the strategy is to prove analyticity is as follows. The same estimate applies to $b(DB_z)$ for $|z| < \varepsilon_p/2$, uniformly in z. We fix the $(\mathbb{H}^p_D, 1)$-atom a. It follows from the molecular estimate that $m_z = b(DB_z)a$ belongs to the Hilbert space H of $L^2(w_Q)$ functions m with $\int_{\mathbb{R}^n} m = 0$, where

$$w_Q(x) = |Q|^{\frac{2}{p}-1}\big(1 + \frac{d(x, 4Q)}{\ell(Q)}\big)^{2s}, \quad \frac{n}{p} - \frac{n}{2} < s < \frac{n}{p} - \frac{n}{2} + \varepsilon$$

and Q is the cube associated to a in the definition. Note that $H \subset L^1$. The bounded compactly supported functions with mean value 0 form a dense subspace of H. For f such a function, $f w_Q \in L^2$ as well as $b(DB_z)a$ since $a \in L^2$. Thus, by the analyticity on L^2, $z \mapsto \langle m_z, f w_Q\rangle$ is analytic. Next, by Cauchy estimates using the uniform bounds in the space H, we have for $|z|$ small enough,

$$|\langle m_z, f w_Q\rangle - \langle m_0, f w_Q\rangle| \lesssim \|b\|_\infty \cdot |z| \cdot \|f\|_H,$$

hence

$$\|b(DB_z)a - b(DB)a\|_H \lesssim \|b\|_\infty \cdot |z|.$$

Since $s > \frac{n}{p} - \frac{n}{2}$, this implies the H^p estimate

$$\|b(DB_z)a - b(DB)a\|_{H^p} \lesssim \|b\|_\infty \cdot |z|.$$

Note that $b(DB_z)a - b(DB)a \in \mathbb{H}^2_D$, hence this is also an estimate in the space $\mathbb{H}^p_D$. Since we know already boundedness of $b(DB_z)-b(DB)$ on $\mathbb{H}^p_D$ (but it can be obtained by extension), we conclude for (49) by density of linear combinations of $(\mathbb{H}^p_D, 1)$-atoms. $\square$

Proof lemma 7.2. — This is basically the same strategy as for proving the square function estimate. Assume $\|b\|_\infty = 1$ to simplify matters. Fix a $(\mathbb{H}^p_D, 1)$-atom a. Choose $\psi \in \Psi^\tau_\sigma(S_\mu)$ with σ, τ large and so that $\int_0^\infty \psi(tz)\,\frac{dt}{t} = 1$ for $z \in S_\mu$. Thus,

$$m = b(DB)a = \int_0^\infty (b\psi_t)(DB)a\,\frac{dt}{t}$$

with $\psi_t(z) = \psi(tz)$. Now write $a = Du$ as in the definition of $(\mathbb{H}^p_D, 1)$-atoms. We show estimates on m. Let Q be the cube associated to a. On $4Q$, by H^∞-calculus

$$\Big(\int_{4Q} |m|^2\Big)^{1/2} \lesssim \Big(\int |a|^2\Big)^{1/2} \le |Q|^{\frac{1}{2}-\frac{1}{p}}.$$

On $S_j(Q)$, $j \ge 2$, we write $(b\psi_t)(DB)a = D(b\psi_t)(BD)u$ and

$$\Big(\int_{S_j(Q)} |m|^2\Big)^{1/2} \lesssim \int_0^\infty \Big(\int_{S_j(Q)} |D(b\psi_t)(BD)u|^2\Big)^{1/2}\,\frac{dt}{t}.$$

We use once more lemma 5.14 and the fact that $\sigma, \tau > 0$ are large enough in the L^q-L^2 estimates of section 3.5 applied to $(b\psi_t)(BD)$ to obtain

$$\Big(\int_{S_j(Q)} |D(b\psi_t)(BD)u|^2\Big)^{1/2} \lesssim t^{-1} t^{\frac{n}{2}-\frac{n}{q}} \langle 2^j\ell(Q)/t\rangle^{-K}\|u\|_q$$

with K large and q chosen so that $p_- < q < p^*$ and $q \le 2$. Plugging this estimate into the t-integral we have

$$\Big(\int_{S_j(Q)} |m|^2\Big)^{1/2} \lesssim (2^j\ell(Q))^{-1}(2^j\ell(Q))^{\frac{n}{2}-\frac{n}{q}}\|u\|_q \lesssim (2^j\ell(Q))^{\frac{n}{2}-\frac{n}{p}} 2^{-j\varepsilon}$$

with $\varepsilon = 1 + \frac{n}{q} - \frac{n}{p} > 0$. It remains to prove $c = \int m = 0$. Indeed, $m - c1_Q \in H^p$ as it is a classical molecule for H^p using the estimates on m. Now, we know that $m \in \mathbb{H}^p_D \subset H^p$, thus $c1_Q \in H^p$ and it is classical (for example, using the characterization by maximal function) that $1_Q \notin H^p$ since its mean value is not 0 and c must be 0. $\square$

Remark 7.3. — It is unclear to us whether m is itself an $(\mathbb{H}^p_D, \epsilon, 1)$-molecule in the sense of our definition. One can indeed write $m = Dv$ with $v = \int_0^\infty (b\psi_t)(BD)u\,\frac{dt}{t}$ and obtain by the same method

$$\|v\|_{L^2(S_j(Q))} \lesssim \|b\|_\infty \left(2^j\ell\,(Q)\right)^{\frac{n}{2}-\frac{n}{p}} 2^{-j\varepsilon}2^j.$$

There is an extra factor 2^j. However, this is sufficient to prove a uniform L^{p^*} bound on v if one needs it.

8. REGULARIZATION VIA SEMIGROUPS

This section will be used in chapter 14 below.

It is well known that classical semigroups have regularization properties: for example, the usual Poisson semigroup on $\mathbb{R}^n$ maps L^1 into L^∞, as easily seen using the Poisson kernel. Here, there is no kernel information. Nevertheless, such regularization holds abstractly in the Hardy spaces.

Theorem 8.1. — *Let $T = BD$ or DB. Let $0 < p \leq q \leq \infty$ and $0 \leq \alpha \leq \beta < \infty$. Fix $t > 0$. Then the operator $e^{-t|T|}$ has extensions with the following mapping properties and bounds*

$$\begin{aligned} H^p_T &\longrightarrow H^q_T \quad \text{with bound} \quad Ct^{-(\frac{n}{p}-\frac{n}{q})}, \\ \dot{\Lambda}^\alpha_T &\longrightarrow \dot{\Lambda}^\beta_T \quad \text{with bound} \quad Ct^{\alpha-\beta}, \\ H^p_T &\longrightarrow \dot{\Lambda}^\alpha_T \quad \text{with bound} \quad Ct^{-\frac{n}{p}-\alpha}. \end{aligned}$$

Moreover, the mapping properties hold with the same bounds when the spaces are replaced by the corresponding pre-Hardy spaces $\mathbb{H}^{\mathcal{T}}_T$ with the possible exception of the first line when $p < q \leq 1$.

Remark 8.2. — The proof will show this result is not limited to BD or DB. It holds for any operator T on $\mathbb{R}^n$ having a Hardy space theory (for example, bisectorial with H^∞-calculus plus L^2 off-diagonal bounds). The bounds are valid for an operator having the scaling of a first order operator. For an operator of order m, then raise the bounds to power $\frac{1}{m}$.

Proof. — *Step 1*: $p \leq 1$ and $q = 2$ in the first line.

We pick an $(\mathbb{H}^p_T, \epsilon, M)$-molecule a with $M > \frac{n}{p} - \frac{n}{2}$. Let ℓ be the side length of the associated cube. Observe that $a \in \mathsf{R}_2(T) \subset \mathbb{H}^2_T$, thus $e^{-t|T|}a \in \mathbb{H}^2_T$ and

$$\|e^{-t|T|}a\|_2 \sim \|e^{-t|T|}a\|_{\mathbb{H}^2_T}.$$

Since $\|a\|_2 \lesssim \ell^{-(\frac{n}{p}-\frac{n}{2})}$ and $e^{-t|T|}$ is uniformly bounded on L^2 we have

$$\|e^{-t|T|}a\|_2 \lesssim \ell^{-(\frac{n}{p}-\frac{n}{2})}.$$

Now we have $a = T^M b$ with $b \in \mathsf{D}_2(T^M)$ and $\|b\|_2 \lesssim \ell^M \ell^{-(\frac{n}{p}-\frac{n}{2})}$. As $(tT)^M e^{-t|T|}$ is uniformly bounded on L^2, we have $\|e^{-t|T|}a\|_2 \lesssim t^{-M}\ell^{M-(\frac{n}{p}-\frac{n}{2})}$. Thus

$$\|e^{-t|T|}a\|_2 \lesssim \inf(\ell^{-(\frac{n}{p}-\frac{n}{2})}, t^{-M}\ell^{M-(\frac{n}{p}-\frac{n}{2})}) \leq t^{-(\frac{n}{p}-\frac{n}{2})}.$$

Next, let $f \in \mathbb{H}^p_T$. Pick a molecular $(\mathbb{H}^p_T, \epsilon, M)$-representation $f = \sum \lambda_j a_j$ which converges in L^2 and also with $\sum |\lambda_j|^p \le 2^p \|f\|^p_{\mathbb{H}^p_T}$. From L^2 continuity of the semigroup we have $e^{-t|T|}f = \sum \lambda_j e^{-t|T|}a_j$, hence

$$\|e^{-t|T|}f\|_2 \lesssim \sum |\lambda_j| t^{-(\frac{n}{p}-\frac{n}{2})} \le \|(\lambda_j)\|_{\ell^p}\, t^{-(\frac{n}{p}-\frac{n}{2})} \le 2\|f\|_{\mathbb{H}^p_T} t^{-(\frac{n}{p}-\frac{n}{2})}.$$

as $p \le 1$. Finally, taking completion we have proved step 1.

Step 2: $p = 2$ and $q = \infty$ in the first line.

This an easy consequence of proposition 4.8. Let $g \in H^2_T = \mathbb{H}^2_T$. Let $f \in \mathbb{H}^1_{T^*}$. Using the first step with T^* (which is of the same type as T),

$$|\langle f, e^{-t|T|}g\rangle| = |\langle e^{-t|T^*|}f, g\rangle| \le \|g\|_{\mathbb{H}^2_T}\|e^{-t|T^*|}f\|_{\mathbb{H}^2_{T^*}} \lesssim \|g\|_{\mathbb{H}^2_T}\|f\|_{\mathbb{H}^1_{T^*}} t^{-(n-\frac{n}{2})}.$$

Thus, $e^{-t|T|}g \in (\mathbb{H}^1_{T^*})^* = H^\infty_T$ by definition of H^∞_T and density of $\mathbb{H}^1_{T^*}$ in $H^1_{T^*}$, and

$$\|e^{-t|T|}g\|_{H^\infty_T} \lesssim \|g\|_{\mathbb{H}^2_T} t^{-(n-\frac{n}{2})}.$$

Step 3: All cases in the first line. Using the semigroup property and combining the first two steps, we have the first line for (p,∞) for any $0 < p < \infty$ and we also know the first line for all pairs (p,p) for $0 < p \le \infty$ from the discussion in section 6. We conclude this line by complex interpolation.

Step 4: The second line. This is the dual of the first line $H^p_{T^*} \to H^q_{T^*}$, where $\alpha = n(\frac{1}{q}-1)$ and $\beta = n(\frac{1}{p}-1)$.

Step 5: The third line. Combine $H^p_T \to H^\infty_T = \dot{\Lambda}^0_T$ with $\dot{\Lambda}^0_T \to \dot{\Lambda}^\alpha_T$ using the semigroup property.

Step 6: The first line with the pre-Hardy spaces. Before we begin recall that this is not immediate from the results above as we do not know whether $\mathbb{H}^p_T = H^p_T \cap H^2_T$ in general. We come back to the definition. Let $f \in \mathbb{H}^p_T$. As $e^{-t|T|}f \in \mathbb{H}^2_T$, we want to show that $\mathbb{Q}_{\psi,T}(e^{-t|T|}f) \in T^q_2$ with the desired bound for some allowable ψ for $\mathbb{H}^q_T$. We can only do this when $q > 1$. We choose ψ matching the conditions of the third and fourth columns for the exponent q in the table before proposition 4.1. By duality in tent spaces and density, it is enough to bound $(\mathbb{Q}_{\psi,T}(e^{-t|T|}f), G)$ by $\|G\|_{T^{q'}_2}$ for any $G \in T^{q'}_2 \cap T^2_2$. By the choice of G, we have

$$(\mathbb{Q}_{\psi,T}(e^{-t|T|}f), G) = \langle f, e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G)\rangle.$$

Now, the choice for ψ implies $\mathbb{S}_{\psi^*,T^*}G \in \mathbb{H}^{q'}_{T^*}$ and using the just proved first or third lines and duality, $e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G) \in H^{p'}_{T^*} = (H^p_T)^*$. We obtain

$$|\langle f, e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G)\rangle| \lesssim t^{-(\frac{n}{p}-\frac{n}{q})}\|f\|_{\mathbb{H}^p_T} \cdot \|G\|_{T^{q'}_2}.$$

Step 7: The third line with the pre-Hardy spaces, that is $\mathbb{H}^p_T \to \mathbb{L}^\alpha_T$. Let $f \in \mathbb{H}^p_T$. As $e^{-t|T|}f \in \mathbb{H}^2_T$, we have to show that $\mathbb{Q}_{\psi,T}(e^{-t|T|}f) \in T^\infty_{2,\alpha}$ with the desired bound for some allowable ψ for $\mathbb{L}^\alpha_T$. We let $\alpha = n(\frac{1}{q}-1)$ for some $q < 1$ and choose ψ matching the conditions of the third and fourth columns for the exponent α in the table before proposition 4.1. By duality in tent spaces and density, it is enough to bound $(\mathbb{Q}_{\psi,T}(e^{-t|T|}f), G)$ by $\|G\|_{T_2^q}$ for any $G \in T_2^q \cap T_2^2$. By the choice of G, we have

$$(\mathbb{Q}_{\psi,T}(e^{-t|T|}f), G) = \langle f, e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G)\rangle.$$

Now, the choice of ψ implies $\mathbb{S}_{\psi^*,T^*}G \in \mathbb{H}^q_{T^*}$, and using the just proved first or third line and duality, $e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G) \in (H^p_T)^*$. We obtain

$$|\langle f, e^{-t|T^*|}(\mathbb{S}_{\psi^*,T^*}G)\rangle| \lesssim t^{-\frac{n}{p}-\alpha}\|f\|_{\mathbb{H}^p_T} \cdot \|G\|_{T_2^q}.$$

Step 8: The second line with the pre-Hardy spaces, that is $\mathbb{L}^\alpha_T \to \mathbb{L}^\beta_T$. The argument is similar to the previous ones and we leave details to the reader. □

Corollary 8.3. — *Let $p \le q$ with $p \le 2$. If both p, q belong to the interval of exponents in $(\frac{n}{n+1}, \infty)$ for which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$, then the semigroup $e^{-t|DB|}$ has a bounded extension from $H^p_{DB} = H^p_D$ to $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with bound $Ct^{-(\frac{n}{p}-\frac{n}{q})}$.*

Proof. — From the previous theorem, the semigroup $e^{-t|DB|}$ extends to a bounded operator from H^p_{DB} to H^q_{DB} with the desired bound as $p \le q$ and it also maps H^p_{DB} to $H^2_{DB} \subset L^2$ as $p \le 2$. By proposition 6.3, we have that $H^q_{DB} \cap L^2 = \mathbb{H}^q_{DB}$ for q in the prescribed interval and the result follows. □

9. NON-TANGENTIAL MAXIMAL ESTIMATES

In this chapter, we establish the following results for $\widetilde{N}_*$ defined in (26).

Theorem 9.1. — *Let $(a, p_+(DB))$ be an interval with $a > \frac{n}{n+1}$ on which $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms. Then for $p \in (a, (p_+)^*)$, we have*

$$\|\widetilde{N}_*(e^{-t|DB|}h)\|_p \sim \|h\|_p$$

for all $h \in \overline{\textsf{R}_2(D)}$ if $p > 1$. If $p \leq 1$, we have

$$\|\widetilde{N}_*(e^{-t|DB|}h)\|_p \sim \|h\|_{H^p}$$

for all $h \in \overline{\textsf{R}_2(D)}$ and B pointwise accretive, or for all $h \in \mathbb{H}^{2,\pm}_{DB}$. This applies for $a = (p_-(DB))_*$.

Remark 9.2. — We think that the hypothesis of pointwise accretivity is not necessary but we are unable to remove it at this time: this is the only result of this memoir where this hypothesis is used. Nevertheless, the validity of the equivalence for $h \in \mathbb{H}^{2,\pm}_{DB}$ suffices for applications to BVPs.

Theorem 9.3. — *Let $(a, p_+(DB^*))$ be an interval with $a \geq 1$ on which $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalence of norms. Then for $1 < p < a'$, we have*

$$\|\widetilde{N}_*(e^{-t|BD|}\mathbb{P}h)\|_p \sim \|\mathbb{P}h\|_p$$

for all $h \in \overline{\textsf{R}_2(BD)}$ if $p \geq 2$ and

$$\|\widetilde{N}_*(e^{-t|BD|}h)\|_p \sim \|h\|_p \sim \|\mathbb{P}h\|_p$$

for all $h \in \overline{\textsf{R}_2(BD)}$ if $p_-(BD) < p < 2$. This applies with $a = \max((p_-(DB^*))_*, 1)$.

Remark 9.4. — The inequality $\|\widetilde{N}_*(e^{-t|T|}h)\|_p \lesssim \|h\|_{\mathbb{H}^p_T}$ holds for $0 < p \leq 2$ when $h \in \overline{\textsf{R}_2(T)}$ for $T = BD$ or DB thanks to lemma 4.15 and the equivalence at $p = 2$ (which we prove next).

Remark 9.5. — We shall also prove

$$\|\widetilde{N}_*(e^{-t|BD|}h)\|_p \lesssim \|h\|_p$$

for $2 < p < (p_+(BD))^*$ and $h \in L^2$, hence in particular $h \in \overline{\textsf{R}_2(BD)}$. But if $p \geq p_+(BD)$ the right hand side is not equivalent to the $\mathbb{H}^p_{BD}$ norm, while $\|\mathbb{P}h\|_p$ is. This is why we have to insert $\mathbb{P}$ in theorem 9.3.

Remark 9.6. — Note that the result in theorem 9.3 for $p<2$ sounds different. Let the r variant of $\widetilde{N}_*$ be defined as

$$\widetilde{N}_*^r(g)(x) := \sup_{t>0}\Big(\fint\!\!\fint_{W(t,x)} |g|^r\Big)^{1/r}, \qquad x\in\mathbb{R}^n,$$

so that $\widetilde{N}_*^2=\widetilde{N}_*$. In fact, one can only prove

$$\|\widetilde{N}_*^r(e^{-t|BD|}\mathbb{P}h)\|_p \sim \|\mathbb{P}h\|_p$$

for all $h\in\overline{\mathsf{R}_2(BD)}$ with $r<p$ if $p<2$. And this is sharp since, as

$$e^{-t|BD|}\mathbb{P}h - e^{-t|BD|}h = \mathbb{P}h - h$$

for all $t>0$, $\widetilde{N}_*^r(e^{-t|BD|}\mathbb{P}h - e^{-t|BD|}h)\sim \boldsymbol{M}_r(\mathbb{P}h-h)$ and $\boldsymbol{M}_r$ is not bounded on L^p if $p\le r$.

Remark 9.7. — We thank M. Mourgoglou for pointing out to us that the results in this chapter hold with the non-tangential maximal function on Whitney regions replaced by the non-tangential maximal function on slices

$$\sup_{t>0}\Big(\fint_{B(x,c_1t)} |e^{-t|T|}h|^2\Big)^{1/2}, \quad x\in\mathbb{R}^n.$$

For the lower bounds, this is trivial as there is a pointwise domination of $\widetilde{N}_*$ by the latter. For the upper bounds, the arguments need some adjustements. The main one is to go from the integral on slices $\fint_{B(x,c_1t)}|\psi(tT)h|^2$ to a solid integral on a Whitney region in order to use square function estimates. This can be done using the method of proof of proposition 2.1 in [**3**], up to using 2 different ψ, which is not a problem. We skip details.

Remark 9.8. — All the results of this section concerning $T=DB$ are valid with $e^{-s|T|}$ replaced $\varphi(sT)$ where $\varphi\in H^\infty(S_\mu)$ with

$$|\varphi(z)|\lesssim |z|^{-\alpha} \quad\text{and}\quad |\varphi(z)-\varphi(0)|\lesssim |z|^{\alpha}$$

for some $\alpha>0$. It suffices to write $\varphi(z)=\varphi(0)e^{-[z]}+\psi(z)$. Concerning $T=BD$, all results hold in the range $p_-(BD)<p<(p_+(BD))^*$ for such φ. For $p\ge (p_+(BD))^*$, we also impose $\varphi\in\mathcal{R}^2_\sigma$ for σ large enough.

9.1. L^2 estimates and Fatou type results

Theorem 9.9. — *Let $T=DB$ or BD. Then one has the estimate*

$$\|\widetilde{N}_*(e^{-t|T|}h)\|_2\sim\|h\|_2,\ \forall h\in\overline{\mathsf{R}_2(T)}. \tag{50}$$

Furthermore, for any $h\in L^2$ (not just $\overline{\mathsf{R}_2(T)}$), we have that the Whitney averages of $e^{-t|T|}h$ converge to h in L^2 sense, that is for almost every $x_0\in\mathbb{R}^n$,

$$\lim_{t\to 0}\fint\!\!\fint_{W(t,x_0)} |e^{-s|T|}h-h(x_0)|^2=0. \tag{51}$$

In particular, this implies the almost everywhere convergence of Whitney averages

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x_0)} e^{-s|T|}h = h(x_0). \tag{52}$$

Proof. — Let us begin with the non-tangential maximal estimate. The bound from below is easy:

$$\|h\|_2^2 = \lim_{t\to 0} \frac{1}{t}\int_t^{2t} \|e^{-s|T|}h\|_2^2\, ds \lesssim \|\widetilde{N}_*(e^{-s|T|}h)\|_2^2.$$

Next, the bound from above for $T = DB$ is due to [**82**, theorem 5.1] (When D has a special form it appeared first in disguise in [**9**].) We provide a different proof in the spirit of the decompositions above. It is easy to check that $e^{-[z]} \in \mathcal{R}_2^2(S_\mu)$: there exist $\phi_\pm \in \mathcal{R}^2(S_\mu)$ such that

$$\psi_\pm(z) := (e^{-[z]} - \phi_\pm(z))\chi^\pm(z) \in \Psi_2^2(S_\mu).$$

Thus, $\widetilde{N}_*(\psi_\pm(tDB)h) \lesssim S(\psi_\pm(tBD)h)$ and the L^2 bound follows from the square function bounds for DB. It remains to check the L^2 bounds for $\widetilde{N}_*(\phi_\pm(tBD)h^\pm)$ where $h^\pm = \chi^\pm(DB)h$. It suffices to do it for $h \in \mathsf{R}_2(D)$ by density. Thus $h^\pm \in \mathsf{R}_2(D)$ and there exist $v^\pm \in \mathsf{D}_2(D) \cap \overline{\mathsf{R}_2(D)}$ such that $h^\pm = Dv^\pm$, and we can write

$$\phi_\pm(tDB)h^\pm = D\phi_\pm(tBD)(v^\pm - c^\pm),$$

where $c^\pm$ is any constant. Fix a Whitney region

$$W(t,x) = (c_0^{-1}t, c_0 t) \times B(x, c_1 t),$$

choose $c^\pm$ as the average of $v^\pm$ on the ball $B(x, c_1 t)$. Using the local coercivity estimate (44), we have, with a slightly enlarged Whitney region $\widetilde{W}(t,x)$ in the right hand side,

$$\begin{aligned}\fint\!\!\fint_{W(t,x)} |\phi_\pm(tDB)h^\pm|^2 \lesssim \fint\!\!\fint_{\widetilde{W}(t,x)} &|BD\phi_\pm(tBD)(v^\pm - c^\pm)|^2 \\ &+ t^{-2}\fint\!\!\fint_{\widetilde{W}(t,x)} |\phi_\pm(tDB)(v^\pm - c^\pm)|^2.\end{aligned}$$

As $\phi_\pm \in \mathcal{R}^2(S_\mu)$, $z\phi^\pm$ and $\phi^\pm$ have L^2 off-diagonal decay with decay as large as one wants, using the usual analysis in annuli and Poincaré inequality for $\frac{2n}{n+2} \le p < 2$ and $p \ge 1$, we obtain

$$\Big(\fint\!\!\fint_{W(t,x)} |\phi_\pm(tDB)h^\pm|^2\Big)^{1/2} \lesssim \boldsymbol{M}_p(\nabla v^\pm)(x).$$

Thus, the L^2 norm of $\widetilde{N}_*(\phi_\pm(tDB)h^\pm)$ is controlled by $\|\nabla v^\pm\|_2$ and we use the coercivity of D on $\mathsf{D}_2(D) \cap \overline{\mathsf{R}_2(D)}$ to get a bound $\|Dv^\pm\|_2 = \|h^\pm\|_2 \lesssim \|h\|_2$. The proof for DB is complete.

The proof for $T = BD$ follows from the result for DB: If $g \in \overline{\textsf{R}_2(BD)}$, then $B^{-1}g = h \in \overline{\textsf{R}_2(DB)}$ with $\|h\|_2 \sim \|g\|_2$ and $e^{-t|BD|}g = Be^{-t|DB|}h$. Thus

$$\|\widetilde{N}_*(e^{-t|BD|}g)\|_2 = \|\widetilde{N}_*(Be^{-t|DB|}h)\|_2 \leq \|B\|_\infty \|\widetilde{N}_*(e^{-t|DB|}h)\|_2 \sim \|h\|_2 \sim \|g\|_2.$$

It remains to show the almost everywhere convergence result. We begin with BD. Let $h \in L^2$. Pick x_0 a Lebesgue point for the condition

$$\lim_{t\to 0} \fint_{B(x_0,t)} |h - h(x_0)|^2 = 0. \tag{53}$$

Write as above,

$$e^{-s|BD|}h = \psi(sBD)h + (I + isBD)^{-1}h$$

with $\psi(z) = e^{-[z]} - (1 + iz)^{-1}$. The quadratic estimate (14) implies that

$$\lim_{t\to 0} \iint\!\!\!\!\!\!-_{W(t,x_0)} |\psi(sBD)h|^2 = 0$$

for almost every $x_0 \in \mathbb{R}^n$. Now the key point is that $Dc = 0$ if c is a constant, thus $(I + isBD)^{-1}[h(x_0)] = h(x_0)$. It follows that

$$(I + isBD)^{-1}h - h(x_0) = (I + isBD)^{-1}(h - h(x_0))$$

so that for arbitrarily large N,

$$\iint\!\!\!\!\!\!-_{W(t,x_0)} |(I + isBD)^{-1}(h - h(x_0))|^2 \lesssim \sum_{j\geq 1} 2^{-jN} t^{-n} \int_{B(x_0,2^j t)} |h - h(x_0)|^2. \tag{54}$$

Breaking the sum at j_0 with $2^{-j_0} \sim \sqrt{t}$ and choosing $N \geq n+1$, we obtain a bound

$$\sup_{\tau \leq \sqrt{t}} \fint_{B(x_0,\tau)} |h - h(x_0)|^2 + \sqrt{t}\,\boldsymbol{M}(|h - h(x_0)|^2)(x_0),$$

where $\boldsymbol{M}$ is the Hardy-Littlewood maximal function. Using the weak type (1,1) of $\boldsymbol{M}$, almost every $x_0 \in \mathbb{R}^n$ satisfy $\boldsymbol{M}(|h|^2)(x_0) < \infty$. Hence, the latter expression goes to 0 as $t \to 0$ at those x_0 meeting all the requirements.

We turn to the proof for $T = DB$. Let $g \in L^2$. If $g \in \textsf{N}_2(DB)$, this is a consequence of the Lebesgue differentiation theorem on $\mathbb{R}^n$ as $e^{-s|DB|}g = g$ is independent of s. We assume next that $g \in \overline{\textsf{R}_2(DB)}$. As

$$\lim_{t\to 0} \iint\!\!\!\!\!\!-_{W(t,x_0)} |g - g(x_0)|^2 = \lim_{t\to 0} \fint_{B(x_0,c_1 t)} |g - g(x_0)|^2 = 0$$

for almost every $x_0 \in \mathbb{R}^n$, it is enough to show the almost everywhere limit

$$\lim_{t\to 0} \iint\!\!\!\!\!\!-_{W(t,x_0)} |e^{-s|DB|}g - g|^2 = 0.$$

We also choose x_0 so that

$$\lim_{t\to 0} \fint_{B(x_0,c_1 t)} |Bg - (Bg)(x_0)|^2 = 0.$$

Write again $e^{-s|DB|}g-g = \psi(sDB)g+(I+isDB)^{-1}g-g$. The quadratic estimate (14) implies that

$$\lim_{t\to 0} \iint_{W(t,x_0)} |\psi(sDB)g|^2 = 0$$

for almost every $x_0 \in \mathbb{R}^n$. Now

$$(I+isDB)^{-1}g - g = -isDh_s$$

with $h_s = B(I+isDB)^{-1}g = (I+isBD)^{-1}(Bg)$ and $Bg \in L^2$. Let

$$\tilde{h}_s := (I+isBD)^{-1}(Bg) - (Bg)(x_0).$$

Applying lemma 5.14 to $u = \tilde{h}_s$ using $Dh_s = D\tilde{h}_s$ and integrating with respect to s implies

$$\begin{aligned}\iint_{W(t,x_0)} |isDh_s|^2 &\lesssim \iint_{\widetilde{W}(t,x_0)} |isBDh_s|^2 + \iint_{\widetilde{W}(t,x_0)} |\tilde{h}_s|^2 \\ &\lesssim \iint_{\widetilde{W}(t,x_0)} |(I+isBD)^{-1}(Bg) - Bg|^2 \\ &\qquad + \iint_{\widetilde{W}(t,x_0)} \Big|Bg - (Bg)(x_0)|^2,\end{aligned}$$

where $\widetilde{W}(t,x_0)$ is a slightly expanded version of $W(t,x_0)$ and, in the last inequality, we have written $\tilde{h}_s = (I+isBD)^{-1}(Bg) - Bg + Bg - (Bg)(x_0)$. The last two integrals have been shown to converge to 0 for almost every $x_0 \in \mathbb{R}^n$ in the argument for BD. This concludes the proof. $\square$

9.2. Lower bounds for $p \neq 2$

A first argument follows from the almost everywhere bounds.

Proposition 9.10. — *Let $T = DB$ or BD and $1 < p < \infty$. Then one has the estimate*

$$\|h\|_p \lesssim \|\widetilde{N}_*(e^{-t|T|}h)\|_p, \ \forall h \in L^2. \tag{55}$$

Proof. — It follows from the almost everywhere limit (52) that

$$|h| \leq \widetilde{N}_*(e^{-t|T|}h) \tag{56}$$

almost everywhere. It suffices to integrate. $\square$

Our second result, inspired by an argument found in [**61**] in the case of second order divergence form operators, yields the following improvement under a supplementary hypothesis.

Proposition 9.11. — *Assume B is pointwise accretive. Let $T = DB$ and $\frac{n}{n+1} < p < \infty$. Then one has the estimate*

$$\|h\|_{H^p} \lesssim \|\widetilde{N}_*(e^{-t|DB|}h)\|_p, \ \forall h \in \overline{\mathsf{R}_2(D)}. \tag{57}$$

There is no corresponding statement for BD for $p \le 1$. It has to do with the cancellations. Note that we assume *a priori* knowledge for h to make sense of the action of the semigroup. As we shall see, if we only have B accretive on the range on D, our argument provides us with the weaker bound

$$\|h\|_{H^p} \lesssim \|\widetilde{N}_*(e^{-t|DB|}h)\|_p + \|\widetilde{N}_*(e^{-t|DB|}(\operatorname{sgn}(DB)h))\|_p.$$

We begin with the following Caccioppoli inequality.

Lemma 9.12. — *Assume B is pointwise accretive and $F, \partial_t F, DBF \in L^2_{\text{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^N)$. Assume F is a solution of*

$$\iint \langle \partial_t F, \partial_t G\rangle + \langle BDBF, DG\rangle = 0, \tag{58}$$

for all compactly supported $G \in L^2_{\text{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^N)$ with $\partial_t G, DG \in L^2_{\text{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^N)$, the inner product being the one of $\mathbb{C}^N$. Then

$$\fint\!\!\fint_{W(t,x_0)} |\partial_t F|^2 + \fint\!\!\fint_{W(t,x_0)} |DBF|^2 \le \frac{C}{t^2} \fint\!\!\fint_{\widetilde{W}(t,x_0)} |F|^2, \tag{59}$$

where $W(t,x_0)$ is a Whitney box and $\widetilde{W}(t,x_0)$ a slightly enlarged Whitney box. The constant C depends on the ratio of enlargements, dimension and accretivity bounds for B. In particular this holds for $F(t,x) = e^{-t|DB|}h(x)$ with $h \in \overline{\mathsf{R}_2(D)}$.

Proof. — Let us begin with the end of the statement. If $h \in \overline{\mathsf{R}_2(D)}$, then by semigroup theory, for fixed t, F and $\partial_t F$ are in L^2, as well as

$$DBF = -\operatorname{sgn}(DB)\partial_t F$$

using the H^∞-calculus. Now we remark that F satisfies the equation $\partial_t^2 F = DBDBF$ because $|DB|^2 = DBDB$. Thus using the self-adjointness of D and the skew-adjointness of ∂_t, we obtain (58).

Let us prove (59) assuming (58). Let $\chi(s,y)$ be a real-valued smooth function with support in $\widetilde{W}(t,x_0)$, value 1 on $W(t,x_0)$ and $|\nabla\chi| \lesssim \frac{1}{t}$. It is enough to prove

$$\frac{\kappa}{2}\iint |\chi DBF|^2 + \frac{\kappa}{2}\iint |\chi\partial_t F|^2 \lesssim t^{-2} \iint_{\widetilde{W}(t,x_0)} |F|^2, \tag{60}$$

where κ is the accretivity constant for B. Let $D_\chi = [D,\chi]$. As in the proof of (44), D_χ is multiplication by a matrix supported on $\widetilde{W}(t,x_0)$ and bounded by Ct^{-1}. First,

$$\kappa \iint |\chi DBF|^2 \le \kappa \iint |D(\chi BF)|^2 + Ct^{-2} \iint_{\widetilde{W}(t,x_0)} |F|^2.$$

Then, the accretivity of B on the range of D yields

$$\kappa \iint |D(\chi BF)|^2 \le \operatorname{Re} \iint \langle BD(\chi BF), D(\chi BF)\rangle$$

and the right hand side can be computed using

$$\begin{aligned}\iint \langle BD(\chi BF), D(\chi BF)\rangle &= \iint \langle BD_\chi BF, D(\chi BF)\rangle + \iint \langle \chi BDBF, D(\chi BF)\rangle \\ &= \iint \langle BD_\chi BF, D_\chi BF\rangle - \iint \langle BD_\chi BF), \chi DBF\rangle \\ &\quad - \iint \langle BDBF, D_\chi(\chi BF)\rangle + \iint \langle BDBF, D(\chi^2 BF)\rangle.\end{aligned}$$

In the last four integrals, the first is on the right order and the second and third are controlled by absorption inequalities isolating χDBF and we arrive at

$$\frac{\kappa}{2}\iint |\chi DBF|^2 \lesssim \operatorname{Re}\iint \langle BDBF, D(\chi^2 BF)\rangle + Ct^{-2}\iint_{\widetilde{W}(t,x_0)} |F|^2. \tag{61}$$

Similarly, using the pointwise accretivity of B,

$$\begin{aligned}\kappa\iint |\chi\partial_t F|^2 &\leq \operatorname{Re}\iint \langle \chi\partial_t F, B\chi\partial_t F\rangle \\ &= \operatorname{Re}\iint \langle \partial_t F, \partial_t(\chi^2 BF)\rangle + 2\operatorname{Re}\iint \langle \chi\partial_t F, \partial_t\chi BF\rangle.\end{aligned}$$

Again, by absorption inequalities, we obtain

$$\frac{\kappa}{2}\iint |\chi\partial_t F|^2 \leq \operatorname{Re}\iint \langle \partial_t F, \partial_t(\chi^2 F)\rangle + Ct^{-2}\iint_{\widetilde{W}(t,x_0)} |F|^2. \tag{62}$$

Combining the two estimates (61) and (62), and using (58), prove (60), hence the lemma. $\square$

REMARK 9.13. — If we only assume the accretivity of B on $\overline{\mathsf{R}_2(D)}$ then it is not clear how to dominate $\iint |\chi\partial_t F|^2$ by an expression involving F. If $F = e^{-t|DB|}h$ then observing that $\partial_t F = -DB(\operatorname{sgn}(DB)F)$ and one can repeat the proof of (61) which we have done on purpose using only the accretivity of B on the range. But this brings an average of $t^{-2}|\operatorname{sgn}(DB)F|^2$ in the right hand side, which means replacing h by $\operatorname{sgn}(DB)h$.

Proof of proposition 9.11. — We use auxiliary functions. Let a, b be the constants such that the function $\rho = a1_{[1,2)} + b1_{[2,3)}$ satisfies $\int \rho(s)\,ds = 1$ and $\int \rho(s)s\,ds = 0$. Define the bounded holomorphic function

$$m(z) = \int_1^3 \rho(s)e^{-s[z]}\,ds$$

in the half-planes $\operatorname{Re} z > 0$ and $\operatorname{Re} z < 0$ and at $z = 0$ with $m(0) = 1$. So one has $m(tDB)$ is well defined by the H^∞-calculus. Let

$$\tilde\rho(t) = -\int_1^t \rho(s)s\,ds = \int_t^\infty \rho(s)s\,ds.$$

Thus $\tilde{\rho}$ has support in $[1,3]$ as well. Integrating by parts, we have

$$\begin{aligned} m'(z) &= -\operatorname{sgn}(z)\int_1^3 \rho(s)se^{-s[z]}\,ds \\ &= \operatorname{sgn}(z)\int_1^3 \tilde{\rho}(s)[z]e^{-s[z]}\,ds = \int_1^3 \tilde{\rho}(s)ze^{-s[z]}\,ds. \end{aligned}$$

Now, set $F_t = e^{-t|DB|}h$, $G_t = m(tDB)h$ and $\widetilde{G}_t = m'(tDB)h$. We have

$$G_t = \int_1^3 \rho(s)F_{st}\,ds, \quad \widetilde{G}_t = \int_1^3 \frac{\tilde{\rho}(s)}{s}(stDBF_{st})\,ds,$$

and it follows from the support in [1,3] of ρ and $\tilde{\rho}$ that

$$\widetilde{N}_*(G) + \widetilde{N}_*(\widetilde{G}) \lesssim \widetilde{N}_*(F) + \widetilde{N}_*(tDBF).$$

Thus, using lemma 9.12 and adjusting the parameters in Whitney boxes, it suffices to prove

$$\|h\|_{H^p} \lesssim \|\widetilde{N}_*(G)\|_p + \|\widetilde{N}_*(\tilde{G})\|_p.$$

Using the formula for G_t, and $F_t \to h$ in $\mathcal{H}$ when $t \to 0$ and $h \in \overline{\mathsf{R}_2(D)}$, we have $G_t \to h$ in L^2 (convergence in the Schwartz distributions suffices for this argument) as $t \to 0$. To evaluate the H^p norm, we use the maximal characterisation of Fefferman and Stein: Let $\varphi(y) = r^{-n}\phi(\frac{x-y}{r}) = \phi_r(x-y)$ for some fixed function ϕ assumed to be C^∞, real-valued, compactly supported in $B(0,c_1)$ with $\int\phi = 1$. It is enough to prove

$$\Big|\int_{\mathbb{R}^n} h\varphi\Big| \lesssim \widetilde{N}_*(G)(x) + \boldsymbol{M}_{\frac{n}{n+1}}(\widetilde{N}_*(\widetilde{G}))(x), \tag{63}$$

since this shows that $\sup_{r>0}|h*\phi_r|$ is controlled by an L^p function as desired. The argument works for $\frac{n}{n+1} < p \leq 1$ by the Fefferman-Stein's theorem, but also for $1 < p < \infty$ by Lebesgue's theorem.

