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A COMMUTATIVE $\mathbb{P}^1$-SPECTRUM REPRESENTING MOTIVIC COHOMOLOGY OVER DEDEKIND DOMAINS

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A COMMUTATIVE $\mathbb{P}^1$-SPECTRUM REPRESENTING MOTIVIC COHOMOLOGY OVER DEDEKIND DOMAINS

Markus Spitzweck

Abstract. – We construct a motivic Eilenberg-MacLane spectrum with a highly structured multiplication over general base schemes which represents Levine's motivic cohomology, defined via Bloch's cycle complexes, over smooth schemes over Dedekind domains. Our method is by gluing p-completed and rational parts along an arithmetic square. Hereby the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser's version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As an application the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of motives with integral coefficients.

Our spectrum is compatible with base change giving rise to a formalism of six functors for triangulated categories of motivic sheaves over general base schemes including the localization triangle.

Further applications are a generalization of the Hopkins-Morel isomorphism and a structure result for the dual motivic Steenrod algebra in the case where the coefficient characteristic is invertible on the base scheme.

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CHAPTER 1

INTRODUCTION

This paper furnishes the construction of a motivic Eilenberg-MacLane spectrum in mixed characteristic. One of our main purposes is to use this spectrum for the definition of triangulated categories of motivic sheaves with integral (and thus also arbitrary) coefficients over general base schemes. These categories will satisfy properties combining and expanding on properties of triangulated categories of motives which have already been constructed. In [**7**] Cisinski-Déglise develop a theory of Beilinson motives yielding a satisfying theory of motives with rational coefficients over general base schemes. This theory is equivalent to an approach due to Morel where one considers modules over the positive rational sphere spectrum, see loc. cit. Voevodsky constructed triangulated categories of motives over a (perfect) field ([**54**], [**39**]) in which integral motivic cohomology of smooth schemes is represented. In the Cisinski-Déglise/Morel category over a regular base the resulting motivic cohomology are Adams-graded pieces of rationalized K-theory, which fits with the envisioned theory of Beilinson. In [**44**] modules over the motivic Eilenberg-MacLane spectrum over a field are considered and it is proved that those are equivalent to Voevodsky's triangulated categories of motives in the characteristic 0 case. This result has recently been generalized to perfect fields [**28**] where one has to invert the characteristic of the base field in the coefficients. Étale motives are developed in [**2**] and [**6**].

We build upon these works and construct motivic categories using motivic stable homotopy theory. More precisely we define objects with a (coherent) multiplication in the category of $\mathbb{P}^1$-spectra over base schemes and consider as in [**44**] their module categories. The resulting homotopy categories are defined to be the categories of motivic sheaves.

This family of commutative ring spectra is cartesian, i.e., for any map between base schemes $X \to Y$ the pullback of the ring spectrum over Y compares via an equivalence to the ring spectrum over X. This is equivalent to saying that all spectra pull back from $\mathrm{Spec}(\mathbb{Z})$. We thus give an affirmative answer to a version of a conjecture due to Voevodsky [**56**, Conjecture 17].

To ensure good behavior of our construction our spectra have to satisfy a list of desired properties. Over fields the spectra coincide with the usual motivic Eilenberg-MacLane spectra (this ensures that over fields usual motivic cohomology is represented in our categories of motivic sheaves). Rationally we recover the theory of Beilinson motives, because the rationalizations of our spectra are isomorphic to the respective Beilinson spectra, and there is a relationship to Levine's motivic cohomology defined using Bloch's cycle complexes in mixed characteristic ([**36**]).

To ensure all of that we first construct a spectrum over any Dedekind domain $\mathscr{D}$ of mixed characteristic satisfying the following properties: It represents Bloch-Levine's motivic cohomology of smooth schemes over $\mathscr{D}$ (Corollary 7.19), it pulls back to the usual motivic Eilenberg-MacLane spectrum with respect to maps from spectra of fields to the spectrum of $\mathscr{D}$ (Theorem 8.22) and it is an E_∞-ring spectrum. (We remark that such an E_∞-structure can always be strictified to a strict commutative monoid in symmetric $\mathbb{P}^1$-spectra by results of [**24**].)

