**Torsten Schoeneberg**

# SEMISIMPLE LIE ALGEBRAS AND THEIR CLASSIFICATION OVER $\mathfrak{p}$-ADIC FIELDS

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# SEMISIMPLE LIE ALGEBRAS AND THEIR CLASSIFICATION OVER $\mathfrak{p}$-ADIC FIELDS

**Torsten Schoeneberg**

***Abstract.* —** We study semisimple Lie algebras over fields of characteristic zero, with emphasis on $\mathfrak{p}$-adic fields and aiming at classification. We first transfer parts of the structure theory of reductive algebraic groups to our setting, with some variations. Classifying invariants are attached to Lie algebras and visualised with Satake-Tits diagrams. We give necessary and sufficient criteria for these diagrams. Over general fields of characteristic zero, we then classify all quasi-split forms, and we adapt an older classification theory for the classical types $A$-$D$ to our language. Finally we focus on $\mathfrak{p}$-adic fields, where we achieve a complete classification by combination of certain well-known properties of these fields with our general results and methods, and we discuss the relation of this with a theorem of Kneser. This extends work by Weisfeiler.

***Résumé* (Algèbres de Lie semi-simples et leur classification sur les corps p-adiques)**

Nous étudions les algèbres de Lie semi-simples sur les corps de caractéristique nulle, où l'accent est mis sur les corps $p$-adiques, l'objectif étant leur classification. Nous transférons d'abord certaines parties de la théorie de la structure des groupes réductifs dans notre contexte, avec quelques variations. Des invariants classifiants sont attachés aux algèbres de Lie et sont visualisés à l'aide de diagrammes de Satake-Tits. Nous donnons des critères nécessaires et suffisants pour ces diagrammes. Sur les corps généraux de caractéristique nulle, nous classifions ensuite toutes les formes quasi-déployées et nous traduisons une théorie ancienne de classification pour les types classiques $A$-$D$ dans notre langue. Nous mettons enfin l'accent sur les corps $p$-adiques, où nous obtenons une classification complète par combinaison de certaines propriétés bien connues sur ces corps avec nos résultats généraux et nos méthodes, et nous abordons la relation de ces résultats avec un théorème de Kneser. Tout cela prolonge un travail de Weisfeiler.

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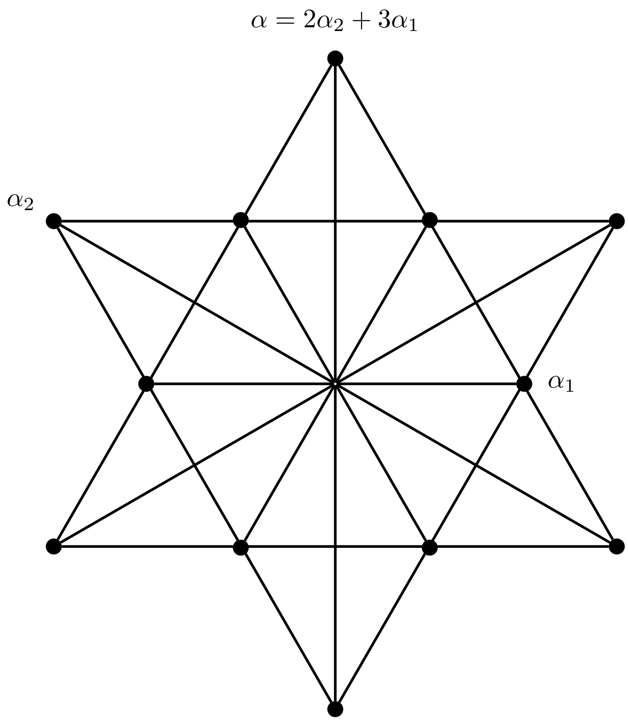
To the memory of Michel Lazard and Boris Weisfeiler

*[. . . ] Und es starben*
*Noch andere viel. Mit eigener Hand*
*Viel traurige, wilden Muths, doch göttlich*
*Gezwungen, zulezt, die anderen aber*
*Im Geschike stehend, im Feld. [. . . ]*

Hölderlin, *Mnemosyne*



# CHAPTER 1

# INTRODUCTION

In 1964, Boris Weisfeiler published the short note [**We1**] on the classification of semisimple Lie algebras over a $\mathfrak{p}$-adic field. It is the purpose of this work to explain this note, append proofs to it, and put it into mathematical context.

Let us first give some mathematical-historical background. The Killing-Cartan classification of semisimple Lie algebras over the complex numbers $\mathbb{C}$ is one of the standard classification theorems of modern mathematics. Both Killing, who had the results, and E. Cartan, who gave streamlined proofs in his *Thèse* of 1894, were interested in "infinitesimal groups" – as they were called then – mainly as a means to study Lie groups. In this spirit, Cartan also achieved the classification over the ground field $\mathbb{R}$. It is no wonder that fundamental concepts in the theory are named after them.

Likewise, the *Weyl group* is named after Weyl, who introduced this powerful tool in his approach to the subject in the 1920s. Cartan had already reduced the classification to that of *root systems*. Whereas he had achieved this classification in a computational way, Weyl's methods paved a new route, which was then taken by Coxeter, Witt and Dynkin. It also became clear that the classification over $\mathbb{C}$ really is one of the *split* semisimple Lie algebras, which are the only ones over an algebraically closed field. With the works of Chevalley and Cartier's "Séminaire Sophus Lie" in the 1950s, this theory seems to have reached its definite form, in which it was presented in Bourbaki's fundamental treatment [**Bo2**] and found its way in many textbooks. Most mathematicians will have heard of the four classical families $A$-$D$ and the five exceptional types $E_6, E_7, E_8, F_4$ and $G_2$.

But Weyl had paved more than one way. He was the one who (orally) introduced the name "Lie algebra", and he promoted the subject as a means in itself which could be treated analogously to associative algebras – which, at the same time, were researched thoroughly by algebraists and number theorists, over various ground fields. Such a study for Lie algebras was initiated independently by Landherr [**Lan1**] and Jacobson [**Jac1**] in 1935. Landherr made the fundamental observation that one can start from the known classification over an algebraically closed field and then investigate *forms* of the respective types: that is, take one type from the Killing-Cartan list and then find, over your ground field $k$, those Lie algebras which fall in this type after scalar extension to an algebraic closure $\bar{k}$. In modern terminology, he achieved this a) for the *inner forms* of type $A$ over a general field of characteristic 0, and b) for all forms of type $A$ over $\mathbb{R}$ (reproving Cartan's results) and all $\mathfrak{p}$-adic fields, thinking in the direction of number fields and a local-global principle like the one Hasse had introduced for associative algebras. Jacobson on the other hand found a method that could deal with (almost) all the other classical types, i.e. $B$-$D$ and the *outer forms* of type $A$, over any field of characteristic 0. Both Landherr's and Jacobson's approach can be described as enveloping the Lie algebras with certain associative algebras and then using known classification results for these. Jacobson and others went on to find similar approaches to the exceptional types, using various other kinds of algebras.

Meanwhile, after the Lie algebras had stepped out of the shadow of Lie groups, the theory of *algebraic groups* had emerged and, with the revolution in algebraic geometry, soon adopted a generality which left Lie algebras behind. With Weil's fundamental paper [**Wei**], the Landherr-Jacobson approach could be seen as a variant of a special case of *Galois cohomology* at work.

Mathematicians now aimed at a classification of semisimple, or more generally reductive, algebraic groups over fields (and then, schemes) as general as possible. It was certainly understood, not only by Weisfeiler, that by their close connection in characteristic 0, this would comprise the classification of semisimple Lie algebras. Many people worked on the group case, where the Borel-Tits paper [**BT1**] was a highlight, not least because of the generality of the ground field. Although Tits had announced most of the results earlier, it was Satake who had first published (in [**Sat2**]) proofs for some central theorems, at least over perfect ground fields. He would later extend his approach in [**Sat3**]. On the other hand, the scope of Tits' "Table II" in [**Tit1**] is intimidating. Satake's and Tits' visualisations of the classification are remarkably similar, and these *Satake-Tits diagrams* will abound in this work. They are also at the heart of Weisfeiler's note.

We proceed as follows.

▹ Chapter 2 collects some preliminaries and gives references for results we will use freely. In particular, we assume the classification of the split case via root systems and Dynkin diagrams to be known.

▹ In Chapter 3 we start with the structure theory of semisimple Lie algebras over any field $k$ of zero characteristic. Our general approach is to translate Satake's method to our setting, which is certainly what Weisfeiler did. A central concept are the *toral subalgebras* which are the analogue of tori. There is, however, one major deviation from Satake's method: namely, we start with a self-contained construction of $k$-rational root systems. This generalisation of the well-known split case is due to Seligman [**Sel2**] and more or less straightforward, its most profound ingredient being the Jacobson-Morozov theorem. Then we translate a bulk of Satake's and Borel-Tits' work to Lie algebras. One major tool is the elementary but fruitful notion of $\Gamma$-*bases* which bring root system combinatorics in balance with Galois actions. A recurring theme is the treatment of the relative Weyl group: its realisation as a subquotient of the absolute Weyl group, and simultaneously as a quotient of certain automorphism groups of the Lie algebra, is a highlight of this chapter. To achieve it, we use the $k$-rational root system and thereby circumvent some technical parts of Satake's method.

The main goal, however, is to attach two invariants to a semisimple Lie algebra, its *index* and its *anisotropic kernel*. In Section 3.1, we define them and show that in a natural manner they only depend on the isomorphism class of the Lie algebra. In Section 3.2 we discuss opposite extremes of these invariants – the *anisotropic* and *(quasi-)split* cases –, which play a major role in the classification, and we give basic but instructive examples. In Section 3.3 we prove the isomorphism theorem which describes how the invariants indeed classify semisimple Lie algebras. In Section 3.4 we finally introduce the Satake-Tits diagrams that visualise large parts of the invariants. At this point, the reader should be able to read Weisfeiler's table.

▷ In Chapter 4 we begin to prove its correctness. The question is now largely translated to: which diagrams can occur over a given field $k$? It turns out in Section 4.1 that one can reduce this question to connected diagrams. In Section 4.2, we bring Galois cohomology into play; it gives a powerful tool as well as the clarifying terminology of *outer and inner forms of various types*. Then we present, in Sections 4.3 and 4.4, some general principles which exclude or produce *admissible* diagrams, or certain forms, over *any* characteristic zero field.

In this generality we also present the classification of the classical types in Section 4.5. The key ideas here are still the ones of Landherr and Jacobson, polished in the spirit of Galois cohomology, so that e.g. the appearance of *Brauer groups* is not too surprising. This approach is covered in the textbooks [**Jac6**] and [**Sel1**], but we only follow it for a while. Namely, their classification translates to one of involutorial algebras, which in turn are classified by certain equivalence classes of (skew-)hermitian spaces. This is good and fine, but we prefer to classify with the theory of Chapter 3 by assigning the corresponding Satake-Tits diagrams. Proceeding this way, some of the more delicate uniqueness issues in the associative-algebraic classification can be avoided. Indeed its full strength will only be needed for inner forms of type $A$, which had already been treated completely in Landherr's very first paper. In Section 4.6, we discuss and construct all quasi-split Lie algebras, whose classification is equivalent to that of quadratic and cubic extensions of the ground field.

▷ In Chapter 5 we turn to special fields. The $\mathfrak{p}$-adic fields are our primary target, but it is useful to look first at C1 fields. In the language of Serre's Galois cohomology [**Ser3**], they are one-dimensional, and with a result by Springer we show in

Section 5.1 that over these fields all semisimple Lie algebras are quasi-split. Now $\mathfrak{p}$-adic fields are not of this kind – cohomologically, they are rather two-dimensional –, but their maximal unramified extensions are, and this suffices to show Weisfeiler's first theorem:

THEOREM (1). — *If $\mathfrak{g}$ is a semisimple Lie algebra over a $\mathfrak{p}$-adic field $k$, then there is a finite unramified extension $\mathfrak{K}|k$ such that $\mathfrak{g}_{\mathfrak{K}}$ is quasi-split.*

This implies that for most forms there exist unramified (in particular, cyclic) splitting extensions. In Section 5.2 we classify the inner forms of type $A$ over $\mathfrak{p}$-adic fields, and then discuss Weisfeiler's second theorem which says that anisotropic forms over $\mathfrak{p}$-adic fields are very rare – we have already encountered all of them:

THEOREM (2). — *All simple anisotropic Lie algebras over a $\mathfrak{p}$-adic field $k$ are derived algebras of $k$-division algebras. In particular, they are inner forms of type $A$.*

We connect it with Kneser's theorem [**Kne**] about the vanishing of a certain cohomology group. On the basis of what we have until there, Kneser's result implies both theorem (2) and Weisfeiler's table, hence the complete classification over $\mathfrak{p}$-adic fields. However, Weisfeiler's work had appeared earlier, and it is not clear from [**We1**] how he arrived at his results, or even whether he concluded the theorem from the table or vice versa.

We undertake the verification in Sections 5.3 to 5.5. Specifically, in 5.3. we collect some technical facts and rule out certain anisotropic exceptional forms. In 5.4 we verify both theorem (2) and the table for the classical types. [1] The special $\mathfrak{p}$-adic ingredient here is a well-known theorem which says that all hermitian, skew-hermitian and related forms in more than very few variables are isotropic; apart from this, we only use our general results from Chapter 4. In Section 5.5 we attack the exceptional types. For some diagrams, suitable Lie algebras are constructed with the techniques of Chapter 4. It turns out that we can exclude all other diagrams with our general methods and the results for the classical types – except for the anisotropic ones. To exclude these, we extend the results of 5.3, which works for almost all the anisotropic forms. The only ones for which we still have to rely on Kneser's theorem are one inner form of type $E_6$, and one trialitarian form of type $D_4$ (although to be fair, the classification in types $E_7$ and $E_8$ rests on the result for $E_6$).

▷ In Chapter 6 we very briefly address the case $k = \mathbb{R}$ and finish with concluding remarks about other approaches.

1. One outer form of type $D_{2m}$ was overlooked by Weisfeiler, see Section 5.4.4.

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# CHAPTER 2

# PREREQUISITES

*Notations*:

▷ $n$ is a (natural) number and $p$ is a prime.

▷ $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{F}_p$ have the usual meanings, likewise $M_{m\times n}(\,.\,)$, $M_n(\,.\,)$ and $\mathrm{GL}_n(\,.\,)$.

▷ $\mathbb{F}_q$ denotes a finite extension of $\mathbb{F}_p$.

▷ We use Gauß' floor function $\lfloor x \rfloor$ and Euler's totient function $\phi(n)$.

▷ For a ring $R$, the group of units is denoted by $R^*$ or $R^\times$. For a vector or matrix $X$, the transpose is denoted by ${}^tX$.

▷ We write $I_n$ for the unit matrix in $\mathrm{GL}_n$, and $E_{ij}$ is the matrix with $(i, j)$-th entry 1 and all other entries 0. A diagonal matrix is denoted by $\mathrm{diag}(x_1, \dots, x_n)$.

## 2.1. Lie algebras and root systems

Definition 2.1.1. — Let $R$ be a commutative unital ring. A *Lie algebra* over $R$ is an $R$-module $\mathfrak{g}$ together with an $R$-bilinear map – the *Lie bracket* –

$$[\,.\,,.\,]:\ \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g},\qquad (x,y)\longmapsto[x,y]$$

satisfying, for all $x, y, z \in \mathfrak{g}$, the relations

$$[x,x]=0\quad\text{and}\quad [x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

We will only deal with the case where $R$ is a field, so that $\mathfrak{g}$ is a vector space, which is most often assumed to be *finite dimensional*. In the following, a ring denoted by the letter $k$ will always be meant to be a field.

For the concepts of *ideals* and *subalgebras*, *products* of Lie algebras, *abelian* (= *commutative*), *nilpotent* and *solvable* Lie algebras, as well as the *centre* $\mathfrak{z}(\mathfrak{g})$, the *radical* $\mathfrak{r}(\mathfrak{g})$, and the *derived* Lie algebra $\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, the *derived series* $\mathcal{D}^i\mathfrak{g}$, *descending central series* $\mathcal{C}^i\mathfrak{g}$, and also *representations*, in particular the *adjoint representation*, and the *Killing form* of a Lie algebra, see [**Bo2**, I]. For subsets $H, K$ of a Lie algebra,

$[H, K]$ denotes the subalgebra generated by all $[h, k]$ for $h \in H, k \in K$. Further, for a subset $H$ of the Lie algebra $\mathfrak{g}$, we have its

▷ *centraliser* $\mathfrak{z}_{\mathfrak{g}}(H) := \{x \in \mathfrak{g} : [x, h] = 0 \text{ for all } h \in H\}$; it is a subalgebra, and if $H$ is an ideal of $\mathfrak{g}$, then so is $\mathfrak{z}_{\mathfrak{g}}(H)$;

▷ *normaliser* $\mathfrak{n}_{\mathfrak{g}}(H) := \{x \in \mathfrak{g} : [x, h] \in H \text{ for all } h \in H\}$; if $H$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{n}_{\mathfrak{g}}(H)$ is the largest among the subalgebras $\mathfrak{n}$ of $\mathfrak{g}$ such that $H$ is an ideal in $\mathfrak{n}$.

We state the concept of semisimplicity, as well as the stronger one of simplicity and the weaker one of reductivity. For the assertions in the following, see [**Bo2**, I.6].

Definition 2.1.2. — Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $k$.

i. $\mathfrak{g}$ is called *simple* if it is not abelian and its only ideals are $\mathfrak{g}$ and $\{0\}$.

   Now let $\mathrm{char}(k) = 0$.

ii. $\mathfrak{g}$ is called *semisimple* if the following equivalent conditions are satisfied:
   (a) $\mathfrak{g}$ is isomorphic to a product of simple Lie algebras.
   (b) The radical $\mathfrak{r}(\mathfrak{g})$ is zero.
   (c) $\mathfrak{g}$ does not contain any non-zero abelian ideal.
   (d) $\mathfrak{g}$ does not contain any non-zero solvable ideal.
   (e) The Killing form on $\mathfrak{g}$ is non-degenerate.

iii. $\mathfrak{g}$ is called *reductive* if the following equivalent conditions are satisfied:
   (a) $\mathfrak{g}$ is isomorphic to the product of an abelian and a semisimple Lie algebra.
   (b) $\mathfrak{g}$ is isomorphic to the product of its centre $\mathfrak{z}(\mathfrak{g})$ and its derived algebra $\mathfrak{D}\mathfrak{g}$.
   (c) $\mathfrak{D}\mathfrak{g}$ is semisimple.
   (d) The adjoint representation of $\mathfrak{g}$ is a direct sum of irreducible representations.
   (e) Radical and centre of $\mathfrak{g}$ are equal: $\mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

iv. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *reductive in* $\mathfrak{g}$ if the representation $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$, $h \mapsto \mathrm{ad}_{\mathfrak{g}} h$ is the direct sum of irreducible representations.

There are no semisimple Lie algebras of dimension 1 or 2, and over an algebraically closed $k$ there is up to isomorphism a unique simple one of dimension 3, namely, $\mathfrak{sl}_2(k)$.

In a semisimple Lie algebra $\mathfrak{g}$, we have well-behaved notions of *semisimple* and *nilpotent elements*, and we have *Jordan decomposition*: every element $x \in \mathfrak{g}$ can be written uniquely as the sum $x = x_s + x_n$ of a semisimple and a nilpotent one commuting with each other, called the *semisimple* and *nilpotent component* of $x$, respectively; further, $\mathrm{ad}_{\mathfrak{g}}(x_s)$ and $\mathrm{ad}_{\mathfrak{g}}(x_n)$ are the semisimple and nilpotent parts of $\mathrm{ad}_{\mathfrak{g}}(x) \in \mathrm{End}_k(\mathfrak{g})$, and can be written as polynomials in $\mathrm{ad}_{\mathfrak{g}}(x)$ without constant term. For all this, see [**Bo2**, I.6.3] and [**Bo1**, VII.5, especially nos. 7–9].(1) A subalgebra is called *nil* if it consists of nilpotent elements.

1. For semisimple and nilpotent components of a vector space endomorphism, the Bourbaki reference in earlier (Hermann) editions was *Algèbre* VIII.9.

We also mention the base change routines that we will often use:

Definition 2.1.3. — Let $K|k$ be a field extension.

i. If $\mathfrak{G}$ is a Lie algebra over $K$, then in the obvious way it can be considered as a Lie algebra over $k$. Call this Lie algebra $R_{K|k}\mathfrak{G}$, the *scalar restriction* of $\mathfrak{G}$ to $k$.
ii. If $\mathfrak{g}$ is a Lie algebra over $k$, then the tensor product $K \otimes_k \mathfrak{g}$ becomes a Lie algebra over $K$ when as Lie bracket $[.,.]_K$ we take the unique $K$-bilinear map satisfying
$$[a \otimes x, b \otimes y]_K = ab \otimes [x, y]$$
for $a, b \in K, x, y \in \mathfrak{g}$. Call this Lie algebra $\mathfrak{g}_K$, the *scalar extension* of $\mathfrak{g}$ to $K$.

Note that in scalar restriction, the usual assumption of finite dimension of the Lie algebra only remains intact for a *finite* extension $K|k$ (or if the algebra is zero).

If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, we can in an obvious way, and will without further notice, identify $\mathfrak{h}_K$ with a ($K$-)subalgebra of $\mathfrak{g}_K$. Also, we will occasionally identify $\mathfrak{g}$ with the ($k$-)subalgebra of $R_{K|k}\mathfrak{g}_K$ given by $1 \otimes \mathfrak{g} \subset \mathfrak{g}_K$.

We will use the notion of *Cartan subalgebras* (cf. [**Bo2**, VII.2]) which in our case are better viewed as the the *maximal toral subalgebras*, see Chapter 3 and especially 3.1.5. With these, the notions of *Borel subalgebras* and *parabolic subalgebras* are defined, first in the split case (cf. [**Bo2**, VIII.3.3–4]) and then in the general case as those that become Borel or parabolic after scalar extension to an algebraic closure (cf. [**Bo2**, VIII.3.5]). In particular, Borel subalgebras do not necessarily exist. In fact, the question of existence of Borel subalgebras will play an important role in our classification, and we will make use of the results of [**Bo2**, VIII.10] which very roughly say that "many nilpotent elements" is equivalent to "smaller parabolic subalgebras".

The *rank* $\mathrm{rk}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and the notion of a *regular element* are defined in [**Bo2**, VII.2]. If $\mathrm{char}(k) = 0$, all Cartan subalgebras of $\mathfrak{g}$ have dimension $\mathrm{rk}(\mathfrak{g})$, and they are precisely the $\mathfrak{g}^0(x)$ for regular elements $x$. Here,
$$\mathfrak{g}^0(x) := \{y \in \mathfrak{g} : (\mathrm{ad}_{\mathfrak{g}}(x)^n)(y) = 0 \text{ for some } n \geq 1\}.$$
For semisimple $x$, we have $\mathfrak{g}^0(x) = \ker(\mathrm{ad}_{\mathfrak{g}}(x))$, and the dimension of this space is $\mathrm{rk}(\mathfrak{g})$ if and only if $x$ is regular; else it is strictly larger.

Remark 2.1.4 (Stabilities under scalar extension). — Let $K|k$ be an extension of fields of characteristic zero, and let $\mathfrak{g}$ be a Lie algebra over $k$.

i. We have $(\mathfrak{z}(\mathfrak{g}))_K = \mathfrak{z}(\mathfrak{g}_K)$; and for any subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we have $(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}))_K = \mathfrak{n}_{\mathfrak{g}_K}(\mathfrak{h}_K)$, see [**Bo2**, I.3.8]. (The analogous statement for the centraliser is wrong in general.)
ii. If $\kappa(.,.)$ is the Killing form of $\mathfrak{g}$, the Killing form of $\mathfrak{g}_K$ is $\kappa_K(a \otimes x, b \otimes y) = ab\,\kappa(x, y)$, see [**Bo2**, I.3.8].
iii. The following properties are satisfied by $\mathfrak{g}$ if and only if they are satisfied by $\mathfrak{g}_K$:

(a) Being semisimple, see [**Bo2**, I.6.10].
(b) Being reductive, see [**Bo2**, I.6.10].
(c) Being nilpotent, see [**Bo2**, I.4.5].
(d) Being solvable, see [**Bo2**, I.5.6].
(e) Being a Cartan subalgebra (of another Lie algebra $\mathfrak{g}'$ resp. $\mathfrak{g}'_K$), see [**Bo2**, VII.2.2, Prop. 3].

iv. If $\mathfrak{g}_K$ is simple, $\mathfrak{g}$ is simple, see [**Bo2**, I.6.10]. The converse is not true in general, cf. Section 4.1; at least by the above, $\mathfrak{g}_K$ is semisimple for simple $\mathfrak{g}$.

Remark 2.1.5 (Stabilities under scalar restriction). — Let $K|k$ be a *finite* extension of fields of characteristic zero, and let $\mathfrak{G}$ be a Lie algebra over $K$. Set $\mathfrak{g} := R_{K|k}\mathfrak{G}$. We have $\mathfrak{z}(\mathfrak{G}) = \mathfrak{z}(\mathfrak{g})$. The following properties are satisfied by $\mathfrak{G}$ if and only if they are satisfied by $\mathfrak{g}$:

i. Being simple, see [**Bo2**, I.6.10].
ii. Being semisimple, see [**Bo2**, I.6.10].
iii. Being nilpotent: follows from [**Bo2**, I.1.9].
iv. Being solvable: follows from [**Bo2**, I.1.9].

We will make use of the *theorems of Engel* (cf. [**Bo2**, I.4.2]), *Levi-Malcev* (cf. [**Bo2**, I.6.8]) and *Jacobson-Morozov* (cf. [**Bo2**, VIII.11]) for which, however, we use a slightly different normalisation:

Definition 2.1.6. — An $\mathfrak{sl}_2$*-triple* in a Lie algebra $\mathfrak{g}$ is a triple $(x, h, y)$ of elements of $\mathfrak{g}$ distinct from $(0, 0, 0)$ and such that

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

(Bourbaki's $y$ is our $-y$.)

Accordingly, for us the Lie algebra $\mathfrak{sl}_2(k)$ has the standard basis:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Jacobson-Morozov theorem states that every nilpotent element $x \neq 0$ in a semisimple Lie algebra $\mathfrak{g}$ can be extended to an $\mathfrak{sl}_2$-triple $(x, h, y)$. An $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is the same as a (necessarily injective) homomorphism $\mathfrak{sl}_2(k) \hookrightarrow \mathfrak{g}$ and thus a way to view $\mathfrak{g}$ as a representation of $\mathfrak{sl}_2(k)$.

From this *representation theory* (cf. [**Bo2**, VIII.1]) we use the following. Let $E$ be a representation of $\mathfrak{sl}_2(k)$, and for an eigenvalue $\lambda$ of $H$ in $E$ (a "weight"), denote the corresponding eigenspace by $E_\lambda$. The following facts are stated in or follow immediately from *loc. cit.* Proposition 2 and its Corollaire (in particular, part iii is Corollaire (ii) with $1 = i \geq p = 2 - \lambda$).

Lemma 2.1.7. — *Let $E$ be a finite-dimensional representation of $\mathfrak{sl}_2(k)$.*

i. *If $\lambda$ is a weight, then it is an integer, and $|\lambda|, |\lambda| - 2, \dots, -|\lambda|$ are weights too. Consequently, the largest and smallest weights are integers symmetric around* 0*, i.e. they are of the form $\lambda$ and $-\lambda$ for $\lambda \in \mathbb{Z}_{\geq 0}$.*
ii. *$E$ is the direct sum of the $E_\lambda$.*
iii. *If $\lambda > 0$ is an eigenvalue of $H$ in $E$, then $X$ maps $E_{\lambda-2}$ onto $E_\lambda$.*

Another nice application is:

LEMMA 2.1.8. — *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an inclusion of semisimple Lie algebras. Then every $x \in \mathfrak{h}$ which is nilpotent in $\mathfrak{h}$ is nilpotent in $\mathfrak{g}$.*

Namely, by Jacobson-Morozov, there is a homomorphism $\mathfrak{sl}_2 \hookrightarrow \mathfrak{h}$ sending $X$ to $x$, and viewing $\mathfrak{g}$ as $\mathfrak{sl}_2$-representation with respect to this gives that $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}(X)$ is nilpotent.

For *automorphisms of (semisimple) Lie algebras*, we follow [**Bo2**, VIII.5]. In particular:

DEFINITION 2.1.9. — We have the chain of subgroups $\mathrm{Aut}_e(\mathfrak{g}) \subset \mathrm{Aut}_0(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$, with strict inclusions in general, where $\mathrm{Aut}_e(\mathfrak{g})$ are the *elementary automorphisms*, that is, finite products of

$$\exp(\mathrm{ad}\, n) := x \longmapsto \sum_{k \geq 0} \frac{(\mathrm{ad}_{\mathfrak{g}}(n))^k}{k!}(x)$$

for nilpotent $n \in \mathfrak{g}$; and $\mathrm{Aut}_0(\mathfrak{g})$ are those automorphisms that become elementary after scalar extension to an algebraic closure. For a subset $H \subseteq \mathfrak{g}$ we set

$$\mathrm{Aut}(\mathfrak{g}, H) = \{f \in \mathrm{Aut}(\mathfrak{g}) : f(H) = H, \quad \mathrm{Aut}_*(\mathfrak{g}, H) = \mathrm{Aut}_*(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g}, H)$$

for $* \in \{0, e\}$.

We presuppose the classification of *split Lie algebras* via root systems as in [**Bo2**, VIII.4]; also, the matrix description of the "classical types" $A$-$D$ in [**Bo2**, VIII.13] or [**Jac6**, IV. 6]. For *root systems*, the general reference is [**Bo2**, VI]. Let us state some basics in the following.

Let $V$ be vector space over a field $k$ with $\mathrm{char}(k) = 0$, and let $V^*$ be its dual. For $\alpha \in V, \alpha^* \in V^*$, we define an element $s_{\alpha,\alpha^*} \in \mathrm{End}_k(V)$ by:

$$s_{\alpha,\alpha^*}(x) := x - \alpha^*(x)\alpha \tag{1}$$

If $\alpha^*(\alpha) = 2$, this is a *reflection* in the sense of [**Bo2**, V.2], in particular $s^2_{\alpha,\alpha^*} = \mathrm{id}_V$.

DEFINITION 2.1.10

i. A *root system* (in $V$) is a subset $R \subset V$ that satisfies:

(RS1) $R$ is finite, generates $V$, and $0 \notin R$.

(RS2) For all $\alpha \in R$, there exists $\check{\alpha} \in V^*$ with $\check{\alpha}(\alpha) = 2$ and $s_{\alpha,\check{\alpha}}(R) \subseteq R$.

(RS3) For all $\alpha \in R$, $\check{\alpha}(R) \subseteq \mathbb{Z}$.

As explained in [**Bo2**, VI.1.1], $\check{\alpha}$ in (RS2) is uniquely determined by $\alpha$ (and thus (RS3) is to be understood). For brevity, set

$$s_\alpha := s_{\alpha,\check{\alpha}}.$$

ii. For a root system $R$ in $V$, we call
 - ▷ $\operatorname{rank}(R) := \dim_k V$ the *rank* of $R$,
 - ▷ $A(R) := \{\sigma \in \operatorname{Aut}_k(V) : \sigma(R) = R\}$ the *automorphism group* of $R$, and
 - ▷ $W(R) :=$ subgroup of $A(R)$ generated by the $s_\alpha$ for $\alpha \in R$ the *Weyl group* of $R$.

iii. The root system $R$ is called *reduced* if whenever $\alpha = c \cdot \beta$ for $\alpha, \beta \in R$, we have $c \in \{1, -1\}$.

We refer to [**Bo2**, VI] for the plentitude of properties that follow from these definitions. In particular, we will freely use the translation of the theory to and from the ground fields $\mathbb{Q}$ and $\mathbb{R}$, the relations and proportions between roots in a root system, and the decomposition of a root system into *irreducible* root systems.

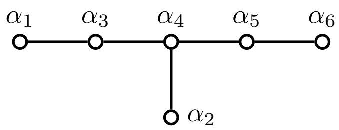
Remark 2.1.11. — Let $R$ be a root system (in the real vector space $V$). For $\sigma \in A(R)$, we have

$$\big(\sigma(\alpha)\check{\ }\big)\big(\sigma(\beta)\big) = \check{\alpha}(\beta)$$

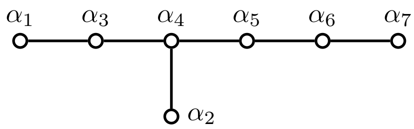
for all $\alpha, \beta \in R$. Namely, there is an $A(R)$-invariant scalar product $\langle .\,, . \rangle$ on $V$ such that

$$\check{\alpha}(.) = 2\frac{\langle \alpha, . \rangle}{\langle \alpha, \alpha \rangle}\cdot$$

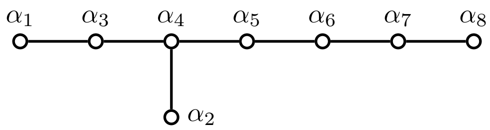
The notions of *positive roots* and *bases* of a root system are used. Recall in particular that $W(R)$ acts simply transitively on the set of bases (cf. [**Bo2**, VI.1.5, Thm 2 and Rem. 4]), and that $W(R)$ is normal in $A(R)$ with $A(R)/W(R) \simeq \operatorname{Aut}(R, \Delta)$ for any basis $\Delta$ of $R$ (cf. [**Bo2**, VI.1.5, Prop. 16 and 4.2, Cor.]); further, that this group identifies with the graph automorphisms of the *Dynkin diagram*. We have the usual list of irreducible reduced root systems/connected Dynkin diagrams $A_n$, $B_n$, $C_n$, $D_n$ $(n \geq 2)$, $E_6, E_7, E_8, F_4, G_2$ (with the repetitions $A_1 = B_1 = C_1$; $B_2 = C_2$; $D_2 = A_1 \times A_1$; $D_3 = A_3$), as well as the non-reduced $BC_n$, and we use the plates at the end of [**Bo2**, VI] for their properties. In particular we adhere to Bourbaki's labelling of the simple roots/vertices, although they look strange for the $E$ types, where the "lower" vertex is $\alpha_2$:




$E_6$




$E_7$




$E_8$

$A(R)/W(R)$ is the symmetric group $S_3$ for $D_4$ (an exceptional phenomenon which will occur as *triality* in the course of this work), of order two for $A_n (n \geq 2), D_n (n \neq 4)$ and $E_6$, and trivial in all other cases.

## 2.2. $\mathfrak{p}$-adic fields, C1 fields, quaternion algebras

We have the field of $p$-adic numbers $\mathbb{Q}_p$ with ring of integers $\mathbb{Z}_p$, and we call $\mathfrak{p}$*-adic field* a finite extension of $\mathbb{Q}_p$. Any finite extension of a $\mathfrak{p}$-adic field is again a $\mathfrak{p}$-adic field. A $\mathfrak{p}$-adic field $k$ is a *local field* of characteristic zero. It is complete with respect to a discrete valuation $v_k : k \to \mathbb{Z} \cup \{\infty\}$ and has some $\mathbb{F}_q$ as residue field. If $k$ is a local field, we denote by $\mathcal{O}_k$ its ring of integers, by $\mathfrak{m}_k$ the maximal ideal of $\mathcal{O}_k$ and by $\pi_k$ a uniformiser, i.e. generator of $\mathfrak{m}_k$. The general reference for local fields is [**Ser2**]. Specifically, the concept of (un)ramified extensions will be important. Any local field $k$ has a *maximal unramified extension* which is unique up to isomorphism (or unique inside a fixed algebraic closure $\bar{k}$) and will be denoted by $k^{nr}$; see [**Ser2**, III.5].

Definition 2.2.1 (Property C1). — We say that a field $k$ *has the C1 property* or *is C1* if the following holds: every homogeneous polynomial in $n$ variables $P \in k[X_1, \ldots, X_n]$ of degree $d \in \{1, \ldots, n-1\}$ has a non-trivial zero, meaning that there is $(0, \ldots, 0) \neq (a_1, \ldots, a_n) \in k^n$ with $P(a_1, \ldots, a_n) = 0$.

At one point we use the

Theorem 2.2.2 (cf. [**Lang**]). — *If $k$ is a local field, $k^{nr}$ is C1.*

*Quaternion algebras* will be of some importance in the classification. See [**Pie**, 1.6–1.7] and [**Inv**, 2.C] for the following basics: a quaternion algebra $Q$ over a field $k$ with $\mathrm{char}(k) \neq 2$ is described (not one-to-one) by two parameters $a, b \in k^*$:

$$Q := \left(\frac{a, b}{k}\right)$$

which is the 4-dimensional $k$-algebra with basis $1, i, j, ij$ and multiplication given by $i^2 = a$, $j^2 = b$ and $ij = -ji$ which implies $(ij)^2 = -ab$. The three-dimensional space spanned by $i, j$ and $ij$ is that of *pure quaternions*. Over a $\mathfrak{p}$-adic field $k$, just as over $\mathbb{R}$, up to isomorphism there is only one quaternion division algebra (i.e. quaternion algebra which is a skew field). The one over $\mathbb{R}$, the Hamilton quaternions, is usually denoted as

$$\mathbb{H} = \Big(\frac{-1,-1}{\mathbb{R}}\Big)$$

while for those over a $\mathfrak{p}$-adic $k$, a standard description (for $p \neq 2$) is

$$\Big(\frac{u,\pi_k}{k}\Big)$$

where $u \in \mathcal{O}_k^* \setminus \mathcal{O}_k^{*2}$.

On a quaternion algebra $Q := (\frac{a,b}{k})$, there is the so-called *standard involution*

$$\gamma : x1 + yi + zj + wij \mapsto x1 - yi - zj - wij,$$

i.e. the $k$-linear map with eigenspaces $k$ and the pure quaternions, of respective eigenvalues 1 and $-1$. It satisfies $\gamma^2 = \mathrm{id}_Q$ and $\gamma(st) = \gamma(t)\gamma(s)$ [!] for all $s, t \in Q$. We will need this in the discussion of the classical types of Lie algebras in Section 4.5.

## 2.3. Quadratic, symplectic and hermitian forms

Our general references for this theory are [**Bo1**, IX] and [**Lam**]. Since we are generally in characteristic $\neq 2$, we have the well-known equivalences between quadratic forms and symmetric bilinear forms. For symplectic (= alternate bilinear) forms, we will use the standard form in [**Bo1**, IX.5.1]. In left/right conventions, we follow Bourbaki and make the following

DEFINITION 2.3.1. — Let $D$ be a skew field and $\sigma : D \to D$ an *involutorial anti-automorphism*, i.e. an additive map satisfying

$$\sigma(ab) = \sigma(b)\sigma(a) \text{ for all } a, b \in D \quad \text{and} \quad \sigma^2 = \mathrm{id}_D\,.$$

Let $V$ be a left $D$-vector space. A bi-additive map $h : V \times V \to D$ satisfying

$$h(ax, by) = ah(x, y)\sigma(b)$$

for all $x, y \in V$ and $a, b \in D$ is called a *$\sigma$-hermitian form* (on $V$) if

$$h(y, x) = \sigma\big(h(x, y)\big)$$

for all $x, y \in V$, and is called a *$\sigma$-skew-hermitian form* (on $V$) if

$$h(y, x) = -\sigma\big(h(x, y)\big)$$

for all $x, y \in V$. Such an $h$ is *non-degenerate* if $h(x, V) = 0$ implies $x = 0$.

Skew-hermitian forms are called “anti-hermitian” by Bourbaki; in any case, they are the $\epsilon$-hermitian forms with $\epsilon = -1$. The notions of *(totally) isotropic* vectors or subspaces are defined in [**Bo1**, IX.4]. There we also find the fundamental

THEOREM 2.3.2 (Witt's decomposition theorem [**Wit**]). — *Let $k$ be a field with $\mathrm{char}(k) \neq 2$, and let $(V, q)$ be a quadratic space over $k$. Then there is a decomposition*

$$(V, q) \simeq (V_0, 0) \oplus (V_h, q_h) \oplus (V_a, q_a)$$

*where $V_0 = \ker(q)$ is the radical, $(V_h, q_h)$ is hyperbolic and $(V_a, q_a)$ is anisotropic, and all three are uniquely determined up to isomorphism. If $V_0 = 0$, the integer $r := \frac{1}{2} \dim_k V_h$ is the maximal dimension of a totally isotropic subspace of $(V, q)$, and is called the* Witt index *of $(V, q)$.*

Loc. cit. also contains its analogue for hermitian and skew-hermitian forms, which also have a well-defined Witt index if they satisfy the "condition T": For every $x \in V$, there is $a \in D$ such that

$$h(x, x) = a + \epsilon\sigma(a)$$

where $\epsilon = 1$ or $-1$ according to whether $h$ is hermitian or skew-hermitian. We do not have to worry about this, because we are in characteristic $\neq 2$ and can set $a = \frac{1}{2} h(x, x)$.

If $b$ is a non-degenerate symmetric bilinear form on an $n$-dimensional $k$-vector space, the *discriminant* $d(b)$ is the well-defined residue in $k^*/k^{*2}$ of $(-1)^{\lfloor \frac{1}{2} n \rfloor}$ times the determinant of a representing matrix of $b$; the sign ensures that hyperbolic spaces have discriminant 1.

# CHAPTER 3

# STRUCTURE THEORY AND THE ISOMORPHISM THEOREM

Let $k$ be a field with $\mathrm{char}(k) = 0$. Unless mentioned otherwise, all Lie algebras are finite dimensional, and all field extensions $K|k$ will be inside a fixed algebraic closure $\bar{k}|k$ with absolute Galois group $\mathcal{G} = \mathrm{Gal}(\bar{k}|k)$. One can check that nothing depends essentially on the choice of $\bar{k}$.

General references for the algebraic group analogue of this chapter are [**BT1**], [**Sat3**], [**Spr2**, 2..5–6] and [**Spr3**, Chap. 15–17].

*Throughout this whole chapter*, let $\mathfrak{g}$ be a *semisimple* Lie algebra over $k$. Let $\kappa(.\,,.)$ be its Killing form.

## 3.1. Definition of the invariants

### 3.1.1. Toral subalgebras and rational root decompositions

Definition 3.1.1 (Weights and roots). — Let $\mathfrak{h}$ be any subalgebra of $\mathfrak{g}$. For any linear form $\alpha \in \mathfrak{h}^* := \mathrm{Hom}_k(\mathfrak{h}, k)$, we set

$$\mathfrak{g}_\alpha := \mathfrak{g}_\alpha(\mathfrak{h}) := \left\{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\right\}.$$

If this subspace is non-zero, we call it the *weight space* of the *weight* $\alpha$ (of $\mathfrak{h}$ in $\mathfrak{g}$). The non-zero weights are called the *roots* of $\mathfrak{h}$ in $\mathfrak{g}$, the – obviously finite – set of these is denoted by $R(\mathfrak{g}, \mathfrak{h})$.

Note that in the above situation, $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$. Also, an immediate calculation with the Lie algebra axioms gives

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{2}$$

for all $\alpha, \beta \in \mathfrak{h}^*$.

For later use, we note:

Lemma 3.1.2. — *For $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, every element of $\mathfrak{g}_\alpha$ is nilpotent in $\mathfrak{g}$.*

*Proof.* — Let $x \in \mathfrak{g}_\alpha$. It suffices to show that $X := \mathrm{ad}_{\mathfrak{g}}\, x$ is nilpotent in the associative algebra $\mathrm{End}_k(\mathfrak{g})$. We will show that the trace $\mathrm{Tr}(X^n)$ is 0 for all $n \geq 1$, which proves the assertion by [**Bo1**, VII.5.5, Cor. 4]. (1)

Choose $h \in \mathfrak{h}$ with $r := \alpha(h) \neq 0$, and set $H := \mathrm{ad}_{\mathfrak{g}}(h)$, $Y := \frac{1}{r}X$. Then we have $X = HY - YH$, and because $Y$ commutes with $X^{n-1}$ (where $X^0 := 1$), we can write $X^n$ as commutator

$$X^n = X^{n-1}(HY - YH) = (X^{n-1}H)Y - Y(X^{n-1}H)$$

which has trace 0 because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any endomorphisms $A, B$. (The argument is a special case of [**Jac6**, II.5, Lemma 4]; for a proof explicitly using scalar extension, see [**Bo2**, VII.1.3, Prop. 10.(iv)].) □

Definition 3.1.3 (Toral and split toral subalgebras)

i. A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is *toral* if it is abelian and all its elements are semisimple.
ii. A toral subalgebra $\mathfrak{a}$ is called *split* (over $k$) if all its elements are ad-diagonalisable over $k$; that is, for every $x \in \mathfrak{a}$, there is a $k$-basis of $\mathfrak{g}$ consisting of eigenvectors for $\mathrm{ad}_{\mathfrak{g}}(x)$ with all eigenvalues in $k$.
iii. A *maximal* (split) toral subalgebra is one which is not strictly contained in another (split) toral subalgebra.

Remark 3.1.4

i. Subalgebras of (split) toral subalgebras are (split) toral.
ii. For an isomorphism $h : \mathfrak{g} \simeq \mathfrak{g}'$, the subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is (maximal/split/maximal split) toral if and only if so is $h(\mathfrak{a}) \subseteq \mathfrak{g}'$.
iii. If $k$ is algebraically closed, every toral subalgebra is split.
iv. The argument in [**Hum**, 8.1] shows that a subalgebra which consists of ad-diagonalisable elements is automatically abelian. Therefore, one could define a split toral subalgebra as a subalgebra consisting of ad-diagonalisable elements. On the other hand, the "abelian" condition is *not* redundant in the definition of toral subalgebras; in fact, we will encounter non-zero semisimple Lie algebras which consist entirely of semisimple elements, see 3.2.1 with example 3.2.5.
v. Maximal (split) toral subalgebras always exist because our Lie algebras have finite dimension and we have the split toral subalgebra $\{0\}$.
vi. The usual *Caveat* about the order of words is in order: whereas a split maximal toral subalgebra is also a maximal split toral subalgebra, not every maximal split toral subalgebra is a split maximal toral subalgebra (it is if and only if it is a maximal toral subalgebra).

Proposition 3.1.5. — *In our setting, the maximal toral subalgebras* $\mathfrak{t}$ *are precisely the* Cartan subalgebras *of* $\mathfrak{g}$. *In particular, they equal their own centraliser and*

1. In earlier editions, this was Corollaire 4 in *Algèbre* VII.3.5.

*normaliser, all maximal toral subalgebras of* $\mathfrak{g}$ *have the same dimension* (*cf.* [**Bo2**, VII.3.3]) *and a split maximal toral subalgebra is a splitting Cartan subalgebra in the sense of* [**Bo2**, VIII.2.1].

*Proof.* — In [**Bo2**, VII], this is left as exercise 3 to §2. That a Cartan subalgebra of $\mathfrak{g}$ is maximal toral follows easily from [**Bo2**, VII.2.4]. The following proof of the converse, for maximal toral $\mathfrak{t}$, is a combination of the proof of [**Sel2**, §1, Prop. 1] with ideas of [**Hum**, 8.1–8.2] and [**Xue**].

*Step 1.* — For any toral subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, we have $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))$. Namely, $\mathrm{ad}_{\mathfrak{g}}(a)$ is semisimple for $a \in \mathfrak{a}$, so $\ker(\mathrm{ad}_{\mathfrak{g}}(a)) = \ker((\mathrm{ad}_{\mathfrak{g}}(a))^2)$. Hence

$$\begin{aligned} n \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})) &\Longrightarrow [a,n] \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \text{ for all } a \in \mathfrak{a} \Rightarrow [a,[a,n]] = 0 \text{ for all } a \in \mathfrak{a} \\ &\Longleftrightarrow n \in \ker((\mathrm{ad}_{\mathfrak{g}}(a))^2) \text{ for all } a \in \mathfrak{a} \\ &\Longleftrightarrow n \in \ker(\mathrm{ad}_{\mathfrak{g}}(a)) \text{ for all } a \in \mathfrak{a} \Longleftrightarrow n \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}), \end{aligned}$$

and the other inclusion is clear. (Similarly one can show $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$.)

*Step 2.* — Keeping the notation of step 1, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ contains the semisimple and nilpotent components (in $\mathfrak{g}$) of its elements. Namely, for $x \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$, write $x = x_s + x_n$; then $\mathrm{ad}_{\mathfrak{g}}(x_s) = (\mathrm{ad}_{\mathfrak{g}}(x))_s$ and $\mathrm{ad}_{\mathfrak{g}}(x_n) = (\mathrm{ad}_{\mathfrak{g}}(x))_n$ can be written as polynomials in $\mathrm{ad}_{\mathfrak{g}}(x)$ without constant term, so that both restrict to the zero map on $\mathfrak{a}$, hence $x_s$ and $x_n$ centralise $\mathfrak{a}$.

*Step 3.* — Any semisimple element $x_s$ of $\mathfrak{z} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ is contained in $\mathfrak{t}$. Namely, if it were not, we would have the toral subalgebra $\mathfrak{t} + kx_s \supsetneq \mathfrak{t}$.

*Step 4.* — $\mathfrak{z}$ is nilpotent. Namely, decomposing $x = x_s + x_n$ for $x \in \mathfrak{z}$, by step 3 we have $\mathrm{ad}_{\mathfrak{z}}(x) = \mathrm{ad}_{\mathfrak{z}}(x_s) + \mathrm{ad}_{\mathfrak{z}}(x_n) = \mathrm{ad}_{\mathfrak{z}}(x_n)$ and this, being the restriction of the nilpotent endomorphism $\mathrm{ad}_{\mathfrak{g}}(x_n)$ to $\mathfrak{z}$, is a nilpotent endomorphism, so that the assertion follows from Engel's theorem.

By steps 1 and 4, $\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}$. In particular, all its elements are semisimple (cf. [**Bo2**, VII.2.4, Thm 2]), which implies $x = x_s$ in the above decompositions. So by step 3, $\mathfrak{t} = \mathfrak{z}$. (For this last step, some of the references use the non-degeneracy of the Killing form on $\mathfrak{z} \times \mathfrak{z}$, which in turn is shown via scalar extension. Implicitly, this is also used in the proof of our Bourbaki citation.) □

REMARK 3.1.6. — From $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$ for a maximal toral $\mathfrak{t}$, it also follows that $\mathfrak{t}$ is a maximal abelian subalgebra of any intermediate algebra $\mathfrak{t} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ (and maximal toral in $\mathfrak{h}$ for semisimple $\mathfrak{h}$ – only in this case we have defined the concept). For any toral subalgebra $\mathfrak{a}$, it follows from $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ that the centraliser commutes with scalar extension: $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))_K = \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{a}_K)$.

Lemma 3.1.7. — *If* $\mathfrak{a}$ *is a*

i. *toral subalgebra*
ii. *split toral subalgebra*
iii. *maximal toral subalgebra*

*of* $\mathfrak{g}$*, then for every extension* $K|k$*,* $\mathfrak{a}_K$ *has the respective property in* $\mathfrak{g}_K$*.*

*Proof*

i. $\mathfrak{a}_K$ is abelian and generated additively by the elements $c \otimes a$ with $c \in K, a \in \mathfrak{a}$. These are obviously semisimple, and the sum of commuting semisimple elements is again semisimple (cf. [**Bo1**, VII.5.8, Cor.]).
ii. Analogously, scalar multiples and sums of commuting diagonalisable elements are diagonalisable (cf. [**Bo1**, VII.5.7, Cor.]).
iii. We have $\mathfrak{n}_{\mathfrak{g}_K}(\mathfrak{a}_K) = (\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}))_K = \mathfrak{a}_K$ by [**Bo2**, I.3.8] and 3.1.5. Thus there is not even any abelian ($K$-)Lie algebra $\mathfrak{b}$ with $\mathfrak{a}_K \subsetneq \mathfrak{b} \subseteq \mathfrak{g}_K$.

    With that proposition, this (and the converse "$\mathfrak{a}_K$ maximal toral in $\mathfrak{g}_K \Rightarrow \mathfrak{a}$ maximal toral in $\mathfrak{g}$") also follows from the corresponding assertion for Cartan subalgebras, [**Bo2**, VII.2.1, Prop. 3]. □

If $\mathfrak{a}$ is split toral, then by definition, all $\mathrm{ad}_{\mathfrak{g}}(a) \in \mathrm{End}_k(\mathfrak{g})$ for $a \in \mathfrak{a}$ are diagonalisable and commute with each other. It is then a well-known fact of linear algebra that, with the notation of 3.1.1, we have a decomposition of $k$-vector spaces:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha \,.$$

We will now collect decisive facts about this *weight decomposition* that generalise standard facts (cf. [**Bo2**, VIII.2]) about splitting Cartan subalgebras – which, as per remark 3.1.5, are the *split maximal* toral subalgebras in our setting. In particular we will see that for a *maximal split* toral $\mathfrak{s}$, $R(\mathfrak{g}, \mathfrak{s})$ is a (possibly non-reduced) root system. One should beware that the case $\mathfrak{s} = 0$ or equivalently, $R(\mathfrak{g}, \mathfrak{s}) = \varnothing$, can occur, see example 3.2.5. Most of the following discussion (prop. 2.1.8 to 2.1.13) is a reformulation of [**Sel2**, I.1].

Proposition 3.1.8. — *Let* $\mathfrak{a}$ *be a split toral subalgebra.*

i. $R(\mathfrak{g}, \mathfrak{a})$ *spans* $\mathfrak{a}^*$ *as* $k$*-vector space.*
ii. *For weights* $\alpha$ *and* $\beta$ *with* $\alpha + \beta \neq 0$*,* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal with respect to* $\kappa$*.*
iii. *For every weight* $\alpha$*, the restriction of* $\kappa$ *to* $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ *is non-degenerate. In particular, if* $\alpha \in R(\mathfrak{g}, \mathfrak{a})$*, then* $-\alpha \in R(\mathfrak{g}, \mathfrak{a})$*, and* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{-\alpha}$ *are dual to each other with respect to* $\kappa$*.*
iv. *For all* $s, s' \in \mathfrak{a}$ *we have* $\kappa(s, s') = \sum_{\beta \in R(\mathfrak{g},\mathfrak{a})} \dim_k(\mathfrak{g}_\beta)\beta(s)\beta(s')$*.*

*Proof*

i. If the $k$-span of $R(\mathfrak{g}, \mathfrak{a})$ were strictly smaller, there would exist an $a \in \mathfrak{a} \setminus \{0\}$ with $\alpha(a) = 0$ for all $\alpha \in R(\mathfrak{g}, \mathfrak{a})$. Then the weight decomposition would imply $[a, x] = 0$ for all $x \in \mathfrak{g}$, that is, $a \in \mathfrak{z}(\mathfrak{g})$; but $\mathfrak{z}(\mathfrak{g})$ is zero because $\mathfrak{g}$ is semisimple.

ii. If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, applying (2) twice shows that for any weight $\gamma$, $\mathrm{ad}_\mathfrak{g}(x) \circ \mathrm{ad}_\mathfrak{g}(y)$ maps $\mathfrak{g}_\gamma$ to $\mathfrak{g}_{\gamma+\alpha+\beta}$, which implies the assertion by the weight decomposition.

iii. The first assertion follows from ii. and the non-degeneracy of $\kappa$, the others are immediate consequences.

iv. This is immediate from the weight decomposition. □

With the following crucial proposition we get our hands on the *coroots*.

PROPOSITION 3.1.9. — *Let $\mathfrak{s}$ be a* maximal *split toral subalgebra. For $\alpha \in R(\mathfrak{g}, \mathfrak{s})$, there is a unique element $H_\alpha \in \mathfrak{s}$ such that*

$$\alpha(H_\alpha) = 2 \quad \textit{and} \quad H_\alpha = [U, V]$$

*for certain $U \in \mathfrak{g}_\alpha, V \in \mathfrak{g}_{-\alpha}$. Moreover, for any $0 \neq X \in \mathfrak{g}_\alpha$ there is $0 \neq Y \in \mathfrak{g}_{-\alpha}$ such that $[X, Y] = H_\alpha$ (and then necessarily, $[H_\alpha, X] = 2X$ and $[H_\alpha, Y] = -2Y$).*

*Proof.* — Let $0 \neq X \in \mathfrak{g}_\alpha$. By 3.1.2, $X$ is nilpotent. By the Jacobson-Morozov theorem, there are $y, z \in \mathfrak{g}$ such that $[z, X] = 2X$, $[z, y] = -2y$ and $[X, y] = z$. Decomposing

$$y = \sum_{\beta \in R(\mathfrak{g},\mathfrak{a}) \cup \{0\}} y_\beta \quad \text{and} \quad z = \sum_{\beta \in R(\mathfrak{g},\mathfrak{a}) \cup \{0\}} z_\beta$$

with respect to the weight decomposition, we see from (2) and $X \in \mathfrak{g}_\alpha$ that $[z_0, X] = 2X$ (in particular, $z_0 \neq 0$), that $[z_\beta, X] = 0$ for $\beta \neq 0$, and that $[X, y_{-\alpha}] = z_0$. Applying [**Bo2**, VIII.11.2, Lemma 6] to $h = z_0$ and $x = X$ gives a "better" $\widetilde{y} \in \mathfrak{g}$ with $[z_0, \widetilde{y}] = -2\widetilde{y}$ and $[X, \widetilde{y}] = z_0$. Again decomposing $\widetilde{y} = \sum_{\beta \in R(\mathfrak{g},\mathfrak{a}) \cup \{0\}} \widetilde{y}_\beta$ and using (2) we get $[X, \widetilde{y}_{-\alpha}] = z_0$ (in particular, $\widetilde{y}_{-\alpha} \neq 0$), and $[z_0, \widetilde{y}_\beta] = -2\widetilde{y}_\beta$ for all weights $\beta$, specifically for $\beta = -\alpha$.