To prove (63), let $\chi(t)$ be an L^∞-normalized, scalar, bump function on $[0,\infty)$: it is C^1, supported in $[0,c_0r)$ with value 1 on $[0,c_0^{-1}r]$ and $\|\chi\|_\infty + r\|\chi'\|_\infty \lesssim 1$. The function $\Phi(s,y) = \varphi(y)\chi(s)$ is an extension of φ to $\mathbb{R}^{1+n}_+$. Thus

$$\int_{\mathbb{R}^n} h\varphi = -\iint_{\mathbb{R}^{1+n}_+}\partial_s(G\Phi) = -\iint_{\mathbb{R}^{1+n}_+} G\partial_s\Phi - \iint_{\mathbb{R}^{1+n}_+}\partial_s G\Phi = \mathrm{I} + \mathrm{II}.$$

Note that the integrand of I is supported in the Whitney box $W(r,x)$, so this integral is dominated by $\widetilde{N}_*G(x)$. For II, observe that

$$\partial_s G = DBm'(sDB)h = DB\widetilde{G}_s.$$

Integrating D by parts, and using the boundedness of B, we obtain

$$\Big|\iint_{\mathbb{R}^{1+n}_+}\partial_s G\Phi\Big| \lesssim \iint_T |\widetilde{G}|\|\nabla_y\Phi\|_\infty \lesssim r^{-n-1}\iint_T |\widetilde{G}|,$$

where $T := (0, c_0 r) \times B(x, c_1 r)$. Then, using the inequality

$$\iint_{\mathbb{R}^{1+n}_+} |u| \lesssim \|\widetilde{N}_* u\|_{\frac{n}{n+1}}$$

found in [**61**] for $u = |\widetilde{G}| 1_T$ and support considerations, we obtain

$$r^{-n-1} \iint_T |\widetilde{G}| \lesssim \Big(r^{-n} \int_{(1+c_0)B(x,c_1 r)} (\widetilde{N}_*(\widetilde{G}))^{\frac{n}{n+1}} \Big)^{\frac{n+1}{n}}$$

and (63) is proved. □

Remark 9.14. — An examination of the argument above shows that one can take the q-variant $\widetilde{N}^q_*$ with any $q \in [1, 2]$.

Proposition 9.15. — *Let $T = DB$ and $\frac{n}{n+1} < p < \infty$. Then one has the estimate*

$$\|h\|_{H^p} \lesssim \|\widetilde{N}_*(e^{-t|DB|} h)\|_p, \quad \forall h \in \mathbb{H}^{2,\pm}_{DB}. \tag{64}$$

Here the difference is that we restrict h in one of the spectral spaces.

Proof. — If $h \in \mathbb{H}^{2,+}_{DB}$, then $F = e^{-t|DB|}h = e^{-tDB}\chi^+(DB)h$ and $\partial_t F = -DBF$. Thus we can run the previous argument with F replacing G and get the inequality (63) with F replacing both G and $\widetilde{G}$.

When $h \in \mathbb{H}^{2,-}_{DB}$, then $F = e^{-t|DB|}h = e^{tDB}\chi^-(DB)h$ and $\partial_t F = DBF$, so that we conclude as above. □

9.3. Some upper bounds for $p \neq 2$

Proposition 9.16. — *Let $T = DB$ or BD and $2 < p < (p_+(T))^*$. Then one has the estimate*

$$\|\widetilde{N}_*(e^{-t|T|}h)\|_p \lesssim \|h\|_p, \quad \forall h \in L^2. \tag{65}$$

Proof. — Write $e^{-t|T|}h = \psi(tT)h + (I + itT)^{-1}h$ where

$$\psi(z) = e^{-[z]} - (1 + iz)^{-1} \in \Psi^1_1(S_\mu).$$

By geometric considerations,

$$\|\widetilde{N}_*(\psi(tT)h)\|_p \lesssim \|\psi(tT)h\|_{T^p_2}$$

and we may apply corollary 5.18 to obtain

$$\|\psi(tT)h\|_{T^p_2} \lesssim \|h\|_p$$

in the given range of p. Next, the L^2 off-diagonal estimates (2.3) for the resolvent $(I + itT)^{-1}$ yields the pointwise estimate

$$\widetilde{N}_*((I + itT)^{-1}h) \lesssim \boldsymbol{M}_2(|h|)$$

which gives an L^p estimate for all $2 < p \leq \infty$. □

Note that the argument for BD provides a proof of the assertion in remark 9.5.

We continue with some upper bounds when $p < 2$.

PROPOSITION 9.17. — 1) *For* $p_-(BD) < p < 2$ *and for all* $h \in \overline{\mathsf{R}_2(BD)}$, *we have*

$$\|\widetilde{N}_*(e^{-t|BD|}h)\|_p \lesssim \|h\|_p.$$

2) *For* $(p_-(DB))_* < p < 2$ *and for all* $h \in \overline{\mathsf{R}_2(D)}$, *we have*

$$\|\widetilde{N}_*(e^{-t|DB|}h)\|_p \lesssim \|h\|_{H^p},$$

where $H^p = L^p$ *if* $p > 1$.

Proof. — The first item follows from lemma 4.15 and theorem 5.3: for $h \in \overline{\mathsf{R}_2(BD)}$ and $p_-(BD) < p < 2$

$$\|\widetilde{N}_*(e^{-t|BD|}h)\|_p \lesssim \|h\|_{\mathbb{H}^p_{BD}} \sim \|\mathbb{P}h\|_p \sim \|h\|_p.$$

The equivalence $\|h\|_p \sim \|\mathbb{P}h\|_p$ for all $h \in \overline{\mathsf{R}_2(BD)}$ in this range of p was obtained in proposition 3.8.

The second item follows from lemma 4.15 and theorem 5.1: for $h \in \overline{\mathsf{R}_2(D)}$,

$$\|\widetilde{N}_*(e^{-t|DB|}h)\|_p \lesssim \|h\|_{\mathbb{H}^p_{DB}} \sim \|h\|_{H^p}. \qquad \square$$

9.4. End of proof of theorem 9.4

For the lower bounds, combine propositions 9.10 and 9.11 when B is pointwise accretive and proposition 9.15 in general. We note that we do not use the assumption on equality of Hardy spaces in the statement.

We turn to upper bounds. So far we have completed the theorem when $p > p_-(DB)$ on applying propositions 9.16 and 9.17, 2). But by theorem 5.1, the argument of proposition 9.17, 2), applies when $p < 2$ is such that $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms. This concludes the proof.

9.5. End of proof of theorem 9.5

Combining propositions 9.10, 9.16 and 9.17 gives all the lower bounds for any $p > 1$ and also the upper bounds in the range $p_-(BD) < p < (p_+(DB))^*$ by specializing to $\mathbb{P}h$ for $h \in \overline{\mathsf{R}_2(BD)}$.

It remains to provide an argument for upper bounds when $p \geq (p_+(DB))^*$ and $p = q'$ where $q > 1$ is assumed such that $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalence of norms. We do this now.

As in the proof of theorem 9.9, observe that our assumption implies for h in $\overline{\mathsf{R}_2(BD)}$,

$$\|\mathbb{P}\chi^{\pm}(BD)h\|_p \lesssim \|\mathbb{P}h\|_p.$$

Now $\phi(z) = e^{-[z]} \in \mathcal{R}^2_\sigma(S_\mu) \cap \Psi^\tau_0(S_\mu)$ for any $\sigma > 0$ and $\tau > 0$. Pick $\phi_\pm \in \mathcal{R}^2(S_\mu)$ such that

$$|\phi(z) - \phi_\pm(z)| = O(|z|^\sigma), \quad \forall z \in S_{\mu\pm}.$$

Then $\psi_\pm(z) := (\phi - \phi_\pm)(z)\chi^\pm(z)$ satisfy $\psi_\pm \in \Psi^2_\sigma(S_\mu)$. Hence, for $h \in \overline{\mathsf{R}_2(BD)}$, using

$$h = \chi^+(BD)h + \chi^-(BD)h = h^+ + h^-,$$

we have the decomposition

$$\phi(tBD)\mathbb{P}h = \psi_+(tBD)\mathbb{P}h + \psi_-(tBD)\mathbb{P}h + \phi_+(tBD)\mathbb{P}h^+ + \phi_-(tBD)\mathbb{P}h^-.$$

From geometric considerations, we deduce from lemma 5.16 if σ is large enough

$$\|\widetilde{N}_*(\psi_+(tBD)\mathbb{P}h)\|_p \lesssim \|\psi_+(tBD)\mathbb{P}h\|_{T^p_2} \lesssim \|\mathbb{P}h\|_p$$

and similarly for the term with ψ_-. Next, the L^2 off-diagonal estimates of lemma 2.3 for the combinations of iterates of resolvents $(I + itT)^{-2}$ yields the pointwise estimate

$$\widetilde{N}_*(\phi_+(tBD)\mathbb{P}h^+) \lesssim \boldsymbol{M}_2(|\mathbb{P}h^+|)$$

Thus, as $p > 2$ and using the assumption on p,

$$\|\widetilde{N}_*(\phi_+(tBD)\mathbb{P}h^+)\|_p \lesssim \|\mathbb{P}h^+\|_p \lesssim \|\mathbb{P}h\|_p.$$

We argue similarly for $\phi_-(tBD)\mathbb{P}h^-$. This finishes the proof.

10. NON-TANGENTIAL SHARP FUNCTIONS FOR BD

As we saw, the non-tangential maximal inequality that involves the pre-Hardy space $\mathbb{H}^p_{BD}$ is with $e^{-t|BD|}\mathbb{P}$, that is taking the semigroup after having projected on $\overline{\mathsf{R}_2(D)}$. The problem with $\mathbb{P}$ is one cannot use kernel estimates in such a context as it is a singular integral operator.

Also when for some reason (for example $p_+ > n$), we want to reach BMO or $\dot{\Lambda}^\alpha$ spaces, the non-tangential maximal function is inappropriate.

We observe that for all $h \in L^2$ and all $t > 0$ we have the following relation

$$e^{-t|BD|}\mathbb{P}h - \mathbb{P}h = e^{-t|BD|}h - h. \tag{66}$$

Indeed, $g = \mathbb{P}h - h \in \mathsf{N}_2(D) = \mathsf{N}_2(BD)$, so that $e^{-t|BD|}g = g$ for all $t > 0$.

We are therefore led to consider

$$\widetilde{N}_\sharp(e^{-t|BD|}h) := \widetilde{N}_*(e^{-t|BD|}h - h),$$

which we name *non-tangential sharp function (of* $e^{-t|BD|}h$*) associated to* BD. Thanks to (66), we have

$$|\widetilde{N}_\sharp(e^{-t|BD|}h) - \widetilde{N}_*(e^{-t|BD|}\mathbb{P}h)| \le \boldsymbol{M}_2(|\mathbb{P}h|).$$

Thus, if $2 < p$, $\widetilde{N}_\sharp(e^{-t|BD|}h)$ and $\widetilde{N}_*(e^{-t|BD|}\mathbb{P}h)$ have same L^p behavior. In particular,

$$\|\widetilde{N}_\sharp(e^{-t|BD|}h)\|_p \lesssim \|\mathbb{P}h\|_p$$

holds in the range of $p > 2$ where the same upper bound holds for $\widetilde{N}_*(e^{-t|BD|}\mathbb{P}h)$. If this range is all $(2,\infty)$ we may wonder what happens at $p = \infty$.

It is also convenient to introduce the $\alpha \ge 0$ variant of $\widetilde{N}_\sharp$:

$$\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)(x) = \sup_{t>0} t^{-\alpha}\Big(\iint_{W(t,x)} |e^{-s|BD|}h - h|^2\Big)^{1/2}.$$

Note that for $\alpha = 0$, this is $\widetilde{N}_\sharp$.

Theorem 10.1. — *Assume that for some q with $\frac{n}{n+1} < q < 2$, we have $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalent norms. If $q > 1$ and $p = q'$, we have*

$$\|\widetilde{N}_\sharp(e^{-t|BD|}h)\|_p \sim \|\mathbb{P}h\|_p, \quad \forall h \in \overline{\mathsf{R}_2(BD)},$$

and if $q \le 1$ and $\alpha = n(\frac{1}{q} - 1)$,

$$\|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\|_\infty \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha}, \quad \forall h \in \overline{\mathsf{R}_2(BD)}.$$

This result rests on two lemmata.

LEMMA 10.2. — *For $2 < p \leq \infty$, we have*

$$\|h\|_{\mathbb{H}^p_{BD}} \lesssim \|\widetilde{N}_\sharp(e^{-t|BD|}h)\|_p, \quad \forall h \in \overline{R_2(BD)},$$

and for $0 \leq \alpha < 1$,

$$\|h\|_{\mathbb{L}^\alpha_{BD}} \lesssim \|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\|_\infty, \quad \forall h \in \overline{R_2(BD)}.$$

LEMMA 10.3. — *For $2 < p \leq \infty$, we have*

$$\|\widetilde{N}_\sharp(e^{-t|BD|}h)\|_p \lesssim \|\mathbb{P}h^+\|_p + \|\mathbb{P}h^-\|_p + \|h\|_{\mathbb{H}^p_{BD}}, \quad \forall h \in \overline{R_2(BD)},$$

and for $0 \leq \alpha < 1$,

$$\|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\|_\infty \lesssim \|\mathbb{P}h^+\|_{\dot{\Lambda}^\alpha} + \|\mathbb{P}h^-\|_{\dot{\Lambda}^\alpha} + \|h\|_{\mathbb{L}^\alpha_{BD}}, \quad \forall h \in \overline{R_2(BD)},$$

where $h^\pm = \chi^\pm(BD)h$.

Let us admit the lemmas and prove the theorem.

As seen many times, if $q > 1$ and $p = q'$, the hypothesis implies that

$$\|\mathbb{P}h^+\|_p + \|\mathbb{P}h^-\|_p \sim \|\mathbb{P}h\|_p \sim \|h\|_{\mathbb{H}^p_{BD}}.$$

If $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$ then

$$\|\mathbb{P}h^+\|_{\dot{\Lambda}^\alpha} + \|\mathbb{P}h^-\|_{\dot{\Lambda}^\alpha} \sim \|\mathbb{P}h\|_{\dot{\Lambda}^\alpha} \sim \|h\|_{\mathbb{L}^\alpha_{BD}}.$$

The conclusion follows right away.

Proof of lemma 10.2. — To prove this result, we introduce the Carleson function

$$C_\alpha F(x) := \sup \Big(\frac{1}{r^{2\alpha}|B(y,r)|} \iint_{T_{y,r}} |F(t,z)|^2 \, \frac{dtdz}{t} \Big)^{1/2},$$

the supremum being taken over all open balls $B(y,r)\ni x$ in $\mathbb{R}^n$ and $T_{y,r} = (0,r)\times B(y,r)$. For $0 \leq \alpha < 1$ and a suitable allowable ψ for both $\mathbb{H}^p_{BD}$ and $\mathbb{L}^\alpha_{BD}$, we shall show the pointwise bound

$$C_\alpha(\psi(tBD)h) \lesssim \boldsymbol{M}_2\big(\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\big), \quad \forall h \in \overline{R_2(BD)}. \tag{67}$$

Admitting this inequality, we have

$$\|h\|_{\mathbb{H}^p_{BD}} \lesssim \|\psi(tBD)h\|_{T^p_2} \lesssim \|C_0(\psi(tBD)h)\|_p \lesssim \|\widetilde{N}_\sharp(e^{-t|BD|}h)\|_p.$$

The first inequality is the lower bound valid for any $\psi \in \Psi(S_\mu)$, the second inequality is from [**40**, theorem 3(a)] and the last one uses (67), the maximal theorem and $p > 2$. Similarly

$$\|h\|_{\mathbb{L}^\alpha_{BD}} \lesssim \|\psi(tBD)h\|_{T^\infty_{2,\alpha}} = \|C_\alpha(\psi(tBD)h)\|_\infty \lesssim \|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\|_\infty.$$

We turn to the proof of (67). We adapt an argument in [**47**], theorem 2.14, to our situation. We choose

$$\widetilde{\psi}(z) = z^N e^{-[z]} \quad \text{and} \quad \psi(z) = \widetilde{\psi}(z)(e^{-[z]} - 1)$$

so that $\widetilde{\psi} \in \Psi_N^\tau(S_\mu)$ and $\psi \in \Psi_{N+1}^\tau(S_\mu)$ for all $\tau > 0$. The integer N will be chosen large. It will be convenient to set $P_t = e^{-t|BD|}$, so that

$$\widetilde{\psi}(tBD) = (tBD)^N P_t \quad \text{and} \quad \psi(tBD) = (tBD)^N P_t(P_t - I).$$

We fix $h \in \overline{\mathsf{R}_2(BD)}$ and $x \in \mathbb{R}^n$. Consider $T_{y,r} = (0,r) \times B(y,r)$ such that $x \in B(y,r)$. Recall that

$$W(t,z) := (c_0^{-1}t, c_0 t) \times B(z, c_1 t),$$

for some fixed constants $c_0 > 1$, $c_1 > 0$. We set $I_t = (c_0^{-1}t, c_0 t)$.

Set $g = h - \fint_{I_r} P_\tau h\, d\tau$ and consider

$$I(y,r) = \iint_{T_{y,r}} |\psi(sBD)g(z)|^2 \frac{dsdz}{s}.$$

Pick $a > 0$ such that the balls $B_k = B(x + akr, \frac{c_1}{2}r)$, $k \in \mathbb{Z}^n$, cover $\mathbb{R}^n$ with bounded overlap. We set $g_k = g 1_{B_k}$. If $B_k \cap 2B(y,r) \neq \varnothing$, which occurs for boundedly (with respect to x, y, r) many k then we use the square function estimate and definition of g_k to obtain

$$\iint_{T_{y,r}} |\psi(sBD)g_k(z)|^2 \frac{dsdz}{s} \lesssim \|g_k\|_2^2 \leq |B_k| \fint\!\!\!\fint_{I_r \times B_k} |h - P_\tau h|^2.$$

If $B_k \cap 2B(y,r) = \varnothing$, which occurs when $|k| \geq K$ for some integer $K \neq 0$, then we can use the L^2 off-diagonal decay (19) for each s to obtain

$$\iint_{T_{y,r}} |\psi(sBD)g_k(z)|^2 \frac{dsdz}{s} \lesssim |k|^{-2(N+1)}\|g_k\|_2^2 \leq |k|^{-2(N+1)}|B_k| \fint\!\!\!\fint_{I_r \times B_k} |h - P_\tau h|^2.$$

For $N + 1 > n$, we obtain (using Minkowski inequality for the integral followed by Cauchy-Schwarz inequality for the sum)

$$I(y,r) \lesssim \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-N-1} |B_k| \fint\!\!\!\fint_{I_r \times B_k} |h - P_\tau h|^2.$$

Now observe that $|B_k| = 2^{-n}|B(z, c_1 r)|$ and if $z \in B_k$, then $B_k \subset B(z, c_1 r)$. Hence

$$\begin{aligned}
|B_k| \fint\!\!\!\fint_{I_r \times B_k} |h - P_\tau h|^2 &\leq 2^n |B_k| \inf_{z \in B_k} \fint\!\!\!\fint_{W(r,z)} |h - P_\tau h|^2 \\
&\leq 2^n r^{2\alpha} |B_k| \inf_{z \in B_k} \widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)^2(z) \\
&\leq 2^n r^{2\alpha} \int_{B_k} \widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)^2(z)\, dz
\end{aligned}$$

and this implies

$$\begin{aligned}
I(y,r) &\lesssim r^{2\alpha} \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-N-1} \int_{B_k} \widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)^2(z)\, dz \\
&\lesssim \boldsymbol{M}_2\big(\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}h)\big)^2(x) r^{n+2\alpha},
\end{aligned}$$

where the last inequality uses the bounded overlap of the balls B_k and requires $N+1>n$. Next, we bound

$$J(y,r)=\iint_{T_{y,r}}\left|\psi(sBD)\Big(\fint_{I_r}P_\tau h\,d\tau\Big)(z)\right|^2\frac{dsdz}{s}.$$

We compute

$$\begin{aligned}\psi(sBD)P_\tau&=(sBD)^N P_{s+\frac{\tau}{2}}(P_{s+\frac{\tau}{2}}-P_{\frac{\tau}{2}})\\&=(sBD)^N P_{s+\frac{\tau}{2}}(P_{s+\frac{\tau}{2}}-I)+(sBD)^N P_{s+\frac{\tau}{2}}(I-P_{\frac{\tau}{2}}).\end{aligned}$$

Let us call $J_1(y,r)$ and $J_2(y,r)$ the integrals corresponding to the first term and second term respectively. We first handle J_2. Use $s\le s+\frac{\tau}{2}$, change variable $s\mapsto s+\frac{\tau}{2}$, and observe that as $\tau\in I_r$ and $0<s<r$, we have $s+\frac{\tau}{2}\in[\frac{c_0^{-1}}{2}r,r+\frac{c_0}{2}r]=J_r$. Thus,

$$\begin{aligned}J_2(y,r)&\lesssim\fint_{I_r}\int_{B(y,r)}\int_{J_r}|\widetilde{\psi}(sBD)(h-P_{\frac{\tau}{2}}h)(z)|^2\frac{ds}{s}dzd\tau\\&=\fint_{I_{\frac{r}{2}}}\int_{B(y,r)}\int_{J_r}|\widetilde{\psi}(sBD)(h-P_\tau h)(z)|^2\frac{ds}{s}dzd\tau.\end{aligned}$$

We use the L^2 off-diagonal estimates for $\widetilde{\psi}(sBD)$ with $N>n$, which are uniform in $s\in J_r$, and obtain the desired bound on $J_2(y,r)$ with the same analysis (change r to $\frac{r}{2}$ in the definition of the balls B_k) as above.

For J_1, we operate the same change of variable to get

$$\begin{aligned}J_1(y,r)&\lesssim\fint_{I_r}\int_{B(y,r)}\int_{J_r}|\widetilde{\psi}(sBD)(P_sh-h)(z)|^2\frac{ds}{s}dzd\tau\\&=\int_{B(y,r)}\int_{J_r}|\widetilde{\psi}(sBD)(P_sh-h)(z)|^2\frac{ds}{s}dz.\end{aligned}$$

Now, we observe that J_r can be covered by a bounded (with respect to r) number of interval $I_{c_0^{2i}\frac{r}{2}}$ We proceed a similar analysis as before for each integral

$$\int_{B(y,r)}\int_{I_{c_0^{2i}\frac{r}{2}}}$$

with the appropriate B_k type balls, use the L^2 off-diagonal estimates for $\widetilde{\psi}(sBD)$ with $N>n$. This leads to the same bound for $J_1(y,r)$ as for $I(y,r)$. We leave details to the reader. $\square$

Proof of lemma 10.3. — We begin with the L^p estimates and proceed exactly as in the proof of theorem 9.3. We have $\phi(z)=e^{-[z]}\in\mathcal{R}^2_\sigma(S_\mu)\cap\Psi_0^\tau(S_\mu)$ for any $\sigma>0$ and $\tau>0$. Pick $\phi_\pm\in\mathcal{R}^2(S_\mu)$ such that

$$|\phi(z)-\phi_\pm(z)|=O(|z|^\sigma),\quad\forall z\in S_{\mu\pm}.$$

Then $\psi_\pm(z):=(\phi-\phi_\pm)(z)\chi^\pm(z)$ satisfy $\psi_\pm\in\Psi^2_\sigma(S_\mu)$. Hence, for $h\in\overline{\mathsf{R}_2(BD)}$, using

$$h=\chi^+(BD)h+\chi^-(BD)h=h^++h^-,$$

we have the decomposition

$$\phi(tBD)\mathbb{P}h - \mathbb{P}h = \psi_+(tBD)\mathbb{P}h + \psi_-(tBD)\mathbb{P}h \\ + \phi_+(tBD)\mathbb{P}h^+ - \mathbb{P}h^+ + \phi_-(tBD)\mathbb{P}h^- - \mathbb{P}h^-.$$

From geometric considerations, we deduce from lemma 5.16 if σ is large enough

$$\|\widetilde{N}_*(\psi_+(tBD)\mathbb{P}h)\|_p \lesssim \|\psi_+(tBD)\mathbb{P}h\|_{T^p_2} \lesssim \|h\|_{\mathbb{H}^p_{BD}}$$

and similarly for the term with ψ_-. Next, the L^2 off-diagonal estimates of lemma 2.3 for the combinations of iterates of resolvent $(I+itT)^{-2}$ yield the pointwise estimate

$$\widetilde{N}_*(\phi_+(tBD)\mathbb{P}h^+ - \mathbb{P}h^+) \lesssim \boldsymbol{M}_2(|\mathbb{P}h^+|).$$

Thus, as $p>2$,

$$\|\widetilde{N}_*(\phi_+(tBD)\mathbb{P}h^+ - \mathbb{P}h^+)\|_p \lesssim \|\mathbb{P}h^+\|_p.$$

We argue similarly for $\phi_-(tBD)\mathbb{P}h^-$. This proves the first estimate since

$$\phi(tBD)\mathbb{P}h - \mathbb{P}h = \phi(tBD)h - h.$$

For the Hölder estimates, we use the same decomposition and observe that $\widetilde{N}_{\sharp,\alpha}(g) \lesssim C_\alpha g$ pointwise. Hence, for σ large enough,

$$\|\widetilde{N}_{\sharp,\alpha}(\psi_+(tBD)\mathbb{P}h)\|_\infty \lesssim \|\psi_+(tBD)\mathbb{P}h\|_{T^\infty_{2,\alpha}} \lesssim \|h\|_{\mathbb{L}^\alpha_{BD}}$$

and similarly for the term with ψ_-. Next, we fix a Whitney box $W(t,x)$ and let $c^\pm$ be the average of $\mathbb{P}h^\pm$ on the ball $B(x,c_1t)$. Then we write

$$\phi_+(sBD)\mathbb{P}h^+ - \mathbb{P}h^+ = \phi_+(sBD)(\mathbb{P}h^+ - c^+) - (\mathbb{P}h^+ - c^+).$$

The L^2 off-diagonal estimates of lemma 2.3 for the combinations of iterates of resolvent $(I+itT)^{-2}$ yield the pointwise estimate

$$\widetilde{N}_{\sharp,\alpha}(\phi_+(sBD)\mathbb{P}h^+)^2(x) \lesssim \sup_{t>0} t^{-\alpha} \fint_{B(x,c_1t)} |\mathbb{P}h^+ - c^+|^2$$

which leads to the estimate

$$\|\widetilde{N}_{\sharp,\alpha}(\phi_+(sBD)\mathbb{P}h^+)\|_\infty \lesssim \|\mathbb{P}h^+\|_{\dot{\Lambda}^\alpha}.$$

The argument for $\phi_-(tBD)\mathbb{P}h^-$ is similar. □

11. SOBOLEV SPACES FOR DB AND BD

So far, we have privileged the L^2 theory: we considered estimates with *a priori* knowledge for h in the closure of the L^2 range. But this is only for convenience. As mentioned in the introduction, we can consider a Sobolev theory as well and relax this *a priori* information on h. This is required for use of energy spaces. For any bisectorial operator with a H^∞-calculus on the closure of its range, there is a Sobolev space theory associated to this operator as developed by means of quadratic estimates in this context in [**21**], extending many earlier works for self-adjoint operators, positive operators... (see the references there). But here, we want a theory that leads to concrete spaces.

For the operator DB, the relevant Sobolev theory is for regularity indices $s \in [-1, 0]$. For $s = 0$, this is already done. We shall do it for $s < 0$ in this section. This has been considered in some special cases for D in relation with the boundary value problems [**83, 20**]. For BD, things are more complicated. There are two options for regularity indices $0 \le s \le 1$: the Sobolev spaces associated to BD or the Sobolev spaces associated to the operators $\mathbb{P}BD$ after projecting by $\mathbb{P}$. The first theory leads to abstract spaces and the second to concrete spaces. They are both useful.

11.1. Definitions and properties

For convenience, we denote by $\mathcal{H}^0_D = \overline{\textsf{R}_2(D)}$ and $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$. Let $S = D|_{\mathcal{H}^0_D}$ with domain $\textsf{D}_2(D) \cap \mathcal{H}^0_D$. Then S is an injective, self-adjoint operator. Recall that $\mathbb{P}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}^0_D$. Let $\mathcal{B}$ be the operator on $\mathcal{H}^0_D$ defined by

$$\mathcal{B}h = \mathbb{P}Bh = \mathbb{P}B\mathbb{P}h$$

for $h \in \mathcal{H}^0_D$. Recall that as B is a strictly accretive operator on $\mathcal{H}^0_D$, the restriction of $\mathbb{P}$ on $B\mathcal{H}^0_D$ is an isomorphism onto $\mathcal{H}^0_D$ and $\mathcal{B}$ is a strictly accretive operator on $\mathcal{H}^0_D$.

Define

$$T : \mathcal{H}^0_D \longrightarrow \mathcal{H}^0_D, \quad T = \mathcal{B}S = \mathbb{P}BD_{|\mathcal{H}^0_D} \quad \text{with } \textsf{D}_2(T) = \textsf{D}_2(S),$$
$$\underline{T} : \mathcal{H}^0_D \longrightarrow \mathcal{H}^0_D, \quad \underline{T} = S\mathcal{B} = D\mathbb{P}B_{|\mathcal{H}^0_D} = DB_{|\mathcal{H}^0_D} \quad \text{with } \textsf{D}_2(\underline{T}) = \mathcal{B}^{-1}\textsf{D}_2(S).$$

Using proposition 2.1 and the comment that follows it, T and $\underline{T}$ are ω-bisectorial operators on $\mathcal{H}^0_D$. Moreover, they are injective. Observe also that

$$V : \overline{\textsf{R}_2(BD)} \to \overline{\textsf{R}_2(BD)}, \quad V = BD|_{\overline{\textsf{R}_2(BD)}} \quad \text{with } \textsf{D}_2(V) = \overline{\textsf{R}_2(BD)} \cap \textsf{D}_2(D)$$

is also an injective ω-bisectorial operator with H^∞-calculus on $\overline{\textsf{R}_2(BD)}$.

We remark that if $\psi \in \Psi(S_\mu)$, we have the intertwining relation

$$\psi(T)\mathbb{P}h = \mathbb{P}\psi(BD)h = \mathbb{P}\psi(V)h, \quad h \in \overline{R_2(BD)}, \tag{68}$$

and

$$\psi(\underline{T})h = \psi(DB)h, \quad h \in \mathcal{H}^0_D. \tag{69}$$

These relations are easily verified for the resolvent and then one uses (8). It follows that the operator norms of $\psi(T)$ and $\psi(\underline{T})$ are bounded by $C_\mu\|\psi\|_\infty$, which guarantees that T and $\underline{T}$ have H^∞-calculus on $\mathcal{H}^0_D$ and the two formulæ above extend to all $b \in H^\infty(S_\mu)$.

We define the Sobolev spaces next. We use the curly style $\mathcal{H}$ to distinguish them from pre-Hardy and Hardy spaces where we use the mathbb style $\mathbb{H}$ or roman style H.

For $s \in \mathbb{R}$, define the inhomogeneous Sobolev space associated with S, $\mathcal{H}^s_S$, as the subspace of $\mathcal{H}^0_D$ for which

$$\|h\|_{S,s} = \left\{\int_0^\infty t^{-2s}\|\psi_t(S)h\|_2^2\,\frac{dt}{t}\right\}^{1/2} < \infty$$

for a suitable $\psi \in \Psi(S_\mu)$, for example $\psi(z) = z^k e^{-[z]}$ and k an integer with $k > \max(s,0)$. We define the homogeneous Sobolev space associated with S, $\dot{\mathcal{H}}^s_S$, as the completion of $\mathcal{H}^s_S$ for $\|h\|_{S,s}$.

Remark that from the spectral theorem

$$\dot{\mathcal{H}}^0_S = \mathcal{H}^0_S = \mathcal{H}^0_D.$$

Next, it can be checked that $\|h\|_{S,s} = c_{\psi,s}\||S|^s h\|_2$ where $|S| = (S^2)^{1/2}$. As $S = D|_{\mathcal{H}^0_D}$, $\mathcal{H}^s_S$ is the closed subspace of the usual inhomogeneous Sobolev space $\mathcal{H}^s$, equal to the image of $\mathcal{H}^s$ under the projection $\mathbb{P}$, and similarly $\dot{\mathcal{H}}^s_S$ is the image of the usual homogeneous Sobolev space $\dot{\mathcal{H}}^s$ under (the extension of) $\mathbb{P}$ (which extends boundedly to $\dot{\mathcal{H}}^s$ as it is a smooth singular integral convolution operator). It is not hard to check that $\dot{\mathcal{H}}^s_S \cap \dot{\mathcal{H}}^0_S = \mathcal{H}^s_S$.

Note that the H^∞- and self-adjoint calculi of S on $\dot{\mathcal{H}}^0_S$ extend to $\dot{\mathcal{H}}^s_S$ and that S extends to an isomorphism between $\dot{\mathcal{H}}^s_S$ and $\dot{\mathcal{H}}^{s-1}_S$. Classically, the intersection of $\dot{\mathcal{H}}^s_S$ is dense in each of them. Here is a precise statement whose proof is left to the reader. Alternately, one can do this using the usual Sobolev spaces $\dot{\mathcal{H}}^s$ and project under $\mathbb{P}$.

Lemma 11.1. — *Let*

$$\theta(z) = ce^{-[z]-[z]^{-1}} \in \bigcap_{\sigma>0,\tau>0}\Psi^\tau_\sigma(S_\mu)$$

with $c^{-1} = \int_0^\infty \theta(t)\,\frac{dt}{t}$. *For any* $s \in \mathbb{R}$ *and* $h \in \dot{\mathcal{H}}^s_S$, $h_k = \int_{1/k}^k \theta(tS)h\,\frac{dt}{t} \in \bigcap_{s'\in\mathbb{R}}\mathcal{H}^{s'}_S$ *and converges to* h *in* $\dot{\mathcal{H}}^s_S$ *as* $k \to \infty$.

Having defined S and the associated Sobolev spaces, we use the more concrete notation $\dot{\mathcal{H}}^s_D = \dot{\mathcal{H}}^s_S$ and similarly for the inhomogeneous spaces.

We also use the notation DB for $\underline{T}$, BD for V, $\mathbb{P}BD$ for T.

We come back to the formal notation when needed for clarity in the proofs.

We define similarly the inhomogeneous Sobolev spaces $\mathcal{H}^s_{DB}$, $\mathcal{H}^s_{BD}$ and $\mathcal{H}^s_{\mathbb{P}BD}$ replacing S by $\underline{T}$, V and T respectively.

PROPOSITION 11.2. — *Let $s \in \mathbb{R}$.*

1) *The quadratic norms are equivalent under changes of suitable non-degenerate ψ.*
2) *The bounded holomorphic functional calculus extends : for any $b \in H^\infty(S_\mu)$, $b(X)$ is bounded on $\mathcal{H}^s_X$ if $X = DB, BD$ or $\mathbb{P}BD$.*
3) *$\mathbb{P} : \mathcal{H}^s_{BD} \to \mathcal{H}^s_{\mathbb{P}BD}$ is an isomorphism.*
4) *$\mathcal{H}^s_{DB}$ and $\mathcal{H}^{-s}_{B^*D}$ are in duality for the L^2 inner product.*
5) *$\mathcal{H}^s_{DB}$ and $\mathcal{H}^{-s}_{\mathbb{P}B^*D}$ are in duality for the L^2 inner product.*

Proof. — 1) is standard and we skip it. 2) is a straightforward consequence of the definitions of the spaces and of the norms. For 3), using the intertwining property (68), and the isomorphism $\mathbb{P} : \mathcal{H}^0_{BD} = \overline{\mathsf{R}_2(BD)} \to \overline{\mathsf{R}_2(D)} = \mathcal{H}^0_{\mathbb{P}BD}$, we obtain

$$\|\psi(\mathbb{P}BD)\mathbb{P}h\|_2 = \|\mathbb{P}\psi(BD)h\|_2 \sim \|\psi(BD)h\|_2$$

for all $h \in \mathcal{H}^0_{BD}$ and $\psi \in \Psi(S_\mu)$. We conclude easily for the isomorphism using the defining norms of the Sobolev spaces. The proof of 4) is a simple consequence of the Calderón reproducing formula so that for suitable ψ, φ we have

$$\langle f, g\rangle = (\mathbb{Q}_{\psi,DB}f, \mathbb{Q}_{\varphi,B^*D}g)$$

for all $f \in \mathcal{H}^0_{DB}$ and $g \in \mathcal{H}^0_{B^*D}$. We skip details. For 5), we use the intertwining property: for all $f \in \mathcal{H}^0_{DB}$ and $h \in \mathcal{H}^0_{\mathbb{P}B^*D}$, writing $h = \mathbb{P}g$ with $g \in \mathcal{H}^0_{B^*D}$

$$\langle f, h\rangle = \langle f, g\rangle = (\mathbb{Q}_{\psi,DB}f, \mathbb{Q}_{\varphi,B^*D}g) = (\mathbb{Q}_{\psi,DB}f, \mathbb{P}\mathbb{Q}_{\varphi,B^*D}g) = (\mathbb{Q}_{\psi,DB}f, \mathbb{Q}_{\varphi,\mathbb{P}B^*D}h)$$

and the conclusion follows easily. □

Now define their completions $\dot{\mathcal{H}}^s_{DB}$, $\dot{\mathcal{H}}^s_{BD}$ and $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$ respectively. So far, these completions are abstract spaces.