The latter property makes it possible to consider the category of highly structured modules over pullbacks of the spectrum from the terminal scheme (the spectrum of the integers), thus defining triangulated categories of motivic sheaves $\mathrm{DM}(X)$ over general base schemes X such that over smooth schemes over Dedekind domains of mixed characteristic the Ext-groups compute Bloch-Levine's motivic cohomology (Corollary 7.20, using Theorem 8.25). For general base schemes we define motivic cohomology to be represented by our spectrum, i.e.,

$$H^i_{\mathrm{mot}}(X, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathsf{SH}(X)}(\mathbf{1}, \Sigma^{i,n} f^* \mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}) \cong \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}(0), \mathbb{Z}(n)[i]).$$

Here $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ is the structure morphism, $\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ is our spectrum over the integers and $\mathbf{1}$ is the sphere spectrum (the unit with respect to the smash product) in the stable motivic homotopy category $\mathsf{SH}(X)$. By the base change property these cohomology groups coincide with Voevodsky's motivic cohomology if X is smooth over a field. We note that the ring structure on our Eilenberg-MacLane spectrum gives the (bigraded) motivic cohomology groups a (graded commutative) ring structure, a property which was (to the knowledge of the author) missing for Levine's motivic cohomology. In particular we obtain a product structure on Chow groups of smooth schemes over Dedekind domains.

By the work of Ayoub [**1**] the base change property enables one to get a full six functor formalism for these categories of motivic sheaves including the localization triangle (Theorem 9.1).

We remark that the spectrum we obtain gives rise to motivic complexes over any base scheme X. More precisely one can extract objects $\mathbb{Z}(n)^X$ in the derived category of Zariski sheaves on the category of smooth schemes over X representing our motivic cohomology. There are unital, associative and commutative multiplication maps $\mathbb{Z}(n)^X \otimes^{\mathbb{L}} \mathbb{Z}(m)^X \to \mathbb{Z}(n+m)^X$ inducing the multiplication on motivic cohomology. (These multiplications are in fact part of a graded E_∞-structure (which follows from the existence of the strong periodization, see Section C), but we do not make this explicit since we have no application for this enhanced structure.) If X is a smooth scheme over a Dedekind domain of mixed characteristic or over a field we have isomorphisms $\mathbb{Z}(0)^X \cong \underline{\mathbb{Z}}$ and $\mathbb{Z}(1)^X \cong \mathcal{O}^*_{/X}[-1]$ (for the latter isomorphism see Theorem 7.10).

Moreover one can extract motivic Eilenberg-MacLane spaces. If X is as above there are isomorphisms $K(\mathbb{Z}(1),2)_X \cong \mathbb{P}^\infty_X$ (Proposition 10.7) and $K(\mathbb{Z}/n(1),1)_X \cong W_{X,n}$ (Proposition 10.8) in the motivic pointed homotopy category $\mathscr{H}_\bullet(X)$ of X. Here $W_{X,n}$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^\infty_X}(-n)$ on $\mathbb{P}^\infty_X$ with the zero section removed (a motivic lens space).

The motivic complexes can also be viewed as objects in the derived category of Zariski sheaves on all base schemes. As such they satisfy cdh-descent which can be shown by the methods of [**5**, §3].

Among our applications is a generalization of the Hopkins-Morel ismorphism (Theorem 10.3), relying on the recent work of Hoyois ([**27**], which in turn relies on work of Hoyois-Kelly-Østvær [**28**]). In certain cases it follows that the Eilenberg-MacLane spectrum is cellular (Corollary 10.4). We obtain a description of the dual motivic Steenrod algebra over base schemes over which the coefficient characteristic is invertible (Theorem 10.26). We note that one can ask if the statement of this theorem is valid over any base scheme (thus asking for a description of the smash product of the mod-p motivic Eilenberg-MacLane spectrum with itself in characteristic p).

The outline of this paper is as follows. In Section 3 we define motivic complexes over small sites and describe their main properties, most notably the localization sequence due to Levine (Theorem 3.1) and the relation to étale sheaves (where the Bloch-Kato conjecture enters) (Theorem 3.9).

In Section 4 an E_∞-spectrum $\mathsf{M}\mathbb{Z}$ is constructed with the main property that it represents motivic cohomology with finite coefficients (which follows from Corollary 4.1.2) and is rationally isomorphic to the Beilinson spectrum.

For the definition we use an arithmetic square, i.e., we first define p-completed spectra for all prime numbers p and glue their product along the rationalization of this product to the Beilinson spectrum (Definition 4.27).

The spectra with finite p-power coefficients which define the p-completed parts are constructed using truncated étale sheaves outside characteristic p and logarithmic de Rham-Witt sheaves at characteristic p.

Our spectrum is constructed in the world of complexes of sheaves of abelian groups and spectrum objects therein. By transfer of structure this also defines (E_∞-or commutative ring) spectra in the world of $\mathbb{P}^1$-spectra in motivic spaces.