We now claim that $H_\alpha := z_0$ has the properties asserted in the second sentence. (Our construction then shows the assertion of the third sentence for the above $X$ with $Y := \widetilde{y}_{-\alpha}$.)

The above construction already shows $\alpha(H_\alpha) = 2$, and that $H_\alpha$ is the Lie product of an element of $\mathfrak{g}_\alpha$ with one of $\mathfrak{g}_{-\alpha}$. Further, $H_\alpha$ is contained in $\mathfrak{g}_0$ which means that it commutes with all elements of $\mathfrak{s}$. Finally, it being the semisimple part of an $\mathfrak{sl}_2$-triple and the representation theory of $\mathfrak{sl}_2$ shows that all eigenvalues of $\mathrm{ad}_\mathfrak{g}(H_\alpha)$ are integers, so it certainly is diagonalisable. Hence with $\mathfrak{s}$ also $\mathfrak{s} + kH_\alpha$ is a split toral subalgebra of $\mathfrak{g}$, and by maximality, we must have $H_\alpha \in \mathfrak{s}$. The uniqueness will follow from the following lemma by noting that for $H, H' \in \mathfrak{a}$ with $\beta(H) = \beta(H')$ for all roots $\beta$, we have that $H - H'$ is central in $\mathfrak{g}$, hence 0. □

LEMMA 3.1.10. — *For $\alpha \in R(\mathfrak{g},\mathfrak{s})$, let $U \in \mathfrak{g}_\alpha, V \in \mathfrak{g}_{-\alpha}$ be such that $H := [U,V]$ is in $\mathfrak{s}$ and satisfies $\alpha(H) = 2$. Let $\beta$ be any weight of $\mathfrak{s}$ in $\mathfrak{g}$. Then there are $q, r \in \mathbb{Z}_{\geq 0}$ such that for $i \in \mathbb{Z}$, we have $\mathfrak{g}_{\beta+i\alpha} \neq \{0\}$ if and only if $-r \leq i \leq q$; further, $\beta(H) = r - q$.*

*Proof.* — $U, H, V$ are an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, with respect to which we consider the $\mathfrak{sl}_2$-representation $E := \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$. Let $q$ (resp. $r$) be maximal in $\mathbb{Z}$ with the property $\mathfrak{g}_{\beta+q\alpha} \neq \{0\}$ (resp. $\mathfrak{g}_{\beta-r\alpha} \neq \{0\}$). The eigenspaces of $H$ in $E$ are the non-zero $E_{\beta(H)+2i} = \mathfrak{g}_{\beta+i\alpha}$ for $i \in \mathbb{Z}$, so by 2.1.7.ii we have

$$\beta(H) + 2q = -\big(\beta(H) - 2r\big),$$

hence $\beta(H) = r - q$; then by part i of the same lemma, the possible eigenspaces above are indeed non-zero for $-r \leq i \leq q$. □

These $H_\alpha$ allow us to derive further results:

PROPOSITION 3.1.11. — *Let $\mathfrak{s}$ be a maximal split toral subalgebra, and for $\alpha \in R(\mathfrak{g},\mathfrak{s})$, let $H_\alpha$ be as above.*

i. *For $\alpha, \beta \in R(\mathfrak{g},\mathfrak{s})$, we have $\beta(H_\alpha) \in \mathbb{Z}$.*
ii. *For $\alpha \in R(\mathfrak{g},\mathfrak{s})$, we have $\kappa(H_\alpha, H_\alpha) \in \mathbb{Z}_{>0}$.*
iii. *For $\alpha \in R(\mathfrak{g},\mathfrak{s})$ and $s \in \mathfrak{s}$, we have:*

$$\alpha(s) = 2\frac{\kappa(H_\alpha, s)}{\kappa(H_\alpha, H_\alpha)}.$$

iv. *If $\alpha_1, \ldots, \alpha_n \in R(\mathfrak{g},\mathfrak{s}) \subset \mathfrak{s}^*$ are $k$-linearly independent, then $H_{\alpha_1}, \ldots, H_{\alpha_r} \in \mathfrak{s}$ are $k$-linearly independent. The $H_\alpha$ for $\alpha \in R(\mathfrak{g},\mathfrak{s})$ span $\mathfrak{s}$ over $k$.*
v. *The restriction of $\kappa$ to $\mathfrak{s} \times \mathfrak{s}$ is non-degenerate.*

*Proof*

i. Restatement of part of the previous lemma.
ii. We have $\kappa(H_\alpha, H_\alpha) = \sum_{\beta\in R(\mathfrak{g},\mathfrak{s})} \dim_k(\mathfrak{g}_\beta)(\beta(H_\alpha))^2$, so the assertion follows from i. and $\alpha(H_\alpha) = 2 > 0$.
iii. Let $\alpha \in R(\mathfrak{g},\mathfrak{s})$. Setting $s' := s - \frac{1}{2}\alpha(s)H_\alpha \in \mathfrak{s}$ (and using ii.), the claim is equivalent to: $\kappa(H_\alpha, s') = 0$.

   To show this, choose an $\mathfrak{sl}_2$-triple $X, H_\alpha, Y$ as in 3.1.9. There is a subset $B \subseteq R(\mathfrak{g},\mathfrak{s})$ such that the roots can be partitioned as

$$R(\mathfrak{g},\mathfrak{s}) = \coprod_{\beta\in B} \{\beta + i\alpha : i \in \mathbb{Z}, \beta + i\alpha \in R(\mathfrak{g},\mathfrak{s})\};$$

   the spaces $\mathfrak{g}(\beta,\alpha) := \sum_{i\in\mathbb{Z},\beta+i\alpha\in R(\mathfrak{g},\mathfrak{s})} \mathfrak{g}_{\beta+i\alpha}$ are stable under $\mathrm{ad}(s')$ as well as under the adjoint action of the $\mathfrak{sl}_2$-triple. Because of the partition, $\kappa(H_\alpha, s')$ is the sum of the traces of $\mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(H_\alpha) \circ \mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(s')$ for $\beta \in B$. But because $\alpha(s') = 0$, such a trace is just $\beta(s')$ multiplied with the trace of $\mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(H_\alpha)$. Finally, this last trace is zero because the endomorphism $\mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(H_\alpha)$ is the commutator of the endomorphisms $\mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(X)$ and $\mathrm{ad}_{\mathfrak{g}(\beta,\alpha)}(Y)$.

iv. Assume we have $\lambda_1, \ldots, \lambda_n \in k$ with $\sum_{i=1}^n \lambda_i H_{\alpha_i} = 0$. Then for all $s \in \mathfrak{s}$, we have

$$0 = \kappa\,(\sum_{i=1}^{n} \lambda_i H_{\alpha_i}, s) \overset{iii.}{=} \frac{1}{2} \sum_{i=1}^{n} \lambda_i \cdot \kappa(H_{\alpha_i}, H_{\alpha_i}) \cdot \alpha_i(s),$$

so by assumption and part ii, all $\lambda_i$ must be zero. The second assertion follows from the first one and 3.1.8.i.

v. This follows from part iii, as $\alpha(s) = 0$ for all $\alpha \in R(\mathfrak{g}, \mathfrak{s})$ implies $s = 0$ for $s \in \mathfrak{s}$.

□

Using the canonical identification of $\mathfrak{s}$ with its bidual $(\mathfrak{s}^*)^*$, we set $\check{\alpha} := H_\alpha$ and see with the remark after (1) that $s_{\alpha,\check{\alpha}}$ is a reflection.

Theorem 3.1.12. — *Let $\mathfrak{s}$ be a maximal split toral subalgebra of $\mathfrak{g}$. Then $R(\mathfrak{g}, \mathfrak{s})$ is a root system (in $\mathfrak{s}^*$).*

*Proof.* — Finiteness and $0 \notin R$ are obvious, so (RS1) follows from 3.1.8.i. For $\alpha, \beta$ in $R(\mathfrak{g}, \mathfrak{s})$, Lemma 3.1.10 gives us non-negative $q, r$ such that

$$s_{\alpha,\check{\alpha}}(\beta) = \beta - \beta(H_\alpha)\alpha = \beta + (q - r)\alpha,$$

and that this is a weight. It is non-zero because as a reflection, $s_{\alpha,\check{\alpha}}$ is injective, so (RS2) follows. Finally, (RS3) is given by 3.1.11.i. □

Let us point out three differences to the "split" situation in [**Bo2**, VIII.2.2] (for all of them to occur, see example 3.2.9):

▷ $R(\mathfrak{g}, \mathfrak{s})$ is not reduced in general: there might be $\alpha \in R(\mathfrak{g}, \mathfrak{s})$ with $2\alpha \in R(\mathfrak{g}, \mathfrak{s})$.

▷ The root spaces $\mathfrak{g}_\alpha$ can have dimension $> 1$.

▷ The space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ can also have dimension $> 1$, in particular might be not generated by the element $H_\alpha$ (but its intersection with $\mathfrak{s}$ is, as follows from later results).

Obviously, the third point can only occur if the second one does. The next proposition shows that the first two points can only occur together, and "why" they do not in the case of a split maximal toral $\mathfrak{s}$.

Proposition 3.1.13

i. *For $\alpha, \beta \in R(\mathfrak{g}, \mathfrak{s})$ with $\beta(H_\alpha) > 0$ and for every $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, we have $[X_\alpha, \mathfrak{g}_{\beta-\alpha}] = \mathfrak{g}_\beta$.*

ii. *For $\alpha \in R(\mathfrak{g}, \mathfrak{s})$ we have* $\dim_k \mathfrak{g}_{2\alpha} < \dim_k \mathfrak{g}_\alpha$.

iii. *If $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s}$ (equivalently, $\mathfrak{s}$ is maximal toral),* $\dim_k \mathfrak{g}_\alpha = 1$ *for all $\alpha \in R(\mathfrak{g}, \mathfrak{s})$. In particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = k \cdot H_\alpha$, and $R(\mathfrak{g}, \mathfrak{s})$ is reduced.*

*Proof*

i. By 3.1.9, $X_\alpha$ is part of an $\mathfrak{sl}_2$-triple $X_\alpha, H_\alpha, Y_\alpha$. Set $\lambda := \beta(H_\alpha)$. In the finite dimensional $\mathfrak{sl}_2$-representation $E := \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$ again we have $E_{\lambda+2i} = \mathfrak{g}_{\beta+i\alpha}$ for $i \in \mathbb{Z}$; now apply 2.1.7.iii.

ii. Trivial if $2\alpha \notin R(\mathfrak{g},\mathfrak{s})$. Else, part i with $\beta = 2\alpha$ shows that for $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, $\operatorname{ad}(X_\alpha)$ induces a $k$-linear surjection $\mathfrak{g}_\alpha \twoheadrightarrow \mathfrak{g}_{2\alpha}$, whose kernel contains $0 \neq X_\alpha$.

iii. Setting $\beta = \alpha$ in part i, we get $\mathfrak{g}_\alpha = [\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}), X_\alpha] = [\mathfrak{s}, X_\alpha] = \alpha(\mathfrak{s})X_\alpha$. □

The root system $R(\mathfrak{g},\mathfrak{s})$ abstractly comes equipped with its Weyl group, which is identified with the Weyl group of its inverse root system formed by the $H_\alpha = \check{\alpha}$ (see the remarks after Proposition 2 in [**Bo2**, VI.1.1]). One can check with 3.1.11 that on the inverse root system, the Killing form $\kappa$ is non-degenerate and invariant under the action of the Weyl group. We will find a realisation of the Weyl group as a subquotient of $\operatorname{Aut}_e(\mathfrak{g},\mathfrak{s})$ (see definition 2.1.9) in the spirit of [**Bo2**, VIII.5.2]. To start with, there is a well-defined group homomorphism, the contragredient of the restriction to $\mathfrak{s}$,

$$\varepsilon_{\mathfrak{s}} : \operatorname{Aut}_e(\mathfrak{g},\mathfrak{s}) \longrightarrow A\big(R(\mathfrak{g},\mathfrak{s})\big), \quad f \longmapsto \big[\alpha \mapsto \alpha \circ f_{|\mathfrak{s}}^{-1} \text{ for } \alpha \in R(\mathfrak{g},\mathfrak{s})\big]$$

whose kernel obviously is $\{f \in \operatorname{Aut}_e(\mathfrak{g},\mathfrak{s}) : f_{|\mathfrak{s}} = \operatorname{id}_{\mathfrak{s}}\}$.

Proposition 3.1.14. — *For $\alpha \in R(\mathfrak{g},\mathfrak{s})$, choose an $\mathfrak{sl}_2$-triple $(X, H_\alpha, Y)$ as in 3.1.9 and set*

$$g_\alpha := \exp(\operatorname{ad} Y) \circ \exp(\operatorname{ad}(-X)) \circ \exp(\operatorname{ad} Y) \in \operatorname{Aut}_e(\mathfrak{g}).$$

*Then $g_\alpha(H_\alpha) = -H_\alpha$, and $g_\alpha = \operatorname{id}$ on the orthogonal complement of $H_\alpha$ in $\mathfrak{s}$ (with respect to $\kappa$).*

*Proof.* — For the second assertion, $s \perp H_\alpha$ for $s \in \mathfrak{s}$ by 3.1.11.iii means $\alpha(s) = 0$, hence $g_\alpha(s) = s$ because $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$. For the first assertion, calculate:

$$\begin{aligned}
&\big(\exp(\operatorname{ad} Y) \circ \exp(\operatorname{ad}(-X)) \circ \exp(\operatorname{ad} Y)\big)(H_\alpha)\\
&\quad = (\exp(\operatorname{ad} Y) \circ \exp\big(\operatorname{ad}(-X))\big)(H_\alpha + 2Y)\\
&\qquad = \exp(\operatorname{ad} Y)\big(-H_\alpha + 2Y + 2X - 2H_\alpha + \tfrac{1}{2}(-4X)\big)\\
&\qquad\quad = \exp(\operatorname{ad} Y)(-H_\alpha + 2Y) = -H_\alpha + 2Y + (-2Y) = -H_\alpha
\end{aligned}$$ □

Corollary 3.1.15. — *For choices as above, let $G$ be the subgroup of $\operatorname{Aut}_e(\mathfrak{g},\mathfrak{s})$ generated by the $g_\alpha$. Then $\varepsilon_{\mathfrak{s}}(G) = W(R(\mathfrak{g},\mathfrak{s}))$.*

We will later improve this result, see 3.1.47 and the subsection around it.

We end this subsection with a conjugacy result on maximal split toral subalgebras in $\mathfrak{g}$. It will be needed "only" to show that the invariants we will define are independent from various choices. The analogous conjugacy statement for maximal $k$-split tori in reductive groups is proven in [**BT1**, 4.21].

Theorem 3.1.16. — *Let $\mathfrak{s}, \mathfrak{s}'$ be maximal split toral subalgebras of $\mathfrak{g}$. Then there is an elementary automorphism $g \in \mathrm{Aut}_e(\mathfrak{g})$ such that $g(\mathfrak{s}') = \mathfrak{s}$.*

*Proof.* — The only statement and proof of (a slightly more general version of) this in the literature seems to be [**Sel2**, I.3]. We give a sketch of this proof. Fix a system of positive roots $P$ in $R(\mathfrak{g}, \mathfrak{s})$ and set

$$\mathfrak{N}(\mathfrak{s}) := \bigoplus_{\lambda \in P} \mathfrak{g}_\lambda, \qquad \mathfrak{P}(\mathfrak{s}) := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \oplus \mathfrak{N}(\mathfrak{s}).\ ^{(2)}$$

*Step 1.* — For every semisimple $x_s \in \mathfrak{P}(\mathfrak{s})$, there is an $h \in \mathrm{Aut}_e(\mathfrak{g})$ with $h(x_s) \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$.

Namely, write a general element $x \in \mathfrak{P}(\mathfrak{s})$ as $u+n_0$ with $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}), n_0 \in \mathfrak{N}(\mathfrak{s})$. Using that the endomorphism $\mathrm{ad}_{\mathfrak{N}(\mathfrak{s})}(u)$ is semisimple (by 3.1.29), one defines $n_i$ and $n_0^{(i)}$ in $\mathfrak{N}(\mathfrak{s})$ inductively so that

$$\exp(\mathrm{ad}\, n_i) \circ \cdots \circ \exp(\mathrm{ad}\, n_1)(x) = u + n_0^{(i)} \mod \mathcal{C}^i \mathfrak{N}(\mathfrak{s}),$$

where the $n_0^{(i)}$ are in the kernel of $\mathrm{ad}_{\mathfrak{N}(\mathfrak{s})}(u)$. Using nilpotence of $\mathfrak{N}(\mathfrak{s})$, after finitely many steps one has $u$ and $n_0^{(r)}$ being the semisimple and nilpotent part of $\exp(\mathrm{ad}\, n_r) \circ \cdots \circ \exp(\mathrm{ad}\, n_1)(x)$; in particular, if $x = x_s$ is semisimple, it is mapped to $u$.

*Step 2.* — If $\mathfrak{s}' \subset \mathfrak{P}(\mathfrak{s})$, there is $h \in \mathrm{Aut}_e(\mathfrak{g})$ with $h(\mathfrak{s}') = \mathfrak{s}$. (3)

Namely, pick a "generic" element $x \in \mathfrak{s}'$, meaning $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}')$, and apply step 1. One gets $\mathfrak{s} \subseteq \mathfrak{z}_{\mathfrak{g}}(h(x)) = \mathfrak{z}_{\mathfrak{g}}(h(\mathfrak{s}'))$. Using maximalities, one infers that $\mathfrak{s} = h(\mathfrak{s}')$.

*Step 3.* — One checks that $\mathfrak{N}(\mathfrak{s})$ is a maximal nil (= consisting of nilpotent elements) subalgebra of $\mathfrak{g}$, and that $\mathfrak{P}(\mathfrak{s})$ is the normaliser of $\mathfrak{N}(\mathfrak{s})$ in $\mathfrak{g}$. One further checks that for any two maximal nil subalgebras $\mathfrak{N}_1, \mathfrak{N}_2$ of $\mathfrak{g}$ with respective normalisers $\mathfrak{P}_1, \mathfrak{P}_2$, one has $(i = 1, 2)$:

$$\mathfrak{P}_i = (\mathfrak{P}_1 \cap \mathfrak{P}_2) + \mathfrak{N}_i$$

*Step 4.* — Let $\mathfrak{N}$ be a maximal nil subalgebra of $\mathfrak{g}$ with normaliser $\mathfrak{P}$. Then there is $h \in \mathrm{Aut}_e(\mathfrak{g})$ with $h^{-1}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})) \subseteq \mathfrak{P}$.

Namely, there is a reductive subalgebra $\mathfrak{M}$ of $\mathfrak{g}$ consisting of semisimple elements and such that $\mathfrak{P} = \mathfrak{M} \oplus \mathfrak{N}$ and $\mathfrak{P}(\mathfrak{s}) = \mathfrak{M} \oplus \mathfrak{N}(\mathfrak{s})$. (To construct such an algebra inside $\mathfrak{P}(\mathfrak{s}) \cap \mathfrak{P}$ with step 3 is the most technical part of the proof, using the embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ and facts about Lie algebras of endomorphisms.) Now first one conjugates the centre $\mathfrak{z}(\mathfrak{M})$ into $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, applying the technique of step 2. (Here again there is a

2. This is in fact a minimal parabolic subalgebra of $\mathfrak{g}$.
3. This is an analogue of Corollaire (ii) of [**Bo2**, VIII.3, Prop. 9] in our situation.

technical digression to construct a suitably "generic" element.) Secondly, one looks at the algebra

$$\mathfrak{C} := \mathfrak{z}_{\mathfrak{P}(\mathfrak{s})}(\mathfrak{z}(\mathfrak{M}))$$

and shows that it is reductive with semisimple part $\mathcal{D}(\mathfrak{C} \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})) \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. Since the semisimple $\mathcal{D}\mathfrak{M}$ is contained in $\mathfrak{C}$, by a part of the Levi-Malcev theorem (cf. [**Bo2**, I.6.8, Cor. 1]) there is an elementary automorphism (of $\mathfrak{C}$, extending to $\mathfrak{g}$) $h$ which conjugates $\mathcal{D}\mathfrak{M}$ into that semisimple part, and stabilises $\mathfrak{z}(\mathfrak{M})$. So we have conjugated all of $\mathfrak{M}$ into $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. With a dimension argument, one shows that in fact $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = h(\mathfrak{M}) \subseteq h(\mathfrak{P})$.

We now define $\mathfrak{N}(\mathfrak{s}')$ and $\mathfrak{P}(\mathfrak{s}')$ analogously, with respect to some system of positive roots $P' \subset R(\mathfrak{g}, \mathfrak{s}')$. By steps 3 and 4, we can assume without loss of generality that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \subseteq \mathfrak{P}(\mathfrak{s}')$.

*Step 5.* — Under this assumption, the roots

$$P(P, P') := \big\{\alpha \in R(\mathfrak{g}, \mathfrak{s}) : \mathfrak{g}_\alpha \cap \mathfrak{P}(\mathfrak{s}') \neq \{0\}\big\}$$

form a system of positive roots, i.e. there is $w \in W(R(\mathfrak{g}, \mathfrak{s}))$ with $P = wP(P, P')$, and we have:

$$\mathfrak{P}(\mathfrak{s}') = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \oplus \bigoplus_{\alpha \in P(P,P')} \mathfrak{g}_\alpha$$

Namely, for this one argues cleverly with (2), 3.1.13.i and $\mathfrak{sl}_2$-triples.

*Step 6.* — As in 3.1.14 and its corollary, there is an $f \in \mathrm{Aut}_e(\mathfrak{g}, \mathfrak{s})$ that induces the $w$ in step 5. It follows:

$$f(\mathfrak{P}(\mathfrak{s}')) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha = \mathfrak{P}(\mathfrak{s})$$

So without loss of generality, we are in the situation of step 2, and therefore finished. □

COROLLARY 3.1.17. — *The integer* $\mathrm{rank}(R(\mathfrak{g}, \mathfrak{s})) := \dim_k(\mathfrak{s})$ *is independent from the choice of* $\mathfrak{s}$. *We call it the* $k$-rank *of* $\mathfrak{g}$.

**3.1.2. Galois actions and the anisotropic kernel.** — Let $\mathfrak{a}$ be a toral subalgebra of $\mathfrak{g}$. Choose a $k$-basis $a_1, \dots, a_n$ of $\mathfrak{a}$; by field theory, there exists an extension $K|k$ such that all the characteristic polynomials of the $\mathrm{ad}_{\mathfrak{g}}(a_i)$ split into linear factors over $K$. This implies that $\mathfrak{a}_K$ is a *split* toral subalgebra of $\mathfrak{g}_K$.

DEFINITION 3.1.18 (Splitting extensions). — An extension $K|k$ such that $\mathfrak{a}_K$ is split toral in $\mathfrak{g}_K$ is called a *splitting extension*, and $K$ a *splitting field*, for the toral subalgebra $\mathfrak{a}$.

It is clear from the preceding construction that a splitting extension can be chosen to be finite and Galois over $k$. It is also clear that every extension of a splitting field is again a splitting field.

After we have come this far using not much scalar extension, we will now unleash the full power of this tool. To begin with, let $\mathfrak{a} \subseteq \mathfrak{t}$ be any inclusion of toral subalgebras of $\mathfrak{g}$ such that $\mathfrak{a}$ is split over $k$, and let $K|k$ be a splitting extension for $\mathfrak{t}$. We have the vector space decomposition:

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\lambda \in R(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\lambda \tag{3}$$

Tensoring this gives:

$$\mathfrak{g}_K = (\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))_K \oplus \bigoplus_{\lambda \in R(\mathfrak{g},\mathfrak{a})} (\mathfrak{g}_\lambda)_K \tag{4}$$

On the other hand, $\mathfrak{a}_K$ is split toral in $\mathfrak{g}_K$, so:

$$\mathfrak{g}_K = \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{a}_K) \oplus \bigoplus_{\lambda \in R(\mathfrak{g}_K,\mathfrak{a}_K)} (\mathfrak{g}_K)_\lambda \tag{5}$$

And finally, because $\mathfrak{t}_K$ is split toral in $\mathfrak{g}_K$:

$$\mathfrak{g}_K = \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{t}_K) \oplus \bigoplus_{\alpha \in R(\mathfrak{g}_K,\mathfrak{t}_K)} (\mathfrak{g}_K)_\alpha \tag{6}$$

These decompositions are strongly related. Recall that for any finite-dimensional vector space $V$ over $k$, we have a canonical inclusion

$$V \hookrightarrow K \otimes_k V, \quad v \longmapsto 1 \otimes v,$$

and a canonical isomorphism

$$K \otimes_k (V^*) \cong (K \otimes_k V)^*, \tag{7}$$

see [**Bo1**, II.6.4], where by abuse of notation the $^*$ denotes $k$-dual on the left, but $K$-dual on the right. Using this, the following lemma establishes identity between (4) and (5).

Lemma 3.1.19. — *The composed map*

$$\mathfrak{a}^* \hookrightarrow (\mathfrak{a}^*)_K \cong (\mathfrak{a}_K)^*$$

*induces a bijection $R(\mathfrak{g},\mathfrak{a}) \simeq R(\mathfrak{g}_K,\mathfrak{a}_K)$. We have $(\mathfrak{g}_\lambda)_K = (\mathfrak{g}_K)_\lambda$ for every $\lambda$ in $R(\mathfrak{g},\mathfrak{a})$ using this identification. Further, we can identify $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))_K$ and $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{a}_K)$ ("case $\lambda = 0$").*

*Proof.* — We have already remarked $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))_K = \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{a}_K)$ in 3.1.6. Calling $1 \otimes \lambda$ the image of $\lambda \in R(\mathfrak{g},\mathfrak{a})$ under the composed map, clearly $(\mathfrak{g}_\lambda)_K \subseteq (\mathfrak{g}_K)_{1\otimes\lambda}$. Comparing (4) and (5) and a consideration of dimensions shows that all these inclusions have to be equalities, which also shows that the map is surjective on the roots. □

The inclusion $\mathfrak{a}_K \subseteq \mathfrak{t}_K$ induces the restriction map

$$\rho : (\mathfrak{t}_K)^* \twoheadrightarrow (\mathfrak{a}_K)^*.$$

This connects (5) with (6):

PROPOSITION 3.1.20. — *We have*

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})_K = \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{t}_K) \oplus \bigoplus_{\substack{\alpha \in R(\mathfrak{g}_K, \mathfrak{t}_K) \\ \rho(\alpha)=0}} (\mathfrak{g}_K)_\alpha$$

*and for every* $\lambda \in R(\mathfrak{g}_K, \mathfrak{a}_K)$,

$$(\mathfrak{g}_\lambda)_K = \bigoplus_{\substack{\alpha \in R(\mathfrak{g}_K, \mathfrak{t}_K) \\ \rho(\alpha)=\lambda}} (\mathfrak{g}_K)_\alpha.$$

*The map* $\rho$ *restricts to a surjection*

$$\rho : R(\mathfrak{g}_K, \mathfrak{t}_K) \cup \{0\} \twoheadrightarrow R(\mathfrak{g}_K, \mathfrak{a}_K) \cup \{0\}.$$

*Proof.* — Using the identification of the foregoing lemma, the "$\supseteq$" in the asserted equations is clear. As before, comparing (5) and (6) and a consideration of dimensions proves all claims. □

COROLLARY 3.1.21. — *For any toral subalgebra* $\mathfrak{a} \subseteq \mathfrak{g}$, *the centraliser* $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ *is a reductive Lie algebra.*

*Proof.* — By 2.1.4, being reductive is stable under scalar extension; by 3.1.6, for a toral subalgebra, the centraliser commutes with scalar extension; thus we can w.l.o.g. assume $\mathfrak{a}$ to be split. Choose a maximal toral subalgebra $\mathfrak{t}$ containing $\mathfrak{a}$, and a splitting field $K$ for $\mathfrak{t}$. Again it suffices to check that $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))_K$ is reductive. By 3.1.20 and 3.1.5,

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})_K = \mathfrak{t}_K \oplus \bigoplus_{\substack{\alpha \in R(\mathfrak{g}_K, \mathfrak{t}_K) \\ \rho(\alpha)=0}} (\mathfrak{g}_K)_\alpha$$

and this is a reductive Lie algebra by [**Bo2**, VIII.3.1, Prop. 2]. □

We now *fix a maximal split toral subalgebra* $\mathfrak{s}$, *a maximal toral subalgebra* $\mathfrak{t}$ *containing* $\mathfrak{s}$ (again the existence of such a $\mathfrak{t}$ is clear by finite dimension), and *a finite Galois splitting extension* $K|k$ *for* $\mathfrak{t}$. We set

$$R := R(\mathfrak{g}_K, \mathfrak{t}_K), \quad \overline{R} := R(\mathfrak{g}_K, \mathfrak{s}_K), \quad R_0 := \{\alpha \in R : \rho(\alpha) = 0\}.$$

By 3.1.12 and 3.1.19, $R$ and $\overline{R}$ are root systems. The elements of $\overline{R}$ are sometimes called *restricted roots*, or *(k)-rational roots*. By 3.1.5, for $R$ we have the classical theory of [**Bo2**, VIII]. Finally, if $R_0$ is not empty, it also is a root system (in the vector space it generates): we obviously have $-\alpha \in R_0$ and $\alpha + \beta \in R \Rightarrow \alpha + \beta \in R_0$ for $\alpha, \beta \in R_0$

and conclude by [**Bo2**, VI.1.7, Prop. 23]. We will have a closer look at $R_0$ later: see the Subsection 3.1.2.1, in particular 3.1.31.

In the theory of algebraic and Lie groups, the roots and weights are constructed from some module of characters usually denoted by variants of the letter $X$. Here we just *define* $X$ to be the additive group (= $\mathbb{Z}$-module) in $(\mathfrak{t}_K)^*$ generated by $R$. By 3.1.8.i, it is a lattice, i.e. the natural map $K \otimes_{\mathbb{Z}} X \to (\mathfrak{t}_K)^*$ is bijective. (4)

We will now construct actions of the absolute Galois group $\mathcal{G}$ on several of the above objects, in particular on roots. Since they will all be induced by the action of $\mathcal{G}$ on $K$, we can and will see them as actions of $\Gamma := \mathrm{Gal}(K|k)$. We start with the action of $\Gamma$ on $\mathfrak{g}_K = K \otimes_k \mathfrak{g}$ (more general, every $\mathfrak{a}_K$ for a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$) given by operation on the first factor. Then the action of $\sigma \in \Gamma$ is $\sigma$-linear, i.e.

$$\sigma(ax+y) = \sigma(a)\sigma(x) + \sigma(y) \text{ for all } a \in K, x, y \in \mathfrak{g}_K,$$

and satisfies:

$$\big[\sigma(x), \sigma(y)\big] = \sigma\big([x,y]\big) \quad \text{for all } x, y \in \mathfrak{g}_K$$

In particular, $\sigma$ acts as automorphism of $R_{K|k}\,\mathfrak{g}_K$. Further, because $\sigma$ is an automorphism of $K$, for a $K$-subspace $U$ of $\mathfrak{g}_K$, $\sigma(U)$ is a $K$-subspace of the same $K$-dimension. The fixed set $(\mathfrak{g}_K)^\Gamma$ is $\mathfrak{g}$, and we define the following, $k$-linear, kind of a trace map:

$$\mathfrak{g}_K \twoheadrightarrow \mathfrak{g}, \quad x \longmapsto \widehat{x} := \sum_{\sigma \in \Gamma} \sigma(x).$$

It is surjective because for $x \in \mathfrak{g}$, $\widehat{x} = |\Gamma| \cdot x$.

We also have $\Gamma$-actions on $(\mathfrak{t}_K)^*$ and $(\mathfrak{s}_K)^*$ per (7), which we will for the moment write in the form ${}^\sigma\lambda$. Choosing e.g. a basis $t_1, \ldots, t_n$ of $\mathfrak{t}$ and the corresponding dual basis $u_1, \ldots, u_n$ of $\mathfrak{t}^*$, the isomorphism (7) writes $\lambda \in (\mathfrak{t}_K)^*$ as

$$\sum_{i=1}^{n} \lambda(1 \otimes t_i) \otimes u_i$$

so that, for $\sigma \in \Gamma$, ${}^\sigma\lambda$ is defined to be the $K$-linear map sending $1 \otimes t_i \in \mathfrak{t}_K$ to $\sigma(\lambda(1 \otimes t_i))$. Abusing notation, this is the map $\sigma \circ \lambda \circ \sigma^{-1}$, where the left $\sigma$ is the original $k$-automorphism of $K$ and the right one is the automorphism of $R_{K|k}\mathfrak{t}_K$.

Now for any weight $\alpha$ of $\mathfrak{t}_K$ in $\mathfrak{g}_K$, $\sigma((\mathfrak{g}_K)_\alpha)$ is a $K$-subspace of $\mathfrak{g}_K$, and for $x \in \sigma((\mathfrak{g}_K)_\alpha)$, say $x = \sigma(y)$ with $y \in (\mathfrak{g}_K)_\alpha$, we have

$$\begin{aligned}
[a \otimes t_i, x] &= \big[\sigma(\sigma^{-1}(a) \otimes t_i), \sigma(y)\big] \\
&= \sigma\big([\sigma^{-1}(a) \otimes t_i, y]\big) = \sigma\big(\alpha(\sigma^{-1}(a) \otimes t_i) \cdot y\big) \\
&= ({}^\sigma\alpha)(a \otimes t_i) \cdot x
\end{aligned}$$

for $a \in K$ and $t_i$ as above. Using that the $\sigma \in \Gamma$ act as automorphisms, we see:

4. This root lattice is often called $Q$ in the group case; it sits inside the weight lattice $P$. Roughly speaking, with $X := Q$ we define Lie algebras to be "of adjoint type"; less roughly said, their automorphism groups really are.

LEMMA 3.1.22

i. *For $\sigma \in \Gamma$ and a weight $\alpha$ of $\mathfrak{t}_K$ in $\mathfrak{g}_K$, ${}^\sigma\alpha$ also is a weight of $\mathfrak{t}_K$ in $\mathfrak{g}_K$. The corresponding weight space is $\sigma((\mathfrak{g}_K)_\alpha)$, whose $K$-dimension is the same as that of $(\mathfrak{g}_K)_\alpha$, namely, 1. The Galois action stabilises $R$, a fortiori restricts to $X$, turning it into a $\Gamma$-module.*
ii. *On $\overline{R}$, the Galois action is trivial.*
iii. *The map $\rho$ is equivariant with respect to the Galois actions. That is (by ii.), $\rho(\sigma(\alpha)) = \rho(\alpha)$ for all $\sigma \in \Gamma, \alpha \in R$.*

*Proof.* — Part i sums up the foregoing discussion. For part ii, analogous considerations apply, and additionally we can use that by 3.1.19, for a $k$-basis $s_1, \dots, s_m$ of $\mathfrak{s}$, we have $\lambda(1 \otimes s_i) \in k = K^\Gamma$ for all $1 \leq i \leq m$ and $\lambda \in \overline{R}$. Part iii follows from the canonicity of the isomorphism (7). □

REMARK 3.1.23. — We will later define another Galois action on the roots as a certain twisting of the above action – see 3.1.41.

From now on, if not noted otherwise, we will consider $R$ as a root system inside the $\mathbb{R}$-vector space $V := \mathbb{R} \otimes_{\mathbb{Z}} X$. The automorphism group $A(R)$ and the Weyl group $W(R)$ are thus subgroups of $\mathrm{Aut}_{\mathbb{R}}(V)$. The Galois action above is identified with a homomorphism $\Gamma \to A(R)$, and we will write $\sigma_A$ for the image of $\sigma \in \Gamma$ under this homomorphism (so ${}^\sigma\chi = \sigma_A(\chi)$ for $\chi \in X$). This enables us to define a Galois action on the Weyl group:

LEMMA 3.1.24. — *For $w \in W(R)$, set ${}^\sigma w := \sigma_A \circ w \circ \sigma_A^{-1}$, where composition is in $A(R)$. This defines an action of $\Gamma$ on the Weyl group.*

*Proof.* — One has ${}^\sigma w \in W(R)$ because $W(R)$ is normal in $A(R)$. Verification of the rest is straightforward. □

To see more structure in $X$ and $R$, we go back to our Lie algebras over $K$. The natural pairing between $\mathfrak{t}_K^*$ and $\mathfrak{t}_K$ – for which we have $({}^\sigma\ell)(\sigma(x)) = \sigma(\ell(x))$ for $\ell \in \mathfrak{t}_K^*$, $x \in \mathfrak{t}_K$ and $\sigma \in \Gamma$ – induces an inclusion-reversing bijection

$$\{\Gamma\text{-stable } K\text{-subspaces of } \mathfrak{t}_K\} \rightleftarrows \{\Gamma\text{-stable } K\text{-subspaces of } (\mathfrak{t}_K)^*\},$$
$$\mathfrak{a} \longrightarrow \mathfrak{a}^\perp := \{\ell \in (\mathfrak{t}_K)^* : \ell(\mathfrak{a}) = 0\},$$

where ${}^\perp L := \{t \in \mathfrak{t}_K : L(t) = 0\} \leftarrow L$.

Note that since $\mathfrak{t}_K$ is abelian, we can write "subalgebras" for "subspaces" on the left. Also, the projection $(\mathfrak{t}_K)^* \twoheadrightarrow (\mathfrak{t}_K)^*/\mathfrak{a}^\perp$ can be identified with the restriction map $(\mathfrak{t}_K)^* \twoheadrightarrow \mathfrak{a}^*$, $\ell \mapsto \ell_{|\mathfrak{a}}$. Since $X$ spans $(\mathfrak{t}_K)^*$ by 3.1.8.i, the above induces an inclusion-reversing bijection

$$\{\Gamma\text{-stable subalgebras of } \mathfrak{t}_K\} \rightleftarrows \{\Gamma\text{-stable saturated submodules of } X\},$$
$$\mathfrak{a} \longrightarrow {}_{\mathfrak{a}}X := \big\{\chi \in X : \chi(\mathfrak{a}) = 0\big\},$$
$$\mathfrak{a}_Y := \{t \in \mathfrak{t}_K : Y(t) = 0\} \longleftarrow Y$$

where a submodule $Y \subseteq X$ is called *saturated* if $X/Y$ has no $\mathbb{Z}$-torsion. Using 3.1.8.i again, we can identify $X/{}_{\mathfrak{a}}X$ with a lattice in $\mathfrak{a}^*$ so that the projection $X \twoheadrightarrow X/{}_{\mathfrak{a}}X$ again corresponds to the restriction $\chi \mapsto \chi_{|\mathfrak{a}}$.

Finally, note that the set on the left is in bijection with the set of $k$-subalgebras of $\mathfrak{t}$: Every $\Gamma$-stable $K$-subalgebra (= $K$-subspace) $\mathfrak{a}$ of $\mathfrak{t}_K$ arises by scalar extension from a $k$-subalgebra (= $k$-subspace) of $\mathfrak{t}$, namely, from $\mathfrak{a}^\Gamma$ (see e.g. [**Spr3**, Prop. 11.1.4]).

Definition 3.1.25. — Define the $\Gamma$-stable saturated submodule

$$X_0 := \Big\{\chi \in X : \sum_{\sigma\in\Gamma} {}^{\sigma}\chi = 0\Big\}.$$

In general we have

$$|\Gamma| \cdot X_0 \subsetneq \big\langle\{{}^{\sigma}\chi - \chi : \chi \in X\}\big\rangle_{\mathbb{Z}} \subsetneq X_0, \tag{8}$$

but the following shows that $X/X_0$ is well-suited as $\Gamma$-coinvariants of $X$. Let $\mathfrak{r} \subseteq \mathfrak{t}$ be a subalgebra. $\mathfrak{a} := \mathfrak{r}_K$ is a $\Gamma$-stable subalgebra of $\mathfrak{t}_K$. We have an induced action of $\Gamma$ on $X/{}_{\mathfrak{a}}X$.

Proposition 3.1.26. — *With the above notation, the following are equivalent:*

i. $X_0 \subseteq {}_{\mathfrak{a}}X$.
ii. *The* $\Gamma$*-action on* $X/{}_{\mathfrak{a}}X$ *is trivial.*
iii. $\mathrm{im}(\alpha_{|\mathfrak{r}}) \subseteq k$ *for all* $\alpha \in R$.
iv. $\mathfrak{r}$ *is a split toral subalgebra of* $\mathfrak{g}$.

*Proof.* — i. $\Rightarrow$ ii. Obviously, $\chi - {}^{\sigma}\chi \in X_0$ for all $\chi \in X$, $\sigma \in \Gamma$. But condition ii. is equivalent to $\chi - {}^{\sigma}\chi \in {}_{\mathfrak{a}}X$ for all $\chi \in X, \sigma \in \Gamma$.

ii. $\Rightarrow$ i. Let $\chi \in X_0$. As remarked, ii. says that $\chi - {}^{\sigma}\chi \in {}_{\mathfrak{a}}X$ for all $\sigma \in \Gamma$, so also

$$\sum_{\sigma\in\Gamma}(\chi - {}^{\sigma}\chi) = |\Gamma| \cdot \chi - \underbrace{\sum_{\sigma\in\Gamma} {}^{\sigma}\chi}_{=0} \in {}_{\mathfrak{a}}X$$

and therefore $\chi \in {}_{\mathfrak{a}}X$ since ${}_{\mathfrak{a}}X$ is saturated.

ii. $\Leftrightarrow$ iii. Choose a $k$-basis $t_1, \dots, t_n$ of $\mathfrak{t}$ such that $t_1, \dots, t_r$ $(r \leq n)$ form a basis of $\mathfrak{r}$. Then using our identifications and the description of the Galois action in 3.1.22, we see that ii. is equivalent to $\alpha(t_i) \in K^\Gamma = k$ for $1 \leq i \leq r$ and all $\alpha \in R$, which is iii.

iii. $\Rightarrow$ iv. Remember the trace $\widehat{x} := \sum_{\sigma\in\Gamma}\sigma(x) \in \mathfrak{g}$ for $x \in \mathfrak{g}_K$. For $\alpha \in R$, $x \in (\mathfrak{g}_K)_\alpha$ and $r \in \mathfrak{r}$, condition iii. implies $[1\otimes r, \widehat{x}] = \alpha(1\otimes r)\widehat{x}$; that is, $\widehat{x}$ is an element of the weight space to the weight $\alpha_{|\mathfrak{r}}$ of $\mathfrak{r}$ in $\mathfrak{g}$. Now decomposing

$$x = \sum_{\alpha\in R\cup\{0\}} x_\alpha$$

we have

$$\widehat{x} = \sum_{\alpha\in R\cup\{0\}} \widehat{x}_\alpha$$

and by $k$-linearity and surjectivity of the trace, it follows that $\mathfrak{g}$ is the sum of the weight spaces of $\mathfrak{r}$ in $\mathfrak{g}$. Hence each $\mathrm{ad}_{\mathfrak{g}}(r)$ for $r \in \mathfrak{r}$ is diagonalisable over $k$.

iv. $\Rightarrow$ iii. This follows from the relations between the weight decompositions with respect to $\mathfrak{r}, \mathfrak{t}, \mathfrak{r}_K$ and $\mathfrak{t}_K$ in 3.1.19 and 3.1.20. □

Corollary 3.1.27

i. *$X_0$ is the annihilator of $\mathfrak{s}_K$ in $X$. Every subalgebra of $\mathfrak{t}$ which is split toral in $\mathfrak{g}$ is contained in $\mathfrak{s}$: in other words, $\mathfrak{s}$ is the unique maximal split toral subalgebra of $\mathfrak{g}$ contained in $\mathfrak{t}$. The map $\rho$ is induced by the projection $X \twoheadrightarrow X/X_0$ which we will also call $\rho$.*

ii. $R_0 = R\cap X_0$.

*Proof*

i. The proposition and the foregoing discussions imply that the annihilator $\mathfrak{a}_{X_0}$ of $X_0$ is of the form $\mathfrak{r}_K$, where $\mathfrak{r}$ is a split toral subalgebra of $\mathfrak{g}$ that contains all other subalgebras of $\mathfrak{t}$ which are split toral in $\mathfrak{g}$. In particular, it contains $\mathfrak{s}$, and by maximality of $\mathfrak{s}$, must be equal to it.

ii. The assertion follows from the first statement in i. □

Note that in general, $X_0$ is *not* generated by $R_0$, see example 3.2.9. The submodule generated by $R_0$ will turn up when we now have a closer look at:

**3.1.2.1.** *The anisotropic kernel.* — By 3.1.21, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ is reductive. More precisely, the derived algebra of $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}))_K$ is the semisimple

$$\underbrace{\Big(\sum_{\alpha\in R_0} K\cdot H_\alpha\Big)}_{\subseteq \mathfrak{t}_K} \oplus \bigoplus_{\alpha\in R_0} (\mathfrak{g}_K)_\alpha$$

and its centre is $\bigcap_{\alpha\in R_0}\ker(\alpha) \subseteq \mathfrak{t}_K$. Write $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{Z}\times\mathcal{D}\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ where $\mathfrak{Z}$ is the centre of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, and the derived algebra $\mathcal{D}\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ is semisimple.

Remark 3.1.28. — We have the chain of inclusions

$$\mathfrak{s} \subseteq \mathfrak{Z} \subseteq \mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$$

(For the second one, remark that $\mathfrak{t}$ is its own centraliser in $\mathfrak{g}$ and a fortiori in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$.) As $\mathfrak{t}$ is abelian, the second inclusion is an equality if the third one is, and we will see the converse shortly. In example 3.2.9, the first inclusion will be proper; in example 3.2.5, the second and third one will be.

Remark 3.1.29. — All elements of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ are semisimple (in $\mathfrak{g}$). All elements of the semisimple Lie algebra $\mathfrak{g}_a := \mathcal{D}\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ are semisimple (in $\mathfrak{g}_a$).

*Proof.* — In step 2 of the proof of 3.1.5 we have seen that with an element $x \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, its semisimple and nilpotent part are also in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. So for the first assertion it suffices to see there is no $x \neq 0$ in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ which is nilpotent in $\mathfrak{g}$. If there were, by a Jacobson-Morozov argument improved with weight decomposition like in the proof of 3.1.9, we would find an $h \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ with $[h, x] = 2x$ on the one hand, but $h$ ad-diagonalisable in $\mathfrak{g}$ on the other hand; since it commutes with $\mathfrak{s}$, the latter implies that $h$ is contained in $\mathfrak{s}$, hence $[h, x] = 0$, a contradiction. For the second assertion note that for $x \in \mathfrak{g}_a$, $\mathrm{ad}_{\mathfrak{g}_a}(x)$ is the restriction of the semisimple endomorphism $\mathrm{ad}_{\mathfrak{g}}(x)$ to the $\mathrm{ad}_{\mathfrak{g}}(x)$-stable subspace $\mathfrak{g}_a$, and therefore itself semisimple. □

Definition 3.1.30. — The semisimple Lie algebra

$$\mathfrak{g}_a := \mathcal{D}\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \qquad (\cong \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})/\mathfrak{Z})$$

is called the *anisotropic kernel* of $\mathfrak{g}$ (with respect to $\mathfrak{s}$). We call $X_a$ (resp. $V_a$) the submodule of $X$ (resp. subspace of $V$) generated by $R_0$. [5] As noted before, we have $X_a \subseteq X_0$ without equality in general. Remark that in the duality setup before definition 3.1.25, $X_a$ is the annihilator of $\mathfrak{Z}_K$ in $X$.

The anisotropic kernels are the bad guys because they generally resist classification. In later chapters we will see that they disappear over C1 fields and are under strict control over $\mathfrak{p}$-adic fields. The opposite cases $\mathfrak{g}_a = \mathfrak{g}$ and $\mathfrak{g}_a = \{0\}$ will be treated in Section 4.2, where we also say something about the terminology.

Proposition 3.1.31. — *The set* $\mathfrak{t}_a := \mathfrak{t} \cap \mathfrak{g}_a$ *is a maximal toral subalgebra of* $\mathfrak{g}_a$, *and*

$$(\mathfrak{t}_a)_K = \sum_{\alpha \in R_0} K \cdot H_\alpha$$

*is split maximal toral in*

$$(\mathfrak{g}_a)_K = (\mathfrak{t}_a)_K \oplus \bigoplus_{\alpha \in R_0} (\mathfrak{g}_K)_\alpha.$$

*Restriction* $\alpha \mapsto \alpha_{|(\mathfrak{t}_a)_K}$ *induces a bijection* $R_0 \simeq R((\mathfrak{g}_a)_K, (\mathfrak{t}_a)_K)$.

5. "$a$" for anisotropic. We found notations like $X^0$ (cf. [**Sat2**], [**Sat3**]) or $X_1$ (cf. [**Spr3**]) more confusing.

*Proof.* — It is clear that $\mathfrak{t}_a$ is a toral subalgebra of $\mathfrak{g}_a$. Because $\mathfrak{Z} \subseteq \mathfrak{t}$, an abelian subalgebra $\mathfrak{a} \supsetneq \mathfrak{t}_a$ in $\mathfrak{g}_a$ would give rise to an abelian subalgebra $\mathfrak{Z} + \mathfrak{a} \supsetneq \mathfrak{t}$ in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, but we had remarked in 3.1.6 that $\mathfrak{t}$ is maximal abelian in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. So $\mathfrak{t}_a$ is maximal toral in $\mathfrak{g}_a$ and consequently $(\mathfrak{t}_a)_K$ is split maximal toral in $(\mathfrak{g}_a)_K$. Finally, we have $(\mathfrak{g}_a)_K = \mathcal{D}((\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}))_K)$ (cf. [**Bo2**, I.1.9]), and according to earlier remarks, $\mathfrak{t}_K \cap (\mathfrak{g}_a)_K = \sum_{\alpha\in R_0} K \cdot H_\alpha$ also is a maximal toral subalgebra of $(\mathfrak{g}_a)_K$, so the evident inclusion $(\mathfrak{t}_a)_K \subseteq \mathfrak{t}_K \cap (\mathfrak{g}_a)_K$ is an equality. The last assertion is an immediate consequence. □

Corollary 3.1.32. — *One has the equivalences*

$$\mathfrak{Z} = \mathfrak{t} \ \Leftrightarrow\ \mathfrak{t}_a = \{0\} \ \Leftrightarrow\ \mathfrak{g}_a = \{0\} \ \Leftrightarrow\ \mathfrak{Z} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \ \Leftrightarrow\ \mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \ \Leftrightarrow\ X_a = \{0\},$$

### 3.1.3. Relative Weyl groups, Γ-linear orders and the twisted Galois action

For later use, we introduce certain subgroups of the Weyl group $W(R)$.

Lemma 3.1.33. — *Mapping $s_\alpha \in \mathrm{GL}(V_a)$ to $s_\alpha \in \mathrm{GL}(V)$ defines a group isomorphism from $W(R_0)$ to the subgroup of $W(R)$ generated by $\{s_\alpha : \alpha \in R_0\}$; call this group $W_0$. It is invariant under the Galois action on $W(R)$ from* 3.1.24.

*Proof.* — The first assertion is true in the general situation of a subspace $U \subseteq V$ generated by a symmetric and closed subsystem of roots as considered in [**Bo2**, VI.7, Prop. 23]). Namely, there is a $W(R)$-invariant scalar product $\langle .\,,.\rangle$ on $V$ (cf. [**Bo2**, VI.1.1, Prop. 3]) and hence (cf. [**Bo2**, V.2.3]), $s_\alpha \in \mathrm{GL}(V)$ is given by

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Writing $U^\perp$ for the orthogonal complement of $U$ in $V$ with respect to $\langle .\,,.\rangle$, we see that for $\alpha \in U$, the spaces $U$ and $U^\perp$ are $s_\alpha$-invariant; moreover, the restriction $s_{\alpha|U}$ is $s_\alpha \in \mathrm{GL}(U)$ (by the identification [**Bo2**, VI.1.7, Prop. 23]), whereas $s_{\alpha|U^\perp} = \mathrm{id}_{U^\perp}$.

The invariance under the Galois action follows from the invariance of $R_0$ under the Galois action. □

Remark 3.1.34. — The proof also shows: The induced actions of $W_0$ on $X/X_a$ and $V/V_a$, a fortiori on $X/X_0$ and $V/V_0$, are trivial.

One of our aims is to find the Weyl group of $\overline{R}$ as a subquotient in $W(R)$. In this direction, up to here we can prove:

Lemma 3.1.35. — *Set $W_\Gamma := \{w \in W(R) : w(X_0) = X_0\}$.*

i. *$W_0$ is a normal subgroup of $W_\Gamma$.*
ii. *We have a homomorphism $\pi : W_\Gamma \to A(\overline{R})$ with $W_0 \subseteq \ker(\pi)$.*

*Proof.* — Part i follows from $ws_\alpha w^{-1} = s_{w(\alpha)}$ for $\alpha \in R_0, w \in W$. For part ii, we had seen earlier that the morphism of Γ-modules

$$\rho : X \twoheadrightarrow X/X_0$$

corresponds to the restriction $\chi \mapsto \chi_{|\mathfrak{s}_K}$ (identifying $X/X_0$ with a lattice in $\mathfrak{s}_K^*$), and induces a surjection $R \twoheadrightarrow \overline{R}$. For $w \in W_\Gamma$, define $\pi(w)(\rho(\chi)) := \rho(w(\chi))$ for any $\chi \in X$. By the definition of $W_\Gamma$, $\pi(w)$ is a well-defined automorphism of $X/X_0$ that stabilises $\overline{R}$. Equality $\pi(W_0) = \{\mathrm{id}\}$ follows from the previous remark. □

We will see in 3.1.48 that in fact $\ker(\pi) = W_0$ and $\mathrm{im}(\pi) = W(\overline{R})$. But for this, we need the notion of a $\Gamma$-*linear order*, introduced by Satake [**Sat2**].

**3.1.3.1.** $\Gamma$-*linear orders.* — Let $\leq$ be a linear group order on $X$, i.e. a total order with $x \leq y \Rightarrow x+z \leq y+z$ for all $x, y, z \in X$. As usual, $x > y$ means the negation of $x \leq y$, and the meaning of $<$ and $\geq$ is clear. Defining one such linear order is equivalent to choosing an additively closed subset $X^+ \subset X$ such that $X = X^+ \sqcup \{0\} \sqcup -X^+$, the correspondence being given by $X^+ = \{\chi \in X : \chi > 0\}$.

Lemma 3.1.36. — *The following are equivalent:*

i. *For $\chi \in X \setminus X_0$ with $\chi > 0$, we have $\chi' > 0$ for all $\chi' \equiv \chi \mod X_0$.*
ii. *For $\chi \in X \setminus X_0$ with $\chi > 0$, we have ${}^\sigma\chi > 0$ for all $\sigma \in \Gamma$.*

*Proof.* — i. $\Rightarrow$ ii. is clear from ${}^\sigma\chi - \chi \in X_0$. For the converse, remark first that ii. implies $\chi > 0 \Leftrightarrow$ all ${}^\sigma\chi > 0 \Leftrightarrow \sum_{\sigma\in\Gamma} {}^\sigma\chi > 0$ for any $\chi \in X$. Now if $\chi$ as in i. is given and $\chi' = \chi + \chi_0$ with $\chi_0 \in X_0$, then

$$\sum_{\sigma\in\Gamma} {}^\sigma\chi' = \sum_{\sigma\in\Gamma} {}^\sigma\chi$$

and this is $> 0$ by ii. □

Definition 3.1.37. — A linear group order on $X$ is called a $\Gamma$-*linear order* if it satisfies the equivalent conditions above.

Remark 3.1.38. — A $\Gamma$-linear order on $X$ clearly induces linear group orders on $X/X_0$ and $X_0$. Conversely, given linear group orders on $X/X_0$ and $X_0$, there is a unique $\Gamma$-linear order on $X$ inducing them. In particular, $\Gamma$-linear orders exist: For a $\mathbb{Z}$-module isomorphic to some $\mathbb{Z}^r$ (as $X_0$ and $X/X_0$ are), the *lexicographic order* on $\mathbb{Z}^r$:

$$(x_1, \dots, x_r) > 0 \iff \text{there is } s \in \{1, \dots, r\} \text{ with } x_1 = \dots = x_{s-1} = 0 \text{ and } x_s > 0$$

transports to a linear group order on it (depending on the chosen isomorphism, i.e. on the choice of basis).