PROPOSITION 11.3. — 1) *For $s \in \mathbb{R}$, for all bounded holomorphic functions $b \in H^\infty(S_\mu)$, $b(\mathbb{P}BD)$ extends to a bounded operator on $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$. In particular, this holds for $\mathrm{sgn}(\mathbb{P}BD)$ which is a bounded self-inverse operator on $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$. Also, $\mathbb{P}BD$ and $|\mathbb{P}BD| = \mathrm{sgn}(\mathbb{P}BD)\mathbb{P}BD$ extend to isomorphisms between $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$ and $\dot{\mathcal{H}}^{s-1}_{\mathbb{P}BD}$. The operator $|\mathbb{P}BD|$ extends to a sectorial operator on $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$ and fractional powers $|\mathbb{P}BD|^\alpha$ are isomorphisms from $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$ onto $\dot{\mathcal{H}}^{s-\alpha}_{\mathbb{P}BD}$.*

2) *$\dot{\mathcal{H}}^s_{\mathbb{P}BD}$ topologically splits as the sum of the two spectral closed subspaces*

$$\dot{\mathcal{H}}^{s,+}_{\mathbb{P}BD} = \mathsf{N}(\mathrm{sgn}(\mathbb{P}BD) - I) = \mathsf{R}(\chi^+(\mathbb{P}BD)),$$
$$\dot{\mathcal{H}}^{s,-}_{\mathbb{P}BD} = \mathsf{N}(\mathrm{sgn}(\mathbb{P}BD) + I) = \mathsf{R}(\chi^-(\mathbb{P}BD)).$$

3) *The same two items hold with $\mathbb{P}BD$ replaced by DB or BD.*
4) *For $0 \le s \le 1$, $\dot{\mathcal{H}}^s_{\mathbb{P}BD} = \dot{\mathcal{H}}^s_D$ and for $-1 \le s \le 0$, $\dot{\mathcal{H}}^s_{DB} = \dot{\mathcal{H}}^s_D$ with equivalence of norms.*
5) *Furthermore, for $-1 \le s < 0$, we have for*
$$\|h\|_{D,s} \approx \left\{ \int_0^\infty t^{-2s} \|e^{-t|DB|}h\|_2^2 \,\frac{dt}{t} \right\}^{1/2}.$$
6) *For all $s \in \mathbb{R}$, $\mathbb{P}$ extends to an isomorphism from $\dot{\mathcal{H}}^s_{BD}$ onto $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$.*
7) *For all $s \in \mathbb{R}$, $\dot{\mathcal{H}}^s_{DB}$ and $\dot{\mathcal{H}}^{-s}_{B^*D}$ are dual spaces for a duality extending the L^2 inner product.*
8) *For all $s \in \mathbb{R}$, $\dot{\mathcal{H}}^s_{DB}$ and $\dot{\mathcal{H}}^{-s}_{\mathbb{P}B^*D}$ are dual spaces for a duality extending the L^2 inner product.*

Proof. — For 1)–5), this is the theory of [**21**], except for the cases $s = -1$ and $s = 1$ of 4), proved in [**20**, proposition 4.4] using the holomorphic functional calculus on L^2 for DB and BD.

Items 6)–8) are easy consequences of the previous proposition and density. □

Corollary 11.4. — *Let $-1 \le s \le 0$. Then $D : \dot{\mathcal{H}}^{s+1}_{\mathbb{P}BD} = \dot{\mathcal{H}}^{s+1}_D \to \dot{\mathcal{H}}^s_D = \dot{\mathcal{H}}^s_{DB}$ is an isomorphism. In particular, for $t > 0$ and $h \in \dot{\mathcal{H}}^{s+1}_{\mathbb{P}BD}$, we have*
$$De^{-t|\mathbb{P}BD|}h = e^{-t|DB|}Dh.$$
Similarly D extends to an isomorphism $\dot{\mathcal{H}}^{s+1}_{BD} \to \dot{\mathcal{H}}^s_{DB}$. In particular, for $t > 0$ and $h \in \dot{\mathcal{H}}^s_{BD}$, we have
$$De^{-t|BD|}h = e^{-t|DB|}Dh.$$

Proof. — Let us consider the first assertion. Take a suitable $\psi \in \Psi(S_\mu)$ and $h \in \mathsf{D}_2(S)$. Then $Dh = Sh$ and
$$\psi(\underline{T})Sh = \psi(DB)Dh = D\psi(BD)h = S\psi(T)h.$$
Then change $\psi(z)$ to $\psi(tz)$ and use the isomorphism property of S, the property 4) in the proposition above and also the density of $\mathsf{D}_2(S) = \mathcal{H}^1_D$ in $\dot{\mathcal{H}}^{s+1}_D$. For the second part, the extension is defined as $D \circ \mathbb{P}$, where $\mathbb{P}$ is the extension given in item 6) of the previous proposition and D is the isomorphism just described. □

Proposition 11.5. — *Let $0 < s \le 1$.*

1) *For any $h \in \dot{\mathcal{H}}^s_{BD}$, $e^{-t|BD|}h - h$ can be defined in L^2 with*
$$\|e^{-t|BD|}h - h\|_2 \le \mathrm{C}t^s.$$
2) *For any $h \in \dot{\mathcal{H}}^s_{\mathbb{P}BD}$, $e^{-t|\mathbb{P}BD|}h - h$ can be defined in L^2 with*
$$\|e^{-t|\mathbb{P}BD|}h - h\|_2 \le \mathrm{C}t^s.$$

3) *For any* $h \in \dot{\mathcal{H}}^s_{BD}$, *with the above definition*

$$\mathbb{P}(e^{-t|BD|}h - h) = e^{-t|\mathbb{P}BD|}\mathbb{P}h - \mathbb{P}h.$$

Proof. — For 1), observe that

$$\phi(z) = \frac{e^{-t[z]} - 1}{[z]^s} \in H^\infty(S_\mu)$$

with bound Ct^s and that $\||BD|^s h\|_2 \sim \|h\|_{BD,s}$ when $h \in \dot{\mathcal{H}}^s_{BD}$. The relation

$$e^{-t|BD|}h - h = \phi(BD)|BD|^s h$$

valid for $h \in \mathcal{H}^s_{BD}$ thus extends to $h \in \dot{\mathcal{H}}^s_{BD}$. The proof for the second item is the same. The third item is the intertwining property of the H^∞-calculi, extended to $\dot{\mathcal{H}}^s_{BD}$ and $\dot{\mathcal{H}}^s_{\mathbb{P}BD}$. □

11.2. A priori estimates

The following lemma tells us that we can use different norms, more suitable to extensions.

Lemma 11.6. — *We have*

$$\|Dh\|_{\dot{W}^{-1,p}} \sim \|h\|_p, \quad \forall p \in (1,\infty)\ \forall h \in \overline{R_2(D)},$$
$$\|Dh\|_{\dot{\Lambda}^{\alpha-1}} \sim \|h\|_{\dot{\Lambda}^\alpha}, \quad \forall \alpha \in [0,1)\ \forall h \in \overline{R_2(D)}.$$

Proof. — First, assume $h \in L^p$. Then $Dh \in \dot{W}^{-1,p}$ and if $g \in \dot{W}^{1,p'}$,

$$|\langle Dh, g\rangle| = |\langle h, Dg\rangle| \le \|h\|_p \|Dg\|_{p'} \lesssim \|h\|_p \|g\|_{\dot{W}^{1,p'}}.$$

We conclude $\|Dh\|_{\dot{W}^{-1,p}} \lesssim \|h\|_p$. For the converse, recall that $\mathcal{S}_0$ is the space of Schwartz functions with compactly supported Fourier transforms away from the origin. By density, we have $\|h\|_p = \sup\{|\langle h, g\rangle|; g \in \mathcal{S}_0, \|g\|_{p'} = 1\}$ and for $g \in \mathcal{S}_0$, we have $\mathbb{P}g \in \mathcal{S}_0$ as well, so

$$|\langle h, g\rangle| = |\langle h, \mathbb{P}g\rangle| = |\langle Dh, D^{-1}\mathbb{P}g\rangle| \lesssim \|Dh\|_{\dot{W}^{-1,p}} \|D^{-1}\mathbb{P}g\|_{\dot{W}^{1,p'}}.$$

Here, we observe that $D^{-1}\mathbb{P}g \in \mathcal{S}_0$ is a Schwartz distribution (using a Fourier transform argument) and as $\nabla D^{-1}\mathbb{P}$ is bounded on $L^{p'}$, we obtain

$$\|D^{-1}\mathbb{P}g\|_{\dot{W}^{1,p'}} \lesssim \|g\|_{p'} = 1.$$

Consider now the second statement. Clearly, $h \in \dot{\Lambda}^\alpha$ implies $Dh \in \dot{\Lambda}^{\alpha-1}$. For the converse, note that if $g \in \mathbb{P}\mathcal{S}_0$, then $D^{-1}g \in \dot{H}^{1,q}$. Indeed, $D^{-1}g \in \mathcal{S}'$, $\nabla D^{-1}g = \nabla D^{-1}\mathbb{P}g \in H^q$. Thus,

$$|\langle h, g\rangle| = |\langle Dh, D^{-1}g\rangle| \le \|Dh\|_{\dot{\Lambda}^{\alpha-1}} \|D^{-1}g\|_{\dot{H}^{1,q}} \lesssim \|Dh\|_{\dot{\Lambda}^{\alpha-1}} \|g\|_{H^q}.$$

By density of $\mathbb{P}(\mathcal{S}_0)$ in H^q_D, this implies that $h \in \dot{\Lambda}^\alpha$ with the desired estimate. □

We continue with the extension of the functional calculus of DB to negative Sobolev spaces of the type $\dot W^{-1,p}$ or negative Hölder spaces $\dot\Lambda^{\alpha-1}$ under the appropriate assumption.

PROPOSITION 11.7. — *Let $q\in(\frac{n}{n+1},p_+(DB^*))$ be such that $\mathbb{H}^q_{DB^*}=\mathbb{H}^q_D$ with equivalence of norms. Let $\mathcal{T}=T_2^{q'},Y=L^{q'},\dot Y^{-1}=\dot W^{-1,q'}$ if $q>1$ and $\mathcal{T}=T^\infty_{2,\alpha},Y=\dot\Lambda^\alpha,\dot Y^{-1}=\dot\Lambda^{\alpha-1}$ with $\alpha=n(\frac1q-1)$ if $q\le 1$. Let $b\in H^\infty(S_\mu)$. Then*

$$\|b(DB)h\|_{\dot Y^{-1}}\lesssim\|b\|_\infty\|h\|_{\dot Y^{-1}},\quad \forall h\in\bigcup_{-1\le s\le 0}\dot{\mathcal{H}}^s_D,,$$

$$\|Db(BD)\tilde h\|_{\dot Y^{-1}}\lesssim\|b\|_\infty\|D\tilde h\|_{\dot Y^{-1}},\quad \forall \tilde h\in\bigcup_{-1\le s\le 0}\dot{\mathcal{H}}^{s+1}_{BD}.$$

Proof. — Let us begin with $h\in\dot{\mathcal{H}}^{-1}_D$. By corollary 11.4, there exists a unique $g\in\dot{\mathcal{H}}^0_{BD}$ with $h=Dg=D\mathbb{P}g$. By similarity, $b(DB)h=Db(BD)g=D\mathbb{P}b(BD)g$. Thus, using lemma 11.6 twice, since $\mathbb{P}g,\mathbb{P}b(BD)g\in\dot{\mathcal{H}}^0_D=\overline{\mathsf{R}_2(D)}$,

$$\|b(DB)h\|_{\dot Y^{-1}}\sim\|\mathbb{P}b(BD)g\|_Y\lesssim\|b\|_\infty\|\mathbb{P}g\|_Y\sim\|b\|_\infty\|h\|_{\dot Y^{-1}}.$$

Next, we assume $h\in\dot{\mathcal{H}}^s_D$ with $-1<s\le 0$. Consider the approximations h_k of lemma 11.1. They belong in particular to $\dot{\mathcal{H}}^{-1}_D$. Thus $\|b(DB)h_k\|_{\dot Y^{-1}}\lesssim\|h_k\|_{\dot Y^{-1}}$ uniformly in k. Now, using Fourier transform and the Mikhlin theorem, $h\mapsto h_k$ is bounded on $\dot Y^{-1}$, uniformly in k. Hence $(b(DB)h_k)$ is a bounded sequence in $\dot Y^{-1}$, thus has a weak-$*$ converging subsequence in $\dot Y^{-1}$, and in particular in the Schwartz distributions. But, by proposition 11.3, $b(DB)$ is bounded on $\dot{\mathcal{H}}^s_D$, hence $b(DB)h_k\to b(DB)h$ in $\dot{\mathcal{H}}^s_D$ so also in the Schwartz distributions. Thus, the limit of the above subsequence is $b(DB)h$ which, therefore, belongs to $\dot Y^{-1}$ with the desired estimate.

Let us turn to the second point. If $\tilde h\in\dot{\mathcal{H}}^{s+1}_{BD}$, then $h=D\tilde h\in\dot{\mathcal{H}}^s_D$ and $Db(BD)\tilde h=b(DB)h$ by the isomorphism property in corollary 11.4. Thus,

$$\|Db(BD)\tilde h\|_{\dot Y^{-1}}=\|b(DB)h\|_{\dot Y^{-1}}\lesssim\|b\|_\infty\|h\|_{\dot Y^{-1}}=\|b\|_\infty\|D\tilde h\|_{\dot Y^{-1}}.\qquad\square$$

The following result is an extension of earlier results with *a priori* Sobolev initial elements instead of just L^2 so far. This result will be especially useful for $s=-\frac12$ later.

THEOREM 11.8. — 1) *Let I be the subinterval in $(\frac{n}{n+1},p_+(DB))$ on which we have*

$$\mathbb{H}^q_{DB}=\mathbb{H}^q_D$$

with equivalent norms. Then the following holds:

For DB we have, for all $h\in\bigcup_{-1\le s\le 0}\dot{\mathcal{H}}^s_D$,

$$\|S(tDBe^{-t|DB|}h)\|_q\sim\|S(t\partial_te^{-t|DB|}h)\|_q\sim\|h\|_{H^q}$$

and

$$\|\widetilde N_*(e^{-t|DB|}h)\|_q\sim\|h\|_{H^q}$$

when $q > 1$, *or* $q \leq 1$ *and* B *pointwise accretive, or* $q \leq 1$ *and* $h \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^{s,\pm}_{DB}$.

2) *If* I^* *designates the same interval but for* DB^* *and* $q \in I^*$, *let* $\mathcal{T} = T_2^{q'}, Y = L^{q'}, \dot{Y}^{-1} = \dot{W}^{-1,q'}$ *if* $q > 1$ *and* $\mathcal{T} = T^{\infty}_{2,\alpha}, Y = \dot{\Lambda}^{\alpha}, \dot{Y}^{-1} = \dot{\Lambda}^{\alpha-1}$ *with* $\alpha = n(\frac{1}{q} - 1)$ *if* $q \leq 1$. *Then, we obtain the following equivalences:*

2a) *Tent space estimate for* BD *in disguise:*

$$\|te^{-t|DB|}h\|_{\mathcal{T}} \sim \|h\|_{\dot{Y}^{-1}}, \quad \forall h \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^{s}_{D}.$$

2b) *Tent space estimate for* BD: *for all* $\tilde{h} \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^{s+1}_{BD}$,

$$\|tDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \sim \|tBDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \sim \|t\partial_t e^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \sim \|D\tilde{h}\|_{\dot{Y}^{-1}}.$$

2c) *Sharp function for* BD: *Finally, if* $1 < q \leq 2$ *we have*

$$\|\widetilde{N}_{\sharp}(e^{-t|BD|}\tilde{h})\|_{p} \sim \|D\tilde{h}\|_{\dot{Y}^{-1}}, \quad \forall \tilde{h} \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^{s+1}_{BD},$$

and in the case $q \leq 1$, *we have*

$$\|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}\tilde{h})\|_{\infty} \sim \|D\tilde{h}\|_{\dot{Y}^{-1}}, \quad \forall \tilde{h} \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^{s+1}_{BD}.$$

Proof. — So far, and thanks to lemma 11.6, all statements have been proved when $h \in \dot{\mathcal{H}}^0_D$ for those involving DB and when $\tilde{h} \in \dot{\mathcal{H}}^0_{BD}$ for those involving BD. Our goal is thus to extend this to more general h or $\tilde{h}$. The argument consists in tedious verifications with adequate approximation procedures.

Proof of 1). — We begin with the quadratic estimates. We fix q in the prescribed interval. Let $\psi \in \Psi(S_\mu)$ for which we have

$$\|\mathbb{Q}_{\psi,DB}h\|_{T^q_2} \lesssim \|h\|_{H^q}$$

for all $h \in \dot{\mathcal{H}}^0_{DB} = \dot{\mathcal{H}}^0_D$. We want to extend it to $h \in \dot{\mathcal{H}}^s_D$ for some $s \in [-1, 0)$. Let h be such. Assume also $\|h\|_{H^q} < \infty$ otherwise there is nothing to prove. Consider the functions $h_k \in \dot{\mathcal{H}}^0_D$ as in lemma 11.1: they converge in $\dot{\mathcal{H}}^s_D$ to h. Classical Hardy space theory also shows convergence in H^q. Now, the estimates apply to h_k. Thus $(\mathbb{Q}_{\psi,DB}h_k)$ is a Cauchy sequence in T^q_2, hence converges to some F in T^q_2. This enforces the convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. As $\dot{\mathcal{H}}^s_D = \dot{\mathcal{H}}^s_{DB}$, it is easy to see that the sequence $(\mathbb{Q}_{\psi,DB}h_k)$ converges also in $L^2_{t,loc}(L^2_x)$ to $\mathbb{Q}_{\psi,DB}h$. Thus $\mathbb{Q}_{\psi,DB}h = F \in T^q_2$ and this concludes the extension.

Conversely, assume that

$$\|h\|_{H^q} \lesssim \|\mathbb{Q}_{\psi,DB}h\|_{T^q_2}$$

for all $h \in \dot{\mathcal{H}}^0_{DB} = \dot{\mathcal{H}}^0_D$ and some $\psi \in \Psi(S_\mu)$. Again, we have to extend it to $h \in \dot{\mathcal{H}}^s_D$ for some $s \in [-1, 0)$. We assume $\|\mathbb{Q}_{\psi,DB}h\|_{T^q_2} < \infty$, otherwise there is nothing to prove. Take $\varphi \in \Psi(S_\mu)$ for which we have the Calderón reproducing formula (23) and also that $\mathbb{S}_{\varphi,DB}$ maps $T^q_2 \cap T^2_2$ into $\mathbb{H}^q_{DB}$. Let χ_k be the indicator function of $[1/k,k]\times B(0,k)$. Then

$$h_k := \mathbb{S}_{\varphi,DB}(\chi_k \mathbb{Q}_{\psi,DB} h) \in \mathbb{H}^q_{DB} = \mathbb{H}^q_D.$$

By taking the limit as $k\to\infty$, h_k converges to some $\tilde h \in H^q_{DB} = H^q_D$. Next, by testing against a Schwartz function g,

$$\langle h_k, g\rangle = \big(\chi_k \mathbb{Q}_{\psi,DB} h, \mathbb{Q}_{\varphi^*,B^*D} g\big) = \int_0^\infty \langle \chi_k \psi(tDB)h, \mathbb{P}\varphi^*(tB^*D)g\rangle \frac{dt}{t}.$$

If $\varphi(z) = z\tilde\varphi(z)$ for some $\tilde\varphi \in \Psi(S_\mu)$, then $\varphi^*(tB^*D)g = t\tilde\varphi^*(tB^*D)(B^*Dg)$. It easily follows using $-1\le s\le 0$ and treating differently the integral for $t<1$ or $t>1$, that

$$\int_0^\infty t^{2s}\|\mathbb{P}\varphi^*(tB^*D)g\|_2^2 \frac{dt}{t} < \infty,$$

(for $s=-1$ use the square functions estimates) while

$$\int_0^\infty t^{-2s}\|\psi(tDB)h\|_2^2 \frac{dt}{t} \lesssim \|h\|^2_{DB,s} \sim \|h\|^2_{D,s}.$$

Thus dominated convergence theorem applies to yield that

$$\langle h_k, g\rangle \to \int_0^\infty \langle \psi(tDB)h, \mathbb{P}\varphi^*(tB^*D)g\rangle \frac{dt}{t} = \langle h, g\rangle.$$

This shows that $h=\tilde h$ in the sense of Schwartz distributions, so that $h\in H^q_D$ with the desired estimate.

Let us look at the extension for non-tangential maximal estimates. The extension to all $h\in \dot{\mathcal{H}}^s_D$ for some $s\in[-1,0)$ of

$$\|\widetilde{N}_*(e^{-t|DB|}h)\|_q \lesssim \|h\|_{H^q}$$

can be handled as for square functions. Conversely, an inspection of the proofs of propositions 9.11 and 9.15 shows the converse in the different cases of the statement.

Proof of 2a) *and* 2b). — We fix $-1\le s\le 0$. The extension for the upper bound $\|te^{-t|DB|}h\|_{\mathcal{T}} \lesssim \|h\|_{\dot Y^{-1}}$, when $h\in\dot{\mathcal{H}}^s_D$, can be done as for 1) when $s<0$. Consider the functions $h_k\in\dot{\mathcal{H}}^0_D$ as in lemma 11.1: they converge in $\dot{\mathcal{H}}^s_D$ to h. It is easy to check that (h_k) is uniformly bounded in $\dot Y^{-1}$ with $\|h_k\|_{\dot Y^{-1}} \lesssim \|h\|_{\dot Y^{-1}}$. Thus it remains to go to the limit for $\|te^{-t|DB|}h_k\|_{\mathcal{T}}$. Convergence in $\dot{\mathcal{H}}^s_D$ implies that $(te^{-t|DB|}h_k)$ converges to $te^{-t|DB|}h$ in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ and, at the same time, as it is a bounded sequence in $\mathcal{T}$, which is a dual space, it has a weakly-$*$ convergent subsequence. Testing against bounded function with compact support in $\mathbb{R}^{1+n}_+$, we conclude that the limit must also be $te^{-t|DB|}h$ and the desired estimate follows.

Now, for 2b), let $\tilde h\in\dot{\mathcal{H}}^{s+1}_{BD}$. Then we know from corollary 11.4 that

$$h = D\tilde h \in \dot{\mathcal{H}}^s_{DB} = \dot{\mathcal{H}}^s_D \quad\text{and}\quad De^{-t|BD|}\tilde h = e^{-t|DB|}D\tilde h = e^{-t|DB|}h.$$

Using what we just did

$$\|tDe^{-t|BD|}\tilde h\|_{\mathcal{T}} \lesssim \|D\tilde h\|_{\dot Y^{-1}}.$$

Using the boundedness of B we also have the upper bound

$$\|tBDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \lesssim \|tDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \lesssim \|D\tilde{h}\|_{\dot{Y}^{-1}}.$$

Finally, $t\partial_t e^{-t|BD|}\tilde{h} = tBDe^{-t|BD|}\mathrm{sgn}(BD)\tilde{h}$, so that

$$\|t\partial_t e^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \lesssim \|D\mathrm{sgn}(BD)\tilde{h}\|_{\dot{Y}^{-1}} \lesssim \|D\tilde{h}\|_{\dot{Y}^{-1}},$$

where the last inequality follows from proposition 11.7.

For the converse inequalities in 2a) and 2b), a moment's reflection tells us that it is enough to show, when $\tilde{h} \in \dot{\mathcal{H}}^{s+1}_{BD}$, that $\|D\tilde{h}\|_{\dot{Y}^{-1}} \lesssim \|tBDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}}$. As the other inequalities follow from this one, set

$$\psi(z) = ze^{-[z]}.$$

Consider φ allowable for $\mathbb{H}^q_{DB^*}$ such that the Calderón formula (23) holds. Let $g \in \mathbb{H}^q_D \cap \dot{\mathcal{H}}^{-s-1}_D = \mathbb{H}^q_{DB^*} \cap \dot{\mathcal{H}}^{-s-1}_{DB^*}$. Hence, for the inner product in tent spaces

$$|(\mathbb{Q}_{\psi,BD}\tilde{h}, \mathbb{Q}_{\varphi^*,DB^*}g)| \lesssim \|\mathbb{Q}_{\psi,BD}\tilde{h}\|_{\mathcal{T}}\|g\|_{\mathbb{H}^q_{DB^*}}.$$

Using the approximations with the functions χ_k above, let

$$\tilde{h}_k = \mathbb{S}_{\varphi,BD}(\chi_k \mathbb{Q}_{\psi,BD}\tilde{h}) \in \mathbb{H}^{\mathcal{T}}_{BD}.$$

Then, using lemma 11.6 and $\tilde{h}_k \in \dot{\mathcal{H}}^0_{BD}$,

$$\|D\tilde{h}_k\|_{\dot{Y}^{-1}} \sim \|\tilde{h}_k\|_Y \lesssim \|\chi_k tBDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}} \lesssim \|tBDe^{-t|BD|}\tilde{h}\|_{\mathcal{T}}.$$

It remains to show that $D\tilde{h}_k$ converges to $D\tilde{h}$ in the sense of distributions as this will imply $\|D\tilde{h}\|_{\dot{Y}^{-1}} \leq \liminf \|D\tilde{h}_k\|_{\dot{Y}^{-1}}$. Let g be a Schwartz function. Then

$$\langle D\tilde{h}_k, g\rangle = \langle \tilde{h}_k, Dg\rangle = \big(\chi_k \mathbb{Q}_{\psi,BD}\tilde{h}, \mathbb{Q}_{\varphi^*,DB^*}(Dg)\big).$$

Then, as $-1 \leq s \leq 0$ and $Dg \in \dot{\mathcal{H}}^{-s-1}_D = \dot{\mathcal{H}}^{-s-1}_{DB^*}$,

$$\int_0^\infty t^{2(s+1)}\|\varphi^*(tDB^*)(Dg)\|_2^2 \,\frac{dt}{t} < \infty,$$

while

$$\int_0^\infty t^{-2(s+1)}\|\psi(tBD)\tilde{h}\|_2^2 \,\frac{dt}{t} \lesssim \|\tilde{h}\|^2_{BD,s+1} \sim \|h\|^2_{D,s}.$$

Thus dominated convergence theorem applies to yield that

$$\langle \tilde{h}_k, Dg\rangle \longrightarrow \big(\mathbb{Q}_{\psi,BD}\tilde{h}, \mathbb{Q}_{\varphi^*,DB^*}(Dg)\big).$$

If $\varphi\psi$ has enough decay at 0 and ∞ then

$$(\mathbb{Q}_{\psi,BD}\tilde{h}, \mathbb{Q}_{\varphi^*,DB^*}(Dg)) = \langle \mathbb{S}_{\varphi,BD}\mathbb{Q}_{\psi,BD}\tilde{h}, Dg\rangle = \langle \tilde{h}, Dg\rangle = \langle D\tilde{h}, g\rangle.$$

Proof of 2c). — As in 2b),

$$\|D\tilde{h}\|_{\dot{Y}^{-1}} \sim \|\psi(tBD)\tilde{h}\|_{\mathcal{T}}$$

for any allowable ψ for $\mathbb{H}^{\mathcal{T}}_{BD}$ and $\tilde{h} \in \dot{\mathcal{H}}^{s+1}_{BD}$. As observed in proposition 11.5, $e^{-t|BD|}\tilde{h} - \tilde{h} \in L^2$ when $\tilde{h} \in \dot{\mathcal{H}}^{s+1}_{BD}$, so that the proof of lemma 10.2 goes through without change. This proves the lower bounds for $\widetilde{N}_\sharp(e^{-t|BD|}\tilde{h})$ and $\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}\tilde{h})$.

As for the upper bounds, let

$$\tilde{h}_\varepsilon = e^{-\varepsilon|BD|}\tilde{h} - e^{-(1/\varepsilon)|BD|}\tilde{h}, \quad \varepsilon > 0.$$

It follows from proposition 11.5 that $\tilde{h}_\varepsilon \in \overline{\mathsf{R}_2(BD)}$, thus we obtain from theorem 9.3 the uniform upper bounds,

$$\|\widetilde{N}_\sharp(e^{-t|BD|}\tilde{h}_\varepsilon)\|_{q'} \lesssim \|\mathbb{P}\tilde{h}_\varepsilon\|_Y$$

in the case $Y = L^{q'}$ and

$$\|\widetilde{N}_{\sharp,\alpha}(e^{-t|BD|}\tilde{h}_\varepsilon)\|_\infty \lesssim \|\mathbb{P}\tilde{h}_\varepsilon\|_Y$$

in the case $Y = \dot{\Lambda}^\alpha$. Remark that $D\tilde{h}_\varepsilon = e^{-\varepsilon|DB|}D\tilde{h} - e^{-(1/\varepsilon)|DB|}D\tilde{h}$, so that by lemma 11.6 and proposition 11.7,

$$\|\mathbb{P}\tilde{h}_\varepsilon\|_Y \sim \|D\tilde{h}_\varepsilon\|_{\dot{Y}^{-1}} \lesssim \|D\tilde{h}\|_{\dot{Y}^{-1}}.$$

As

$$e^{-t|BD|}\tilde{h}_\varepsilon - \tilde{h}_\varepsilon = e^{-\varepsilon|BD|}(e^{-t|BD|}\tilde{h} - \tilde{h}) - e^{-(1/\varepsilon)|BD|}(e^{-t|BD|}\tilde{h} - \tilde{h}),$$

$e^{-t|BD|}\tilde{h}_\varepsilon - \tilde{h}_\varepsilon$ converges in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ to $e^{-t|BD|}\tilde{h} - \tilde{h}$. A linearisation of the non-tangential sharp function, together with Fatou's lemma in the case where $Y = L^p$, $p < \infty$, yields the conclusion. We skip easy details. □

12. APPLICATIONS TO ELLIPTIC PDE'S

In this chapter, we are given $L = -\operatorname{div} A\nabla$ as in the introduction (t-independent, bounded and accretive on $\mathcal{H}^0 = \mathcal{H}^0_D$, coefficients). We first discuss representations of solutions in the class $\mathcal{E}$. Then, we prove here theorem 1.1 and theorem 1.2 with some further estimates.

12.1. A priori results for conormal gradients of solutions in $\mathcal{E}$

We recall that $\mathcal{E} = \bigcup_{-1\le s\le 0} \mathcal{E}_s$ where

$$\mathcal{E}_s = \begin{cases} \{u; \|\widetilde{N}_*(\nabla u)\|_2 < \infty\}, & \text{if } s = 0,\\ \{u; \|S(t^{-s}\nabla u)\|_2 < \infty\}, & \text{otherwise.}\end{cases}$$

Recall from [**8, 83**] that conormal gradients

$$F(t,x) = \nabla_A u(t,x) = \begin{bmatrix} \partial_{\nu_A} u(t,x)\\ \nabla_x u(t,x)\end{bmatrix} \in L^2_{\text{loc}}(\mathbb{R}^{1+n}_+)$$

(we omit the target space of F in the notation) of weak solutions $u \in \mathcal{E}_s$ of $Lu = 0$ on $\mathbb{R}^{1+n}_+$ satisfy the equation (in distributional sense at first, and eventually in strong semigroup sense)

$$\partial_t F + DBF = 0, \tag{70}$$

and have a trace on $\mathbb{R}^n$ and semigroup representation $\nabla_A u|_{t=0} \in \dot{\mathcal{H}}^{s,+}_{DB} \subset \dot{\mathcal{H}}^s_D$,

$$\nabla_A u(t,.) = e^{-t|DB|}\nabla_A u|_{t=0} = e^{-t|DB|}\chi^+(DB)\nabla_A u|_{t=0} = e^{-tDB}\nabla_A u|_{t=0}, \tag{71}$$

where

$$D := \begin{bmatrix} 0 & \operatorname{div}_x\\ -\nabla_x & 0\end{bmatrix}, \qquad \textsf{D}(D) = \begin{bmatrix} \textsf{D}(\nabla)\\ \textsf{D}(\operatorname{div})\end{bmatrix} \subset L^2(\mathbb{R}^n, \mathbb{C}^N),\ N = m(1+n),$$

and

$$B = \hat{A} := \begin{bmatrix} 1 & 0\\ c & d\end{bmatrix}\begin{bmatrix} a & b\\ 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b\\ ca^{-1} & d - ca^{-1}b\end{bmatrix} \tag{72}$$

whenever we write

$$A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

and L in the form

$$L = -\begin{bmatrix} \partial_t & \nabla_x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \partial_t \\ \nabla_x \end{bmatrix}.$$

Here, D and B satisfy the necessary requirements and the semigroup $e^{-t|DB|}$ is appropriately interpreted as in section 11.

Conversely, for any $h \in \dot{\mathcal{H}}^{s,+}_{DB}$, the $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ function

$$F(t,x) = e^{-t|DB|}h(x) = e^{-tDB}\chi^+(DB)h(x)$$

is the conormal gradient of a weak solution $u \in \mathcal{E}_s$ of

$$Lu = 0$$

on $\mathbb{R}^{1+n}_+$ and $h = \nabla_A u|_{t=0}$. Note that u is unique modulo constants. Note also that u is a continuous function of $t \geq 0$ valued in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. See [**8**] for $s = -1$ and [**23**, remark 8.9] for all $s \in [-1, 0]$.

It is convenient to use the notation $v = \left[\begin{smallmatrix} v_\perp \\ v_\parallel \end{smallmatrix}\right]$ for vectors in $\mathbb{C}^{m(1+n)}$, where $v_\perp \in \mathbb{C}^m$ is called the scalar part and $v_\parallel \in \mathbb{C}^{mn} = (\mathbb{C}^m)^n$ the tangential part of v. With this notation, for any $s \in \mathbb{R}$,

$$\dot{\mathcal{H}}^s_D = \begin{bmatrix} \dot{\mathcal{H}}^s_\perp \\ \dot{\mathcal{H}}^s_\parallel \end{bmatrix}. \tag{73}$$

Given the definition of D, we have

$$\mathbb{P} = \begin{bmatrix} I & 0 \\ 0 & RR^* \end{bmatrix},$$

where R is the array of Riesz transforms on $\mathbb{R}^n$ acting componentwise on $\mathbb{C}^m$-valued functions and R^* is its adjoint. It follows that

$$\dot{\mathcal{H}}^s_\perp = \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m) \quad \text{and} \quad \dot{\mathcal{H}}^s_\parallel = R\dot{\mathcal{H}}^s_\perp,$$

which is also denoted by $\dot{\mathcal{H}}^s_\nabla(\mathbb{R}^n; \mathbb{C}^{mn})$ in [**23**].

Let $u \in \mathcal{E}_s$ be a solution to $Lu = 0$ in $\mathbb{R}^{1+n}_+$. Using that $D : \dot{\mathcal{H}}^{s+1,+}_{\mathbb{P}BD} \to \dot{\mathcal{H}}^{s,+}_{DB}$ is an isomorphism, there exists a unique $U(0,.) \in \dot{\mathcal{H}}^{s+1,+}_{\mathbb{P}BD} \subset \dot{\mathcal{H}}^{s+1}_D$ such that

$$DU(0,.) := -\nabla_A u|_{t=0} \in \dot{\mathcal{H}}^{s,+}_{DB}.$$

Then, define

$$U(t,.) = e^{-t|\mathbb{P}BD|}U(0,.) = e^{-t\mathbb{P}BD}\chi^+(\mathbb{P}BD)U(0,.), \quad t \geq 0,$$

accordingly to proposition 11.5 with $U(t,\,.\,) - U(0,.) \in L^2$. Using that $\mathbb{P}$ extends to an isomorphism $\dot{\mathcal{H}}^{s+1,+}_{BD} \to \dot{\mathcal{H}}^{s+1,+}_{\mathbb{P}BD}$, there exists a unique $v(0,.) \in \dot{\mathcal{H}}^{s+1,+}_{BD}$ such that

$$U(0,.) = \mathbb{P}v(0,.) \tag{74}$$

and this v satisfies

$$Dv(0,.) = DU(0,.) = -\nabla_A u|_{t=0},$$

where $Dv(0,.)$ is taken in the appropriate sense. One defines, in $\dot{\mathcal{H}}^{s+1,+}_{BD}$,

$$v(t,.) = e^{-t|BD|}v(0,.) = e^{-tBD}\chi^+(BD)v(0,.), \quad t \geq 0,$$

accordingly to proposition 11.5, so that $v(t,\,.\,) - v(0,.) \in L^2$, and one has

$$U(t,.) = \mathbb{P}v(t,.)$$

in $\dot{\mathcal{H}}^{s+1}_D$ and in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+) \cap C([0,\infty); \dot{\mathcal{H}}^s_D)$

$$Dv(t,.) = DU(t,.) = -\nabla_A u(t,.). \tag{75}$$

In fact, U and v share the same first component as $\mathbb{P}$ is the identity on scalar parts and their tangential parts satisfy for all $t \geq 0$,

$$(U(t,.))_{\parallel} = (\mathbb{P}v)_{\parallel}(t,.) = ((RR^*v_{\parallel})(t,.)), \text{ in } \dot{\mathcal{H}}^{s+1}_{\parallel}$$

or, equivalently,

$$(R^*U_{\parallel})(t,.) = (R^*v_{\parallel})(t,.), \text{ in } \dot{\mathcal{H}}^{s+1}_{\perp}.$$

Here, $RR^*v_{\parallel}$ is meant as the appropriate extension of the tangential part of $\mathbb{P}$ acting on v, so $R^*v_{\parallel}$ is to be interpreted in this way. It tells us that any estimate on $U_{\parallel}$ is thus an estimate on $R^*v_{\parallel}$.

We finish this discussion with the pointwise relation between u, $U_{\perp}$ and $v_{\perp}$. Recall that $u \in \mathcal{E}_s$ and is continuous as a function of t valued in $L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^m)$. Also $U_{\perp} = v_{\perp} \in \dot{\mathcal{H}}^{1/2}_{\perp}$ at $t = 0$. They can be regarded as L^2_{loc} functions and they agree up to a constant. We decide to set the constant to be 0. Moreover,

$$U(t,\,.\,) - U(0,.) = \mathbb{P}(v(t,.) - v(0,.))$$

belongs to L^2 and is continuous as a function of t. As $\mathbb{P}$ is the identity on scalar parts, we have the equality $U_{\perp} = v_{\perp}$ in $C([0,\infty); L^2_{\mathrm{loc}}(\mathbb{R}^n))$. Following the proof in [**8**] where the case $s = -1$ is treated (we changed signs compared to [**8**]), there exists a constant $c \in \mathbb{C}^m$ such that for all $t \geq 0$

$$u(t,.) = (U(t,.))_{\perp} + c = (v(t,.))_{\perp} + c \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^n),$$

(it is no longer modulo constants) so that we have the following representations for u in $C([0,\infty); L^2_{\mathrm{loc}}(\mathbb{R}^n))$ with $h = v(0,.) \in \dot{\mathcal{H}}^{s+1,+}_{BD}$,

$$u(t,.) - c = (e^{-t|\mathbb{P}BD|}\mathbb{P}h)_{\perp} = (e^{-t|BD|}h)_{\perp} = (\mathbb{P}e^{-t|BD|}h)_{\perp}.$$

Thus U and v are potential vectors for the solution u. Both are useful.

If, furthermore, $s = -1$, *i.e.* $h = v(0,.) \in \overline{R_2(BD)}$, then

$$e^{-t|\mathbb{P}BD|}\mathbb{P}h = \mathbb{P}e^{-t|BD|}\mathbb{P}h,$$

so we also have $u(t,.) - c = (\mathbb{P}e^{-t|BD|}\mathbb{P}h)_{\perp} = (e^{-t|BD|}\mathbb{P}h)_{\perp}$.

Let us mention a consequence of this discussion.