In order to prove that $\mathsf{M}\mathbb{Z}$ represents integrally Bloch-Levine's motivic cohomology we define in Section 5 a second motivic spectrum $\mathcal{M}$ which by definition represents Bloch-Levine's integral motivic cohomology (and which will finally be isomorphic to $\mathsf{M}\mathbb{Z}$). To do that we introduce a strictification process for Bloch-Levine's cycle complexes to get a strict presheaf on smooth schemes over a Dedekind domain. Hereby we rely heavily on a moving lemma due to Levine (Theorem 5.8). Using a localization sequence for the pair $(\mathbb{A}^1, \mathbb{G}_m)$ we obtain bonding maps arranging the motivic complexes into a $\mathbb{G}_m$-spectrum (see Section 5.3). This section also contains the construction of an étale cycle class map (inspired by the construction in [**36**]) which is compatible with certain localization sequences (Proposition 5.2.3).

After treating motivic complexes over a field (Section 6) we give our comparison statements in Section 7. First we compute the exceptional inverse image of $\mathscr{M}$ with respect to the inclusion of a closed point into our Dedekind scheme (Theorem 7.4). Theorem 7.14 states that the rationalization $\mathscr{M}_{\mathbb{Q}}$ is just the Beilinson spectrum. Our main comparison statement is Theorem 7.18 which asserts a canonical isomorphism between $\mathscr{M}$ and $\mathsf{M}\mathbb{Z}$ as spectra.

Section 8 discusses base change. Here the Bloch-Kato filtration on p-adic vanishing cycles plays a key role to obtain the part of base change where the characteristics of the base field and of the coefficients coincide.

We treat the motivic functor formalism in Section 9. Section 10 contains the applications to the Hopkins-Morel isomorphism and the dual motivic Steenrod algebra.

The first two appendices discuss (semi) model structures on sheaf categories and algebra objects therein and definitions and properties of pullbacks of algebraic cycles.

Our motivic Eilenberg-MacLane spectrum is strongly periodizable in the sense of [**50**] (Theorem C.2, Remark 10.2). This shows that geometric mixed Tate sheaves with integral coefficients over a number ring or similar bases which satisfy a weak version of the Beilinson-Soulé vanishing conjecture can be modeled as representations of an affine derived group scheme along the lines of [**49**] (Corollary C.4).

We finally remark that it should be possible to generalize our strictification process in Section 5 to define a homotopy coniveau tower over Dedekind domains as in [**35**]. We will come back to this question in future work.

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CHAPTER 2

PRELIMINARIES AND NOTATION

For a site $\mathcal{S}$ and a category $\mathcal{C}$ we denote by $\mathrm{Sh}(\mathcal{S},\mathcal{C})$ the category of sheaves on $\mathcal{S}$ with values in $\mathcal{C}$. If R is a commutative ring we set $\mathrm{Sh}(\mathcal{S},R) := \mathrm{Sh}(\mathcal{S},\mathrm{Mod}_R)$, where Mod_R denotes the category of R-modules.

For a Noetherian separated scheme S of finite Krull dimension (such schemes we will call henceforth base schemes) we denote by Sch_S the category of separated schemes of finite type over S and by Sm_S the full subcategory of Sch_S of smooth schemes over S.

For $t \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$ we denote by $\mathrm{Sm}_{S,t}$ the site Sm_S equipped with the topology t.

For S and t as above we denote by S_t the site consisting of the full subcategory of Sm_S of étale schemes over S equipped with the topology t.

If m is invertible on S we write $\mathbb{Z}/m(r)^S$ for the sheaf $\mu_m^{\otimes r}$ on $S_{\text{ét}}$. If it is clear from the context we also write $\mathbb{Z}/m(r)$.

We let $\epsilon\colon \mathrm{Sm}_{S,\text{ét}} \to \mathrm{Sm}_{S,\mathrm{Zar}}$ and $\epsilon\colon S_{\text{ét}} \to S_{\mathrm{Zar}}$ be the canonical maps of sites.

If X is a presheaf of sets on Sm_S we let $R[X]_t$ be the sheaf of R-modules on $\mathrm{Sm}_{S,t}$ freely generated by X. If $Y \hookrightarrow X$ is a monomorphism we let $R[X,Y]_t := R[X]_t/R[Y]_t$.

For S the spectrum of a Dedekind domain we let Smc_S be the full subcategory of Sch_S of schemes X over S such that each connected component of X is either smooth over S or smooth over a closed point of S.