Given any linear group order on $X$, the roots in $R$ which are positive with respect to this order form a positive system of roots in the sense of [**Bo2**, VI.1.7]; therefore there is a unique basis $\Delta$ of $R$ such that the positive roots with respect to this basis are exactly the positive roots with respect to the given linear order. A basis associated in this way with a $\Gamma$-linear order is called a $\Gamma$-*basis.*

From now on, *fix a $\Gamma$-basis $\Delta$ of $R$.*

In general, for a root system $R$ with basis $D$ and closed subsystem $R'$, it is not true that $D \cap R'$ is a basis of $R'$. In our situation, the $\Gamma$-linear order remedies this:

Proposition 3.1.39. — *Let $\Delta_0$ be the basis of $R_0$ with respect to the induced order. Then $\Delta_0 = \Delta \cap R_0 = \Delta \cap X_0$.*

*Proof.* — It suffices to show that every (w.l.o.g.: positive) $\alpha \in R_0$ is a linear combination of the elements of $\Delta \cap R_0$. Let $\delta_1, \dots, \delta_r$ be the different elements of $\Delta$ arranged such that $\Delta \cap R_0 = \{\delta_i : i < s\}$ for a certain $0 \le s \le r+1$. Write $\alpha = \sum_{i=1}^{r} n_i \delta_i$ with all $n_i \in \mathbb{Z}_{\ge 0}$. If we had $n_k \ge 1$ for some $k \ge s$, then

$$\delta_k \equiv \delta_k - \alpha = \Big( \sum_{\substack{1 \le i \le r \\ i \ne k}} -n_i \delta_i \Big) - (n_k - 1)\delta_k \mod X_0,$$

but $\delta_k \notin X_0$ is positive and the right hand side is not, contradicting 3.1.36. □

Let $\overline{\Delta} := \rho(\Delta \setminus \Delta_0) = \rho(\Delta) \setminus \{0\}$. We will see in 3.1.45 that this is a basis of $\overline{R}$.

Lemma 3.1.40. — *Let $\Delta'$ be another $\Gamma$-basis.*

i. *$\Delta = \Delta'$ if and only if $\Delta_0 = \Delta_0'$ and $\overline{\Delta} = \overline{\Delta'}$.*

ii. *If $\overline{\Delta} = \overline{\Delta'}$, then there is a $w_0 \in W_0$ with $w_0(\Delta) = \Delta'$ (and unique in $W(R)$ with this property).*

*Proof.* — For the non-trivial direction in i, the conditions imply that all $\alpha \in \Delta'$ are positive with respect to a $\Gamma$-linear order defining $\Delta$, hence with respect to $\Delta$, so the bases must coincide. For part ii, by the proposition and 3.1.33 there is $w_0 \in W_0$ with $w_0(\Delta_0) = \Delta_0' = w_0(\Delta) \cap R_0 = (w_0(\Delta))_0$; on the other hand,

$$\overline{w_0(\Delta)} = \rho\big(w_0(\Delta) \setminus \Delta_0'\big) = \pi(w_0)\big(\rho(\Delta)\big) \setminus \{0\} = \rho(\Delta) \setminus \{0\} = \overline{\Delta} = \overline{\Delta}',$$

so $w_0(\Delta) = \Delta'$ by part i. The uniqueness assertion follows from $W(R)$ acting simply transitively on bases. □

**3.1.3.2.** *The twisted Galois action.* — We will now define a new action of $\Gamma$ on the roots, as follows:

Let $\sigma \in \Gamma$. By 3.1.22 and the characterisation of bases in [**Bo2**, VI.1.7, Cor. 3], $\sigma$ acts on $R$ in such a way that $\sigma_A(\Delta)$ is another basis (remember $\sigma_A$ is the element of $A(R)$ through which $\sigma$ acts). Hence there is a unique element $w_\sigma \in W(R)$ such that $w_\sigma(\sigma_A(\Delta)) = \Delta$. We will soon show the crucial fact that $w_\sigma \in W_0$.

Definition 3.1.41. — For $\chi \in X$, set [(6)]

$$t(\sigma)(\chi) := w_\sigma(\sigma_A(\chi))$$

6. The twisted action is denoted as $\sigma^*$ in [**Tit1**, 2.3], ${}_\Delta\sigma$ in [**BT1**, 6.2], $[\sigma]$ in [**Sat3**, II 2.1] (where $w_\sigma$ is our $w_\sigma^{-1}$), $\varphi(\sigma)$ in [**Spr2**, 2.6.1], $\tau(\sigma)$ in [**Spr3**, 15.5.2].

Lemma 3.1.42. — *This defines a group homomorphism* $t : (\mathcal{G} \twoheadrightarrow)\Gamma \to \mathrm{Aut}_{\mathbb{Z}}(X)$ *such that each* $t(\sigma)$ *stabilises* $R$ *and* $\Delta$*; so we can and will also view* $t$ *as homomorphism to the subgroup* $Aut(R, \Delta) \simeq A(R)/W(R)$ *of* $A(R)$.

*Proof.* — For the first assertion, recall the $\Gamma$-action on $W(R)$ by conjugation from 3.1.24. The uniqueness statement before the definition implies that $w_{\sigma\tau} = w_\sigma{}^\sigma(w_\tau)$ for any $\sigma, \tau \in \Gamma$, meaning that $\sigma \mapsto w_\sigma$ is a 1-cocycle $\Gamma \to W(R)$; this gives $t(\sigma\tau) = w_{\sigma\tau}\sigma_A\tau_A = w_\sigma\sigma_A w_\tau\tau_A = t(\sigma)t(\tau)$. The rest follows directly from 3.1.22 and the definitions. □

Proposition 3.1.43 (Properties of $t$)

i. $t(\Gamma)$ *stabilises* $\Delta_0$*; a fortiori, also* $\Delta \setminus \Delta_0$, $R_0$ *and* $R \setminus R_0$.
ii. *For* $\chi \in X$ *and* $\sigma \in \Gamma$, *we have*
$$t(\sigma)(\chi) \equiv {}^\sigma\chi \mod X_a \quad \textit{and} \quad {}^\sigma\chi \equiv \chi \mod X_0.$$
iii. *For* $\chi, \psi \in X \setminus X_0$, *we have* $\rho(\chi) = \rho(\psi)$ *if there is* $\sigma \in \Gamma$ *with* $t(\sigma)(\chi) = \psi$.
iv. *Write* $\Delta = \{\delta_1, \ldots, \delta_r\}$ *such that* $\Delta_0 = \{\delta_1, \ldots, \delta_{s-1}\}$ (*as in the proof of* 3.1.39). *Then for* $i \geq s$ *and* $\sigma \in \Gamma$, *we have* $\sigma_A(\delta_i) = t(\sigma)(\delta_i) + \sum_{j=1}^{s-1} n_j\delta_j$ *for certain* $n_j \in \mathbb{Z}_{\geq 0}$.
v. (*Converse of* iii *for simple roots.*) *With the notations of iv, if* $\rho(\delta_i) = \rho(\delta_k)$ *for* $i, k \geq s$, *then there is* $\sigma \in \Gamma$ *with* $t(\sigma)(\delta_i) = \delta_k$.

*Proof*

i. Let $\sigma \in \Gamma$. With the same argument as before definition 3.1.41, there is a unique $w_{0,\sigma} \in W_0$ such that $w_{0,\sigma}(\sigma_A(\Delta_0)) = \Delta_0$. So it suffices to show the following *claim*: Using the identification of 3.1.33, this $w_{0,\sigma}$ is the $w_\sigma$ defined before.

   As seen earlier, $w_{0,\sigma}$ (being an element of $W_0$) induces the identity on $X/X_0$, which by 3.1.36 implies that $w_{0,\sigma}(\alpha) > 0$ and $\notin R_0$ for $\alpha \in R \setminus R_0$ with $\alpha > 0$. Analogously $\sigma_A(\alpha) > 0$ and $\notin R_0$ for $\alpha \in R \setminus R_0$ with $\alpha > 0$. Thus $w_{0,\sigma} \circ \sigma_A$ stabilises the set of positive elements of $R$ which are not in $R_0$. By definition, it stabilises the positive elements of $R_0$, so it stabilises the set of all positive elements. Thus, $w_{0,\sigma}(\sigma_A(\Delta))$ is a basis of $R$ consisting of positive elements with respect to the basis $\Delta$ and hence must itself be $\Delta$, and our claim follows from the uniqueness property used to define $w_\sigma$.
ii. The first claim follows from $w_\sigma \in W_0$ and 3.1.34, the second from 3.1.27.
iii. With part ii this is immediate from 3.1.26 and 3.1.27.
iv. By part i, $t(\sigma)(\delta_i) = \delta_k$ for some $k \geq s$, so by part ii, ${}^\sigma\delta_i \in R$ is of the form
$$\delta_k + \sum_{j=1}^{s-1} n_j\delta_j$$
   with $n_j \in \mathbb{Z}$. $\Delta$ is a basis and one coefficient is positive, so all $n_j \geq 0$.

v. For any $\chi, \psi \in V \setminus V_0$, $\rho(\chi) = \rho(\psi)$ implies (and is in fact equivalent to) $\sum_{\sigma\in\Gamma} \sigma_A(\chi) = \sum_{\sigma\in\Gamma} \sigma_A(\psi)$. Thus iv implies
$$\sum_{\sigma\in\Gamma} t(\sigma)(\delta_i) = \sum_{\sigma\in\Gamma} t(\sigma)(\delta_k) + \sum_{j\le s-1} m_j\delta_j$$
for certain $m_j \in \mathbb{Z}$. Since $t(\Gamma)$ stabilises $\{\delta_j : j \ge s\}$ by i, and $\Delta$ is a vector space basis of $V$, one $t(\sigma)(\delta_i)$ has to be $\delta_k$ (and all $m_j = 0$). □

Remark 3.1.44. — The twisted Galois action on roots can also be defined (on $\Delta$, and then linearly extended) via the Galois action on parabolic subalgebras, and the correspondence of [**Bo2**, VIII.3.4, Rem.] between elements of $\Delta$ and certain parabolic subalgebras. Cf. [**Tit1**, 2.3] and [**BT1**, 6.2].

Corollary 3.1.45
i. *$V_0$ is generated by $\Delta_0 \cup \{\delta - \varepsilon : \delta, \varepsilon \in \Delta \setminus \Delta_0, \rho(\delta) = \rho(\varepsilon)\}$.*
ii. *$\overline{\Delta}$ is a basis of $\overline{R}$.*

*Proof*
i. By (8), $V_0$ is generated by $\{{}^\sigma\chi - \chi : \chi \in V, \sigma \in \Gamma\}$ and a fortiori ($\Delta$ spans $V$) by $\{{}^\sigma\delta - \delta : \delta \in \Delta, \sigma \in \Gamma\}$. Since ${}^\sigma\delta \in R \cap X_0 = R_0 \subset X_a = \mathbb{Z}\Delta_0$ for $\delta \in \Delta_0$, it is spanned by $\Delta_0 \cup \{{}^\sigma\delta - \delta : \delta \in \Delta \setminus \Delta_0, \sigma \in \Gamma\}$, so by part iv of the previous proposition, by $\Delta_0 \cup \{t(\sigma)(\delta) - \delta : \delta \in \Delta \setminus \Delta_0, \sigma \in \Gamma\}$ which is contained in the set in the claim.
ii. It suffices to show that the distinct elements of $\overline{\Delta}$, call them $\lambda_1, \dots, \lambda_r$, are linearly independent in $V/V_0$. For each of them fix one $\delta_{j(i)} \in \Delta \setminus \Delta_0$ with $\rho(\delta_{j(i)}) = \lambda_i$. Then the set $\Delta_0 \cup \{\delta - \delta_{j(i)} : 1 \le i \le r, \delta_{j(i)} \ne \delta \in \Delta \setminus \Delta_0, \rho(\delta) = \rho(\delta_{j(i)})\}$ is linearly independent (because the $\delta \in \Delta$ are), and by part i generates $V_0$. Since its cardinality is $|\Delta| - r = \dim_k V - r$, we have $r = \dim_k V/V_0$. □

**3.1.3.3.** *Relative Weyl groups revisited.* — We can now complete our earlier investigations of the Weyl group. Recall the groups $W_\Gamma, W_0$ and the map $\pi$ from 3.1.35; we want to show that $\pi$ induces an isomorphism $W_\Gamma/W_0 \simeq W(\overline{R})$. Along the way, we improve 3.1.15.

Recall from [**Bo2**, VIII.5.2] that there is a well-defined group homomorphism
$$\varepsilon_{\mathfrak{t}_K} : \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K) \twoheadrightarrow W(R), \quad s \longmapsto [\chi \mapsto \chi \circ s^{-1}_{|\mathfrak{t}_K} \text{ for } \chi \in X]$$
(the contragredient of the restriction to $\mathfrak{t}_K$) which induces an isomorphism
$$\mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K)/\ker(\varepsilon_{\mathfrak{t}_K}) \cong W(R), \tag{9}$$
where $\ker(\varepsilon_{\mathfrak{t}_K}) = \{s \in \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K) : s_{|\mathfrak{t}_K} = \mathrm{id}_{\mathfrak{t}_K}\}$.

Lemma 3.1.46

i. *For* $s \in \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K)$,

$$\varepsilon_{\mathfrak{t}_K}(s) \in W_\Gamma \iff s \in \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K),$$
$$\varepsilon_{\mathfrak{t}_K}(s) \in W_0 \iff s_{|\mathfrak{s}_K} = \mathrm{id}_{\mathfrak{s}_K}.$$

ii. $\mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K) =$

$$\big(\mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K) \cap \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K)\big) \cdot \big\{s \in \mathrm{Aut}_e(\mathfrak{g}_K) : s_{|\mathfrak{s}_K} = \mathrm{id}_{\mathfrak{s}_K}\big\}.$$ [7]

iii. *There is a canonical isomorphism*

$$\mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K)/\{s \in \mathrm{Aut}_e(\mathfrak{g}_K) : s_{|\mathfrak{s}_K} = \mathrm{id}_{\mathfrak{s}_K}\} \cong W_\Gamma/W_0.$$

*Proof.* — Part i follows from the duality of $\mathfrak{s}_K$ and $X_0$ in the sense of 3.1.27. Part iii is a consequence of (9), the first two parts and one of the isomorphism theorems.

In part ii, the inclusion "$\supseteq$" is trivial. So let $s \in \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K)$; then $s(\mathfrak{t}_K)$ is another maximal toral subalgebra of $\mathfrak{g}_K$, and is contained in $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$ (because $s$ stabilises $\mathfrak{s}_K$ and a fortiori its centraliser). So by the conjugacy theorem 3.1.16 [8] applied to $\mathfrak{t}_K$ and $s(\mathfrak{t}_K)$ in $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$, there is $s_1 \in \mathrm{Aut}_e(\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K))$ with $s_1(s(\mathfrak{t}_K)) = \mathfrak{t}_K$, so $s_1 s$ in $\mathrm{Aut}_e(\mathfrak{g}_K)$ stabilises both $\mathfrak{t}_K$ and $\mathfrak{s}_K$. The proof is finished with the following fact: $s_1$, like any element of $\mathrm{Aut}_e(\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K))$, can naturally be extended to an element of $\{s \in \mathrm{Aut}_e(\mathfrak{g}_K) : s_{|\mathfrak{s}_K} = \mathrm{id}_{\mathfrak{s}_K}\}$.

To see this, let $n \in \mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$ be $\mathrm{ad}_{\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)}$-nilpotent; write $n = n_{\mathfrak{Z}} + n_{\mathcal{D}}$ for the decomposition $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K) = \mathfrak{Z} \times \mathcal{D}$ into centre and semisimple part. Then $\exp(\mathrm{ad}_{\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)} n) = \exp(\mathrm{ad}_{\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)} n_{\mathcal{D}})$ on $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$, and $n_{\mathcal{D}}$ is nilpotent in $\mathcal{D}$. By 2.1.8, $\mathrm{ad}_{\mathfrak{g}_K}(n_{\mathcal{D}})$ is nilpotent, and exp of it obviously extends $\exp(\mathrm{ad}_{\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)} n)$ (which fixes $\mathfrak{s}_K$ pointwise). □

We can now put things together with 3.1.15. The homomorphism $\varepsilon_{\mathfrak{s}}$ via our identifications extends to

$$\varepsilon_{\mathfrak{s}_K} : \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{s}_K) \longrightarrow A(\overline{R}), \quad s \longmapsto [\chi \mapsto \chi \circ s^{-1}_{|\mathfrak{s}_K} \text{ for } \chi \in X/X_0].$$

Its kernel is $\{s \in \mathrm{Aut}_e(\mathfrak{g}_K) : s_{|\mathfrak{s}_K} = \mathrm{id}_{\mathfrak{s}_K}\}$, and we had seen that its image contains $W(\overline{R})$. Going through our identifications shows that $\pi$ is the composition of $\varepsilon_{\mathfrak{s}_K}$ with the (inverse of the) isomorphism from part iii of the last lemma.

Lemma 3.1.47. — *One has* $\mathrm{im}(\varepsilon_{\mathfrak{s}_K}) = W(\overline{R})$.

7. This is the analogue of $\mathcal{N}(S) = \mathcal{N}(S,T) \cdot \mathcal{Z}(S)$ in [**BT1**, 5.5]; cf. [**Sat3**, II.2.1.5].

8. Being in the split case, we can also apply [**Bo2**, VIII.3.3] here. $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$ being reductive, not necessarily semisimple, is not a problem, because its centre is contained in $\mathfrak{t}_K$ and fixed by $s$ as well as by every automorphism of $\mathfrak{z}_{\mathfrak{g}_K}(\mathfrak{s}_K)$.

*Proof.* — Because we know already that the image contains $W(\overline{R})$, and because $A(\overline{R})$ is the semidirect product of $W(\overline{R})$ and $\mathrm{Aut}(\overline{R}, B)$ for an arbitrary basis $B$ of $\overline{R}$, it suffices to show (for our chosen $\Gamma$-basis $\Delta$):

$$\varepsilon_{\mathfrak{s}_K}(s) \in \mathrm{Aut}(\overline{R}, \overline{\Delta}) \Longrightarrow \varepsilon_{\mathfrak{s}_K}(s) = \mathrm{id}\,.$$

With the identifications of the previous lemma, we can assume w.l.o.g. that $s \in \mathrm{Aut}_e(\mathfrak{g}_K, \mathfrak{t}_K)$ and look at $\varepsilon_{\mathfrak{t}_K}(s) \in W_\Gamma$. Then we have

$$\overline{\varepsilon_{\mathfrak{t}_K}(s)(\Delta)} = \overline{\Delta}$$

by assumption, and $\varepsilon_{\mathfrak{t}_K}(s)(\Delta_0)$ is another basis of $R_0$. So there is a (unique) $w \in W_0$ such that $w\varepsilon_{\mathfrak{t}_K}(s)(\Delta_0) = \Delta_0$, whereas ($W_0$ acting trivially on $X/X_0$)

$$\overline{w\varepsilon_{\mathfrak{t}_K}(s)(\Delta)} = \overline{\Delta}.$$

On the other hand, one easily sees that $W_\Gamma$ sends $\Gamma$-bases to $\Gamma$-bases, so by 3.1.40.i, we have $w\varepsilon_{\mathfrak{t}_K}(s)(\Delta) = \Delta$. The action of $W(R)$ on bases is simple, so $\varepsilon_{\mathfrak{t}_K}(s) = w^{-1} \in W_0$, hence $\varepsilon_{\mathfrak{s}_K}(s) = \mathrm{id}$. □

Proposition 3.1.48. — *One has:*

i. $\pi$ *induces an isomorphism* $W_\Gamma/W_0 \cong W(\overline{R})$.

ii. *For* $w \in W_\Gamma$*, the following are equivalent:*

  (a) $w \in W_0$.

  (b) $\pi(w) = \mathrm{id}$.

  (c) $\pi(w)(\overline{\Delta}) = \overline{\Delta}$.

iii. $W_\Gamma$ *operates simply transitively on the set of all* $\Gamma$*-bases.*

*Proof.* — Part i follows from the foregoing discussion, and part ii from i (the non-trivial $(c) \Rightarrow (a)$ also follows directly from 3.1.40). For part iii, we have remarked in the previous proof that $W_\Gamma$ does indeed act on the set of all $\Gamma$-bases. The action is simple because so is the action of $W(R)$ on the set of all bases. The transitivity follows with part i and 3.1.40 from the transitivity of the actions of $W(\overline{R})$ and $W_0$ on the sets of bases of $\overline{R}$ and $R_0$, respectively, and the triviality of the $W_0$-action on $\overline{R}$. □

Remark 3.1.49. — For the analogues in the group case, see 6.10 and parts of 5.3 in [**BT1**], as well as [**Sat2**, §2] and [**Sat3**, II.2.1–2.3]. There, a different route is chosen: Not knowing a priori that $\overline{R}$ is a root system, the group $W_\Gamma/W_0$ is directly identified with $\mathcal{N}(S)/\mathcal{Z}(S)$ for a maximal split torus $S$. Then by an intricate investigation into this group's action on (minimal) parabolic subgroups, it is shown that $\overline{R}$ is a root system having $W_\Gamma/W_0$ as Weyl group.

In a more general context, the question when an analogous subquotient of the Weyl group is again a Weyl group of a root system is analysed in [**Sch**], see also [**Hec**]. It turns out that parts i and iii of the proposition are kind of equivalent, and that they impose strong conditions on the Galois action and other symmetry properties of the root system, which we will use later for the classification.

Whereas realising the Weyl group in the form $N(T)/Z(T)$ for a suitable torus $T$ is common in group theory, it seems that its realisation as quotient of automorphism groups of Lie algebras has only been done in [**Bo2**, VIII.5], covering only the split case.

**3.1.4. The invariants and their independence from choices.** — We can now define our invariants:

▷ The tuple $(X, R, \Delta, \Delta_0, t)$ [9] is called the *index* of our semisimple Lie algebra $\mathfrak{g}$ (for now: with respect to all the choices we have made). We will show that, up to a natural notion of equivalence, the index depends only on $\mathfrak{g}$ (in fact, only on its isomorphism class) and not on the choices.

▷ Recall from 3.1.30 the *anisotropic kernel* $\mathfrak{g}_a$ of our semisimple Lie algebra $\mathfrak{g}$ (for now: with respect to the chosen $\mathfrak{s}$). The following lemma is obvious from its definition:

LEMMA 3.1.50. — *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two semisimple Lie algebras over $k$ with $\mathfrak{s}$ (resp. $\mathfrak{s}'$) chosen as before. Suppose there is an isomorphism $f : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ with $f(\mathfrak{s}) = \mathfrak{s}'$. Then $f$ induces an isomorphism $f_a : \mathfrak{g}_a \xrightarrow{\sim} \mathfrak{g}'_a$.*

With the conjugacy theorem 3.1.16 we conclude:

COROLLARY 3.1.51. — *Up to an elementary automorphism in $\mathfrak{g}$, $\mathfrak{g}_a$ is independent of the choice of $\mathfrak{s}$. Up to isomorphism, it only depends on the isomorphism class of $\mathfrak{g}$.*

Next, we will see in which way the index depends on $\mathfrak{g}$ and the choices.

DEFINITION 3.1.52 (Congruence of indices). — Let $(X, R, \Delta, \Delta_0, t), (X', R', \Delta', \Delta'_0, t')$ be two indices. An isomorphism of abelian groups $h : X \xrightarrow{\sim} X'$ such that

i. $h(R) = R', h(\Delta) = \Delta', h(\Delta_0) = \Delta'_0$;

ii. $h(\alpha)\check{} = \check{\alpha} \circ h^{-1}$ for all $\alpha \in R$ or equivalently, $(h(\alpha))\check{}(h(\beta)) = \check{\alpha}(\beta)$ for all $\alpha, \beta \in R$ [10];

iii. $t'(g) = h \circ t(g) \circ h^{-1}$ for every $g \in \mathcal{G}$

is called a *congruence* $h : (X, R, \Delta, \Delta_0, t) \to (X', R', \Delta', \Delta'_0, t')$, and if such $h$ exists, the two indices are called *congruent*. Obviously, congruence defines an equivalence relation on any set of indices.

Choosing, firstly, different splitting fields $K$ and $L$ (as long as both are finite and Galois) in an obvious way does not affect the index. To see this, one immediately reduces to the case $K \subseteq L$, and this is easily checked.

Next, let us vary $\mathfrak{s}$, $\mathfrak{t}$ and $\Delta$.

9. There is some redundancy here; e.g. $X$ can be recovered from $R$, different from the case of algebraic groups. We keep it like this since we often need both $X$ and $R$ anyway.

10. In fact, condition ii follows from i (cf. [**Win**, 3.7.3.1.2/4 and 3.7.3.4.3]), but we wanted to state it explicitly.

Proposition 3.1.53. — *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two semisimple Lie algebras over $k$ with $\mathfrak{s}, \mathfrak{t}, \Delta$ (resp. $\mathfrak{s}', \mathfrak{t}', \Delta'$) chosen as before, such that $K$ is a finite Galois splitting field for both $\mathfrak{t}$ and $\mathfrak{t}'$. Suppose there is an isomorphism (of $K$-Lie algebras) $F : \mathfrak{g}_K \xrightarrow{\sim} \mathfrak{g}'_K$ such that $F(\mathfrak{t}_K) = \mathfrak{t}'_K$ and $F(\mathfrak{s}_K) = \mathfrak{s}'_K$. Then in the following cases we have a congruence of the corresponding indices:*

i. *$F = \mathrm{id}_K \otimes f$ for an isomorphism $f : \mathfrak{g} \simeq \mathfrak{g}'$.*

ii. *$\mathfrak{g} = \mathfrak{g}'$ and $F \in \mathrm{Aut}_0(\mathfrak{g}_K)$.*

*Proof.* — The map $\phi : X' \xrightarrow{\sim} X, \chi \mapsto \chi \circ (F_{|\mathfrak{t}_K})$ is an isomorphism that satisfies

▷ $\phi(R') = R$, and

▷ $\phi(X'_0) = X_0$ (by $F(\mathfrak{s}_K) = \mathfrak{s}'_K$ and 3.1.27).

Call $\psi := \phi^{-1}$ its inverse. A moment's consideration with the defining property of $H_{\psi(\alpha)}$ (in 3.1.9) shows $H_{\psi(\alpha)} = F(H_\alpha)$, in other words $(\psi(\alpha))\check{} = \check{\alpha} \circ \psi^{-1}$. The second point above implies that $\psi(\Delta)$ is a $\Gamma$-basis of $R'$. So by 3.1.48, there is a unique $w' \in W'_\Gamma$ such that $w'(\psi(\Delta)) = \Delta'$, and by 3.1.39 a fortiori $w'(\psi(\Delta_0)) = \Delta'_0$. We claim that $h := w' \circ \psi$ is a congruence (from $(X, \dots)$ to $(X', \dots)$), and with 2.1.11, the one thing left to check is what it does with the Galois actions.

So let $\sigma \in \Gamma$ be given. We have $w_\sigma \in W_0$ (and $w'_\sigma \in W'_0$) being defined uniquely in $W(R)$ (and $W(R')$) by $w_\sigma(\sigma_{A(R)}(\Delta)) = \Delta$ (and $w'_\sigma(\sigma_{A(R')}(\Delta')) = \Delta'$). Setting

$$^\sigma h := \sigma_{A(R')} \circ h \circ \sigma_{A(R)}^{-1},$$

that is, $(^\sigma h)(\chi) = {}^\sigma(h({}^{\sigma^{-1}}\chi))$, one calculates

$$\left(h \circ w_\sigma \circ (^\sigma h)^{-1}\right)\left(\sigma_{A(R')}(\Delta')\right) = \Delta'$$

so if the element in the left bracket is in $W(R')$, it must be $w'_\sigma$. Look at the two cases:

i. Here we have $^\sigma F := \sigma \circ F \circ \sigma^{-1} = F$, a fortiori $^\sigma\phi = \phi$, i.e. $\sigma_{A(R)}\psi^{-1} = \psi^{-1}\sigma_{A(R')}$. Hence

$$\begin{aligned} h w_\sigma \sigma_{A(R)} h^{-1} \sigma_{A(R')}^{-1} &= w'\psi\, w_\sigma\, \sigma_{A(R)} \psi^{-1} w'^{-1} \sigma_{A(R')}^{-1} \\ &= \underbrace{w'}_{\in W(R')} \ \psi w_\sigma \psi^{-1} \underbrace{\sigma_{A(R')} w'^{-1} \sigma_{A(R')}^{-1}}_{\in W(R')} \end{aligned}$$

and the middle term is in $W(R')$, too: Indeed, $\psi\check{(\alpha)} = \check{\alpha} \circ \psi^{-1}$ implies $\psi s_\alpha \psi^{-1} = s_{\psi(\alpha)}$ and the claim follows.

ii. Let $e := (\mathfrak{g}_K, \mathfrak{t}_K, \Delta, (X_\alpha)_{\alpha \in \Delta})$ be an épinglage (cf. [**Bo2**, VIII.4, Def. 1]). Then $\sigma \in \Gamma$ transforms it into the épinglage $\sigma(e) := (\mathfrak{g}_K, \mathfrak{t}_K, \sigma_{A(R)}(\Delta), (\sigma(X_\alpha))_{\sigma_{A(R)}(\alpha)})$. On the other hand, the given $F$ transforms it into the épinglage

$$e' := \left(\mathfrak{g}_K, \mathfrak{t}'_K, \psi(\Delta), (F(X_\alpha))_{\psi(\alpha)}\right).$$

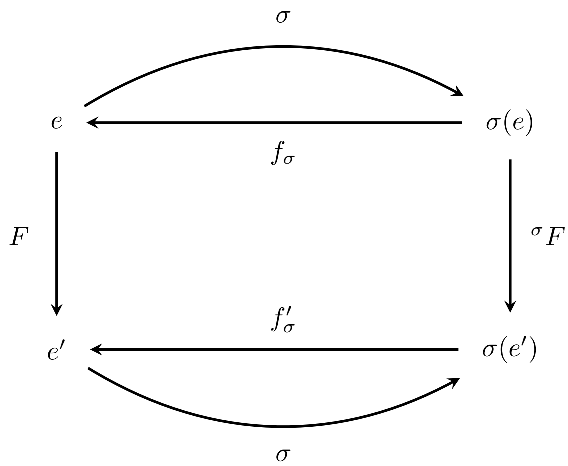
This, in turn, is transformed by $\sigma$ into

$$\sigma(e') := \left(\mathfrak{g}_K, \mathfrak{t}'_K, \sigma_{A(R')}\psi(\Delta), (\sigma(F(X_\alpha)))_{\sigma_{A(R')}\psi(\alpha)}\right).$$

Now by [**Bo2**, VIII.5.3, Prop. 5], there are unique $f_\sigma \in \mathrm{Aut}_0(\mathfrak{g}_K, \mathfrak{t}_K)$ and $f'_\sigma \in \mathrm{Aut}_0(\mathfrak{g}_K, \mathfrak{t}_K)$ such that $f_\sigma(\sigma(e)) = e$ and $f'_\sigma(\sigma(e')) = e'$. By the definition of $w_\sigma$ and $w'_\sigma$ and *ibid.* 2, Prop. 4, we have

$$\epsilon_{\mathfrak{t}_K}(f_\sigma) = w_\sigma \quad \text{and} \quad \epsilon_{\mathfrak{t}'_K}(f'_\sigma) = w'_\sigma.$$

Further note the following: It is easily seen that ${}^\sigma F \in \mathrm{Aut}_0(\mathfrak{g}_K)$ (use that $\sigma \circ \exp(\mathrm{ad}\, n) \circ \sigma^{-1} = \exp(\mathrm{ad}(\sigma(n))))$. Also, $F \circ f_\sigma$ and $f'_\sigma \circ {}^\sigma F$ both transform $\sigma(e)$ to $e'$, and since they are both in $\mathrm{Aut}_0(\mathfrak{g}_K)$, they must be equal by *ibid.* . 3, Prop. 5; or in other words, not just the outer but also the inner diagram in




is commutative. This implies in particular

$$\psi \circ w_\sigma = w'_\sigma \circ {}^\sigma\psi \tag{10}$$

hence

$$\psi \circ w_\sigma \circ \sigma_{A(R)} = w'_\sigma \circ {}^\sigma\psi \circ \sigma_{A(R)} = w'_\sigma \circ \sigma_{A(R')} \circ \psi.$$

Inserting this into our element shows what we want:

$$\begin{aligned} h w_\sigma \sigma_{A(R)} h^{-1} \sigma_{A(R')}^{-1} &= w'(\psi\, w_\sigma\, \sigma_{A(R)}) \psi^{-1} w'^{-1} \sigma_{A(R')}^{-1} \\ &= w'(w'_\sigma \sigma_{A(R')} \psi) \psi^{-1} w'^{-1} \sigma_{A(R')}^{-1} \\ &= \underbrace{w' w'_\sigma}_{\in W(R')} \underbrace{\sigma_{A(R')} w'^{-1} \sigma_{A(R')}^{-1}}_{\in W(R')} \in W(R'). \end{aligned}$$

In both cases we conclude that for $\chi \in X$,

$$\begin{aligned} \big(t'(\sigma) \circ h\big)(\chi) = w'_\sigma\big(\sigma_A(h(\chi))\big) &= h\Big(w_\sigma\big(\sigma_A(h^{-1}\big(\sigma_A^{-1}(\sigma_A(h(\chi)))\big))\big)\Big) \\ &= h\big(w_\sigma(\sigma_A(\chi))\big) = \big(h \circ t(\sigma)\big)(\chi), \end{aligned}$$

so that indeed $t'(\sigma) = h \circ t(\sigma) \circ h^{-1}$. □

Remark 3.1.54

i. Certainly the above proposition holds in more cases, but we were not able to prove it in full generality in which its group analogue is claimed in [**Sat2**, p. 228] and [**Sat3**, pp. 85 et *seq.*]. We see a gap in this claim exactly where we had to go into the cases above.

ii. The independence statement in [**BT1**, 6.2] (whose equation (3) should be read "modulo conjugation") corresponds to our case ii. It is proved via parabolic subgroups and uses in a crucial way that they all are conjugate to "standard" parabolic subgroups; cf. [**Tit1**, 2.3]. One could translate this to our setting, where the equivalence classes of parabolic subalgebras under the action of $\mathrm{Aut}_0(\mathfrak{g}_K)$ have a nice description in [**Bo2**, VIII.5.3, Rem. 4]. The key fact for both this and our proof (whose heart is equation (10)) is the assertion of [**Bo2**, VIII.5.3, Prop. 5] that the action of $\mathrm{Aut}_0(\mathfrak{g}_K)$ on the épinglages of $\mathfrak{g}_K$ is simply transitive. We feel unsure about the last lines in Bourbaki's proof, but the assertion can certainly be proven e.g. using VIII.3.3, Corollaire to Prop. 10 instead of VII.3.2, Thm 1, without further scalar extension.

Definition 3.1.55 (Congruence associated with an isomorphism)

In the setting of the above proposition, the congruence $w' \circ \psi$ defined in its proof is called the *congruence associated with* $F$.

Applying the proposition to $F = \mathrm{id}$ already shows that for fixed $\mathfrak{s}$ and $\mathfrak{t}$, the index of $\mathfrak{g}$ up to congruence does not depend on the choice of $\Delta$, that is, the choice of the $\Gamma$-linear order. More precisely it shows that for two such choices, the congruence is given by a specific element of the Weyl group.

Proposition 3.1.56. — *Let* $\mathfrak{s}, \mathfrak{t}$ *respectively* $\mathfrak{s}', \mathfrak{t}'$ *in* $\mathfrak{g}$ *be chosen as before, and let* $K$ *be a splitting field for both* $\mathfrak{t}$ *and* $\mathfrak{t}'$. *Then there is an* $F \in \mathrm{Aut}_e(\mathfrak{g}_K)$ *such that*

$$F(\mathfrak{s}_K) = \mathfrak{s}'_K \quad \textit{and} \quad F(\mathfrak{t}_K) = \mathfrak{t}'_K.$$

*Proof.* — Theorem 3.1.16 gives us an elementary automorphism $h \in \mathrm{Aut}_e(\mathfrak{g})$ with $h(\mathfrak{s}) = \mathfrak{s}'$. A fortiori, $h(\mathfrak{z}_\mathfrak{g}(\mathfrak{s})) = \mathfrak{z}_\mathfrak{g}(\mathfrak{s}')$, and both $h(\mathfrak{t})$ and $\mathfrak{t}'$ are maximal toral (= Cartan) subalgebras of $\mathfrak{g}$, contained in $\mathfrak{z}_\mathfrak{g}(\mathfrak{s}')$ and hence Cartan subalgebras of $\mathfrak{z}_\mathfrak{g}(\mathfrak{s}')$. It follows that $h_K(\mathfrak{t}_K)$ and $\mathfrak{t}'_K$ are both split Cartan subalgebras of the reductive Lie algebra $\mathfrak{z}_\mathfrak{g}(\mathfrak{s}')_K$. From [**Bo2**, VIII.3.3] we get $\eta \in \mathrm{Aut}_e((\mathfrak{z}_\mathfrak{g}(\mathfrak{s}'))_K)$ with $\eta(h_K(\mathfrak{t}_K)) = \mathfrak{t}'_K$. Because $\eta$ is elementary, firstly, it extends to an automorphism $\widehat{\eta}$ of $\mathfrak{g}_K$, and secondly, it fixes $\mathfrak{s}'_K$. Set $F := \widehat{\eta} \circ h_K$. □

Combining this with 3.1.53 shows that the index of $\mathfrak{g}$ depends, up to congruence, not on the choices made. Further:

Theorem 3.1.57. — *Let $f : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ be an isomorphism of Lie algebras. Then for appropriate choices, there is a congruence from the respective index of $\mathfrak{g}$ to that of $\mathfrak{g}'$*

$$h : (X, R, \Delta, \Delta_0, t) \xrightarrow{\sim} (X', R', \Delta', \Delta_0', t')$$

*and an isomorphism $f_a : \mathfrak{g}_a \xrightarrow{\sim} \mathfrak{g}_a'$ of the anisotropic kernels, which are compatible in the sense that the induced map*

$$h_{|X_a} : (X_a, R_0, \Delta_0, \Delta_0, t_{|X_a}) \longrightarrow (X_a', R_0', \Delta_0', \Delta_0', t'_{|X_a'})$$

*is the congruence associated with $(f_a)_K := \mathrm{id} \otimes f_a : (\mathfrak{g}_a)_K \xrightarrow{\sim} (\mathfrak{g}_a')_K$.*

*Proof.* — If we choose $\mathfrak{s}, \mathfrak{t}, \Delta$ and a splitting field $K$ for $\mathfrak{t}$ as before, then $\mathfrak{s}' := f(\mathfrak{s})$ is a maximal split toral subalgebra of $\mathfrak{g}'$, $\mathfrak{t}' := f(\mathfrak{t})$ is a maximal toral one containing $\mathfrak{s}'$. Call $\psi$ as usual the inverse of the contragredient of $F := \mathrm{id} \otimes f : \mathfrak{g}_K \to \mathfrak{g}_K'$. With $F$ also $\psi$ commutes with the (untwisted) Galois actions, so that $\Delta' := \psi(\Delta)$ is a $\Gamma$-basis (of $R' \subset X'$, defined with respect to $\mathfrak{t}_K'$). A congruence between the indices defined with respect to these choices is just given by $\psi$ (no twisting with a Weyl group element needed), and taking as $f_a$ of course the restriction to $\mathfrak{g}_a$ of $f$, and $\Delta_0$ as $\Gamma$-basis of $R_0$, the compatibility is obvious. □

The converse of this is the "isomorphism theorem" from this chapter's title. We will state and prove it in Section 3.3. Before doing this, let us look at some special cases and examples.

## 3.2. Special cases and examples

In this section, we will consider opposite extreme cases of what our invariants can be. [**Spr3**, 16.2.2] whose proof, however, needs slight corrections. They will be very useful for the classification. Keep the notation of the previous sections, i.e. choose $\mathfrak{s}, \mathfrak{t}, \Delta$, etc. as before.

### 3.2.1. The anisotropic case

Lemma 3.2.1. — *The following are equivalent:*

i. $\Delta_0 = \Delta$,
ii. $R_0 = R$,
iii. $X_0 = X$,
iv. $\overline{R} = \varnothing$,
v. $\mathfrak{s} = \{0\}$,
vi. $\mathfrak{g}_a = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{g}$,
vii. *every element of $\mathfrak{g}$ is semisimple,*
viii. *$\mathfrak{g}$ contains no non-zero nilpotent element,*
ix. *$\mathfrak{g}$ contains no parabolic subalgebra distinct from $\mathfrak{g}$.*

*Proof.* — Equivalences and implication i $\Leftrightarrow$ ii $\Leftrightarrow$ iii $\Leftrightarrow$ iv $\Leftrightarrow$ v $\Rightarrow$ vi are clear. Algebra $\mathfrak{g}$ being semisimple implies v $\Leftarrow$ vi. We have vii $\Leftrightarrow$ viii from the Jordan decomposition, and by 3.1.2, viii $\Rightarrow$ iv. Remark 3.1.29 gives vi $\Rightarrow$ vii (alternatively, v $\Rightarrow$ viii by Jacobson-Morozov).

Finally, the equivalence viii $\Leftrightarrow$ ix is exercise 6.a of [**Bo2**, VIII.10]. Direction "$\Leftarrow$" follows by contraposition from [**Bo2**, VIII.10, Cor. 2] applied to $\mathfrak{n} = k{\cdot}X$ for a non-zero nilpotent $X \in \mathfrak{g}$, because the radical of $\mathfrak{g}$ and hence the set called "$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{g})$" in *loc. cit.* (defined in the same paragraph) is zero. For the converse, assume there is a parabolic subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$. Then by [**Bo2**, VIII.3.4, Prop. 13] the largest nilpotent ideal of $\mathfrak{p}_{\bar{k}}$ is a non-zero sum of root spaces in $\mathfrak{g}_{\bar{k}}$, which (cf. *ibid.*. Prop. 9) consists of nilpotent elements of $\mathfrak{g}_{\bar{k}}$. By [**Bo2**, I.4.5], this ideal is the scalar extension of the largest nilpotent ideal in $\mathfrak{p}$. In particular, under the usual identifications, its intersection with $\mathfrak{p}$ is non-zero, and consists of elements which are nilpotent in $\mathfrak{g}$. □

Definition 3.2.2. — If these equivalent conditions are satisfied, $\mathfrak{g}$ is called *anisotropic.*

Parts vii to ix (and v, saying that $\{0\}$ is the unique split toral subalgebra in $\mathfrak{g}$) show directly that this is independent from the choices of $\mathfrak{s}, \mathfrak{t}$ etc. One readily sees e.g. from 3.1.29 that the anisotropic kernel of a semisimple $\mathfrak{g}$ is indeed anisotropic, and of course an anisotropic Lie algebra is its own anisotropic kernel. Also, part viii combined with lemma 2.1.8 gives

Corollary 3.2.3. — *A semisimple subalgebra of an anisotropic Lie algebra is anisotropic.*

Remark 3.2.4. — The terminology comes from anisotropic quadratic forms; the connection will occur in the next example and become striking in 4.5.14; see also [**Sat3**, I.4.4, Example 1], [**BT1**, 4.24]. [**Sat3**] and [**We1**] use the word *compact.* This is common usage in the theory of real Lie algebras (cf. [**Hel**, Chap. III, X], [**OVi**], [**Oni**], [**Bo2**, IX]) and ultimately stems from the fact that over $\mathbb{R}$, these Lie algebras correspond to compact Lie groups $\mathrm{Int}(\mathfrak{g})$. For the connection between anisotropy and compactness in algebraic groups over local fields cf. [**BT1**, § 9] and [**Sat3**, I.4.4.4].

Example 3.2.5 (Anisotropic form of $A_1$). — Let $K|k$ be a quadratic extension, say $K = k(y)$ with $y^2 =: a \in k$. Denote the non-trivial element of $\Gamma = \mathrm{Gal}(K|k)$ by $\sigma$:

$$\sigma(r + sy) = r - sy \quad \text{for } r, s \in k.$$

Assume further that there is an element $b \in k$ which is not a norm of the extension $K|k$. Consider, inside $R_{K|k}\,\mathfrak{sl}_2(K)$, the 3-dimensional subalgebra $\mathfrak{g}$ given by all elements

$$\begin{pmatrix} ry & s + ty \\ b(s - ty) & -ry \end{pmatrix}$$

with $r, s, t \in k$. One easily sees that $\mathfrak{g}_K$ identifies with $\mathfrak{sl}_2(K)$ (in particular, $\mathfrak{g}$ is simple), a general element of which we denote as

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

subject to $x_{11} = -x_{22}$. We claim that $\mathfrak{g}$ is anisotropic, for which we verify 3.2.1.viii. Assume that

$$x = \begin{pmatrix} ry & s+ty \\ b(s-ty) & -ry \end{pmatrix}$$

is a nilpotent element of $\mathfrak{g}$. Then $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent and so is

$$\mathrm{ad}_{\mathfrak{g}_K}(1 \otimes x) = \mathrm{id} \otimes \mathrm{ad}_{\mathfrak{g}}(x),$$

hence (cf. [**Bo2**, I.6.3, Thm 3]) so is $1 \otimes x = x$ viewed as element of $\mathfrak{sl}_2(K)$. This implies by definition (cf. [**Bo2**, I.6.3, Def. 3]) that the matrix $x$ itself is nilpotent, so its characteristic polynomial must be $T^2$, that is, $-r^2y^2 - b(s^2 - t^2y^2) = 0$ which is equivalent to

$$(s = t = 0 = r) \quad \text{or} \quad \Big(b = \frac{-r^2y^2}{s^2 - t^2y^2} = N_{K|k}\Big(\frac{ry}{s+ty}\Big)\Big)$$

The second case was excluded, so $x = 0$ and our claim is proved.

NB. — Our choice of $a, b$ is made so that the quadratic form $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$ (or equivalently, the quadratic form $X_3^2 - bX_1^2 - aX_2^2$) is anisotropic over $k$. This in turn is equivalent to the quaternion algebra

$$D := \Big(\frac{a, b}{k}\Big)$$

being a division algebra, cf. [**Pie**, 1.6]. In fact, our $\mathfrak{g}$ is the derived algebra (= semisimple part) of the associative $k$-algebra $D$ viewed as Lie algebra with commutator bracket. For $K|k = \mathbb{C}|\mathbb{R}$ and $a = b = -1$, we have the Hamilton quaternions as $D$, and $\mathfrak{g}$ is the compact real form of $\mathfrak{sl}_2(\mathbb{C})$. For $k$ a $\mathfrak{p}$-adic field, one can take as $K|k$ the unique unramified extension of degree 2 and as $b$ a uniformiser of $k$. For more in this direction, see Section 4.4 and compare example (a) in Remark 4.5.20.i.

As $\mathfrak{g}$ is anisotropic, the unique split toral subalgebra is $\mathfrak{s} = \{0\}$. Any one-dimensional subspace is a maximal toral subalgebra, but to have $K$ as splitting field, we choose

$$\mathfrak{t} = \Big\{\begin{pmatrix} ry & 0 \\ 0 & -ry \end{pmatrix} : r \in k\Big\}$$

$\mathfrak{g}_K$ has the well-known root $\alpha(x) = x_{11} - x_{22} \quad (= 2x_{11})$ with respect to the split maximal toral subalgebra

$$\mathfrak{t}_K = \Big\{\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} : x_{11} = -x_{22}\Big\}.$$

We have $\rho(\alpha) = \alpha_{|\mathfrak{s}_K} = 0$ and the root spaces

$$(\mathfrak{g}_K)_\alpha = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, (\mathfrak{g}_K)_{-\alpha} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

Let us look at the Galois action on a root space: Writing

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \frac{x}{2} \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}}_{\in \mathfrak{g}} + \frac{x}{2y} \cdot \underbrace{\begin{pmatrix} 0 & y \\ -by & 0 \end{pmatrix}}_{\in \mathfrak{g}}$$

we see with $\sigma(y) = -y$ that

$$\sigma\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \frac{\sigma(x)}{2} \cdot \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} - \frac{\sigma(x)}{2y} \cdot \begin{pmatrix} 0 & y \\ -by & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b\sigma(x) & 0 \end{pmatrix}$$

so that $\sigma((\mathfrak{g}_K)_\alpha) = (\mathfrak{g}_K)_{-\alpha}$, hence ${}^\sigma\alpha = -\alpha$. Further:

i. $R = R_0 = \{\pm\alpha\}$ (type $A_1$)
ii. $\Delta = \Delta_0$ for both possible bases (automatically $\Gamma$-bases) $\{\alpha\}$ and $\{-\alpha\}$
iii. $X = X_0 = X_a = \mathbb{Z}\alpha$
iv. $\{0\} = \mathfrak{s} = \mathfrak{Z} \subsetneq \mathfrak{t} \subsetneq \mathfrak{g}_a = \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) = \mathfrak{g}$
v. $W(R) = W(R)_\Gamma = W_0 = \{\pm \operatorname{id}\}$
vi. $w_\sigma = -\operatorname{id}$ so that $t(\sigma) = \operatorname{id}$.

### 3.2.2. The quasi-split case

Lemma 3.2.6. — *The following are equivalent:*

i. $\Delta_0 = \varnothing$.
ii. $R_0 = \varnothing$.
iii. $X_a = \{0\}$.
iv. $\mathfrak{g}_a = \{0\}$.
v. $\mathfrak{z}_\mathfrak{g}(\mathfrak{s}) = \mathfrak{t}$.
vi. $\mathfrak{g}$ *contains a Borel subalgebra.*

*Proof.* — i ⇔ ii ⇔ iii ⇔ iv ⇔ v amends 3.1.32. Now if these are satisfied,

$$\mathfrak{g} = \underbrace{\mathfrak{z}_\mathfrak{g}(\mathfrak{s})}_{\mathfrak{t}} \oplus \bigoplus_{\lambda \in \overline{R}} \mathfrak{g}_\lambda$$

and with 3.1.20 one sees that for a system of positive roots $\overline{R}^+$ in $\overline{R}$,

$$\mathfrak{t} \oplus \bigoplus_{\lambda \in \overline{R}^+} \mathfrak{g}_\lambda$$

is a Borel subalgebra of $\mathfrak{g}$. Conversely, assume there is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Then its largest nilpotent ideal $\mathfrak{N}$ has $\mathfrak{b}$ as normaliser in $\mathfrak{g}$; this follows from stability of the normaliser and the largest nilpotent ideal (cf. [**Bo2**, I.5.5]) under scalar extension.

Hence $\mathfrak{N}$ is the intersection with $\mathfrak{g}$ (or the set of all $\mathrm{Gal}(\bar{k}|k)$-invariants) of $\mathfrak{N}_{\bar{k}}$ = sum of all positive root spaces with respect to some toral subalgebra and order of roots. Certainly $\mathfrak{N}$ is a nil subalgebra in $\mathfrak{g}$, and if it were not maximal nil in $\mathfrak{g}$, by [**Bo2**, VIII.10, Cor. 2] there would exist a parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$ whose radical strictly contains $\mathfrak{N}$; so that the radical of $\mathfrak{q}_{\bar{k}}$ would strictly contain $\mathfrak{N}_{\bar{k}}$, which is absurd by a dimension comparison. Now if $\mathfrak{s}, \mathfrak{t}, K$ are chosen as usual, by step 4 of the proof of 3.1.16 we can assume that $\mathfrak{s} \subseteq \mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \subseteq \mathfrak{b}$ and hence $\mathfrak{b}_K$ must be of the form $\mathfrak{t}_K \oplus \bigoplus_{\alpha \in P} (\mathfrak{g}_K)_\alpha$ for a system of positive roots $P \subset R$. Since on the other hand, all $(\mathfrak{g}_K)_\alpha$ for $\alpha \in R_0$ are in $\mathfrak{b}_K$, necessarily $R_0 = \varnothing$. □

Definition 3.2.7. — If these equivalent conditions are satisfied, $\mathfrak{g}$ is called *quasi-split.*

Remark 3.2.8. — The property of being *split* is stronger, as the next example shows. Characterisations of split Lie algebras will be given in 3.2.12.

Example 3.2.9 (Quasi-split form of $A_2$ [12]). — Let $K|k$ be a quadratic extension, say $K = k(y)$ with $y^2 \in k$. Denote the non-trivial element of $\Gamma = \mathrm{Gal}(K|k)$ by $\sigma$:

$$\sigma(a + by) = a - by \quad \text{for } a, b \in k.$$

Consider, inside $R_{K|k}\, \mathfrak{sl}_3(K)$, the 8-dimensional subalgebra $\mathfrak{g}$ given by all elements

$$\begin{pmatrix} a+by & c+dy & ey \\ f+gy & -2by & -c+dy \\ hy & -f+gy & -a+by \end{pmatrix}$$

with $a, b, c, d, e, f, g, h \in k$. One easily sees that $\mathfrak{g}_K$ identifies with $\mathfrak{sl}_3(K)$, a general element of which we denote as

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

subject to $x_{11} + x_{22} + x_{33} = 0$. We claim that

$$\mathfrak{s} := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} : a \in k \right\} \quad \text{and} \quad \mathfrak{t} := \left\{ \begin{pmatrix} a+by & 0 & 0 \\ 0 & -2by & 0 \\ 0 & 0 & -a+by \end{pmatrix} : a, b \in k \right\}$$

are maximal split toral and maximal toral subalgebras of $\mathfrak{g}$, respectively. Indeed,

$$\mathfrak{t}_K = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} : x_{11} + x_{22} + x_{33} = 0 \right\}.$$

12. This is essentially the example from [**Sel2**, p. 13], generalised from $\mathbb{C}|\mathbb{R}$ and with slightly different normalisations.

is known to be maximal toral in $\mathfrak{g}_K$, so $\mathfrak{t}$ is maximal toral in $\mathfrak{g}$ by the proof of 3.1.7. On the other hand, one sees that $\mathfrak{s}$ is split toral and that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{t}$, so any split toral subalgebra containing $\mathfrak{s}$ must be in $\mathfrak{t}$. But $\mathfrak{t}$ itself is not split, and $\dim_k(\mathfrak{s}) = 1 < 2 = \dim_k(\mathfrak{t})$ proves our claim.

$\mathfrak{g}_K$ has the well-known roots $\alpha_1(x) = x_{11} - x_{22}$, $\alpha_2(x) = x_{22} - x_{33}$ with respect to $\mathfrak{t}_K$. We have $\rho(\alpha_1) = \alpha_{1|\mathfrak{s}_K} = \alpha_{2|\mathfrak{s}_K} = \rho(\alpha_2) =: \lambda_K$. Further:

i. $R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ (type $A_2$),
ii. $\overline{R} = \{\pm\lambda_K, \pm 2\lambda_K\}$ (type $BC_1$),
iii. $R_0 = \Delta_0 = \varnothing$,
iv. $X = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$,
v. $X_0 = \mathbb{Z}(\alpha_1 - \alpha_2)$ (so indeed $\rho(\alpha_1) = \rho(\alpha_2)$ under $\rho : X \twoheadrightarrow X/X_0$),
vi. $X_a = \{0\}$,
vii. $\{0\} = \mathfrak{g}_a \subsetneq \mathfrak{s} \subsetneq \mathfrak{Z} = \mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$,
viii. ${}^{\sigma}\alpha_1 = \alpha_2$, ${}^{\sigma}\alpha_2 = \alpha_1$,
ix. the two $\Gamma$-bases are $\Delta = \{\alpha_1, \alpha_2\}$ and $\Delta' = \{-\alpha_1, -\alpha_2\}$ ,
x. $W(R)_\Gamma = \{\mathrm{id}, s_{\alpha_1+\alpha_2}\}, W_0 = \{\mathrm{id}\}$,
xi. $w_\sigma = \mathrm{id}$ so that $t(\sigma) = \sigma_A$ is the transposition of $\alpha_1$ and $\alpha_2$.

Root spaces and coroots "upstairs" are well known:

$$(\mathfrak{g}_K)_{\alpha_1} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathfrak{g}_K)_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathfrak{g}_K)_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$H_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_{\alpha_1+\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and $H_\alpha$ is a $K$-basis of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for each $\alpha \in R$. But "below", for the rational root

$$\lambda : \mathfrak{s} \longrightarrow k, \quad \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \longmapsto a$$

we have

$$\mathfrak{g}_\lambda = \begin{pmatrix} 0 & c+dy & 0 \\ 0 & 0 & -c+dy \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ f+gy & 0 & 0 \\ 0 & -f+gy & 0 \end{pmatrix},$$

$$H_\lambda = -H_{-\lambda} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathfrak{g}_{2\lambda} = \begin{pmatrix} 0 & 0 & ey \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{2\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular $\dim_k(\mathfrak{g}_\lambda) = \dim_k(\mathfrak{g}_{-\lambda}) = 2$ as predicted by 3.1.13. In the same vein, we have

$$\Bigg[\underbrace{\begin{pmatrix}0 & y & 0\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}}_{\in\mathfrak{g}_\lambda}, \underbrace{\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}}_{\in\mathfrak{g}_{-\lambda}}\Bigg] = \begin{pmatrix}y & 0 & 0\\ 0 & -2y & 0\\ 0 & 0 & y\end{pmatrix} \in \mathfrak{t}\setminus\mathfrak{s}$$

so in fact $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] = \mathfrak{t}$ has dimension 2 and is not generated by $H_\lambda$. Also note how the root spaces are "skew" with respect to each other: although $(\mathfrak{g}_\lambda)_K = (\mathfrak{g}_K)_{\alpha_1} \oplus (\mathfrak{g}_K)_{\alpha_2}$, we have $\mathfrak{g}_\lambda \cap (\mathfrak{g}_K)_{\alpha_i} = \{0\}$ for $i = 1, 2$.