Lemma 12.1. — *Assume $u \in \mathcal{E}_s$, $-1 \le s \le 0$, is a weak solution of $Lu = 0$. Assume q is such that $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ with equivalence of norms. Let $p = q'$ if $q > 1$. Then*

$$\|\nabla_A u|_{t=0}\|_{\dot W^{-1,p}} < \infty$$

if, and only if, there exists $h \in \dot{\mathcal{H}}^{s+1,+}_{BD} \cap H^{p,+}_{BD}$ with $Dh = \nabla_A u|_{t=0}$, and we have

$$\|\nabla_A u|_{t=0}\|_{\dot W^{-1,p}} \sim \|\mathbb{P}h\|_p.$$

Let $\alpha = n(\frac{1}{q} - 1)$ if $q \le 1$. Then

$$\|\nabla_A u|_{t=0}\|_{\dot\Lambda^{\alpha-1}} < \infty$$

if, and only if, there exists $h \in \dot{\mathcal{H}}^{s+1,+}_{BD} \cap \dot\Lambda^{\alpha,+}_{BD}$ with $Dh = \nabla_A u|_{t=0}$, and we have

$$\|\nabla_A u|_{t=0}\|_{\dot\Lambda^{\alpha-1}} \sim \|\mathbb{P}h\|_{\dot\Lambda^{\alpha}}.$$

Proof. — Let us consider the case $q > 1$. Remark that $\dot{\mathcal{H}}^{s+1}_{BD}$ is the dual of $\dot{\mathcal{H}}^{-s-1}_{DB^*} = \dot{\mathcal{H}}^{-s-1}_{D}$ and H^p_{BD} is the dual of $H^q_{DB^*} = H^q_D$ with identical dualities when restricted to dense subspaces. The intersection $\dot{\mathcal{H}}^{-s-1}_D \cap H^q_D$ is well-defined within the Schwartz distributions, dense in each factor, and the intersection of duals $\dot{\mathcal{H}}^{s+1}_{BD} \cap H^p_{BD}$ makes sense (as a subspace of the sum). If $h \in \dot{\mathcal{H}}^{s+1,+}_{BD} \cap H^{p,+}_{BD}$ with $\nabla_A u|_{t=0} = Dh = D\mathbb{P}h$, then

$$\|\nabla_A u|_{t=0}\|_{\dot W^{-1,p}} = \|D\mathbb{P}h\|_{\dot W^{-1,p}} \lesssim \|\mathbb{P}h\|_p$$

by an argument similar to that of lemma 11.6. Conversely, let $g \in \dot{\mathcal{H}}^{-s-1}_D \cap H^q_D$. Then $D^{-1}g \in \dot{\mathcal{H}}^{-s}_D \cap \dot W^{1,q}$. Indeed, if $g \in \dot{\mathcal{H}}^{-s-1}_D \cap H^q_D$, then

$$\nabla D^{-1} g = \nabla D^{-1}\mathbb{P} g \in \dot{\mathcal{H}}^{-s-1}_D \cap H^q_D.$$

Thus, the map $g \mapsto \langle \nabla_A u|_{t=0}, D^{-1}g\rangle$ is defined on $\dot{\mathcal{H}}^{-s-1}_{DB^*} \cap H^q_{DB^*}$ and defines $h \in \dot{\mathcal{H}}^{s+1}_{BD} \cap H^p_{BD}$ with $Dh = \nabla_A u|_{t=0}$ and one has $\mathbb{P}h \in \dot{\mathcal{H}}^{s+1}_D \cap H^p_D$ so that

$$\|\mathbb{P}h\|_p \lesssim \|\nabla_A u|_{t=0}\|_{\dot W^{-1,p}}.$$

Applying the projector $\chi^+(DB)$ leaves $\nabla_A u|_{t=0}$ unchanged, thus it follows that $h = \chi^+(BD)h$ (in both spaces).

In the case $q \le 1$, we argue as above and replace $\dot W^{1,q}$ by $\dot H^{1,q}$. □

Remark 12.2. — This proof reveals that one can make the Sobolev and Hardy space theories consistent in the appropriate ranges of exponents.

12.2. A priori comparisons of various norms

We may now translate theorem 11.8 in the context of solutions of $Lu = 0$ in $\mathbb{R}^{1+n}_+$. We remark that if L is associated to B, then the operator L^*, with coefficients A^*, is associated to $\widetilde{B} = \widehat{A^*} = NB^*N$, with $N = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. As $DN + ND = 0$ and N preserves $\overline{\mathsf{R}_2(D)}$, we see

$$D\widetilde{B} = -N(DB^*)N = N^{-1}(-DB^*)N,$$

as $N = N^{-1}$ For the functional calculi of $D\widetilde{B}$ and DB^*, we obtain

$$b(D\widetilde{B}) = Nb(-DB^*)N$$

for all $b \in H^\infty(S_\mu)$. Therefore, we see that $h \in \mathbb{H}^{q,\pm}_{D\widetilde{B}}$ if and only if $Nh \in \mathbb{H}^{q,\mp}_{DB^*}$. Also, $\mathbb{H}^q_{DB^*} = \mathbb{H}^q_D$ if and only if $\mathbb{H}^q_{D\widetilde{B}} = \mathbb{H}^q_D$. More directly, proposition 4.8 applies to the pair of spaces $(\mathbb{H}^{\mathcal{T}}_{D\widetilde{B}}, \mathbb{H}^{\mathcal{T}^*}_{BD})$ for the pairing $\langle Nf, g\rangle$ on $\overline{\mathsf{R}_2(D)} \times \overline{\mathsf{R}_2(BD)}$. Similarly, $h \in \dot{\mathcal{H}}^{s,\pm}_{D\widetilde{B}}$ if and only if $Nh \in \dot{\mathcal{H}}^{s,\mp}_{DB^*}$ and $\dot{\mathcal{H}}^{s}_{D\widetilde{B}}, \dot{\mathcal{H}}^{-s}_{BD}$ are dual spaces for this pairing (or, rather, its extension). Hence all statements proved before adapt to this new pairing.

Theorem 12.3. — *We set*

$$p_\pm(L) = p_\pm(DB) = p_\pm(BD)$$

and I_L be the subinterval of $(\frac{n}{n+1}, p_+(L))$ for which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms.

▷ *For any $q \in I_L$, we have that all weak solutions of*

$$Lu = 0$$

with $u \in \mathcal{E}$, satisfy

$$\|\widetilde{N}_*(\nabla u)\|_q \sim \|\nabla_A u|_{t=0}\|_{H^q} \sim \|S(t\partial_t \nabla u)\|_q, \tag{76}$$

where $H^q = L^q$ if $q > 1$.

▷ *For any $q \in I_L$, we have that all weak solutions of*

$$L^* u = 0$$

with $u \in \mathcal{E}$, satisfy with $p = q'$ if $q > 1$ and $\alpha = n(\frac{1}{q} - 1)$ if $q \leq 1$,

$$\|S(t\,\nabla u)\|_p \sim \|\nabla_{A^*} u|_{t=0}\|_{\dot{W}^{-1,p}}, \tag{77}$$

$$\|t\,\nabla u\|_{T^\infty_{2,\alpha}} \sim \|\nabla_{A^*} u|_{t=0}\|_{\dot{\Lambda}^{-1,\alpha}} \sim \|\widetilde{N}_{\sharp,\alpha}(v)\|_\infty. \tag{78}$$

For those p with $p > 2$, we also have

$$\|\nabla_{A^*} u|_{t=0}\|_{\dot{W}^{-1,p}} \sim \|\widetilde{N}_\sharp(v)\|_p. \tag{79}$$

Finally, we note the a priori "$N < S$" inequality. For p as above, up to an additive normalizing constant c, we have

$$\|\widetilde{N}_*(u-c)\|_p \lesssim \|S(t\,\nabla u)\|_p. \tag{80}$$

Proof. — The only thing to prove is (80). Assume $\|S(t\,\nabla u)\|_p < \infty$, otherwise there is nothing to prove. Since $u \in \mathcal{E}$, we know that

$$u(t,.) - c = (e^{-t|\widetilde{B}D|}h)_\perp = v_\perp$$

for some $h \in \dot{\mathcal{H}}^{s+1,+}_{\widetilde{B}D}$, which by lemma 12.1 can also be chosen in $H^{p,+}_{\widetilde{B}D}$ for p in the specified range, and some $c \in \mathbb{C}^m$, and we have

$$\|S(t\,\nabla u)\|_p \sim \|\nabla_{A^*} u|_{t=0}\|_{\dot{W}^{-1,p}} \sim \|\mathbb{P}h\|_p.$$

Approximate h by $h_k \in \mathbb{H}^{p,+}_{\widetilde{B}D}$ (one first approximates h in $\mathbb{H}^p_{\widetilde{B}D}$ and then, apply $\chi^+(\widetilde{B}D)$), then this gives a solution u_k by

$$u_k(t,.) - c = (e^{-t|\widetilde{B}D|}h_k)_\perp = (\mathbb{P}e^{-t|\widetilde{B}D|}h_k)_\perp$$

and theorem 9.3 implies

$$\|\widetilde{N}_*(u_k(t,.) - c)\|_p \lesssim \|\mathbb{P}h_k\|_p.$$

By the isomophism property of $\mathbb{P}$, $\mathbb{P}h_k$ converges to $\mathbb{P}h$ in L^p and also u_k converges to u in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. It is then easy to conclude using lemma 12.1. □

Remark 12.4. — The comparison (77) and the first comparison in (78) were used in [**23**]. Note that for $\alpha = 0$, this is a Carleson measure/BMO comparison.

Remark 12.5. — Let us mention that under the De Giorgi condition on $L^*_\parallel$ in section 13, we have a range $(1-\varepsilon', 2+\varepsilon)$ for (76), a range $(2-\varepsilon, \infty)$ for p in (77), (79) and (80), and a range $[0, \varepsilon)$ for (78). Again, this is *a priori* for weak solutions $u \in \mathcal{E}$.

12.3. Boundary layer potentials

Following [**81**], the boundary layer operators are identified as follows: for $t \neq 0$, $\nabla_A \mathcal{S}_t$ and $\mathcal{D}_t$ are defined as L^2 bounded operators by, for $f \in L^2(\mathbb{R}^n; \mathbb{C}^m)$,

$$\nabla_A \mathcal{S}_t f := \begin{cases} +e^{-tDB}\chi^+(DB)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] & \text{if } t > 0, \\ -e^{+tDB}\chi^-(DB)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] & \text{if } t < 0, \end{cases} \tag{81}$$

and

$$\mathcal{D}_t f := \begin{cases} -\big(e^{-tBD}\chi^+(BD)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t > 0, \\ +\big(e^{+tBD}\chi^-(BD)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t < 0. \end{cases} \tag{82}$$

We recall that for any $h \in L^2$, $(\mathbb{P}h)_\perp = (h)_\perp$, hence

$$\mathcal{D}_t f := \begin{cases} -\big(\mathbb{P}e^{-tBD}\chi^+(BD)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t > 0, \\ +\big(\mathbb{P}e^{+tBD}\chi^-(BD)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t < 0. \end{cases} \tag{83}$$

Now that we have the Sobolev space $\dot{\mathcal{H}}^s_D$, (81) makes sense for

$$f \in \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m) = \dot{\mathcal{H}}^s_\perp, \quad -1 \le s \le 0$$

and we can even define $\mathcal{S}_t$ consistently from $\dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m)$ to $\dot{\mathcal{H}}^{s+1}(\mathbb{R}^n; \mathbb{C}^m)$ by

$$\mathcal{S}_t f := \begin{cases} -\big(D^{-1}e^{-tDB}\chi^+(DB)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t > 0, \\ +\big(D^{-1}e^{+tDB}\chi^-(DB)\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]\big)_\perp & \text{if } t < 0. \end{cases} \tag{84}$$

We remark that D^{-1} can be indifferently thought as a $\dot{\mathcal{H}}^s_D \to \dot{\mathcal{H}}^{s+1}_D$ or $\dot{\mathcal{H}}^s_{DB} \to \dot{\mathcal{H}}^{s+1}_{BD}$ map. As we take scalar components the conclusion is the same.

Similarly the right hand side of (83) makes sense for $f \in \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m)$ for $0 \le s \le 1$ by the results of section 11. Indeed, $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \dot{\mathcal{H}}^s_D = \mathbb{P}\dot{\mathcal{H}}^s_{BD}$ and $\mathbb{P}$ is the identity on the scalar part. Hence $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \dot{\mathcal{H}}^s_{BD}$. We define $\mathcal{D}_t f$ by (83) for such f.

Note that we may let $t \to 0$ from above or below using the strong continuity of the semigroups (In Sobolev spaces, this follows from the sectoriality of their generators as observed in proposition 11.3) to obtain the jump relations. Those were proved in [**3**] under De Giorgi-Nash assumptions on L and L^*. Let us see that. From (81) we have for all $f \in \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m)$, $-1 \le s \le 0$,

$$\nabla_A \mathcal{S}_{0+} f - \nabla_A \mathcal{S}_{0-} f = (\chi^+(DB) + \chi^-(DB))\begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{85}$$

which encodes the jump relation of the conormal derivative of $\mathcal{S}_t$ across the boundary and the continuity of the tangential gradient of $\mathcal{S}_t$ across the boundary. We used that $\chi^+(DB) + \chi^-(DB) = I$ on $\dot{\mathcal{H}}^s_D \ni \left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$. For the double layer, we have for $f \in \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m)$, $0 \le s \le 1$,

$$\mathcal{D}_{0+} f - \mathcal{D}_{0-} f = -\Big(\mathbb{P}(\chi^+(BD) + \chi^-(BD))\begin{bmatrix} f \\ 0 \end{bmatrix}\Big)_\perp = -\Big(\begin{bmatrix} f \\ 0 \end{bmatrix}\Big)_\perp = -f. \tag{86}$$

We used that $\chi^+(BD) + \chi^-(BD) = I$ on $\dot{\mathcal{H}}^s_{BD} \ni \left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$, by the results of chapter 11.

Finally, we have the usual duality relations of single layer potentials and double layer potentials. Denote for a moment $\mathcal{S}_t = \mathcal{S}^A_t$. Then, in the $L^2(\mathbb{R}^n; \mathbb{C}^m)$ sesquilinear duality, for $f \in \dot{\mathcal{H}}^s(\mathbb{R}^n; \mathbb{C}^m)$ and $g \in \dot{\mathcal{H}}^{-s-1}(\mathbb{R}^n; \mathbb{C}^m)$, $-1 \le s \le 0$,

$$\langle g, \mathcal{S}^A_t f \rangle = \langle \mathcal{S}^{A^*}_{-t} g, f \rangle. \tag{87}$$

We provide the proof for convenience using the duality $\langle N\widetilde{h}, h\rangle$ for vectors and the relation between A^* and $\widetilde{B}$. We may assume $t > 0$. We have

$$\begin{aligned}
\langle g, \mathcal{S}_t^A f\rangle &= +\Big\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, -D^{-1}e^{-tDB}\chi^+(DB)\begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= -\Big\langle N\begin{bmatrix} g \\ 0 \end{bmatrix}, D^{-1}e^{-tDB}\chi^+(DB)\begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= +\Big\langle ND^{-1}\begin{bmatrix} g \\ 0 \end{bmatrix}, e^{-tDB}\chi^+(DB)\begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= +\Big\langle Ne^{t\widetilde{B}D}\chi^-(\widetilde{B}D)D^{-1}\begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= +\Big\langle ND^{-1}e^{tD\widetilde{B}}\chi^-(D\widetilde{B})\begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle = \langle \mathcal{S}_{-t}^{A^*} g, f\rangle.
\end{aligned}$$

Similarly, one has that, writing $\mathcal{D}_t^A = \mathcal{D}_t$ for a moment, for $f \in \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m)$ and $g \in \dot{\mathcal{H}}^{-s}(\mathbb{R}^n;\mathbb{C}^m)$, $0 \le s \le 1$,

$$\langle g, \mathcal{D}_t^A f\rangle = \langle \partial_{\nu_{A^*}} \mathcal{S}_{-t}^{A^*} g, f\rangle. \tag{88}$$

The proof is similar to the above one. Assume again $t > 0$. We have

$$\begin{aligned}
\langle g, \mathcal{D}_t^A f\rangle &= \Big\langle N\begin{bmatrix} g \\ 0 \end{bmatrix}, -e^{-tBD}\chi^+(BD)\begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= -\Big\langle Ne^{tD\widetilde{B}}\chi^-(D\widetilde{B})\begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle \\
&= +\Big\langle N\nabla_{A^*}\mathcal{S}_{-t}^{A^*}\begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix}\Big\rangle = \langle \partial_{\nu_{A^*}} \mathcal{S}_{-t}^{A^*} g, f\rangle.
\end{aligned}$$

The proof with $t < 0$ is left to the reader.

The extension of the semigroups to Hardy spaces H^p_{DB} and H^p_{BD} and identification with usual spaces made in section 6 yield the following result.

THEOREM 12.6. — *Let I_L be the interval in $(\frac{n}{n+1}, p_+(L))$ on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms and I_{L^*} be the interval in $(\frac{n}{n+1}, p_+(L^*))$ on which $\mathbb{H}^q_{D\widetilde{B}} = \mathbb{H}^q_D$ with equivalence of norms.*

1) *For $\in I_L$, we have the estimate*

$$\sup_{t>0} \|\nabla_A \mathcal{S}_t f\|_{H^q} \lesssim \|f\|_{H^q}, \quad \forall f \in \bigcup_{-1\le s\le 0} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

where $H^q = L^q$ if $q > 1$, and $\nabla_A \mathcal{S}_t f$ converges strongly in H^q as $t \to 0^+$. In particular, $\mathcal{S}_t$, $\partial_{\nu_A}\mathcal{S}_t$ and $\partial_t \mathcal{S}_t$ extend to uniformly bounded operators

$$\mathcal{S}_t : H^q \longrightarrow \dot{H}^{1,q}, \quad \partial_{\nu_A}\mathcal{S}_t : H^q \longrightarrow H^q$$

and

$$\partial_t \mathcal{S}_t : L^q \longrightarrow L^q, \quad \textit{if, moreover, } q > 1,$$

with strong limit as $t \to 0+$.

2) *For $q \in I_L$, we have the estimate*

$$\sup_{t>0} \|\nabla_A \mathcal{D}_t f\|_{H^q} \lesssim \|\nabla f\|_{H^q} = \|f\|_{\dot{H}^{1,q}}, \quad \forall f \in \bigcup_{0\le s\le 1} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

where $H^q = L^q$ if $q > 1$, and $\nabla_A \mathcal{D}_t f$ converges strongly in H^q as $t \to 0^+$. In particular, $\mathcal{D}_t$ extends to uniformly bounded operators

$$\mathcal{D}_t : \dot{H}^{1,q} \longrightarrow \dot{H}^{1,q},$$

with strong limit as $t \to 0+$.

3) *For $q \in I_{L^*}$, we have the estimate*

$$\sup_{t>0} \|\mathcal{S}_t f\|_{L^p} \lesssim \|f\|_{\dot{W}^{-1,p}}, \quad \forall f \in \bigcup_{-1\le s\le 0} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

where $p = q'$ if $q > 1$, and $\mathcal{S}_t f$ converges strongly in $\dot{W}^{-1,p}$ as $t \to 0^+$, and

$$\sup_{t>0} \|\mathcal{S}_t f\|_{\dot{\Lambda}^\alpha} \lesssim \|f\|_{\dot{\Lambda}^{\alpha-1}}, \quad \forall f \in \bigcup_{-1\le s\le 0} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

if $q \le 1$ and $\alpha = n(\frac{1}{q} - 1)$ and $\mathcal{S}_t f$ converges for the weak-$*$ topology of $\dot{\Lambda}^\alpha$ if $t \to 0^+$. In particular, for those specified p and α, $\mathcal{S}_t$ extends by density to uniformly bounded operators

$$\mathcal{S}_t : \dot{W}^{-1,p} \longrightarrow L^p$$

with strong limit as $t \to 0+$ and by duality to bounded operators

$$\mathcal{S}_t : \dot{\Lambda}^{\alpha-1} \longrightarrow \dot{\Lambda}^\alpha,$$

with weak-$*$ limit as $t \to 0+$.

4) *For $q \in I_{L^*}$, we have the estimate*

$$\sup_{t>0} \|\mathcal{D}_t f\|_{L^p} \lesssim \|f\|_{L^p}, \quad \forall f \in \bigcup_{0\le s\le 1} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

where $p = q'$ if $q > 1$, and $\mathcal{D}_t f$ converges strongly in L^p as $t \to 0^+$, and

$$\sup_{t>0} \|\mathcal{D}_t f\|_{\dot{\Lambda}^\alpha} \lesssim \|f\|_{\dot{\Lambda}^\alpha}, \quad \forall f \in \bigcup_{0\le s\le 1} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$

if $q \le 1$ and $\alpha = n(\frac{1}{q} - 1)$ and $\mathcal{D}_t f$ converges for the weak-$*$ topology of $\dot{\Lambda}^\alpha$ if $t \to 0^+$. In particular, for those specified p and α, $\mathcal{D}_t$ extends by density to uniformly bounded operators

$$\mathcal{D}_t : L^p \longrightarrow L^p$$

with strong limit as $t \to 0+$ and by duality to bounded operators

$$\mathcal{D}_t : \dot{\Lambda}^\alpha \longrightarrow \dot{\Lambda}^\alpha$$

with weak-$*$ limit as $t \to 0+$.

5) *For any integer $k \geq 0$, the same estimates than for $\mathcal{S}_t$ hold for $(t\partial_t)^k\mathcal{S}_t$ in the specified ranges of the above items. The same estimates than for $\mathcal{D}_t$ hold for $(t\partial_t)^k\mathcal{D}_t$ in the specified ranges of the above items.*
6) *The above items holds changing t to $-t$.*
7) *The jump relations* (85) *and* (86) *hold in all the topologies above where $\mathcal{S}_t$ and $\mathcal{D}_t$ are bounded respectively.*

According to corollary 13.3, this improves the known results obtained in [**61**] for operators with De Giorgi-Nash conditions as far as convergence at the boundary is concerned (strong convergence is obtained: it was known only for $p = 2$ combining [**8**] and [**81**]) and also with a weaker hypothesis (only an assumption on $L_\parallel^*$ or $L_\parallel$). Also these boundedness results are new without De Giorgi-Nash conditions. Let us now isolate the results concerning square functions and non-tangential maximal estimates for boundary layers.

Theorem 12.7. — *Let I_L be the interval in $(\frac{n}{n+1}, p_+(L))$ on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms and I_{L^*} be the interval in $(\frac{n}{n+1}, p_+(L^*))$ on which $\mathbb{H}^q_{D\widetilde{B}} = \mathbb{H}^q_D$ with equivalence of norms.*

1) *For $q \in I_L$, we have the estimate*
$$\|\widetilde{N}_*(\nabla\mathcal{S}_{\pm t}f)\|_q \sim \|t\partial_t\nabla\mathcal{S}_{\pm t}f\|_{T^q_2} \lesssim \|f\|_{H^q},$$
$$\|\widetilde{N}_*(\nabla\mathcal{D}_{\pm t}f)\|_q \sim \|t\partial_t\nabla\mathcal{D}_{\pm t}f\|_{T^q_2} \lesssim \|\nabla_x f\|_{H^q} \sim \|f\|_{\dot{H}^{1,q}},$$
where $H^q = L^q$ if $q > 1$.
2) *For $q \in I_{L^*}$, $q > 1$ and $p = q'$ then*
$$\|\widetilde{N}_*(\mathcal{S}_{\pm t}f)\|_p \lesssim \|t\,\nabla\mathcal{S}_{\pm t}f\|_{T^p_2} \lesssim \|f\|_{\dot{W}^{-1,p}},$$
$$\|\widetilde{N}_*(\mathcal{D}_{\pm t}f)\|_p \lesssim \|t\,\nabla\mathcal{D}_{\pm t}f\|_{T^p_2} \lesssim \|f\|_{L^p},$$
3) *For $q \in I_{L^*}$, $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$, then*
$$\|\widetilde{N}_{\sharp,\alpha}(\mathcal{S}_{\pm t}f)\|_\infty \lesssim \|t\,\nabla\mathcal{S}_{\pm t}f\|_{T^\infty_{2,\alpha}} \lesssim \|f\|_{\dot{\Lambda}^{\alpha-1}},$$
$$\|\widetilde{N}_{\sharp,\alpha}(\mathcal{D}_{\pm t}f)\|_\infty \lesssim \|t\,\nabla\mathcal{D}_{\pm t}f\|_{T^\infty_{2,\alpha}} \lesssim \|f\|_{\dot{\Lambda}^{\alpha}},$$

For statements concerning $\mathcal{S}_{\pm t}$ we a priori assume
$$f \in \bigcup_{-1\leq s\leq 0} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m),$$
and for statements concerning $\mathcal{D}_{\pm t}$,
$$f \in \bigcup_{0\leq s\leq 1} \dot{\mathcal{H}}^s(\mathbb{R}^n;\mathbb{C}^m).$$
Here, ∇ is the full gradient (∂_t, ∇_x). Alternately, it can be replaced by the conormal gradient $(\partial_{\nu_A}, \nabla_x)$. The non-tangential sharp functions are meant as the corresponding non-tangential maximal functions for $\mathcal{S}_{\pm t}f - \mathcal{S}_{\pm 0}f$ or $\mathcal{D}_{\pm t}f - \mathcal{D}_{\pm 0}f$. Also in 2),

if $p > 2$, the corresponding quantities $\|\widetilde{N}_\sharp(.)\|_p$ are equivalent to the T^p_2 terms in the middle.

As proved in [**23**], there is a generalized boundary layer representation for the conormal gradients of solutions in $\mathcal{E}$. This can be integrated to give the "usual" boundary layer representation for the solution itself. It improves the results found in [**23**] and [**55**]. Theorem 8.1 in [**32**] proved under De Giorgi-Nash assumptions on L and L^* is of the same spirit.

Corollary 12.8. — *Let I_{L^*} be the interval in $(\frac{n}{n+1}, p_+(L^*))$ on which $\mathbb{H}^q_{D\widetilde{B}} = \mathbb{H}^q_D$ with equivalence of norms. Let $u \in \mathcal{E}_s$, $-1 \leq s \leq 0$, be a solution of*

$$Lu = -\operatorname{div} A \nabla u = 0$$

in $\mathbb{R}^{1+n}_+$. Let $p \in (1, \infty)$ with $q = p' \in I_{L^*}$ such that $u|_{t=0} \in L^p(\mathbb{R}^n; \mathbb{C}^m)$ and $\partial_{\nu_A} u|_{t=0} \in \dot{W}^{-1,p}(\mathbb{R}^n; \mathbb{C}^m)$. Then the abstract boundary layer representation

$$u(t, x) = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_t(u|_{t=0})(x)$$

holds for all $t \geq 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m)$. In particular, $\sup_{t \geq 0} \|u(t, .)\|_{L^p(\mathbb{R}^n;\mathbb{C}^m)} < \infty$.

Proof. — Let $s \in [-1, 0]$ for which $u \in \mathcal{E}_s$. By corollary 8.4 in [**23**], we have

$$\nabla_A u(t, .) = \nabla_A \mathcal{S}_t(\partial_{\nu_A} u|_{t=0}) - \nabla_A \mathcal{D}_t(u|_{t=0}).$$

The equality holds in $\mathcal{E}_s \cap C([0, \infty); \dot{\mathcal{H}}^{s,+}_{DB})$. Thus, we have

$$u(t, x) = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_t(u|_{t=0})(x) + c, \quad t > 0, \tag{89}$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$, but also in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m)$ for each $t > 0$ as the right hand side belongs to $L^p(\mathbb{R}^n; \mathbb{C}^m) + \mathbb{C}^m$ by the boundedness properties of the boundary layers established in theorem 12.6 and the left hand side is in $L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m)$ as $u \in \mathcal{E}_s$. We also point out that c is independent of t because both sides are weak solutions with the same conormal gradient at the boundary. One can pass to the limit in $t \to 0$, after testing against a $C^\infty_0(\mathbb{R}^n; \mathbb{C}^m)$ function. For the right hand side, we use the strong limits in the theorem above and for the left hand side, this is because $t \mapsto u(t, .)$ is continuous at 0 in $L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m)$ as $u \in \mathcal{E}_s$ (this observation is remark 8.9 in [**23**]). One obtains $u|_{t=0}(x) = \mathcal{S}_0(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_{0^+}(u|_{t=0})(x) + c$. As all the functions belong to $L^p(\mathbb{R}^n; \mathbb{C}^m)$, we conclude that $c = 0$. □

Remark 12.9. — Note that (89) holds under the sole assumption that $u \in \mathcal{E}_s$. So for Hölder or BMO spaces, the equality holds in those spaces.

12.4. The block case

Consider

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

that is, A is block diagonal. In this case, B is also block diagonal with

$$B = \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix}.$$

12.4.1. The case $a = 1$. — We assume $a = 1$. The Hardy space theory for $1 < p < \infty$ was explicitly developed in [**66**]. The limitation to $p > 1$ is due to the fact that these authors work with UMD-valued functions. Remark that

$$DB = \begin{bmatrix} 0 & \operatorname{div} d \\ -\nabla & 0 \end{bmatrix}, \quad (DB)^2 = \begin{bmatrix} -\operatorname{div} d\nabla & 0 \\ 0 & -\nabla \operatorname{div} d \end{bmatrix}.$$

In particular, $(DB)^2$ is sectorial with angle ω (instead of 2ω if B is an arbitrary matrix with angle of accretivity ω). Also $(DB)^2$ has an H^∞-calculus on $L^2(\mathbb{R}^n; \mathbb{C}^N)$. Set

$$L = -\operatorname{div} d\nabla \quad \text{and} \quad M = -\nabla \operatorname{div} d,$$

both defined as ω-sectorial operators on $L^2(\mathbb{R}^n; \mathbb{C}^m)$ and $L^2(\mathbb{R}^n; \mathbb{C}^{nm})$ with H^∞-calculus. Note that $M = 0$ on $\mathsf{N}(\operatorname{div} d)$ and that the Hodge decomposition

$$L^2(\mathbb{R}^n; \mathbb{C}^{nm}) = \overline{\mathsf{R}_2(\nabla)} \oplus \mathsf{N}(\operatorname{div} d)$$

is consistent with the splitting

$$L^2(\mathbb{R}^n; \mathbb{C}^{n(1+m)}) = \overline{\mathsf{R}_2(DB)} \oplus \mathsf{N}(DB) = \begin{bmatrix} L^2(\mathbb{R}^n; \mathbb{C}^m) \\ \overline{\mathsf{R}_2(\nabla)} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathsf{N}(\operatorname{div} d) \end{bmatrix}.$$

It was shown in [**27**] that the interval $(p_-(DB), p_+(DB))$ is the largest interval of p such that one has the corresponding Hodge decomposition for L^p, which is also $(q_+(L^*)', q_+(L))$ where $q_+(L)$ was introduced in [**7**].

Since DB admits L^2 off-diagonal estimates to any order, so does $(DB)^2$ and, as $(DB)^2$ is diagonal, so do L and M. So both L and M enjoy a Hardy space theory. Only the decay of the allowable ψ changes because of the second order nature of L and M. Explicit conditions on ψ can be found [**66**] (see also [**60**]). Using even (with respect to $z \mapsto -z$) allowable ψ for all these Hardy spaces $\mathbb{H}^p$ below, we obtain that

$$f = \begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix} \in \mathbb{H}^p_{DB} \Longleftrightarrow f_\perp \in \mathbb{H}^p_L \text{ and } f_\parallel \in \mathbb{H}^p_M, \text{with } \|f\|_{\mathbb{H}^p_{DB}} \sim \|f_\perp\|_{\mathbb{H}^p_L} + \|f_\parallel\|_{\mathbb{H}^p_M}.$$

Using the $\mathbb{H}^p_{DB}$ theory for $0 < p < \infty$, we have that $\operatorname{sgn}(DB)$ is bounded on $\mathbb{H}^p_{DB}$. We note that this is equivalent to

$$\|L^{1/2}u\|_{\mathbb{H}^p_L} \sim \|\nabla u\|_{\mathbb{H}^p_M}, \quad \forall u \in \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m).$$

Indeed, pick $f \in \mathbb{H}^2_{DB} = \mathbb{H}^2_D$ so that $f_\perp \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ and $f_\parallel = \nabla g_\perp$ for $g_\perp \in \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$. Also, one can write $f_\perp = L^{1/2}h_\perp$ with $h_\perp \in \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ by the solution of the Kato problem for operators and systems [**13**], [**16**]. Then as $|DB|$ is the diagonal operator with entries $L^{1/2}$, $M^{1/2}$, we have

$$\operatorname{sgn}(DB)f = \begin{bmatrix} L^{-1/2}\operatorname{div} d\nabla g_\perp \\ -M^{-1/2}\nabla f_\perp \end{bmatrix} = \begin{bmatrix} -L^{1/2}g_\perp \\ -\nabla L^{-1/2}f_\perp \end{bmatrix} = \begin{bmatrix} -L^{1/2}g_\perp \\ -\nabla h_\perp \end{bmatrix}.$$

For the last line, we used the equality $(I+t^2M)^{-1}\nabla f = \nabla(I+t^2L)^{-1}f$ for all $f \in W^{1,2}$, extended to $f \in L^2$ (by extending the resolvents), and

$$\begin{aligned} M^{-1/2}\nabla f &= \frac{2}{\pi}\int_0^\infty (I+t^2M)^{-1}tM^{1/2}M^{-1/2}\nabla f\,\frac{dt}{t} \\ &= \frac{2}{\pi}\int_0^\infty \nabla L^{-1/2}tL^{1/2}(I+t^2L)^{-1}f\,\frac{dt}{t} = \nabla L^{-1/2}f, \end{aligned}$$

where, classically, the integrals converge strongly in L^2 by the H^∞-calculus for L and M and since both operators are bounded on L^2 (for the one on the left, one can see that by duality). Thus we may apply the equality to $f_\perp \in L^2$. Thus

$$\|\operatorname{sgn}(DB)f\|_{\mathbb{H}^p_{DB}} \sim \|L^{1/2}g_\perp\|_{\mathbb{H}^p_L} + \|\nabla h_\perp\|_{\mathbb{H}^p_M}$$

while

$$\|f\|_{\mathbb{H}^p_{DB}} \sim \|L^{1/2}h_\perp\|_{\mathbb{H}^p_L} + \|\nabla g_\perp\|_{\mathbb{H}^p_M}.$$

As $h_\perp$ and $g_\perp$ are arbitrary and unrelated in $\dot W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$, this shows the announced equivalence. [(1)]

Proposition 12.10. — *Let $p \in (\frac{n}{n+1},\infty)$. If $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms then $\mathbb{H}^p_L = H^p \cap L^2$ and $\mathbb{H}^p_M = H^p \cap \nabla\dot W^{1,2}$ and $\|L^{1/2}u\|_{H^p} \sim \|\nabla u\|_{H^p}$ for all $u \in \dot W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$, where H^p is the classical Hardy space if $p \le 1$ and L^p is $p>1$.*

Proof. — Recall that $\mathbb{H}^p_D = H^p \cap \mathbb{P}(L^2)$ and $\mathbb{P}(L^2) = L^2(\mathbb{R}^n;\mathbb{C}^m) \oplus \nabla\dot W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$. Thus, $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ if and only if $\mathbb{H}^p_L = H^p \cap L^2$ and $\mathbb{H}^p_M = H^p \cap \nabla\dot W^{1,2}$ so that they are both subspaces of H^p. The conclusion for the Riesz transform $\nabla L^{-1/2}$ follows right away. □

The interval of L^p boundedness of the Riesz transform $\nabla L^{-1/2}$ is characterized in [**7**] as the interval $(q_-(L), q_+(L))$, which is the largest open interval on which $\sqrt{t}\,\nabla e^{-tL}$ is bounded on L^p, uniformly in $t>0$. And it is also known that $q_-(L) = p_-(L)$ where $(p_-(L), p_+(L))$ is the largest open interval on which e^{-tL} is bounded on L^p, uniformly in $t>0$. It was shown in [**60**] (in the case of equations: $m=1$) that for $1<p<\infty$, $H^p_L = L^p$ if and only if $p \in (p_-(L), p_+(L))$. When $0<p\le 1$, [**60**] proves that $\|f\|_{H^p} \lesssim \|f\|_{\mathbb{H}^p_L}$ and, when $(p_-(L))_* < p \le 1$, that $\|L^{1/2}u\|_{\mathbb{H}^p_L} \sim \|\nabla u\|_{H^p}$ when $u \in \dot W^{1,2}(\mathbb{R}^n)$. But H^p_L is not identified when $p \le 1$. The possibility of identifying H^p_L for $p \le 1$ seems new. It turns out that the number

1. The direction from boundedness of $\operatorname{sgn}(DB)$ to the statement for $L^{1/2}$ has been known for long: it is for example in [**21**]. It is explicitly in [**66**] in this context. The converse was pointed out to us by A. McIntosh.

$p_-(L)$ may not be the relevant critical exponent for this. We isolate a number of interesting facts in this corollary.

COROLLARY 12.11. — *Let I be the interval in $(\frac{n}{n+1},\infty)$ on which $\mathbb{H}^p_{DB}=\mathbb{H}^p_D$ with equivalence of norms. Then, $I\cap(1,\infty)\subset(q_-(L),q_+(L))$. As $q_+(L)=p_+(DB)$, we also conclude that $\sup I=p_+(DB)$. Also, if $p\in I\cap(1,\infty)$, then e^{-tL} is bounded on L^p uniformly in $t>0$. Finally, if $\inf I<p\le 1$, $H^p_L=H^p$.*

A large part of [**60**] is concerned with developing the H^p_L theory, for the full range $0<p<\infty$ together with variants involving regularity indices. See also [**67**] for $0<p\le 1$. See also non-tangential maximal estimates in [**78**] towards solving the associated second order PDE $\partial_t^2 u+\operatorname{div} d\nabla u=0$, which can be seen as a special case of (1). Some larger ranges of exponents are obtained there, probably due to the "diagonal" structure of the PDE (no cross terms in t and x).

12.4.2. The case $a\neq 1$. — The full block diagonal case with $a\neq 1$ can be treated similarly. In this situation,

$$L=-\operatorname{div} d\nabla a^{-1}\quad\text{and}\quad M=-\nabla a^{-1}\operatorname{div} d,$$

which are 2ω-sectorial operators on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ with H^∞-calculus on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ as diagonal components of $(DB)^2$. The same discussion applies concerning the links between $\mathbb{H}^p_{DB}$, $\mathbb{H}^p_L$ and $\mathbb{H}^p_M$ and that $\|L^{1/2}u\|_{\mathbb{H}^p_L}\sim\|\nabla(a^{-1}u)\|_{\mathbb{H}^p_M}$. Thus if $\mathbb{H}^p_{DB}=\mathbb{H}^p_D$, then $\mathbb{H}^p_L=H^p\cap L^2$ and $H^p_L=H^p$ (again, this is by convention L^p if $p>1$). Remark also that if $\mathbb{H}^p_{DB}=\mathbb{H}^p_D$ and $p>1$, then the resolvent of L and semigroup generated by $L^{1/2}$ are bounded on L^p (There may be no semigroup generated by $-L$ if $2\omega\ge\pi/2$).

If $\mathbb{H}^p_{DB}=\mathbb{H}^p_D$, by similarity, we obtain a characterization of the Hardy space associated to $-a^{-1}\operatorname{div} d\nabla$ as $a^{-1}H^p$.

In boundary dimension $n=1$, M and L are of the same type because div and ∇ both become $\frac{d}{dx}$. Although not formulated in the language of the current article, it was shown in [**28**] that $H^p_L=H^p$ for all $p\in(\frac12,\infty)$ (in the case of equations, that is when $m=1$). The same thus holds for M replacing L and therefore $H^p_{DB}=H^p$ for those p. The proof there extends to arbitrary systems with $m>1$. Nevertheless, this follows directly on applying proposition 3.11 for any m as the symbol of D is invertible on $\mathbb{R}\setminus\{0\}$.

13. SYSTEMS WITH DE GIORGI TYPE CONDITIONS

We are given $B = \widehat{A}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \begin{bmatrix} 0 & \mathrm{div} \\ -\nabla & 0 \end{bmatrix}$ as before, which corresponds to the second order system

$$L = -\operatorname{div} A\nabla.$$

Let $L_\parallel = -\operatorname{div} d\,\nabla$ where d is the lower right coefficient in A. This operator acts on the boundary $\mathbb{R}^n$ of $\mathbb{R}^{1+n}_+$. Classical elliptic theory implies there exists $\lambda(L_\parallel) \in (0, n]$ such that the following holds:

For any $\lambda \in [0, \lambda(L_\parallel))$, there exists a constant $C \geq 0$ such that for any ball $B(x_0, R)$, for any $v \in W^{1,2}(B(x_0, R))$ weak solution in $B(x_0, R)$ of $L_\parallel v = 0$ and for all $0 < \rho \leq R$

$$\int_{B(x_0,\rho)} |\nabla v|^2 \leq C \left(\frac{\rho}{R}\right)^\lambda \int_{B(x_0,R)} |\nabla v|^2. \tag{90}$$

The constant C depends on L^∞ bounds and accretivity of d on $\mathcal{R}_2(\nabla)$, λ and $\lambda(L_\parallel)$.

Definition 13.1 (From [**5**]). — $L_\parallel$ satisfies the De Giorgi condition if $\lambda(L_\parallel) > n-2$.

It is equivalent to the fact that weak solutions of $L_\parallel$ are locally bounded and Hölder continuous with exponent less than

$$\alpha(L_\parallel) = \frac{\lambda(L_\parallel) - n + 2}{2}.$$

See [**56**] for explicit proofs.

This condition holds for any $L_\parallel$ as above if $n \leq 2$, for real d and their L^∞ perturbations when $m = 1, n \geq 3$. It also holds if d is constant for any n, m (with $\lambda_+(L^*_\parallel) = n$) and if d is an L^∞ perturbation of a constant (with any $\lambda(L_\parallel) < n$).

Theorem 13.2. — *Assume that $L^*_\parallel$ satisfies the De Giorgi condition. For $p_\parallel < p \leq 1$, with $p_\parallel = \frac{n}{n+\alpha(L^*_\parallel)}$, any $(\mathbb{H}^p_D, 1)$-atom α and integer $M \geq M(n)$, we have*

$$\|tDB(I + itDB)^{-M}\alpha\|_{T^p_2} \lesssim 1$$

with implicit constant depending only on n, m, M, the L^∞ and accretivity bounds of B, and the constants in the De Giorgi condition for $L^*_\parallel$.

It is quite striking that no regularity is imposed on the weak solutions of $L_\parallel$, nor any condition on the other coefficients a, b, c of L.

Corollary 13.3. — *Assume that $L_\|^*$ satisfies the De Giorgi condition. Then we have $\mathbb{H}^p_{DB} = \mathbb{H}^p_D$ with equivalence of norms for $p_\| < p < p_+(DB)$.*

We remark that this identification is obtained without knowing kernel bounds.

Proof. — The case $2 < p < p_+(DB)$ is from the general theory and there is nothing new. We consider $p < 2$.

Remark that $\psi(z) = z(1+iz)^{-M} \in \Psi_1^{M-1}(S_\mu)$ is allowable for $\mathbb{H}^p_{DB}$ for any $p \in (\frac{n}{n+1}, 2)$ if $M - 1 > \frac{n}{2} + 1$. The theorem above tells that for $p_\| < p \leq 1$ and $(\mathbb{H}^p_D, 1)$-atoms α, $\alpha \in \mathbb{H}^p_{DB}$ and $\|\alpha\|_{\mathbb{H}^p_{DB}} \lesssim 1$. A density argument provides that $\mathbb{H}^p_D \subset \mathbb{H}^p_{DB}$ with continuous inclusion. By complex interpolation (arguing as in the proof of corollary 4.14) this holds for $1 < p < 2$. Now the converse inclusion and continuity bound were known from corollary 4.17 for $\frac{n}{n+1} < p < 2$. □

Thus, by duality, all the *a priori* estimates for weak solutions of $Lu = 0$ with $u \in \mathcal{E}$ apply to this situation assuming $L_\|$ satisfies the De Giorgi condition with exponent $\lambda(L_\|) > n-2$. For example, we have, normalizing u by an additive constant in the first inequality,

$$\begin{aligned}
\|\widetilde{N}_*(u)\|_p &\lesssim \|S(t\nabla u)\|_p, \quad \forall p \in (2-\varepsilon, \infty),\\
\|\widetilde{N}_\sharp(u)\|_p &\lesssim \|S(t\nabla u)\|_p, \quad \forall p \in (2, \infty),\\
\|\widetilde{N}_{\sharp,\alpha}(u)\|_\infty &\lesssim \|t\nabla u\|_{T^\infty_{2,\alpha}}, \quad \forall \alpha \in [0, \alpha(L)),\ \alpha(L) = \frac{\lambda(L_\|) - n + 2}{2}.
\end{aligned}$$

The first inequality was known if L is a real and scalar operator [**54**]. However there is a subtle difference. In that work, the *a priori* assumption $u \in \mathcal{E}$ is not required and p can be any positive number: the proof in this specific situation uses the $p = 2$ case in [**8**] and good lambda arguments requiring the converse inequality

$$\|S(t\nabla u)\|_p \lesssim \|\widetilde{N}_*(u)\|_p$$

(which [**54**] proves using changes of variables, so it is not clear at all whether this can extend to complex situations) and finiteness of $\|\widetilde{N}_*(u)\|_p$ (which can even be replaced by the usual non-tangential function by interior regularity estimates). So, in fact, [**54**] proves that on the class of weak solutions with $\|\widetilde{N}_*(u)\|_p < \infty$, one has

$$\|S(t\nabla u)\|_p \sim \|\widetilde{N}_*(u)\|_p$$

for any $0 < p < \infty$. Here, we show that when $u \in \mathcal{E}$, then

$$\|\widetilde{N}_*(u)\|_p \lesssim \|S(t\nabla u)\|_p$$

when $2 - \varepsilon < p < \infty$ (Note that the *a priori* information $u \in \mathcal{E}$ will be removed in [**24**]: the combination of all this shows that the two classes of weak solutions (one with square function control and the other with non-tangential maximal control) are identical (up to additive normalisation) in this range of p and this class of operators (real and scalar)).

The latter two inequalities seem new even when L is a real and scalar operator.

13.1. Preliminary computations

We begin with some computation. As before, we write

$$f \in L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m}) \quad \text{as} \quad f = \begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix}$$

with $f_\perp \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ and $f_\parallel \in L^2(\mathbb{R}^n; \mathbb{C}^{nm})$. We also write L^2 from now on without precision.

For $t \in \mathbb{R}$ set $R_t = (I + itDB)^{-1}$ and

$$L_t = \begin{bmatrix} 1 & it \operatorname{div}_x \end{bmatrix} \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \begin{bmatrix} 1 \\ it \nabla_x \end{bmatrix}.$$

Lemma 13.4. — *Let $f \in L^2$ and $t \in \mathbb{R}$. Then the equation $R_t f = F$ is equivalent to the system*

$$F_\perp = au_t + bF_\parallel, \tag{91}$$

$$F_\parallel = it \nabla_x u_t + f_\parallel, \tag{92}$$

with

$$u_t = L_t^{-1} \begin{bmatrix} 1 & it \operatorname{div}_x \end{bmatrix} \begin{bmatrix} f_\perp - bf_\parallel \\ -df_\parallel \end{bmatrix}.$$

Proof. — Let g, G defined by $f = \overline{A} g$ and $F = \overline{A} G$ with

$$\overline{A}(x) = \begin{bmatrix} a(x) & b(x) \\ 0 & 1 \end{bmatrix}.$$

Then, by [**9**, lemma 2.53] (see [**8**, lemma 9.3] for a direct proof in this context), $R_t f = F$ is equivalent to

$$G_\perp = u_t, \tag{93}$$

$$G_\parallel = it \nabla_x u_t + g_\parallel. \tag{94}$$

It suffices to note that $F_\perp = aG_\perp + bG_\parallel$ and $F_\parallel = G_\parallel$. □

Lemma 13.5. — *Assume $f \in L^2$ has the form $f = \left[\begin{smallmatrix} f_\perp \\ it \nabla h \end{smallmatrix}\right]$ with $f_\perp \in L^2$ and $h \in W^{1,2}$. Then the equation $R_t f = F$ is equivalent to $F = \left[\begin{smallmatrix} F_\perp \\ it \nabla H \end{smallmatrix}\right]$ with $F_\perp \in L^2$ and $H \in W^{1,2}$ given by*

$$\begin{bmatrix} F_\perp \\ H \end{bmatrix} = \mathcal{R}_t \begin{bmatrix} f_\perp \\ h \end{bmatrix}$$

with $\mathcal{R}_t$ being the 2×2 matrix of operators

$$\mathcal{R}_t = \begin{bmatrix} aL_t^{-1} & T_t \\ L_t^{-1} & U_t \end{bmatrix},$$

where

$$U_t = L_t^{-1}(a + it\,\mathrm{div}\,c),$$

and

$$T_t = -a + (a + itb\,\nabla)L_t^{-1}(a + it\,\mathrm{div}\,c).$$

Here, a, b, c, d mean multiplication by the corresponding functions $a(x), b(x), c(x), d(x)$.

Proof. — Write

$$u_t = L_t^{-1}(f_\perp - itb\,\nabla h - it\,\mathrm{div}\,d\nabla h)$$

and using the definition of $L_t h$ we obtain

$$u_t = -h + L_t^{-1}(f_\perp + ah + it\,\mathrm{div}\,ch).$$

Thus (92) is equivalent to

$$F_\parallel = it\,\nabla L_t^{-1}(f_\perp + ah + it\,\mathrm{div}\,ch) = it\,\nabla(L_t^{-1}f_\perp + U_t h)$$

because $-it\,\nabla h + f_\parallel = 0$, and (91) is equivalent to

$$F_\perp = aL_t^{-1}f_\perp + T_t h. \qquad \square$$

13.2. Proof of theorem 13.2

We start the proof of the theorem. Let $\alpha = D\beta$ be an $(\mathbb{H}^p_D, 1)$-atom. This means that α, β are both supported in a ball Q, with

$$\|\alpha\|_2 \le |Q|^{\frac{1}{2}-\frac{1}{p}} \quad \text{and} \quad \|\beta\|_2 \le r(Q)|Q|^{\frac{1}{2}-\frac{1}{p}},$$

with $r(Q)$ the radius of Q. Note that $\alpha_\perp$ is the divergence of $\beta_\parallel$. In particular, $\alpha_\perp$ is a classical L^2-atom (valued in $\mathbb{C}^m$) for the Hardy space H^p and each component has mean value 0. Also $\alpha_\parallel$ is a gradient field.

Call $C_k(T_Q)$ the following regions in $\mathbb{R}^{1+n}_+$. For $k \ge 0$,

$$R_k(T_Q) = (0, 2^k r(Q)] \times 2^k Q, \quad C_0(T_Q) = R_1(T_Q) \quad \text{and} \quad C_k(T_Q) = R_{k+1}(T_Q) \setminus R_k(T_Q)$$

for $k > 0$. It is enough to show that

$$\iint_{C_k(T_Q)} |tDBR_t^M \alpha|^2 \,\frac{dtdx}{t} \lesssim |2^k Q|^{1-\frac{2}{p}} 2^{-k\varepsilon} \tag{95}$$

for some $\varepsilon > 0$ and M large enough.

For simplicity we assume that Q is the unit ball centered at 0. All estimates are affine invariant because all assumptions in the theorem are stable under affine changes of variables so this is no loss of generality.

First for $k = 0$, (95) holds as a consequence of the square function estimate (14) for DB and the size of $\|\alpha\|_2$.

For $k > 0$, we note that $itDBR_t^M\alpha = R_t^{M-1}\alpha - R_t^M\alpha$ and it is enough to treat each term separately. Hence we have to show that

$$\iint_{C_k} |R_t^M\alpha|^2 \,\frac{dtdx}{t} \lesssim 2^{k(n-\frac{2n}{p}-\varepsilon)} \tag{96}$$

for large enough M, where we set $C_k = C_k(T_Q)$.

The part of the integral in (95) where $t \le 1$ can be treated using the L^2 off-diagonal decay of R_t^M (11)

$$\int_{2^{k+1}Q\setminus 2^kQ} |R_t^M\alpha|^2\,dx \lesssim (2^k/t)^{-N}\|\alpha\|_2^2 \tag{97}$$

for all N. Thus integrating this estimate in $t \in (0,1]$ yields a bound 2^{-kN}.

For the remaining part, when $t > 1$, we claim assuming M large enough and all N, we have that for $1 \le t < 2^k$, we have

$$\int_{2^{k+1}Q\setminus 2^kQ} |R_t^M\alpha|^2\,dx \lesssim (2^k/t)^{-N}t^{n-\frac{2n}{p}-\varepsilon} \tag{98}$$

and for $2^k \le t \le 2^{k+1}$,

$$\int_{2^{k+1}Q} |R_t^M\alpha|^2\,dx \lesssim t^{n-\frac{2n}{p}-\varepsilon}. \tag{99}$$

Then, integrating in the corresponding t intervals the above estimates concludes the proof of (97).

To end the proof of the theorem, it remains to prove the claim. This is where we use fully that α is an $(\mathbb{H}^p_D, 1)$-atom and the above calculations. Write

$$\alpha = f, \quad f^{(k)} = R_t^k f.$$

Since $f = \left[\begin{smallmatrix} f_\perp \\ it\nabla h\end{smallmatrix}\right]$ with $h = -(it)^{-1}\beta_\perp$, we have $f^{(k)} = \left[\begin{smallmatrix} f_\perp^{(k)} \\ it\nabla h^{(k)}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} f_\perp^{(k)} \\ h^{(k)}\end{smallmatrix}\right] = \mathcal{R}_t^k\left[\begin{smallmatrix} f_\perp \\ h\end{smallmatrix}\right]$. Fix $t > 0$. Since

$$L_t h^{(k+1)} = f_\perp^{(k)} + ah^{(k)} + it\operatorname{div} ch^{(k)},$$

the usual Caccioppoli argument for the (non homogeneous) operator L_t yields

$$\int_{B_t} |it\nabla h^{(k+1)}|^2 dx \le C\int_{cB_t} (|h^{(k+1)}|^2 + |f_\perp^{(k)}|^2 + |h^{(k)}|^2)dx$$

for any $c > 1$ and some $C > 0$ independent of the ball B_t of radius within $t/2$ and $2t$, k and depending only on the L^∞ and accretivity bounds of A. From

$$|f^{(k+1)}|^2 = |f_\perp^{(k+1)}|^2 + |it\nabla h^{(k+1)}|^2$$

and using a bounded covering by balls of radius $\sim t$, we see that it is enough to prove (98) and (99) by replacing $R_t^M\alpha$ by $\mathcal{R}_t^M\left[\begin{smallmatrix} f_\perp \\ h\end{smallmatrix}\right]$ (up to fattening slightly C_k to a similar type of region, which we ignore in the sequel as this is only a cosmetic change in the estimates). Hence, it suffices to prove assuming M large enough that, for all N and $1 \le t < 2^k$, we have

$$\int_{2^{k+1}Q\setminus 2^kQ} |\mathcal{R}_t^M \begin{bmatrix} f_\perp \\ h \end{bmatrix}|^2 dx \lesssim (2^k/t)^{-N} t^{n-\frac{2n}{p}-\varepsilon} \tag{100}$$

and for $2^k \le t \le 2^{k+1}$,

$$\int_{2^{k+1}Q} |\mathcal{R}_t^M \begin{bmatrix} f_\perp \\ h \end{bmatrix}|^2 dx \lesssim t^{n-\frac{2n}{p}-\varepsilon}. \tag{101}$$

To do this, we proceed to an analysis of the iterates of the adjoint of $\mathcal{R}_t$, starting from L^2 using the scales of Morrey spaces and Campanato spaces (here for functions defined on $\mathbb{R}^n$ and valued in $\mathbb{C}^m$) following [**5**]. For $0 \le \lambda \le n$, define the Morrey space $L^{2,\lambda}(\mathbb{R}^n;\mathbb{C}^m) = L_0^{2,\lambda} \subset L^2_{\mathrm{loc}}$ by the condition

$$\|f\|_{L_0^{2,\lambda}} \equiv \sup_{x\in\mathbb{R}^n,\, 0<R\le 1} \Big(R^{-\lambda}\int_{B(x,R)} |f|^2\Big)^{1/2} < \infty,$$

where $B(x,r)$ denotes the Euclidean ball of center x and radius $r>0$. For $0 \le \lambda \le n+2$, one defines the Campanato space $L_1^{2,\lambda}(\mathbb{R}^n;\mathbb{C}^m) = L_1^{2,\lambda} \subset L^2_{\mathrm{loc}}$ by

$$\|f\|_{L_1^{2,\lambda}} \equiv \sup_{x\in\mathbb{R}^n,\, 0<R\le 1} \Big(R^{-\lambda}\int_{B(x,R)} |f-(f)_{x,R}|^2\Big)^{1/2} < \infty.$$

The notation $(u)_{x,R}$ stands for the mean value of u over the ball $B(x,R)$. The space $L^2 \cap L_i^{2,\lambda}$ is equipped with the norm $\|f\|_2 + \|f\|_{L_i^{2,\gamma}}$. We also denote by $\mathcal{L}_i^{2,\lambda}$ the corresponding homogeneous spaces when dropping the constraint that $R \le 1$.

Here are a few facts for the appropriate ranges of λ.

(a) $L_i^{2,\lambda_1} \subset L_i^{2,\lambda_2}$ if $\lambda_1 > \lambda_2$.

(b) $L^2 \cap L_1^{2,\lambda} \equiv L^2 \cap L_0^{2,\lambda}$ if $\lambda < n$.

(c) $L^2 \cap L_i^{2,\lambda} \equiv L^2 \cap \mathcal{L}_i^{2,\lambda}$.

(d) $L_0^{2,\lambda}$ is preserved by multiplication by bounded functions.

In particular the higher the λ, the better the regularity in these scales. We have the following lemma.

Lemma 13.6. — *For M large enough (depending only on dimension) and $0 \le \lambda < \lambda(L_\parallel^*)$ $(\le n)$, we have that $\mathcal{R}_t^{*M}$ maps $L^2 \times L^2$ into $\mathcal{L}_1^{2,\lambda+2} \times \mathcal{L}_0^{2,\lambda}$ for all $t \ne 0$. Furthermore,*

▷ *the operator norm of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathcal{R}_t^{*M}$ from $L^2 \times L^2$ into $\mathcal{L}_1^{2,\lambda+2}$ is bounded by $C|t|^{-\lambda/2-1}$ and*

▷ *the operator norm of $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathcal{R}_t^{*M}$ from $L^2 \times L^2$ into $\mathcal{L}_0^{2,\lambda}$ is bounded by $C|t|^{-\lambda/2}$.*

Assuming this lemma, we argue as follows to prove (100) and (101). First, the De Giorgi condition and $p_\parallel < p$ means that we can take $\lambda = n-2+2\alpha$ for some $\alpha > n(\frac{1}{p}-1)$ in the previous lemma and the sought ε will be $2\alpha - 2n(\frac{1}{p}-1)$. Next, we prove (101) by dualizing against $g \in L^2 \times L^2$, supported in $2^{k+1}Q$, with norm 1. Then

$$\Big\langle \mathcal{R}_t^M \begin{bmatrix} f_\perp \\ h \end{bmatrix}, g \Big\rangle = \Big\langle f_\perp, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{R}_t^{*M} g \Big\rangle + \Big\langle h, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{R}_t^{*M} g \Big\rangle.$$

For the first term, since $f_\perp$ has mean value 0 on Q, we can subtract the mean value on Q of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{R}_t^{*M} g$ then use Cauchy-Schwarz inequality and the $\mathcal{L}_1^{2,\lambda+2}$ estimate which leads to a bound

$$\|f_\perp\|_2 C t^{-\lambda/2-1} \|g\|_2 \le C t^{-\lambda/2-1}.$$

For the second term, we merely use Cauchy-Schwarz inequality and the $\mathcal{L}_0^{2,\lambda}$ estimate which leads to a bound

$$\|h\|_2 C t^{-\lambda/2} \|g\|_2 \le C t^{-\lambda/2-1}$$

using that $\|h\|_2 \le t^{-1}$. This proves (101).

To prove (100) we need to incorporate some decay in the bounds of the above lemma. This is done using the standard exponential perturbation argument. Let φ be a real-valued, Lipschitz function. We also assume φ bounded but do not use its bound. Let

$$\mathcal{R}_{t,\varphi} = \exp(-\varphi/t)\mathcal{R}_t \exp(\varphi/t).$$

A simple computation shows that this operator has the same form and properties as $\mathcal{R}_t$ with d unchanged and a, b, c modified by an additive $O(\|\nabla\varphi\|_\infty)$ term. Also since the higher order coefficient of $\exp(-\varphi/t)L_t \exp(\varphi/t)$ is the same as the one of L_t, we also have the De Giorgi condition on the adjoint of the higher order term. Thus, we have the same bounds for $\mathcal{R}_{t,\varphi}^M$ uniformly for $\|\nabla\varphi\|_\infty \le \delta$ for some $\delta > 0$ depending solely on L^∞ and accretivity bounds, and on the De Giorgi condition of $L_\parallel^*$. Having fixed Q (the unit ball) and $k \ge 1$, we choose

$$\varphi(x) = \delta \inf(d(x, Q), N)$$

for a fixed $N \ge 2^{k+1}$. Hence, $\|\nabla\varphi\|_\infty \le \delta$ and $\inf \varphi = \delta(2^k - 1)$ on $2^{k+1}Q \setminus 2^k Q$. Using the support condition of $f_\perp, h$ and the definition of φ, we obtain that

$$\mathcal{R}_t^M \begin{bmatrix} f_\perp \\ h \end{bmatrix} = \mathcal{R}_t^M \begin{bmatrix} \exp(\varphi/t) f_\perp \\ \exp(\varphi/t) h \end{bmatrix} = \exp(\varphi/t)\mathcal{R}_{t,\varphi}^M \begin{bmatrix} f_\perp \\ h \end{bmatrix}.$$

Using the bounds for $\mathcal{R}_{t,\varphi}^M$, we obtain powers of t as above, multiplied by the supremum on $2^{k+1}Q \setminus 2^k Q$ of $\exp(-\varphi/t)$, that is $\exp(-\delta(2^k - 1)/t)$. This proves (100). The proof of the theorem is complete modulo that of the last lemma.

For later use, we record the following estimate that comes from a modification of the above arguments.

Corollary 13.7. — *Assume $\lambda(L_\|^*) > n-2$ and let $p_\| < p \le 1$. If α is a $(\mathbb{H}^p_D, 1)$-atom associated to the ball Q, then for any other ball Q', we have for large enough M (depending only on dimension and $\lambda(L_\|^*)$)*

$$\int_{Q'} |R_t^M \alpha|^2 \, dx \lesssim e^{-\delta \frac{\operatorname{dist}(Q',Q)}{t}} t^{n-\frac{2n}{p}-\varepsilon} \tag{102}$$

for all $t > 0$ and some $\delta > 0$ and $\varepsilon > 0$.

13.3. Proof of lemma 13.6

First by scaling it suffices to assume $t = 1$. Since the Morrey and Campanato spaces of the statement are the homogeneous ones, the powers of t follow automatically by a rescaling argument (which yields operators with the same hypotheses). We thus drop the index t in the notation. From fact (c), it suffices to work in the inhomogeneous spaces. It follows from [**5**, theorem 3.10] (this is done for real equations but the proof applies *mutatis mutandi* to complex systems with Gårding inequality) that for $\lambda \ge 0$ we have the boundedness properties

$$\begin{aligned} L^{*-1} &: L^2 \cap L_1^{2,\lambda} \longrightarrow L^2 \cap L_1^{2,\lambda'}, \quad 0 \le \lambda' \le \lambda + 4, \quad \lambda' < \lambda(L_\|^*), \\ \nabla L^{*-1} &: L^2 \cap L_1^{2,\lambda} \longrightarrow L^2 \cap L_1^{2,\lambda'}, \quad 0 \le \lambda' \le \lambda + 2, \quad \lambda' < \lambda(L_\|^*), \\ L^{*-1} \operatorname{div} &: L^2 \cap L_1^{2,\lambda} \longrightarrow L^2 \cap L_1^{2,\lambda'}, \quad 0 \le \lambda' \le \lambda + 2, \quad \lambda' < \lambda(L_\|^*), \\ \nabla L^{*-1} \operatorname{div} &: L^2 \cap L_1^{2,\lambda} \longrightarrow L^2 \cap L_1^{2,\lambda'}, \quad 0 \le \lambda' \le \lambda, \qquad\quad \lambda' < \lambda(L_\|^*). \end{aligned}$$

Note that U^* is a combination of the first two lines, so there is a gain of 2 at most. However for T^*, we must use the fourth line so there is no gain. Since

$$\mathcal{R}^* = \begin{bmatrix} L^{*-1}a^* & L^{*-1} \\ T^* & U^* \end{bmatrix},$$

starting from $g^{(0)} \in L^2 \times L^2$ and letting $g^{(k+1)} = \mathcal{R}^* g^{(k)}$ for $k \ge 0$, we argue as follows using facts (b) and (d). As $g^{(0)} \in (L^2 \cap L_1^{2,0}) \times (L^2 \cap L_1^{2,0})$, we see that $g^{(1)} \in (L^2 \cap L_1^{2,2}) \times (L^2 \cap L_1^{2,0})$. Next, we see $g^{(2)} \in (L^2 \cap L_1^{2,4}) \times (L^2 \cap L_1^{2,2})$ unless $\lambda(L_\|^*) \le 2$ in which case we stop and have obtained $g^{(2)} \in (L^2 \cap L_1^{2,\lambda+2}) \times (L^2 \cap L_1^{2,\lambda})$ for all $\lambda < \lambda(L_\|^*)$ (because of (a)). In the case $\lambda(L_\|^*) > 2$, we see that $g^{(3)} \in (L^2 \cap L_1^{2,6}) \times (L^2 \cap L_1^{2,4})$ unless $\lambda(L_\|^*) \le 4$ in which case we stop and have obtained $g^{(3)} \in (L^2 \cap L_1^{2,\lambda+2}) \times (L^2 \cap L_1^{2,\lambda})$ for all $\lambda < \lambda(L_\|^*)$. Since $\lambda(L_\|^*) \le n$, we must stop in a finite number of steps.

13.4. Openness

We want to prove the analog statement to proposition 7.1, in the range found in corollary 13.3, namely

Proposition 13.8. — *Fix $p \in (p_{\|}, p_+(DB))$. Then for any B' with $\|B - B'\|_\infty$ small enough (depending on p), $\mathbb{H}^p_{DB'} = \mathbb{H}^p_D$ with equivalence of norms. Furthermore, for any $b \in H^\infty(S_\mu)$ with $\omega_B < \mu < \pi/2$, we have*

$$\|b(DB) - b(DB')\|_{\mathcal{L}(H^p_D)} \lesssim \|b\|_\infty \|B - B'\|_\infty. \tag{103}$$

The proof is the same as for proposition 7.1. Indeed, from [**5**], we know that the De Giorgi condition is an open condition of the coefficients of $L_{\|}^*$. Thus corollary 13.3 applies to any perturbation of the corresponding DB. Then $H^\infty(S_\mu)$-functions of DB' are bounded on H^p_D uniformly for $\|B-B'\|_\infty$ small enough. Thus, the estimate (103) holds directly for $1 < p$ by the theory of analytic functions valued in Banach spaces. For $p \le 1$, it suffices to prove the atom to molecule estimate as in lemma 7.2. This is the only point requiring a specific argument.

For some $\varepsilon > 0$ depending only on p and n, then for all $(\mathbb{H}^p_D, 1)$-atoms α, with associated cube Q and all $j \ge 0$,

$$\|b(DB)\alpha\|_{L^2(S_j(Q))} \lesssim \|b\|_\infty \left(2^j \ell(Q)\right)^{\frac{n}{2} - \frac{n}{p}} 2^{-j\varepsilon}$$

and moreover $\int b(DB)\alpha = 0$.

To show this we argue as follows. For each integer M, there are constants $c_{M\pm}$ such that

$$\psi(z) = c_{M\pm}(iz)^M(1+iz)^{-2M}(iz)(1+iz)^{-M}$$

if $z \in S_{\mu\pm}$ satisfies $\int_0^\infty \psi(tz)\,\frac{dt}{t} = 1$ for all $z \in S_\mu$. Thus we can resolve $b(DB)$ as $\int_0^\infty (b\psi_t)(DB)\,\frac{dt}{t}$. As before, it is no loss of generality to assume that the ball associated to α is the unit ball. For M large enough, for all $t > 0$ and arbitrary integer N and $j \ge 2$,

$$\|(itDB)(I + itDB)^{-M}\alpha\|^2_{L^2(S_j(Q))} \lesssim \langle 2^j/t\rangle^{-N} t^{n - \frac{2n}{p} - \varepsilon}.$$

This is also valid for $S_j(Q)$ replaced by $4Q$. This is the estimate (102). Next, the L^2 off-diagonal estimates (19) apply to $b(DB)(itDB)^M(1 + itDB)^{-2M}$ to give

$$\|1_E\, b(DB)(itDB)^M(I + itDB)^{-2M} 1_F u\|_2 \lesssim \|b\|_\infty \langle \mathrm{dist}(E,F)/t\rangle^{-M} \|u\|_2$$

for all $t > 0$, Borel sets $E, F \subset \mathbb{R}^n$ and $u \in L^2$ with support in F. It is an easy computation to obtain

$$\|(b\psi_t)(DB)\alpha\|_{L^2(S_j(Q))} \lesssim \langle 2^j/t\rangle^{-M} t^{\frac{n}{2} - \frac{n}{p} - \frac{\varepsilon'}{2}}$$

for large enough M and $0 < \varepsilon' < \varepsilon$. With this in hand, one can estimate the t-integral upon taking M large enough and get the desired bound for $\int_{S_j(Q)} |b(DB)\alpha|^2$ when $j \ge 2$. The integral of $\int_{4Q} |b(DB)\alpha|^2$ is controlled as usual using the H^∞-calculus. We skip further details.

14. APPLICATION TO PERTURBATION OF SOLVABILITY FOR THE BOUNDARY VALUE PROBLEMS

Here, we continue some developments started in [**23**]. Some words are necessary. At the time [**23**] was written, theorems 1.1 and 1.2 of this memoir were known from the present authors. Part of theorem 1.1 was reproved in [**23**] under some De Giorgi conditions allowing a more direct argument bypassing Hardy space estimates (parts of this proof was due to other authors as mentioned in the introduction) and theorem 1.2 was quoted in [**23**] as well as the development on boundary layers from [**61**]. While writing the present article, we have improved the development on boundary layers as presented in section 12.3.

In [**23**] the goal was to prove extrapolation of solvability results for boundary value problems using a method "à la Calderón-Zygmund". For example, it was shown that the solvability of the regularity (resp. Neumann) problem in L^p, $1 < p \leq 2$, [1] with energy solutions can be pushed down to obtain solvability in L^q with $1 < q < p$ and also H^q with $q_0 < q \leq 1$ where q_0 is derived from the De Giorgi-Nash conditions used there, which involved interior and boundary regularity for the system (1) and its adjoint. Also extrapolation for the Dirichlet problems and Neumann problems in negative Sobolev spaces (going up the scale of exponents this time) was deduced thanks to Regularity/Dirichlet and Neumann/Neumann duality principles (see [**23**] for explanations).

It is not clear at this time what could be the similar results as in [**23**] in our general framework. First, we do not use here interior regularity. Secondly, those results require some kind of boundary regularity.

Instead, we can prove an extrapolation result "à la Šneĭberg", namely theorem 1.3, which does not require any boundary regularity. Also we establish a stability result in the coefficients.

14.1. Proof of theorem 1.3

We begin with the regularity problem.

For $\frac{n}{n+1} < q < \infty$ and $X = H^q$, one can formulate two notions of solvability as follows. First, $(R)^L_X$ is solvable for the energy class if there exists $C_X < \infty$ such that for any $f \in H^q_\parallel \cap \dot{\mathcal{H}}^{-1/2}_\parallel$ the energy solution u of $\operatorname{div} A\nabla u = 0$ on $\mathbb{R}^{1+n}_+$ with regularity

1. The limitation $p \leq 2$ is inherent to the method used there but can be lifted to $p < p_+(DB)$ once we have the needed boundedness .

data $\nabla_x u|_{t=0} = f$ satisfies

$$\|\widetilde{N}_*(\nabla_A u)\|_q \leq C_X \|f\|_{H^q_\parallel}.$$

We say that $(R)^L_X$ is solvable if there exists a constant $C_X < \infty$ such that for any $f \in H^q_\parallel$ there exists a weak solution u of

$$\operatorname{div} A\nabla u = 0$$

in $\mathbb{R}^{1+n}_+$ with regularity data $\nabla_x u|_{t=0} = f$ (in the prescribed sense below) and

$$\|\widetilde{N}_*(\nabla_A u)\|_q \leq C_X \|f\|_{H^q_\parallel}.$$

This means that solvability is existence of a solution with prescribed boundary trace and interior estimate.

Although one can formulate these problems for all q, they take meaningful sense in the restricted range I_L. We recall that I_L is the interval in $(\frac{n}{n+1}, p_+(L))$ on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms. For q in this range, the map

$$N_\parallel : H^q_{DB} \longrightarrow H^q_\parallel, \quad h \longmapsto h_\parallel$$

is well-defined and bounded.

To prove theorem 1.3, the first lemma tells us that we can build solutions from our semigroup approach. This is a feature of this method.

Lemma 14.1. — *Assume $q \in I_L$. Let $S^+_q(t)$ be the extension of the semigroup $e^{-t|DB|}$, $t \geq 0$, to $H^{q,+}_{DB}$. Let $h \in H^{q,+}_{DB}$. Then, the function*

$$(t, x) \longmapsto S^+_q(t)h(x)$$

is the conormal gradient of a weak solution u (uniquely determined up to a constant) of

$$\operatorname{div} A\nabla u = 0$$

on $\mathbb{R}^{1+n}_+$ with

$$\|\widetilde{N}_*(\nabla_A u)\|_q \leq C_q \|h\|_{H^q}.$$

Moreover, this solution is such that $(\nabla_A u)(t, .)$ converges to h in strong H^q topology as $t \to 0$.

Proof. — For q in this range, we know that $H^{q,+}_{DB}$ is a closed subspace of $H^q_{DB} = H^q_D$ with H^q topology. When h belongs to the dense class $\mathbb{H}^{q,+}_{DB}$, we know that

$$F = e^{-t|DB|}h$$

satisfies the non-tangential maximal estimates. Passing to completion for $h \in H^{q,+}_{DB}$, we have

$$\|\widetilde{N}_*(S^+_q(t)h)\|_q \leq C_q \|h\|_{H^q},$$

and in particular, $S^+_q(t)h(x) \in L^2_{\text{loc}}$. Also for $h \in \mathbb{H}^{q,+}_{DB}$, we knew that F was an L^2_{loc} and a solution to

$$\partial_t F + DBF = 0$$

in the weak sense, so it is preserved by taking limit in L^2_{loc}. Thus there exists a weak solution u (uniquely determined up to a constant) of

$$\operatorname{div} A\nabla u = 0$$

on $\mathbb{R}^{1+n}_+$ such that $\nabla_A u(t,x) = S_q^+(t)h(x)$ in L^2_{loc} sense. Finally, we have seen the strong convergence of $S_q^+(t)$ on $H^{q,+}_{DB}$ (this is easy from the one of the extended semigroup $S_q(t)$ on H^q_{DB}). So the strong limit as $t \to 0$ is granted. □

Lemma 14.2. — *Let $q \in I_L$ and $X = H^q$. If $(R)^L_X$ is solvable for the energy class then $N_\parallel : H^{q,+}_{DB} \to H^q_\parallel$ is an isomorphism. If $N_\parallel$ is surjective onto $H^q_\parallel$ then $(R)^L_X$ is solvable with strong limit as $t \to 0$ for $\nabla_\parallel u(t,.)$ in H^q topology.*

Admitting this lemma, we can finish the proof of theorem 1.3 by applying the result of Šneĭberg [**84**] in the Banach case and Kalton-Mitrea [**69**] in the quasi-Banach case. Indeed, the spaces $H^{q,+}_{DB}$ are complex interpolation spaces: we know this for H^q_{DB} and the spectral spaces $H^{q,+}_{DB}$ are the images of H^q_{DB} under the bounded extension of the projection $\chi^+(DB)$. Thus $N_\parallel : H^{p,+}_{DB} \to H^p_\parallel$ is invertible for p in a neighborhood of q. This implies that $(R)^L_{H^p}$ is solvable for p in this neighborhood, applying the second part of the previous lemma.

Proof of lemma 14.2. — Let us prove the second statement first. By the open mapping theorem (see [**69**] for the quasi-Banach version of it), there exists a constant $C > 0$ such that for all $f \in H^q_\parallel$, one can find $h \in H^{q,+}_{DB}$, with

$$N_\parallel h = f \quad \text{and} \quad \|h\|_{H^q} \lesssim C\|f\|_{H^q}.$$

Applying lemma 14.1 with h yields a solution.

We now prove the first part. On the energy class, we know there is a Dirichlet to Neumann map $\Gamma_{DN} : \dot{\mathcal{H}}^{-1/2}_\parallel \to \dot{\mathcal{H}}^{-1/2}_\perp$ that is bounded and invertible by existence and uniqueness of energy solutions with prescribed Dirichlet or Neumann data. See [**20**] for a proof in this context. Also, we have

$$N_\parallel \circ (\Gamma_{DN}, I) = I_{\dot{\mathcal{H}}^{-1/2}_\parallel} \quad \text{and} \quad (\Gamma_{DN}, I) \circ N_\parallel = I_{\dot{\mathcal{H}}^{-1/2,+}_{DB}}.$$

Here we use the same name for the map $N_\parallel : \dot{\mathcal{H}}^{-1/2,+}_{DB} \to \dot{\mathcal{H}}^{-1/2}_\parallel$. We know $(R)^L_X$ is solvable for the energy class if and only if there exists $C > 0$ such that

$$\|\Gamma_{DN} f\|_{H^q} \lesssim \|f\|_{H^q}$$

for all $f \in H^q_\parallel \cap \dot{\mathcal{H}}^{-1/2}_\parallel$ by [**23**], lemma 10.4. As $H^q_\parallel \cap \dot{\mathcal{H}}^{-1/2}_\parallel$ is dense in $H^q_\parallel$, this means that Γ_{DN} extends to a bounded operator from $H^q_\parallel$ into $H^q_\perp$. As $H^{q,+}_{DB} \cap \dot{\mathcal{H}}^{-1/2,+}_{DB}$ is also dense in $H^{q,+}_{DB}$ (see the argument below for convenience), this means that the operator (Γ_{DN}, I) extends to a bounded operator from $H^q_\parallel$ into $H^{q,+}_{DB}$. Extending the above operator identities shows that this extension is the inverse of $N_\parallel : H^{q,+}_{DB} \to H^q_\parallel$.