Finally it might be worthwhile to see the Galois action on a root space explicitly: Writing

$$\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \frac{x}{2}\cdot\underbrace{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}}_{\in\mathfrak{g}} + \frac{x}{2y}\cdot\underbrace{\begin{pmatrix}0 & y & 0\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}}_{\in\mathfrak{g}}$$

we see with $\sigma(y) = -y$ that

$$\sigma\left(\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) = \frac{\sigma(x)}{2}\cdot\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix} - \frac{\sigma(x)}{2y}\cdot\begin{pmatrix}0 & y & 0\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\sigma(x)\\ 0 & 0 & 0\end{pmatrix}$$

so that indeed $\sigma((\mathfrak{g}_K)_{\alpha_1}) = (\mathfrak{g}_K)_{\alpha_2}$.

Remark 3.2.10. — This example can be generalised to construct non-split quasi-split Lie algebras of type $A_n$ for arbitrary $n \geq 2$. (For $n = 1$, the construction degenerates to the split $\mathfrak{sl}_2(k)$.) Namely, let $K = k(y)$ be as above, $d = n + 1$ and consider inside $R_{K|k}\mathfrak{sl}_d(K)$ those matrices $(x_{i,j})$ that satisfy $x_{i,j} = -\sigma(x_{d+1-j,d+1-i})$; that is, those traceless $d \times d$ matrices over $K$ such that each entry is the negative conjugate of the one mirrored at the secondary diagonal (in particular, the entries on the secondary diagonal are $k$-multiples of $y$); or in yet other words, the traceless $d \times d$ matrices $(x_{i,j})$ over $K$ satisfying $(x_{i,j}) \cdot H + H \cdot {}^t(\sigma(x_{i,j})) = 0$ where $H$ is the $d \times d$ matrix with entries 1 on the secondary diagonal and 0 else. A maximal split toral subalgebra is

$$\mathfrak{s} := \big\{\operatorname{diag}(x_{1,1}, \ldots, x_{d/2,d/2},\ -x_{d/2,d/2}, \ldots, -x_{1,1}) : x_{i,i} \in k\big\}$$

or

$$\mathfrak{s} := \big\{\operatorname{diag}(x_{1,1}, \ldots, x_{n/2,n/2},\ 0,\ -x_{n/2,n/2}, \ldots, -x_{1,1}) : x_{i,i} \in k\big\}$$

according to whether $n$ is odd or even. One calculates that for odd $n$, the rational root system $\overline{R}$ is of type $C_{d/2}$, whereas for even $n$, it is of type $BC_{n/2}$ (we will show this with an easy method in 3.4.1). We have $w_\sigma = \mathrm{id}$ so that the twisted and untwisted Galois action coincide, and $\sigma$ transposes $\alpha_i$ and $\alpha_{n-i+1}$, where $\alpha_i$ are the conventionally named roots of $\mathfrak{sl}_d(K)$. [(13)]

13. Cf. [**Bo2**, VIII.13, exercise 16.a] and 4.5.22 for much more generality.

Remark 3.2.11. — For different choices of the quadratic extension $K|k$, the above gives non-isomorphic Lie algebras. In the index, the difference is in the kernel of the twisted Galois action $t : \mathrm{Gal}(\bar{k}|k) \to \mathrm{Aut}(R, \Delta)$, whereas the image of $t$ is always the same. We call the fixed field of the kernel of $t$ the "fixed field of $t$" for short.

### 3.2.3. The split case

Lemma 3.2.12. — *The following are equivalent:*

i. $\mathfrak{g}$ *is quasi-split, and the twisted Galois action is trivial (i.e.* $t \equiv \mathrm{id}$*).*
ii. *The non-twisted Galois action on* $X$ *is trivial.*
iii. $X_0 = \{0\}$.
iv. $\mathfrak{s} = \mathfrak{t}$.
v. $R = \overline{R}$.
vi. $\mathfrak{t}$ *is split toral.*
vii. $(\mathfrak{g}, \mathfrak{t})$ *is a split semisimple Lie algebra in the sense of* [**Bo2**, VIII.2].

*Proof.* — i $\Rightarrow$ ii by 3.1.43.ii and 3.2.6.iii. The equivalence iii $\Leftrightarrow$ iv follows from 3.1.27, and ii $\Leftrightarrow$ iii as well as iv $\Leftrightarrow$ v $\Leftrightarrow$ vi $\Leftrightarrow$ vii $\Leftrightarrow$ i are clear. □

Remark 3.2.13 (Splitting fields). — As a matter of convention, we call a *splitting field* for our semisimple $\mathfrak{g}$ a field $K$ such that there is a maximal split toral $\mathfrak{s}$ in $\mathfrak{g}$, and a maximal toral $\mathfrak{t}$ containing $\mathfrak{s}$, such that $K$ is a splitting field for $\mathfrak{t}$. *Caveat*: this does *not* mean that any toral subalgebra, nor even every other $\mathfrak{t}'$ of this kind, is split by $K$.

Any splitting field for $\mathfrak{g}$ must contain the fixed field of $t$. So if this fixed field is a splitting field itself – e.g. for quasi-split $\mathfrak{g}$ – it is the unique *minimal* splitting field, meaning that every splitting field for any choice $\mathfrak{t} \supseteq \mathfrak{s}$ has to contain it. Not all Lie algebras have such a minimal splitting field. In fact different non-split toral subalgebras $\mathfrak{t}$, even if chosen with respect to the same $\mathfrak{s}$, might even have linearly disjoint splitting extensions. For example in 5.2.2 we will see Lie algebras for which every extension of a given degree $d$ is a splitting field (for some $\mathfrak{t}$ chosen with respect to $\mathfrak{s} = 0$ there).

## 3.3. The isomorphism theorem

In this section we prove the converse of 3.1.57. In this sense, the description of the isomorphism class of a semisimple Lie algebra is reduced to the description of its index and anisotropic kernel. For *split* semisimple Lie algebras, the statement reduces to a variant of the classical uniqueness theorem as formulated by Chevalley and others, see e.g. [**Bo2**, VIII.4.4]. In fact we will use this result for a splitting extension, and then derive the general version by Galois descent, including the notorious Hilbert 90. Our proof follows [**Sat2**] and [**Sat3**, II.2.4], i.e. translates it to Lie algebras.

Let $\mathfrak{g}, \mathfrak{g}'$ be two semisimple Lie algebras over $k$. Fix choices $\mathfrak{s}, \mathfrak{s}', \dots$ as before, including a common finite Galois splitting field $K$ for both $\mathfrak{t}$ and $\mathfrak{t}'$. With respect to these choices, we have the indices $(X, R, \Delta, \Delta_0, t)$ and $(X', R', \Delta', \Delta_0', t')$, and the anisotropic kernels $\mathfrak{g}_a$ and $\mathfrak{g}_a'$.

Assume that there is a congruence

$$h : (X, R, \Delta, \Delta_0, t) \longrightarrow (X', R', \Delta', \Delta_0', t')$$

and an isomorphism

$$f_a : \mathfrak{g}_a \xrightarrow{\sim} \mathfrak{g}_a'$$

which are compatible in the sense that the induced map

$$h_{|X_a} : (X_a, R_0, \Delta_0, \Delta_0, t_{|X_a}) \longrightarrow (X_a', R_0', \Delta_0', \Delta_0', t'_{|X_a'})$$

is the congruence associated with $(f_a)_K := \mathrm{id} \otimes f : (\mathfrak{g}_a)_K \xrightarrow{\sim} (\mathfrak{g}_a')_K$.

THEOREM 3.3.1 (Satake, Tits, Weisfeiler). — *Under the above assumptions, there is an isomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ inducing $f_a$ and such that $h$ is associated with*

$$f_K := \mathrm{id} \otimes f : \mathfrak{g}_K \longrightarrow \mathfrak{g}_K'.$$

A special case is:

COROLLARY 3.3.2. — *If $\mathfrak{g}$ is quasi-split, its isomorphism class is uniquely determined by its index.*

The proof of the theorem will take up the rest of this section.

**3.3.1. Requisites.** — For each $\alpha \in R$, choose a basis of $(\mathfrak{g}_K)_\alpha$, i.e. a non-zero element $e_\alpha \in (\mathfrak{g}_K)_\alpha$, in such a way that $[e_\alpha, e_{-\alpha}] = H_\alpha$ (this is possible by 3.1.9). Make an analogous choice of $e'_{\alpha'}$ for $\alpha' \in R'$. These choices remain fixed throughout this section. By 3.1.22, for each $(\sigma, \alpha) \in \Gamma \times R$ there is a unique $\xi_{\sigma,\alpha} \in K^*$ defined by the relation:

$$\sigma(e_\alpha) = \xi_{\sigma,\alpha} \cdot e_{\sigma_A(\alpha)} \tag{11}$$

Now for $\sigma, \tau \in \Gamma$ we have $\sigma(\tau(e_\alpha)) = \sigma(\xi_{\tau,\alpha}) \cdot \sigma(e_{\tau_A(\alpha)}) = \sigma(\xi_{\tau,\alpha}) \cdot \xi_{\sigma,\tau_A(\alpha)} \cdot e_{\sigma_A \tau_A(\alpha)}$, so that:

$$\xi_{\sigma\tau,\alpha} = \sigma(\xi_{\tau,\alpha}) \cdot \xi_{\sigma,\tau_A(\alpha)} \tag{12}$$

(So for each $\alpha \in R$, the map $\sigma \mapsto \xi_{\sigma,\alpha}$ is a cocycle from the subgroup $\Gamma_\alpha := \{\sigma \in \Gamma : \sigma(\alpha) = \alpha\}$ to $K^*$, but we will not need this.) Define $\xi'_{\sigma,\alpha'} \in K^*$ analogously.

Next, suppose we are given an isomorphism $F : \mathfrak{g}_K \xrightarrow{\sim} \mathfrak{g}_K'$ such that $F(\mathfrak{t}_K) = \mathfrak{t}_K'$ (write $F : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}_K', \mathfrak{t}_K')$ for short). We get the isomorphism

$$\psi : X \xrightarrow{\sim} X', \quad \chi \longmapsto \chi \circ \left((F_{|\mathfrak{t}_K})^{-1}\right)$$

with $\psi(R) = R'$. This in turn gives, for $\alpha \in R$, unique $\eta_\alpha \in K^*$ defined by

$$F(e_\alpha) = \eta_\alpha \cdot e'_{\psi(\alpha)}. \tag{13}$$

From the root space decomposition it is clear that if two isomorphisms $F$ and $G$ give rise to the same tuple $(\psi, (\eta_\alpha)_{\alpha\in R})$, we have $F = G$, so $F$ is uniquely determined by this tuple. Write

$$F \longleftrightarrow \big(\psi, (\eta_\alpha)_{\alpha\in R}\big).$$

We will later see some sufficient conditions for a tuple to occur, i.e. to arise from an isomorphism. The proof of the following remark contains a necessary one:

Remark 3.3.3. — For any basis $B$ of the root system $R$, an isomorphism $F$ as above is already determined by $\psi$ and $(\eta_\alpha)_{\alpha\in B}$.

*Proof.* — The chosen $e_\alpha$ give rise to structure constants $N_{\alpha,\beta} \in K^*$ defined by

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} \cdot e_{\alpha+\beta} \quad \text{for } \alpha, \beta \in R \text{ with } \alpha + \beta \in R; \tag{14}$$

also remember $[e_\alpha, e_{-\alpha}] = H_\alpha$. Analogously, define $N'_{\alpha',\beta'}$. It is then not hard to show that $F$ commuting with the Lie brackets and $F(H_\alpha) = H_{\psi(\alpha)}$ implies

$$\eta_{\alpha+\beta} = \eta_\alpha \cdot \eta_\beta \cdot \frac{N'_{\psi(\alpha),\psi(\beta)}}{N_{\alpha,\beta}}, \quad \eta_{-\alpha} = \eta_\alpha^{-1}$$

for $\alpha, \beta \in R$ with $\alpha + \beta \in R$, from which the claim follows. □

For $\sigma \in \Gamma$ and an isomorphism $F$ as above, set ${}^\sigma F := \sigma \circ F \circ \sigma^{-1}$ – the $\sigma$'s meaning the $k$-automorphisms of $\mathfrak{g}'_K$ and $\mathfrak{g}_K$ – which is again an isomorphism $(\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$. The next lemma shows how: i) this Galois action and ii) composition translate to the tuple. For $\phi \in \mathrm{Hom}_{\mathbb{Z}}(X, X')$, we define ${}^\sigma\phi := \sigma_{A'}\,\phi\,\sigma_A^{-1}$ with the $\sigma_A \in A(R)$, $\sigma_{A'} \in A(R')$ again meaning the elements defined by the Galois actions on $X$ and $X'$.

Lemma 3.3.4

i. *For $\sigma$ and $F$ as above, we have*

$${}^\sigma F \longleftrightarrow \Big({}^\sigma\psi, \Big(\sigma(\eta_{\sigma_A^{-1}(\alpha)}) \cdot \frac{\xi'_{\sigma,\psi(\sigma_A^{-1}(\alpha))}}{\xi_{\sigma,\sigma_A^{-1}(\alpha)}}\Big)_{\alpha\in R}\Big).$$

ii. *For isomorphisms*

$$(\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{F} (\mathfrak{g}'_K, \mathfrak{t}'_K) \xrightarrow{F'} (\mathfrak{g}''_K, \mathfrak{t}''_K)$$

*(with $X'', R'', (e''_\alpha)_{\alpha\in R''}$, etc.) we have*

$$F' \circ F \longleftrightarrow (\psi' \circ \psi, (\eta'_{\psi(\alpha)} \cdot \eta_\alpha)_{\alpha\in R})$$

*Proof.* — Let $\alpha \in R$. For $\chi \in X$ we have

$$\begin{aligned}({}^{\sigma}\psi)(\chi) &= {}^{\sigma}(\psi({}^{\sigma^{-1}}\chi)) = {}^{\sigma}(\psi(\sigma^{-1}\circ\chi\circ\sigma)) \\ &= {}^{\sigma}(\sigma^{-1}\circ\chi\circ\sigma\circ(F_{|\mathfrak{t}_K})^{-1}) = \chi\circ\sigma\circ(F_{|\mathfrak{t}_K})^{-1}\circ\sigma^{-1} \\ &= \chi\circ({}^{\sigma}F_{|\mathfrak{t}_K})^{-1}.\end{aligned}$$

where the internal $\sigma$'s with which the maps are composed are the $k$-automorphisms of $\mathfrak{t}_K, \mathfrak{t}'_K$ and $K$. Further,

$$\begin{aligned}({}^{\sigma}F)(e_\alpha) &\overset{(11)}{=} \sigma(F(\xi_{\sigma^{-1},\alpha}\cdot e_{\sigma_A^{-1}(\alpha)})) \\ &= \sigma(\xi_{\sigma^{-1},\alpha})\cdot\sigma(F(e_{\sigma_A^{-1}(\alpha)})) \\ &\overset{(13)}{=} \sigma(\xi_{\sigma^{-1},\alpha})\cdot\sigma(\eta_{\sigma_A^{-1}(\alpha)}\cdot e'_{\psi(\sigma_A^{-1}(\alpha))}) \\ &= \sigma(\xi_{\sigma^{-1},\alpha})\cdot\sigma(\eta_{\sigma_A^{-1}(\alpha)})\cdot\sigma(e'_{\psi(\sigma_A^{-1}(\alpha))}) \\ &\overset{(11)}{=} \sigma(\xi_{\sigma^{-1},\alpha})\cdot\sigma(\eta_{\sigma_A^{-1}(\alpha)})\cdot\xi'_{\sigma,\psi(\sigma_A^{-1}(\alpha))}\cdot e'_{\sigma_{A'}\psi\sigma_A^{-1}(\alpha)},\end{aligned}$$

and $\sigma(\xi_{\sigma^{-1},\alpha})$ is the inverse of $\xi_{\sigma,\sigma^{-1}(\alpha)}$ by (12). (Note $e'_{\sigma_{A'}\psi\sigma_A^{-1}(\alpha)} = e'_{({}^{\sigma}\psi)(\alpha)}$.) This shows i. For ii., just remark

$$F'\big(F(e_\alpha)\big) = F'(\eta_\alpha\, e'_{\psi(\alpha)}) = \eta_\alpha\, F'(e'_{\psi(\alpha)}) = \eta_\alpha\, \eta'_{\psi(\alpha)}\, e''_{\psi'(\psi(\alpha))}. \qquad \square$$

Set $T_X := \mathrm{Hom}_{\mathbb{Z}}(X, K^*)$, see [**Bo2**, VIII.5..2] – our $X$ is the root lattice called $Q(R)$ there. There is an embedding $i : T_X \hookrightarrow \mathrm{Aut}(\mathfrak{g}_K, \mathfrak{t}_K)$ called "$f$" in *loc. cit.*: for $\Phi \in T_X$ and any weight $\alpha \in R \cup \{0\}$, the restriction of $i(\Phi)$ to the weight space $(\mathfrak{g}_K)_\alpha$ is given by scalar multiplication with $\Phi(\alpha)$. In fact our tuples, in the case of $F \in \mathrm{Aut}(\mathfrak{g}_K, \mathfrak{t}_K)$, are explicit versions of the exact sequence

$$1 \to T_X \longrightarrow \mathrm{Aut}(\mathfrak{g}_K, \mathfrak{t}_K) \longrightarrow A(R) \to 1,$$

with respect to the same chosen basis $(e_\alpha)_{\alpha\in R}$ for domain and target. (Vaguely said, $T_X$ is a multiplicative ersatz torus, and in the group setting, $i(\Phi)$ would be the inner automorphism given by conjugation with $\Phi$.)

LEMMA 3.3.5. — *Let $F \in \mathrm{Aut}(\mathfrak{g}_K, \mathfrak{t}_K)$. Then for $\Phi \in T_X$, we have $F = i(\Phi)$ if and only if $F \leftrightarrow (\mathrm{id}, (\Phi(\alpha))_{\alpha\in R})$. In particular, given any basis $B$ of $R$ and arbitrary $\eta_\alpha \in K^*$ for $\alpha \in B$, there are unique $\eta_\alpha$ for $\alpha \in R \setminus B$ and a unique $F \in \mathrm{Aut}(\mathfrak{g}_K, \mathfrak{t}_K)$ such that $F \leftrightarrow (\mathrm{id}, (\eta_\alpha)_{\alpha\in R})$. This $F$ is in $i(T_X)$.*

*Proof.* — That $i(\Phi) \leftrightarrow (\mathrm{id}, (\Phi(\alpha))_{\alpha\in R})$ for $\Phi \in T_X$ is clear; and we had already remarked that $F$ is uniquely determined by its tuple. For the second assertion, $B$ is a $\mathbb{Z}$-basis of $X$, so there is a unique $\Phi \in T_X$ with $\Phi(\alpha) = \eta_\alpha$ for $\alpha \in B$. Then by the first part, $F := i(\Phi)$ has a tuple of the prescribed form, with $\Phi(\alpha) = \eta_\alpha$ for all $\alpha \in R$. The uniqueness then follows from 3.3.3. $\square$

To end this subsection, we explicitly remark the following:

PROPOSITION 3.3.6. — *$F : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ is of the form $\mathrm{id} \otimes f$ with $f : (\mathfrak{g}, \mathfrak{t}) \xrightarrow{\sim} (\mathfrak{g}', \mathfrak{t}')$ if and only if ${}^\sigma F = F$ for all $\sigma \in \Gamma$.*

*Proof.* — ${}^\sigma F = F$ for all $\sigma \in \Gamma$ means that $F$ is $\Gamma$-equivariant, which implies that by restriction it induces an isomorphism (of Lie algebras over $k$)

$$f : \mathfrak{g} = (\mathfrak{g}_K)^\Gamma \simeq (\mathfrak{g}'_K)^\Gamma = \mathfrak{g}'.$$

with $f(\mathfrak{t}) = f((\mathfrak{t}_K)^\Gamma) = f(\mathfrak{t}_K)^\Gamma = (\mathfrak{t}'_K)^\Gamma = \mathfrak{t}'$. We naturally identify $F$ with $\mathrm{id} \otimes f$. The other direction is trivial. □

**3.3.2. Construction of the isomorphism.** — Keeping assumptions and notations, we now construct an isomorphism from $\mathfrak{g}_K$ to $\mathfrak{g}'_K$ and then improve it until we can let it descend to an isomorphism $f$ as in the theorem. We start with a refined version of the isomorphism theorem for split semisimple Lie algebras.

PROPOSITION 3.3.7. — *Let $\psi : X \xrightarrow{\sim} X'$ be an isomorphism with $\psi(R) = R'$. Let $B$ be any basis of $R$, and for any $\alpha \in B$, let $\eta_\alpha \in K^*$. Then there are $\eta_\alpha \in K^*$ for $\alpha \in R \setminus B$ and an isomorphism $F : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ such that $F \leftrightarrow (\psi, (\eta_\alpha)_{\alpha \in R})$.*

*Proof.* — By the mentioned theorem – see e.g. [**Bo2**, VIII.4.4, hm 2.i] – there is an isomorphism $F_1 : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ such that $\psi(\chi) = \chi \circ F^{-1}_{1|\mathfrak{t}_K}$ for all $\chi \in X$. Consequently, we have $F_1 \leftrightarrow (\psi, (\theta_\alpha)_{\alpha \in R})$ for certain $\theta_\alpha \in K^*$. There is a unique $\Phi \in T_X$ with $\Phi(\alpha) = \theta_\alpha^{-1} \eta_\alpha$ for all $\alpha \in B$. Setting $\eta_\alpha := \Phi(\alpha)\theta_\alpha$ for all $\alpha \in R$ and using 3.3.5 and 3.3.4.ii, we see that $F := F_1 \circ i(\Phi)$ is an isomorphism $(\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ satisfying $F \leftrightarrow (\psi, (\eta_\alpha)_{\alpha \in R})$. □

So in order to get a good $F$, we need $\psi$ and $\eta_\alpha$ with good properties. For this, we will now make use of the extra data we have got.

Let the isomorphism $(f_a)_K : (\mathfrak{g}_a)_K \xrightarrow{\sim} (\mathfrak{g}'_a)_K$ (sending $(\mathfrak{t}_a)_K$ to $(\mathfrak{t}'_a)_K$) correspond to the tuple $(\psi_0, (\eta^0_\alpha)_{\alpha \in R_0})$.

PROPOSITION 3.3.8. — *There is an isomorphism $F : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ with $F \leftrightarrow (\psi, (\eta_\alpha)_{\alpha \in R})$, satisfying the following properties:*

i. (a) *$\psi_{|X_a} = \psi_0$;*
   (b) *$\eta_\alpha = \eta^0_\alpha$ for $\alpha \in R_0$;*

ii. *there is $w' \in W'_\Gamma$ such that $w'\psi = h$;*

iii. (a) *${}^\sigma\psi = \psi$ for all $\sigma \in \Gamma$;*
   (b) *For all $(\sigma, \alpha) \in \Gamma \times R$,*

$$\xi'_{\sigma, \psi(\alpha)} = \xi_{\sigma,\alpha} \cdot \frac{\eta_{\sigma_A(\alpha)}}{\sigma(\eta_\alpha)}.$$

*Proof*

*First step.* — By definition (3.1.55), the congruence associated with $(f_a)_K$ is given by the map $w'\,\psi_0$ for a unique $w' \in (W_0')_\Gamma = W_0'$; by assumption, it is equal to $h_{|X_a}$. So define

$$\psi := w'^{-1}h : X \xrightarrow{\sim} X'$$

which certainly satisfies $\psi(R) = R'$, i.(a) and ii. We show that we also have iii.(a).

Let $\sigma \in \Gamma$. Remember that ${}^\sigma\psi$ was defined as $\sigma_{A'}\psi\sigma_A^{-1}$, where $\sigma_{A'}$ means the element in $A(R')$ and $\sigma_A$ the one in $A(R)$ through which $\sigma$ operates on $X'$ and $X$. We naturally extend everything to the vector spaces $V$ and $V'$. On these, we have $A(R)$- (respectively, $A(R')$-)invariant scalar products, hence a decomposition $V = V_a \oplus V_a^\perp$ such that $V_a^\perp$ can $A(R)$-equivariantly be identified with $V/V_a$ (and analogous for $V'$). Further, property ii in the definition 3.1.52 of a congruence shows that we have $h(V_a^\perp) = (V_a')^\perp$.

Now it suffices to check $\sigma$-invariance for the restrictions $\psi_{|V_a}$ and $\psi_{|V_a^\perp}$. The first one is extended from $\psi_0$, which is $\sigma$-invariant by 3.3.4.i because $(f_a)_K$ is.

From our congruence we know $t'(\sigma) = h \circ t(\sigma) \circ h^{-1}$; on the other hand, we had (see the proof of 3.1.43.i)

$$t(\sigma) = w_\sigma \sigma_A \qquad \big(\text{resp. } t'(\sigma) = w'_\sigma \sigma_{A'}\big)$$

with $w_\sigma \in W_0$, $w'_\sigma \in W'_0$. Since (by 3.1.34) $W_0$ acts trivially on $V_a^\perp$, and $W_0'$ trivially on $(V_a')^\perp$, we get

$$\sigma_{A'} \circ h_{|V_a^\perp} = (h \circ \sigma_A)_{|V_a^\perp}$$

and, because $w'$ was in $W_0'$ as well,

$$\sigma_{A'} \circ \psi_{|V_a^\perp} = (\psi \circ \sigma_A)_{|V_a^\perp}$$

which proves our claim.

*Second step.* — We define $\eta_\alpha$ for $\alpha \in \Delta$ so that i.(b) and iii.(b) are satisfied. Of course $\eta_\alpha := \eta_\alpha^0$ for $\alpha \in \Delta_0$. For a preliminary consideration, choose arbitrary $\widetilde{\eta}_\alpha \in K^*$ for $\alpha \in \Delta \setminus \Delta_0$. The previous proposition shows the existence of an isomorphism $\widetilde{F} : (\mathfrak{g}_K, \mathfrak{t}_K) \xrightarrow{\sim} (\mathfrak{g}'_K, \mathfrak{t}'_K)$ such that $\widetilde{F} \leftrightarrow (\psi, (\widetilde{\eta}_\alpha)_{\alpha\in R})$ with certain $\widetilde{\eta}_\alpha$ for $\alpha \in R \setminus \Delta$. Using i.(a) and 3.3.3 we see that the restriction of such $\widetilde{F}$ to $(\mathfrak{g}_a)_K$ coincides with $(f_a)_K$, so that necessarily $\widetilde{\eta}_\alpha = \eta_\alpha^0$ for $\alpha \in R_0$. Further, using 3.3.4 and ${}^\sigma\psi = \psi$, we get

$${}^\sigma\widetilde{F} \circ \widetilde{F}^{-1} \longleftrightarrow \big(\mathrm{id}, (\widetilde{\zeta}_{\sigma,\sigma_A^{-1}(\alpha)})_{\alpha\in R}\big)$$

with

$$\widetilde{\zeta}_{\sigma,\alpha} := \frac{\xi'_{\sigma,\psi(\alpha)}}{\xi_{\sigma,\alpha}} \cdot \frac{\sigma(\widetilde{\eta}_\alpha)}{\widetilde{\eta}_{\sigma_A(\alpha)}} \tag{15}$$

so that iii.(b) is equivalent to $\widetilde{\zeta}_{\sigma,\alpha} = 1$ for all $(\sigma,\alpha) \in \Gamma \times R$. (NB: This is already satisfied for $(\sigma,\alpha) \in \Gamma \times R_0$ because the $\eta^0_\alpha$ came from $(f_a)_K \leftrightarrow (\psi_0, (\eta^0_\alpha)_{\alpha\in R_0})$, and $\mathrm{id}_{(\mathfrak{g}_a)_K} = {}^\sigma(f_a)_K \circ (f_a)_K^{-1} \leftrightarrow (\mathrm{id}_{X_a}, (\widetilde{\zeta}_{\sigma,\sigma_A^{-1}(\alpha)})_{\alpha\in R_0})$.)

Now 3.3.5 shows that for $\sigma \in \Gamma$ there is a $\Phi_\sigma \in T_X$ with ${}^\sigma\widetilde{F} \circ \widetilde{F}^{-1} = i(\Phi_\sigma)$, hence

$$\widetilde{\zeta}_{\sigma,\sigma_A^{-1}(\alpha)} = \Phi_\sigma(\alpha)$$

for all $\alpha \in R$. Because $\Phi_\sigma$ (a fortiori $\Phi_\sigma \circ \sigma_A$) is a group homomorphism, it follows that

- ▷ $\widetilde{\zeta}_{\sigma,\alpha} = \widetilde{\zeta}_{\sigma,\beta}$ if $\alpha - \beta \in X_a$ (use $\sigma_A^{-1}(R_0) = R_0$ and the NB above);
- ▷ it suffices to choose $(\widetilde{\eta}_\alpha)_{\alpha\in\Delta\setminus\Delta_0}$ such that $\widetilde{\zeta}_{\sigma,\alpha} = 1$ for $\alpha \in \Delta \setminus \Delta_0$ and $\sigma \in \Gamma$.

From (12) and (15) one also calculates (for $\sigma, \tau \in \Gamma, \alpha \in R$):

$$\widetilde{\zeta}_{\sigma\tau,\alpha} = \widetilde{\zeta}_{\sigma,\tau_A(\alpha)} \cdot \sigma(\widetilde{\zeta}_{\tau,\alpha}) \tag{16}$$

Now fix a first choice of $\widetilde{\eta}_\alpha \in K^*$ for $\alpha \in \Delta \setminus \Delta_0$. Fix any such $\alpha$. Set

$$\overline{\Gamma}_\alpha := \big\{\sigma \in \Gamma : t(\sigma)(\alpha) = \alpha\big\},$$

a normal subgroup of $\Gamma$. Then 3.1.43.ii (or iv), the first bullet above and (16) show that $\sigma \mapsto \widetilde{\zeta}_{\sigma,\alpha}$ is a cocycle $\overline{\Gamma}_\alpha \to K^*$. Thus by Hilbert 90, there is $\mu_\alpha \in K^*$ such that $\widetilde{\zeta}_{\sigma,\alpha} = \sigma(\mu_\alpha)\mu_\alpha^{-1}$ for $\sigma \in \overline{\Gamma}_\alpha$.

Set $\eta_\alpha := \mu_\alpha^{-1}\widetilde{\eta}_\alpha$, keeping the other $\widetilde{\eta}_\bullet$ on $\Delta \setminus \{\alpha\}$. In this second choice, we have newly determined $\eta_\beta$ for all roots $\beta$ of the form $\sum_{\delta\in\Delta} n_\delta\delta$ with $n_\alpha \neq 0$; in particular $\eta_{\sigma_A(\alpha)} = \mu_\alpha^{-1}\widetilde{\eta}_{\sigma_A(\alpha)}$ for all $\sigma \in \overline{\Gamma}_\alpha$ by 3.1.43.iv and the relations in 3.3.3. So for this improved choice we have

$$\zeta_{\sigma,\alpha} = \frac{\xi'_{\sigma,\psi(\alpha)}}{\xi_{\sigma,\alpha}} \cdot \frac{\sigma(\mu_\alpha^{-1}\widetilde{\eta}_\alpha)}{\mu_\alpha^{-1}\widetilde{\eta}_{\sigma_A(\alpha)}} = \widetilde{\zeta}_{\sigma,\alpha} \cdot \frac{\mu_\alpha}{\sigma(\mu_\alpha)} = 1$$

for all $\sigma \in \overline{\Gamma}_\alpha$. The next step will be to extend this to all $\sigma \in \Gamma$.

Let $\sigma_1, \dots, \sigma_m$ be a complete set of representatives of $\Gamma/\overline{\Gamma}_\alpha$ with $\sigma_1 = \mathrm{id}$. Consider the orbits, with respect to the $t$-action, of $\Gamma$ in $\Delta \setminus \Delta_0$: the one which contains $\alpha$ is $\{\alpha, t(\sigma_2)\alpha, \dots, t(\sigma_m)\alpha\}$. Pick $j \in \{2, \dots, m\}$ and define

$$\beta := ({}^{\sigma_j^{-1}}w_{\sigma_j})(\alpha) = (\sigma_{jA}^{-1} \circ w_{\sigma_j} \circ \sigma_{jA})(\alpha).$$

Since $w_{\sigma_j} \in W_0$, also ${}^{\sigma_j^{-1}}w_{\sigma_j} \in W_0$ (3.1.33), and thus by 3.1.34,

$$\beta = \alpha + \sum_{\delta\in\Delta_0} n_\delta\delta \text{ with certain } n_\delta \in \mathbb{Z}_{\geq 0}\,.$$

On the other hand, $t(\sigma_j)\alpha$ is an element of $\Delta \setminus (\Delta_0 \cup \{\alpha\})$. So when we now make a third choice, which differs from the second one in defining:

$$\eta_{t(\sigma_j)\alpha} := \sigma_j(\eta_\beta) \cdot \frac{\xi'_{\sigma_j,\psi(\beta)}}{\xi_{\sigma_j,\beta}}$$

where the things on the right hand side are from the second choice, then $\eta_\beta$ remains the same in the third as in the second choice. This is made so that in this third choice, by (15) and using $\sigma_{jA}(\beta) = t(\sigma_j)\alpha$, we get

$$\zeta_{\sigma_j,\beta} = \frac{\sigma_j(\eta_\beta)}{\eta_{\sigma(\beta)}} \cdot \frac{\xi'_{\sigma_j,\psi(\beta)}}{\xi_{\sigma_j,\beta}} = 1$$

and hence also $\zeta_{\sigma_j,\alpha} = 1$ because $\beta - \alpha \in X_a$. Writing a general $\sigma$ as product of some $\sigma_j$ with something in $\overline{\Gamma}_\alpha$ and using the cocycle condition (16), we now have achieved $\zeta_{\sigma,\alpha} = 1$ for all $\sigma \in \Gamma$.

But plugging this result into (16) once more, we see that we even have $\zeta_{\sigma,\tau(\alpha)} = 1$ for every $\sigma, \tau \in \Gamma$, and so by $\tau(\alpha) - t(\tau)(\alpha) \in X_a$ and the first bullet, we have reached

$$\zeta_{\sigma,\alpha'} = 1$$

where $\alpha'$ is any root in the $t$-orbit of $\alpha$, and $\sigma$ is arbitrary.

Now if the orbit is the whole set $\Delta \setminus \Delta_0$, we are finished. If not, pick a new $\alpha$ from another orbit and repeat the procedure – there is no interference with the earlier adjustments. After finitely many steps, the proof is complete, and the $F$ given by the $\psi$ and $\eta_\alpha$ is constructed and satisfies ${}^\sigma F = F$ for any $\sigma \in \Gamma$. □

The proof of the theorem is finished as follows. The above $F$ by 3.3.6 descends to an isomorphism $f : (\mathfrak{g}, \mathfrak{t}) \xrightarrow{\sim} (\mathfrak{g}', \mathfrak{t}')$ which is of the form described in the theorem: by part i of the proposition, it induces $f_a$, and by part ii, $\mathrm{id} \otimes f \cong F$ induces the congruence $h$.

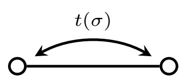
## 3.4. Visualisation: Satake-Tits diagrams

A convenient way to visualise an index $(X, R, \Delta, \Delta_0, t)$ was introduced by Satake in [**Sat1**]. Closely resembling is the visualisation in [**Tit1**], so the name *Satake-Tits diagram* seems appropriate. What one does is to take the Dynkin diagram of the root system $R$ (in $V$, with respect to the basis $\Delta$) and to decorate it by

▷ colouring black those vertices whose corresponding roots are in $\Delta_0$ (Tits instead encircles those in $\Delta \setminus \Delta_0$; confusingly, in the table of [**Spr3**, p. 320], the vertices of $\Delta \setminus \Delta_0$ are blackened);

▷ drawing arrows which show how the $t$-action permutes the vertices (Tits instead encircles the ones in a common $t$-orbit). Indeed, by 3.1.42 and [**Bo2**, VI.4.2, Cor.], each $t(\sigma)$ "is" a diagram automorphism.

So the Satake-Tits diagrams of our examples 3.2.5 and 3.2.9 are:

 and 


Of course, equivalent indices give the same diagram. A Lie algebra is anisotropic (resp. quasi-split, resp. split) if and only if all vertices in its diagram are black (resp.

white, resp. white and there are no arrows). Since in all cases except for the type $D_4$, the automorphism group of a connected Dynkin diagram has order $\leq 2$, the arrows do not get very complicated in many cases. Indeed if $[K:k]=2$ for $K$ the fixed field of $t$, we can and will do without the labelling of the arrows (they would all be the non-trivial automorphism of $K|k$), and instead add the field $K$ to the description. This information describes the index up to equivalence.

Remark 3.4.1. — We call a Satake-Tits diagram *closed* if every pair of vertices in it is linked by a combination of edges and/or arrows. We will see in Section 4.1 that a Lie algebra with closed Satake-Tits diagram arises by scalar restriction from an "absolutely simple one" whose underlying Dynkin diagram is connected; and in 4.3.5 that a partition of a Satake-Tits diagram in its closed components corresponds to a decomposition of the Lie algebra in its simple factors. So a Lie algebra is simple (resp. absolutely simple) if its Satake-Tits diagram is closed (resp. has connected underlying Dynkin diagram).

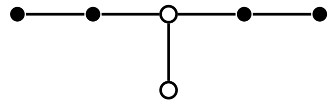
We end this section and chapter with a useful application of Satake-Tits diagrams.

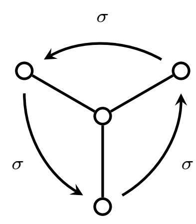
**3.4.1. Application: how to compute the rational root system.** — We describe a sometimes convenient way to read off the $k$-rational roots $\overline{R}$ from the Satake-Tits diagram of a Lie algebra $\mathfrak{g}$. The method is given by Tits in [**TiW**, 42.3.5], cf. [**Tit1**, 2.5]. Assume that the $k$-rank of $\mathfrak{g}$ is $\geq 1$ (in other words, $\overline{R}$ is not empty), and that the underlying Dynkin diagram is connected, so that $R$ is irreducible.

Lemma 3.4.2. — *The root system $\overline{R}$ is irreducible. Fixing a $\Gamma$-basis $\Delta$, $\rho$ projects the highest root to the highest root of $\overline{R}$ with respect to $\overline{\Delta}$.*

*Proof* (*see* [**Spr3**, 15.5.6]*; for the* highest root $\alpha \in R$*, see* [**Bo2**, VI.1.8]). — One sees that for any positive root $\beta \in R$, there is a finite sequence $\beta = \beta_0, \beta_1, \ldots, \beta_n = \alpha$ such that $\beta_i - \beta_{i-1} = n_i \delta_i$ for certain $n_i \geq 0$ and $\delta_i \in \Delta$. Reduction of this sequence gives an analogous one in $\overline{R}$ which shows that in fact $\rho(\alpha)$ is the unique highest root in $\overline{R}$, so that $\overline{R}$ is irreducible. □

Using that $\overline{R}$ is an irreducible root system, it can now be determined by the surjectivity of $\rho : R \twoheadrightarrow \overline{R}$, 3.1.43, 3.1.45, and the plates at the end of [**Bo2**, VI]. Let us show the method with two typical examples: We will later encounter the diagrams





of respective types $E_6$ and $D_4$. By 3.1.43.v and 3.1.45, the $k$-rank of $\mathfrak{g}$ (= rank of $\overline{R}$) is the number of $t$-orbits containing white vertices (Tits calls these the "distinguished orbits"). So in both examples, the rational root system has rank 2. For the $E_6$ type, $\rho(\alpha_2)$ and $\rho(\alpha_4)$ ($\alpha_2$ is the "lower vertex" in Bourbaki's odd-looking labelling, $\alpha_4$ is the one above it) are a basis of $\overline{R}$; Bourbaki's plate shows that the root system $R$ of type $E_6$ has highest root

$$\alpha := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

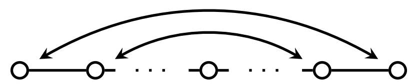
so $\overline{R}$ has highest root $\rho(\alpha) = 2\rho(\alpha_2) + 3\rho(\alpha_4)$. The only root system of rank 2 which contains such a combination of basis roots is $G_2$, so this must be it, with $\rho(\alpha_4)$ being the shorter root.

For the $D_4$ type, analogously, one basis root of $\overline{R}$ is $\rho(\alpha_2)$ ($\alpha_2$ is the "middle" vertex), and the other is $\rho(\alpha_1) = \rho(\alpha_3) = \rho(\alpha_4)$: being in one $t$-orbit, they all have the same image. Now $R$ has highest root

$$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$$

so $\overline{R}$ contains $3\rho(\alpha_1)+2\rho(\alpha_2)$, and again $\overline{R}$ must be of type $G_2$, now with the common projection of the "outer" roots as the short one.

One more example: We had seen the quasi-split forms of $A_n$. Say $n \geq 3$ is odd, $d := n + 1$. The Satake-Tits diagram is



$\overline{R}$ has basis

$$\delta_1 := \rho(\alpha_1) = \rho(\alpha_n), \delta_2 := \rho(\alpha_2) = \rho(\alpha_{n-1}), \dots,$$
$$\delta_{d/2-1} := \rho(\alpha_{d/2-1}) = \rho(\alpha_{d/2+1}), \delta_{d/2} := \rho(\alpha_{d/2}),$$

and we see that it has highest root

$$\sum_{i=1}^{d/2-1} 2\delta_i + \delta_{d/2}$$

and also contains

$$(0 \cdot \delta_1 +) \sum_{i=2}^{d/2-1} 2\delta_i + \delta_{d/2}.$$

The first information still gives us the possibilities $B_{d/2}$ and $C_{d/2}$, with $\delta_{d/2}$ being the leftmost or rightmost vertex in the usual Dynkin diagram labelling. But then the second information rules out $B_{d/2}$ (except for $d = 4$, where $B_2 = C_2$), so $\overline{R}$ is of type $C_{d/2}$ as claimed in 3.2.10.

This method can be used to *exclude* many diagrams, because they would give rise to impossible rational root systems. We do not need it in this generality, but will exploit it in the case of $k$-rank 1, see 4.3.3.

# CHAPTER 4

# CLASSIFICATION OVER GENERAL FIELDS

Interpreting the isomorphism theorem as a *uniqueness* statement, we can no longer avoid the question of *existence*. The question is: Which diagrams and which anisotropic kernels can occur over a given field $k$? For example, if $k$ is algebraically closed, one knows that there cannot be any arrows or black vertices (i.e. no anisotropic kernels). In Sections 4.1–4.4 we develop methods to exclude or construct various diagrams, by a combination of *algebraic* and *combinatorial* reasoning. Section 4.5 gives a general theory for the classical types $A$-$D$, and Section 4.6 deals with the quasi-split forms. Both suggest, as earlier examples did, that for a complete classification one has to take account of the *arithmetic* of the given field, which will be done in the next chapter.

In this chapter, $\mathrm{char}(k) = 0$ if not noted otherwise. All Lie algebras are understood to be of finite dimension. We say that an index or a Satake-Tits diagram is *$k$-admissible* if there is a semisimple Lie algebra over $k$ which has this index or diagram.

## 4.1. Reduction to absolutely simple Lie algebras

It is obvious that to classify semisimple Lie algebras, we only have to classify simple ones. But we can reduce further.

In this section, our ground field $k$ is only assumed to be perfect, with algebraic closure $\bar{k}$. [1]

Definition 4.1.1. — A Lie algebra $\mathfrak{g}$ over $k$ is called *absolutely simple* if $\mathfrak{g}_{\bar{k}}$ is simple over $\bar{k}$, or equivalently, $\mathfrak{g}_K$ is simple over $K$ for every extension $K|k$.

It turns out that every simple Lie algebra is the scalar restriction of an absolutely simple one. This was announced by Albert and proven – for a larger class of non-associative algebras than Lie algebras – with different methods by Landherr

1. The perfectness assumption is used only in the proposition.

(cf. [**Lan1**]) and Jacobson (cf. [**Jac2**]) in the 1930s. We present here a nice little theory around this fact, generally following Jacobson, and along the way solving [**Bo2**, VIII.4, Exercise 4].

For a Lie algebra $\mathfrak{g}$ over $k$, let $A(\mathfrak{g})$ be the (associative, unital) $k$-subalgebra of $\mathrm{End}_k(\mathfrak{g})$ generated by all $\mathrm{ad}_{\mathfrak{g}}(x)$, $x \in \mathfrak{g}$. Remark straightaway that for any field extension $L|k$, $a \otimes \mathrm{ad}_{\mathfrak{g}}(x) \mapsto \mathrm{ad}_{\mathfrak{g}_L}(a \otimes x)$ defines a natural isomorphism of associative $L$-algebras:

$$L \otimes_k A(\mathfrak{g}) \cong A(\mathfrak{g}_L) \tag{17}$$

Also remark that $\mathfrak{g}$ is a (left) $A(\mathfrak{g})$-module, and that an ideal of $\mathfrak{g}$ is the same as an $A(\mathfrak{g})$-submodule. Further define:

$$K := K(\mathfrak{g}) := \big\{ s \in \mathrm{End}_k(\mathfrak{g}) : s \circ \mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(x) \circ s \text{ for all } x \in \mathfrak{g} \big\}$$

which as finite dimensional $k$-vector space is the same as $\mathfrak{z}_{\mathfrak{gl}(\mathfrak{g})}(\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}))$; but we view it as associative $k$-algebra and remark that as such it identifies with $\mathrm{End}_{A(\mathfrak{g})}(\mathfrak{g})$.

Now let $\mathfrak{g}$ be simple. Then $K$ is a skew field by Schur's lemma. In fact, it is a field; namely, since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ it suffices to see that two elements $s, t \in K$ commute on a commutator $[x, y]$ for $x, y \in \mathfrak{g}$. But

$$s\big(t([x, y])\big) = s\big([x, ty]\big) = [sx, ty] = t\big([sx, y]\big) = t\big(s([x, y])\big)$$

where we have used, from left to right, that $t$ commutes with $\mathrm{ad}_{\mathfrak{g}}(x)$, $s$ with $-\mathrm{ad}_{\mathfrak{g}}(ty)$, $t$ with $\mathrm{ad}_{\mathfrak{g}}(sx)$ and $s$ with $-\mathrm{ad}_{\mathfrak{g}}(y)$. We call $K$ the *centroid* of $\mathfrak{g}$ and remark that $\mathfrak{g}$ has a natural structure as Lie algebra over $K$. When viewing it as such, we denote it by ${}^K\mathfrak{g}$.

We claim that the centroid of ${}^K\mathfrak{g}$ is $K$. Indeed, it is

$$\begin{aligned} &\big\{ s \in \mathrm{End}_K({}^K\mathfrak{g}) : s \circ \mathrm{ad}_{{}^K\mathfrak{g}}(x) = \mathrm{ad}_{{}^K\mathfrak{g}}(x) \circ s \text{ for all } x \in {}^K\mathfrak{g} \big\} \\ &\qquad \subseteq \big\{ s \in \mathrm{End}_k(\mathfrak{g}) : s \circ \mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(x) \circ s \text{ for all } x \in \mathfrak{g} \big\} = K \end{aligned}$$

and the other inclusion is clear. Further, the inclusion $A(\mathfrak{g}) \subseteq \mathrm{End}_k(\mathfrak{g})$ factors through natural maps $A(\mathfrak{g}) \hookrightarrow \mathrm{End}_K({}^K\mathfrak{g}) \hookrightarrow \mathrm{End}_k(\mathfrak{g})$, and the first arrow is bijective by Jacobson's density theorem [**Bo1**, VIII.5.5]. (2) Consequently, the following are equivalent:

i. $\mathfrak{g}$ is simple and $K = k$,

ii. $A(\mathfrak{g}) = \mathrm{End}_k(\mathfrak{g})$.

In this case we call $\mathfrak{g}$ *central simple*. So e.g. ${}^K\mathfrak{g}$ is central simple if $\mathfrak{g}$ is simple. It follows from (17) that every scalar extension of a central simple Lie algebra is again central simple, a fortiori absolutely simple. But we have much more:

PROPOSITION 4.1.2. — *Let $\mathfrak{g}$ be a simple Lie algebra and $L|k$ a Galois extension containing the centroid $K$. Then $\mathfrak{g}_L \simeq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ where $r = [K : k]$ and the $\mathfrak{g}_i$ are*

2. Or §4.2 in earlier editions. The theorem is absent from [**Jac2**], as he only proved it eight years later!

*absolutely simple Lie algebras over L. In particular, $\mathfrak{g}$ is central simple if and only if it is absolutely simple.*

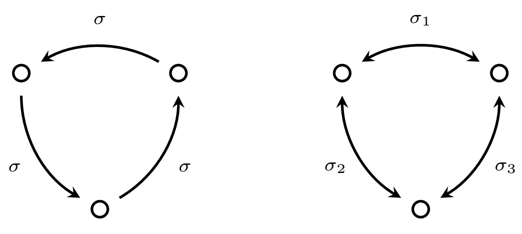
*Proof.* — Writing $K = k[X]/(f)$ where $f$ is a minimal polynomial of a primitive element of $K|k$, we have $L \otimes_k K \cong \prod_{i=1}^r L_i$ (as $L$-algebras) where the $L_i$ are all $L$ but with an $L$-action twisted via certain elements $\sigma_i : L \simeq L_i$ of the Galois group $\mathrm{Gal}(L|k)$, permuting the zeros of $f \in L[X]$. In particular, $r = [K : k]$. Then with (17),

$$A(\mathfrak{g}_L) \cong L \otimes_k \mathrm{End}_K({}^K\mathfrak{g}) \cong \mathrm{End}_{L\otimes_k K}\big((L \otimes_k K) \otimes_K ({}^K\mathfrak{g})\big)$$
$$\cong \mathrm{End}_{\prod_{i=1}^r L_i}\Big(\bigoplus_{i=1}^r ({}^K\mathfrak{g})_{L_i}\Big) \cong \prod_{i=1}^r \mathrm{End}_{L_i}\big(({}^K\mathfrak{g})_{L_i}\big).$$

Calling $e_i$ the $i$-th idempotent in the last product, the $A(\mathfrak{g}_L)$-module $e_i \cdot \mathfrak{g}_L$ is a simple ideal $\mathfrak{g}_i$ in $\mathfrak{g}_L$, which is in fact the simple $L$-Lie algebra deduced from $({}^K\mathfrak{g})_L$ by scalar extension (i.e. twisting the $L$-action) with $\sigma_i$. (It can be shown that as soon as $L$ is a splitting field for $\mathfrak{g}$, the $\mathfrak{g}_i$ are all isomorphic as Lie algebras over $L$.) □

As promised, we can write our simple Lie algebra as scalar restriction of an absolutely simple one: $\mathfrak{g} = R_{K|k}({}^K\mathfrak{g})$. Now let $\mathrm{char}(k) = 0$. The proposition and its proof show how the Satake-Tits diagram of $\mathfrak{g}$ looks like: It consists of $[K : k]$ copies of the Satake-Tits diagram of ${}^K\mathfrak{g}$ (whose underlying Dynkin diagram is irreducible), and these copies are connected by arrows. To be more precise, for a finite Galois splitting field $L$, one should index the copies by $L_i$ corresponding to $L \otimes_k K \cong \prod_{i=1}^r L_i$; the arrows between the copies correspond to the way $\mathrm{Gal}(L|k)$ interchanges the $L_i$ and hence the $\mathfrak{g}_i$ in the proposition. In particular, a simple Lie algebra has closed Satake-Tits diagram in the sense of 3.4.1.

As an example, start with a split simple (automatically absolutely simple) Lie algebra $\mathfrak{G}$ over a finite extension $K|k$ and consider $\mathfrak{g} := R_{K|k}\mathfrak{G}$. The Satake-Tits diagram will consist of $[K : k]$ copies of the Dynkin diagram of $\mathfrak{G}$ with arrows between them; so if $[K : k] > 1$, $\mathfrak{g}$ is non-split quasi-split (but also: not absolutely simple). More generally, every finite scalar restriction of a quasi-split Lie algebra is quasi-split. So the Satake-Tits diagram of $R_{K|k}\,\mathfrak{sl}_2(K)$ for a Galois resp. non-Galois cubic extension $K|k$ are:




Here, $\sigma$ is a generator of $\mathrm{Gal}(K|k) \simeq \mathbb{Z}/3$, resp. $\sigma_i$ are the three transpositions in $S_3 \simeq \mathrm{Gal}(L|k)$, $L$ being the normal closure of $K$. We will use this in Section 4.6.

### 4.2. Galois cohomology and forms of certain types

General references for this section are [**Ser2**, VII, Appendix and X], [**Ser3**, III.1] and [**Wei**]. The variants in [**Jac6**, X] and [**Sel1**, IV..1–2] have their origin in Jacobson's pre-cohomological version (cf. [**Jac3**, 6]). Compare also [**Sat3**, I.3].

Let $K|k$ be a field extension and $\mathfrak{G}$ a Lie algebra over $K$. A *$K|k$-form of $\mathfrak{G}$* is a pair $(\mathfrak{g}, F)$ consisting of a Lie algebra $\mathfrak{g}$ over $k$ and an isomorphism $F : \mathfrak{G} \xrightarrow{\sim} \mathfrak{g}_K$ of Lie algebras over $K$. Sometimes we drop the specific $F$ and just say that $\mathfrak{g}$ is a $K|k$-form of $\mathfrak{G}$.

Now let $\mathfrak{g}_1$ and $\mathfrak{g}$ be Lie algebras over $k$ such that $(\mathfrak{g}_1, F)$ is a $K|k$-form of $\mathfrak{g}_K$, and suppose $K|k$ is Galois. Every $\sigma \in \mathrm{Gal}(K|k)$ acts on both $(\mathfrak{g}_1)_K$ and $\mathfrak{g}_K$, and we set ${}^{\sigma}F := \sigma \circ F \circ \sigma^{-1}$ which is again a $K$-isomorphism from $\mathfrak{g}_K$ to $(\mathfrak{g}_1)_K$. This is compatible with the notation before and in 3.3.4, and in the special case $\mathfrak{g} = \mathfrak{g}_1$ gives a left $\mathrm{Gal}(K|k)$-action on $\mathrm{Aut}(\mathfrak{g}_K)$. In the general case, set

$$a_\sigma := F^{-1} \circ {}^{\sigma}F.$$

Then it is easily checked that $a_\sigma \in \mathrm{Aut}(\mathfrak{g}_K)$ and that $a_{\sigma\tau} = a_\sigma({}^{\sigma}a_\tau)$ for all $\sigma, \tau$ in $\mathrm{Gal}(K|k)$, so that $\sigma \mapsto a_\sigma$ is a cocycle. Next, if $(\mathfrak{g}_1, F)$ and $(\mathfrak{g}_2, F')$ are two $K|k$-forms of $\mathfrak{g}_K$ as above, and if there is a $k$-isomorphism of Lie algebras $f : \mathfrak{g}_2 \xrightarrow{\sim} \mathfrak{g}_1$, for the cocycles $a_\sigma$ and $b_\sigma$ corresponding to $F$ and $F'$ as above, we have

$$b_\sigma = \phi^{-1} \circ a_\sigma \circ ({}^{\sigma}\phi)$$

for all $\sigma \in \mathrm{Gal}(K|k)$, where $\phi := F^{-1} \circ f_K \circ F' \in \mathrm{Aut}(\mathfrak{g}_K)$. In other words, the corresponding cocycles are cohomologous.

Theorem 4.2.1. — *The above assignment induces a bijection from the set of isomorphism classes of $K|k$-forms of $\mathfrak{g}_K$ to $H^1(\mathrm{Gal}(K|k), \mathrm{Aut}(\mathfrak{g}_K))$.*

Although this is proven in much greater generality e.g. in [**Ser2**, X.2], let us stress the following down-to-earth approach to the inverse of this map. Given a cocycle

$$a : \mathrm{Gal}(K|k) \longrightarrow \mathrm{Aut}(\mathfrak{g}_K), \quad \sigma \longmapsto a_\sigma,$$

we get for each $\sigma \in \mathrm{Gal}(K|k)$ a map $u_a(\sigma) := a_\sigma \circ \sigma$ (where the second $\sigma$ denotes the canonical action on $\mathfrak{g}_K$). Then $u_a(\sigma)$ is a $\sigma$-semilinear, bijective map from $\mathfrak{g}_K$ to itself, compatible with the Lie bracket, we have $u_a(\mathrm{id}) = \mathrm{id}$ and $u_a(\sigma\tau) = u_a(\sigma) \circ u_a(\tau)$. One calls the $u_a$'s *Galois semi-automorphisms* and says they define the *Galois action twisted with the cocycle $a$*. Define $\mathfrak{g}(a)$ to be the set of elements of $\mathfrak{g}_K$ which are fixed by all $u_a(\sigma), \sigma \in \mathrm{Gal}(K|k)$. Then $\mathfrak{g}(a)$ is a Lie algebra over $k$, and one shows that the $K$-span of $\mathfrak{g}(a)$, which identifies with $\mathfrak{g}(a)_K$, is all of $\mathfrak{g}_K$. Hence $\mathfrak{g}(a)$ is a $K|k$-form of $\mathfrak{g}_K$. If $b$ is a cocycle cohomologous to $a$, say $b_\sigma = \phi^{-1} \circ a_\sigma \circ ({}^{\sigma}\phi)$ for some $\phi \in \mathrm{Aut}(\mathfrak{g}_K)$, then $\phi$ restricts to an isomorphism $\mathfrak{g}(b) \xrightarrow{\sim} \mathfrak{g}(a)$ of Lie algebras over $k$.