To conclude, we show that $H^{q,+}_{DB} \cap \dot{\mathcal{H}}^{-1/2,+}_{DB}$ is dense in $H^{q,+}_{DB}$ in the H^q_{DB} topology as this topology is equivalent to the H^q topology. As

$$H^{q,+}_{DB} \cap \dot{\mathcal{H}}^{-1/2,+}_{DB} = \chi^+(DB)(H^q_{DB} \cap \dot{\mathcal{H}}^{-1/2}_{DB}),$$

it suffices to show that $\mathbb{H}^q_{DB} \cap \dot{\mathcal{H}}^{-1/2}_{DB}$ is dense in $\mathbb{H}^q_{DB}$ (which is dense in H^q_{DB}). Let $h \in \mathbb{H}^q_{DB}$. Pick a Calderón reproducing formula

$$h = \int_0^\infty \psi(tDB)h \,\frac{dt}{t}$$

which converges in $\mathbb{H}^q_{DB}$ by construction of these spaces for an appropriate ψ. Observe that for fixed $t > 0$, $\psi(tDB)h \in \dot{\mathcal{H}}^0_{DB}$ and if $\psi(z) = z\widetilde{\psi}(z)$, we have $\psi(tDB)h \in \dot{\mathcal{H}}^{-1}_{DB}$. Thus, $\psi(tDB)h \in \dot{\mathcal{H}}^{-1/2}_{DB}$. This concludes the argument for the density. $\square$

Let us turn to the Neumann problem. We say that $(N)^L_X$ is solvable for the energy class if there exists $C_X < \infty$ such that for any $f \in H^q \cap \dot{\mathcal{H}}^{-1/2}$ the energy solution u of $\operatorname{div} A\nabla u = 0$ on $\mathbb{R}^{1+n}_+$ with regularity data $\partial_{\nu_A} u|_{t=0} = f$ satisfies

$$\|\widetilde{N}_*(\nabla_A u)\|_q \le C_X \|f\|_{H^q}.$$

We say that $(N)^L_X$ is solvable if there exists a constant $C_X < \infty$ such that for any $f \in H^q$ there exists a weak solution u of $\operatorname{div} A\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ with regularity data $\partial_{\nu_A} u|_{t=0} = f$ (in the prescribed sense below) and

$$\|\widetilde{N}_*(\nabla_A u)\|_q \le C_X \|f\|_{H^q}.$$

This means that solvability is existence of a solution with prescribed boundary trace and interior estimate.

The proof of theorem 1.3 for the Neumann problem on $X = H^q$ is the same with same range for q, changing $N_\parallel$ to $N_\perp$ where $N_\perp h = h_\perp$, and using the following lemma, the proof of which is entirely analogous to the previous one with the Neumann to Dirichlet map $\Gamma_{ND} : \dot{\mathcal{H}}^{-1/2}_\perp \to \dot{\mathcal{H}}^{-1/2}_\parallel$ replacing the Dirichlet to Neumann map Γ_{DN} (one being the inverse of the other).

Lemma 14.3. — *If $(N)^L_X$ is solvable for the energy class then $N_\perp : H^{q,+}_{DB} \to H^q_\perp = H^q$ is an isomorphism. If this map is surjective then $(N)^L_X$ is solvable with strong limit at $t = 0$ for $\partial_{\nu_A} u(t,.)$ in H^q topology.*

Let us turn to the Dirichlet problem (formulated with square functions as in the introduction). We argue in the dual range of the interval in $(\frac{n}{n+1}, p_+(L))$ on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms. By the results in section 11.2, it is convenient to introduce new spaces. For $1 < p < \infty$, we let $\dot{W}^{-1,p}_D$ be the image of $\dot{W}^{-1,p}$ under (the bounded extension of) $\mathbb{P}$. Thanks to lemma 11.6, it can be identified to the image of $\overline{R_p(D)} = H^p_D$ under D, which becomes an isomorphism. We now assume $p = q'$ with q as above. Thanks to proposition 11.7, H^∞ functions of $D\widetilde{B}$ act boundedly on $\dot{W}^{-1,p}_D$. Also, by theorem 4.20 and corollary 4.21, we can see that D extends to an isomorphism from $H^p_{\widetilde{B}D}$ onto $\dot{W}^{-1,p}_D$ and the relation $Db(\widetilde{B}D) = b(D\widetilde{B})D$ valid on an appropriate dense subspace of $H^p_{\widetilde{B}D}$ for $b \in H^\infty(S_\mu)$ extends by density to $H^p_{\widetilde{B}D}$. Thus we can define

$$\dot{W}^{-1,p,\pm}_{D\widetilde{B}} = DH^{p,\pm}_{\widetilde{B}D} = D\chi^+(\widetilde{B}D)H^p_{\widetilde{B}D}$$

and a strongly continuous semigroup on $\dot{W}^{-1,p,\pm}_{D\widetilde{B}}$, which extends $(e^{-t|D\widetilde{B}|})_{t\geq 0}$. All this is consistent as long as $p = q'$ because we work in the ambient space of Schwartz distributions.

If $q \leq 1$ and $\sigma = n(\frac{n}{q} - 1)$, we can define $\dot{\Lambda}^{\alpha-1}_D$ and $\dot{\Lambda}^{\alpha-1,\pm}_{D\widetilde{B}}$ as images of $\Lambda^\alpha_{\widetilde{B}D}$ and $\dot{\Lambda}^{\alpha,\pm}_{\widetilde{B}D}$ under the extension of D, which is an isomorphism (this uses again theorem 4.20 and corollary 4.21). Again, by similarity, the boundedness and regularity of semigroups carry to this setting. So the semigroup extending $e^{-t|D\widetilde{B}|}$ by this construction is weakly-$*$ continuous. The natural predual in the duality defined in section 12.2 is $\dot{H}^{1,q}_{BD}$ which is defined via completion of the space $\mathbb{H}^2_{BD}$ for the norm

$$\|t^{-1}\psi(tBD)h\|_{T^q_2}$$

for appropriate ψ. This is routinely done as for the Hardy spaces we have developed with much details and we skip those here. But, as $q \in I_L$, this space identifies to $\dot{H}^{1,q}_D$ under the projection $\mathbb{P}$. So the weak-$*$ continuity is against any distribution in $\dot{H}^{1,q}_D$ or even in $\dot{H}^{1,q}$ (because the D null distributions in $\dot{H}^{1,q}$ are annihilated by $\dot{\Lambda}^{\alpha-1}_D$ elements).

We mention, that in the range of p and α specified above ($p = q'$ or $\alpha = n(\frac{1}{q} - 1)$, the scalar parts of $H^p_{\widetilde{B}D}$ elements are in fact L^p functions. Similarly the scalar parts of $\dot{\Lambda}^\alpha_{BD}$ elements are $\dot{\Lambda}^\alpha$ functions.

For $Y = L^p$ or $\dot{\Lambda}^\alpha$ with $1 < p < \infty$ or $0 \geq \alpha < 1$, and $\mathcal{T} = T^p_2$ or $T^\infty_{2,\alpha}$, one can formulate two notions of solvability for the Dirichlet problem as follows. First, $(D)^{L^*}_Y$ is solvable for the energy class if there exists $C_Y < \infty$ such that for any $f \in Y \cap \dot{\mathcal{H}}^{1/2}_\perp$ the energy solution u of $\operatorname{div} A^*\nabla u = 0$ on $\mathbb{R}^{1+n}_+$ with Dirichet data $u|_{t=0} = f$ satisfies

$$\|t\,\nabla_{A^*}u\|_{\mathcal{T}} \leq C_Y\|f\|_Y \sim C_Y\|\nabla f\|_{\dot{Y}^{-1}}.$$

We say that $(D)^{L^*}_Y$ is solvable if for any $f \in Y$ there exists a solution u of $\operatorname{div} A^*\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ with regularity data $u|_{t=0} = f$ (in the prescribed sense below) and

$$\|t\,\nabla_{A^*}u\|_{\mathcal{T}} \leq C_Y\|f\|_Y \sim C_Y\|\nabla f\|_{\dot{Y}^{-1}}.$$

This means that solvability is existence of a solution with prescribed boundary trace and interior estimate.

Although one can formulate these problems for all p or α, they take meaningful sense in the restricted dual range of I_L. We recall that I_L is the interval in $(\frac{n}{n+1}, p_+(L))$ on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms. For q in this range, the map

$$N_\parallel : \dot{Y}^{-1,+}_{D\widetilde{B}} \longrightarrow \dot{Y}^{-1}_\parallel, \quad h \longmapsto h_\parallel$$

is well-defined and bounded. It is convenient to set $\dot{Y}_{\parallel}^{-1}$, the space of distributions of the form ∇f in $\dot{Y}^{-1}$.

To prove theorem 1.3 for the Dirichlet problem, the first lemma tells us that we can build solutions from our semigroup approach.

Lemma 14.4. — *Assume $q \in I_L$. Let $\widetilde{S}^+_{Y^{-1}}(t)$ be the extension of the semigroup $e^{-t|D\widetilde{B}|}$, $t \geq 0$, to $\dot{Y}^{-1,+}_{D\widetilde{B}}$ described above. Let $h \in \dot{Y}^{-1,+}_{D\widetilde{B}}$. Then, the function*

$$(t,x) \longmapsto \widetilde{S}^+_{Y^{-1}}(t)h(x)$$

is the conormal gradient of a weak solution u (uniquely determined up to a constant) of

$$\operatorname{div} A^*\nabla u = 0$$

on $\mathbb{R}^{1+n}_+$ with

$$\|t\,\nabla_{A^*}u\|_{\mathcal{T}} \leq C_Y\|h\|_{\dot{Y}^{-1}}.$$

Moreover this solution, is such that $(\nabla_{A^*}u)(t,.)$ converges to h at $t \to 0$ in the strong topology of $\dot{Y}^{-1}$ if $q > 1$ and in the weak-$*$ topology of $\dot{Y}^{-1}$ if $q \leq 1$. Moreover, in the case $q > 1$ and $p = q'$,

$$t \longmapsto u(t,.) \in C_0([0,\infty); L^p(\mathbb{R}^n;\mathbb{C}^m)) + \mathbb{C}^m.$$

If one normalizes the constant to be 0 (by either imposing $u|_{t=0} \in L^p(\mathbb{R}^n;\mathbb{C}^m)$ or by imposing that the solution converges to 0 at ∞ is some weak sense), then this solution satisfies the layer potential representation as in corollary 12.8 taking the bounded extensions of the layer potentials for L^*, $\mathcal{S}_t^{A^*}$ from $\dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m)$ to $L^p(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{D}_t^{A^*}$ on $L^p(\mathbb{R}^n;\mathbb{C}^m)$ proved in theorem 12.6, 3)and 4). Finally, one has the non-tangential maximal estimate $\|\widetilde{N}_*u\|_p \lesssim \|t\,\nabla u\|_{T^p_2}$ (again the constant is imposed to be 0).

Proof. — The first part of the proof is again is consequence of the construction and the estimates, once we see that $(t,x) \mapsto \widetilde{S}^+_{Y^{-1}}(t)h(x)$ is an L^2_{loc} function on $\mathbb{R}^{1+n}_+$. We see this and skip other details. By construction it is a tempered distribution on $\mathbb{R}^{1+n}_+$. If $q > 1$, then the semigroup extends by density from $\mathbb{H}^{2,+}_{D\widetilde{B}} \cap \dot{W}^{-1,p,+}_{D\widetilde{B}}$ and on such a dense space we have seen that $(t,x) \mapsto t\widetilde{S}^+_{Y^{-1}}(t)h(x)$ belongs to T^p_2. The density argument yields convergence in T^p_2, thus in L^2_{loc}. For $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$, $(t,x) \mapsto t\widetilde{S}^+_{Y^{-1}}(t)h(x)$ is build as a weak-$*$ limit in $T^\infty_{2,\alpha}$, hence it also has the L^2_{loc} property.

Let us turn to the second part of the proof. By assumption, $h_{\parallel} = \nabla f$ for some $f \in L^p(\mathbb{R}^n;\mathbb{C}^m)$. Also $h_{\perp} \in \dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m)$. Then we have

$$\nabla_{A^*}u(t,.) = \nabla_{A^*}\mathcal{S}_t^{A^*}h_{\perp} - \nabla_{A^*}\mathcal{D}_t^{A^*}f,$$

where $\mathcal{S}_t^{A^*}$ and $\mathcal{D}_t^{A^*}$ are understood as the appropriate extensions. To see this, we proceed exactly as in the proof of corollary 8.4 in [**23**], starting from the fact that $\nabla_{A^*}u(t,.)$ is defined by the semigroup representation using the abstract definitions of the layer potentials and density arguments. Once this is established, the rest of the proof is similar to that of corollary 12.8 for the convergence issues. We skip details. The non-tangential maximal estimate follows from a similar approximation argument as for the proof of (80). □

Then the result concerning the solvability of Dirichlet problems is the following one.

Lemma 14.5. — *Let $q \in I_L$ and Y be as above. If $(D)_Y^{L^*}$ is solvable for the energy class then*

$$N_\| : \dot{Y}^{-1,+}_{D\widetilde{B}} \longrightarrow \dot{Y}^{-1}_\|$$

is an isomorphism. If $N_\|$ is surjective onto $\dot{Y}^{-1}_\|$ then $(D)_Y^{L^*}$ is solvable with limit as $t \to 0$ for $u(t,.)$ in L^p topology if $q > 1$ and $p = q'$ or with limit as $t \to 0$ for $u(t,.)$ in $\dot{\Lambda}^\alpha$ weak-$*$ topology if $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$.

Proof. — The first part of the proof proceeds as the one of lemma 14.2 with the Dirichlet to Neumann map $\Gamma_{DN} : \dot{\mathcal{H}}^{-1/2}_\| \to \dot{\mathcal{H}}^{-1/2}_\perp$. We have that $(D)_Y^{L^*}$ is solvable for the energy class if and only if there exists $C > 0$ such that $\|\Gamma_{DN} g\|_{\dot{Y}^{-1}} \lesssim \|g\|_{\dot{Y}^{-1}_\|}$ for all $g \in \dot{Y}^{-1}_\| \cap \dot{\mathcal{H}}^{-1/2}_\|$. This is a reformulation of [**23**], corollary 11.3. Then similar density arguments show that the extension of the map (Γ_{DN}, I) is the desired inverse of $N_\|$. The second part is an application of the open mapping theorem again. □

The proof of theorem 1.3 is now done as the one for the regularity problem.

We finish with the Neumann problem on negative Sobolev/Hölder spaces. Again $\dot{Y}^{-1} = \dot{W}^{-1,p}$ or $\dot{\Lambda}^{\alpha-1}$. First, $(N)^{L^*}_{Y^{-1}}$ is solvable for the energy class if there exists $C_Y < \infty$ such that for any $f \in \dot{Y}^{-1} \cap \dot{\mathcal{H}}^{-1/2}$ the energy solution u of

$$\operatorname{div} A^*\nabla u = 0$$

on $\mathbb{R}^{1+n}_+$ with Neumann data $\partial_{\nu_{A^*}} u|_{t=0} = f$ satisfies

$$\|t\, \nabla_{A^*} u\|_{\mathcal{T}} \leq C_Y \|f\|_{\dot{Y}^{-1}}.$$

We say that $(N)^{L^*}_{Y^{-1}}$ is solvable if for any $f \in \dot{Y}^{-1}$ there exists a weak solution u of $\operatorname{div} A^*\nabla u = 0$ in $\mathbb{R}^{1+n}_+$ with Neumann data $\partial_{\nu_{A^*}} u|_{t=0} = f$ (in the prescribed sense below) and

$$\|t\, \nabla_{A^*} u\|_{\mathcal{T}} \leq C_Y \|f\|_{\dot{Y}^{-1}}.$$

This means that solvability is existence of a solution with prescribed boundary trace and interior estimate.

The proof of theorem 1.3 for the Neumann problem on a negative Sobolev/Hölder space is the same as for the Dirichlet problem with same range for q: we know how to construct solutions by lemma 14.4. Next, changing $N_\parallel$ to

$$N_\perp : \dot{Y}^{-1,+}_{D\widetilde{B}} \longrightarrow \dot{Y}^{-1}_\perp = \dot{Y}^{-1}, \quad N_\perp h = h_\perp,$$

we use the following lemma, the proof of which is entirely analogous to the previous one with the Neumann to Dirichlet $\Gamma_{ND} : \dot{\mathcal{H}}^{-1/2}_\perp \to \dot{\mathcal{H}}^{-1/2}_\parallel$ map replacing the Dirichlet to Neumann map Γ_{DN} (one being the inverse of the other).

Lemma 14.6. — *Let $q \in I_L$ and Y be as above. If $(N)^{L^*}_{Y^{-1}}$ is solvable for the energy class then*

$$N_\perp : \dot{Y}^{-1,+}_{D\widetilde{B}} \longrightarrow \dot{Y}^{-1}$$

is an isomorphism. If $N_\perp$ is surjective onto $\dot{Y}^{-1}$ then $(N)^{L^*}_{Y^{-1}}$ is solvable with limit as $t \to 0$ for $\partial_{\nu_{A^*}} u(t,.)$ in strong topology of $\dot{W}^{-1,p}$ if $q > 1$ and $p = q'$ or with limit as $t \to 0$ for $\partial_{\nu_{A^*}} u(t,.)$ in weak-$*$ topology on $\dot{\Lambda}^{\alpha-1}$ if $q \leq 1$ and $\alpha = n(\frac{1}{q} - 1)$.

Remark 14.7. — Concerning the Dirichlet problem under De Giorgi type condition on $L_\parallel$, this theorem covers the case of BMO data. In this case, this shows that if the Dirichlet problem for L is solvable for the energy class with BMO data, then it is solvable (may be not for the energy class) with L^p data for unspecified large p's. This result for real, non-necessarily t-independent equations, is in [**44**] and we extend it here to more general systems when the coefficients are t-independent. In case of real equations, solvability for the energy class is reached due to the harmonic measure techniques used.

14.2. Stability in the coefficients

We now establish stability under perturbation of the coefficients in the t-independent coefficients class. We do this for the regularity problem. For each of the other 3 boundary value problems, there will be similar statement and proof which we shall not include and leave to the reader. This can be compared to prior results established in the literature for systems in the upper half-space with t-independent coefficients (see [**42, 37, 72, 69, 31, 55**], etc) or bi-lipschitz diffeomorphic images of this situation. The only point is that we do not know how to obtain solvability in the energy class in the conclusion but only prove solvability.

Theorem 14.8. — *Let I_L be the interval $(p_-(DB))_*, p_+(DB))$ of theorem 5.1 or $(p_\parallel, p_+(DB))$ of corollary 13.3 on which $\mathbb{H}^q_{DB} = \mathbb{H}^q_D$ with equivalence of norms and set $X = H^q$. If $(R)^L_X$ is solvable for the energy class then $(R)^{L'}_X$ is solvable where $L' = -\operatorname{div} A'\nabla$ has t-independent coefficients with $\|A - A'\|_\infty$ small enough depending on X.*

Proof. — The assumption allows us to apply proposition 7.1 or proposition 13.8. In both situations, if $q \in I_L$ we have,

$$\|\chi^+(DB') - \chi^+(DB)\|_{\mathcal{L}(H^q_D)} \leq C\|A - A'\|_\infty$$

for small enough $\|A - A'\|_\infty$, where $B = \widehat{A}, B' = \widehat{A}'$. As $\chi^+(DB')$ is a projector, it implies that it is an isomorphism from $H^{q,+}_{DB}$ onto $H^{q,+}_{DB'}$ with uniform bounds for small enough $\|A - A'\|_\infty$. Next, $N_\parallel \chi^+(DB') : H^{q,+}_{DB} \to H^q_\parallel$ is a perturbation of $N_\parallel = N_\parallel \chi^+(DB) : H^{q,+}_{DB} \to H^q_\parallel$ in operator norm. As solvability of $(R)^L_X$ for the energy class implies $N_\parallel : H^{q,+}_{DB} \to H^q_\parallel$ is invertible by lemma 14.2, it follows that $N_\parallel \chi^+(DB') : H^{q,+}_{DB} \to H^q_\parallel$ is invertible for small enough $\|A - A'\|_\infty$. Combining these two informations, we obtain that $N_\parallel : H^{q,+}_{DB'} \to H^q_\parallel$ is invertible uniformly for $\|A - A'\|_\infty$ small enough. This implies solvability of $(R)^{L'}_X$ by lemma 14.2. □

Remark 14.9. — Although it seems natural to expect it, we are not able to remove the assumption on I_L at this time.

PART II

L^p-L^q THEORY FOR HOLOMORPHIC FUNCTIONS OF PERTURBED FIRST ORDER DIRAC OPERATORS

Sebastian Stahlhut

INTRODUCTION

In this article, we are interested in L^p-L^q estimates for operators defined by the functional calculus of certain first order differential operators of Dirac type. Let us start with an example.

In one dimension, the operator

$$\Big(\operatorname{Id} - it\frac{d}{dx}\Big)^{-1},$$

where $-i\frac{d}{dx}$ is defined as a self-adjoint operator in $L^2(\mathbb{R})$ and $t \in \mathbb{R}^*$, is known to have a kernel $\frac{1}{2|t|}e^{-\frac{|x-y|}{|t|}}$, hence it is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for any $1 \leq p \leq q \leq \infty$.

In higher dimensions the operators

$$(\operatorname{Id} + itD)^{-1},$$

where $D = \begin{pmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{pmatrix}$, are examples of bounded operators on $L^p(\mathbb{R}^n, \mathbb{C}^N)$ but not from $L^p(\mathbb{R}^n, \mathbb{C}^N)$ to $L^q(\mathbb{R}^n, \mathbb{C}^N)$ $p, q \in (1, \infty)$ with $p < q$. However, the operator

$$tD(\operatorname{Id} + itD)^{-2}$$

satisfies for all Borel sets $E, F \subset \mathbb{R}^n$, all $u \in L^p(\mathbb{R}^n, \mathbb{C}^N)$ and for certain values $K \in [0, \infty)$ and $q > p$,

$$\|tD(\operatorname{Id} + itD)^{-2}\chi_E u\|_{L^q(F)} \lesssim |t|^{\frac{n}{q} - \frac{n}{p}}\Big(1 + \frac{d(E,F)}{|t|}\Big)^{-K}\|u\|_{L^p(E)},$$

where $d(E, F) := \inf\{|x - y|; x \in E,\, y \in F\}$ is the distance between the sets E and F and χ_E denotes the characteristic function of E.

Here, we want to explore this phenomenon for perturbed first order Dirac operators DB and BD (see below for definitions). The off-diagonal estimates are important, when one seeks to prove, for example,

$$\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty \int_{B(x,t)} \big|\, tDB(\operatorname{Id} + itDB)^{-2}u(y)\big|^2 \frac{dydt}{t^{n+1}}\Big)^{p/2} dx\Big)^{\frac{1}{p}} \lesssim \|u\|_{L^p}$$

for certain values of p, where $B(x,t) \subset \mathbb{R}^n$ denotes the ball of radius t and center x. This was shown in Part I of the present memoir.

The notion of L^2-L^2 off-diagonal estimates arises from [**43**] and [**51**]. Such off-diagonal estimates were proved and used for second order elliptic operators for the solution of the Kato square root problem in [**13**] and used to compensate the lack of pointwise kernel estimates. In [**58**] such L^2-L^2 off-diagonal estimates were used to prove certain L^p bounds Riesz transform associated to second order elliptic operators. It was in [**36**] that the L^p-L^q version of those were used to prove L^p estimates in absence of pointwise bounds. In [**36**], [**6**], [**60**], [**59**], [**57**] and [**7**], L^p-L^q off-diagonal estimates for semigroup and resolvent of elliptic second order differential operators were used to prove square root estimates, boundedness for square functions, certain maximal functions, Riesz transforms, etc. Similar work was done in the context of self-adjoint operators in [**57**], [**19**] and [**38**]. Another line of developments was a generalized Calderón-Zygmund theory for operators which do not satisfy kernel estimates, where one used off-diagonal estimates as replacement (cf. [**36**], [**7**], [**17**], [**18**], [**34**], [**49**], [**35**], etc.). As consequence of the solution of the Kato square root problem, interest arises also in square function estimates for first order Dirac operators. In [**29**] and [**10**] vertical square function estimates were proved using L^2-L^2 off-diagonal estimates for the resolvent. In [**22**] such off-diagonal estimates for the resolvent were used to develop a Hardy space theory associated to Hodge-Dirac operators on manifolds. In [**63**] and [**65**] L^p-L^p off-diagonal estimates for the resolvent of first order Dirac operators were applied to prove an extrapolation theorem for R-bisectoriality and to prove equivalence of R-bisectoriality and holomorphic functional calculus on intervals of Lebesgue exponents. Ajiev [**1**] even introduced an L^p-L^q theory for first order Dirac operators under certain restrictions, which we remove here.

Our plan is as follows.

In chapter 15, we introduce the basic notions.

In chapter 16, we discuss our main results.

In a first section, we give sufficient conditions for L^p-L^q off-diagonal estimates and L^p-L^q boundedness for operators in the functional calculus of these perturbed first order Dirac operators in terms of decay properties at 0 and ∞ for the associated holomorphic functions. In particular, we give a relation between the decay properties for the associated holomorphic functions and the number K above. These results will be given in propositions 16.3 and 16.9 below. Corollary 16.11 gives a version for bounded holomorphic functions, which have no decay at 0. These results are partially contained in the work of [**1**] when the range of the perturbed first order Dirac operator is stable under multiplication by smooth cut-off functions/cut-off functions.

In a second section, we discuss when this is the case. We give a condition in proposition 16.19 that shows that for the operators D and DB, Ajiev's results may not be always applicable, in particular not for $D = \begin{pmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{pmatrix}$ as above, whereas ours are.

In a third section, we give a necessary condition for L^p-L^q boundedness when $p < q$, which highlights the connection of L^p-L^q boundedness to kernel/range decompositions. In particular, this condition shows that the operators

$$(\mathrm{Id} + itD)^{-1}, \quad (\mathrm{Id} + itDB)^{-1} \quad \text{and} \quad (\mathrm{Id} + itBD)^{-1}$$

are not L^p-L^q bounded for the particular $D = \begin{pmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{pmatrix}$ and $p < q$. Also, this condition shows that the semi-groups

$$e^{-t|D|}, \quad e^{-t|DB|} \quad \text{and} \quad e^{-t|BD|},$$

where $|.| = \sqrt{(.)^2}$, are not L^p-L^q bounded for this particular D.

The fourth section concerns analytic extensions of our results to complex times t. Finally, we give an application of L^p-L^q boundedness to estimates for fractional operators related to D, DB and BD in the fifth section.

15. SETTING

15.1. Definitions and notation

Let $1 < q < \infty$, $n, N \in \mathbb{N}^*$. By an unbounded operator on $L^q := L^q(\mathbb{R}^n, \mathbb{C}^N)$ we mean a linear map $T : \boldsymbol{D}_q(T) \to L^q$ with domain $\boldsymbol{D}_q(T) \subset L^q$. We denote the null space by $\boldsymbol{N}_q(T)$ and the range by $\boldsymbol{R}_q(T)$. We say that T admits a kernel/range decomposition in L^q whenever

$$L^q = \boldsymbol{N}_q(T) \oplus \overline{\boldsymbol{R}_q(T)}, \tag{104}$$

where the sum is topological and $\overline{\boldsymbol{R}_q(T)}$ is the closure of $\boldsymbol{R}_q(T)$ in L^q. A class of operators which admit a kernel/range decomposition are bisectorial operators. We say that a linear operator T is bisectorial of type $\omega \in [0, \frac{\pi}{2})$ if T is closed, the spectrum of T is contained in a bisector

$$S_\omega := \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg \lambda| \leq \omega \text{ or } |\arg(-\lambda)| \leq \omega\} \cup \{0\}$$

and for each $\nu \in (\omega, \frac{\pi}{2})$ there exists a constant $C_\nu > 0$ such that

$$\|(\mathrm{Id} + \lambda T)^{-1}\|_{L^q \to L^q} \leq C_\nu \tag{105}$$

for all $\lambda \in \mathbb{C} \backslash S_\nu$. The bound (105) allows one to define a functional calculus. To $\sigma > 0$, $\tau > 0$ and $\nu \in (0, \frac{\pi}{2})$ we define $\Psi_\sigma^\tau(\dot{S}_\nu)$ to be the set of all holomorphic functions $\psi : \dot{S}_\nu \to \mathbb{C}$ such that

$$|\psi(\lambda)| \lesssim \frac{|\lambda|^\sigma}{1 + |\lambda|^{\sigma+\tau}} \tag{106}$$

for all $\lambda \in \dot{S}_\nu := \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg \lambda| < \omega \text{ or } |\arg(-\lambda)| < \omega\}$. Moreover, we define

$$\Psi(\dot{S}_\nu) := \bigcup_{\sigma, \tau > 0} \Psi_\sigma^\tau(\dot{S}_\nu)$$

and set $\psi(0) = 0$ when $\psi \in \Psi(\dot{S}_\nu)$. Having these definitions in hand we can define a functional calculus as follows. Let T be a bisectorial operator of type $\omega \in [0, \frac{\pi}{2})$ on L^q, then for each $\nu \in (\omega, \frac{\pi}{2})$ the Dunford integral

$$\psi(T) := \frac{1}{2\pi i} \int_{\partial S_\theta} \psi(\lambda)(\mathrm{Id} - \lambda^{-1} T)^{-1} \frac{d\lambda}{\lambda} \tag{107}$$

defined an improper Riemann integral converges normally in $\mathcal{L}(L^q)$ for all $\theta \in (\omega, \nu)$, where $\partial S_\theta := \{\pm t e^{\pm i\theta} : t \in (0, \infty)\}$ is oriented counterclockwise on the four branches surrounding S_ω. We say that a bisectorial operator T of type ω has a bounded holomorphic functional calculus, if for each $\nu \in (\omega, \frac{\pi}{2})$ there exists a constant $C_\nu > 0$ such that for all $\psi \in \Psi(\dot{S}_\nu)$ and all $u \in L^q$ holds

$$\|\psi(T)u\|_{L^q} \leq C_\nu \|\psi\|_{H^\infty(\dot{S}_\nu)} \|u\|_{L^q}.$$

Whenever this is the case the bounded holomorphic functional calculus may be extended to the class $H^\infty(\dot{S}_\nu)$ by a limiting procedure and to the class $H^\infty(\dot{S}_\nu, \{0\})$ using the kernel/range decomposition of T. Indeed in this case, for $f \in H^\infty(\dot{S}_\nu, \{0\})$ we define

$$f(T)u := f(0)u_N + f|_{\dot{S}_\nu}(T)u_R \tag{108}$$

where u_N, u_R denote the projections of u onto null space and closure of the range of T according to (104). Here, $H^\infty(\dot{S}_\nu)$ is the set of all bounded holomorphic functions $f : \dot{S}_\nu \to \mathbb{C}$ with norm

$$\|f\|_{H^\infty(\dot{S}_\nu)} := \sup_{z \in \dot{S}_\nu} |f(z)|$$

and $H^\infty(\dot{S}_\nu, \{0\})$ is the set of all bounded functions $f : \dot{S}_\nu \cup \{0\} \to \mathbb{C}$ with norm

$$\|f\|_{H^\infty(\dot{S}_\nu, \{0\})} := \sup_{z \in \dot{S}_\nu \cup \{0\}} |f(z)|$$

such that the restriction $f|_{\dot{S}_\nu}$ is holomorphic. For more details on kernel/range decompositions, bisectorial operators and functional calculus we refer the reader to [**41**], [**76**], [**53**], [**79**] and the references therein.

We fix p, q with $1 < p \leq q < \infty$ in the sequel. In the following we are interested in the following three boundedness properties for families of operators.

Definition 15.1 (Boundedness for families of operators). — Let $\mathcal{A} \subset \mathbb{C}\backslash\{0\}$ be a subset of the complex plane and $U_p \subset L^p$, $U_q \subset L^q$ be closed subspaces. We say that a family of operators $\{T_\lambda\}_{\lambda \in \mathcal{A}}$ is U_p-U_q *bounded* if for all $\lambda \in \mathcal{A}$ and all $u \in U_p$ holds $T_\lambda u \in U_q$ and there exists a constant $C_{p,q} > 0$ such that for all $\lambda \in \mathcal{A}$ and all $u \in U_p$ holds

$$\|T_\lambda u\|_{L^q} \leq C_{p,q} |\lambda|^{\frac{n}{q} - \frac{n}{p}} \|u\|_{L^p}.$$

Definition 15.2 (Off-diagonal estimates for families of operators)

Let $\mathcal{A} \subset \mathbb{C}\backslash\{0\}$ be a subset of the complex plane. We say that a family of operators $\{T_\lambda\}_{\lambda \in \mathcal{A}}$ *satisfies L^p-L^q off-diagonal estimates of order* $K \in [0, \infty)$ if there exists a constant $C_{K,p,q} > 0$ such that for all Borel sets $E, F \subset \mathbb{R}^n$, all $\lambda \in \mathcal{A}$ and all $u \in L^p$ holds

$$\|\chi_E T_\lambda(\chi_F u)\|_{L^q} \leq C_{K,p,q} |\lambda|^{\frac{n}{q} - \frac{n}{p}} (1 + \frac{d(E,F)}{|\lambda|})^{-K} \|\chi_F u\|_{L^p},$$

where $d(E,F) := \inf_{x \in E, y \in F} |x-y|_{\mathbb{R}^n}$ denotes the distance between E and F and χ_E denote the characteristic functions of a set E.

DEFINITION 15.3 (Biparameter off-diagonal estimates for families of operators)

Let $\mathcal{A}, \mathcal{B} \subset \mathbb{C}\backslash\{0\}$ be two subsets of the complex plane. We say that a family of operators $\{T_{\lambda_1,\lambda_2}\}_{(\lambda_1,\lambda_2)\in\mathcal{A}\times\mathcal{B}}$ satisfies L^p-L^q *biparameter off-diagonal estimates in* (λ_1, λ_2) *of order* $K \in [0, \infty)$ if there exists a constant $C_{K,p,q} > 0$ such that for all Borel sets $E, F \subset \mathbb{R}^n$, all $(\lambda_1, \lambda_2) \in \mathcal{A} \times \mathcal{B}$ and all $u \in L^p$ holds

$$\|\chi_E T_{\lambda_1,\lambda_2}(\chi_F u)\|_{L^q} \leq C_{K,p,q}|\lambda_1|^{\frac{n}{q}-\frac{n}{p}}\Big(1 + \frac{d(E,F)}{|\lambda_2|}\Big)^{-K} \|\chi_F u\|_{L^p}.$$

15.2. First order Dirac operators

We are interested in families of operators defined by the bounded holomorphic functional calculus of the following special class of bisectorial operators.

ASSUMPTION 15.4. — *Let $n, N \in \mathbb{N}^*$. Let D be a first order differential operator on $\mathbb{R}^n$ acting on functions valued in $\mathbb{C}^N$ that satisfies the conditions* (D0), (D1) *and* (D2) *in* [**63**]. *These are:*

1) *D has the representation*

$$D = -i\sum_{j=1}^{n} \widehat{D}_j \partial_j$$

with matrices $\widehat{D}_j \in \mathcal{L}(\mathbb{C}^N)$,

2) *There exists $\kappa > 0$ such that the symbol*

$$\widehat{D}(\xi) = \sum_{j=1}^{n} \widehat{D}_j \xi_j$$

satisfies $\kappa|\xi||e| \leq |\widehat{D}(\xi)e|$ for all $\xi \in \mathbb{R}^n$ and all $e \in \boldsymbol{R}(\widehat{D}(\xi))$,

3) *There exists $\omega_D \in [0, \frac{\pi}{2})$ such that the spectrum of the symbol satisfies*

$$\sigma(\widehat{D}(\xi)) \subset S_{\omega_D}.$$

Further, let B be the operator defined via pointwise multiplication by the matrix function $B(x)$, $x \in \mathbb{R}^n$, with $B \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$. We assume additionally one of the following equivalent conditions:

4) *Assume B satisfies the coercivity condition $\|Bu\|_{L^2} \gtrsim \|u\|_{L^2}$ for all $u \in \overline{\boldsymbol{R}_2(D)}$ and there exists $\omega \in [0, \frac{\pi}{2})$ such that BD is bisectorial of type ω on L^2,*

5) *Assume B^* satisfies the coercivity condition $\|B^*u^*\|_{L^2} \gtrsim \|u^*\|_{L^2}$ for all $u^* \in \overline{\boldsymbol{R}_2(D^*)}$ and there exists $\omega \in [0, \frac{\pi}{2})$ such that DB is bisectorial of type ω on L^2.*

In the sequel, we shall systematically assume without mention that Assumption 15.4 *holds in all statements involving DB or BD.*

EXAMPLE 15.5. — The operators D and B appearing in the works [**11**] and [**8**] satisfy Assumption 15.4. Further examples are in [**65**] and [**63**].

Here, we do not assume that D is self-adjoint or that B satisfies a strict accretivity condition. The equivalence of the conditions 4) and 5) was proven in [**27**]. The first consequence of Assumption 15.4 is the following proposition due to [**65**].

PROPOSITION 15.6. — *Let $1 < q < \infty$.*

1) *D is a bisectorial operator of type ω_D with bounded holomorphic functional calculus in L^q.*
2) *$L^q = \boldsymbol{N}_q(D) \oplus \overline{\boldsymbol{R}_q(D)}$, i.e. D admits a kernel/range decomposition on $L^q(\mathbb{R}^n, \mathbb{C}^N)$,*
3) *$\boldsymbol{N}_q(D)$ and $\overline{\boldsymbol{R}_q(D)}$, $1 < q < \infty$, are complex interpolation families.*
4) *D satisfies the coercivity condition*
$$\|\nabla u\|_{L^q(\mathbb{R}^n, \mathbb{C}^n \otimes \mathbb{C}^N)} \lesssim \|Du\|_{L^q(\mathbb{R}^n, \mathbb{C}^N)} \quad \text{for all } u \in \boldsymbol{D}_q(D) \cap \overline{\boldsymbol{R}_q(D)} \subset W^{1,q}.$$
Here, we use the notation ∇u for $\nabla \otimes u$ and $\|u\|_{W^{1,q}} = \|u\|_{L^q} + \|\nabla u\|_{L^q}$.
5) *The same properties hold for the adjoint D^*.*

By [**63**] and [**27**, proposition 2.1], it follows that the operators BD and DB have a meaning as unbounded operators in L^q with natural domains $\boldsymbol{D}_q(D)$ and $B^{-1}\boldsymbol{D}_q(D)$, the preimage of $\boldsymbol{D}_q(D)$ under B. Moreover, assumption 15.4 implies the existence of an open interval $\mathcal{I}_2 \subset (1, \infty)$ containing 2 such that for all $q \in \mathcal{I}_2$ holds $\|Bu\|_{L^q} \geq C\|u\|_{L^q}$ whenever $u \in \overline{\boldsymbol{R}_q(D)}$ and $\|B^*u^*\|_{L^{q'}} \geq C\|u^*\|_{L^{q'}}$ whenever $u^* \in \overline{\boldsymbol{R}_{q'}(D^*)}$. This was shown in [**27**] and [**63**] and used to extrapolate R-bisectoriality. As L^2-bisectoriality self-improves to L^2-R-bisectoriality [(1)] we get from the works [**27**, theorem 5.1], [**65**, lemma 2.4, theorem 2.5], [**65**, corollary 8.17] and [**70**, theorem 5.3] (to recall the main ingredients) the following theorem.

THEOREM 15.7. — *There exists an open interval $\mathcal{I}_{D,B} = (p_-(D,B), p_+(D,B))$ containing 2 and maximal in $\mathcal{I}_2$ such that the following properties for $T \in \{BD, DB\}$ and $q \in \mathcal{I}_{D,B}$ hold:*

1) *T admits a kernel/range decomposition on $L^q(\mathbb{R}^n, \mathbb{C}^N)$,*
2) *T is a bisectorial operator of type ω on $L^q(\mathbb{R}^n, \mathbb{C}^N)$,*
3) *T has a bounded holomorphic functional calculus on $L^q(\mathbb{R}^n, \mathbb{C}^N)$,*
4) *for each $\nu \in (\omega, \frac{\pi}{2})$ the family $\{(\mathrm{Id} + \lambda T)^{-1}\}_{\lambda \in \mathbb{C} \setminus S_\nu}$ satisfies $L^q - L^q$ off-diagonal estimates of order K for all $K \in [0, \infty)$.*

Moreover, all the properties 1), 2), 3) and 4) fail whenever $q = p_\pm(D,B) \in \mathcal{I}_2$.