Now let $\mathrm{char}(k) = 0$ and let all extensions be inside a fixed algebraic closure $\bar{k}|k$. As noted in Chapter 3, for every semisimple $\mathfrak{g}$ over $k$ there is a finite Galois extension $K|k$

such that $\mathfrak{g}_K$ is split over $K$. The split semisimple Lie algebras, in turn, are classified and can be defined over any prime field (for us, $\mathbb{Q}$). We can express this as follows: For each (isomorphism class of) irreducible root system $R \in \{A_n, B_n, \ldots, G_2\}$, fix a split Lie algebra $\mathfrak{S}(R)$ of corresponding type over $\mathbb{Q}$. (This corresponds to fixing "base points" in cohomology sets.) Then for every *absolutely simple* Lie algebra $\mathfrak{g}$ over $k$ there is a finite Galois extension $K|k$ such that $\mathfrak{g}$ is a $K|k$-form of one $\mathfrak{S}(R)_K$.[3] In this case, we say that $\mathfrak{g}$ is a *form of type* $R$, and it is easily seen that this type is indeed well-defined and depends only on (much less than) the $k$-isomorphism class of $\mathfrak{g}$; in fact, it "is" the $\bar{k}$-isomorphism class of $\mathfrak{g}_{\bar{k}}$, as $\mathfrak{g}$ is a $\bar{k}|k$-form of $\mathfrak{S}(R)_{\bar{k}}$.

Putting this information together and using the limit procedures for infinite Galois cohomology, we get:

Theorem 4.2.2. — *The isomorphism classes of forms of type $R$ over $k$ are in bijection with the pointed set*

$$H^1\big(\operatorname{Gal}(\bar{k}|k), \operatorname{Aut}(\mathfrak{S}(R)_{\bar{k}})\big),$$

*the class of the split form being the distinguished point. Those forms which have the Galois splitting extension $K|k$ correspond to the ones in*

$$H^1\big(\operatorname{Gal}(K|k), \operatorname{Aut}(\mathfrak{S}(R)_K)\big)$$

*which by inflation identifies with a subset of the above.*

This translates our problem to a computation of Galois cohomology sets. Of course this is an equally difficult problem, but the cohomology machinery gives new methods to attack it. The classification of the *split* Lie algebras, which provides the underlying Dynkin diagrams to the Satake-Tits diagrams in our general approach, here pops up via their respective automorphism groups. The other ingredient are Galois groups, in which the field arithmetic is encoded.

Remark 4.2.3

i. For $k = \mathbb{R}$, the Galois group $\mathbb{Z}/2$ looks innocent, and still one gets a rich theory. But the messy-looking classification using certain "involutions" in real or complex Lie groups, going back to E. Cartan and to be found e.g. in [**OVi**, Chap. 5.1] or [**Oni**, §3], finds a nice interpretation in a cohomological theorem by Borel and Serre, [**Ser3**, III.4.5].
ii. A posteriori we will see that splitting fields for absolutely simple Lie algebras often have small degrees; in fact, over a $\mathfrak{p}$-adic field $k$, all forms of type different from $A_n$ are split by extensions of degree $\leq 6$, and the splitting field can often be chosen to be unramified. But to arrive at this conclusion, we first have to exploit the general theory.

3. In fact it follows from the previous section that a *simple* $\mathfrak{g}$ is a $K|k$-form of some finite product of copies of $\mathfrak{S}(R)_K$ for a *unique* irreducible $R$.

DEFINITION 4.2.4. — A $K|k$-form of the split Lie algebra $\mathfrak{S}(R)_K$ is called an *inner form* if a corresponding cocycle has image in $\mathrm{Aut}_0(\mathfrak{S}(R)_K)$, i.e. if its corresponding element is in the image of the map

$$H^1\big(\operatorname{Gal}(K|k), \mathrm{Aut}_0(\mathfrak{S}(R)_K)\big) \longrightarrow H^1\big(\operatorname{Gal}(K|k), \mathrm{Aut}(\mathfrak{S}(R)_K)\big)$$

induced by inclusion. Otherwise, it is called an *outer form.*

With the methods of the next subsection and [**Bo2**, VIII.5] one can show e.g. that an absolutely simple Lie algebra is an inner form of a split Lie algebra if and only if there are arrows in its Satake-Tits diagram. Note that $\mathrm{Aut}_0(\mathfrak{S})$ is of index $\leq 2$ in $\mathrm{Aut}(\mathfrak{S})$ for split simple $\mathfrak{S}$ except for those of type $D_4$, where the quotient is the symmetric group $S_3$. In this case, a form whose corresponding cocycle has images of order 3 or 6 modulo $\mathrm{Aut}_0(\mathfrak{S}(D_4))$ is called *of trialitarian type.*

**4.2.1. Application: An existence statement.** — Connecting the cohomological viewpoint with parts of our preparation for the isomorphism theorem in Chapter 3, we will later be able to prove *existence* of certain forms. The idea of the following is in [**Sat3**, II.3.1]; although it is necessarily a little technical, we hope that our presentation can add to clarification.

In our preparation for the isomorphism theorem we had expressed certain isomorphisms of Lie algebra as tuples. We will now do this for the automorphisms which are in the image of a cocycle. Let $\mathfrak{g}'$ be a Lie algebra over $k$ and fix $\mathfrak{s}', \mathfrak{t}', K|k$ as usual, giving its index $(X', R', \dots)$. Then $\mathfrak{g}'$ is a $K|k$-form of $\mathfrak{S}(R)_K$. Because $\mathfrak{S}(R)$ is actually defined over $\mathbb{Q}$, we can choose a maximal split toral $\mathfrak{t}$ (giving rise to a root system $R$ and lattice $X$) and bases for the corresponding root spaces $(e_\alpha)_{\alpha\in R}$ defined over $\mathbb{Q}$. Because the split maximal toral (= split Cartan) subalgebras in $\mathfrak{S}(R)_K$ are conjugate, we can assume the isomorphism $F : \mathfrak{S}(R)_K \to \mathfrak{g}'_K$ satisfies $F(\mathfrak{t}_K) = \mathfrak{t}'_K$. Let $\sigma \mapsto a_\sigma$ be the corresponding cocycle. We recall the notations of Section 3.3.1 and have a very special case: Because $\mathfrak{t}$ and the $e_\alpha$ are defined over $\mathbb{Q}$, the Galois action on $X$ is trivial, i.e. all $\sigma_A = \mathrm{id}$, and all $\xi_{\sigma,\alpha} = 1$ in (11). Further, calling $\psi : X \to X'$ the contragredient of $F$, we also just define $e'_{\psi(\alpha)} := F(e_\alpha)$ for all $\alpha \in R$, so that by definition (see (13)) we have

$$F \longleftrightarrow \big(\psi, (1)_{\alpha\in R}\big)$$

Then it follows immediately from 3.3.4 that

$$\begin{aligned} F^{-1} &\longleftrightarrow \big(\psi^{-1}, (1)_{\alpha\in R'}\big), \\ {}^\sigma F &\longleftrightarrow ({}^\sigma\psi, (\xi'_{\sigma,\psi(\alpha)})_{\alpha\in R}), \\ a_\sigma = F^{-1}\circ {}^\sigma F &\longleftrightarrow (a^*_\sigma, (\xi'_{\sigma,\psi(\alpha)})_{\alpha\in R}), \end{aligned}$$

where $a^*_\sigma = \psi^{-1} \circ {}^\sigma\psi$ is the contragredient of $a_\sigma$ and the $\xi'_\bullet$ are defined by $\sigma(e'_\alpha) = \xi'_{\sigma,\alpha} \cdot e'_{\sigma_{A'}(\alpha)}$. Indeed ${}^\sigma\psi = \sigma_{A'} \circ \psi$ and $a^*_\sigma = \psi^{-1} \circ \sigma_{A'} \circ \psi$ by triviality of the $\sigma_A$. From now on we stay in $X'$, i.e. identify $X$ and $X'$ via $\psi$, so that $a^*_\sigma = \sigma_{A'}$. (One could say that $a_\sigma$ is the linear map whose "matrix" in the good basis $e_\alpha$ is the "matrix" of the semilinear map $\sigma$ on $\mathfrak{g}'_K$ with respect to the image, via $F$, of this basis.)

*Fact 1.* — The $\xi'_\bullet$ satisfy

$$\xi'_{\sigma\tau,\alpha} = \sigma(\xi'_{\tau,\alpha}) \cdot \xi'_{\sigma,\tau_{A'}(\alpha)} \tag{18}$$

(for all $\alpha \in R, \sigma, \tau \in \Gamma$) by (12).

*Fact 2.* — $a^*_\sigma$ can be recovered from the index and $a^*_{\sigma|X'_a}$. Namely, for $\chi' \in X'$ and $\sigma \in \Gamma$ we have $t'(\sigma)(\chi') = w'_\sigma(\sigma_{A'}(\chi'))$ so that $a^*_\sigma = \sigma_{A'} = (w'_\sigma)^{-1}t'(\sigma)$. We know $t'(\sigma)$ from the index and $w'_\sigma$ from the restriction to $X'_a$.

*Fact 3.* — Knowing $a^*_\sigma$, the tuple $(\xi'_{\sigma,\alpha})_{\alpha\in R}$ is determined by $(\xi'_{\sigma,\alpha})_{\alpha\in R_0}$. Namely, this follows from the isomorphism theorem.

So with the information of its index $(X', R', \Delta', \Delta'_0, t')$, and assuming its anisotropic kernel is given as a form of $\mathfrak{S}(R'_0)_K$ (where $K|k$ is finite Galois and contains the fixed field of $t$), one can actually construct the cocycle $a$ and hence the Lie algebra $\mathfrak{g}'$ as form of the split Lie algebra $\mathfrak{S}(R')_K$. This suggests to try the process with something that "could be an index":

Definition 4.2.5. — A *possible index* is a tuple $(X, R, \Delta, \Delta_0, t)$ where $R$ is a root system spanning the free $\mathbb{Z}$-module $X$, $\Delta$ is a basis of $R$, $\Delta_0$ a subset of $\Delta$ and $t$ a homomorphism from $\mathrm{Gal}(\bar{k}|k)$ to $A(R)$, such that $\Delta$ and $\Delta_0$ are stable under each $t(\sigma)$. Call $R_0$ the subsystem generated by $\Delta_0$, $X_a$ the submodule spanned by $R_0$.

Let a possible index $(X, R, \Delta, \Delta_0, t)$ be given and assume the following:

i. We have an anisotropic $k$-Lie algebra $\mathfrak{g}_a$ given as a form of the split Lie algebra $\mathfrak{S}(R_0)_K$, with index $(X_a, R_0, \Delta_0, \Delta_0, t_{|X_a})$, where $K|k$ is finite Galois and contains the fixed field of $t$.

   As above this gives a tuple $(a^*_{\sigma\,|X_a}, (\xi_{\sigma,\alpha})_{\alpha\in R_0})$, and elements $w_\sigma \in W(R_0) \subseteq W(R)$ defined by $t(\sigma)(\chi) = w_\sigma(\sigma_A(\chi))$ for $\chi \in X_a$. Define, on all of $X$

$$a^*_\sigma := (w_\sigma)^{-1} \circ t(\sigma)$$

ii. The $(\xi_{\sigma,\alpha})_{\alpha\in R_0}$ can be extended to $(\xi_{\sigma,\alpha})_{\alpha\in R}$ such that:

   a) each $(a^*_\sigma, (\xi_{\sigma,\alpha})_{\alpha\in R})$ comes from a (necessarily unique) automorphism of $\mathfrak{S}(R)_K$, call it $a_\sigma$;

   b) (18) is satisfied.

Proposition 4.2.6. — *Under these conditions, there is a $K|k$-form $\mathfrak{g}$ of the split Lie algebra $\mathfrak{S}(R)_K$ which has index $(X, R, \Delta, \Delta_0, t)$ and anisotropic kernel $\mathfrak{g}_a$. In particular, the index is $k$-admissible.*

*Proof.* — With 3.3.4 one calculates that condition (ii) defines a cocycle $\sigma \mapsto a_\sigma$ from $\Gamma$ with values in $\mathrm{Aut}(\mathfrak{S}(R)_K)$. With the construction after theorem 4.2.1, this cocycle defines a form $(\mathfrak{g}, F)$ of type $R$, so its index is $(X, R, ?, ?, ?)$. The Galois action on $\mathfrak{g}$ is described by $a_\sigma$, in particular $\sigma_A = a_\sigma^*$. Define $X_0$ with respect to this $\Gamma$-action; from anisotropy of $\mathfrak{g}_a$ and the $\Gamma$-action restricting to the original one on $X_a = \mathbb{Z}\cdot\Delta_0$, it follows that $\Delta_0 \subseteq (\Delta \cap X_0)$. Now we claim that for all $\sigma \in \Gamma$, $a_\sigma^*(\delta) = w_\sigma^{-1} \circ t(\sigma)(\delta)$ is positive (with respect to $\Delta$) for $\delta \in \Delta \setminus \Delta_0$. Namely, $\delta' := t(\sigma)(\delta) \in \Delta \setminus \Delta_0$ by assumption, and $w_\sigma^{-1}(\delta') = \delta' + \sum_{\delta_i \in \Delta_0} n_i\delta_i$ for certain $n_i \in \mathbb{Z}_{\geq 0}$ because $w_\sigma$ sends roots to roots and induces the identity on $X/X_a$. The claim is proven. It implies that indeed $\Delta_0 = \Delta \cap X_0$, since for $\delta \notin \Delta_0$ it implies that $\sum_{\sigma\in\Gamma} \sigma_A(\delta)$ is $\neq 0$, so that $\delta \notin X_0$. It also implies that $\Delta$ is a $\Gamma$-basis. It is then immediate that the twisted action with respect to this coincides with the given $t$. □

## 4.3. Necessary and sufficient conditions on the index

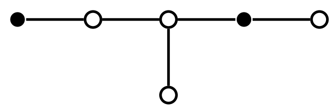
In this section we exploit some facts which, although rather simple for the most part, will reduce the number of admissible indices by large. Unless noted otherwise, $\mathfrak{g}$ is a semisimple $k$-Lie algebra with index $(X, R, \Delta, \Delta_0, t)$ with respect to $\mathfrak{s}, \mathfrak{t}, K$ as always, $\mathrm{char}(k) = 0$ and all extensions of $k$ are in a fixed algebraic closure $\bar{k}$ with $\mathcal{G} = \mathrm{Gal}(\bar{k}|k)$.

**4.3.1. The opposition involution.** — For any root system $R$ with basis $D$, the *opposition involution* of $D$ is the element $I \in A(R)$ defined as follows: there is a unique $w_I \in W(R)$ such that $w_I(D) = -D$. Set $I(\alpha) := -w_I(\alpha)$ for $\alpha \in R$. This defines an automorphism of the Dynkin diagram of $(R, D)$ which stabilises each irreducible component. If $R$ is irreducible, it is the identity, except in the cases $A_n$ ($n \geq 2$), $E_6$ and $D_n$ ($n$ odd (!)) where it is the unique non-trivial involution of the Dynkin diagram.

Lemma 4.3.1. — *Let $I$ be the opposition involution of $\Delta$. Then $I$ stabilises $\Delta_0$ and commutes with all $t(\sigma)$, $\sigma \in \mathcal{G}$.*

*Proof.* — We show that $w_I \in W_\Gamma$. There is $w_1 \in W_\Gamma$ such that $\pi(w_1) \in W(\overline{R})$ maps $\overline{\Delta}$ to $-\overline{\Delta}$. Then $w_1(\Delta \setminus \Delta_0) = -(\Delta \setminus \Delta_0)$. Since $w_1(\Delta_0)$ is another basis of $R_0$, there further is $w_2 \in W_0$ with $w_2(w_1(\Delta_0)) = -\Delta_0$. Then $w_2w_1(\Delta) = -\Delta$ and hence $w_I = w_2w_1 \in W_\Gamma$. This implies that $I$ stabilises $X_0$ and thus $\Delta_0$. Now let $\sigma \in \mathcal{G}$. Since $t(\sigma) \in A(R)$, we have $(t(\sigma))^{-1}\, w_I\, t(\sigma) \in W(R)$, but since $t(\sigma)$ stabilises $\Delta$ and is linear, this element also maps $\Delta$ to $-\Delta$ and hence is $w_I$. □

So roughly said, the black (and white) vertices in a Satake-Tits diagram of the mentioned types have to be symmetric. E.g. the following is impossible:



**4.3.2. Admissible subindices.** — In this subsection, stability under $t(\mathcal{G})$ and containment of (parts of the) anisotropic kernel in subdiagrams of a Satake-Tits diagram will allow us to descend corresponding subalgebras from $\mathfrak{g}_K$ to $\mathfrak{g}$, which gives us admissibility criteria.

First, let $\Delta'$ be a subset of $\Delta$ which contains $\Delta_0$ and is stable under every $t(\sigma)$. Set $X' := X \cap \mathbb{Q} \cdot \Delta'$, $R' := R \cap X'$, and denote by $t'$ the induced twisted Galois action on $X'$. $R'$ is the subsystem of $R$ generated by (i.e. consisting of roots which are non-negative or non-positive combinations of elements of) $\Delta'$, and we have $X' = \mathbb{Z}R'$, cf. [**Bo2**, VI.1.7].

Set $\overline{R}' := \rho(R') \setminus \{0\}$ which is the same as the subsystem of $\overline{R}$ generated by $\overline{\Delta}' = \rho(\Delta') \setminus \{0\}$. The conditions on $\Delta'$ ensure by 3.1.43.v that $R \cap \rho^{-1}(\overline{R}' \cup \{0\}) = R'$. Further, set

$$\mathfrak{s}_0 := \bigcap_{\lambda \in \overline{R}'} \ker(\lambda) = \bigcap_{\lambda \in \overline{\Delta'}} \ker(\lambda) \subseteq \mathfrak{s}.$$

By 3.1.21, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0)$ is reductive, so $\mathfrak{g}' := \mathcal{D}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0))$ is semisimple. We have

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \oplus \bigoplus_{\lambda \in \overline{R}'} \mathfrak{g}_\lambda \quad \text{and} \quad \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0)_K = \mathfrak{z}_{\mathfrak{g}_K}((\mathfrak{s}_0)_K) = \mathfrak{t}_K \oplus \bigoplus_{\alpha \in R'} (\mathfrak{g}_K)_\alpha.$$

One also sees $\mathfrak{s} \cap \mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0)) = \mathfrak{s}_0$ and thus

$$\mathfrak{s} \cong \mathfrak{s}_0 \times (\mathfrak{s} \cap \mathfrak{g}') \subseteq \mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0)) \times \mathfrak{g}' \cong \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}_0).$$

LEMMA 4.3.2 (Erasing lemma). — *The Lie algebra $\mathfrak{g}'$ is semisimple with maximal split toral subalgebra $\mathfrak{s}' := \mathfrak{s} \cap \mathfrak{g}' = \sum_{\lambda \in \overline{R}'} kH_\lambda$ and maximal toral subalgebra $\mathfrak{t}' := \mathfrak{t} \cap \mathfrak{g}'$. With respect to these, its index is $(X', R', \Delta', \Delta_0, t')$.*

*Proof.* — (In fact, $\mathfrak{g}'$ is the standard Levi subalgebra of the parabolic subalgebra corresponding to $\Delta'$ via [**Bo2**, VIII.3.4, Rem.]. 3.1.2.1 was the case $\Delta' = \Delta_0$.) We have $\mathfrak{s} \cap \mathfrak{g}' = \sum_{\lambda \in \overline{R}'} kH_\lambda$ by definition of the $H_\lambda$ (3.1.9) and 3.1.11.iv. One also sees that $\mathfrak{g}'$ contains the anisotropic kernel $\mathfrak{g}_a$ as subalgebra,

$$\mathfrak{g}' \supseteq \mathfrak{g}_a \oplus \sum_{\lambda \in \overline{R}'} kH_\lambda \oplus \bigoplus_{\lambda \in \overline{R}'} \mathfrak{g}_\lambda \quad \text{and} \quad \mathfrak{g}'_K = \sum_{\alpha \in R'} KH_\alpha \oplus \bigoplus_{\alpha \in R'} (\mathfrak{g}_K)_\alpha$$

as vector spaces. It follows that $\mathfrak{t}'_K = \sum_{\alpha \in R'} KH_\alpha$ is maximal toral in $\mathfrak{g}'_K$, hence $\mathfrak{t}'$ is maximal toral in $\mathfrak{g}'$ and a fortiori, $\mathfrak{s}'$ is maximal split toral in $\mathfrak{g}'$. The rest now follows easily, using $X_0 \subseteq X'$. □

This puts severe restrictions on indices. In a $k$-admissible Satake-Tits diagram it means the following: Erase the $t(\mathcal{G})$-orbit of any set of white vertices (i.e. the vertices and all edges touching them); then what is left must again be a $k$-admissible diagram.

There is a kind of a converse of 4.3.2 which allows us to build $k$-admissible diagrams from ones with smaller $k$-rank:

Proposition 4.3.3 (Patching proposition). — *Let* $(X, R, \Delta, \Delta_0, t)$ *be a possible index* (4.2.5). *Assume there are subsets* $\Delta', \Delta''$ *of* $\Delta$ *both of which are invariant under* $t(\mathcal{G})$ *and such that*

$$\Delta = \Delta' \cup \Delta'', \quad \Delta' \cap \Delta'' = \Delta_0.$$

*Like in the beginning of this subsection, we can form the "possible subindices"* $(X', R', \Delta', \Delta_0, t')$ *and* $(X'', R'', \Delta'', \Delta_0, t'')$. *Let* $\mathfrak{g}_a$ *be an anisotropic Lie algebra over* $k$, *given as* $K|k$*-form of* $\mathfrak{S}(R_0)_K$, *with index* $(X_a, R_0, \Delta_0, \Delta_0, t_{|X_a}) = (X'_a, R'_0, \Delta_0, \Delta_0, t_{|X'_a}) = (X''_a, R''_0, \Delta_0, \Delta_0, t_{|X''_a})$, *and* $K|k$ *finite Galois and containing the fixed field of* $t$. *Then if both possible subindices are* $k$*-admissible with Lie algebras* $\mathfrak{g}', \mathfrak{g}''$ *split over* $K$ *and having* $\mathfrak{g}_a$ *as anisotropic kernels, then* $(X, R, \Delta, \Delta_0, t)$ *is* $k$*-admissible with Lie algebra* $\mathfrak{g}$ *having anisotropic kernel* $\mathfrak{g}_a$.

*Proof.* — We want to apply 4.2.6. Like in Section 4.2.1, for all $\sigma \in \Gamma := \mathrm{Gal}(K|k)$, the Lie algebras $\mathfrak{g}_a$, $\mathfrak{g}'$ and $\mathfrak{g}''$ define $a^*_{\sigma|X'}$ and $a^*_{\sigma|X''}$ which both extend $a^*_{\sigma|X_a}$ and by $\Delta' \cap \Delta'' \subseteq \Delta_0$ patch well to a common $a^*_\sigma$ on $X$. Further they define tuples of scalars $(\xi^0_{\sigma,\alpha})_{\alpha\in R_0}$, $(\xi'_{\sigma,\alpha})_{\alpha\in R'}$ and $(\xi''_{\sigma,\alpha})_{\alpha\in R''}$ which coincide for common $\alpha$ (necessarily in $R_0$) and each satisfy (18) for their respective $\alpha$'s. Now for $\alpha \in \Delta$ we set $\xi_{\sigma,\alpha} := \xi'_{\sigma,\alpha}$ or $\xi''_{\sigma,\alpha}$ according to whether $\alpha \in \Delta'$ or $\Delta''$. By 3.3.7 for $\psi = a^*_\sigma : X \to X$ these $(\xi_{\sigma,\alpha})_{\alpha\in\Delta}$ can be extended to $(\xi_{\sigma,\alpha})_{\alpha\in R}$ such that there exists an $a_\sigma \in \mathrm{Aut}(\mathfrak{S}(R)_K)$ with $a_\sigma \leftrightarrow (a^*_\sigma, (\xi_{\sigma,\alpha})_{\alpha\in R})$. (Note that certainly $\xi^0_{\sigma,\alpha} = \xi_{\sigma,\alpha}$ for $\alpha \in R_0$ because of the assumption on the anisotropic kernel, but we do not claim these new $\xi$'s to coincide with any of the above $\xi'$ or $\xi''$ for $\alpha \in R \setminus \Delta$.) To apply 4.2.6, it remains to check that $\sigma \mapsto a_\sigma$ is a cocycle or equivalently, these $\xi_{\sigma,\alpha}$ satisfy (18) for all $\alpha \in R$. They certainly do for $\alpha \in \Delta$ and we will see that this implies the stronger claim. Namely, with 3.3.4 one computes that for $\sigma, \tau \in \Gamma$:

$$a^{-1}_{\sigma\tau} a_\sigma({}^\sigma a_\tau) \leftrightarrow (\mathrm{id}, (\xi^{-1}_{\sigma\tau,\alpha} \cdot \xi_{\sigma,\tau_A(\alpha)} \cdot \sigma(\xi_{\tau,\alpha}))_{\alpha\in R})$$

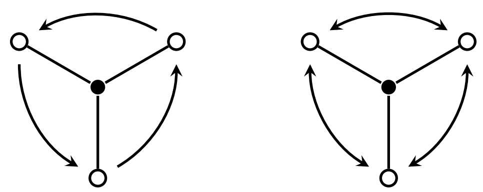
so that by 3.3.5, we have $\xi^{-1}_{\sigma\tau,\alpha} \cdot \xi_{\sigma,\tau_A(\alpha)} \cdot \sigma(\xi_{\tau,\alpha}) = \Phi(\alpha)$ for some $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(X, K^*)$. But as said, $\Phi_{|\Delta} \equiv 1$ and thus $\Phi \equiv 1$ and everything is proven. □

Remark 4.3.4. — It is possible to generalise this to finitely many sub-bases $\Delta'$, and also to relax a little the condition that they all contain $\Delta_0$: It is only necessary that in the Satake-Tits diagram, the subdiagrams contain with each black vertex all black vertices linked to it by a (possibly double or triple) edge, cf. [**Sat3**, II.3.1].

Remark 4.3.5. — Let $\mathfrak{g} = \prod_{i=1}^r \mathfrak{g}_i$ be a semisimple Lie algebra decomposed into its simple factors. With Section 4.1 one sees that this corresponds to a decomposition of the Satake-Tits diagram into closed components (cf. 3.4.1). Conversely, assume we have a partition of the Satake-Tits diagram of a semisimple $\mathfrak{g}$ into closed subdiagrams $S_i$. Then a combination of the erasing lemma, the patching proposition (in the remarked generalised version) and the isomorphism theorem shows that $\mathfrak{g}$ is the direct product of subalgebras $\mathfrak{g}_{S_i}$ with indicated diagrams, which are then necessarily simple.

**4.3.3. The case of $k$-rank 1.** — Let $\mathfrak{g}$ be an absolutely simple Lie algebra over $k$ whose $k$-rank is 1. This means that in its Satake-Tits diagram, there is either exactly one white vertex, or exactly two with an arrow between them.

(A priori there could also be the case of a diagram of type $D_4$ where the outer vertices are white and in one $t(\mathcal{G})$-orbit, while the inner vertex $\alpha_2$ is black:



where the action is via $\mathbb{Z}/3\mathbb{Z}$ and $S_3$, respectively. But this is impossible since $R$ contains its highest root $\alpha := \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ and thus the root system $\overline{R}$ would contain $\rho(\alpha_1)$ and $\rho(\alpha) = 3 \cdot \rho(\alpha_1)$.)

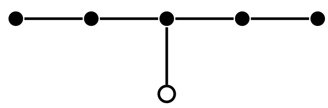
Let $\alpha$ (resp. $\alpha, \beta$) be the root(s) in $\Delta \setminus \Delta_0$. In this situation, we have:

Lemma 4.3.6. — *If* $\sum_{\delta \in \Delta} n_\delta \delta$ *is any positive root in $R$, then* $n_\alpha \in \{1, 2\}$ *(resp.* $n_\alpha = n_\beta = 1$*).*

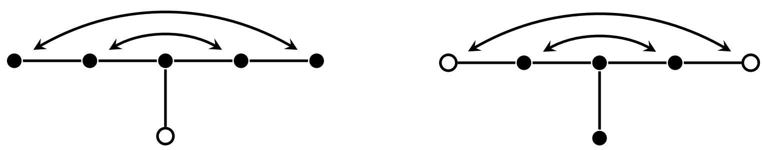
*Proof.* — $\overline{R}$ is a root system of rank 1 with basis $\overline{\Delta} = \{\rho(\alpha)\}$ (resp. $= \{\rho(\alpha) = \rho(\beta)\}$). It is either $A_1$ (which enforces the case $n_\alpha = 1$) or $BC_1$, which implies the statement. □

This little fact rules out many indices in the exceptional types. Going through the plates at the end of [**Bo2**, VI], we see which single white vertex (or orbit) is possible:

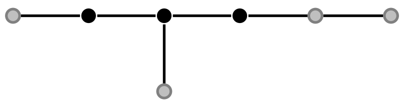
i. For inner forms of type $E_6$, invoking also 4.3.1, the only possibility is $\alpha = \alpha_2$:



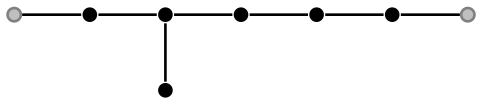
ii. For outer forms of type $E_6$, the possibilities are $\alpha = \alpha_2$ and $\{\alpha, \beta\} = \{\alpha_1, \alpha_6\}$:



iii. For $E_7$, the possibilities for a single white vertex are $\alpha_1, \alpha_2, \alpha_6$ and $\alpha_7$:

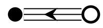


iv. For $E_8$, the possibilities for a single white vertex are $\alpha_1$ and $\alpha_8$:



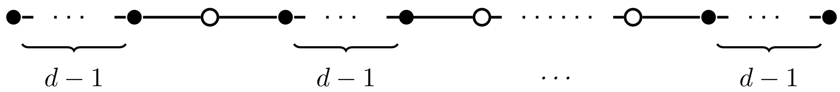
v. For $F_4$, the possibilities for a single white vertex are $\alpha_1$ and $\alpha_4$:

vi. For $G_2$, the only possibility for a single white vertex is $\alpha_2$:

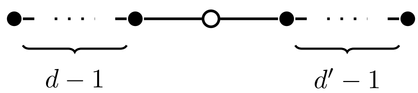


### 4.3.4. An $\mathbf{A_n}$ application

Proposition 4.3.7 (see Tits [**Tit1**]). — *An inner form of type $A_n$ with $k$-rank $r$ can exist only if $d := (n+1)/(r+1)$ is an integer. Its Satake-Tits diagram has to be*




*Proof.* — Induction on $r$ which is the number of white vertices. For $r = 0$ the assertion is trivial, and for $r = 1$, 4.3.1 implies that $n$ has to be odd and the white vertex must be the central one. Now assume $n_0 \geq r_0 \geq 2$ and the assertion shown for $r < r_0$ and arbitrary $n$. Given a $k$-admissible diagram of inner type $A_{n_0}$ with exactly $r_0$ white vertices, say the first white vertex from the left is the $d$-th vertex of all. Erase it. By the erasing lemma 4.3.2, what remains on the right of this has to be a $k$-admissible diagram, of inner type $A_{n_0-d}$ and with $k$-rank $r_0 - 1$. By the induction hypothesis, $d' := (n_0 - d + 1)/r$ is an integer, the abridged diagram to the right looks as above with $d'$ for $d$, and in particular, the second white vertex from the left in the original diagram is the $(d + d')$-th vertex of all. Erase this one, and again by 4.3.2, what is on the left of it,




is a $k$-admissible diagram. So by the case of $k$-rank 1 we must have $d = d'$. Hence $d(r_0 + 1) = n_0 - d + 1 + d = n_0 + 1$, i.e. $d = (n_0 + 1)/(r_0 + 1)$ and the whole diagram is of the asserted form. □

We will describe these forms explicitly in the next section and 4.5.21.

### 4.4. Skew fields and anisotropic types

In this section, we show a method to produce absolutely simple anisotropic Lie algebras which generalises example 3.2.5. It will turn out later that over a $\mathfrak{p}$-adic field $k$, these are the only absolutely simple anisotropic forms.

For this section and the next, recall the concept of *central simple algebra* (CSA for short) over a field $k$, and how it leads to the *Brauer group* $\mathrm{Br}(k)$ (cf. [**Ser2**, X..4–7], [**Pie**, 12.5], [**Rei**, Chap. 7]). For a CSA $A$ over $k$, we have its invariants

▷ *degree* $\mathrm{Deg}(A) = \sqrt{\dim_k A}$ and

▷ *index* $\mathrm{Ind}(A) = \mathrm{Deg}(D)$ where $D$ is a division algebra equivalent to $A$ in $\mathrm{Br}(k)$.

Further there is the

▷ *exponent* (or *period*) $\mathrm{Exp}(A)$ which is the order of the class of $A$ in $\mathrm{Br}(k)$. It is known (cf. [**Pie**, 14.4, Prop. b.ii]) that $\mathrm{Exp}(A)$ and $\mathrm{Ind}(A)$ have the same prime divisors.

We also make use of the *reduced characteristic polynomial*, and the *reduced norm* which is denoted as Nrd, cf. [**Pie**, Chap. 16], [**Rei**, Section 9].

When viewing an associative $k$-algebra as a Lie algebra over $k$ with respect to the commutator bracket $[a, b] := ab-ba$, we denote it by the corresponding Fraktur letter: So the associative algebra $A$ when viewed as Lie algebra is called $\mathfrak{A}$, etc.

Let $k$ be any field and $X \in M_n(k)$. On the $n^2$-dimensional $k$-vector space $M_n(k)$, consider the operator $\mathrm{ad}(X) \in \mathrm{End}_k(M_n(k))$ given by

$$\mathrm{ad}(X)(Y) := XY - YX.$$

The following lemma says that the eigenvalues of $\mathrm{ad}(X)$ are the pairwise differences of the eigenvalues of $X$.

Lemma 4.4.1. — *Assume that over an algebraic closure $\bar{k}$, the characteristic polynomial of $X$ decomposes as $\chi_X(T) = \prod_{i=1}^{n}(T - \lambda_i)$. Then the characteristic polynomial of $\mathrm{ad}(X)$ over $\bar{k}$ decomposes as*

$$\chi_{\mathrm{ad}(X)}(T) = \prod_{1\le i,j\le n} \big(T - (\lambda_i - \lambda_j)\big).$$

*Proof.* — For $X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ this is easy, the elementary matrix $E_{ij}$ being an eigenvector for $\lambda_i - \lambda_j$, and the general version can be derived by scalar extension to $\bar{k}$ and Jordan forms. Compare e.g. [**Bo2**, VII.2.2, Example 3] for a conceptual or [**Lan1**, p. 47] for a computational proof. □

Now let $D$ be a $k$-central division algebra with $n := \mathrm{Deg}(D)$. Its (Lie-algebraic) centre $\mathfrak{z} = \mathfrak{z}(\mathfrak{D})$ is its (associative-algebraic) centre $k$. Assume that $k$ is perfect.

Proposition 4.4.2. — *Every nilpotent element of $\mathfrak{D}$ is contained in $\mathfrak{z}$.*

*Proof.* — It is known that there exists a finite field extension $K|k$ in $\bar{k}$ such that we have a $K$-algebra isomorphism

$$K \otimes_k D \simeq M_n(K)$$

with regard to which we identify $D = 1 \otimes D$ with a sub-$k$-algebra of the right hand side. Let now $a \in D$ such that $a \in \mathfrak{D}$ is nilpotent; this means that $\mathrm{ad}(a) \in \mathrm{End}_K(M_n(K))$ is nilpotent when restricted to $D$. But $D$ generates $M_n(K)$ as $K$-vector space, so $\mathrm{ad}(a)$ is nilpotent, whence its characteristic polynomial is:

$$\chi_{\mathrm{ad}(a)}(T) = T^{(n^2)}$$

The lemma implies that the characteristic polynomial of $a \in M_n(K)$ is

$$\chi_{M_n(K),a}(T) = (T - \lambda)^n$$

with some $\lambda \in \bar{k}$. Note that by definition, $\chi_a := \chi_{M_n(K),a}$ is the *reduced characteristic polynomial* of $a \in D$, which depends neither on the choice of the splitting field $K$ nor on the splitting isomorphism $K \otimes_k D \simeq M_n(K)$, and whose coefficients are already in $k$: cf. [**Pie**, Prop. 16.1], [**Rei**, Thm 9.3]. On the other hand, the minimum polynomial $\min_{k,a}(T)$ divides $\chi_a(T)$ in $k[T]$ (cf. [**Pie**, Lemma 16.3b and Exercise 16.3.3], [**Rei**, Exercise 9.1]). Thus in $\bar{k}[T]$ we have $\min_{k,a}(T) = (T - \lambda)^m$ for some $1 \leq m \leq n$. But since $D$ is a skew field and $k$ is perfect, $a$ is contained in a separable field extension $k(a)|k$ inside $D$. By Galois theory necessarily $m = 1$ and $a = \lambda \in k = \mathfrak{z}$. (The argument works more generally if $\mathrm{char}(k)$ does not divide $n$.) □

Corollary 4.4.3. — *Let* $\mathrm{char}(k) = 0$. $\mathfrak{D}/\mathfrak{z} \simeq \mathcal{D}\mathfrak{D}$ *is either zero (namely, if and only if* $D = k$*, i.e.* $n = 1$*) or an anisotropic form of type* $A_{n-1}$.

*Proof.* — Assume $k \subsetneq D$. Then with the splitting field $K$ from above, we get $\mathfrak{D}_K \simeq \mathfrak{gl}_n(K)$, $\mathfrak{z}_K = \mathfrak{z}(\mathfrak{D}_K)$ and thus $(\mathfrak{D}/\mathfrak{z})_K \simeq \mathfrak{D}_K/\mathfrak{z}_K \simeq \mathfrak{sl}_n(K)$. If $x \in \mathfrak{D}/\mathfrak{z}$ is nilpotent, then so is $\mathrm{ad}_{\mathfrak{D}}(\widehat{x})$ for any representative $\widehat{x}$ of $x$ in $\mathfrak{D}$, so $\widehat{x} \in \mathfrak{z}$ by the proposition, hence $x = 0$. So $\mathfrak{D}/\mathfrak{z}$ is anisotropic. □

Another corollary is [**Bo2**, VIII.10, Exercise 7b] and will be taken up in 5.1.4:

Corollary 4.4.4. — *If over a field* $k$ *with* $\mathrm{char}(k) = 0$ *all semisimple Lie algebras are quasi-split, then every finite-dimensional division algebra* $D$ *over* $k$ *is a field; or in other words,* $\mathrm{Br}(K) = 0$ *for every finite extension* $K|k$.

*Proof.* — If $D$ is not a field, let $K \subsetneq D$ be its centre. Apply the previous corollary for $D$ over $K$ to get the simple $R_{K|k}(\mathfrak{D}/\mathfrak{z})$ which contains no nilpotent element and thus is anisotropic. □

## 4.5. Involutorial algebras and the classical types

For the "classical types" $A$-$D$ there is a theory which relates forms of Lie algebras (and algebraic groups) to objects of associative algebra which in turn can be classified. This unified approach was laid out in [**Wei**], concentrating on the group case. The corresponding theory for Lie algebras can be found in our main source [**Sel1**, IV.3] but goes back to Landherr and Jacobson; for a version with little Galois cohomology, cf. [**Jac6**, X] and exercises 16 and 17 to [**Bo2**, VIII.13]. A modern resource for the vast theories surrounding this is [**Inv**]. From a Galois cohomological viewpoint, the upshot is that the automorphism groups of certain objects are naturally equivalent, which induces the correspondences.

In this section, $k$ is generally assumed to be of characteristic $\neq 2$, although this is mostly for simplicity. When we go into details with Lie algebras, we restrict to $\mathrm{char}(k) = 0$.

Definition 4.5.1 (Involutorial algebras, first and second kind)

i. An *involutorial $k$-algebra* is a pair $(A, \iota)$ consisting of a finite dimensional $k$-algebra $A$ and a $k$-linear map $\iota : A \to A$ satisfying $\iota^2 = \mathrm{id}_A$ and $\iota(xy) = \iota(y)\iota(x)$ for all $x, y \in A$. In this context, $\iota$ is called an *involution* (or "involutorial anti-automorphism"). For any field extension $K|k$, we have the scalar extension $(K \otimes_k A, \mathrm{id} \otimes \iota)$ which we call $(A_K, \iota_K)$ and which is an involutorial $K$-algebra. An *isomorphism* of involutorial $k$-algebras $f : (A, \iota) \to (B, j)$ is of course an isomorphism of $k$-algebras $f : A \to B$ with $j \circ f = f \circ \iota$.

ii. An involutorial $k$-algebra $(A, \iota)$ is called *simple* if $A$ contains no proper $\iota$-stable ideal. It is called *absolutely simple* if $(A_K, \iota_K)$ is simple for all extensions $K|k$.

iii. An absolutely simple involutorial $k$-algebra $(A, \iota)$ is called
   - ▹ *of the first kind* if $A$ is a central simple algebra over $k$. Otherwise, it is called
   - ▹ *of the second kind*. Then either
     - ▹▹ $A$ contains a proper ideal $B$, in which case it is easily seen that $A = B \oplus \iota(B)$ and $B$ is a central simple $k$-algebra: such an $(A, \iota)$ is called *of type $AI$*. Or
     - ▹▹ $A$ is simple, in which case one sees that its centre $C$ is a quadratic field extension of $k$ such that $\iota_{|C}$ is the non-trivial automorphism of $C|k$. Such an $(A, \iota)$ is called *of type $AII$*.

Note that $(A_K, \iota_K)$ is of the same kind as $(A, \iota)$, and also of type $AI$ if $(A, \iota)$ is. On the other hand, $(A_C, \iota_C)$ is of type $AI$ if $(A, \iota)$ is of type $AII$ with centre $C$. Also note that for the second kind, our terminology coincides with the one of [**Inv**, 2.B], but deviates slightly from the general terminology of [**Inv**], where what we call an involutorial $k$-algebra of type $AII$ would be a central simple algebra over $C$ with involution of the second kind (the ground field is different).

THEOREM 4.5.2 (Wedderburn, Albert). — *Let $(A, \iota)$ be an involutorial $k$-algebra of the first kind (resp. of type AII). Then there is a unique $n \in \mathbb{N}_{\geq 1}$ and a division algebra $D$ with centre $k$ (resp. $C$), unique up to isomorphism, such that $A \simeq \mathrm{End}_D(D^n)$, and on $D$ there is an involution $I$ such that $I_{|k} = \mathrm{id}_k$ (resp. $I_{|C} = \iota_{|C}$); in particular, $(D, I)$ is an involutorial $k$-algebra of the first kind (resp. of type AII).*

*Proof.* — The Wedderburn part is found in any book about algebras. For the involution, see [**Alb**, Chap. II, Thm 5]. Remark that it is *not necessarily* so that $I$ is the restriction of $\iota$ to $D$ (or even that $\iota$ restricts to $D$) under a specific isomorphism $A \simeq \mathrm{End}_D(D^n)$, but there is one for which it is. Compare [**Inv**, Thm 3.1] for a more general variant. □

This is complemented by the following crucial theorem. Recall the notions from Section 2.3.

THEOREM 4.5.3. — *Let $(D, \sigma)$ be an involutorial $k$-algebra such that $D$ is a skew field. If it is of the first kind, there is a one-to-one correspondence*

$$\left.\begin{array}{r}\textit{non-degenerate } \sigma\textit{-hermitian and}\\ \sigma\textit{-skew-hermitian forms on } D^m\\ (\textit{up to a factor in } k^*)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{involutions of the first kind}\\ \textit{on the } k\textit{-CSA } \mathrm{End}_D(D^m)\end{array}\right.$$

*If it is of the second kind (necessarily of type AII), there is a one-to-one correspondence*

$$\left.\begin{array}{r}\textit{non-degenerate}\\ \sigma\textit{-hermitian forms on } D^m\\ (\textit{up to a factor in } k^*)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{involutions } \iota \textit{ of the second kind}\\ \textit{on the } C\textit{-CSA } \mathrm{End}_D(D^m)\\ \textit{satisfying } \iota_{|C} = \sigma_{|C}\end{array}\right.$$

*Both correspondences are given by mapping a form $h$ to $\iota : A \mapsto A^*$, where $A^*$ is the (left) adjoint to $A$ with respect to $h$.*

*Proof.* — This is [**Inv**, Thm. 4.2] applied to our situation ($E = D$, $M = D^n$, $\theta = \sigma$). (The left/right conventions in [**Inv**] are different from ours/Bourbaki's, but there is no essential difference.) □

**4.5.1. The first kind: Types $B$, $C$ and $D$.** — Let $(A, \iota)$ be of the first kind, so that $A$ is a CSA over $k$, and let $n = \mathrm{Deg}(A)$.

REMARK 4.5.4. — $\mathrm{Exp}(A)$ is 1 or 2. Consequently, $\mathrm{Ind}(A)$ is a power of 2; if $n$ is odd, it is 1 which means that $A$ is split.

*Proof.* — The involution gives an isomorphism of $k$-algebras $A \simeq A^{\mathrm{op}}$, and as is well-known, the class of $A^{\mathrm{op}}$ in $\mathrm{Br}(k)$ is the inverse of the class of $A$. □

Set

$$\mathrm{Sym}(A,\iota) := \{a \in A : \iota(a) = a\},$$
$$\mathrm{Skew}(A,\iota) := \{a \in A : \iota(a) = -a\}.$$

The following are straightforward:

Lemma 4.5.5 (Basic properties)

i. $A = \mathrm{Sym}(A,\iota) \oplus \mathrm{Skew}(A,\iota)$ *as* $k$*-vector spaces.*
ii. $\mathrm{Skew}(A,\iota)$ *is stable under the commutator bracket, thus a Lie subalgebra of* $\mathfrak{A}$ (*remember, this is the notation for* $A$ *viewed as Lie algebra*)*; when viewed as such, call it* $\mathfrak{S}(A,\iota)$.
iii. *For any field extension* $K|k$, *we have a natural identification* $\mathfrak{S}(A_K,\iota_K) = (\mathfrak{S}(A,\iota))_K$.

Now let us first look at the split case $A = M_n(k)$. By 4.5.3 (with $(D,\sigma) = (k,\mathrm{id})$), $\iota$ corresponds to a nondegenerate bilinear form $b$ on $k^n$ which is well-defined up to multiplication with a scalar in $k^*$ and either symmetric or skew-symmetric. In the first case, call $\iota$ *of orthogonal type*, in the second case, *of symplectic type.*

In the non-split case, take the same definition after extension with a splitting field, by which we mean an extension $K|k$ such that $A_K \simeq M_n(K)$. The type does not depend on the choice of such a splitting field and isomorphism. Indeed we have (cf. [**Inv**, Prop. 2.6]):

Lemma 4.5.6. — $\iota$ *is of orthogonal type if and only if*

$$\dim_k(\mathrm{Skew}(A,\iota)) = \frac{n(n-1)}{2}.$$

$\iota$ *is of symplectic type if and only if*

$$\dim_k(\mathrm{Skew}(A,\iota)) = \frac{n(n+1)}{2}.$$

*Moreover, if* $\iota$ *is of symplectic type,* $n$ *is even.*

(The last remark just comes from the fact that on an odd-dimensional vector space, all skew-symmetric forms are degenerate.)

From now on, let $\mathrm{char}(k) = 0$.

▷ For *odd* $n$ ($\geq 3$), $A$ is split by 4.5.4 and $\iota$ is of *orthogonal type.* For an appropriate extension $K|k$ (in fact, it can be chosen as a composite of quadratic extensions), there is an isomorphism $A_K \simeq M_n(K)$ such that the symmetric bilinear form $b_K$ corresponding to $\iota_K$ is given by $b_K(x,y) = {}^t xSy$ where

$$S := \begin{pmatrix} 0 & I_{\nu-1} & \\ I_{\nu-1} & 0 & \\ & & 1 \end{pmatrix}$$

with $n = 2\nu{+}1$. The involution $\iota_K$ is then given by $A \mapsto S^{-1}({}^tA)S$. Thus $\mathfrak{S}(A_K, \iota_K) = (\mathfrak{S}(A, \iota))_K$ is, by the classical description, [4] a split Lie algebra of type $B_\nu$.

▷ For *even* $n$ and $\iota$ of *symplectic type*, as soon as $A_K$ is split, by [**Bo1**, IX.5.1] there is an isomorphism $A_K \simeq M_n(K)$ such that the symplectic form $b_K$ is given by $b_K(x, y) = {}^txSy$ where

$$S := \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}$$

with $n = 2\nu$. Analogous to above, $(\mathfrak{S}(A, \iota))_K$ is a split Lie algebra over $K$ of type $C_\nu$.

▷ For *even* $n$ ($\geq 4$) and $\iota$ of *orthogonal type*, after extension with a splitting field of $A$, and then (analogously to type $B$) a series of quadratic extensions, for $b_K$ we now have a matrix

$$S := \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix}$$

and $(\mathfrak{S}(A, \iota))_K$ is split of type $D_\nu$ for $n = 2\nu$.

We thus have:

Proposition 4.5.7. — $\mathfrak{S}(A, \iota)$ *is an absolutely simple Lie algebra of respective type* $B$, $C$ *or* $D$ *in the cases considered above.*

The great thing is that this construction gives *all* $k$-forms of absolutely simple Lie algebras of the respective types (except for trialitarian types $D_4$). The crucial facts we need for this, first noted by Jacobson, are the next two lemmata.

Lemma 4.5.8. — $\mathrm{Skew}(A, \iota)$ *generates* $A$ *as associative* $k$*-algebra* (*i.e. the smallest sub-*$k$*-algebra of* $A$ *containing it is the whole algebra* $A$.)

*Proof.* — See [**Inv**, Lemma 2.26 (1)]. One reduces by scalar extension to standard split cases, which are done by direct matrix calculations. □

Lemma 4.5.9. — *Let* $\mathfrak{S}(A, \iota)_K$ *be split as above. If it is not of type* $D_4$*, then every* $\mathfrak{f} \in \mathrm{Aut}((\mathfrak{S}(A, \iota)_K)$ *is induced by a unique automorphism* $f$ *of* $(A_K, \iota_K)$*, which in turn is of the form* $X \mapsto F^{-1}XF$ *for* $F \in \mathrm{GL}_n(K)$ *with* $\iota_K(F)F = S^{-1}({}^tF)SF \in K^*{\cdot}I_n$*.* [5] *For type* $D_4$*, the same holds for* $\mathfrak{f}$ *which are of order* $\leq 2$ *modulo* $\mathrm{Aut}_0((\mathfrak{S}(A, \iota)_K)$.

*Proof.* — See e.g [**Sel1**, Lemma IV.3.1] except for the $D_4$ case which is easily amended. One uses the explicit description of automorphisms of the split Lie algebras (cf. [**Bo2**, VIII.13]) to see that they are of the form $X \mapsto F^{-1}XF$ for $F$ as described; with the previous lemma, their extension to automorphisms of $(A, \iota)$ is unique. (One

4. See e.g. [**Jac6**, IV.6]. There are different but equivalent normalisations in [**Bo2**, VIII.13]. Without coordinates, the split forms of types $B$ and $D$ are those where $b$ corresponds to a quadratic form of maximal Witt index $\nu = \lfloor \frac{1}{2}n \rfloor$.

5. $F$ is a (*matrix of*) *similitude* with respect to ($S$ or) $b_K$, i.e. $b_K(Fx, Fy) \in K^* \cdot b_K(x, y)$.

can even descend to an analogous result for non-split Lie algebras, cf. [**Jac3**, Thm 3] and [**Inv**, Prop. 2.25].) □

This shows by the general Galois cohomology machinery that the forms of $\mathfrak{S}(A, \iota)$ and the forms of $(A, \iota)$ are in correspondence:

THEOREM 4.5.10 (Jacobson, Weil). — *Let $\mathfrak{g}$ be an absolutely simple Lie algebra over $k$ of type $B_\nu$, $C_\nu$ or $D_\nu$ (in the case $D_\nu$, $\nu \geq 2$ and non-trialitarian). Then there is an involutorial $k$-algebra of the first kind $(A, \iota)$ as described in the respective cases above such that $\mathfrak{g} \simeq \mathfrak{S}(A, \iota)$.*

*Proof.* — Let $K|k$ be a finite Galois splitting field for $\mathfrak{g}$. By the discussion after 4.2.1 we have a cocycle $\sigma \mapsto a_\sigma$ from $\mathrm{Gal}(K|k)$ into $\mathrm{Aut}(\mathfrak{S}(M_n(K), \iota_K))$, with $\iota_K$ given via a matrix $S$ as in the descriptions of the split form of the respective type; and $\mathfrak{g}$ is isomorphic to the fixed elements of $\mathfrak{S}(M_n(K), \iota_K)$ under all $a_\sigma \circ \sigma$. Now by the lemma, we view the cocycle as one from $\mathrm{Gal}(K|k)$ to $\mathrm{Aut}(M_n(K), \iota_K)$, and define $A$ to be the fixed set in $M_n(K)$ of all $a_\sigma \circ \sigma$, $\sigma \in \mathrm{Gal}(K|k)$. One checks immediately that $(A, \iota_{K|A})$ is an involutorial $k$-algebra of the respective type, and with 4.5.8 one sees that $\mathfrak{g} \simeq \mathfrak{S}(A, \iota_{K|A})$. □

REMARK 4.5.11. — We do not address the question how *unique* such an $(A, \iota)$ is – which is dealt with extensively in Jacobson's work – because here we are only interested in existence, which provides us with $k$-admissible diagrams. In most cases (see the later remark 4.5.20 for typical exceptions), the $(A, \iota)$ to a form $\mathfrak{g}$ is unique up to isomorphism; the condition "up to isomorphism" translates to equivalence modulo scaling ("cogredience" in Jacobson's terminology) of certain (skew-)hermitian forms. But for a $\mathfrak{p}$-adic field $k$, admissibility conditions and the isomorphism theorem will imply that we almost only need a more careful look at the "uniqueness" of the analogous construction for type $A$, taken up later.

Now we connect the theorem with the classification of Chapter 3: what is the index of a Lie algebra $\mathfrak{S}(A, \iota)$?

**4.5.1.1.** *The index of of $\mathfrak{S}(A, \iota)$.* — Let $d := \mathrm{Ind}(A)$ and $m = n/d$. We have an isomorphism $A \simeq \mathrm{End}_D(D^m)$, where $D$ is a $k$-division algebra Brauer equivalent to $A$. By 4.5.2, there is an involution of the first kind on $D$. In the split case $D = k$ it is the identity, and we had already seen that the involutions of orthogonal/symplectic type on $A$ are induced by symmetric bilinear/symplectic forms on $k^m$. If, on the other hand, $d \geq 2$, we assume that the involution on $D$ is of symplectic type [6] and call it $\gamma$. Then we have the following refined version of 4.5.3, also part of [**Inv**, Thm 4.2.1]:

6. This is always possible by composing with an inner automorphism (cf. [**Inv**, Prop. 2.7.3]) and will be the canonical choice in later applications, where $D$ is a quaternion algebra.

THEOREM 4.5.12. — *There are one-to-one correspondences*

$$\left.\begin{array}{r}\textit{non-degenerate}\\ \gamma\textit{-skew-hermitian forms on } D^m\\ (\textit{up to a factor in } k^*)\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\textit{involutions of the first kind}\\ \textit{of orthogonal type}\\ \textit{on the } k\textit{-CSA } \mathrm{End}_D(D^m)\end{array}\right.$$

$$\left.\begin{array}{r}\textit{non-degenerate}\\ \gamma\textit{-hermitian forms on } D^m\\ (\textit{up to a factor in } k^*)\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\textit{involutions of the first kind}\\ \textit{of symplectic type}\\ \textit{on the } k\textit{-CSA } \mathrm{End}_D(D^m)\end{array}\right.$$

Now let $h(.\,,.)$ be a symmetric bilinear form on $k^m$ or a $\gamma$-(skew-)hermitian form on $D^m$. Write $V$ for $k^m$ or $D^m$ in these cases, and let $\iota$ be the left adjoint with respect to $h$ on $A$, that is, $h(v, aw) = h(\iota(a)v, w)$ for all $a \in A, v, w \in V$.

LEMMA 4.5.13. — *If $h(.\,,.)$ is anisotropic and $\mathfrak{g} := \mathfrak{S}(A,\iota)$ is semisimple (which by our earlier results is the case for $n \geq 3$), then $\mathfrak{g}$ is anisotropic.* [7]

*Proof.* — $V$ naturally is a $\mathfrak{g}$-module as $\mathfrak{g}$ is a subalgebra of $\mathfrak{A}$. Assume that there is a nilpotent element $x \neq 0$ in $\mathfrak{g}$. By Jacobson-Morozov, $x$ is part of an $\mathfrak{sl}_2$-triple $(x, H, y)$ in $\mathfrak{g}$, so that $V$ also becomes a finite dimensional $\mathfrak{sl}_2(k)$-module, and decomposes into eigenspaces $V_\lambda$ with $\lambda \in \mathbb{Z}$ for the operation of $H$. Since $0 \neq H \in A$, we can take $\lambda$ different from 0. Then for all $v \in V_\lambda \neq \{0\}$ we have

$$-\lambda h(v,v) = h(-\lambda v, v) = h(-Hv, v) = h(\iota(H)v, v) = h(v, Hv) = h(v, \lambda v) = \lambda h(v,v)$$

which implies $h(v,v) = 0$, a contradiction to $h$ being anisotropic. □

Now let us first investigate the *case $d = 1$ or equivalently, the CSA $A$ is split over $k$.* $h(.\,,.)$ is a symmetric bilinear form which we now call $b(.\,,.)$. $\mathfrak{g}$ is of type $B_{\lfloor \frac{1}{2}n \rfloor}$ for odd $n$ and of type $D_{n/2}$ for even $n$. By Witt decomposition, we can set $V = k^n$ together with a matrix

$$S := \begin{pmatrix} 0 & I_r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & S_a \end{pmatrix}$$

such that $r$ is the Witt index of $b(v,w) = ({}^t v)Sw$, and $S_a \in M_{n-2r,n-2r}(k)$ defines the anisotropic form $b_a(v_a, w_a) = ({}^t v_a)S_a w_a$ on $k^{n-2r}$. The corresponding involution (the left adjoint with respect to $b$) on $A = M_n(k)$ is given by

$$\iota(X) = {}^t S^{-1} \cdot {}^t X \cdot {}^t S.$$

A computation with the relation $\iota(X) \overset{!}{=} -X$ shows that our $\mathfrak{g}$ consists of those matrices of the form

7. The converse is also true, as will become clear in the following.

$$\begin{pmatrix} A & B & G \\ C & -({}^tA) & H \\ E & F & J \end{pmatrix}$$

where $A, B, C \in M_r(k)$, $E, F \in M_{n-2r\times r}(k)$, $G, H \in M_{r\times n-2r}(k)$, with the following relations: ${}^tB = -B$, ${}^tC = -C$ (i.e. $B$ and $C$ are skew-symmetric), ${}^tF \cdot {}^tS_a = -G$, ${}^tE \cdot {}^tS_a = -H$, and ${}^tS_a^{-1} \cdot {}^tJ \cdot {}^tS_a = -J$. If we assume, as we always can, $S_a$ to be a diagonal matrix $\mathrm{diag}(b_1, \dots, b_{n-2r})$, the last relation means that the $(i, j)$-entry of $J$ is its $(j, i)$-entry times $-b_i^{-1}b_j$; in particular, the diagonal of $J$ is zero.

We visibly have the $r$-dimensional $k$-split toral subalgebra

$$\mathfrak{s} := \left\{ \mathrm{diag}(a_1, \dots, a_r, -a_1, \dots, -a_r, 0, \dots, 0) : a_i \in k \right\}$$

and will see that it is maximal split toral. Indeed, one checks that its centraliser – which contains any larger toral subalgebra – is

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \left\{ \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & 0 & & \\ & & a_r & & & & \\ & & & -a_1 & & & \\ & & & & \ddots & & \\ & 0 & & & & -a_r & \\ & & & & & & \boxed{J} \end{pmatrix} : J \text{ as above}, a_i \in k \right\}.$$

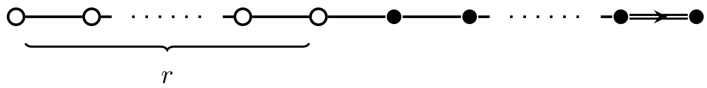
So the matrices $J$ in the lower right corner give the Lie algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})/\mathfrak{s}$. For $n - 2r \geq 3$, we had seen in the lemma that this is an anisotropic semisimple Lie algebra, hence $\mathfrak{s}$ is maximal split toral in $\mathfrak{g}$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s} \oplus \mathfrak{g}_a$ for the anisotropic kernel $\mathfrak{g}_a$ of $\mathfrak{g}$ (with respect to $\mathfrak{s}$), which consists of the matrices with entries 0 outside of the block $J$. Setting $\mathfrak{t} := \mathfrak{s} \oplus \mathfrak{t}_a$ for $\mathfrak{t}_a$ maximal toral in $\mathfrak{g}_a$, and fixing the usual root basis in a split extension, one concludes that in the Satake-Tits diagram, the vertices $\alpha_1, \dots, \alpha_r$ are white, the remaining ones are black.

In the cases $n - 2r = 0$ and $n - 2r = 1$ (where $J = 0$), we get the split forms of type $D_r$ and $B_r$, respectively.

In the remaining case $n = 2r + 2$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})/\mathfrak{s}$ is one-dimensional, in particular abelian, and we must have a non-split quasi-split form of type $D_{n/2}$ with maximal toral subalgebra $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. The anisotropic part of the space, with $S_a = \mathrm{diag}(b_1, b_2)$, corresponds to the form $b_1X^2 + b_2Y^2$, with discriminant $d = -b_1b_2 \cdot k^{*2} \notin k^{*2}$. (Note that on an even-dimensional vector space, the discriminant is invariant under scaling.) Visibly $K := k(\sqrt{-b_1b_2})$ is the minimal splitting field, and the non-trivial element of $\mathrm{Gal}(K|k)$ under the twisted action $t$ must correspond to the non-trivial automorphism of the Dynkin diagram that flips the two outer vertices $\alpha_r$ and $\alpha_{r+1}$.

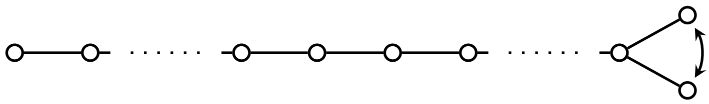
Before taking up the case $d > 1$, with these results we can already settle

**4.5.1.2.** *Type $B_\nu$.* — Because by 4.5.4 the CSA $A$ is automatically split. We thus see from the above discussion that the corresponding Lie algebra of type $B_\nu$ has the following Satake-Tits diagram with total number of vertices $\nu = \frac{n-1}{2}$:

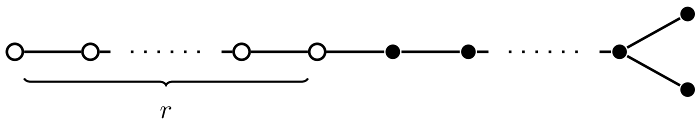



Remark 4.5.14. — This is the most direct connection to the theory and nomenclature of quadratic forms, and might explain why Tits calls his version of the isomorphism theorem a "Witt-type theorem": In this example, all the information of the (Satake-Tits) index is in the (Witt) index $r$; and the isomorphism class of the anisotropic kernel (of the Lie algebra) is determined by the isomorphism class of the anisotropic kernel (of the quadratic form). *Witt's decomposition theorem appears here as suggestive special case of the isomorphism theorem.*

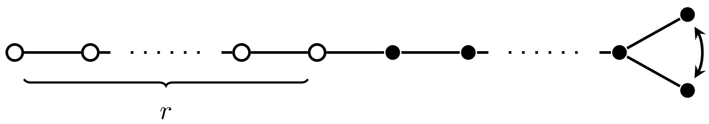
**4.5.1.3.** *Type $D_\nu$.* — Here, the CSA $A$ might be split or not. If it is, the previous discussion applies. For $r = \nu = \frac{1}{2}n2$, we have "the" split Lie algebra. If $r = \nu - 1$, we get the quasi-split forms



with minimal splitting field $k(\sqrt{d})$ where $d$ is a representative in $k$ of the discriminant of the bilinear form that defines $(A, \iota)$; each quadratic extension can occur. As soon as $r \leq \nu - 2$, the Satake-Tits diagram is either




or




and this depends on the discriminant (which, recall, on an even-dimensional space is invariant under scaling):

Lemma 4.5.15. — *Let $\nu \geq 2, n = 2\nu$ and $(A, \iota)$ be given by a symmetric bilinear form $b$ on $k^n$ with Witt index $r$ as above. The Lie algebra $\mathfrak{g} = \mathfrak{S}(A, \iota)$ is an outer form of type $D_\nu$ (i.e. there is an arrow in the Satake-Tits diagram) if and only if the discriminant $d(b)$ is not in $k^{*2}$. In this case, the fixed field of the twisted Galois action $t$ is $k(\sqrt{d}\,)$ where $d$ is a representative of $d(b)$ in $k^*$.*

*Proof*. — By induction on $\mu := \nu - r$ which is half the dimension of the anisotropic kernel of $b$ (and of $\mathfrak{g}_a$ for $\mu \geq 2$). We know the assertion for $\mu = 0$ (split case) and $\mu = 1$ (non-split quasi-split case) where $d(b)$ was necessarily in $k^{*2}$ resp. *not* in $k^{*2}$. Now let $\mu \geq 2$. Resuming the notations of the general discussion, we can assume that the $2\mu \times 2\mu$-matrix $S_a$ is diagonal, say $S_a = \mathrm{diag}(b_1, b_2, \dots, b_{2\mu})$. By definition, $d(b)$ is the residue of $(-1)^\mu \cdot b_1 \cdot \dots \cdot b_{2\mu} \bmod k^{*2}$.

Remark that for any pair of indices $1 \leq i \neq j \leq 2\mu$, the scalar $-b_i b_j$ or equivalently $-b_i/b_j$ cannot be a square in $k^*$, since otherwise the form defined by $S_a$ would have the isotropic vector ${}^t(0, \dots, 0, 1, 0, \dots, 0, x, 0, \dots, 0)$ with 1 the $i$-th and $x := \sqrt{-b_i/b_j}$ the $j$-th entry.

Set $K(i,j) := k(\sqrt{-b_i b_j}\,)$. Then at least the corresponding two-dimensional space in $V_{K(i,j)}$ is hyperbolic with respect to $b_{K(i,j)}$, and the discriminant is

$$d(b_{K(i,j)}) = (-1)^{\mu-1} \cdot \prod_{\ell \neq i,j} b_\ell \equiv d(b) \bmod K(i,j)^{*2}.$$

Now there are two cases:

- ▷ There is a pair $(i,j)$ such that $d(b_{K(i,j)}) \notin K(i,j)^{*2}$. Then certainly $d(b) \notin k^{*2}$ on the one hand, and on the other hand by the induction hypothesis, $\mathfrak{S}(A,\iota)_{K(i,j)}$ is an outer form, a fortiori $\mathfrak{S}(A,\iota)$ is an outer form and all assertions "descend".
- ▷ $d(b_{K(i,j)}) \in K(i,j)^{*2}$ for all pairs $(i,j)$. We make a further distinction:
  - ▷▷ $d(b) \notin k^{*2}$. But above we saw $d(b) \in K(i,j)^{*2}$ for all pairs, and $K(i,j)|k$ is quadratic, so $k(\sqrt{-b_i b_j}\,) = K(i,j) = k(\sqrt{d}\,)$ for $d = (-1)^\mu \prod b_\ell$. This means that all $-b_i b_j$ are congruent (to $d$ and thus to each other) mod $k^{*2}$. Then $1 \equiv (-b_i b_j)(-b_j b_\ell) \equiv b_i b_\ell \bmod k^{*2}$, hence all $b_i$ are in the same square class, hence by scaling and equivalence we can assume that all $b_i = 1$. With $d(b) \notin k^{*2}$ this is impossible unless $\mu$ is odd and $-1 \notin k^{*2}$, a typical "real" case which can be checked by hand (the non-trivial automorphism of $k(\sqrt{-1}\,)|k$ induces the opposition involution on the diagram).
  - ▷▷ $d(b) \in k^{*2}$. We have to show that our Lie algebra is an inner form. By the induction hypothesis, we know that all $\mathfrak{S}(A,\iota)_{K(i,j)}$ are inner forms. So if we had a non-trivial $t$-action, its fixed field would have to be a proper extension of $k$ contained in all $K(i,j)$, so they would all be equal. Like before we conclude that without loss of generality, all $b_i = 1$, and the $t$-fixed field and all $K(i,j)$ must be $k(\sqrt{-1}\,)$, in particular $-1 \notin k^{*2}$ which now gives a contradiction for odd $\mu$. For even $\mu$, one again checks that the non-trivial automorphism induces the opposition involution on the diagram which for even $\mu$, however, is the identity, contradicting our assumption. □

To complete the discussion of type $D_\nu$, we now sketch the *case* $d > 1$ *or equivalently, $A$ is non-split.* Then $A \simeq \mathrm{End}_D(D^m)$ and we see that by the first case of 4.5.12, our

Lie algebras come from (and, cf. 4.5.11, are classified "up to scaling in $k$" by) $\gamma$-skew-hermitian forms on $D^m$, where we assume $\gamma$ to be of symplectic type. Let such a form have index $r$, then our Lie algebra is given by those matrices $X \in A = M_m(D)$ such that $\iota(X) = {}^tH^{-1} \cdot {}^t(\gamma(X)) \cdot {}^tH = -X$ where $H$ is a $\gamma$-skew-hermitian matrix. The coordinates can be chosen such that

$$H := \begin{pmatrix} 0 & -I_r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & H_a \end{pmatrix}.$$

where $H_a$ defines an anisotropic skew-hermitian form on $D^{m-2r}$. Then $\mathfrak{g}$ consists of the matrices

$$\begin{pmatrix} A & B & G \\ C & -{}^t(\gamma(A)) & K \\ E & F & J \end{pmatrix}$$

where $A, B, C \in M_r(D)$, $E, F \in M_{m-2r\times r}(D)$, $G, K \in M_{r\times m-2r}(D)$ are subject to the following relations: ${}^tB = \gamma(B)$, ${}^tC = \gamma(C)$ (i.e. $B$ and $C$ are $\gamma$-hermitian), ${}^t\gamma(F) \cdot {}^tH_a = G$, ${}^t\gamma(E) \cdot {}^tH_a = -K$, and ${}^tH_a^{-1} \cdot {}^t\gamma(J) \cdot {}^tH_a = -J$. If we assume, as we always can, $H_a$ to be a diagonal matrix $\operatorname{diag}(b_1, \dots, b_{n-2r})$ – where necessarily the $b_i \in \operatorname{Skew}(D, \gamma)$ –, the last relation means that the $(i, j)$-entry of $J$ is the $\gamma$-conjugate of its $(j, i)$-entry times $-b_i^{-1}b_j$; in particular, the diagonal of $J$ consists of elements of $\operatorname{Skew}(D, \gamma)$.

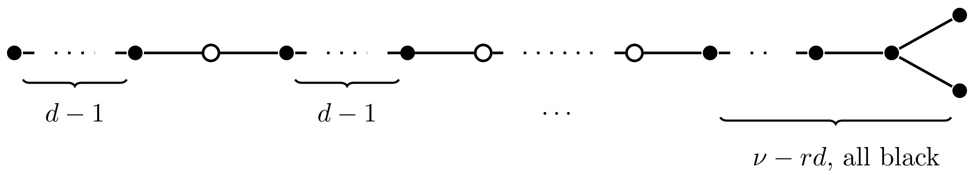
A split toral subalgebra of $k$-dimension $r$ is given by

$$\mathfrak{s} := \{\operatorname{diag}(a_1, \dots, a_r, -a_1, \dots, -a_r, 0, \dots, 0) : a_i \in k\}$$

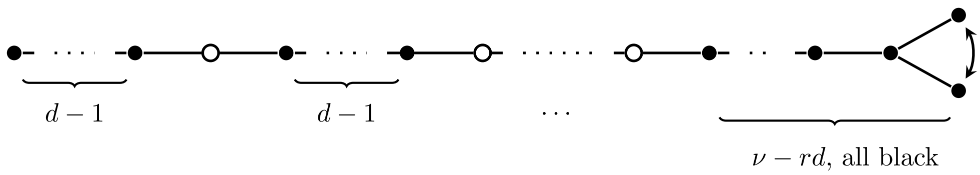
and its centraliser $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ consists of the matrices

$$\begin{pmatrix} c_1 & & & & & & \\ & \ddots & & & 0 & & \\ & & c_r & & & & \\ & & & -\gamma(c_1) & & & \\ & & & & \ddots & & \\ & 0 & & & & -\gamma(c_r) & \\ & & & & & & \boxed{J} \end{pmatrix}$$

with $J$ as above and $c_i \in D$. Let $\mathfrak{u}$ be those matrices as above with all $c_i \in D \setminus k$, so that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s} \oplus \mathfrak{u}$. By 4.5.13, as soon as $2r < m$ (for $d > 2$) or $2r < m - 1$ (for $d = 2$), the matrices with zeros outside of $J$ describe an anisotropic subalgebra $\mathfrak{a}$, but the anisotropic kernel is bigger: it is all of $\mathfrak{u}$, as one can show with the investigation of skew fields in Section 4.4 (compare the later discussion of type $AI$), and it decomposes as $(\mathcal{D}\mathfrak{D})^r \oplus \mathfrak{a}$. This $\mathfrak{a}$ on its own is a form of type $D_\mu$ with $\mu = \frac{1}{2}(n - 2rd) = \nu - rd$. One infers that the general Satake-Tits diagram is




or

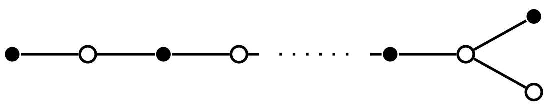



and again the occurrence of an arrow depends on whether the discriminant of the $\gamma$-skew-hermitian form, which in this case can be well-defined as

$$(-1)^m \cdot \operatorname{Nrd}\big(\det(h(v_i, v_j))\big) \mod k^{*2}$$

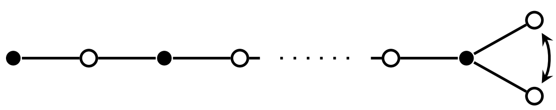
for a $D$-basis $v_1, \ldots, v_m$ of $V$ (cf. [**Tsu**, 1.3]), is in $k^{*2}$ or not. If not, there is an arrow and its fixed field is the corresponding quadratic extension.

In the special case $m = 2r$, equivalently $\nu = rd$, the anisotropic space $V_a$ vanishes and this leads to a diagram with white vertices $\alpha_d, \alpha_{2d}, \ldots, \alpha_{rd}$, and all others black. For example for $d = 2$ we get:



Note the consistency with 4.3.1: Since $d$ is a power of 2 by 4.5.4, the rank $\nu = rd$ is *even*, so the opposition involution does *not* flip the two rightmost vertices $\alpha_{\nu-1}$ (black) and $\alpha_\nu$ (white).

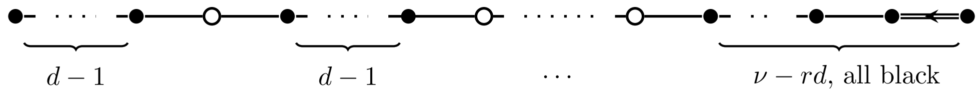
The last exceptional case is $d = 2$ and $\nu = m = 2r + 1$ (hence $r \geq 1$, since a type $D_1$ does not exist). Here $D = Q$ is a quaternion division algebra and $\gamma$ is the *standard involution* from Section 2.2. Indeed, by [**Inv**, Prop. 2.21], this is the unique symplectic involution on $Q$, whereas the orthogonal ones are those of the form $\iota_u(x) = u{\cdot}\gamma(x){\cdot}u^{-1}$ where $u$ is an invertible element of $\operatorname{Skew}(Q, \gamma)$ which is uniquely determined by $\iota_u$ up to a factor in $k^*$. Now the space $V_a$ is of $Q$-dimension 1, the anisotropic form $h_a$ on it is given by $h_a(v, w) = vH_a\gamma(w)$ with a pure quaternion $H_a = q_a \neq 0$, unique up to scaling with $k^*$, and the condition $q_a^{-1} \cdot \gamma(J) \cdot q_a = -J$ is equivalent to $J \in k \cdot q_a$. One calculates that the Satake-Tits diagram is



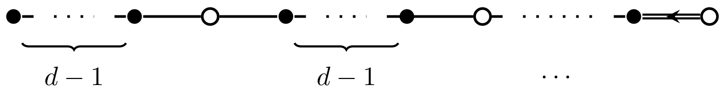
and the fixed field of the arrow is the well-defined quadratic extension $k(\sqrt{a}\,)$ where $a = -\operatorname{Nrd}(q_a)$, a representative of the discriminant.

**4.5.1.4.** *Type* $C_\nu$. — As noted in the split cases after lemma 4.5.6, if the CSA $A$ is split, then $\mathfrak{S}(A, \iota)$ is the split form of type $C_\nu$ with $n = 2\nu$.

Otherwise, if $d \geq 2$, for type $C_\nu$ we are interested in the second case of 4.5.12 and see that our Lie algebras now come from $\gamma$-hermitian forms. The discussion is very similar to the one above, and we just give the result: If such a form has index $r$, the Satake-Tits diagram of the corresponding Lie algebra is




respectively




for $\nu > rd$ respectively $\nu = rd$. Again $d$ is a power of 2 by 4.5.4, and of course $d$ divides $n = 2\nu$.

As a final remark, the case $d > 1$ is strongly restricted for fields of interest:

Remark 4.5.16. — If in $\mathrm{Br}(k)$ there is only one element of order 2, then by 4.5.2 and 4.5.4, in the above cases necessarily $D \simeq D^{\mathrm{op}}$ is "the" division algebra representing this element. Specifically, if $k$ is $\mathbb{R}$ or a $\mathfrak{p}$-adic field, $D$ is the quaternion division algebra over $k$ and $d = 2$ (if not 1) in types $C_\nu$ and $D_\nu$ above.

**4.5.2. The second kind: Type** $A$. — Let $(A, \iota)$ be of the second kind. If it is of type $AI$ with $A \simeq B \oplus \iota(B)$, we set $n := \mathrm{Deg}(B)$; if it is of type $AII$ with centre $C$ set $n := \mathrm{Deg}(A)$ with $A$ viewed as CSA over $C$. So in both cases we have $\dim_k A = 2n^2$, and $n$ is "stable under scalar extension". Setting again

$$\mathrm{Skew}(A, \iota) := \{a \in A : \iota(a) = -a\}$$

we see that this is stable under the commutator bracket and thus a Lie algebra $\mathfrak{S}(A, \iota)$ over $k$; also, $\mathfrak{S}(A_K, \iota_K) \cong (\mathfrak{S}(A, \iota))_K$. Here we have $\dim_k \mathfrak{S}(A, \iota) = n^2$ (for both types). $\mathrm{Skew}(A, \iota)$ generates $A$ as $k$-algebra as soon as $n > 1$ for type $AI$ and $n > 2$ for type $AII$, cf. [**Jac6**, X.4, Lemma 4].

Let us first look at an $(A, \iota)$ of type $AI$, i.e. $A \simeq B \oplus \iota(B)$ for a central simple $k$-algebra $B$. Then $\iota(B) \simeq B^{\mathrm{op}}$ and we can identify $(A, \iota)$ with $(B \times B^{\mathrm{op}}, (a, b^{\mathrm{op}}) \mapsto (b, a^{\mathrm{op}}))$ (cf. [**Inv**, 2.14]); or specifically if $B \simeq M_n(k)$, we can also arrange $\iota(X, Y) = ({}^tY, {}^tX)$. It is obvious that in this case, the skew elements identify with the "anti-diagonal" $(b, -b^{\mathrm{op}})$ (resp. $(X, -{}^tX)$) and thus with a copy of $B$; hence the corresponding Lie algebra is $\mathfrak{S}(A, \iota) \simeq \mathfrak{B}$. If $n \geq 2$ and $B \simeq M_r(D)$ with $D$ a central $k$-division algebra of degree $d$ (so $n = rd$), we thus have

$$\mathfrak{S}(A, \iota) \simeq \mathfrak{gl}_r(D) \text{ (resp. } \mathfrak{D} \text{ for } r = 1),$$
$$\mathfrak{g} := \mathcal{D}\mathfrak{S}(A, \iota) \simeq \mathcal{D}\mathfrak{gl}_r(D) \text{ (resp. } \mathcal{D}\mathfrak{D} \text{ for } r = 1)$$

and if $K|k$ splits $D$ (i.e. $D_K \simeq M_d(K)$) we have

$$\mathfrak{g}_K \simeq \mathfrak{sl}_n(K)$$

so that $\mathfrak{g}$ is a form of type $A_{n-1}$.

Now since for $(A,\iota)$ of type $AII$ with centre $C$, $(A_C,\iota_C)$ is of type $AI$ and thus can be further split as above, we see that $\mathcal{D}\mathfrak{S}(A,\iota)$ is an absolutely simple Lie algebra of type $A_{n-1}$. Again we have a converse [**Sel1**, Lemma and Thm 4.3.1]:

Theorem 4.5.17 (Landherr, Jacobson, Weil). — *Let $n \geq 2$.*

i. *Let $(A_K,\iota_K) \simeq (M_n(K)\times M_n(K), (x,y)\mapsto ({}^t y, {}^t x))$ so that $\mathcal{D}\mathfrak{S}(A,\iota)_K \simeq \mathfrak{sl}_n(K)$. Every $\mathfrak{f} \in \mathrm{Aut}((\mathfrak{S}(A,\iota)_K)$ is induced by a unique automorphism $f$ of $(A_K,\iota_K)$, which in turn is of the form*

$$(X,Y) \longmapsto (F^{-1}XF, {}^t(F^{-1t}YF)) \quad or \quad (X,Y) \longmapsto (F^{-1}YF, {}^t(F^{-1t}XF))$$

*for an $F \in \mathrm{GL}_n(K)$.*

ii. *Like in* 4.5.10 *it follows: let $\mathfrak{g}$ be an absolutely simple Lie algebra of type $A_{n-1}$. Then there is an involutorial $k$-algebra of the second kind $(A,\iota)$ with $\dim_k A = 2n^2$ such that $\mathfrak{g} \simeq \mathcal{D}\mathfrak{S}(A,\iota)$.*

Remark that $x \mapsto -{}^t x$, the non-trivial element of $\mathrm{Aut}(\mathfrak{sl}_n(K))/\mathrm{Aut}_0(\mathfrak{sl}_n(K))$ for $n \geq 3$, corresponds to $(X,Y) \mapsto (Y,X)$ under i.

Corollary 4.5.18. — *If $\mathfrak{g} \simeq \mathcal{D}\mathfrak{S}(A,\iota)$ as in part ii of the theorem, $\mathfrak{g}$ is an inner form (of the split $\mathcal{D}\mathfrak{S}(A_K,\iota_K)$ as in part i) if and only if $(A,\iota)$ is of type $AI$.*

*Proof.* — If $\mathfrak{g}$ is inner, then by what we just remarked the automorphisms in the image of a corresponding cocycle are all of the first form described in the theorem: They stabilise both factors. Then so do their corresponding Galois semi-automorphisms. Hence their fixed point set $A$ visibly is a product of two non-trivial factors. Conversely, assume $A = A_1 \times A_2$ with non-trivial factors and identify it with a subset of a split $A_K$ as in the theorem. $A_{1K}$ and $A_{2K}$ must correspond to the two factors $M_n(K)$; but if in the image of a cocycle corresponding to $A$ there were an automorphism that flips the factors, so would the corresponding Galois semi-automorphism; and for every element $(a_1,a_2) \in A_1 \times A_2$, being fixed by it, $a_1$ would be determined by $a_2$ which is absurd. □

The distinction between $AI$ and $AII$ goes back to Landherr, who originally (cf. [**Lan1**, p. 50]) distinguished the corresponding Lie algebras by characteristic polynomials of their matrix representations. — As promised in 4.5.11 we are now also interested in how unique an $(A,\iota)$ for $\mathfrak{g}$ as in the theorem is:

Proposition 4.5.19. — *Let $(A_1,\iota_1)$ and $(A_2,\iota_2)$ be absolutely simple involutorial $k$-algebras of the second kind, of $k$-dimensions $2n_1^2$ resp. $2n_2^2$, $n_i \geq 2$, such that there is an isomorphism of $k$-Lie algebras $\mathfrak{f} : \mathfrak{S}(A_1,\iota_1) \simeq \mathfrak{S}(A_2,\iota_2)$. Then $\mathfrak{f}$ is induced by* *an isomorphism $f : (A_1, \iota_1) \simeq (A_2, \iota_2)$, and $f$ is unique except for the case that the algebras are of type $AII$ and $n_i = 2$. In particular, if the $(A_i, \iota_i)$ are of type $AI$ and given as $A_i \simeq B_i \times B_i^{\mathrm{op}}$, then either $B_1 \simeq B_2$ or $B_1 \simeq B_2^{\mathrm{op}}$.*

*Proof.* — See [**Jac6**, X.4, Thms 10 and 11] or even [**Lan1**, Satz 3] (for type $AI$), cf. also [**Inv**, Prop. 2.25]. The strategy – which works for both types like we (or rather Weil) defined them, rendering Jacobson's distinction unnecessary – is to embed everything in a common split $(A_K, \iota_K)$ resp. the Lie algebra of its skew elements. Then, balancing cleverly the distinct Galois actions, one reduces to the case of automorphisms of this, which we had settled in the theorem. Now one descends back to the given forms. This shows existence of the isomorphism, and uniqueness follows again because the associative algebras in question are generated by their skew elements. □

Remark 4.5.20

i. In low degrees, there exist exceptional isomorphisms among the first kind ("$B_2 = C_2$") which we have circumvented by not investigating uniqueness there at all, see 4.5.11. There also exist exceptional isomorphisms intertwining the two kinds ("$A_1 = B_1 = C_1, D_2 = A_1 \times A_1, A_3 = D_3$") which we have circumvented by restricting the proposition above to "inside" the second kind. Since in the end we will classify not by involutorial algebras, but by indices and anisotropic kernels, our results not only suffice, but even will imply e.g. the following identities of anisotropic forms over a $\mathfrak{p}$-adic field $k$, where $Q$ is its quaternion division algebra with standard involution $\gamma$:

   (a) ("$A_1 = C_1 = B_1$") the derived Lie algebra of $Q$ over $k$ is isomorphic to $\mathfrak{S}(Q, \gamma)$ (this is immediate!), and to $\mathfrak{S}(M_3(k), \iota)$ where $\iota$ is the adjoint with respect to any anisotropic quadratic form on $k^3$ (specifically, the reduced norm on the pure quaternions; every other is cogredient to this); [8];

   (b) ("$A_3 = D_3$") the derived Lie algebra of a 16-dimensional $k$-central division algebra (there are two anti-isomorphic ones, which by the proposition give the same Lie algebra) is isomorphic to $\mathfrak{S}(M_3(Q), \iota)$ where $\iota$ comes from any anisotropic $\gamma$-skew-hermitian form on $Q^3$ (in fact there is only one). For starters one sees that both Lie algebras have dimension 15.

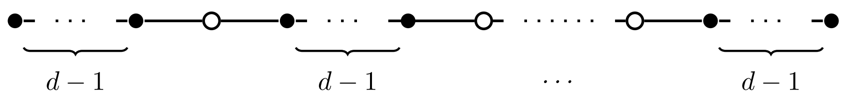
ii. Another instance where the classification with involutorial algebras falls short is related to triality (cf. [**Jac6**, X, Exercise 3]): Over $k = \mathbb{R}$, the Lie algebras $\mathfrak{S}(M_4(\mathbb{H}), \iota)$ (with $\iota$ coming from any skew-hermitian form on $\mathbb{H}^4$) and $\mathfrak{S}(M_8(\mathbb{R}), \iota)$ (with $\iota$ coming from a quadratic form on $\mathbb{R}^8$ of Witt index 2) are isomorphic, although the involutorial algebras are not. This will also appear over $\mathfrak{p}$-adic fields $k$, see 5.4.7. One look at the Satake-Tits diagrams gives an idea:

8. This is true verbatim for $k = \mathbb{R}, Q = \mathbb{H}$; and is nothing else than example 3.2.5.

Turning back to type A, we can now classify both types and describe their diagrams.

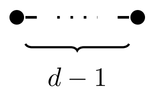
**4.5.2.1.** *The inner forms/Type AI*

Proposition 4.5.21 (cf. 4.3.7). — *If $\mathfrak{g}$ is an inner form of type $A_n$ over $k$, then there is a $k$-central division algebra $D$, $\mathrm{Deg}(D) = d$, and an $r \in \mathbb{Z}_{\geq 0}$ such that $d(r+1) = n+1$ and $\mathfrak{g}$ is isomorphic to the derived algebra of $M_{r+1}(D)$ viewed as Lie algebra over $k$. The isomorphism class of $\mathfrak{g}$ determines $D$ up to isomorphism or anti-isomorphism. Conversely, such derived algebra of $M_{r+1}(D)$ is an inner form of type $A_n$. Its rational root system is of type $A_r$, and its Satake-Tits diagram is:*




*In particular, every anisotropic inner form of type $A_n$ is isomorphic to the derived Lie algebra of a $k$-central division algebra $D$ of degree $n+1$; two such division algebras $D_1, D_2$ have isomorphic derived Lie algebras if and only if $D_1 \simeq D_2$ or $D_1 \simeq D_2^{\mathrm{op}}$.*

*Proof.* — All assertions except for the one about the diagram and the rational roots are combinations of 4.5.17.ii and 4.5.19. To compute the diagram, note the following: Inside the derived algebra of $M_{r+1}(D)$, the diagonal matrices with entries in $k$ are a split toral subalgebra $\mathfrak{s}$ of dimension $r$. On the other hand, the diagonal matrices with entries in in $D \setminus k^*$ form an anisotropic subalgebra isomorphic to a product of $r$ copies of $\mathcal{D}\mathfrak{D}$. Such a $\mathcal{D}\mathfrak{D}$, in turn, by 4.4.3 and 4.5.18 has Satake-Tits diagram

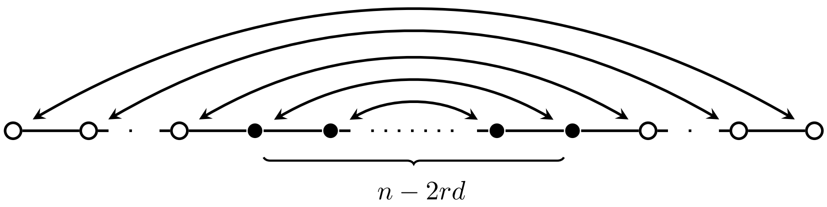



Finally one sees that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ consists exactly of all diagonal matrices in $\mathfrak{g}$, hence is the $k$-direct sum of $\mathfrak{s}$ and the named anisotropic subalgebra, from which it follows that $\mathfrak{s}$ is indeed maximal split toral, the $k$-rank of our algebra is $r$, the named anisotropic subalgebra is its anisotropic kernel (with respect to $\mathfrak{s}$), and the Satake-Tits diagram is as described. (For this, one could also invoke 4.3.7 as soon as one knows that the $k$-rank is $r$.) The rational roots are computed as in 3.4.1. □

**4.5.2.2.** *The outer forms/Type AII.* — Here again we content ourselves with a description of the admissible diagrams, since for $\mathfrak{p}$-adic $k$, we will get a uniqueness/isomorphism statement for free.

Let $(A, \iota)$ be of type $AII$, specifically, $A \simeq M_{m+1}(D)$ with $D$ a central division algebra over $C$ of degree $d$, $C|k$ a quadratic extension, $m+1 = \frac{n+1}{d}$, $n \geq 2$, and there is an involution $\sigma$ of the second kind on $D$, so that $\sigma_{|C}$ is the non-trivial element of $\mathrm{Gal}(C|k)$. By the second part of 4.5.3, the Lie algebras $\mathfrak{S}(A,\iota)$ are classified (not necessarily one-to-one) by the $\sigma$-hermitian forms on $D^{n+1/d}$.

Proposition 4.5.22. — *Let such a form have Witt index $r$. Then $\mathcal{D}\mathfrak{S}(A,\iota)$ has Satake-Tits diagram*




*with $C$ being the fixed field of the twisted Galois action.*

*Proof.* — (We restrict to the case $d=1$ i.e. $D=C$. This is obviously the only case if $k=\mathbb{R}$, but even so for $k$ a $\mathfrak{p}$-adic field, as will turn out in 5.4.3. Cf. [**TiW**, 42.3.4] for a geometric interpretation of the group analogue. See also [**Bo2**, VIII.13, Exercise 16.b].) Fix an element $y$ with $C=k(y)$, $y^2\in k$. In generalisation of 3.2.10 and similar to the types $B-D$, our Lie algebra $\mathfrak{g}=\mathcal{D}\mathfrak{S}(A,\iota)$ is given by those traceless (!) matrices $X\in A=M_{n+1}(C)$ such that $\iota(X)={}^tH^{-1}\cdot{}^t(\sigma(X))\cdot{}^tH=-X$ where $H$ is a $\sigma$-hermitian matrix. The coordinates can be chosen such that

$$H:=\begin{pmatrix}0 & I_r & 0\\ I_r & 0 & 0\\ 0 & 0 & H_a\end{pmatrix}.$$

where $H_a$ defines an anisotropic form on $C^{n-2r}$.

NB. In 3.2.10, we had maximal possible Witt index $r=\lfloor\frac{n+2}{2}\rfloor$ and chose the (equivalent but) different coordinatisation

$$H=\begin{pmatrix}0 & & 1\\ & 1 & \\ & \iddots & \\ 1 & & 0\end{pmatrix}.$$

Now $\mathfrak{g}$ (with our new general coordinates) consists of the matrices

$$\begin{pmatrix}A & B & G\\ D & -{}^t(\sigma(A)) & K\\ E & F & J\end{pmatrix}$$

with trace 0 and where $A,B,D\in M_r(C)$, $E,F\in M_{n-2r\times r}(C)$, $G,H\in M_{r\times n-2r}(C)$ are subject to the following relations: ${}^tB=-\sigma(B)$, ${}^tD=-\sigma(D)$ (i.e. $B$ and $D$ are $\sigma$-skew-hermitian), ${}^t\sigma(F)\cdot{}^tH_a=-G$, ${}^t\sigma(E)\cdot{}^tH_a=-K$, and ${}^tH_a^{-1}\cdot{}^t\sigma(J)\cdot{}^tH_a=-J$. If we assume, as we always can, $H_a$ to be the diagonal matrix $\mathrm{diag}(b_1,\dots,b_{n-2r})$ –

where necessarily the $b_i \in k^-$, the last relation means that the $(i,j)$-entry of $J$ is the $\sigma$-conjugate of its $(j,i)$-entry times $-b_i^{-1}b_j$; in particular, the diagonal of $J$ consists of $k$-multiples of $y$. Again one checks that a split toral subalgebra of $k$-dimension $r$ is given by

$$\mathfrak{s} := \big\{ \operatorname{diag}(a_1, \dots, a_r, -a_1, \dots, -a_r, 0, \dots, 0) : a_i \in k \big\}$$

and its centraliser $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ consists of the traceless matrices

$$\begin{pmatrix} c_1 & & & & & & \\ & \ddots & & & 0 & & \\ & & c_r & & & & \\ & & & -\sigma(c_1) & & & \\ & & & & \ddots & & \\ & 0 & & & & -\sigma(c_r) & \\ & & & & & & \boxed{J} \end{pmatrix}$$

with $c_i \in C$ and $J$ as above. No element of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \setminus \mathfrak{s}$ is ad-diagonalisable over $k$, hence $\mathfrak{s}$ is maximal split toral. One further checks that a maximal toral $\mathfrak{t}$ containing $\mathfrak{s}$ is given by those elements in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ where $J$ is diagonal (note that indeed $\dim_k(\mathfrak{t}) = n$). With an argument similar to 4.5.13, as soon as $2r \leq n-1$, the traceless matrices $J$ describe an anisotropic subalgebra which identifies with $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ modulo its centre. This is the anisotropic kernel $\mathfrak{g}_a$, [9] and the $C$-rank of $(\mathfrak{g}_a)_C$ is $n-2r$.

To "see" the swapping of the roots, one can introduce the toral subalgebra

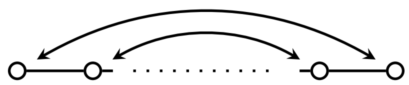
$$\mathfrak{u} := \Big\{ \operatorname{diag}(b_1 y, \dots, b_r y, b_1 y, \dots, b_r y, 0, \dots, 0) : b_i \in k, \sum_{i=1}^{r} b_i = 0 \Big\}.$$

Then $\mathfrak{s} \oplus \mathfrak{u} \oplus \mathfrak{t}_a = \mathfrak{t}$ as long as $2r \leq n$. Call $\alpha_i$ the linear form sending a general element of $\mathfrak{s}$ as above to $a_i - a_{i+1}$, and $\beta_i$ the one sending a general element of $\mathfrak{u}$ as above to $b_i - b_{i+1}$. One can check that $(\alpha_i \pm y\beta_i)_{1 \leq i \leq r}$ are part of a root basis of $R(\mathfrak{g}_C, \mathfrak{t}_C)$, and $\sigma$ switches the sign: they are indeed the white vertices in the diagram. If $2r = n+1$, so the form is quasi-split, one has to set $\beta_r = 0$, mirroring the trace 0 condition and producing the lone middle vertex $\alpha_r$. □

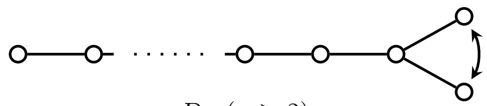
## 4.6. Quasi-split forms

Absolutely simple, non-split quasi-split Lie algebras can a priori exist in the following types:

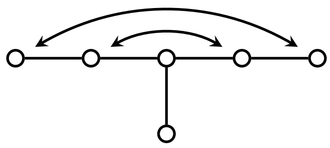
9. After minor sign changes and the trace 0 condition, this is the decisive difference to the discussion for type $D_\nu$. It comes from the commutativity of $C$, opposed to the divison algebra $D$ there having $k$ as centre.



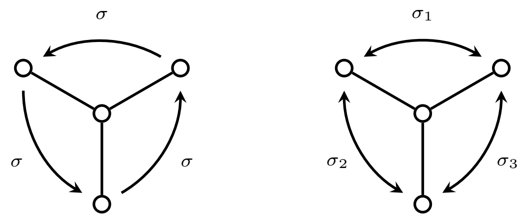
$A_n$ $(n \geq 2)$



$D_n$ $(n \geq 2)$



$E_6$



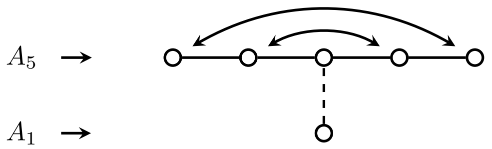

$D_4$ (trialitarian types)

As noted in Section 3.4, the isomorphism class of a semisimple Lie algebra over $k$ having such a diagram is uniquely determined by it together with the fixed field of $t$, which is a quadratic (resp. cubic or degree 6 for the trialitarian types) splitting field for the Lie algebra, in fact its unique minimal splitting field (inside a fixed $\bar{k}$).

Those of *type* $A_n$ were described in 3.2.10 and, with other coordinates, in the previous section. We saw that the isomorphism classes of absolutely simple, non-split, quasi-split $k$-Lie algebras of type $A_n$, $n \geq 2$, correspond bijectively to the quadratic extensions $K|k$ (inside our fixed $\bar{k}$).

Those of *non-trialitarian type* $D_n$ have been dealt with in the general discussion in Section 4.5.1. Their isomorphism classes are in correspondence with anisotropic quadratic forms in two variables up to scaling and equivalence, which (by $b_1X_1^2 + b_2X_2^2 \mapsto -b_1b_2$) is again the set $k^*/k^{*2}$ or that of quadratic extensions of $k$. The rational root system of such a form is easily computed to be of type $B_{n-1}$.

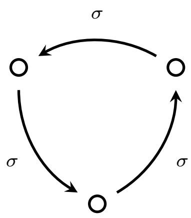
A quasi-split form of *type* $E_6$ for a given quadratic extension $K|k$ can be patched from the corresponding quasi-split form of type $A_5$ and a copy of $\mathfrak{sl}_2(k)$ with the patching Proposition 4.3.3:

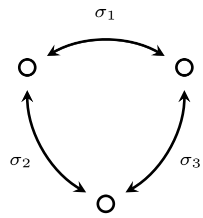



We see that again these forms are in correspondence with quadratic extensions of $k$, because those of type $A_5$ are. The rational root system of such a form is of type $F_4$ as the method of Section 3.4.1 shows.

For the *trialitarian types*, it should be clear what is meant in the diagram: $t(\sigma)$ is a generator for $\mathbb{Z}/3$, and the $t(\sigma_i)$ are the three transpositions in $S_3$, for certain elements $\sigma$ resp. $\sigma_i \in \mathrm{Gal}(\bar{k}|k)$ that restrict to the fixed field of $t$, which is a Galois cubic extension $K|k$ resp. an extension $K|k$ with Galois group isomorphic to $S_3$. The rational root system was computed to be of type $G_2$ in 3.4.1. We show that similar to the quadratic cases above, every Galois extension $K|k$ with Galois group of order 3 or isomorphic to $S_3$, respectively, can occur as fixed field of the $t$-action.

What one patches here is a copy of $\mathfrak{sl}_2(k)$ – giving the central vertex – with a suitable scalar restriction $R_{K_0|k}\,\mathfrak{sl}_2(K_0)$. Namely, $K_0 = K$ for the $\mathbb{Z}/3$ case, whereas for the $S_3$ case we take as $K_0$ any of the three (non-Galois) subextensions of $K|k$ of degree 3. By Section 4.1 the diagrams are, respectively,




and $K$ is the fixed field of the twisted Galois action.

When one knows the split forms, the classification of quasi-split forms is thus entirely equivalent to the arithmetic-algebraic problem to determine all Galois extensions of $k$ with Galois group isomorphic to $\mathbb{Z}/2, \mathbb{Z}/3$ and $S_3$, the last being in turn equivalent to that of non-Galois cubic extensions. This is also equivalent to the determination of $H^1(\mathrm{Gal}(\bar{k}|k), A(R)/W(R))$ (= the homomorphisms of $\mathrm{Gal}(\bar{k}|k)$ to $A(R)/W(R)$ up to conjugacy), which fits in the Galois cohomological approach.

# CHAPTER 5

# CLASSIFICATION OVER SPECIAL FIELDS

Again, $k$ is a field of characteristic zero, and every extension of it is understood to be in a fixed algebraic closure $\bar{k}|k$.

## 5.1. C1 fields and unramified splitting for $\mathfrak{p}$-adic fields

Lemma 5.1.1 (Springer, [**Spr1**]). — *If $k$ is C1, every non-zero semisimple Lie algebra contains a non-zero nilpotent element.*

*Proof.* — Let $\mathfrak{g}$ be a semisimple Lie algebra of dimension $n \geq 1$. For $x \in \mathfrak{g}$, the characteristic polynomial of $\mathrm{ad}_{\mathfrak{g}}(x) \in \mathrm{End}_k(\mathfrak{g})$ with variable $T$ is of the form

$$\sum_{i=r}^{n} p_i(x) T^i$$

where $r = \mathrm{rk}(\mathfrak{g})$ and for $i < n$, $p_i$ is a homogeneous polynomial map from $\mathfrak{g}$ to $k$ of degree $n-i$, meaning that after a choice of basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, there are homogenous polynomials $P_i \in k[X_1, \ldots, X_n]$ of degree $n-i$ such that $p_i(\sum_{i=1}^{n} c_i x_i) = P_i(c_1, \ldots, c_n)$. By the C1 property, in particular there is a non-zero $y \in \mathfrak{g}$ with $p_r(y) = 0$ or in other words, $y$ is non-regular.

We prove the assertion by induction over $n = \dim_k \mathfrak{g}$. For $n = 0 \Leftrightarrow \mathfrak{g} = 0$, there is nothing to show. There are no semisimple Lie algebras of dimension 1 or 2, and if $n = 3$, then $\mathfrak{g}_{\bar{k}} \simeq \mathfrak{sl}_2(\bar{k})$. We identify $\mathfrak{g}$ with a subalgebra of $R_{\bar{k}|k}\mathfrak{g}_{\bar{k}}$ and use that the characteristic polynomial "remains the same" (cf. [**Bo2**, VII.2.2, Rem. 2]) so that by *ibid.*, example 2, $p_2 \equiv 0$ and hence the $y$ constructed above is nilpotent.

Now let $n > 3$ and assume the assertion is true for all Lie algebras of dimension $< n$. Pick a non-regular $y \neq 0$ as above; by Jordan decomposition, w.l.o.g. $y$ is semisimple. Consider $\mathfrak{g}_1 := \mathfrak{z}_{\mathfrak{g}}(y) = \ker(\mathrm{ad}_g(y)) = \mathfrak{g}^0(y)$. By [**Bo2**, VII.1, Prop. 11], $\mathfrak{g}_1$ is reductive, by [**Bo2**, VII.3.3, Cor.] it contains an element $z$ which is regular in $\mathfrak{g}$. If we had $[y, z] = 0$, then by the Jacobi identity $\ker(\mathrm{ad}_{\mathfrak{g}}(y)) = \ker(\mathrm{ad}_{\mathfrak{g}}(z))$, but this is a Cartan subalgebra whose dimension is $\mathrm{rk}(\mathfrak{g}) < \dim_k(\mathfrak{g}_1)$. Thus the derived algebra $\mathcal{D}\mathfrak{g}_1$ is non-zero, semisimple, and its dimension is $< n$. By the induction hypothesis it contains an $x \neq 0$ which is nilpotent in $\mathcal{D}\mathfrak{g}_1$ and hence (2.1.8) in $\mathfrak{g}$. □

This has a strong consequence:

PROPOSITION 5.1.2. — *If k is C1, every semisimple Lie algebra is quasi-split.*

Namely, this follows from Springer's lemma and the following fact.

PROPOSITION 5.1.3. — *The following are equivalent (for the field k):*

i. *Every non-zero semisimple Lie algebra contains a non-zero nilpotent element.*
ii. *Every semisimple Lie algebra contains a Borel subalgebra (i.e. is quasi-split).*
iii. *The only anisotropic semisimple Lie algebra is* $\{0\}$.

*Proof.* — This can be shown directly (cf. [**Bo2**, VIII.10, Exercise 7.a]) but follows easily from our previous results. Indeed, i $\Leftrightarrow$ iii is immediate from 3.2.1.viii. By part ix there, an anisotropic Lie algebra with a Borel subalgebra must be equal to it, hence solvable, hence zero, so ii $\Rightarrow$ iii. Finally, if iii is valid, then in particular every semisimple $\mathfrak{g}$ has trivial anisotropic kernel, so is quasi-split by 3.2.6.iv. □

REMARK 5.1.4. — Corresponding statements for algebraic groups are largely due to Serre [**Ser1**], Springer, and Steinberg [**Ste**]. Indeed the following are equivalent for our field $k$ (remember $\mathrm{char}(k) = 0$ and every extension is understood to be an intermediate extension of $\bar{k}|k$):

i. The cohomological dimension of $\mathrm{Gal}(\bar{k}|k)$ is 1.
ii. For every extension $K|k$, the Brauer group $\mathrm{Br}(K)$ is trivial.
iii. For every extension $K|k$ and every finite Galois extension $L|K$, the $\mathrm{Gal}(L|K)$-module $L^*$ is cohomologically trivial.
iv. In the situation of iii, the norm $N_{L|K} : L^* \to K^*$ is surjective.
v. $H^1(k, G)$ is trivial for every semisimple (or connected linear) algebraic group $G$.
vi. Every linear algebraic group contains a Borel subgroup defined over $k$.
vii. Every non-trivial semisimple algebraic group contains a unipotent element $\neq 1$.
viii. $k$ has the properties of 5.1.3.

This is the property $\dim(k) \leq 1$ in [**Ser3**], and the above equivalences are Proposition 5 in II, 3.1 and Theorems 1, $1'$, 2 in III, 2 there. It is shown in *ibid.* II, 3.2 that the property C1 implies iv and thus all of the above. Springer's lemma above showed that C1 implies viii, and "viii $\Rightarrow$ ii" was 4.4.4.

Proposition 5.1.2 combined with Lang's theorem 2.2.2 implies:

COROLLARY 5.1.5 (Weisfeiler's Theorem 1). — *For a semisimple Lie algebra $\mathfrak{g}$ over a $\mathfrak{p}$-adic field $k$, there exists a finite unramified extension $\mathfrak{K}|k$ such that $\mathfrak{g}_{\mathfrak{K}}$ is quasi-split.*

*Proof*. — Since the maximal unramified extension $k^{nr}$ is C1, 5.1.2 says that $\mathfrak{g}_{k^{nr}}$ contains a Borel subalgebra $\mathfrak{B}$. Let $b_1, \ldots, b_n$ be a $k^{nr}$-basis of $\mathfrak{B}$. Since we can write

$$\mathfrak{g}_{k^{nr}} = \bigcup_{\substack{k \subseteq \mathfrak{K} \subseteq k^{nr} \\ \mathfrak{K}|k \text{ finite}}} \mathfrak{g}_{\mathfrak{K}}$$

there is a finite unramified extension $\mathfrak{K}|k$ such that $\mathfrak{g}_{\mathfrak{K}}$ contains all $b_i$. The vector space $\mathfrak{b}$ spanned by them in $\mathfrak{g}_{\mathfrak{K}}$ satisfies $\mathfrak{b} = \mathfrak{B} \cap \mathfrak{g}_{\mathfrak{K}}$ and hence is a subalgebra, and by definition a Borel subalgebra, of $\mathfrak{g}_{\mathfrak{K}}$. □

## 5.2. Type $AI$ and Kneser's theorem

We had seen in the general discussion how the inner forms of type $A$ are completely described by the central division algebras over $k$. Now for a $\mathfrak{p}$-adic field $k$, one has the well-known (cf. [**Rei**, Sections 14 and 31])

Theorem 5.2.1 (cf. Hasse [**Has**]). — $\mathrm{Br}(k) \simeq \mathbb{Q}/\mathbb{Z}$. *For $d \geq 1$, there are exactly $\phi(d)$ isomorphism classes of $k$-central division algebras of degree $d$. Each of them contains a copy of, and is split by, any field extension $K|k$ of degree $d$, and there are $\lfloor \frac{\phi(d)+1}{2} \rfloor$ classes up to isomorphism and anti-isomorphism.*

Combined with 4.5.21 this gives a complete classification of inner forms of type $A$. In particular:

Proposition 5.2.2. — *For $d \geq 2$, there are $\lfloor \frac{\phi(d)+1}{2} \rfloor$ anisotropic inner forms of type $A_{d-1}$ over $k$ up to isomorphism. Each of them is split by any extension of degree $d$.* (1)

So for $d \in \{2, 3, 4, 6\}$ we can and will speak of "the" anisotropic (inner) form of $\mathfrak{sl}_d$. We will see that $\mathfrak{sl}_d$ has no outer anisotropic forms. In fact the following theorem, due independently to Weisfeiler and Kneser, states that much more is true.

Theorem 5.2.3 (Weisfeiler's Theorem 2). — *Over a $\mathfrak{p}$-adic field $k$, all absolutely simple anisotropic Lie algebras are inner forms of type $A$.*

These are the ones from above, classified by the $k$-central division algebras. The remarkable assertion is that *no other anisotropic forms exist.*

Kneser actually noted the corresponding statement for algebraic groups: see [**Kne**, Satz 3], (2) where it follows from his main

Theorem 5.2.4 (Kneser). — *For a $\mathfrak{p}$-adic field $k$ and a connected, simply connected semisimple algebraic group $G$ defined over $k$, we have $H^1(\mathrm{Gal}(\bar{k}|k), G) = 1$.*

1. This means that for any such extension $K|k$, there is a maximal toral subalgebra of $\mathfrak{g}$ which is split by $K$. Indeed, the maximal abelian subalgebras of $\mathfrak{D}$ "are" the maximal (= $d$-dimensional) subfields of $D$, and their intersections with $\mathcal{D}\mathfrak{D}$ are the maximal toral subalgebras of this.

2. Bruhat and Tits complemented their opus magnum with a completely different and more general proof in [**BT2**, Section 4].

In fact, *the whole classification of semisimple algebraic groups, or Lie algebras, over a* $\mathfrak{p}$*-adic field* $k$ *can be derived from Kneser's theorem.* This is sketched in [**Kne**, II.4–5] and also done in [**Sat3**, II.3.3]. The idea is that the vanishing of Kneser's cohomology group gives upper bounds on the size of $H^1(\mathrm{Gal}(\bar{k}|k), \mathrm{Aut}(X_{\bar{k}}))$, where $X$ is the object of interest. We sketch the procedure, for an absolutely simple Lie algebra $\mathfrak{g}$ of type $R$. There are algebraic groups $G$ and $G_0$ defined over $k$ such that $G(K) = \mathrm{Aut}(\mathfrak{g}_K)$ and $G_0(K) = \mathrm{Aut}_0(\mathfrak{g}_K)$ for every subextension $K|k$ of $\bar{k}$. The group $G_0$ is connected and of adjoint type (cf. [**Bo2**, VIII.5.4]), and there is an exact sequence

$$1 \to Z \longrightarrow \widetilde{G}_0 \longrightarrow G_0 \to 1$$

where $\widetilde{G}_0$ satisfies the conditions for Kneser's theorem and (the dual of) its centre $Z$ can be identified with $P(R)/Q(R)$, which is listed for all types in the plates of [**Bo2**, VI]. With [**Ser3**, I.5.7] we get an exact sequence

$$1 \to H^1(k, G_0) \to H^2(k, Z). \tag{19}$$

We first look at the *case* $G = G_0$ where we only have *inner forms*: e.g. for the types $E_8$, $F_4$ and $G_2$, $Z$ and hence $H^2(k, Z)$ are trivial, so there is only the split form. For type $E_7$, again $G_0 = G$, but now $Z$ has two elements and so has $H^2(k, Z)$ (as can be seen via Tate duality). We will indeed show the existence of one non-split form of type $E_7$ over $k$ in 5.5.5. Hence, this and the split one are all of them. In the *general case* we have $G \simeq G_0 \rtimes S$ (semidirect product) where $S \simeq A(R)/W(R)$ is the automorphism group of the Dynkin diagram (cf. [**Bo2**, VIII.5.3, Cor. 1]). This gives a surjective $\phi$ in the exact sequence

$$H^1(k, G_0) \longrightarrow H^1(k, G) \xrightarrow{\ \phi\ } H^1(k, S)$$

of [**Ser3**, I.5.5]. $H^1(k, S)$ classifies exactly the quasi-split forms, which is the Galois cohomological interpretation of our results in 4.6. Indeed this sequence can be translated to the statement that every form can be viewed as an "inner form of a uniquely determined quasi-split form". (The "inner forms" in our terminology are then the inner forms of the split form, i.e. the image of the left = kernel of the right map in the sequence). With [**Ser3**, I.5.5, Cor. 2] for an $\alpha \in H^1(k, S)$ (corresponding to one quasi-split form), the set $\phi^{-1}(\alpha)$ (corresponding to the "forms of this form") can be identified with the quotient of $H^1(k, {}_aG_0)$ under a certain action of $H^0(k, {}_aS)$, where $a$ is an arbitrary cocycle representing an arbitrary $\bar{a}$ with $\phi(\bar{a}) = \alpha$, and the subscripts denote twisting with this cocycle. In particular, the inner forms (in our terminology) are given by orbits of $S$ in $H^1(k, G_0)$. With (19) one can transport this to the orbits of an $S$-action on $H^2(k, Z)$, which further identifies with (the Galois fixed set of the dual of) $Z$ by Tate duality. In type $E_6$, $Z$ is a group of three and $S$ a group of two elements, the $S$-action permutes the two non-trivial elements of $Z$, so we expect (at most) one non-split inner form and will indeed see it in 5.5.4. For the outer forms of type $E_6$, and also e.g. for the trialitarian types $D_4$, it turns out that already the twisted $H^2(k, {}_aZ)$ is trivial, hence in these cases there exist *no outer forms except for the quasi-split ones* we have described in 4.6. Tables which summarise the results for all types are found in [**Kne**, p. 254] and [**Sat3**, p. 121].