For a more complete and more general version of this theorem we refer the reader to [**86**]. Finally let us make a few remarks in relation to theorem 15.7.

1. We do not introduce this notion here, as it is not of further interest.

Remark 15.8. — For $T \in \{BD, DB\}$ and $q \in \mathcal{I}_{D,B}$ let T_q be the L^q-realization. Then

1) $(BD)_q = BD$ with domain $\boldsymbol{D}_q(BD) = \boldsymbol{D}_q(D)$.
2) $(DB)_q = DB$ with domain $\boldsymbol{D}_q(DB) = B^{-1}\boldsymbol{D}_q(D)$.
3) $T_p = T_q$ on $\boldsymbol{D}_p(T) \cap \boldsymbol{D}_q(T)$.
4) $f(T_p) = f(T_q)$ on $L^p \cap L^q$ for $f \in H^\infty\big(\dot{S}_\nu, \{0\}\big)$.
5) $\mathbb{P}_{\overline{\boldsymbol{R}_p(T)}} = \mathbb{P}_{\overline{\boldsymbol{R}_q(T)}}$ and $\mathbb{P}_{\boldsymbol{N}_p(T)} = \mathbb{P}_{\boldsymbol{N}_q(T)}$ on $L^p \cap L^q$, where $\mathbb{P}_{\overline{\boldsymbol{R}_p(T)}}$ denotes the projection onto $\overline{\boldsymbol{R}_p(T)}$ along $\boldsymbol{N}_p(T)$ and $\mathbb{P}_{\boldsymbol{N}_p(T)}$ denotes the projection onto $\boldsymbol{N}_p(T)$ along $\overline{\boldsymbol{R}_p(T)}$.

Proposition 15.9. — *For $q \in \mathcal{I}_{D,B}$,*

$$\mathbb{P}_{\overline{\boldsymbol{R}_q(D)}} : \overline{\boldsymbol{R}_q(BD)} \longrightarrow \overline{\boldsymbol{R}_q(D)}$$

is an isomorphism with inverse $\mathbb{P}_{\overline{\boldsymbol{R}_q(BD)}} : \overline{\boldsymbol{R}_q(D)} \to \overline{\boldsymbol{R}_q(BD)}$.

Proof. — Let $h \in \overline{\boldsymbol{R}_q(BD)}$. Then $h - \mathbb{P}_{\overline{\boldsymbol{R}_q(D)}}h \in \boldsymbol{N}_q(BD) = \boldsymbol{N}_q(D)$ according to (104) for D and [**27**, proposition 2.1 (3)]. Thus $\mathbb{P}_{\overline{\boldsymbol{R}_q(BD)}}\big(h - \mathbb{P}_{\overline{\boldsymbol{R}_q(D)}}h\big) = 0$ and we see that $\mathbb{P}_{\overline{\boldsymbol{R}_q(BD)}} : \overline{\boldsymbol{R}_q(D)} \to \overline{\boldsymbol{R}_q(BD)}$ is the left inverse of $\mathbb{P}_{\overline{\boldsymbol{R}_q(D)}} : \overline{\boldsymbol{R}_q(BD)} \to \overline{\boldsymbol{R}_q(D)}$. Reversing the roles of D and BD shows that $\mathbb{P}_{\overline{\boldsymbol{R}_q(BD)}} : \overline{\boldsymbol{R}_q(D)} \to \overline{\boldsymbol{R}_q(BD)}$ is the right inverse of $\mathbb{P}_{\overline{\boldsymbol{R}_q(D)}} : \overline{\boldsymbol{R}_q(BD)} \to \overline{\boldsymbol{R}_q(D)}$. □

Remark 15.10 (Similarity Property). — For $q \in \mathcal{I}_{D,B}$ we know that

$$B : \overline{\boldsymbol{R}_q(D)} \longrightarrow \overline{\boldsymbol{R}_q(BD)}$$

is an isomorphism by [**27**, proposition 2.1, (2)]. In particular, for $f \in H^\infty(\dot{S}_\nu)$ we have $f(DB) = B^{-1}f(BD)B$ on $\overline{\boldsymbol{R}_q(D)}$ and $f(BD) = Bf(DB)B^{-1}$ on $\overline{\boldsymbol{R}_q(BD)}$.

Remark 15.11 (The interval for the adjoint operators). — Let $\mathcal{A}' := \{q/(q-1) | q \in \mathcal{A}\}$ for a subset $\mathcal{A} \subset (1,\infty)$. By theorem 15.7 and [**27**, corollary 2.6] we have

$$(\mathcal{I}_{D,B})' = \mathcal{I}_{D^*,B^*}.$$

Remark 15.12 (The interval for $B = \mathrm{Id}$). — Assumption 15.4 and proposition 15.6 imply $\mathcal{I}_{D,\mathrm{Id}} = (1,\infty)$ and $\mathcal{I}_{D,B} \subset \mathcal{I}_{D,\mathrm{Id}}$. Thus, whenever we are allowed to use the conclusion of theorem 15.7 for $T \in \{DB, BD\}$, we are also allowed to use the conclusion of theorem 15.7 for D.

Under assumption 15.4 we are allowed to use kernel/range decompositions and it will be helpful in the sequel to have some properties for the range and the null space. These observations were made in [**63**].

Lemma 15.13. — [**63**, Section 3.3] *For $p, q \in \mathcal{I}_{D,B}$ and $T \in \{BD, DB\}$ the following statements are true:*

1) *We have with respect to L^p-topology the direct sum decomposition*

$$L^p \cap L^q = [\boldsymbol{N}_p(T) \cap \boldsymbol{N}_q(T)] \oplus [\overline{\boldsymbol{R}_p(T)} \cap \overline{\boldsymbol{R}_q(T)}]$$

2) $\boldsymbol{N}_p(T) \cap \boldsymbol{N}_q(T)$ *is dense in* $\boldsymbol{N}_p(T)$ *with respect to L^p-topology.*

3) $\boldsymbol{R}_p(T) \cap \boldsymbol{R}_q(T)$ *is dense in* $\overline{\boldsymbol{R}_p(T)}$ *with respect to L^p-topology.*

4) $\boldsymbol{N}_p(T) \cap L^q \subset \boldsymbol{N}_q(T)$. *Hence* $\boldsymbol{N}_p(T) \cap \overline{\boldsymbol{R}_q(T)} = \{0\}$.

16. THE L^p-L^q THEORY

16.1. L^p-L^q estimates in terms of decay properties of holomorphic functions

There are connections of the bounded holomorphic functional calculus on L^q and $L^q - L^q$ off-diagonal estimates as described in the next lemma, which comes essentially from [**22**, lemma 3.6]. Compare also [**66**, lemma 7.3] and [**60**, lemma 2.28]. Before, let us define for $f \in H^\infty\big(\dot{S}_\nu, \{0\}\big)$ the function $f_t \in H^\infty\big(\dot{S}_\nu, \{0\}\big)$ by

$$f_t(\lambda) := f(t\lambda), \quad \lambda \in \dot{S}_\nu \cup \{0\}.$$

So, one can define families of bounded operators $\{f_t(T)\}_{t>0}$ via the family of functions $\{f_t\}_{t>0}$.

Proposition 16.1 (L^q-L^q off-diagonal estimates). — *Let $T \in \{DB, BD\}$. Denote by*

$$\omega = \omega_{DB} = \omega_{BD}$$

the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Let $\sigma > 0$ and $\tau > 0$ be positive real numbers and $q \in \mathcal{I}_{D,B} = (p_-(D,B), p_+(D,B))$. Suppose $\psi \in \Psi_\sigma^\tau\big(\dot{S}_\nu\big)$ and $g \in H^\infty(\dot{S}_\nu)$. Then the family of operators $\{g(T)\psi_t(T)\}_{t>0}$ satisfies L^q-L^q off-diagonal estimates of order σ.

This is interesting in view of the following example.

Example 16.2 (Off-diagonal estimates and the semigroup)

Here, let us denote

$$\mathrm{sgn} z := \mathrm{sgn}(\mathrm{Re}\, z), \quad \widetilde{z} := \mathrm{sgn}(z)z, \quad z \in \dot{S}_\nu, \quad |T| := \mathrm{sgn}(T)T.$$

1) Let $T \in \{DB, BD\}$. Denote by $\omega = \omega_{DB} = \omega_{BD}$ the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Then the decomposition

$$e^{-t|T|} = \big(e^{-t|T|} - (\mathrm{Id} + itT)^{-1}\big) + (\mathrm{Id} + itT)^{-1}$$

shows that the semigroup $\{e^{-t|T|}\}_{t>0}$ satisfies L^q-L^q off-diagonal estimates of any order $K \in [0,1]$ as the function

$$\psi(z) := e^{-|z|} - (1+iz)^{-1}, \quad z \in \dot{S}_\nu,$$

satisfies $\psi \in \Psi_1^1\big(\dot{S}_\nu\big)$.

2) The example $n = N = 1$ and $D = -i\frac{d}{dx}$ shows that we can not gain more in general. The kernel of the semigroup $\{e^{-t|-i\frac{d}{dx}|}\}$ is the Poisson kernel

$$p_t(x) := \frac{1}{\pi}\frac{t}{|x|^2+t^2}.$$

Thus the semigroup does not satisfy $L^2 - L^2$ off-diagonal estimates of order $K > 1$ in general.

For certain operators in the functional calculus of DB (or BD resp.) we obtain even L^p-L^q off-diagonal estimates and L^p-L^q boundedness. More precisely we have

Proposition 16.3 (L^p-L^q off-diagonal estimates). — *Let $T \in \{DB, BD\}$. Denote by*

$$\omega = \omega_{DB} = \omega_{BD}$$

the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Suppose $p, q \in \mathcal{I}_{D,B}$ such that $p < q$ and $\tau > \frac{n}{p} - \frac{n}{q}$. Then there exists $c := c_{p,q} > 0$ such that for all $0 \le K < \frac{\sigma}{c}$ one has:

For all $\psi \in \Psi_\sigma^\tau(\dot{S}_\nu)$ and all $g \in H^\infty(\dot{S}_\nu)$ the family $\{g(T)\psi_t(T)\}_{t>0}$ satisfies L^p-L^q off-diagonal estimates of order K. Moreover, one can choose

$$c_{p,q} = \Big(1 - \Big(\frac{1}{p} - \frac{1}{q}\Big)\Big(\frac{1}{p_-(D,B)} - \frac{1}{p_+(D,B)}\Big)^{-1}\Big)^{-1}. \tag{109}$$

Proof. — First we prove the following claim for L^p-L^q-boundedness, which is a special case of the lemma taking $E = F = \mathbb{R}^n$ and $K = 0$.

Claim 16.4. — *Suppose $p, q \in \mathcal{I}_{D,B}$ such that $p < q$. Let $\psi \in \Psi_\sigma^\tau(\dot{S}_\nu)$, where $\sigma > 0$ and $\tau > \frac{n}{p} - \frac{n}{q}$, and $g \in H^\infty(\dot{S}_\nu)$. Then the family $\{g(T)\psi_t(T)\}_{t>0}$ is L^p-L^q-bounded.*

The proof of claim 16.4 is organized in several steps. The first step is

Claim 16.5. — *Suppose $q \in \mathcal{I}_{D,B}$ and $p \in [q_*, q] \cap \mathcal{I}_{D,B}$, where the lower Sobolev exponent is defined by $q_* = \frac{qn}{q+n}$. Then for all $\lambda \in \mathbb{C}\backslash S_\nu$, the operator $(\mathrm{Id} + \lambda DB)^{-1}$ is bounded from $\overline{\boldsymbol{R}_p(D)}$ to $\overline{\boldsymbol{R}_q(D)}$ with*

$$\|(\mathrm{Id} + \lambda DB)^{-1}u\|_{L^q} \lesssim |\lambda|^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p}.$$

Proof of claim 16.5. — We first consider estimates for the resolvent of BD and use the similarity property to pass to DB later on. As

$$(\mathrm{Id} + iBD)^{-1} : \overline{\boldsymbol{R}_p(BD)} \longrightarrow \overline{\boldsymbol{R}_p(BD)},$$

$$(\mathrm{Id} + iBD)^{-1} : \overline{\boldsymbol{R}_p(BD)} \longrightarrow \boldsymbol{D}_p(BD)$$

we deduce

$$\mathbb{P}_{\overline{\boldsymbol{R}_p(D)}}(\mathrm{Id} + iBD)^{-1} : \overline{\boldsymbol{R}_p(D)} \to \overline{\boldsymbol{R}_p(D)} \cap \boldsymbol{D}_p(D) \subset W^{1,p}.$$

Thus, by Sobolev embedding theorem, we obtain $\mathbb{P}_{\overline{\boldsymbol{R}_p(D)}}(\mathrm{Id}+iBD)^{-1}u \in L^q$ for all $u \in \overline{\boldsymbol{R}_p(BD)}$ with

$$\|\mathbb{P}_{\overline{\boldsymbol{R}_p(D)}}(\mathrm{Id}+iBD)^{-1}\|_{L^q} \leq C\|u\|_{L^p}.$$

Now if, moreover, $u \in \overline{\boldsymbol{R}_q(BD)}$ then $(\mathrm{Id}+iBD)^{-1}u \in \overline{\boldsymbol{R}_q(BD)}$. By the variant of remark 15.8 for D we have $\mathbb{P}_{\overline{\boldsymbol{R}_p(D)}} = \mathbb{P}_{\overline{\boldsymbol{R}_q(D)}}$ on $L^p \cap L^q$. From that we deduce

$$\mathbb{P}_{\overline{\boldsymbol{R}_p(D)}}(\mathrm{Id}+iBD)^{-1}u = \mathbb{P}_{\overline{\boldsymbol{R}_q(D)}}(\mathrm{Id}+iBD)^{-1}u \in \overline{\boldsymbol{R}_p(D)} \cap \overline{\boldsymbol{R}_q(D)}$$

for all $u \in \overline{\boldsymbol{R}_p(BD)} \cap \overline{\boldsymbol{R}_q(BD)}$. Since $\mathbb{P}_{\overline{\boldsymbol{R}_q(D)}} : \overline{\boldsymbol{R}_q(BD)} \to \overline{\boldsymbol{R}_q(D)}$ is an isomorphism by proposition 15.9, we get

$$\|(\mathrm{Id}+iBD)^{-1}u\|_{L^q} \lesssim \|\mathbb{P}_{\overline{\boldsymbol{R}_q(D)}}(\mathrm{Id}+iBD)^{-1}u\|_{L^q} \lesssim \|u\|_{L^p}$$

for all $u \in \overline{\boldsymbol{R}_p(D)} \cap \overline{\boldsymbol{R}_q(D)}$. By remark 15.10 we know that $B : \overline{\boldsymbol{R}_p(D)} \to \overline{\boldsymbol{R}_p(BD)}$ and $B : \overline{\boldsymbol{R}_q(D)} \to \overline{\boldsymbol{R}_q(BD)}$ are isomorphisms. Thus the similarity property in remark 15.10 yields

$$\|(\mathrm{Id}+iDB)^{-1}u\|_{L^q} \lesssim \|u\|_{L^p}$$

for all $u \in \overline{\boldsymbol{R}_p(D)} \cap \overline{\boldsymbol{R}_q(D)}$. Now, we use a rescaling argument and note that for $\lambda \in \mathbb{C}\backslash\overline{S}_\nu$, B_λ defined by multiplication of $B_\lambda(x) := -ie^{i\arg\lambda}B(|\lambda|x)$ has the same properties as B with uniform bounds in $\arg\lambda$. Let

$$u_\lambda(x) := u(|\lambda|x).$$

Then we have as above

$$\|(\mathrm{Id}+iDB_\lambda)^{-1}u_\lambda\|_{L^q} \lesssim \|u_\lambda\|_{L^p}$$

and substitution $|\lambda|x \mapsto x$ yields the estimate

$$\|(\mathrm{Id}+\lambda DB)^{-1}u\|_{L^q} \lesssim |\lambda|^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p}.$$

for all $u \in \overline{\boldsymbol{R}_p(D)} \cap \overline{\boldsymbol{R}_q(D)}$. By density, the operator $(\mathrm{Id}+\lambda DB)^{-1}$ has the desired extension to $\overline{\boldsymbol{R}_p(D)}$. □

The second step is:

Claim 16.6. — *Suppose $q \in \mathcal{I}_{D,B}$ and $p \in [q_*, q] \cap \mathcal{I}_{D,B}$. Let $\psi \in \Psi_\sigma^\tau(\dot{S}_\nu)$, where $\sigma > 0$ and $\tau > \frac{n}{p} - \frac{n}{q}$, and $g \in H^\infty(\dot{S}_\nu)$. Then we have*

$$\|g(DB)\psi_t(DB)u\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p}$$

for all $t > 0$ and all $u \in L^p \cap L^q$ (By density even for all $u \in L^p$.)

Proof of claim 16.6. — If $u \in \overline{\boldsymbol{R}_p(D)} \cap \overline{\boldsymbol{R}_q(D)}$ we have for each $\theta \in (\omega, \nu)$

$$\begin{aligned} \|g(DB)\psi_t(DB)u\|_{L^q} &\lesssim \int_{\partial S_\theta} |g(\lambda)\|\psi(t\lambda)\|\|(\mathrm{Id}-\lambda^{-1}DB)^{-1}u\|_{L^q}|\frac{d\lambda}{\lambda}| \\ &\lesssim t^{\frac{n}{q}-\frac{n}{p}}\int_{\partial S_\theta} |\psi(t\lambda)\|t\lambda|^{\frac{n}{p}-\frac{n}{q}}|\frac{d\lambda}{\lambda}|\|u\|_{L^p} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p} \end{aligned} \tag{110}$$

by claim 16.5, (107) and the decay properties of ψ. For $u \in L^p \cap L^q$ we can use the the decomposition in lemma 15.13, 1), associated to the operator DB and $\psi_t(DB)\widetilde{u} = 0$ for all $\widetilde{u} \in \boldsymbol{N}_p(DB) \cap \boldsymbol{N}_q(DB)$. $\square$

The third step is the proof of claim 16.4 in the case $T = DB$.

Let us denote $q_0 := q$ and $q_l := (q_{l-1})_*$ for $l \in \mathbb{N}^*$ and $k := \inf\{l \in \mathbb{N}^* : q_l \leq p\}$. Further, we set

$$\begin{aligned}
\delta &:= \frac{1}{k+1}\Big(\tau - \Big(\frac{n}{p} - \frac{n}{q}\Big)\Big),\\
m_l &:= 1 + \delta = \frac{n}{q_l} - \frac{n}{q_{l-1}} + \delta \quad \text{for } 1 \leq l < k,\\
m_k &:= \frac{n}{p} - \frac{n}{q_{k-1}} + \delta = \frac{n}{p} - \frac{n}{q_{k-1}} + \delta.
\end{aligned}$$

Then we factorize

$$\psi(z) = \Big(\prod_{l=1}^{k}\Big(\frac{1+\tilde{z}}{1+\tilde{z}}\Big)^{m_l}\Big) \cdot \Big(\frac{1+\tilde{z}}{\tilde{z}}\Big)^{\frac{k\sigma}{k+1}} \cdot \Big(\frac{\tilde{z}}{1+\tilde{z}}\Big)^{\frac{k\sigma}{k+1}} \cdot \psi(z) =: \zeta(z)\prod_{l=1}^{k}\xi^l(z)$$

where $\tilde{z} := \operatorname{sgn}(z)z$ and

$$\zeta(z) := \Big(\prod_{l=1}^{k}(1+\tilde{z})^{m_l}\Big)\Big(\frac{1+\tilde{z}}{\tilde{z}}\Big)^{\frac{k\sigma}{k+1}}\psi(z), \quad \xi^l(z) := \Big(\frac{\tilde{z}}{1+\tilde{z}}\Big)^{\frac{\sigma}{k+1}}\Big(\frac{1}{1+\tilde{z}}\Big)^{m_l}.$$

We observe that each ξ^l satisfies the conditions of claim 16.6: $\xi^l \in \Psi_{\sigma_l}^{\tau_l}(\dot{S}_\nu)$ where $\sigma_l > 0$, $\tau_l > \frac{n}{q_l} - \frac{n}{q_{l-1}}$, $\sigma_k > 0$, $\tau_k > \frac{n}{p} - \frac{n}{q_{k-1}}$ and $\zeta \in \Psi(\dot{S}_\nu)$. Hence, we have

$$\xi_t^l(DB) : L^{q_{l-1}} \longrightarrow L^{q_l}, \tag{111}$$

$$\xi_t^k(DB) : L^{q_{k-1}} \longrightarrow L^p, \tag{112}$$

$$\zeta_t(DB) : L^p \longrightarrow L^p. \tag{113}$$

Now, claim 16.4 in the case $T = DB$ follows by iteration of claim 16.6.

The fourth step is to deduce claim 16.4 in the case $T = BD$.

From the case $T = DB$ just proved, the similarity property

$$g(BD)\psi_t(BD) = Bg(DB)\psi_t(DB)B^{-1}$$

on $\overline{\boldsymbol{R}_q(BD)}$ and $\overline{\boldsymbol{R}_p(BD)}$, the boundedness and coercivity of B on $\overline{\boldsymbol{R}_q(D)}$ and $\overline{\boldsymbol{R}_p(D)}$ we get

$$\|g(BD)\psi_t(BD)u\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|B^{-1}u\|_{L^p} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p}$$

for all $u \in \overline{\boldsymbol{R}_q(BD)} \cap \overline{\boldsymbol{R}_p(BD)}$. In the general case $u \in L^q \cap L^p$ we can use the decomposition in lemma 15.13, 1) associated to the operator BD and $\psi_t(BD)\widetilde{u} = 0$ for all $\widetilde{u} \in \boldsymbol{N}_p(BD) \cap \boldsymbol{N}_q(BD)$. By density we conclude the assertion

$$\|g(BD)\psi_t(BD)u\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p}$$

for all $u \in L^p$. So, claim 16.4 is completely proved.

Now, we turn to the conclusion of proposition 16.3 using claim 16.4.

First by normalizing, we may assume $\|\psi\|_{H^\infty(\dot{S}_\nu)} = \|g\|_{H^\infty(\dot{S}_\nu)} = 1$. We combine L^r-L^r off-diagonal estimates and L^{p_0}-L^{q_0} boundedness to conclude L^p-L^q off-diagonal estimates by interpolation, where $p, q, r, p_0, q_0 \in \mathcal{I}_{D,B}$. Since we use L^{p_0}-L^{q_0} boundedness we have to make sure that the family of holomorphic functions has enough decay at infinity to use L^{p_0}-L^{q_0} boundedness. So, we define

$$\zeta_t^\alpha(z) := g(z)(1+t\tilde{z})^\alpha \psi_t(z),$$

where we recall $\tilde{z} = \operatorname{sgn}(\operatorname{Re} z)z$ for $z \in \dot{S}_\nu$ and $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha < \tau$ and observe that for fixed $t > 0$ the operator $g(T)\psi_t(T)$ is embedded in the analytic family $\{\zeta_t^\alpha(T)\}_\alpha$. Polar coordinates and $\arg(1+t\tilde{z}) \in (-\nu,\nu)$ yields

$$\sup_{z\in\dot{S}_\nu} |(1+t\tilde{z})^\alpha| \leq e^{\nu|\operatorname{Im}\alpha|}|tz|^{\operatorname{Re}\alpha}.$$

Using polar coordinates we can calculate that

$$|\zeta_t^\alpha(z)| \lesssim e^{\nu|\operatorname{Im}\alpha|}\inf\{|tz|^\sigma, |tz|^{\operatorname{Re}\alpha-\tau}\}$$

and consequently the symbol satisfies $\|\zeta_t^\alpha\|_{H^\infty(\dot{S}_\nu)} \lesssim 1$. Thus we can deduce from proposition 16.1

$$\|\chi_F \zeta_t^\alpha(T)(\chi_E u)\|_{L^r} \lesssim e^{\nu|\operatorname{Im}\alpha|}\Big(1+\frac{d(E,F)}{t}\Big)^{-\sigma}\|\chi_E u\|_{L^r}$$

for all $r \in \mathcal{I}_{D,B}$ and all $\alpha \in \mathbb{C}$ such that $\tau - \operatorname{Re}\alpha > \left(\frac{n}{p}-\frac{n}{q}\right) - \operatorname{Re}\alpha > 0$. Now, let $p_0, q_0 \in \mathcal{I}_{D,B}$. We have for all $\tau - \operatorname{Re}\alpha > \left(\frac{n}{p}-\frac{n}{q}\right) - \operatorname{Re}\alpha > \left(\frac{n}{p_0}-\frac{n}{q_0}\right)$

$$\|\chi_F \zeta_t^\alpha(T)(\chi_E u)\|_{L^{q_0}} \lesssim e^{\nu|\operatorname{Im}\alpha|}t^{\frac{n}{q_0}-\frac{n}{p_0}}\|\chi_E u\|_{L^{p_0}}.$$

by claim 16.4. Next, we will use Stein's interpolation theorem for the analytic family of operators $\{\zeta_t^\alpha(T)\}_\alpha$ with

$$\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{p_0} \qquad\qquad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{q_0}. \tag{114}$$

and $\theta := \left(\frac{n}{p}-\frac{n}{q}-\operatorname{Re}\alpha\right)\left(\frac{n}{p_0}-\frac{n}{q_0}\right)^{-1}$ at $\operatorname{Re}\alpha = 0$. This yields

$$\|\chi_F \zeta_t^\alpha(T)(\chi_E u)\|_{L^q} \lesssim M_{\operatorname{Im}\alpha} t^{c_1\left(\frac{n}{q_0}-\frac{n}{p_0}\right)}\Big(1+\frac{d(E,F)}{t}\Big)^{-c_0\sigma}\|\chi_E u\|_{L^p}$$

when $\operatorname{Re}\alpha = 0$. The constants c_0, c_1 are related to the formula in [**52**, theorem 1.3.7]. Choosing $\alpha = 0$ yields

$$\|\chi_F g(T)\psi_t(T)(\chi_E u)\|_{L^q} \lesssim t^{c_1(\frac{n}{q_0}-\frac{n}{p_0})}\Big(1+\frac{d(E,F)}{t}\Big)^{-c_0\sigma}\|\chi_E u\|_{L^p} \tag{115}$$

By [**52**, theorem 1.3.7, exercise 1.3.8] we know that

$$c_0 = 1-\theta = 1-\Big(\frac{1}{p}-\frac{1}{q}\Big)\Big(\frac{1}{p_0}-\frac{1}{q_0}\Big)^{-1},$$
$$c_1 = \theta = \Big(\frac{1}{p}-\frac{1}{q}\Big)\Big(\frac{1}{p_0}-\frac{1}{q_0}\Big)^{-1}.$$

Thus, (115) reads

$$\|\chi_F g(T)\psi_t(T)(\chi_E u)\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\Big(1+\frac{d(E,F)}{t}\Big)^{-(1-\theta)\sigma}\|\chi_E u\|_{L^p}. \tag{116}$$

Since p, q are fixed in the relation and r is choosen depending on p_0, q_0, the parameter $\theta = \theta(p_0, q_0)$ is determined by p_0, q_0. In order to minimize the factor $\big(1+\frac{d(E,F)}{t}\big)^{-\sigma(1-\theta)}$ in (116), we minimize θ using (114). Indeed, we get by (114) the relation

$$\theta = \theta(p_0,q_0) := \big(\frac{1}{p}-\frac{1}{q}\big)\big(\frac{1}{p_0}-\frac{1}{q_0}\big)^{-1}$$

and observe that

$$\inf\{\theta(p_0,q_0) | p_0, q_0 \in \mathcal{I}_{D,B}\} = \Big(\frac{1}{p}-\frac{1}{q}\Big)\Big(\frac{1}{p_-(D,B)}-\frac{1}{p_+(D,B)}\Big)^{-1}.$$

As we are allowed to choose $p_-(D,B) < p_0 < q_0 < p_+(D,B)$ arbitrary in (114) we get the estimate

$$\big\|\chi_F g(T)\psi_t(T)(\chi_E u)\big\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}\Big(1+\frac{d(E,F)}{t}\Big)^{-K}\|\chi_E u\|_{L^p}.$$

for each $K \in [0,\infty)$ such that $\sigma > K c_{p,q}$, where

$$c_{p,q} := \Big(1-\Big(\frac{1}{p}-\frac{1}{q}\Big)\Big(\frac{1}{p_-(D,B)}-\frac{1}{p_+(D,B)}\Big)^{-1}\Big)^{-1}. \qquad \square$$

The next example shows that there are families of operators with finite σ in the functional calculus, which satisfy off-diagonal estimates of each order $K \in [0,\infty)$, showing that the condition $\sigma > cK$ is sufficient but not necessary.

Example 16.7 (L^p-L^q off-diagonal estimates of arbitrary order)
Let $T \in \{BD, DB\}$ and $\alpha, M \in \mathbb{N}^*$ with $0 < \alpha \le M$. Then the family

$$\big\{(itT)^\alpha(\mathrm{Id}+itT)^{-M}\big\}_{t>0}$$

satisfies L^p-L^q off-diagonal estimates of order K for each $K \in [0,\infty)$ whenever $p, q \in \mathcal{I}_{D,B}$ with $p < q$ such that $M-\alpha > \frac{n}{p}-\frac{n}{q}$.

Remark 16.8. — We do not know if the condition $\tau > \frac{n}{p}-\frac{n}{q}$ is necessary in proposition 16.3.

Sometimes, it is appropriate to have the following variant of proposition 16.3 as used in Part I (see proposition 3.13).

Proposition 16.9 (L^p-L^q biparameter off-diagonal estimates)
Let $T \in \{DB, BD\}$. Denote by

$$\omega := \omega_{DB} = \omega_{BD}$$

the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. *Suppose* $p, q \in \mathcal{I}_{D,B}$ *such that* $p < q$ *and let* $\sigma > 0$, $\tau > \frac{n}{p} - \frac{n}{q}$. *Then there exists* $c := c_{p,q} > 0$ *such that for* $0 \le K < \frac{M}{c}$ *one has: Suppose that* $\psi \in \Psi_\sigma^\tau(\dot{S}_\nu)$ *and* $\varphi \in H^\infty(\dot{S}_\nu)$ *are functions such that* φ *satisfies*

$$|\varphi(\lambda)| \lesssim \inf\{|\lambda|^M, 1\}$$

for all $\lambda \in \dot{S}_\nu$. *Then the family* $\{\psi_t(T)\varphi_r(T)\}_{t \ge r > 0}$ *satisfies* L^p-L^q *biparameter off-diagonal estimates in* (t, r) *of order* K. *Moreover, one can choose*

$$c_{p,q} = \Big(1 - \Big(\frac{1}{p} - \frac{1}{q}\Big)\Big(\frac{1}{p_-(D,B)} - \frac{1}{p_+(D,B)}\Big)^{-1}\Big)^{-1}. \tag{117}$$

Proof. — The conclusion of proposition 16.9 follows by analytic interpolation as in proposition 16.3: in fact, we use interpolation between claim 16.4 (that is, proposition 16.3 in the case $K = 0$) and the next claim.

Claim 16.10. — *With the assumption above and* $p = q$ *the family* $\{\psi_t(T)\varphi_r(T)\}_{t \ge r > 0}$ *satisfies* L^q-L^q *biparameter off-diagonal estimates in* (t, r) *of order* M.

Proof of claim 16.10. — W.l.o.g. assume $\|\psi\|_{H^\infty(\dot{S}_\nu)} \le 1$. Let $u \in L^q$ with $\operatorname{supp} u \subset E$. We have for each $\theta \in (\omega, \nu)$

$$\begin{aligned}
\|\psi_t(T)\varphi_r(T)u\|_{L^q(F)} &\lesssim \int_{\partial S_\theta} |\varphi(r\lambda)| \cdot |\psi(t\lambda)| \cdot \|(\mathrm{Id} - \lambda^{-1}T)^{-1}u\|_{L^q(F)} \Big|\frac{d\lambda}{\lambda}\Big| \\
&\lesssim \int_{\partial S_\theta} |\varphi(\lambda)| \cdot \Big|\psi\Big(\frac{t\lambda}{r}\Big)\Big| \Big(1 + \frac{d(E,F)}{r}|\lambda|\Big)^{-K} \Big|\frac{d\lambda}{\lambda}\Big| \cdot \|u\|_{L^q(E)}, \\
&\lesssim \Big(1 + \frac{d(E,F)}{r}\Big)^{-M} \|u\|_{L^q(E)} \qquad (118)
\end{aligned}$$

where $K \in [0, \infty)$ will be chosen below. For the proof of (118) we consider two cases. On the one hand, if $\frac{d(E,F)}{r} \le 1$ we have

$$\int_{\partial S_\theta} |\varphi(\lambda)| \cdot \Big|\psi\Big(\frac{t\lambda}{r}\Big)\Big| \Big(1 + \frac{d(E,F)}{r}|\lambda|\Big)^{-K} \Big|\frac{d\lambda}{\lambda}\Big| \le \int_{\partial S_\theta} |\varphi(\lambda)| \cdot \inf\{1, |\lambda|^{-\tau}\} \Big|\frac{d\lambda}{\lambda}\Big| \lesssim 1.$$

In fact, the last estimate follows by splitting the contour integral at $|\lambda| = 1$ and using that

$$\Big|\psi\Big(\frac{t\lambda}{r}\Big)\Big| \lesssim \begin{cases} \|\psi\|_{H^\infty(\dot{S}_\nu)} \le 1, & \text{if } |\lambda| \le 1, \\ |\frac{t\lambda}{r}|^{-\tau} \le |\lambda|^{-\tau}, & \text{if } |\lambda| \ge 1. \end{cases}$$

On the other hand if $x := \frac{d(E,F)}{r} \ge 1$, we split

$$\int_{\partial S_\theta} |\varphi(\lambda)| \cdot \Big|\psi\Big(\frac{t\lambda}{r}\Big)\Big| \Big(1 + \frac{d(E,F)}{r}|\lambda|\Big)^{-K} \Big|\frac{d\lambda}{\lambda}\Big|$$

into three parts according to $|\lambda| \le \frac{1}{x}$, $\frac{1}{x} \le |\lambda| \le 1$ and $|\lambda| \ge 1$. From this the estimate

$$\int_{\partial S_\theta} |\varphi(\lambda)| \cdot \Big|\psi\Big(\frac{t\lambda}{r}\Big)\Big| \Big(1 + \frac{d(E,F)}{r}|\lambda|\Big)^{-K} \Big|\frac{d\lambda}{\lambda}\Big| \lesssim \Big(\frac{d(E,F)}{r}\Big)^{-M} \tag{119}$$

easily follows, required we choose $K > M$. In fact, for the first part we use the inequality $(1+\frac{d(E,F)}{r}|\lambda|)^{-K} \le 1$ and for the second and third part we estimate

$$\Big(1+\frac{d(E,F)}{r}|\lambda|\Big)^{-K} \le \Big(\frac{d(E,F)}{r}\Big)^{-K} |\lambda|^{-K}$$

for the same choice of $K > M$ and evaluate the three integrals associated to the three parts. The addition of the three evaluated parts is bounded by the right hand side in (119). $\square$

The lemma is proved. $\square$

For the semigroup $e^{-t|T|}$ for $T \in \{BD, DB\}$ we can only prove $\overline{\boldsymbol{R}_p(T)} - \overline{\boldsymbol{R}_q(T)}$ boundedness whenever $p, q \in \mathcal{I}_{D,B}$ with $p \le q$. More precisely, we prove that $f_t(T)$ maps $\overline{\boldsymbol{R}_p(T)}$ to $\overline{\boldsymbol{R}_q(T)}$, whenever the holomorphic function f has enough decay at infinity. We will apply this result to prove a Hardy-Littlewood-Sobolev inequality for fractional operators $|T|^{-\alpha}$.

Corollary 16.11 (L^p-L^q theory for bounded holomorphic functions)
Let $T \in \{DB, BD\}$. Denote by $\omega := \omega_{DB} = \omega_{BD}$ the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Suppose $p, q \in \mathcal{I}_{D,B}$ such that $p \le q$, and let $g \in H^\infty(\dot{S}_\nu)$ and f be holomorphic function with

$$|f(\lambda)| \lesssim \inf\{1, |\lambda|^{-M}\}$$

for all $\lambda \in \dot{S}_\nu$, where $M > \frac{n}{p} - \frac{n}{q}$. Then the family $\{g(T)f_t(T)\}_{t>0}$ is $\overline{\boldsymbol{R}_p(T)} - \overline{\boldsymbol{R}_q(T)}$ bounded. In particular, the semigroup $\{e^{-t|T|}\}_{t>0}$ is $\overline{\boldsymbol{R}_p(T)} - \overline{\boldsymbol{R}_q(T)}$ bounded for all $p, q \in \mathcal{I}_{D,B}$ with $p \le q$.

Proof. — The first part can be proved using McIntosh convergence lemma and ideas from the proof in proposition 16.3. We leave this to the interested reader. The statement for the semigroup follows from the special choice $g = 1$ and $f(z) = e^{-\tilde{z}}$, where $\tilde{z} = \operatorname{sgn}(\operatorname{Re} z) z$ as usual. $\square$

Remark 16.12. — In the situation of corollary 16.11, the family $\{g(T)f_t(T)\}_{t>0}$ is L^p-L^q bounded whenever $(fg)(0) = 0$. This follows from corollary 16.11 and (108). We treat the case $(fg)(0) \neq 0$ in section 16.2.

16.2. Stability under multiplication by cut-off functions and the relation to Ajiev's work

Let us begin this section with our definition of stability under multiplication by smooth cut-off functions/cut-off functions.

Definition 16.13. — Let U_q be a closed subspace of L^q, $1 \le q < \infty$.

- We say U_q is *stable under multiplication by cut-off functions* if for any $u \in U_q$ and any characteristic function χ_E for a Borel measurable set $E \subset \mathbb{R}^n$ one has $\chi_E u \in U_q$.
- We say U_q is *stable under multiplication by smooth cut-off functions* if for any $u \in U_q$ and any smooth complex-valued function ζ with compact support one has $\zeta u \in U_q$.

Remark 16.14 (Equivalence). — We observe that both notions are equivalent. Indeed, if U_q is stable by cut-off functions it is also stable under multiplication by simple functions. Then by an approximation argument and the closedness of U_q it follows that U_q is stable under multiplication by smooth cut-off functions. Conversely, if U_q is stable under multiplication by smooth cut-off functions, then it follows from the closedness of U_q and a mollifier approximation argument that U_q is stable under multiplication by cut-off functions.

Remark 16.15 (Relation to Ajiev's work). — 1) A combination of [**1**, theorem 4.6 (a)] and [**1**, theorem 4.14] imply corollary 16.11 for a subclass of function pairs (f, g).

2) A combination of [**1**, theorem 4.6 (b)], [**1**, theorem 4.14], [**1**, lemma 4.13] and [**1**, remark 2] imply proposition 16.3 for a subclass of function pairs (ψ, g), provided that the range $\overline{\boldsymbol{R}_p(T)}$ is stable under multiplication by (smooth) cut-off functions.

We never used these notions. However, to compare with [**1**] we investigate whether or not $\overline{\boldsymbol{R}_p(D)}$ is stable under multiplication by (smooth) cut-off functions.

Definition 16.16. — Let D as in assumption 15.4 above and $p \in (1, \infty)$.