We believe, however, that this approach is not the only possible way. First, using results about (skew-)hermitian forms over $\mathfrak{p}$-adic fields, we can completely classify the classical types with our earlier results, and theorem 5.2.3 for these types follows from this. This will be done in Section 5.4. (In [**Kne**, I], Kneser proves his theorem for the classical types using the same results about hermitian forms.) In the further course of the classification in Section 5.5, we will see that each possible diagram can either be excluded by our previous results or shown to be admissible – *except for the anisotropic forms*. Thus the validity of Weisfeiler's table is reduced, without using Kneser's theorem, to the seemingly simpler statement:

Reduced Theorem. — *There are no anisotropic Lie algebras of exceptional type over* $\mathfrak{p}$*-adic fields.*

To prove this, we need some more machinery which we will set up in the next Section 5.3. We will then combine that with our previous results and Weisfeiler's first theorem (5.1.5) to prove the Reduced Theorem in every case except for one form in type $E_6$ and one in type $D_4$.

Kneser himself suggests equivalence of his and Weisfeiler's results in [**Kne**, II, footnote 4]. The paper [**Tit4**] deals with anisotropic forms, but for $\mathfrak{p}$-adic fields it also refers to Kneser's theorem.

Remark 5.2.5. — For $k = \mathbb{R}$ the situation is very different. Here it is a classical result due to E. Cartan that *for every type, there exists a unique anisotropic form.* But only one of them is an inner form of type $A$, in fact $A_1$, the derived Lie algebra of Hamilton's quaternions; whereas e.g. the anisotropic forms of type $A_n$, $n \geq 2$, are outer forms.

## 5.3. Tools for ruling out anisotropic forms: The prime degree case

Recall Weisfeiler's first theorem 5.1.5 which said that we have *unramified*, in particular *cyclic* splitting extensions, at least for the *inner forms* of split Lie algebras (cf. definition 4.2.4). A cyclic extension $K|k$ can further be "filtered" to a tower of fields $k \subset K_0 \subset K_1 \subset \cdots \subset K_d = K$ such that every step $K_r|K_{r-1}$ is of prime degree. These are the ones we are going to attack in this section. Already basic arithmetic constraints rule out most prime degree extensions as splitting fields, as we will see, not only in the $\mathfrak{p}$-adic case. The main goal of this section then is a big step towards the Reduced Theorem in the preceding section: we show that over $\mathfrak{p}$-adic fields, there are *no anisotropic Lie algebras of exceptional type which are split by an unramified extension of prime degree.*

We start with a technical lemma. Let $R$ be a root system and $V$ be the $\mathbb{R}$-vector space spanned by it.

LEMMA 5.3.1. — *Let $\ell$ be a prime, and $\sigma \in A(R)$ such that $\sigma \in \mathrm{GL}(V)$ has minimal polynomial $T^{\ell-1} + T^{\ell-2} + \cdots + T + 1$ (in particular $\sigma^\ell = \mathrm{id}$). Then for every root $\alpha \in R$, after possibly replacing $\sigma$ by some $\sigma^j$ (with $1 \leq j \leq \ell - 1$), the roots*

$$\alpha_1 := \alpha, \alpha_2 := \sigma(\alpha), \ldots, \alpha_{\ell-1} := \sigma^{\ell-2}(\alpha)$$

*satisfy the relations of a basis of root system of type $A_{\ell-1}$ (i.e. correspond to the nodes in the associated Dynkin diagram).*

*Proof.* — The result is trivially true for $\ell = 2$, so let $\ell \geq 3$. We choose an $A(R)$-invariant scalar product $\langle .\, , . \rangle$ on $V$. All $\sigma^j(\alpha)$ have the same length, without loss of generality let this length be 1. $\alpha, \sigma(\alpha), \ldots, \sigma^{\ell-2}(\alpha)$ are linearly independent, and we have

$$\sigma^{\ell-1}(\alpha) = -\sum_{i=0}^{\ell-2} \sigma^i(\alpha)$$

Multiplying out $\langle \sigma^{\ell-1}(\alpha), \sigma^{\ell-1}(\alpha) \rangle = 1$, using the $\sigma$-invariance and $\langle \alpha, \alpha \rangle = 1$, gives

$$(\ell - 1) + 2\sum_{i=1}^{\ell-2} (\ell - 1 - i)\langle \alpha, \sigma^i(\alpha) \rangle = 1 \tag{20}$$

Since $\ell \geq 3$, this means that for some $1 \leq j \leq \ell - 2$, $\langle \alpha, \sigma^j(\alpha) \rangle$ must be negative. Replace $\sigma$ by this $\sigma^j$, so we have $\langle \alpha, \sigma(\alpha) \rangle < 0$ and indeed by [**Bo2**, VI.1.3], $\langle \alpha, \sigma(\alpha) \rangle = -\frac{1}{2}$ because the roots have the same length 1. But then by $\sigma$-invariance, $\langle \sigma^{\ell-1}(\alpha), \alpha \rangle = -\frac{1}{2}$ too and thus

$$\sigma^{\ell-1}(\alpha) + \alpha = -\sum_{i=1}^{\ell-2} \sigma^i(\alpha) \tag{21}$$

is a root which further is of length 1 again ([loc. cit.]). We are finished here if $\ell = 3$. To proceed in the case $\ell > 3$, note first that inserting $\langle \alpha, \sigma(\alpha) \rangle = -\frac{1}{2}$ into (20) gives

$$2\sum_{i=2}^{\ell-2} (\ell - 1 - i)\big\langle \alpha, \sigma^i(\alpha) \big\rangle = 0.$$

Further note that

$$\big\langle \alpha, \sigma^i(\alpha) \big\rangle = \big\langle \alpha, \sigma^{\ell-i}(\alpha) \big\rangle \tag{22}$$

for all $0 \leq i \leq \ell$ by $\sigma$-invariance and $\sigma^\ell = \mathrm{id}$. Coupling these pairs reduces our equation further to

$$(\ell - 2) \cdot \sum_{i=2}^{\ell-2} \big\langle \alpha, \sigma^i(\alpha) \big\rangle = 0 \tag{23}$$

where we can cancel the constant too. Applying $\sigma^{-1}$ to (21), we have the root

$$\sum_{i=0}^{\ell-3} \sigma^i(\alpha) \tag{24}$$

of length 1, which multiplied out gives

$$(\ell-2) + 2\sum_{i=1}^{\ell-3}(\ell-2-i)\big\langle \alpha, \sigma^i(\alpha)\big\rangle = 1$$

or, using $\langle \alpha, \sigma(\alpha)\rangle = -\frac{1}{2}$ and coupling the pairs (22) again,

$$(\ell-3)\cdot \sum_{i=2}^{\ell-3}\langle \alpha, \sigma^i(\alpha)\rangle = 0$$

where again cancelling the constant (we assume $\ell > 3$ now) and comparing with (23) gives:

$$\big\langle \alpha, \sigma^{\ell-2}(\alpha)\big\rangle = \big\langle \alpha, \sigma^2(\alpha)\big\rangle = 0. \tag{25}$$

From this one infers that

$$\sigma^{\ell-1}(\alpha) + \alpha + \sigma(\alpha) = -\sum_{i=2}^{\ell-2}\sigma^i(\alpha) \tag{26}$$

is again a root of length 1. This process can be repeated, showing iteratively that

$$\langle \alpha, \sigma^i(\alpha)\rangle = 0$$

for $i \in \{2, \ldots, \ell-2\}$. This proves the claim. □

Actually, we do not even need $\ell$ to be a prime – we note the following generalisation for use in later sections:

Corollary 5.3.2. — *Let $n \in \mathbb{N}$ and $\sigma \in A(R)$ such that $\sigma \in \mathrm{GL}(V)$ has minimal polynomial $T^{n-1} + T^{n-2} + \cdots + T + 1$ (in particular $\sigma^n = \mathrm{id}$). Then for every root $\alpha \in R$ such that the $\sigma$-conjugates of $\alpha$ span an $(n-1)$-dimensional subspace of $V$, after possibly replacing $\sigma$ by some $\sigma^j$ (with $gcd(j,n) = 1$), the roots*

$$\alpha_1 := \alpha, \alpha_2 := \sigma(\alpha), \ldots, \alpha_{n-1} := \sigma^{n-2}(\alpha)$$

*satisfy the relations of a basis of root system of type $A_{n-1}$ (i.e. correspond to the nodes in the associated Dynkin diagram).*

*Proof.* — By induction on the number of divisors of $n$, the lemma being the start of the induction. Then for general $n$, one imitates the proof until the place after equation (20) where we get

$$\langle \alpha, \sigma^j(\alpha)\rangle < 0$$

for some $1 \leq j \leq n-2$. If $\gcd(j,n)$ were not 1, we could w.l.o.g. assume that $j$ is a divisor of $n$ and use the induction hypothesis on $\sigma^j$ to see that the sets

$$\begin{aligned}&\{\alpha, \sigma^j(\alpha), \sigma^{2j}(\alpha), \dots, \sigma^{(d-1)j}(\alpha)\},\\ &\{\sigma(\alpha), \sigma^{j+1}(\alpha), \sigma^{2j+1}(\alpha), \dots, \sigma^{(d-1)j+1}(\alpha)\},\\ &\ \vdots\\ &\{\sigma^{j-1}(\alpha), \sigma^{2j-1}(\alpha), \sigma^{3j-1}(\alpha), \dots, \sigma^{(n-1)}(\alpha)\}\end{aligned}$$

(where $d := n/j$) would each span root systems of type $A_{d-1}$; and there are $j$ of them. But that would mean the span of all $\sigma$-conjugates of $\alpha$ would only have dimension $j \cdot (d-1) = n - j$, contradiction.

So $j$ does not divide $n$, and the rest of the proof is exactly the same as in the lemma. □

Remark 5.3.3. — i. This is consistent with table 3 in [**Car**]. Also, a posteriori, $\sigma$ is a Coxeter element of the sub-root-system of type $A_{n-1}$.

ii. Note that in the situation of the lemma, $\sigma$ splits $V$ into $\sigma$-invariant subspaces each of dimension $\ell - 1$. In particular, $\mathrm{rk}(R)$ has to be a multiple of $\ell - 1$.

iii. Note however that although the $\sigma$-invariant subspace spanned by the $\sigma$-conjugates of any root $\alpha$ now necessarily contains a sub-root system of type $A_{\ell-1}$, the intersection of $R$ with that subspace might be a larger root system. An example is a root system of type $G_2$ (cf. [**Bo2**, Plate X], or this work's dedication) with $\sigma$ being the rotation by 120°.

The lemma puts severe restrictions on anisotropic forms of exceptional types with splitting fields of prime degree:

Proposition 5.3.4. — *Let $k$ be a field with* $\mathrm{char}(k) = 0$*, and let $\mathfrak{g}$ be an absolutely simple, anisotropic Lie algebra of exceptional type over $k$, with a* cyclic *splitting extension $K|k$ of* prime *degree $\ell$. Let $(X, R, \Delta, \Delta_0 = \Delta, t)$ be the index of $\mathfrak{g}$ with respect to $\mathfrak{s} = \{0\}$ and a maximal toral $\mathfrak{t}$ which is split by the extension $K$. Let $\sigma$ be a generator of $\Gamma = \mathrm{Gal}(K|k)$ and resume notations like $\sigma_A, X_0, V$, etc. from Chapter 3. Then:*

- ▷ *If $\ell = 2$, $\sigma_A = -\mathrm{id}$.*
- ▷ *If $\ell = 3$, $R$ cannot be of type $E_7$.*
- ▷ *If $\ell = 5$, $R$ must be of type $E_8$.*
- ▷ *The case $\ell > 5$ cannot occur.*

*Proof.* — Since $X = X_0 = \{\chi \in X : \sum_{i=0}^{\ell-1} {\sigma_A}^i(\chi) = 0\}$, the element $\sigma_A \in A(R)$ has no eigenvector of eigenvalue 1 in $V = \mathbb{R}X$ (this is called an "elliptic element"), and it is of order $\ell$. This already implies the assertion for $\ell = 2$, and more generally, the minimal polynomial of $\sigma_A \in \mathrm{GL}(V)$ is the cyclotomic polynomial $T^{\ell-1} + T^{\ell-2} + \cdots + T + 1$, so we are in the situation of the preceding lemma and remark. For odd $\ell$ and $R$ not of type $D_4$, we also see that $\sigma_A \in W(R)$ because $|A(R)/W(R)| \leq 2$; for $R$ of type $D_4$, the same holds true for $\ell > 3$. Then $D_4$, $F_4$ and $G_2$ are excluded for $\ell > 3$ because their Weyl groups have orders $2^6 \cdot 3$, $2^7 \cdot 3^2$ and $2^2 \cdot 3$. $E_6$ is excluded for $\ell > 5$ since its Weyl group is of order $2^7 \cdot 3^4 \cdot 5$.

Finally, part ii of the remark implies $\mathrm{rk}(R)/(\ell - 1) \in \mathbb{Z}$, which excludes type $E_6$ for $\ell = 5$ and type $E_7$ for $\ell \geq 3$. □

Now we attack these remaining cases over $\mathfrak{p}$-adic fields, starting with $\ell = 2$. The following has an analogue in [**We3**], especially §§8 and 12, but we were not able to verify all statements there. We take a different approach which however uses the same key fact about the norms of the unramified quadratic extension.

Proposition 5.3.5 (Case $\ell = 2$). — *Let $\mathfrak{g}$ be a semisimple Lie algebra over a $\mathfrak{p}$-adic field $k$ which is split by the unramified* quadratic *extension $K|k$. If the root system of $\mathfrak{g}_K$ contains roots $\gamma, \delta$ such that $\gamma + \delta$ is a root, then $\mathfrak{g}$ cannot be anisotropic.*

*In particular, if $\mathfrak{g}$ is absolutely simple and anisotropic, it must be of type $A_1$, i.e. the anisotropic form of $\mathfrak{sl}_2$.*

*Proof.* — Let $\sigma$ be the non-trivial element of $\mathrm{Gal}(K|k)$. Assume $\mathfrak{g}$ is anisotropic. In our general terminology of choices, necessarily $\mathfrak{s} = \{0\}$. Choose a maximal toral $\mathfrak{t}$ defined over $k$ and split over $K$.

We resume notation from Section 3.3.1 but now fix a *Chevalley basis* consisting of $H_\alpha$ and bases $e_\alpha$ of the root spaces, which means that the structure constants $N_{\alpha,\beta}$ defined in (14) are in $\mathbb{Z} \setminus \{0\}$ (such a basis exists by [**Che**], cf. [**Bo2**, VIII.4.4]). We have $N_{\alpha,\beta} = -N_{\beta,\alpha}$ for all roots $\alpha, \beta$ as well as $H_{-\alpha} = -H_\alpha$. Also, we have the crucial relation

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} \tag{27}$$

cf. [**Che**, p. 23]. (Our sign conventions follow [**Che**] instead of Bourbaki, that is, we have $[e_\alpha, e_{-\alpha}] = H_\alpha$. Remember that we chose our basis of $\mathfrak{sl}_2$ accordingly in 2.1.6.)

Now the root lattice satisfies $X = X_0$ and therefore $\sigma_A = -\mathrm{id}$ on $X$; in particular, $\sigma_A(\alpha) = -\alpha$ for all roots $\alpha$. We get elements $\xi_\alpha \in K^*$ defined by

$$\sigma(e_\alpha) = \xi_\alpha \, e_{-\alpha}$$

Because $\sigma(H_\alpha) = [\sigma(e_\alpha), \sigma(e_{-\alpha})] = \xi_\alpha \xi_{-\alpha} H_{\sigma_A(\alpha)}$ and ${}^\sigma\alpha(\sigma(H_\alpha)) = \sigma(\alpha(H_\alpha)) = 2$, we have $\sigma(H_\alpha) = H_{\sigma_A(\alpha)} = H_{-\alpha} = -H_\alpha$ and $\xi_{-\alpha} = \xi_\alpha^{-1}$ for all $\alpha \in R$. We also have $\sigma(\xi_\alpha) \cdot \xi_{-\alpha} = 1$ as in (12), hence $\xi_\alpha \in k^*$, and for each root $\alpha$, the copy of $\mathfrak{sl}_2(K)$

$$\begin{aligned}\mathfrak{G}(\mathfrak{t}, \alpha) &:= KH_\alpha + (\mathfrak{g}_K)_\alpha + (\mathfrak{g}_K)_{-\alpha}\\ &= KH_\alpha + Ke_\alpha + Ke_{-\alpha}\end{aligned}$$

is $\sigma$-stable, hence descends to $k$ where it becomes $\mathfrak{g}(\mathfrak{t},\alpha) := \mathfrak{G}(\mathfrak{t},\alpha)^{\sigma=\mathrm{id}}$ and necessarily is the anisotropic form of $\mathfrak{sl}_2$. Picking $y \in K^*$ with $\sigma(y) = -y$, it has the $k$-basis

$$yH_\alpha, \quad e_\alpha + \xi_\alpha e_{-\alpha}, \quad y(e_\alpha - \xi_\alpha e_{-\alpha})$$

and visibly is isomorphic to the form described in example 3.2.5, the $\xi_\alpha$ here identifying with the "$b$" there (with our sign conventions; with Bourbaki's conventions instead, it would be $-b$). We had seen in that example that this $\mathfrak{sl}_2$-form being anisotropic is equivalent to $\xi_\alpha \notin N_{K|k}(K)$ which, $K|k$ being unramified, means that

$$\text{for all roots } \alpha, \text{ we have:} \quad \xi_\alpha \equiv \pi_k \mod \mathcal{O}_k^* \tag{28}$$

In other words, *for all roots $\alpha$, the constant $\xi_\alpha$ must be of odd valuation in $k$.* But if there are roots $\gamma,\delta$ whose sum is a root, calculating $\sigma([e_\gamma, e_\delta]) = \sigma(N_{\gamma,\delta}\, e_{\gamma+\delta})$ in two different ways gives

$$N_{-\gamma,-\delta} \cdot \xi_\gamma \xi_\delta = N_{\gamma,\delta} \cdot \xi_{\gamma+\delta}$$

and thus by (27),

$$\xi_{\gamma+\delta} = -\xi_\gamma \xi_\delta \tag{29}$$

which contradicts (28). □

Remark 5.3.6. — Our carefulness about signs in the above proof seems unnecessary (as $-1$ is of valuation 0 anyway), but it becomes important when one wants to see why the analogous statement in the case $k = \mathbb{R}$ is *not* true, and indeed anisotropic forms exist in *every* type, all split by the quadratic $\mathbb{C}|\mathbb{R}$. There, in our sign convention, we would have $\xi_\alpha \equiv -1 \bmod \mathbb{R}^{*2}$ for all roots $\alpha$, and (29) would be perfectly OK; in Bourbaki's sign convention, we would instead have all $\xi_\alpha \in \mathbb{R}^{*2}$ (that is, $> 0$), but in that convention, we have $N_{-\alpha,-\beta} = N_{\alpha,\beta}$ instead of (27) and thus $\xi_{\gamma+\delta} = \xi_\gamma\xi_\delta$ instead of (29), again without contradiction.

Note that the condition in Proposition 5.3.5 – that $R$ contains two roots whose sum is a root of the same length – can be reformulated as $R$ containing a sub-root-system of type $A_2$. We need a bit more than that that to derive a contradiction in the next case, $\ell = 3$.

Proposition 5.3.7 (Case $\ell = 3$). — *Let $\mathfrak{g}$ be a semisimple Lie algebra over a $\mathfrak{p}$-adic field $k$ which is split by the unramified* cubic *extension $K|k$. If the root system of $\mathfrak{g}_K$ contains a sub-root-system of type $A_3$, then $\mathfrak{g}$ cannot be anisotropic.*

*In particular, if $\mathfrak{g}$ is absolutely simple of exceptional type and anisotropic, it would need to be of type $G_2$ (which however will be excluded in the next proposition).*

*Proof.* — Let $\sigma$ be a generator of $\mathrm{Gal}(K|k)$, and again fix $\mathfrak{t}$ and a Chevalley basis of $e_\alpha$'s and $H_\alpha$'s as before. Since $\sigma_A$ is elliptic of order 3, we now have for each root $\alpha$ that: $\alpha$ and $\sigma(\alpha)$ are a basis of the sub-root system consisting of $\{\pm\alpha, \pm\sigma(\alpha), \mp\sigma^2(\alpha) = \pm(\alpha+\sigma(\alpha)\}$ which is of type $A_2$, and the corresponding Chevalley basis vectors $e_?$

generate a subalgebra of $\mathfrak{g}_K$ isomorphic to $\mathfrak{sl}_3(K)$. Again define $\xi_\alpha$ by $\sigma(e_\alpha) = \xi_\alpha e_{\sigma(\alpha)}$ for all $\alpha \in R$. Since the Galois-invariants of each of those subalgebras are subalgebras of the anisotropic $\mathfrak{g}$, they must each be the anisotropic $k$-form of $\mathfrak{sl}_3$. As an analogue to equation (28) in the quadratic case, one can show [(3)] that this implies:

$$\text{For all roots } \alpha, \quad \xi_\alpha^2 \cdot \xi_{\sigma(\alpha)} \equiv \pi_k \text{ or } \pi_k^2 \bmod \mathcal{O}_k^*$$

or in other words, the valuation $v_k(\xi_\alpha^2 \cdot \xi_{\sigma(\alpha)}) \in \mathbb{Z}$ has to be non-trivial modulo 3 for every root $\alpha \in R$. Let us abbreviate the invariant $r(\alpha) := \xi_\alpha^2 \cdot \xi_{\sigma(\alpha)}$. Note that replacing $\sigma$ by $\sigma^2$ would switch for each $\alpha$ whether $r(\alpha)$ is congruent to $\pi$ or $\pi^2$.

On the other hand, using that all occurring Chevalley structure invariants $N_{?,?}$ are $\pm 1$ because all the roots have the same length, one computes

$$r(\alpha + \beta) = \pm r(\alpha) r(\beta)$$

for all roots $\alpha, \beta$ of the same length such that $\alpha + \beta$ is a root of the same length, which means

$$v_k\big(r(\alpha+\beta)\big) = v_k\big(r(\alpha)\big) + v_k\big(r(\beta)\big).$$

Now let $\alpha_1, \alpha_2, \alpha_3$ be an ordered basis of the sub-root-system of type $A_3$ existing by hypothesis. This means that all consecutive sums ($\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_3$) are roots, all of the same length as any $\alpha_i$. Abbreviate $v(1) := v_k(r(\alpha_1))$, $v(2) := v_k(r(\alpha_2))$ and $v(3) := v_k(r(\alpha_3))$. Now an application of Dirichlet's box principle leads to a contradiction: Not all of the integers

$$\begin{aligned} & v(1), v(2), v(3) \\ v_k(r(\alpha_1+\alpha_2)) &= v(1) + v(2), \\ v_k(r(\alpha_2+\alpha_3)) &= v(2) + v(3) \quad \text{and} \\ v_k(r(\alpha_1+\alpha_2+\alpha_3)) &= v(1) + v(2) + v(3) \end{aligned}$$

can be $\not\equiv 0 \bmod 3$ at the same time.

For the final assertion, it is readily checked that all exceptional root systems except those of type $G_2$ contain subsystems of type $A_3$ (except for type $F_4$, their bases even contain bases of such subsystem; in $F_4$ on the other hand, the long and short roots are systems of type $D_4$ which in turn contain various copies of $A_3$). $\square$

3. Computations with matrices as in 3.2.5, from which we derived (28), become a bit awkward now. But one can use our knowledge about the Lie algebra of type $A_2$ in question being the derived algebra of a cyclic CSA and invoke the theory of [**Rei**, 30.4 and 31]. To this end, note that the decisive generating element in the enveloping associative algebra, written as a matrix, is

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \xi_\alpha \\ \xi_\alpha \xi_{\sigma(\alpha)} & 0 & 0 \end{pmatrix};$$

and whether the diagonal matrix $x^3 = \xi_\alpha^2 \cdot \xi_{\sigma(\alpha)} \cdot I$, or rather its diagonal entry, is a norm of $K|k$ or not, decides whether that CSA contains nilpotent elements or not, i.e. whether our Lie algebra is not anisotropic or is anisotropic. Compare also [*loc. cit.*, 14.6].

We can amend the case $G_2$ "by hand" – in fact, we have the very general

Proposition 5.3.8. — *There is no anisotropic Lie algebra over any* (*characteristic* 0) *field* $k$ *which is of type* $G_2$ *and split by a cyclic extension of odd degree.*

*Proof.* — (Cf. [**We2**, § 2].) Assume there is one and fix a maximal toral $\mathfrak{t}$ as usual. A look at the root system (cf. [**Bo2**, Plate X], or this work's dedication) shows that the Galois group must act through a rotation of 120°, and hence factors through a cubic extension. The "long roots" form a closed, symmetric and irreducible subsystem, hence $\mathfrak{g}_K$ has a simple subalgebra $\mathfrak{H}$ which consists of the sum of $\mathfrak{t}_K$ and the corresponding root spaces. Now $\mathfrak{H}$ with the adjoint action naturally acts on the 6-dimensional sum $V$ of the root spaces of the "short roots"; indeed $\mathfrak{t}_K$ leaves each root space invariant, and a long root space $(\mathfrak{g}_K)_\beta$ sends a short root space $(\mathfrak{g}_K)_\alpha$ to 0 if $\alpha+\beta\notin R$, or else to $(\mathfrak{g}_K)_{\alpha+\beta}$ which is again a short root space, as the plate shows. So we have a 6-dimensional representation of $\mathfrak{H}$. Also, $V$ is a vector space complement of $\mathfrak{H}$ in $\mathfrak{g}_K$. Inspecting the plate one sees further that the short roots decompose into two disjoint sets (in the form of equilateral triangles) stable under "adding a long root, if the result is a root"; correspondingly, $V$ is the direct sum of two (3-dimensional) subspaces $V_1$, $V_2$ invariant under the adjoint action of $\mathfrak{H}$, corresponding to those two sets of short roots. These representations are non-trivial homomorphisms $\mathfrak{H}\to\mathfrak{sl}_3(K)$ which are necessarily isomorphisms (by simplicity and dimension counting).

$\mathfrak{H}$ is invariant under our Galois action because the set of the long roots (hence the sum of the corresponding root spaces) and $\mathfrak{t}_K$ are; $V_1$ and $V_2$ are invariant under our Galois action because the two equilateral triangles of the short roots are. So taking fixed sets under the Galois (=: $\Gamma$) action, $\mathfrak{h}:=\mathfrak{H}^\Gamma$ is an 8-dimensional simple subalgebra of $\mathfrak{g}$, and $v_i:=V_i^\Gamma$ are three-dimensional $k$-vector spaces whose direct sum is a complement of $\mathfrak{h}$ in $\mathfrak{g}$, on which $\mathfrak{h}$ acts with the restricted adjoint action. This gives non-trivial homomorphisms $\mathfrak{h}\to\mathfrak{sl}(v_i)\simeq\mathfrak{sl}_3(k)$ which again are necessarily isomorphisms. In other words, the simple subalgebra $\mathfrak{h}$ is split, in contradiction to $\mathfrak{g}$ being anisotropic. □

After this digression, we settle the remaining case $\ell=5$, where according to Proposition 5.3.4 only type $E_8$ can occur anyways. But it is easy now to guess the generalisation of the method for $\ell=2$ and 3:

Proposition 5.3.9 (Case $\ell=5$). — *Let* $\mathfrak{g}$ *be a semisimple Lie algebra over a* $\mathfrak{p}$*-adic field* $k$ *which is split by the unramified extension* $K|k$ *of degree* 5. *If the root system of* $\mathfrak{g}_K$ *contains a sub-root-system of type* $A_5$, *then* $\mathfrak{g}$ *cannot be anisotropic. In particular, no* $\mathfrak{g}$ *of exceptional type is anisotropic and split by the unramified extension of degree* 5.

*Proof.* — The same principle as in the proof of Proposition 5.3.7. This time, one first attaches to each root $\alpha$ a Galois-invariant subalgebra of type $A_4$ whose enveloping matrix algebra contains the decisive element

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \xi_\alpha & 0 & 0 \\ 0 & 0 & 0 & \xi_\alpha \xi_{\sigma(\alpha)} & 0 \\ 0 & 0 & 0 & 0 & \xi_\alpha \xi_{\sigma(\alpha)} \xi_{\sigma^2(\alpha)} \\ \xi_\alpha \xi_{\sigma(\alpha)} \xi_{\sigma^2(\alpha)} \xi_{\sigma^3(\alpha)} & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $x^5 = \xi_\alpha^4 \xi_{\sigma(\alpha)}^3 \xi_{\sigma^2(\alpha)}^2 \xi_{\sigma^3(\alpha)} \cdot I$, so that in this case the invariant

$$r(\alpha) := \xi_\alpha^4 \xi_{\sigma(\alpha)}^3 \xi_{\sigma^2(\alpha)}^2 \xi_{\sigma^3(\alpha)}$$

due to anisotropy is not allowed to be a norm of $K|k$, which boils down to:

$$\text{For all roots } \alpha, \quad v_k\big(r(\alpha)\big) \not\equiv 0 \bmod 5.$$

But again $v_k(r(?))$ is additive on the positive roots in the sub-root-system in the hypothesis, say with ordered basis $(\alpha_1, \dots, \alpha_5)$, and Dirichlet's box principle forces $r(?)$ to be of valuation divisible by 5 for at least one of the consecutive sums

$$? = \sum_{i=j}^{k} \alpha_i \quad \text{for } 1 \leq j \leq k \leq 5,$$

which is a contradiction.

Finally, according to Proposition 5.3.4, the only exceptional type that possibly could exist here is $E_8$, which however does contain various copies of $A_5$. □

Summing up this section, we have proved:

Theorem 5.3.10. — *Let $\mathfrak{g}$ be an absolutely simple Lie algebra of exceptional type over a $\mathfrak{p}$-adic field $k$, which is split by an unramified extension $K|k$ of prime degree. Then $\mathfrak{g}$ cannot be anisotropic.*

Note that the technique developed apparently can be generalised to prove:

Proposition 5.3.11. — *Let $\mathfrak{g}$ be a semisimple Lie algebra over a $\mathfrak{p}$-adic field, split by the unramified extension of prime degree $\ell$. If the root system of $\mathfrak{g}$ contains a subsystem of type $A_\ell$, $\mathfrak{g}$ cannot be anisotropic.*

Caveat: The anisotropic $\mathfrak{g}$ of inner type $A$ which *is* split by degree $\ell$ extensions is not of type $A_\ell$, but of type $A_{\ell-1}$ – the one that featured in Lemma 5.3.1, and was used in the proofs to derive the norm-/valuation-criterion on what we called $r(\alpha)$.

This is of limited use though, since for the exceptional types, only the three cases $\ell \leq 5$ left open by Proposition 5.3.4 are to be considered anyway, whereas for the classical types, we can exclude anisotropy by different, general arguments: which we will do in the next section.

## 5.4. Types AII, B, C and D over $\mathfrak{p}$-adic fields

For a $\mathfrak{p}$-adic field $k$, almost everything about these types follows from

THEOREM 5.4.1 (Witt). — *A quadratic form in* 5 *or more variables over* $k$ *is isotropic.*

*Proof.* — See [**Wit**, Satz 16] or [**Lam**, Chap. 6, Thm 2.12]. □

This was extended by Jacobson (cf. [**Jac5**]), Ramanathan, Tsukamoto (cf. [**Tsu**]) and others to the following. For a common proof, see [**Kne**, I Anhang]:

PROPOSITION 5.4.2. — *The following are isotropic over a* $\mathfrak{p}$*-adic field* $k$*:*

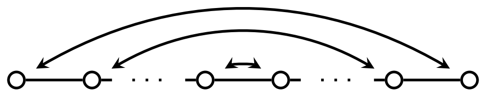
i. *Hermitian forms in* 3 *or more variables over a quadratic extension* $C|k$ (*see type* AII).
ii. *Hermitian forms in* 2 *or more variables over the quaternion division algebra* $Q$ *with standard involution* $\gamma$ (*see type* $C$).
iii. *Skew-hermitian forms in* 4 *or more variables over the quaternion division algebra* (*see type* $D$).

**5.4.1. Type AII.** — The general discussion of the outer forms of type $A$ (4.5.22) showed that they are classified by $\sigma$-hermitian forms on vector spaces over $D$, where $(D, \sigma)$ is a division algebra over $k$ of type $AII$; meaning that there is a quadratic extension $C|k$ such that $D$ is a $C$-central division algebra, and $\sigma$ induces the non-trivial automorphism of $C|k$. As promised in the general discussion, our task is simplified by:

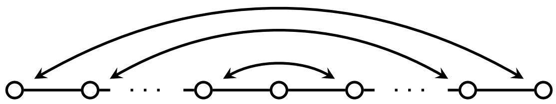
PROPOSITION 5.4.3 (Landherr). — *If* $k$ *is a* $\mathfrak{p}$*-adic field with* $D$*,* $\sigma$ *and* $C$ *as above, then* $C = D$*, so* $\sigma$ *is the non-trivial element of* $\mathrm{Gal}(C|k)$*.*

*Proof.* — There is a complicated proof in [**Lan1**, §4], and a nicer one in [**Lan2**, p. 211 et seq.] which is close to one by Jacobson and generalises to the following: First one shows that $(D_C, \sigma_C)$ is isomorphic to $(D \times D^{\mathrm{op}}, (x, y^{\mathrm{op}}) \mapsto (y, x^{\mathrm{op}}))$ as involutorial $C$-algebra (cf. [**Inv**, Prop. 2.15]). Consequently, $D \simeq D^{\mathrm{op}}$ and the class of $D$ has order $\leq 2$ in $\mathrm{Br}(C)$. Thus if $D \neq C$, $D$ would be the quaternion division algebra over $C$. Secondly, one shows that any quaternion algebra $Q$ over $C$ with $\sigma$ as given is of the form $Q_0 \otimes_k C$ for a $k$-quaternion algebra $Q_0$ (cf. [**Inv**, Prop. 2.22]). But for $\mathfrak{p}$-adic $k$, every quadratic extension splits every quaternion algebra, so that $Q_0 \otimes_k C$ cannot be a skew field. □

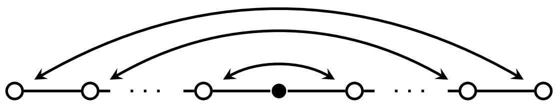
So the outer forms of type $A_n$ ($n \geq 2$) are given by hermitian forms on $C^{n+1}$, where $C$ runs through the different quadratic extensions of $k$. By 5.4.2.i the Witt index $r$ of such a form has to satisfy $n+1-2r \leq 2$, and thus the Satake-Tits diagram (whose number of black vertices was $n-2r$) will be



if $n$ is even and either



or

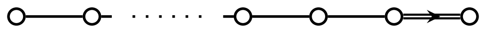


if $n$ is odd. Their rational root systems, determined as in 3.4.1, are $BC_{n/2}$, $C_{(n+1)/2}$ and $BC_{(n+1)/2}$, respectively. The anisotropic kernel in the last case, by the erasing lemma, is an inner form of type $A_1$ and thus the anisotropic form of $\mathfrak{sl}_2$. They all have $C$ as fixed field of the twisted Galois action $t$, and indeed as minimal splitting field. For the third diagram, any other quadratic extension $C' \neq C$ splits the anisotropic kernel, so that $\mathfrak{g}_{C'}$ has the second diagram, with minimal splitting field $C.C'$.

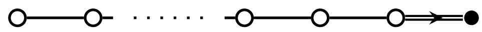
So by  he isomorphism theorem, there are as many isomorphism classes of outer forms of type $A_n$ as quadratic extensions of $k$ for even $n$, and twice as many for odd $n > 1$.

### 5.4.2. $\mathbf{B_n}$

Proposition 5.4.4. — *A Lie algebra of type $B_n$ over $k$ has Satake-Tits diagram*

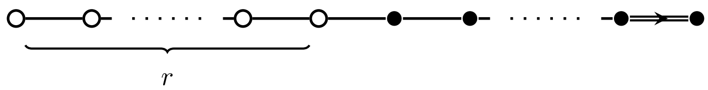


*or*



*In the non-split case, the anisotropic kernel is the anisotropic form of $\mathfrak{sl}_2$, and the rational root system is of type $B_{n-1}$. Any quadratic extension is a splitting field.*

*Proof.* — By the general discussion of 4.5.1.2 we know that such a Lie algebra has the diagram

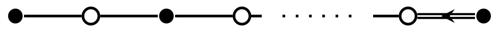



and is given by a quadratic form with Witt index $r$ on $k^{2n+1}$. If we had $r < n-1$, we see by the discussion there (or by the erasing Lemma 4.3.2) that there would exist an anisotropic form in $2(n-r)+1 \geq 5$ variables.

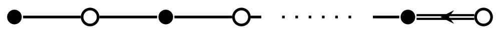
The anisotropic kernel is determined by the erasing lemma and discussion of type $A_1$. This also gives the assertion about the splitting field, and the rational root system is easily calculated like in Section 3.4.1. □

### 5.4.3. $\mathbf{C_n}$

PROPOSITION 5.4.5. — *A Lie algebra of type $C_n$ over $k$ is either split, or its Satake-Tits diagram is either*



*or*



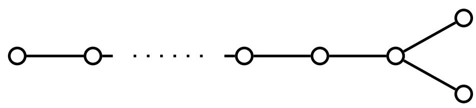
*according to whether $n$ is odd or even. In the non-split case, the anisotropic kernel is a product of $\lfloor \frac{n+1}{2} \rfloor$ copies of the anisotropic form of $\mathfrak{sl}_2$, and the rational root system is of type $BC_r$ for odd $n$ and of type $C_r$ for even $n$, where $r = \lfloor \frac{n}{2} \rfloor$. Any quadratic extension is a splitting field.*

*Proof.* — We had seen already in the general discussion of 4.5.1.4 that $d = 1$ gives the split form. Otherwise, as noted in 4.5.16, $d = 2$ and $D = Q$ is the quaternion division algebra with its standard involution $\gamma$. By 5.4.2.ii, all non-degenerate hermitian forms are equivalent to the one given by the unit matrix, and hence for the Witt index $r$ we have $2r = n - 1$ if $n$ is odd and $2r = n$ if $n$ is even, which gives the diagrams. $\lfloor \frac{n+1}{2} \rfloor = n - r$ is the number of black vertices.

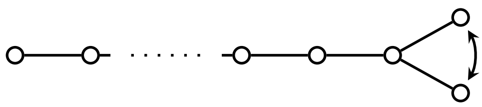
The anisotropic kernel is determined by the erasing lemma and discussion of type $A_1$. This also gives the assertion about the splitting field, and the rational root system is easily calculated like in Section 3.4.1. □

### 5.4.4. $\mathbf{D_n}$

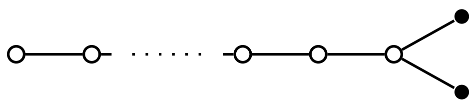
PROPOSITION 5.4.6. — *A Lie algebra of type $D_n$ over $k$ which comes from an involutorial $k$-algebra whose underlying CSA is split has Satake-Tits diagram*



*or*



*or*



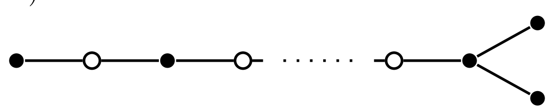
*In the last two cases, the rational root systems are of type $B_{n-1}$ and $B_{n-2}$, respectively. The anisotropic kernel in the last case is a product of the anisotropic form of $\mathfrak{sl}_2$ with itself, and it is split by any quadratic extension. The first and third one are thus completely described up to isomorphism; for the second diagram, there is one isomorphism class of Lie algebras for every quadratic extension of $k$, and this quadratic extension splits it.*

*Proof.* — Proceeding as for type $B$, we can exclude the case $r < n-2$. The first diagram is the case $r = n$, the split form. For $r = n-1$ we have the quasi-split forms discussed in Section 4.5.1. The final case is $r = n-2$. By [**Lam**, Chap. 6, Cor. 2.15] up to isomorphism, there is a unique 4-dimensional anisotropic quadratic space $(k^4, q)$, corresponding to $S_a = \mathrm{diag}(1, -u, -\pi, \pi u)$ (where $\pi$ is a uniformiser and $u \in \mathcal{O}_k$ such that $k(\sqrt{u})$ is unramified; for $p \neq 2$, any $u \in \mathcal{O}_k \setminus \mathcal{O}_k^2$ will do) whose discriminant is the square $(\pi u)^2$. Its diagram is as described in the general discussion, and the anisotropic kernel and rational root system are determined as usual. □

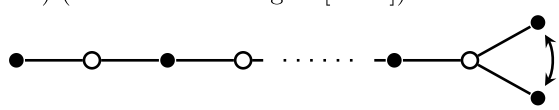
It remains to discuss the forms of type $D_n$ where $d = \mathrm{Ind}(A) = 2$ for the corresponding involutive algebra $(A, \iota)$. Analogous to the discussion for type $C$, this is equivalent to the description of skew-hermitian forms over the quaternion division algebra $Q$ with its standard involution. By 5.4.2.iii now necessarily $n - 2r \leq 3$ for the dimension $n$ (over $Q$) and the Witt index $r$.

The anisotropic skew-hermitian forms are determined in [**Tsu**, Thm 3]: There is exactly one in the dimensions 3 and 0, giving the first and last of the following diagrams. In dimensions 2 and 1 there is one for each quadratic extension, the correspondence being given by the discriminant [4] in $k^*/k^{*2}$, and the corresponding quadratic extension $K|k$ is the fixed field of the $t$-action in the middle two cases. The cases are:

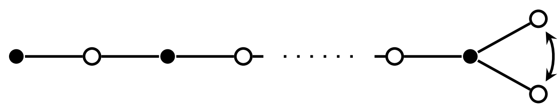
$n = 2r + 3$ ($n$ odd):



$n = 2r + 2$ ($n$ even) (this one is missing in [**We1**]):

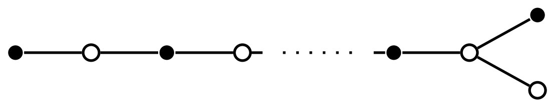


$n = 2r + 1$ ($n$ odd):



$n = 2r$ ($n$ even):

4. Defined as in the general discussion of 4.5.1.3 via the reduced norm, cf. [**Tsu**, 1.3].



The anisotropic kernels are, from top to bottom:

▷ a product of $r$ copies of the anisotropic form of $\mathfrak{sl}_2$ with the anisotropic form of $\mathfrak{sl}_4$;

▷ a product of $r$ copies of the anisotropic form of $\mathfrak{sl}_2$ with $R_{K|k}\mathfrak{a}$, where $K|k$ is the mentioned fixed field of $t$ and $\mathfrak{a}$ is the anisotropic $K$-form of $\mathfrak{sl}_2$;

▷ a product of $r$ copies of the anisotropic form of $\mathfrak{sl}_2$;

▷ ditto.

Finally, with the method of Section 3.4.1, the rational root systems are seen to be of type $B_r$ in the first three cases and of type $C_r$ in the last.

Remark 5.4.7. — For $n = 4$, the phenomenon from 4.5.20.ii occurs and the case $d = 1, r = 2$ reappears as $d = 2, r = 2$. So $\mathfrak{S}(M_4(Q), \iota_1) \simeq \mathfrak{S}(M_8(k), \iota_2)$ with

$$\iota_1 : X \mapsto \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \cdot {}^t(\gamma(X)) \cdot \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

and $\iota_2$ coming from the quadratic form on $k^8$ of Witt index 2 (i.e. with anisotropic kernel the 4-dimensional anisotropic space). All other diagrams that we listed, and hence their Lie algebras, are visibly distinct.

## 5.5. Exceptional Lie algebras over $\mathfrak{p}$-adic fields

We now classify, over a $\mathfrak{p}$-adic field $k$, the absolutely simple Lie algebras of the *exceptional types*, by which we mean

▷ those of types $G_2, F_4, E_6, E_7, E_8$, and

▷ the so-called *trialitarian forms* which are those of type $D_4$ where the twisted Galois action does not factor through a quotient of order $\leq 2$.

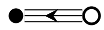
The main method is to use the results of Section 4.3, combined with the results for the classical types, to exclude most possible Satake-Tits diagrams. It should be noted that large parts of this (e.g. everything where only the result about inner types of $A_n$ in 4.3.7 is used) work just as well over arbitrary fields, as Tits' list in [**Tit1**] suggests. The remaining diagrams are then shown to be admissible, except for the anisotropic ones. For these, we exploit the results in the prime degree case of Section 5.3. This way, we can avoid using Kneser's theorem except for one case in type $E_6$ and one in type $D_4$ (although to be clear, one should note that the cases $E_7$ and $E_8$ rely on $E_6$).

### 5.5.1. $\mathbf{G_2}$

Proposition 5.5.1. — *Every Lie algebra of type $G_2$ over $k$ is split.*

a) With Kneser's theorem: The fundamental group $P(R)/Q(R)$ is trivial and hence by Kneser's theorem there is only the split form.

b) Without Kneser's theorem: by 4.3.6, the only other possibilities are anisotropic ones, or a form with diagram



Assume first we had such a Lie algebra. It is split by the unramified quadratic extension $K|k$ (since its anisotropic kernel is), so fix $\mathfrak{s}$, $\mathfrak{t}$ as usual with $\mathfrak{t}_K$ split, and call $\sigma$ the non-trivial element of $\Gamma := \mathrm{Gal}(K|k)$. Looking at the root system (cf. [**Bo2**, Plate X], or this work's dedication) shows that the Galois action would have to consist precisely of the reflection $s_{\alpha_1}$. In particular we would have $\beta - \sigma_A(\beta) = \alpha_1 \in R$ for $\beta = 2\alpha_1 + \alpha_2$. This is impossible by the following lemma, which looks $\mathfrak{p}$-adic but is inspired by an analogous assertion over the field $\mathbb{R}$ in [**Ara**, Prop. 1.3].

Lemma 5.5.2. — *Let $\mathfrak{g}$ be a semisimple Lie algebra with $\mathfrak{s}$, $\mathfrak{t}$ as always such that $\mathfrak{t}$ is split by the unramified quadratic extension $K|k$. Let $\sigma$ be the non-trivial element of* $\mathrm{Gal}(K|k)$*. Then $\beta - \sigma_A(\beta)$ is not a root, for any root $\beta$.*

*Proof.* — $\sigma_A = -\mathrm{id}$ on $X_0$, so the assertion is clear for $\beta \in R_0$. For the general case, we resume notation from Sections 3.3.1 and 5.3: Again we fix a Chevalley basis consisting of $H_\alpha$ and bases $e_\alpha$ of the root spaces, with the structure constants $N_{\alpha,\beta}$ defined in (14) being integers. We have $N_{\alpha,\beta} = -N_{\beta,\alpha}$ for all roots $\alpha, \beta$. Again we define $\xi_\alpha$'s by $\sigma(e_\alpha) = \xi_\alpha e_{\sigma_A(\alpha)}$. Because $\sigma(H_\alpha) = [\sigma(e_\alpha), \sigma(e_{-\alpha})] = \xi_\alpha \xi_{-\alpha} H_{\sigma_A(\alpha)}$ and ${}^\sigma\alpha(\sigma(H_\alpha)) = \sigma(\alpha(H_\alpha)) = 2$, we have $\sigma(H_\alpha) = H_{\sigma_A(\alpha)}$ and $\xi_{-\alpha} = \xi_\alpha^{-1}$ for all $\alpha \in R$.

Now assume there is $\beta \in R$ with $\beta - \sigma_A(\beta) \in R$. Then $N_{\beta,-\sigma_A(\beta)} \in \mathbb{Z} \setminus \{0\}$. Computing $[\sigma(e_\beta), e_{-\beta}] = \sigma([e_\beta, \sigma(e_{-\beta})])$ in two different ways gives

$$\xi_\beta N_{\sigma_A(\beta),-\beta} \cdot e_{\sigma_A(\beta)-\beta} = \sigma(\xi_{-\beta}) \xi_{\beta-\sigma_A(\beta)} N_{\beta,-\sigma_A(\beta)} \cdot e_{\sigma_A(\beta)-\beta}$$

and hence

$$\xi_\beta = -\sigma(\xi_\beta^{-1}) \cdot \xi_{\beta-\sigma_A(\beta)}$$

or, setting $\gamma = \beta - \sigma_A(\beta)$:

$$\xi_\gamma = -N_{K|k}(\xi_\beta)$$

But on the other hand, the subalgebra $KH_\gamma + Ke_\gamma + Ke_{-\gamma}$ is stable under the Galois operation, hence descends to a form of $\mathfrak{sl}_2(K)$ which is anisotropic because $\gamma \in R_0$, and the argument in the proof of Proposition 5.3.5 shows that $\xi_\gamma \notin \pm N_{K|k}(K)$ or in other words, its $\pi_k$-adic valuation is odd, hence we have a contradiction and the lemma is proven. □

So there can be *only split or anisotropic forms.*

Now assume there is an anisotropic one. Choose an unramified splitting field $K|k$. Since $K|k$ is cyclic, we can "filter" the extension to a tower of fields $k = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that each $K_{i+1}|K_i$ is unramified of prime degree.

Now since for all $i$, the scalar extension $\mathfrak{g}_{K_i}$ is either split or anisotropic, and of course scalar extensions of a split form remain split, there must be exactly one step such that $\mathfrak{g}_{K_i}$ is anisotropic but $(\mathfrak{g}_{K_i})_{K_{i+1}} = \mathfrak{g}_{K_{i+1}}$ is split. But then $\mathfrak{g}_{K_i}$ is an anisotropic form of type $G_2$ over the $\mathfrak{p}$-adic field $K_i$, with $K_{i+1}$, an unramified extension of prime degree, as splitting field: which we excluded in Section 5.3. The proposition is proven.

### 5.5.2. $\mathbf{F_4}$

Proposition 5.5.3. — *Every Lie algebra of type $F_4$ over $k$ is split.*

*Proof.* — a) Again $P(R)/Q(R)$ is trivial and hence by Kneser's theorem, only the split form exists.

b) Without using Kneser's theorem: First, all diagrams containing a white vertex are easily ruled out. Namely, if the rightmost vertex $\alpha_4$ is white and the form is not split, by applying the erasing Lemma 4.3.2 to that vertex and comparing with type $B_3$, the only possibility is

which is excluded by erasing $\alpha_2$ and seeing an impossible diagram of type $A_2$ to its right. So $\alpha_4$ is black. If $\alpha_1$ were white, erasing it and looking at type $C_3$ gives the only option

which is impossible by erasing $\alpha_3$. So $\alpha_1$ is black. By 4.3.6 we are left with

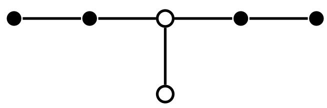
which again is impossible as erasing one white vertex shows. □

So again, there are *only split or anisotropic forms.*

The same procedure as for type $G_2$, filtering an unramified splitting field to subextensions of prime degrees, then excludes anisotropic forms via Section 5.3, and the proposition is proven.

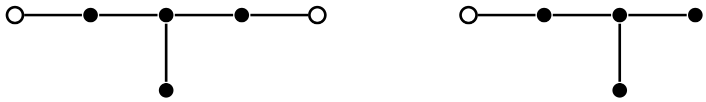
### 5.5.3. $\mathbf{E_6}$, inner forms

Proposition 5.5.4. — *Over $k$, there are two inner forms of Lie algebras of type $E_6$: The split one, and one whose Satake-Tits diagram is*

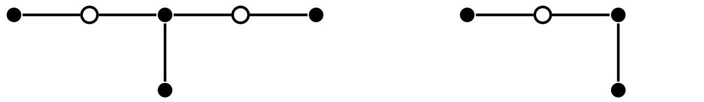


*and whose anisotropic kernel is the product of two copies of the anisotropic form of* $\mathfrak{sl}_3$*. The rational root system is of type* $G_2$*, and a splitting field is the unramified cubic extension.*

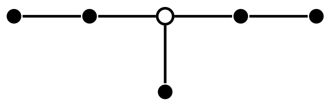
*Proof.* — Below is a list of all possible diagrams, except for the anisotropic one, which are allowed by 4.3.1 (on the left). Most of them are then directly excluded by an application of the erasing Lemma 4.3.2: we signify a part of the diagram which, according to that lemma (that is, after erasing certain white vertices and possibly restricting to a connected component) would have to be $k$-admissible, but is not, according to our discussion of the classical types.



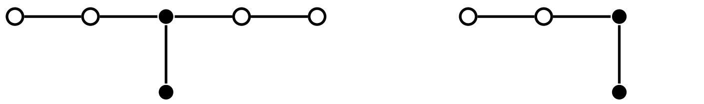
not admissible of type $D_5$



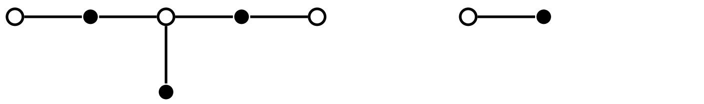
not admissible of type $A_4$



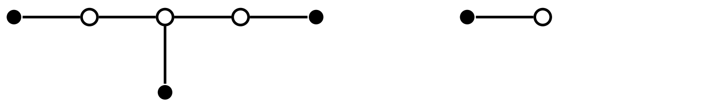
not admissible by 4.3.6



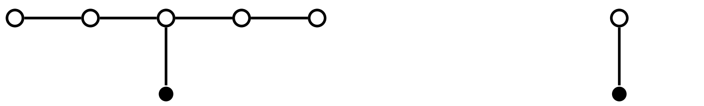
not admissible of type $A_4$



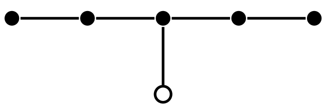
not admissible of type $A_2$



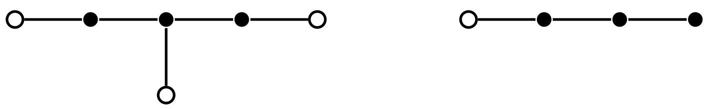
not admissible of type $A_2$



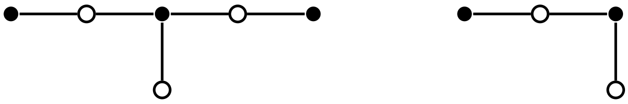
not admissible of type $A_2$



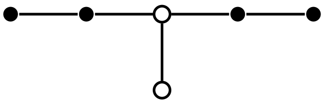
(see below)



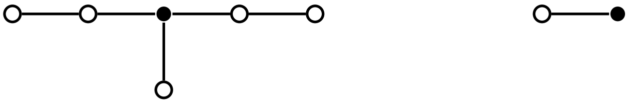
not admissible of type $A_4$



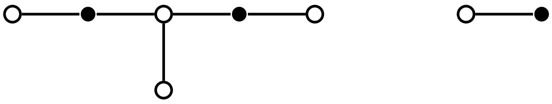
not admissible
of type $A_4$



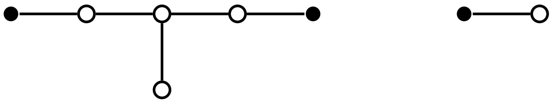
admissible!



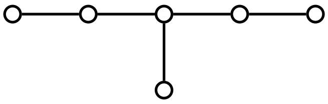
not admissible
of type $A_2$



not admissible
of type $A_2$

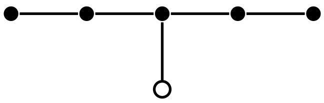


not admissible
of type $A_2$

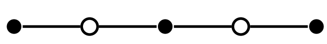


admissible!