1) We define V_p to be the linear subspace of $\mathbb{C}^N$ generated by $⨍_B v$ for all balls $B \subset \mathbb{R}^n$ and all $v \in \overline{\boldsymbol{R}_p(D)}$.
2) We define $W_{p'}$ to be the linear subspace of $\mathbb{C}^N$ generated by $⨍_B w$ for all balls $B \subset \mathbb{R}^n$ and all $w \in \boldsymbol{N}_{p'}(D^*)$

Remark 16.17. — By the Lebesgue differentiation theorem any $v \in \overline{\boldsymbol{R}_p(D)}$ takes values in V_p almost everywhere. Similarly, any $w \in \boldsymbol{N}_{p'}(D^*)$ takes values in $W_{p'}$ almost everywhere. Thus, V_p is the space of almost everywhere values of all elements in $\overline{\boldsymbol{R}_p(D)}$ and $W_{p'}$ is the space of values of all elements in $\boldsymbol{N}_{p'}(D^*)$.

Remark 16.18. — Suppose $1 < p, q < \infty$. Then $V_p = V_q$ and $W_{p'} = W_{q'}$. This follows from the density statements in lemma 15.13. Thus, we may set $V = V_p$ and $W = W_{p'}$ for one $p \in (1, \infty)$.

Proposition 16.19 (Stability under multiplication by smooth cut-off functions)
Let D as in assumption 15.4 *above and $p \in (1, \infty)$.*

1) *If* $\overline{\boldsymbol{R}_p(D)} = L^p$, *then* $\overline{\boldsymbol{R}_p(D)}$ *is stable under multiplication by smooth cut-off functions.*
2) *If* $\overline{\boldsymbol{R}_p(D)} \neq L^p$, *then* $\overline{\boldsymbol{R}_p(D)}$ *is stable under multiplication by smooth cut-off functions if and only if* $V \perp W$ *for the* $\mathbb{C}^N$ *inner product.*

This implies that if 1) *or* 2) *holds for one* p, *it holds for all* p.

Proof. — Assertion 1) is evident, so we turn to the proof of assertion 2).

Since $\overline{\boldsymbol{R}_p(D)}$ is the polar set to $\boldsymbol{N}_{p'}(D^*)$, we have that $\overline{\boldsymbol{R}_p(D)}$ is stable by multiplication of smooth cut-off functions if and only if $\langle \xi v, w\rangle = 0$ for all $v \in \overline{\boldsymbol{R}_p(D)}$, all $w \in \boldsymbol{N}_{p'}(D^*)$ and all smooth cut-off functions ζ. We claim that this is equivalent to $v\overline{w} = 0$ almost everywhere for all $v \in \overline{\boldsymbol{R}_p(D)}$ and all $w \in \boldsymbol{N}_{p'}(D^*)$. As v, w are arbitrary, this is equivalent to $V \perp W$.

Direction $\Rightarrow$: If $\langle \zeta v, w\rangle = 0$ for all such v, w, ζ then we have $\langle \zeta_\xi v, w\rangle = 0$ in particular for all $\xi \in \mathbb{R}^n$, where $\zeta_\xi(x) := e^{-ix\cdot\xi}\zeta(x)$. Let us denote by $\mathcal{F}$ the Fourier transform. Then this implies $\mathcal{F}(\zeta v\overline{w})(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ by definition of the Fourier transform. As $\zeta v\overline{w} \in L^1$, we deduce that $\zeta v\overline{w} = 0$ almost everywhere. Choosing all possible ζ, this concludes the proof of the first direction.

Direction $\Leftarrow$: If $v\overline{w} = 0$ almost everywhere for all $v \in \overline{\boldsymbol{R}_p(D)}$ and all $w \in \boldsymbol{N}_{p'}(D^*)$ then $\zeta v\overline{w} = 0$ almost everywhere for all these v, w and smooth cut-off ζ, hence $\langle \zeta v, w\rangle = 0$. This shows that ζv belongs to the polar set of $\boldsymbol{N}_{p'}(D^*)$, hence $\zeta v \in \overline{\boldsymbol{R}_p(D)}$. This concludes the proof of the stated equivalence and of the lemma. □

EXAMPLE 16.20. — We claim that the spaces $\overline{\boldsymbol{R}_p(DB)} = \overline{\boldsymbol{R}_p(D)}$ associated to the operators DB and D in [**11**] and [**8**] are not stable under multiplication by smooth cut-off functions. Indeed, for $D = \begin{pmatrix} 0 & \mathrm{div} \\ -\nabla & 0 \end{pmatrix}$, we have

$$\boldsymbol{N}_{p'}(D^*) = \boldsymbol{N}_{p'}(D) = \{u = (0, g) \in L^{p'}(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n])\,;\, \mathrm{div} g = 0\} \neq \{0\},$$

hence $W \neq \{0\}$. Next, we have that

$$\overline{\boldsymbol{R}_p(D)} = \{u = (f, g) \in L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n])\,;\, g = \nabla h, h \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{C}^m)\}.$$

Let $c \in \mathbb{C}^m$ and $\xi \in \mathbb{C}^m \otimes \mathbb{C}^n = (\mathbb{C}^m)^n$. Taking $f \in L^p$ which is constant with value c on some ball and $h \in \dot{W}^{1,p}$ with $h(x) = \sum_{i=1}^n x_i \xi_i$ in the same ball. We see that $(c, \xi_1, \ldots, \xi_n) \in V$. Thus, $\mathbb{C}^N \subset V$ (with $N = m(1+n)$). The claim follows as we are in the situation of (2) in proposition 16.19 and W is not orthogonal to V.

This shows that Ajiev's results do not apply to the main motivating example.

16.3. L^p-L^q estimates and the relation to the kernel/range decomposition

From the next proposition and example we will learn more about the relation of kernel/range decomposition and the L^p-L^q boundedness of the related operators in the functional calculus. The proposition shows that $f(0) = 0$ is a necessary condition for functions f to have L^p-L^q boundedness of the associated operator, whenever the null space is not equal $\{0\}$. Before we state the proposition we make a definition.

DEFINITION 16.21 (Not bounded). — Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator. We say T is *not bounded from $\mathcal{X}$ to $\mathcal{Y}$* and write $T : \mathcal{X} \nrightarrow \mathcal{Y}$ if there exists $u \in \mathcal{X}$ such that $Tu \notin \mathcal{Y}$ or if there exists no constant $C > 0$ such that for all $u \in \mathcal{X}$ holds $\|Tu\|_{\mathcal{Y}} \leq C\|u\|_{\mathcal{X}}$.

PROPOSITION 16.22 (Necessary Condition). — *Let $T \in \{DB, BD\}$. Denote by*

$$\omega := \omega_{DB} = \omega_{BD}$$

the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Suppose there exists $r \in \mathcal{I}_{D,B}$ such that $\boldsymbol{N}_r(T) \neq \{0\}$ and let $f \in H^\infty(\dot{S}_\nu, \{0\})$ with $f(0) \neq 0$. Then for all $p, q \in \mathcal{I}_{D,B}$ such that $p \neq q$ we have

$$f(T) : \boldsymbol{N}_p(T) \nrightarrow L^q.$$

In particular, for all $p, q \in \mathcal{I}_{D,B}$ such that $p \neq q$ we have $f(T) : L^p \nrightarrow L^q$.

Proof. — First, we note by lemma 15.13, (2)), that $\boldsymbol{N}_r(T) \neq \{0\}$ for one $r \in \mathcal{I}_{D,B}$ is equivalent to $\boldsymbol{N}_r(T) \neq \{0\}$ for all $r \in \mathcal{I}_{D,B}$. Thus we can assume

$$\boldsymbol{N}_p(T) \neq \{0\} \neq \boldsymbol{N}_q(T)$$

for particular $p, q \in \mathcal{I}_{D,B}$. Since $f(T)u = f(0)u$ for all $u \in \boldsymbol{N}_p(T)$ we observe that the statement $f(T) : \boldsymbol{N}_p(T) \nrightarrow L^q(\mathbb{R}^n, \mathbb{C}^N)$ is equivalent to $f(T) : \boldsymbol{N}_p(T) \nrightarrow \boldsymbol{N}_q(T)$, which we prove next.

We begin with the case $T = DB$ and let $p, q \in \mathcal{I}_{D,B}$. Since

$$(\boldsymbol{N}_p(DB))^* = \boldsymbol{N}_{p'}(B^*D^*) \quad \text{and} \quad (\boldsymbol{N}_q(DB))^* = \boldsymbol{N}_{q'}(B^*D^*),$$

we observe that $f(DB) : \boldsymbol{N}_p(DB) \to \boldsymbol{N}_q(DB)$ is equivalent to

$$f^*(B^*D^*) : \boldsymbol{N}_{q'}(B^*D^*) \longrightarrow \boldsymbol{N}_{p'}(B^*D^*)$$

by duality, where $f^*(\lambda) := \overline{f(\overline{\lambda})}$ for $\lambda \in \dot{S}_\nu \cup \{0\}$. Now, recall also that $p, q \in \mathcal{I}_{D,B}$ is equivalent to $p', q' \in \mathcal{I}_{D^*,B^*}$ by remark 15.11. Thus it suffices to consider the case $T = BD$.

We turn to the case $T = BD$ and assume that for $p, q \in \mathcal{I}_{D,B}$ the operator $f(BD)$ defined by the bounded holomorphic functional calculus maps $\boldsymbol{N}_p(BD)$ to $\boldsymbol{N}_q(BD)$, with quantitative estimate

$$\|f(BD)u\|_{L^q} \leq C\|u\|_{L^p}, \quad \forall u \in \boldsymbol{N}_p(BD),$$

where C is of course independent of u. Since $f(BD)u = f(0)u$ this estimate turns into

$$|f(0)|\|u\|_{L^q} \leq C\|u\|_{L^p}, \quad \forall u \in \boldsymbol{N}_p(BD). \tag{120}$$

Since $\boldsymbol{N}_p(BD) = \boldsymbol{N}_p(D)$ is the null space of a constant coefficient partial differential operator we observe that $u \in \boldsymbol{N}_p(BD)$ is equivalent to $u_s \in \boldsymbol{N}_p(BD)$ for all $s > 0$ by chain rule, where $u_s(x) := u(sx)$ for all $s > 0$ and all $x \in \mathbb{R}^n$. This means the null space $\boldsymbol{N}_p(BD)$ is invariant by rescaling. Thus, if we fix $u \in \boldsymbol{N}_p(BD)$ such that $u \neq 0$ we get the inequality

$$|f(0)|\|u_s\|_{L^q} \leq C\|u_s\|_{L^p}, \quad \forall s > 0,$$

from (120) above. But by substitution, this inequality is equivalent to the inequality

$$s^{\frac{n}{q}-\frac{n}{p}} \leq \frac{C\|u\|_{L^p}}{|f(0)|\|u\|_{L^q}}, \quad \forall s > 0, \tag{121}$$

for our fixed $u \in \boldsymbol{N}_p(BD)$ with $u \neq 0$. If $p < q$ we get a contradiction in (121) as $s \to 0$. If $p > q$ we get a contradiction in (121) as $s \to \infty$. □

Example 16.23. — In the proof of the last proposition we have seen that operators $f(BD)$ do not regularize the null space $\boldsymbol{N}_p(BD)$ whenever $f(0) \neq 0$ and $\boldsymbol{N}_p(BD) \neq \{0\}$. In the special case of block form operators $BD = \begin{pmatrix} 0 & -\operatorname{div} \\ A\nabla & 0 \end{pmatrix}$ as in [**27**, Section 6] we have

$$\boldsymbol{N}_p(BD) = \{u = (0, g) \in L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n])\,;\, \operatorname{div} g = 0\}.$$

The interpretation in this special case is that $f(BD)$ does not regularize the tangential part of functions $(0, g)$ which satisfy $\operatorname{div} g = 0$. In this connection, we note also that the space

$$\boldsymbol{N}_p(\operatorname{div}) = \{g \in L^p(\mathbb{R}^n; \mathbb{C}^m \otimes \mathbb{C}^n)\,;\, \operatorname{div} g = 0\}.$$

is invariant by rescaling.

Corollary 16.24 (Null space equal zero). — *Let $T \in \{DB, BD\}$. Denote by*

$$\omega := \omega_{DB} = \omega_{BD}$$

the type of bisectoriality and let $\nu \in (\omega, \frac{\pi}{2})$. Further, suppose there exists $r \in \mathcal{I}_{D,B}$ such that $\boldsymbol{N}_r(T) = \{0\}$.

1) *For all $p, q \in \mathcal{I}_{D,B}$ satisfying $p \leq q$ the semigroup $\{e^{-t|T|}\}_{t>0}$ is L^p-L^q bounded.*
2) *For all $p, q \in \mathcal{I}_{D,B}$ such that $0 \leq \frac{n}{p} - \frac{n}{q} < 1$ the family $\{(\operatorname{Id} + itT)^{-1}\}_{t>0}$ satisfies L^p-L^q off-diagonal estimates of order K for each $K \in [0, \infty)$.*

Proof. — The easy details are left to the reader. □

Corollary 16.24 really shows the link between $L^p - L^q$ estimates for $p < q$ and the triviality of the null space.

Example 16.25 (cf. Part I, proposition 3.11). — Let $T = BD$. Moreover, suppose $n = 1$ and $\widehat{D}(\xi)$ is invertible for all $\xi \neq 0$. Then we have $\mathcal{I}_{D,B} = (1, \infty)$ and for all $p \in \mathcal{I}_{D,B}$, $\boldsymbol{N}_p(BD) = \boldsymbol{N}_p(D) = \{0\}$. The reader checks that the proof goes through for B satisfying only coercivity instead of strict accretivity. In fact, the coercivity of B

suffices to deduce invertibility of $B \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^N))$ from the Lebesgue differentiation theorem.

16.4. Analytic extensions

Sometimes one is interested in complex times for the results above. So, for appropriate $z \in \mathbb{C}\backslash\{0\}$ and $f \in H^\infty(\dot{S}_\nu, \{0\})$ we define

$$f_z(\lambda) := f(z\lambda), \quad \lambda \in \dot{S}_\nu \cup \{0\}$$

and treat this topic in the next remark.

Remark 16.26 (Analytic extension). — One can extend the results ...

- ... in proposition 16.3 to families $\{g(T)\psi_z(T)\}_{z\in\dot{S}_\beta}$, $\beta \in [0, \frac{\pi}{2} - \omega)$, provided there exists $\epsilon \in (0, \frac{\pi}{2} - \omega - \beta)$ such that $\psi \in \Psi^\tau_\sigma(\dot{S}_{\omega+\epsilon+\beta})$ and $g \in H^\infty(\dot{S}_{\omega+\epsilon+\beta})$,
- ... in Example 16.7 to families $\{(izT)^\alpha(\mathrm{Id} + izT)^{-M}\}_{z\in\dot{S}_\beta}$, $\beta \in [0, \frac{\pi}{2} - \omega)$,
- ... in corollary 16.11 to families $\{g(T)f_z(T)\}_{z\in\dot{S}_\beta}$, $\beta \in [0, \frac{\pi}{2} - \omega)$, provided there exists $\epsilon \in (0, \frac{\pi}{2} - \omega - \beta)$ such that $g, f \in H^\infty(\dot{S}_{\omega+\epsilon+\beta}, \{0\})$ and f satisfies $|f(\lambda)| \lesssim \inf\{1, |\lambda|^{-M}\}$ for all $\lambda \in \dot{S}_{\omega+\epsilon+\beta}$. In particular, the family $\{e^{-z|T|}\}_{z\in\dot{S}^+_\beta}$, $\beta \in [0, \frac{\pi}{2} - \omega)$ is $\overline{\boldsymbol{R}_p(T)} - \overline{\boldsymbol{R}_q(T)}$ bounded.
- ... in corollary 16.24 to the families $\{e^{-z|T|}\}_{z\in\dot{S}^+_\beta}$ and $\{(\mathrm{Id} + izT)^{-1}\}_{z\in\dot{S}_\beta}$, $\beta \in [0, \frac{\pi}{2} - \omega)$.

Proof. — One can adapt the strategies in [**7**, Chapter 3.6]. Details are left to the interested reader. □

16.5. An Application

Here, we will essentially follow [**7**, Section 6.2] [1] to prove L^p-L^q estimates for the fractional operators $|DB|^{-\alpha}$ and $|BD|^{-\alpha}$ with some simplifications in the final limiting argument. We begin with the definition of $|T|^{-\alpha}$ for $T \in \{DB, BD\}$ and $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < \infty$. Fix p, q with $\operatorname{Re}\alpha = \frac{n}{p} - \frac{n}{q}$. For $h \in \boldsymbol{R}_p(T) \cap \overline{\boldsymbol{R}_q(T)}$, define

$$|T|^{-\alpha}h := \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t|T|}h\,dt = \frac{1}{\Gamma(\alpha)} \lim_{(\epsilon,R)\to(0,\infty)} \int_\epsilon^R t^{\alpha-1}e^{-t|T|}h\,dt, \tag{122}$$

and observe that the improper Riemann integral converges in the strong sense in $\overline{\boldsymbol{R}_q(T)}$ with respect to L^q topology. Indeed, convergence at 0 follows from $h \in \overline{\boldsymbol{R}_q(T)}$ and $\operatorname{Re}\alpha > 0$, and convergence at ∞ follows from $\operatorname{Re}\alpha = \frac{n}{p} - \frac{n}{q}$ and $\|e^{-t|T|}h\|_{L^q} \lesssim t^{\frac{n}{q}-\frac{n}{p}}t^{-1}$ by writing $h = Tf$ with $f \in L^p$ and using $\overline{\boldsymbol{R}_p(T)}$ to $\overline{\boldsymbol{R}_q(T)}$ boundedness of the semigroup.

1. We mention there are some inaccuracies in this argument that our argument fixes.

The result we want to prove in this section is the following Hardy-Littlewood-Sobolev inequality, which is the analogue to [**7**, Section 6.2].

THEOREM 16.27 (Hardy-Littlewood-Sobolev inequality). — *Let* $T \in \{DB, BD\}$. *Suppose*

$$p_-(D,B) < p < q < p_+(D,B).$$

Then $|T|^{-\alpha}$ *has a bounded extension from* $\overline{\boldsymbol{R}_p(T)}$ *to* $\overline{\boldsymbol{R}_q(T)}$ *whenever* $\operatorname{Re}\alpha = \frac{n}{p} - \frac{n}{q}$.

Proof. — Fix $\operatorname{Re}\alpha := \frac{n}{p} - \frac{n}{q}$. Set

$$\mathbb{T}_{\epsilon,R}h := \frac{1}{\Gamma(\alpha)} \int_\epsilon^R t^{\alpha-1} e^{-t|T|} h dt$$

for $h \in \overline{\boldsymbol{R}_p(T)}$ and $0 < \epsilon < R < \infty$. The first step is to establish the weak type $p - q$ estimate for $\mathbb{T}_{\epsilon,R}$ from $\overline{\boldsymbol{R}_p(T)}$ to $L^{q,\infty}$ uniformly in ϵ, R. Choose q_0, q_1 with $p < q_0 < q < q_1 < p_+(D,B)$. Since the semigroup $e^{-t|T|}$ is bounded from $\overline{\boldsymbol{R}_p(T)}$ to $\overline{\boldsymbol{R}_{q_0}(T)}$ and to $\overline{\boldsymbol{R}_{q_1}(T)}$ we get for $h \in \overline{\boldsymbol{R}_p(T)}$ with $\|h\|_{L^p} = 1$,

$$\begin{aligned}\Big\| \int_b^R t^{\alpha-1} e^{-t|T|} h\, dt \Big\|_{L^{q_1}} &\le \int_b^R t^{\frac{n}{p}-\frac{n}{q}-1} \|e^{-t|T|}h\|_{L^{q_1}}\, dt \\ &\le C \int_b^R t^{\frac{n}{p}-\frac{n}{q}-1} t^{\frac{n}{q_1}-\frac{n}{p}}\, dt \|h\|_{L^p} \le C b^{\frac{n}{q_1}-\frac{n}{q}},\end{aligned}$$

and similarly

$$\Big\| \int_\epsilon^b t^{\alpha-1} e^{-t|T|} h\, dt \Big\|_{L^{q_0}} \le C b^{\frac{n}{q_0}-\frac{n}{q}},$$

uniformly for ϵ, b, R such that $0 < \epsilon < b < R < \infty$. Hence, for all $\lambda > 0$ we get from Tchebycheff's inequality

$$\begin{aligned}|\{|\mathbb{T}_{\epsilon,R}h| > \lambda\}| &\le \Big|\Big\{\Big| \int_b^R t^{\alpha-1} e^{-t|T|} h\, dt\Big| > \frac{\lambda}{2}\Big\}\Big| + \Big|\Big\{\Big| \int_\epsilon^b t^{\alpha-1} e^{-t|T} h\, dt\Big| > \frac{\lambda}{2}\Big\}\Big| \\ &\le C\lambda^{-q_1} b^{q_1(\frac{n}{q_1}-\frac{n}{q})} + C\lambda^{-q_0} b^{q_0(\frac{n}{q_0}-\frac{n}{q})}.\end{aligned}$$

Thus, if we choose $b^{-\frac{n}{q}} = \lambda$, we get

$$|\{|\mathbb{T}_{\epsilon,R}h| > \lambda\}| \le C\lambda^{-q}, \quad \forall \lambda \in \big(R^{-\frac{n}{q}}, \epsilon^{-\frac{n}{q}}\big).$$

Similarly, one proves in the case $\lambda \le R^{-\frac{n}{q}}$

$$|\{|\mathbb{T}_{\epsilon,R}h| > \lambda\}| \le C\lambda^{-q_0} R^{q_0(\frac{n}{q_0}-\frac{n}{q})} \le C\lambda^{-q}$$

and in the case $\lambda \ge \epsilon^{-\frac{n}{q}}$

$$|\{|\mathbb{T}_{\epsilon,R}h| > \lambda\}| \le C\lambda^{-q_1} \epsilon^{q_1(\frac{n}{q_1}-\frac{n}{q})} \le C\lambda^{-q}$$

to deduce the inequality

$$|\{|\mathbb{T}_{\epsilon,R}h| > \lambda\}| \le C\lambda^{-q}, \quad \forall \lambda \in (0,\infty).$$

The second step is to proceed by real interpolation. Observe that the spaces $\overline{\boldsymbol{R}_p(D)}$ are real interpolation spaces for $1 < p < \infty$ (This is shown in [**63**].)

For $p_-(D,B) < p < p_+(D,B)$, we have $\overline{\boldsymbol{R}_p(DB)} = \overline{\boldsymbol{R}_p(D)}$ and that $\overline{\boldsymbol{R}_p(DB)}$ and $\overline{\boldsymbol{R}_p(BD)}$ are similar spaces under multiplication by B (remark 15.10). Thus, the real interpolation property holds for $\overline{\boldsymbol{R}_p(T)}$ when $p_-(D,B) < p < p_+(D,B)$ for $T \in \{DB, BD\}$. Consider now a pair (p,q) with

$$\operatorname{Re}\alpha = \frac{n}{p} - \frac{n}{q} \quad \text{and} \quad p_-(D,B) < p < q < p_+(D,B).$$

It is possible to pick two pairs (p_0,q_0) and (p_1,q_1) with the same properties and, in addition, $p_0 < p < p_1$ and $q_0 < q < q_1$. By real interpolation, the weak type $p_i - q_i$ estimates yield the strong type $p - q$ estimate, in the sense that $\mathbb{T}_{\epsilon,R}$ maps $\overline{\boldsymbol{R}_p(T)}$ to L^q, uniformly over $0 < \epsilon < R < \infty$.

The last step is a limiting argument. Assume $h \in \boldsymbol{R}_p(T) \cap \overline{\boldsymbol{R}_q(T)}$. We know that $\mathbb{T}_{\epsilon,R}h$ converges strongly to $|T|^{-\alpha}h$ in $\overline{\boldsymbol{R}_q(T)}$ by construction as $\epsilon \to 0$ and $R \to \infty$. As we just showed

$$\sup_{0<\epsilon<R<\infty} \|\mathbb{T}_{\epsilon,R}h\|_{L^q} \leq C\|h\|_{L^p},$$

we deduce

$$\big\||T|^{-\alpha}h\big\|_{L^q} \leq C\|h\|_{L^p}.$$

By density of $\boldsymbol{R}_p(T) \cap \overline{\boldsymbol{R}_q(T)}$ in $\overline{\boldsymbol{R}_p(T)}$ for the L^p topology (lemma 15.13, item 3), $|T|^{-\alpha}$ has a bounded extension from $\overline{\boldsymbol{R}_p(T)}$ to L^q. To see it maps into $\overline{\boldsymbol{R}_q(T)}$, we observe that for $h \in \boldsymbol{R}_p(T) \cap \overline{\boldsymbol{R}_q(T)}$, $|T|^{-\alpha}h$ is by construction the limit in L^q of elements in $\overline{\boldsymbol{R}_q(T)}$. Thus this remains by density for all $h \in \overline{\boldsymbol{R}_p(T)}$. □

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BIBLIOGRAPHY

[1] Ajiev (Sergey S.) – *Extrapolation of the functional calculus of generalized Dirac operators and related embedding and Littlewood-Paley-type theorems. I*, J. Aust. Math. Soc., t. **83** (2007), pp. 297–326.

[2] Albrecht (D.), Duong (X.) & McIntosh (A.) – Operator theory and harmonic analysis, pp. 77–136, in *Instructional Workshop on Analysis and Geometry, Part III (Canberra 1995)*, vol. 34, 1996.

[3] Alfonseca (M.), Auscher (P.), Axelsson (A.), Hofmann (S.) & Kim (S.) – *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Adv. Math., t. **226** (2011), pp. 4533–4606.

[4] Anh (Bui The) & Li (Ji.) – *Orlicz-Hardy spaces associated to operators satisfying bounded H_∞ functional calculus and Davies-Gaffney estimates* (English summary), J. Math. Anal. Appl., t. **373** (2011), pp. 485–501.

[5] Auscher (P.) – *Regularity theorems and heat kernel for elliptic operators*, J. London Math. Soc. (2), t. **54** (1996), pp. 284–296.

[6] ———, *On L^p estimates for square roots of second order elliptic operators on $\mathbb{R}^n$*, Publ. Mat., t. **48** (2004), pp. 159–186.

[7] ———, *On necessary and sufficient conditions for L^p-estimates of Riesz Transforms Associated to Elliptic Operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc., vol. 186, 2007.

[8] Auscher (P.) & Axelsson (A.) – *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I*, Invent. Math., t. **184** (2011), pp. 47–115.

[9] Auscher (P.), Axelsson (A.) & Hofmann (S.) – *Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems*, J. Funct. Anal., t. **255** (2008), pp. 374–448.

[10] Auscher (P.), Axelsson (A.) & McIntosh (A.) – *On a quadratic estimate related to the Kato conjecture and boundary value problems*, Contemp. Math., t. **205** (2010), pp. 105–129.

[11] ———, *Solvability of elliptic systems with square integrable boundary data*, Ark. Mat., t. **48** (2010), pp. 253–287.

[12] Auscher (P.), Axelsson (A.) & Russ (E.) – *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal., t. **18** (2008), pp. 192–248.

[13] Auscher (P.), Hofmann (S.), Lacey (M.), McIntosh (A.) & Tchamitchian (P.) – *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2), t. **156** (2002), pp. 633–654.

[14] ———, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2), t. **156** (2002), pp. 633–654.

[15] Auscher (P.), Hofmann (S.) & Martell (J.-M.) – *Vertical versus conical square functions*, Trans. Amer. Math. Soc., t. **364** (2012), pp. 5469–5489.

[16] Auscher (P.), Hofmann (S.), McIntosh (A.) & Tchamitchian (P.) – *The Kato square root problem for higher order elliptic operators and systems on* $\mathbb{R}^n$, J. Evol Equ., t. **1** (2001), pp. 361–385.

[17] Auscher (P.) & Martell (J.-M.) – *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Adv. Math., t. **212** (2007), pp. 225–276.

[18] ———, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part II: Off-diagonal estimates on spaces of homogeneous type*, J. Evol. Equ., t. **7** (2007), pp. 265–316.

[19] Auscher (P.), McIntosh, A. & Morris (A.) – *Calderón reproducing formulas and applications to Hardy spaces*, Rev. Math. Iberoamericana, t. **31** (2015), pp. 865–900.

[20] Auscher (P.), McIntosh (A.) & Mourgoglou (M.) – *On* L^2 *solvability of BVPs for elliptic systems*, J. Fourier Anal. Appl., t. **19** (2013), pp. 478–494.

[21] Auscher (P.), McIntosh (A.) & Nahmod (A.) – *Holomorphic functional calculi of operators, quadratic estimates and interpolation*, Indiana J. Math., t. **46** (1997), pp. 375–403.

[22] Auscher (P.), McIntosh (A.) & Russ (E.) – *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal., t. **18** (2008), pp. 192–248.

[23] Auscher (P.) & Mourgoglou (M.) – *Boundary layers, Rellich estimates and extrapolation of solvability for elliptic systems*, Proc. London Math. Soc., to appear, doi-10.1112/plms/pdu008.

[24] ———, *Representation and uniqueness for boundary value elliptic problems via first order systems*, arXiv:1404.2687v1.

[25] Auscher (P.) & Rosén (A.) – *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems II*, Analysis and PDE, t. **5** (2012), pp. 983–1061.

[26] Auscher (P.) & Stahlhut (S.) – *A priori estimates for boundary value elliptic problems via first order systems*, in preperation.

[27] ———, Remarks on functional calculus for perturbed first order Dirac operators, p. 31–43 in *Operator theory in harmonic and non-commutative analysis*, Oper. Theory Adv. Appl.

[28] Auscher (P.) & Tchamitchian (P.) – *Calcul fonctionnel précisé pour des opérateurs elliptiques complexes en dimension un et applications à certaines équations elliptiques complexes en dimension deux*, Ann. Inst. Fourier, t. **45** (1995), pp. 721–778.

[29] Axelsson (A.), Keith (S.) & McIntosh (A.) – *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math., t. **163** (2006), pp. 455–497.

[30] Badr (N.) – *Real interpolation of Sobolev spaces*, Math. Scand., t. **105** (2009), pp. 235–264.

[31] Barton (A.) – *Elliptic partial differential equations with almost real coefficients*, Memoir Amer. Math. Soc., t. **223** (2013).

[32] Barton (A.) & Mayboroda (S.) – *Layer Potentials and Boundary-Value Problems for Second Order Elliptic Operators with Data in Besov Spaces*, ArXiv:1309.5404v1 [math.AP], to appear in Memoir Amer. Math. Soc.

[33] Bernal (A.) – Some results on complex interpolation of T_q^p spaces, Interpolation spaces and related topics, in *Israel Mathematical Conference Proceedings*, Ramat-Gan, t. **5** (1992), pp. 1–10.

[34] Bernicot (F.) – *A $T(1)$-theorem in relation to a semigroup of operators and applications to new paraproducts*, Trans. Amer. Math. Soc., t. **364** (2012), pp. 6071–6108.

[35] Bernicot (F.) & Frey (D.) – *Pseudodifferential operators associated with a semigroup of operators*, J. Fourier Anal. Appl., t. **20** (2014), pp. 91–118.

[36] Blunck (S.) & Kunstmann (P. C.) – *Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus*, Rev. Mat. Iberoamericana, t. **19** (2003), pp. 919–942.

[37] Brown (R.) – *The Neumann problem on Lipschitz domains in Hardy spaces of order less than one*, Pacific J. Math., t. **171** (1995), pp. 389–407.

[38] Bui (The Anh) & Duong (Xuan Thinh) – *Weighted Hardy spaces associated to operators and boundedness of singular integrals*, preprint, arXiv:1202.2063v2.

[39] Coifman (R.), McIntosh (A.) & Meyer (Y.) – *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Math., t. **116** (1982), pp. 361–387.

[40] Coifman (R. R.), Meyer (Y.) & Stein (E. M.) – *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal., t. **62** (1985), pp. 304–335.

[41] Cowling (M.), Doust (I.), McIntosh (A.) & Yagi (A.) – *Banach space operators with a bounded H^∞ functional calculus*, J. Austral. Math. Soc. (Series A), t. **60** (1996), pp. 51–89.

[42] Dahlberg (B.) & Kenig (C.) – *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. Math., t. **125** (1987), pp. 437–465.

[43] Davies (E. B.) – *Heat kernel bounds, conservation of probability and the Feller property*, J. Anal. Math., t. **58** (1992), pp. 99–119.

[44] Dindos (M.), Kenig (C.) & Pipher (J.) – *BMO solvability and the A_∞ condition for elliptic operators*, J. Geom. Anal., t. **21** (2011), pp. 78–95.

[45] Dindos (M.) & Kirsch (J.) – *The regularity problem for elliptic operators with boundary data in Hardy-Sobolev space HS^1*, Math. Res. Lett., t. **19** (2012), pp. 699–717.

[46] Duong (X.T.) & Li (J.) – *Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus*, J. Funct. Anal., t. **264** (2013), pp. 1409–1437.

[47] Duong (X.T.) & Yan (L.) – *New Function Spaces of BMO Type, the John-Nirenberg Inequality, Interpolation and Applications*, Comm. Pure Applied Math., t. **LVIII** (2005), pp. 1375–1420.

[48] Fefferman (C.) & Stein (E. M.) – *H^p spaces of several variables*, Acta Math., t. **129** (1972), pp. 137–193.

[49] Frey (D.) & Kunstmann (P. C.) – *A $T(1)$-theorem for non-integral operators*, Math. Annalen, t. **357** (2013), pp. 215–278.

[50] Frey (D.), McIntosh (A.) & Portall (P.) – *Conical square functions estimates and functional calculus for perturbed hodge-dirac operators in L^p*, arXiv:1407.4774.

[51] Gaffney (M. P.) – *The conservation property of the heat equation on riemannian manifolds*, Comm. Pure Appl. Math., t. **12** (1959), pp. 1–11.

[52] Grafakos (L.) – *Classical Fourier Analysis,* 2nd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008.

[53] Haase (M.) – *Functional Calculus for Sectorial Operators*, Birkhäuser Verlag, 2006.

[54] Hofmann (S.), Kenig (C.), Mayboroda (S.) & Pipher (J.) – *Square function/Non-tangential maximal estimates and the Dirichlet problem for non-symmetric elliptic operators*, J. Amer. Math. Soc., t. **28** (2015), pp. 483–529.

[55] Hofmann (S.), Kenig (S.), C. Mayboroda & Pipher (J.) – *The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients*, Math. Ann., t. **361** (2015), pp. 863–907.

[56] Hofmann (S.) & Kim (S.) – *The green function estimates for strongly elliptic systems of second order*, Manuscripta Math., t. **124** (2007), pp. 139–172.

[57] Hofmann (S.), Lu (G.), Mitrea (D.), Mitrea (M.) & Yan (L.) – *Hardy spaces associated to non-negative self-adjoint operators satisfying David-Gaffney estimates*, Mem. Amer. Math. Soc., 2011.

[58] Hofmann (S.) & Martell (J. M.) – *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators*, Publ. Mat., t. **47** (2003), pp. 497–515.

[59] Hofmann (S.) & Mayboroda (S.) – *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann., t. **344** (2009), pp. 37–116, *Corrigendum* arXiv:0907.0129v2 [math.AP].

[60] Hofmann (S.), Mayboroda (S.) & McIntosh (A.) – *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces*, Ann. Sci. Éc. Norm. Supér. (4), t. **44** (2011), pp. 723–800.

[61] Hofmann (S.), Mitrea (M.) & Morris (A.) – *The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ coefficients*, Proc. London Math. Soc., t. **111** (2015), pp. 681–716.

[62] Hofmann (Steve) – *Local $T(b)$ theorems and applications in PDE. Harmonic analysis and partial differential equations,* pp. 29-52, Contemp. Math., vol. 505, Amer. Math. Soc., Providence, RI, 2010.

[63] Hytönen (T.) & McIntosh (A.) – Stability in p of the H^∞-calculus of first-order systems in L^p, in *The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, Proc. Centre Math. Appl. Austral. Nat. Univ., 44, Austral. Nat. Univ., Canberra*, 2010, pp. 167–181.

[64] Hytönen (T.), McIntosh (A.) & Portal (P.) – *Kato's square root problem in Banach spaces*, J. Funct. Anal., t. **254** (2008), pp. 675–726.

[65] ______, *Holomorphic functional calculus of Hodge-Dirac operators in L^p*, J. Evol. Equ., t. **11** (2011), pp. 71–105.

[66] Hytönen (T.), van Neerven (J.) & Portal (P.) – *Conical square functions estimates in UMD Banach spaces and applications to H^∞-functional calculi*, J. Anal. Math., t. **106** (2008), pp. 317–351.

[67] Jiang (R.) & Yang (D.) – *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, J. Func. Anal., t. **258** (2010), pp. 1167–1224.

[68] Kalton (N.) – *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math., t. **84** (1986), pp. 297–324.

[69] Kalton (N.) & Mitrea (M.) – *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc., t. **350** (1998), pp. 3903–3922.

[70] Kalton (N.) & Weis (L.) – *The H^∞-calculus and sums of closed operators*, Math. Ann., t. **321** (2001), pp. 319–345.

[71] Kenig (C.) & Meyer (Y.) – The Cauchy integral on Lipschitz curves and the square root of second order accretive operators are the same, in Peral (ed.), *Recent progress in Fourier analysis*, Math. Studies, vol. 111, North Holland, 1985.

[72] Kenig (C.) & Pipher (J.) – *The Neumann problem for elliptic equations with non smooth coefficients*, Invent. Math., t. **113** (1993), pp. 447–509.

[73] Kenig (C.) & Rule (D.) – *The regularity and Neumann problem for non-symmetric elliptic operators*, Trans. Amer. Math. Soc., t. **361** (2009), pp. 125–160.

[74] Koskela (P.) & Saksman (E.) – *Pointwise characterizations of Hardy-Sobolev functions*, Math. Res. Lett., t. **15** (2008), pp. 727–744.

[75] Le Merdy (C.) – *On square functions associated to sectorial operators*, Bull. Soc. Math. France, t. **132** (2004), pp. 137–156.

[76] ———, Square functions, bounded analytic semigroups, and applications, pp. 191–220, in *Perspectives in operator theory*, vol. 75, Banach Center Publ., Polish Acad. Sci., Warsaw, 2007.

[77] Martell (J.-M.) – *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math., t. **161** (2004), pp. 113–145.

[78] Mayboroda (S.) – *The connections between Dirichlet, Regularity and Neumann problems for second order elliptic operators with complex bounded measurable coefficients*, Adv. Math., t. **225** (2010), pp. 1786–1819.

[79] McIntosh (A.) – Operators which have an H_∞ functional calculus, pp. 210–231, in *Miniconference on operator theory and partial differential equations (North Ryde, 1986)*, vol. 14, Proc. Centre Math. Anal. Austral. Nat. Univ, Canberra., 1986.

[80] Pazy (A.) – *Semigroups of linear operators and applications to partial differential equations*, Springer, 1983.

[81] Rosén (A.) – *Layer potentials beyond singular integral operators*, Pub. Math., t. **57** (2013), pp. 429–454.

[82] ———, *Square function and maximal function estimates for operators beyond divergence form equations*, J. Evol. Equ., t. **13** (2013), pp. 651–674.

[83] ———, Cauchy non-integral formulas, p. 163–178 in *Harmonic Analysis and Partial Differential Equations*, Contemp. Math., vol. 612, Amer. Math. Soc., 2014.

[84] Šneĭberg (I. Ja.) – *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled., t. **9(2(32))** (1974), pp. 214–229, 254–255.

[85] Stahlhut (S.) – *L^p-L^q theory for holomorphic functions of perturbed first order operators*, in preparation.

[86] ———, *Problèmes aux limites pour les systèmes elliptiques*, Ph.D. Thesis, Université Paris-Sud 11, 2014.

[87] Stein (E. M.) – *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, NJ, 1970.