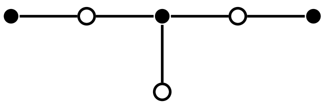
To exclude the diagram



we note that if it were $k$-admissible – being given by the Lie algebra $\mathfrak{g}$, say – then the anisotropic kernel $\mathfrak{g}_a$ would be of inner type $A_5$ and thus be the anisotropic form of $\mathfrak{sl}_6$. This means that it is split by the unramified extension of degree 6, which is the composite field of the unramified extensions of degree 3 and 2, call them $K$ and $L$. Then $(\mathfrak{g}_a)_K$, being a non-split inner form of type $A_5$ which is split by the unramified quadratic extension $L.K|K$, would have Satake-Tits diagram

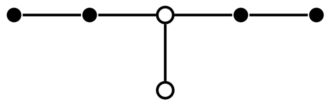


and consequently the diagram of $\mathfrak{g}_K$ would have to be

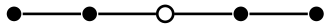


which was excluded in the list above.

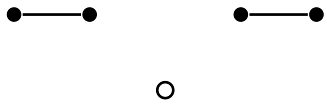
Let us now describe the non-split admissible form



It exists by the patching Proposition 4.3.3. Concretely, we patch: Firstly, the admissible form of type $A_5$



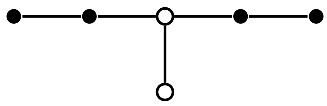
and secondly, a product of $\mathfrak{sl}_2(k)$ with two copies of the anisotropic form of $\mathfrak{sl}_3$:



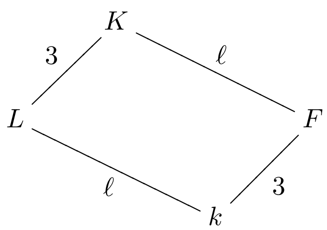
Both have anisotropic kernel as described and so has their patching. The assertion about the splitting field is then clear, and the rational root system was calculated in Section 3.4.1.

We have thus dealt with all possible forms except for anisotropic ones. Kneser's and Satake's table show that there are two inner forms of type $E_6$, so no anisotropic ones can occur.

Without Kneser's theorem, we can at least get the following information: assume $\mathfrak{g}$ is an anisotropic inner form of type $E_6$, and choose an unramified splitting field of minimal degree $K|k$. Again filter the extension as $k = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that each $K_{i+1}|K_i$ is unramified of prime degree. Look at the forms $\mathfrak{g}_{K_i}$. Now let $0 \leq i < n$ be the first index such that $\mathfrak{g}_{K_{i+1}}$ is not anisotropic. By Section 5.3, $\mathfrak{g}_{K_{i+1}}$ cannot be split, thus it has to be the form with diagram



Further, due to minimality and what we know about this form, necessarily $n = i+2$ and $[K_n : K_{i+1}] = 3$. Let us replace $k$ by $K_i$ and call $L := K_{i+1}$, then we have a diagram of fields

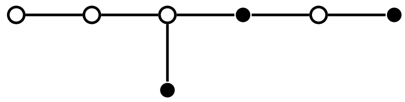



where $\ell := [L : K]$ is a prime and $F$ is the unramified cubic extension of $k$. But then, which form is $\mathfrak{g}_F$? Again due to minimality, it cannot be the split form; and again due to Section 5.3, it cannot be anisotropic; so it must be the form with the same diagram as above. Hence, due to minimality, necessarily $\ell = 3$ and actually $F = L$.

So what we would like to rule out is an anisotropic form, split by the unramified extension of degree 9, whose scalar extension to the intermediate unramified extension of degree 3 would be the non-split non-anisotropic form mentioned above. Sadly, we were not able to lead this to a contradiction. (The method we used so far will however help in cases $E_7$ and $D_4$, see below.) □

### 5.5.4. $\mathbf{E_7}$

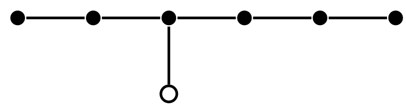
Proposition 5.5.5. — *Over $k$, there are two forms of Lie algebras of type $E_7$: The split one, and one whose Satake-Tits diagram is*



*and whose anisotropic kernel is the product of three copies of the anisotropic form of $\mathfrak{sl}_2$. The rational root system is of type $F_4$. A splitting field is the unramified quadratic extension.*

*Proof.* — As before we rule out other diagrams with at least one white vertex. If the "lower vertex" $\alpha_2$ is white, the "upper row" has to be a $k$-admissible diagram of inner type $A_6$, so the only non-split possibility is:

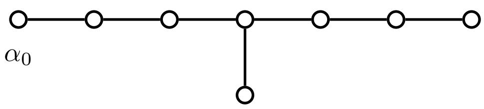
(*)



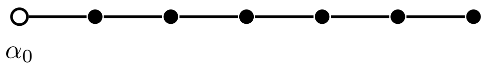
To exclude (*), we generalise the method of the erasing Lemma 4.3.2; cf. [**PSt**, Section 5, Lemma 3]. First note the

Lemma. — Let $R$ be an irreducible root system with basis $B = \{\alpha_1, \dots, \alpha_r\}$, and let $\alpha_0$ be the negative of its highest root. Then for any proper subset $B' \subsetneq B$, the intersection of the $\mathbb{Z}$-span of $B' \cup \{\alpha_0\}$ with $R$ is a root subsystem $R'$ of $R$ (in the vector space it generates) with basis $B' \cup \{\alpha_0\}$.

The proof is straightforward; for the last assertion one uses that the coefficient of any $\alpha_j$ ($j \geq 1$) in any root is less or equal to its coefficient in the highest root. The lemma implies: if one erases a non-empty set of vertices from the *completed Dynkin diagram* of $R$, what remains is a (possibly non-connected) Dynkin diagram of a root system. One can indeed check this case-by-case in the plates of [**Bo2**, VI]. Now the completed Dynkin diagram of type $E_7$ is:

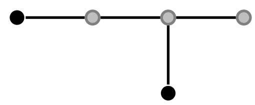


Assume there were a Lie algebra $\mathfrak{g}$ over $k$ having $(*)$ as Satake-Tits diagram for the usual choices including a $\Gamma$-basis $\Delta$. Form the root subsystem $R'$ as in the lemma for $\Delta' := \Delta \setminus \{\alpha_2\}$. Setting $\overline{R}' = \rho(R') \setminus \{0\}$, with 3.4.2 one checks that this is the subsystem of $\overline{R}$ generated by $\rho(\Delta' \cup \{\alpha_0\}) \setminus \{0\}$ (which is just $\{\rho(\alpha_0)\}$ in this case) and that $\rho^{-1}(\overline{R}' \cup \{0\}) = R'$. Proceeding then as in the erasing lemma, one would get a semisimple Lie algebra $\mathfrak{g}'$ over $k$ which would be generated by the anisotropic kernel $\mathfrak{g}_a$ and the root space of $\rho(\alpha_0)$, and whose Satake-Tits diagram would be:

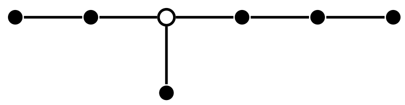


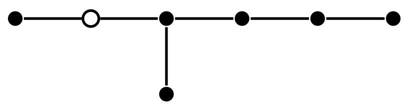

This is an impossible form of type $A_7$, so $(*)$ is excluded (over any field $k$). Hence we can assume from now on that the lower vertex $\alpha_2$ is black. Then the "rightmost vertex" $\alpha_7$ is black, since if not, erasing it gives a non-admissible diagram of type $E_6$.

*Case 1.* — The "leftmost vertex" $\alpha_1$ is black. If the second to the right vertex $\alpha_6$ were white, on the left of it there would be an admissible form of $D_5$ which looks like

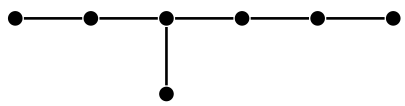


where the gray vertices can be white or black. Comparing with our list there we see that this is impossible: so $\alpha_6$ is black. The same game for $\alpha_5$ and forms of type $A_4$ shows that $\alpha_5$ must be black, then repeating it for $\alpha_4$ and $A_2$ shows that we are left with the diagrams



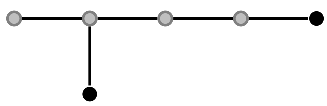


and

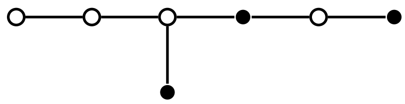


The first two (of $k$-rank 1) are excluded by 4.3.6. The anisotropic one is left aside.

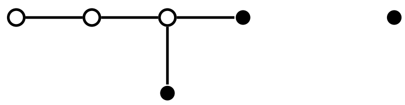
*Case 2.* — The "leftmost vertex" $\alpha_1$ is white. Then erasing it must give an admissible diagram of type $D_6$ which looks like



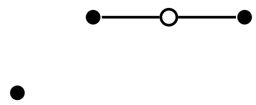
where the gray vertices are either black or white. Comparing with our list for type $D_6$ we see that the only possibility corresponds to the one named in the proposition:



This exists by the patching Proposition 4.3.3. Concretely, we patch: Firstly, the direct product of an admissible form of type $D_5$ and the anisotropic form of $\mathfrak{sl}_2$:



Secondly, the direct product of an admissible form of type $A_3$ and the anisotropic form of $\mathfrak{sl}_2$:



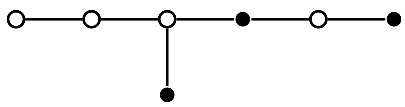
Both have a threefold product of the anisotropic form of $\mathfrak{sl}_2$ as anisotropic kernel and so has our patched form of type $E_7$. The assertion about the splitting field is clear and the rational roots are easily computed like in Section 3.4.1.

We have thus dealt with all possible forms except for anisotropic ones. Kneser's and Satake's table show that there are two forms of type $E_7$, so no anisotropic ones can occur. □

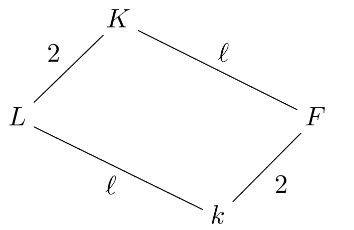
We can, however, rule out anisotropic forms without Kneser's theorem:

PROPOSITION 5.5.6. — *There is no anisotropic form of type $E_7$ over $k$.*

*Proof.* — Assume $\mathfrak{g}$ were one and choose an unramified splitting field of minimal degree $K|k$. Again filter the extension as $k = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that each $K_{i+1}|K_i$ is unramified of prime degree, and let $0 \leq i < n$ be the first index such that $\mathfrak{g}_{K_{i+1}}$ is not anisotropic. By Section 5.3, $\mathfrak{g}_{K_{i+1}}$ cannot be split, thus it has to be the form with diagram



Further, due to minimality and what we know about this form, necessarily $n = i+2$ and $[K_n : K_{i+1}] = 2$. Let us replace $k$ by $K_i$ and call $L := K_{i+1}$, then we have a diagram of fields




where $\ell := [L : K]$ is a prime and $F$ is the unramified quadratic extension of $k$. But which form is $\mathfrak{g}_F$ then? Again due to minimality, it cannot be the split form; and again due to Section 5.3, it cannot be anisotropic; so it must be the form with the same diagram as above. Hence, due to minimality, necessarily $\ell = 2$ and actually $F = L$.

So we have an anisotropic $\mathfrak{g}$ which is split by the unramified extension of degree 4. Make the usual choices of $\mathfrak{t}, \mathfrak{s}$, $R$, $\ldots$, and let $\sigma$ be a generator of $\mathrm{Gal}(K|k)$. Then the minimal polynomial of $\sigma_A$ as element of $\mathrm{GL}(V)$ (where $V$ is the $\mathbb{R}$-vector space spanned by the root system $R$) must divide $T^3 + T^2 + T + 1 = (T+1)(T^2+1)$.

Choose an $A(R)$-invariant scalar product on $V$. We will now look at the cyclic $\mathbb{R}[\sigma_A]$-module generated by any given root $\alpha$, which will be of $\mathbb{R}$-dimension 1, 2 or 3. In each case we will derive a contradiction.

1) If the $\mathbb{R}[\sigma_A]$-module generated by $\alpha$ is one-dimensional over $\mathbb{R}$ – equivalently, $\sigma_A(\alpha) = -\alpha$ – the space spanned by $\alpha$ and thus also its orthogonal complement $\langle\alpha\rangle^\perp$ are Galois-stable and thus define subalgebras of $\mathfrak{g}$ which are necessarily anisotropic. But the roots orthogonal to any given root in $E_7$ form a root system of type $D_6$, and we know from the discussion of the classical types that no Lie algebra of this type can be anisotropic. So $\sigma_A$ cannot be multiplication by $-1$ on any root.

2) I the $\mathbb{R}[\sigma_A]$-module generated by $\alpha$ is two-dimensional over $\mathbb{R}$ – equivalently, $\sigma_A^2(\alpha) = -\alpha$ – it is readily seen that $\alpha$ and $\sigma(\alpha)$ must be orthogonal to each other; in this case, the roots orthogonal to their span form a root system of type $A_1 \times D_4$, and again we know that no anisotropic form of classical type $D_4$ (no triality since $\mathrm{ord}(\sigma_A) = 4$)) exists.

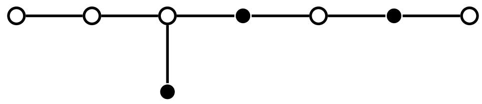
3) So for each root $\alpha$, the $\sigma_A$-invariant subspace spanned by its $\sigma_A$-conjugates in $V$ must be 3-dimensional, and the minimal polynomial of $\sigma_A$ is indeed $T^3 + T^2 + T + 1$. Then by corollary 5.3.2, the $\sigma_A$-conjugates of any root $\alpha$ are the basis of a subsystem of type $A_3$. But the roots orthogonal to such a (Galois-invariant) subsystem in $E_7$ form a (Galois-invariant) subsystem of type $A_1 \times A_2$, which is impossible: one can see that there is no way in which $\sigma_A$ with said minimal polynomial can operate on $A_1 \times A_2$ without stabilising both factors, which would give a 1- and a 2-dimensional Galois-invariant subsystem, contradicting the above results about their non-existence. □

### 5.5.5. $\mathbf{E_8}$

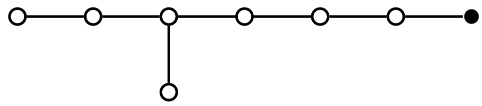
PROPOSITION 5.5.7. — *Every Lie algebra of type $E_8$ over $k$ is split.*

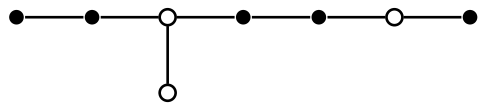
*Proof.* — The root system of type $E_8$ has trivial $P(R)/Q(R)$, so by Kneser's theorem, only one form can exist, and this is the split one. Still, our usual procedures allow us to rule out the other diagrams directly.

Assume first we had a diagram with at least one white vertex. If the "rightmost vertex" $\alpha_8$ were white, by the classification of $E_7$ the only option for a non-split form would be

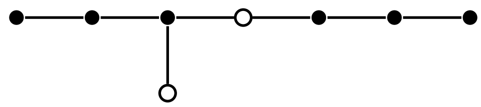


but the segment formed by $\alpha_7$ and $\alpha_8$ is not admissible in type $A_2$. So $\alpha_8$ is black. If $\alpha_7$ were white, the possibilities (by admissible diagrams of inner type $E_6$) are

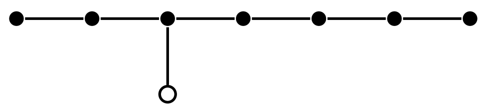




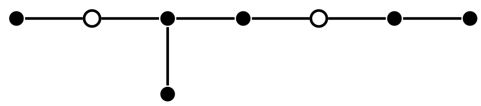
which are both excluded e.g. by erasing the "lower vertex" $\alpha_2$. So $\alpha_7$ is black. Now if the lower vertex $\alpha_2$ were white, the "upper row" would have to be an admissible form of type $A_7$ ending in two black vertices. This gives the possibilities



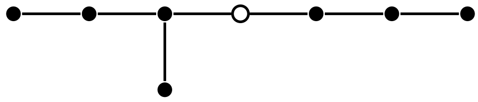
and



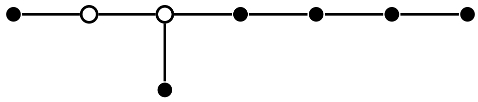
The first is excluded by erasing $\alpha_5$ and looking at the left part, the second is impossible by 4.3.6. So $\alpha_2$ is black. If the "third to the right" vertex $\alpha_6$ were white, by the classification of type $D_5$, the only option is



which is seen to be impossible by erasing the left white vertex. So $\alpha_6$ is black. If $\alpha_5$ were white, by the classification of type $A_4$ the only option is



which is impossible by 4.3.6. So $\alpha_5$ is black. If the leftmost $\alpha_1$ were white, erasing it would give a non-admissible form of type $D_7$, so $\alpha_1$ is black. Since by 4.3.6 there is no form of $k$-rank 1 left, and



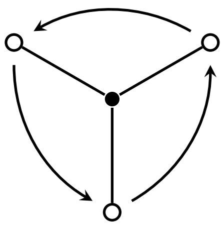
is impossible – as erasing any of the white vertices shows –, we are left with all-black anisotropic forms.

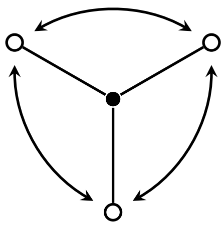
But since we know now that only anisotropic and split forms can exist, the technique of reducing to an unramified splitting field of prime extension, which is settled by Section 5.3, works exactly as in the cases $G_2$ and $F_4$. ☐

### 5.5.6. $\mathbf{D_4}$, trialitarian types

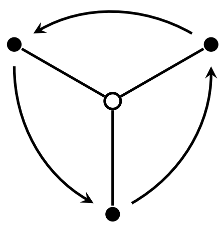
Proposition 5.5.8. — *Except for the quasi-split forms described in Section 4.6, no other Lie algebras of trialitarian type exist over $k$.*

We had already noted in Section 4.3.3 that the diagrams

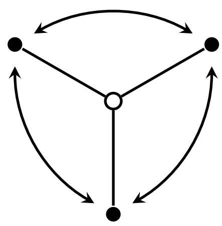




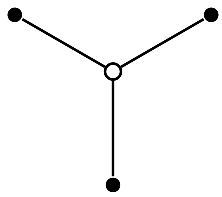
are not admissible over any field. Assume we had a $\mathfrak{g}$ with diagram



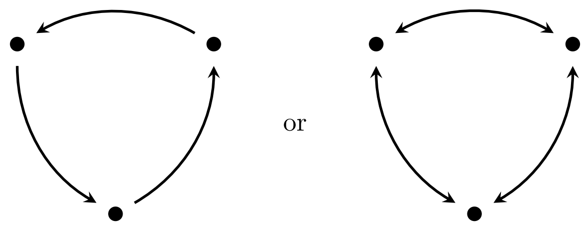
or



where $K :=$ the fixed field of the twisted Galois action would be an extension of $k$ with $\mathrm{Gal}(K|k)$ cyclic of order 3, resp. isomorphic to the symmetric group $S_3$. Then $\mathfrak{g}_K$ would have the diagram

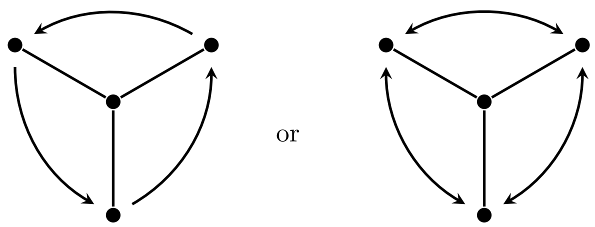


– a non-trialitarian form of type $D_4$ – which however is not admissible over $\mathfrak{p}$-adic fields according to the list in Subsection 5.4.4. [5] (That the diagram of $\mathfrak{g}_K$ would be as described is not as trivial as it might seem; but it follows from Section 4.1 (cf. Remark 3.4.1) that the anisotropic kernel $\mathfrak{g}_a$ of such $\mathfrak{g}$, with diagram



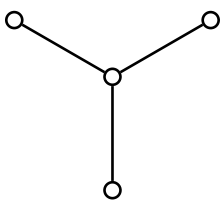

would necessarily be the scalar restriction $R_{K|k}\mathfrak{a}$ of $\mathfrak{a}$ := the anisotropic $K$-form of $\mathfrak{sl}_2$, and we would have $(\mathfrak{g}_a)_K \simeq \mathfrak{a}^3$. From this one can conclude.)

Now let us rule out all anisotropic forms, whose diagrams would be




(case I) (case II)

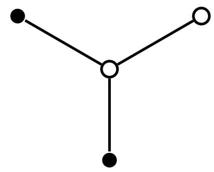
*Case I* (*Twisted Galois action via* $\mathbb{Z}/3$). — Again let $K$ be the fixed field of the twisted Galois action. Then the diagram for $\mathfrak{g}_K$ contains no arrows and thus $\mathfrak{g}_K$, due to our results so far (including the classification of non-trialitarian types), is either the split form



(case Ia)

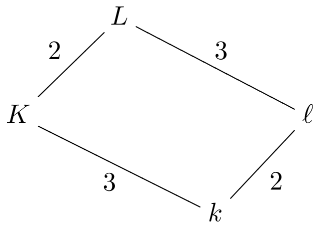
5. According to the list in [**Tit1**], such forms *do* exist over number fields; correspondingly, the described diagram of a non-trialitarian form *is* admissible over number fields.

or has, up to permutation of the outer vertices, the diagram

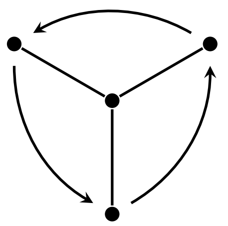


(case Ib)

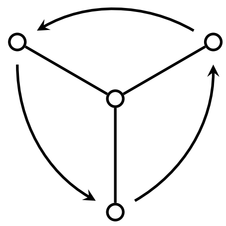
With Galois theory, we will now reduce case Ib to Ia. Namely, $\mathfrak{g}_K$ with this diagram is split by any quadratic extension $L|K$; choose one such that $\mathrm{Gal}(L|k) \simeq \mathrm{Gal}(L|K) \times \mathrm{Gal}(K|k)$, i.e. $L = K[x]$ with $x^2 \in k$, $x \notin K$. Set $\ell := k(x)$:




Then our results so far imply that the Lie algebra $\mathfrak{g}_\ell$ would have diagram



or



(case Ib.1) (case Ib.2)

and have $L$ as a splitting field, with $\mathrm{Gal}(L|l) \simeq \mathrm{Gal}(K|k) \simeq \mathbb{Z}/3$ being the Galois group that acts on the diagram. Thus in case Ib.1, $\mathfrak{g}_\ell$ with its cubic splitting field $L$ is an instance of case Ia. Also:

Lemma 5.5.9. — *Case* Ib.2 *cannot occur.*

*Proof.* — We use the method that allowed us to exclude anisotropic forms in type $E_7$ in Proposition 5.5.6. Namely, we make the usual choices, including a generator $\sigma \in \mathrm{Gal}(L|k) \simeq \mathbb{Z}/6 \simeq \mathbb{Z}/2 \times \mathbb{Z}/3$. The minimal polynomial of $\sigma_A \in A(R)$ must divide

$$T^5 + T^4 + T^3 + T^2 + T + 1 = (T+1)(T^2+T+1)(T^2-T+1)$$

and again we look at the $R[\sigma_A]$-module in $V = \mathbb{R}X$ generated by one given root $\alpha$.

1) If it is one-dimensional over $\mathbb{R}$, the roots orthogonal to it form a root system of type $A_1 \times A_1 \times A_1$, and without loss of generality, we can pick the three orthogonal roots in that product to be basis roots $\alpha_1, \alpha_3$ and $\alpha_4$; whereas the root $\alpha$ we started with would be $-\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$, the negative of the highest root. But using that $\sigma_A$

is elliptic and of order 6, one concludes that modulo renumbering $\alpha_1, \alpha_3$ and $\alpha_4, \sigma_A$ would have to map

$$\alpha_1 \longmapsto -\alpha_3 \longmapsto \alpha_4 \longmapsto -\alpha_1 \longmapsto \alpha_3 \longmapsto -\alpha_4$$

and operate as $-\operatorname{id}$ on $\{\pm\alpha_2, \pm\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \pm\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. This last set, however, is a root system of type $A_2$, and the subalgebra of $\mathfrak{g}$ generated by it would be anisotropic, but split by the quadratic extension $\ell = L^{\sigma^2=id}|k$ (and also be an outer form), which is impossible by our classification of type $A$.

So on no root $\alpha$, $\sigma_A$ operates as $-\operatorname{id}$.

2) Now assume the $\mathbb{R}[\sigma_A]$-module generated by a root $\alpha$ is two-dimensional over $\mathbb{R}$. Considering the minimal polynomial, we must either have

$$\sigma_A^2(\alpha) + \sigma_A(\alpha) + \alpha = 0 \quad \text{or} \quad \sigma_A^2(\alpha) - \sigma_A(\alpha) + \alpha = 0.$$

In both cases, $\alpha$ and $\sigma_A(\alpha)$ would be the basis of a sub-root-system of type $A_2$; in the second case however, it is easily seen (similar to Lemma 5.3.1) that $\sigma_A$ would be the rotation by $60°$, which is not an element of the Weyl group, thus again we would find an anisotropic *outer* form of type $A_2$ as subalgebra in $\mathfrak{g}$, which is impossible. So if the $\mathbb{R}[\sigma_A]$-module spanned by a root $\alpha$ is two-dimensional, we have $\sigma_A^2(\alpha)+\sigma_A(\alpha)+\alpha = 0$. This means, however, that $\sigma_A^3 = \operatorname{id}$ on these submodules; so if this would be the case for all $\alpha \in R$, the whole algebra would be split by the cubic extension $K = L^{\sigma^3=\mathrm{id}}|k$, a contradiction.

So there must be a root $\alpha \in R$ such that the $\mathbb{R}[\sigma_A]$-module generated by it is of dimension 3 or 4 as $\mathbb{R}$-vector space.

3) If its dimension is 3, the roots in that subspace must form a sub-root-system of rank 3. Since we are in type $D_4$, the possibilities for such a sub-root-system would be

$$(A_1)^3 \text{ and } A_3.$$

(There is no subsystem of type $A_1 \times A_2$ in a root system of type $D_4$.) In the first case (which means that we have three orthogonal roots), there is a fourth root in $R$ orthogonal to all of them, and we are back in the excluded case of a one-dimensional $\mathbb{R}[\sigma_A]$-module. The second case is impossible because an anisotropic form of type $A_3$, is not split by an extension of degree 6.

4) Remains the case that for every root $\alpha$, the $\sigma_A$-conjugates of $\alpha$ span the whole space $V$. In this case, the minimal polynomial of $\sigma_A$ would have to have degree 4, hence be

$$(T^2 + T + 1)(T^2 - T + 1) = T^4 + T^2 + 1.$$

Applying the proof of Lemma 5.3.1 to $\sigma_A^2$ we see that for any root $\alpha$, $\alpha$ and $\sigma_A^2(\alpha)$ are a basis of a root system of type $A_2$. Choosing an $A(R)$-invariant scalar product $\langle .\,, . \rangle$ on $V$ and fixing the (unique) root length as $\langle \beta, \beta \rangle = 1$ for every root $\beta$, we have that $\langle \alpha, \sigma_A(\alpha) \rangle = \langle \sigma_A(\alpha), \sigma_A^2(\alpha) \rangle \in \{0, \pm\frac{1}{2}\}$. However, the orthogonal complement of any sub-root-system of type $A_2$ in $D_4$ contains no roots, so $\langle \alpha, \sigma_A(\alpha) \rangle = 0$ is excluded.

But if $\langle \alpha, \sigma_A(\alpha)\rangle = \frac{1}{2}$ (i.e. the angle between them is 60°), one computes (using $\langle \alpha, \sigma_A^2(\alpha)\rangle = -\frac{1}{2}$)

$$\langle \sigma_A(\alpha) - \alpha - \sigma_A^2(\alpha), \sigma_A(\alpha) - \alpha - \sigma_A^2(\alpha)\rangle = 0$$

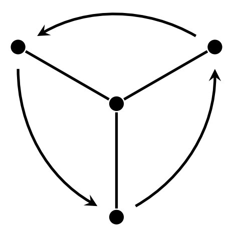
hence $\sigma_A(\alpha) = \alpha + \sigma_A^2(\alpha)$. With a similar computation for $\sigma_A^3(\alpha)$, one sees that the $\sigma$-conjugates of $\alpha$ remain in a two-dimensional subspace, in contradiction to our assumption. Analogously, if $\langle \alpha, \sigma_A(\alpha)\rangle = -\frac{1}{2}$ (i.e., the angle is 120°), we have

$$\langle \sigma_A(\alpha) + \alpha + \sigma_A^2(\alpha), \sigma_A(\alpha) + \alpha + \sigma_A^2(\alpha)\rangle = 0,$$

hence $\sigma_A(\alpha) = -\alpha - \sigma_A^2(\alpha)$, and again one arrives at a contradiction. □

The upshot of all this is:

Corollary 5.5.10. — *If, over every $\mathfrak{p}$-adic field, there is no anisotropic form with diagram*

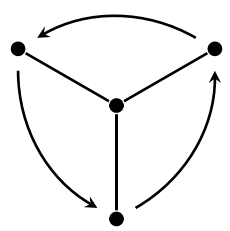


*and split by the cubic fixed field of the twisted Galois action (case Ia above), then there is no anisotropic form with that diagram over any $\mathfrak{p}$-adic field at all.*

**Case II (Twisted Galois action via $S_3$)**

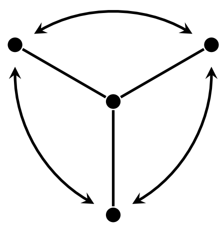
We can reduce this case to the same case I.a above, showing

Proposition 5.5.11. — *If, over every $\mathfrak{p}$-adic field, there is no anisotropic form with diagram*



*and split by the cubic fixed field of the twisted Galois action (case Ia above), then there is no anisotropic form of type $D_4$ over any $\mathfrak{p}$-adic field at all.*

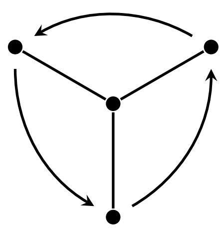
*Proof.* — We have to rule out diagrams



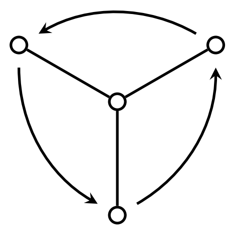
where the Galois group operates via all of the symmetric group $S_3$. Assume a form $\mathfrak{g}$ over a $\mathfrak{p}$-adic field $k$ with such a diagram given. Let $K$ be the fixed field of the twisted Galois action, so that $\mathrm{Gal}(K|k) \simeq S_3$, and let $F$ be the fixed field of the alternating subgroup $A_3$, so that $F|k$ is the unique quadratic subextension of $K|k$:

$$\begin{array}{c} K \\ \Big| \, A_3 \simeq \mathbb{Z}/3 \\ F \\ \Big| \, 2 \\ k \end{array}$$

In the diagram of $\mathfrak{g}_F$, the twisted Galois action operates via $A_3 \simeq \mathbb{Z}/3$, so it is



or



The first case is excluded by our assumption and the last corollary. So we are in the second case. Then $K$ is a splitting field for $\mathfrak{g}$. As usual, choose $\mathfrak{t}, \mathfrak{s}, R, \ldots$ and an $A(R)$-invariant scalar product on $V$, the vector space around the root system. Let $\tau$ be a 3-cycle and $\sigma$ a transposition in $S_3$. Note that due to anisotropy ($X = X_0 = X_a$), we have

$$v + \tau_A(v) + \tau_A^2(v) + \sigma_A(v) + \sigma_A\tau_A(v) + \sigma_A\tau_A^2(v) = 0$$

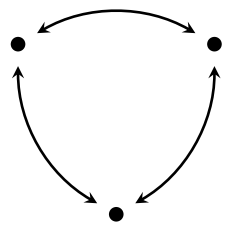
for all $v \in V$, in particular there is no $v \neq 0$ fixed by both $\tau_A$ and $\sigma_A$. We look at the $\mathbb{R}[S_3]$-modules generated by a root $\alpha$ (and what we just said amounts to: this module is not allowed to contain the trivial representation).

1) If one such module is one-dimensional over $\mathbb{R}$, $\tau_A$ operates trivially and $\sigma_A$ as $-\mathrm{id}$ on it (in the language of $S_3$-representations, it is the sign representation). The roots in its orthogonal complement are a system of type $(A_1)^3$, and $\tau_A$ must permute the factors in this triple cyclically. If $\sigma_A$ operated trivially on that triple, then for $v \neq 0$ in it, $v + \tau_A(v) + \tau_A^2(v) \neq 0$ would be a vector fixed by all $S_3$, which is impossible.

One easily sees (using that $\sigma\tau\sigma = \tau^2$ which excludes $\sigma_A = -\operatorname{id}$ on the triple) that choosing any root $\beta$ in the triple, with

$$\beta_1 := \beta, \quad \beta_2 := \tau_A(\beta), \quad \beta_3 = \tau_A^2(\beta),$$

$\sigma_A$ must fix one $\beta_i$ and $\sigma_A(\beta_j) = -\beta_k$ for the other two indices $j, k \neq i$. The Lie subalgebra of $\mathfrak{g}$ generated by the root spaces of this $(A_1)^3$ thus has the full $S_3 \simeq \operatorname{Gal}(K|k)$ operating on it, and its diagram is



It follows from Section 4.1 (cf. remark 3.4.1) that this Lie algebra is the scalar restriction $R_{K|k}\mathfrak{a}$ of $\mathfrak{a}$ := the anisotropic $K$-form of $\mathfrak{sl}_2$. In particular, it would become $\mathfrak{a}^3$ after scalar extension to $K$, hence not be split by $K$, contradicting our assumption.

2) By the representation theory of $S_3$, a two-dimensional module must be the so-called standard representation. The roots in it would be a subsystem of type $A_2$, with $\tau_A$ operating as rotation by 120° in one direction, and $\sigma_A$ operating as a reflection $s_\beta$ associated with one of the contained roots $\beta$ (i.e. $S_3$ operates as the full Weyl group $W(A_2)$). However, the relations between the roots in a system of type $D_4$ make it impossible to define actions of $\tau_A$ and $\sigma_A$ on all of $V$ consistently. (Any root outside our subsystem of type $A_2$ will be non-orthogonal to several roots inside it; and all these angles will have to be held invariant under the operations of $\sigma_A$ and $\tau_A$, often already prescribing how these elements will act on the roots; going through the calculations, one always reaches a contradiction.)

3) If the $\mathbb{R}[S_3]$-module generated by a root $\alpha$ is three-dimensional, the root system in it must be of rank 3, so of type $(A_1)^3$ or $A_3$ (there is no subsystem of type $A_2 \times A_1$ in $D_4$). The first case leads, via orthogonal complement, back to a one-dimensional module already excluded; whereas the anisotropic subalgebra corresponding to a root system of type $A_3$ would not be split by an extension of degree 6.

4) According to the representation theory of $S_3$, there is no cyclic $\mathbb{R}[S_3]$-module of $\mathbb{R}$-dimension $> 3$ without containing the trivial representation. □

Remark 5.5.12. — For the quasi-split trialitarian types, it remains to add to the description in 4.6 which Galois extensions $K|k$ with $\operatorname{Gal}(K|k)$ isomorphic to $\mathbb{Z}/3$ or $S_3$ exist for a $\mathfrak{p}$-adic field $k$.

i. Cubic extensions of $k$ are either unramified or totally ramified. The unique unramified one is Galois. The ramified ones are given (not one-to-one) by Eisenstein polynomials $X^3 + aX^2 + bX + c \in \mathcal{O}_k[X]$ where $a, b, c$ are divisible by $\pi_k$, and $\pi_k^2 \nmid c$. Its discriminant is $\Delta = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$, and it is known that the corresponding extension is Galois if and only if $\Delta \in k^{*2}$.

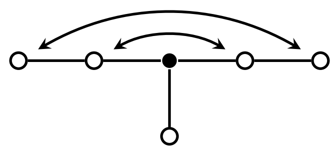
If $p \neq 3$, this can be arranged if and only if $k$ contains a primitive third root of unity; equivalently $3 \mid q - 1$, where $q$ is the order of the residue field of $k$; or equivalently, $-3 \in k^{*2}$. In this case, the $\mathbb{Z}/3$-extensions are in obvious bijection with the subgroups of $k^*/(k^*)^3$ of order 3; in particular $k(\sqrt[3]{\pi_k})$ is a ramified Galois cubic extension of $k$.

ii. An extension $K|k$ with $\mathrm{Gal}(K|k) \cong S_3$ is the Galois closure of any of its three subextensions $L|k$ with $[L : k] = 3$, which are not Galois; and conversely, any non-Galois cubic extension $L|k$ has such a $K|k$ as normal closure. As noted in part i., a non-Galois cubic $L|k$ is given by an Eisenstein polynomial of degree 3 whose discriminant $\Delta$ is *not* a square in $k$. We then have $K = L(\sqrt{\Delta})$, and the inertia degree $f(K|k)$ is 1 or 2, according to whether $f(k(\sqrt{\Delta})|k)$ is 1 or 2, i.e. whether $v_k(\Delta)$ is odd or even. As Weisfeiler's table claims, both cases can occur: if $k$ does *not* contain a primitive third root of unity, the splitting field $K$ of $X^3 - \pi_k$ is an example, for which $f(K|k) = 2$ if $p \neq 3$ and $f(K|k) = 1$ if $k = \mathbb{Q}_3$.

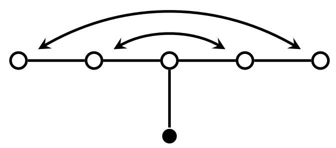
### 5.5.7. $\mathbf{E_6}$, outer forms

PROPOSITION 5.5.13. — *The only outer forms of type $E_6$ over $k$ are the quasi-split ones which we described in Section 4.6 as being in bijection with $k^*/k^{*2}$.*

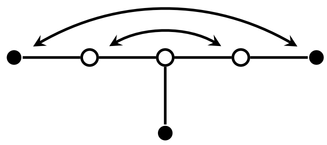
As noted in Section 5.2, the cohomological approach shows that the outer forms of type $E_6$ are quasi-split. But we can do this without Kneser's theorem. First, we rule out other diagrams with white vertices. If the "lower vertex" $\alpha_2$ is white, erasing it must give an admissible diagram of outer type $A_5$ with at least one black vertex, so the whole diagram is



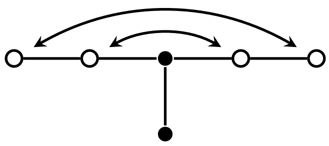
But then erasing the two proper orbits leaves a non-admissible diagram of type $A_2$. So $\alpha_2$ is black. Now the same reasoning rules out

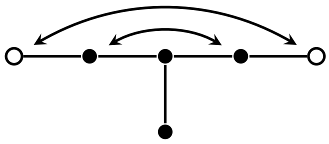


and

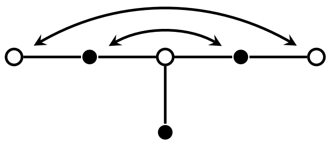


Erasing the outer proper orbit in

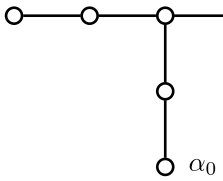




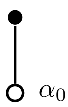
and comparing with the classification of type $D_4$ shows that these diagrams are not $k$-admissible. Neither is



as can be seen, for example, by the generalised erasing method with which we excluded the diagram (*) of type $E_7$ in 5.5.5: The completed Dynkin diagram for $E_6$ is




and erasing the upper middle vertex in the diagram in question leaves a connected component

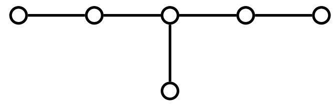



which does not occur in type $A_2$ over any field.

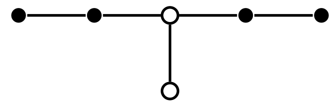
Invoking 4.3.1 and 4.3.6 we see that the all-black *anisotropic form* is the only possibility left. So the proposition is reduced to the following one, which we can prove with our methods (although, to be fair, relying on the non-existence of anisotropic *inner* forms of type $E_6$):

Proposition 5.5.14. — *There is no anisotropic outer form of type $E_6$ over $k$.*

*Proof.* — Assume there is one, and let $K$ be the quadratic extension of $k$ which is the fixed field of the twisted action. $\mathfrak{g}_K$ is an inner form of type $E_6$, hence by their classification either the split one



or the one with diagram:



In the first case, $\mathfrak{g}_K$ split, we choose $\mathfrak{t}, \mathfrak{s}, R \dots$ and a generator $\sigma \in \mathrm{Gal}(K|k)$ as usual and see that $\sigma_A = -\,\mathrm{id}$, hence any given root $\alpha \in R$ is part of a Galois-stable $A_1$ (namely, $\pm\alpha$); but the roots orthogonal to $\alpha$ (with respect to an $A(R)$-invariant scalar product) form a root system of type $A_5$ and are again stable under $\sigma_A$, giving rise to an anisotropic subalgebra of type $A_5$ which is split by a quadratic extension, contradicting what we know about type $A$.

So $\mathfrak{g}_K$ must be the non-split form with the diagram above, which is split by any extension $L|K$ of degree 3. Choose one such that $\mathrm{Gal}(L|k) \simeq \mathbb{Z}/6 \simeq \mathbb{Z}/2 \times \mathbb{Z}/3$ (the unramified extension would do).

Again we choose $\mathfrak{t}, \mathfrak{s}, R, \dots$, a generator $\sigma \in \mathrm{Gal}(L|k)$, and an $A(R)$-invariant scalar product on $V$. We know that $\sigma_A \in GL(V)$ is of order 6 and operates without eigenvalue 1, so its minimal polynomial has to be a divisor of

$$T^5 + T^4 + T^3 + T^2 + T + 1 = (T+1)(T^2+T+1)(T^2-T+1)$$

and we look at the cyclic $\mathbb{R}[\sigma_A]$-modules generated by roots, case by case arriving at a contradiction:

1) Not all those modules can be of $\mathbb{R}$-dimension 1, since that would imply $\sigma_A = -\,\mathrm{id}$.

2) If there was a two-dimensional one, the roots contained in it would be a system either of type $A_1 \times A_1$ or of type $A_2$.

In the first case, one easily sees $\sigma_A^2(\alpha)$ is either $\alpha$ or $-\alpha$. If it is $\alpha$, $\alpha + \sigma_A(\alpha)$ is fixed by $\sigma_A$, hence 0, so $\sigma_A = -\,\mathrm{id}$ and the module is not two-dimensional; if it is $-\alpha$, we have $\sigma_A^2 = -\,\mathrm{id}$ and $\sigma_A$ has $\pm\sqrt{-1}$ as eigenvalues over an algebraic closure, in contradiction to what we know about the minimal polynomial.

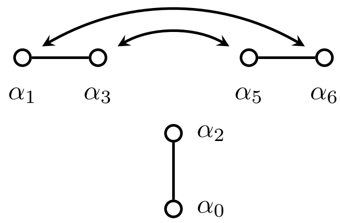
In the second case, on our two-dimensional module, $\sigma_A$ has minimal polynomial $T^2+T+1$, i.e. operates by rotating the roots by 120°. (If it were $T^2-T+1$, $\sigma_A$ would rotate by 60° and we would have an outer form, which cannot be anisotropic.) The roots orthogonal to any system of type $A_2$ in $E_6$ form a system of type $A_2 \times A_2$, Galois-stable in our case. If each factor in this pair were Galois-stable, again $\sigma_A$ would have minimal polynomial $T^2 + T + 1$ on both of them, hence on all $V$, which is impossible since it is of order 6. So $\sigma_A$ has to "switch" the factors. Actually, choose any $\beta$ in that $A_2 \times A_2$; then we can already name the roots conveniently, so that $\sigma_A$ maps

$$\begin{aligned}\alpha_1 := \beta \longmapsto \alpha_6 := \sigma_A(\beta) \longmapsto \alpha_3 := \sigma_A^2(\beta)\\ \longmapsto \alpha_5 := \sigma_A^3(\beta) \longmapsto -\alpha_1 - \alpha_3 \longmapsto -\alpha_5 - \alpha_6 \longmapsto \alpha_1\end{aligned}$$

and

$$\alpha_0 := \alpha \longmapsto \alpha_2 := \sigma_A(\alpha) \longmapsto -\alpha_2 - \alpha_0$$

in the diagram




This diagram is related to a basis of the whole root system by dropping $\alpha_0$ and introducing a root $\alpha_4$ with the relation

$$\alpha_0 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6.$$

But this relation forces $\sigma_A$ to send the root $\alpha_4$ to $-\alpha_2 - \alpha_3 - 2\alpha_5 - \alpha_6$, which is not a root, contradicting $\sigma_A(R) = R$.

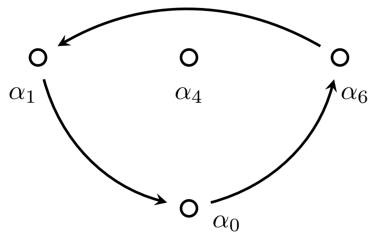
3) A three-dimensional module generated by a root would contain a root system of rank 3 corresponding to an anisotropic form. Type $A_1 \times A_2$ is excluded by easy combinatorics; type $A_3$ would not be split by our field extension of degree 6. Remains the case $A_1 \times A_1 \times A_1$. Orthogonal to such a subsystem in $E_6$ would be a fourth $A_1$, stable under $\sigma_A$ and hence with $\sigma_A = -\operatorname{id}$ on it. On our three-dimensional $A_1 \times A_1 \times A_1$, one readily sees that $\sigma_A^3 = \pm\operatorname{id}$, and "+" is impossible since otherwise for any root in there, $\alpha + \sigma(\alpha) + \sigma^2(\alpha) \neq 0$ would be fixed by $\sigma$. We name basis roots conveniently so that $\sigma_A$ maps

$$\alpha_1 \mapsto -\alpha_0 \mapsto \alpha_6 \mapsto -\alpha_1 \mapsto \alpha_0 \mapsto -\alpha_6$$

and

$$\alpha_4 \mapsto -\alpha_4$$

in the diagram




Now those roots can be amended by roots $\alpha_3, \alpha_5$ and $\alpha_2$ uniquely determined by being at angle 120° to both $\alpha_1$ and $\alpha_4$ (resp. $\alpha_4$ and $\alpha_6$, resp. $\alpha_4$ and $\alpha_0$), so that $\alpha_1, \ldots, \alpha_6$ are again a basis of $R$ and $\alpha_0$ is the negative of the highest root in that basis. One checks that this forces $\sigma_A$ to act on them as follows:

$$\alpha_3 \longmapsto -\alpha_2 \longmapsto \alpha_5 \longmapsto -\alpha_3 \longmapsto \alpha_2 \longmapsto -\alpha_5$$

so that, like in the case $D_4$ (and indeed we have a subsystem of type $D_4$ here with basis $\alpha_2, \ldots, \alpha_5$), we find that $\sigma_A$ operates as $-\mathrm{id}$ on the subsystem consisting of

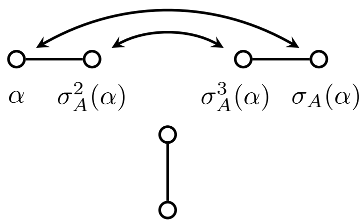
$$\{\pm\alpha_4, \pm(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5), \pm(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)\}$$

This subsystem is of type $A_2$ and would give rise to an anisotropic form of this type, but an *outer* form (since $-\mathrm{id} \notin W(A_2)$), which does not exist. Contradiction.

4) On a four-dimensional cyclic module, $\sigma_A$ would have to operate with minimal polynomial

$$(T^2 + T + 1)(T^2 - T + 1) = T^4 + T^2 + 1.$$

Exactly as in the case $D_4$ (see step 4 in the proof of Lemma 5.5.9) one shows that $\alpha$ and $\sigma_A^2(\alpha)$ form a basis of a sub-root-system of type $A_2$; if $\sigma_A(\alpha)$ was not orthogonal to $\alpha$, one would arrive at a contradiction (the module would have $\mathbb{R}$-dimension 2) like there. So necessarily $\sigma_A(\alpha), \sigma_A^3(\alpha)$ are the basis of a root system of type $A_2$ orthogonal to the other one, so the roots in our module are a system of type $A_2 \times A_2$. But orthogonal to this, and Galois-stable, is another subsystem of type $A_2$, and we are back in the case we ruled out in step 2:



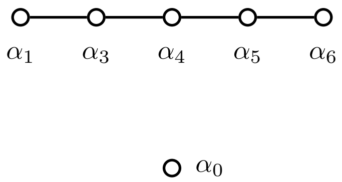

5) A five-dimensional module generated by a root would contain a root system of rank 5 corresponding to an anisotropic form. According to our classification of the classical types, there is no anisotropic form of type $D_5$. A subsystem of type $D_4 \times A_1$ does not exist in $R$, all other sub-root systems of rank 5 would be products of some $A_n$

with the $n$'s adding up to 5. Except for $A_5$, all of these are easily ruled out by basic combinatorics, or reduced to cases already treated; remark also that there are now 5 mutually orthogonal roots in $R$, hence no subsystem of type $(A_1)^5$.

As for the case $A_5$, the orthogonal complement of such a sub-root-system is a Galois-invariant subsystem of type $A_1$ on which $\sigma_A = -\operatorname{id}$; name one of its roots $\alpha_0$. On the $A_5$ factor, being anisotropic and split by our degree 6 extension, by corollary 5.3.2 we can name the basis roots $\alpha_1, \alpha_3, \ldots, \alpha_6$ so that $\sigma_A$ maps

$$\alpha_1 \longmapsto \alpha_3 \longmapsto \alpha_4 \longmapsto \alpha_5 \longmapsto \alpha_6 \longmapsto -\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6.$$

The diagram to look at here is




and again we can switch to a basis of our original $R$ by dropping $\alpha_0$ and introducing $\alpha_2$ subject to

$$\alpha_0 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6.$$

With this one computes that $\sigma_A$ maps

$$\alpha_2 \mapsto -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \mapsto -\alpha_2$$

so that the $\mathbb{R}[\sigma_A]$-module generated by $\alpha_2$ is of $\mathbb{R}$-dimension 2 and thus already shown to be impossible: It would be an outer form of type $A_2$ again.

6) Since the degree of the minimal polynomial of $\sigma_A$ is $\leq 5$, no cyclic $\mathbb{R}[\sigma_A]$-module of $\mathbb{R}$-dimension $\geq 6$ exists.

□

# CHAPTER 6

# CONCLUDING REMARKS

## 6.1. The field $k = \mathbb{R}$

As noted in the introduction, already E. Cartan had classified the semisimple Lie algebras over $\mathbb{R}$, and the theory of real Lie algebras is strongly connected to that of real Lie groups. A comprehensive treatment is [**Hel**]. For Lie algebras, [**OVi**, Chapter 5] uses methods close to ours, including split toral ("diagonalizable") subalgebras, rational root systems and Satake-Tits diagrams, and combines them with classical "real" methods, namely the Cartan involution and Cartan decomposition.

Araki's paper [**Ara**] uses only a few basic lemmata – some of which, however, are peculiar to $\mathbb{R}$ or at least real closed fields – and then goes through all possible Satake-Tits diagrams with lots of combinatorial reasoning. A very streamlined version of this is M. Sugiura's appendix to [**Sat3**], where much is built on Cartan's theorem quoted in 5.2.5 that *for every type there exists a unique anisotropic ("compact") real form.* With the notation of Section 4.2.1, this compact real form corresponds to the cocycle given by $a_\sigma \leftrightarrow (-\operatorname{id}, (-1)_{\alpha\in\Delta})$ where of course $\sigma$ is the complex conjugation. The twisted Galois action $t(\sigma)$ on this compact form is always given by the opposition involution from Section 4.3.1, so that the compact forms are outer forms for the types $A_n$ ($n \geq 2$), $D_n$ ($n$ odd) and $E_6$, and inner forms for all other types. (The parity distinction for type $D_n$ was visible in the proof of 4.5.15.)

For comprehensive lists of Satake-Tits diagrams, rational root systems and the connection to Cartan's labelling of the forms, see [**OVi**, Table 9] and [**Oni**, Table 5].

## 6.2. $k$-rational approaches (Allison, Seligman)

Let $k$ be a field with $\operatorname{char}(k) = 0$. Allison's paper [**All**] proposes an isomorphism theorem for central simple (= absolutely simple, see Section 4.1) Lie algebras which is "$k$-rational" in the sense that both its statement and proof do not make use of scalar extension. It starts from maximal split toral subalgebras $\mathfrak{s}$ (called $\mathcal{T}$ there) and corresponding $k$-rational root systems $\overline{R}$, like in our Section 3.1.1. The precise statement of the criterion for isomorphism of two such Lie algebras $(\mathfrak{g}, \mathfrak{s})$ and $(\mathfrak{g}', \mathfrak{s}')$ is a little technical, but it can be summarised as

▷ an isomorphism $(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}), \mathfrak{s}) \xrightarrow{\simeq} (\mathfrak{z}_{\mathfrak{g}'}(\mathfrak{s}'), \mathfrak{s}')$, which induces an isomorphism of the rational root systems $\overline{R} \simeq \overline{R}'$, which further

- ▷ is compatible with some extra structures corresponding to the $k$-rational Weyl groups, and
- ▷ for "very few" roots $\alpha \in \overline{R}$ and corresponding $\alpha' \in \overline{R}'$, an isomorphism of the root spaces, compatible with everything above.

"Very few" here means that only one root for every root length in $\overline{R}$ has to be taken, and so "very few" has cardinality 1 if $\overline{R}$ is of type $A, D$ or $E$, cardinality 3 if $\overline{R}$ is of type $BC_n$ with $n \geq 2$, and cardinality 2 in all other cases.

Note that isomorphism of the anisotropic kernels is included in the first point. The compatibility with the Weyl group structure allows the reduction to few root spaces, since the roots are permuted by the Weyl group.

Along the way, some interesting results on Lie algebras are achieved via their rational root systems. For example [**All**, Cor. 6.6] says that an absolutely simple $\mathfrak{g}$ with rational root system $\overline{R}$ of type $D$ or $E$ is automatically split (i.e. $R = \overline{R}$); one can indeed check directly that our method from Section 3.4.1 can never produce a root system of these types (except in the split case). The "rational" method uses much more calculation with ideals and subalgebras. See Seligman's [**Sel2**] and [**Sel3**] for an extensive version of this approach.

## 6.3. Explicit constructions for exceptional types

Besides the connection of the classical types to certain associative algebras like we have presented it, there are similar constructions for the exceptional types. One highlight is "Freudenthal's magic square" which in a way gives a construction for all exceptional types at once; except for Freudenthal's works, see e.g. [**Tit2**] and [**Vin**]. In [**GPe**], the Satake-Tits diagrams for some forms of type $E_6$ in this construction are computed.

We mention one recent invention: In [**Inv**, X], the authors construct *trialitarian algebras* which produce the trialitarian types of our Lie algebras. The idea is close in spirit to the construction for the classical types. Like it was necessary there, in the type $AII$, to allow as underlying objects *pairs* of associative algebras (which were "linked" by a switching involution), here one has *triples* of algebras which are linked by an extra structure. This allows to catch the extra operation of the symmetric group $S_3$ and gives further insights into phenomena like the one mentioned in 4.5.20.ii and 5.4.7: Here it turns out that each of the algebras $(M_4(Q), \iota_1)$ and $(M_8(k), \iota_2)$ is a factor of the Clifford algebra of the other.

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We study semisimple Lie algebras over fields of characteristic zero, with emphasis on $\mathfrak{p}$-adic fields and aiming at classification. We first transfer parts of the structure theory of reductive algebraic groups to our setting, with some variations. Classifying invariants are attached to Lie algebras and visualised with Satake-Tits diagrams. We give necessary and sufficient criteria for these diagrams. Over general fields of characteristic zero, we then classify all quasi-split forms, and we adapt an older classification theory for the classical types $A$-$D$ to our language. Finally we focus on $\mathfrak{p}$-adic fields, where we achieve a complete classification by combination of certain well-known properties of these fields with our general results and methods, and we discuss the relation of this with a theorem of Kneser. This extends work by Weisfeiler.

Nous étudions les algèbres de Lie semi-simples sur les corps de caractéristique nulle, où l'accent est mis sur les corps $p$-adiques, l'objectif étant leur classification. Nous transférons d'abord certaines parties de la théorie de la structure des groupes réductifs dans notre contexte, avec quelques variations. Des invariants classifiants sont attachés aux algèbres de Lie et sont visualisés à l'aide de diagrammes de Satake-Tits. Nous donnons des critères nécessaires et suffisants pour ces diagrammes. Sur les corps généraux de caractéristique nulle, nous classifions ensuite toutes les formes quasi-déployées et nous traduisons une théorie ancienne de classification pour les types classiques $A$-$D$ dans notre langue. Nous mettons enfin l'accent sur les corps $p$-adiques, où nous obtenons une classification complète par combinaison de certaines propriétés bien connues sur ces corps avec nos résultats généraux et nos méthodes, et nous abordons la relation de ces résultats avec un théorème de Kneser. Tout cela prolonge un travail de Weisfeiler.