For an $\mathbb{F}_p$-scheme Y we let $W_n\Omega_Y^\bullet$ be the de Rham-Witt complex of Y. It is a complex of sheaves on $Y_{\text{ét}}$ with a multiplication. These complexes assemble to a complex of sheaves on the category of all $\mathbb{F}_p$-schemes. There are canonical epimorphisms $W_{n+1}\Omega_Y^\bullet \twoheadrightarrow W_n\Omega_Y^\bullet$ respecting the multiplication.

For Y as above let $\mathrm{dlog}\colon \mathcal{O}_Y^* \to W_n\Omega_Y^1$ be defined by $x \mapsto \frac{d\underline{x}}{\underline{x}}$, where $\underline{x} = (x,0,0,\ldots)$ is the Teichmüller representative of x.

The logarithmic de Rham-Witt sheaf $W_n\Omega^r_{Y,\log}$ is defined to be the subsheaf of $W_n\Omega_Y^r$ generated étale locally by sections of the form $\mathrm{dlog}x_1\ldots\mathrm{dlog}x_r$. Also $W_n\Omega^0_{Y,\log}$ is the constant sheaf on the abelian group $\mathbb{Z}/p^n$.

These sheaves assemble to a subcomplex $W_n\Omega^\bullet_{Y,\log}$ of $W_n\Omega^\bullet_Y$.

The $W_n\Omega^r_{Y,\log}$ assemble to a sheaf ν_n^r on the category of all $\mathbb{F}_p$-schemes. We set $\nu_n^r = 0$ for $r < 0$. There are natural epimorphisms $\nu^r_{n+1} \twoheadrightarrow \nu^r_n$.

We will also denote restrictions of ν_n^r to certain sites, e.g., to Y_{Zar} or $\mathrm{Sm}_{k,t}$, k some field of characteristic p, by ν_n^r.

For a base scheme S we let $\mathsf{SH}(S)$ be the stable motivic homotopy category and $\mathscr{H}_\bullet(S)$ the pointed $\mathbb{A}^1$-homotopy category of S.

If $\mathscr{A}$ is an abelian category we denote by $\mathsf{D}(\mathscr{A})$ its derived category. We denote by $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, R))$ the $\mathbb{A}^1$-localization of $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, R))$.

We sometimes use the notation f_*, f^* for a (non-derived or derived) push forward or pullback between sheaf categories corresponding to sites induced by a scheme morphism f. The precise sites which are used can always be read off from the source and target categories.

E_∞-structures are understood with respect to (the image of) the linear isometries operad (see [**14**, I.3.]). To be more precise, taking normalized chains of the linear isometries operad yields an operad E in $\mathrm{Cpx}_{\geq 0}(\mathsf{Ab})$, and all our algebras will live in categories $\mathcal{C}$ which receive a symmetric monoidal functor s from $\mathrm{Cpx}_{\geq 0}(\mathsf{Ab})$, and then we will call an algebra over $s(E)$ an E_∞-algebra in $\mathcal{C}$.

Let us give a statement about sheaf operations for the small sites which we will use throughout the text.

LEMMA 2.1. – *Let R be a commutative ring and $f\colon X \to Y$ a morphism between base schemes. Then for $t \in \{\text{ét}, \mathrm{Nis}\}$ the pullback functor $f^*\colon \mathrm{Sh}(Y_t, R) \to \mathrm{Sh}(X_t, R)$ is exact. If moreover f is a closed immersion then the same functor for $t = \mathrm{Zar}$ is also exact, and for $t \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$ the functor $f_*\colon \mathrm{Sh}(X_t, R) \to \mathrm{Sh}(Y_t, R)$ is exact and the functor $f^*f_*\colon \mathrm{Sh}(X_t, R) \to \mathrm{Sh}(X_t, R)$ is naturally isomorphic to the identity (via the unit of the adjunction), so the same holds on the level of derived categories for the endofunctor f^*f_* of $\mathsf{D}(\mathrm{Sh}(X_t, R))$.*

Proof. – We check the first statement on stalks: Let $U \to X$ be ĕtale an $u \in U$ a point. Let y be the image of u in Y and let $\mathcal{O}$ be the (strict) henselization of the local ring of U at u. Let k be the residue field of $\mathcal{O}$. Then the value of $f^*(F)$ ($F \in \mathrm{Sh}(Y_t, R)$) at $\mathcal{O}$ is the value of F at the henselization of $\mathcal{O}_{Y,y}$ relative to the k-point $\mathcal{O}_{Y,y} \to k$. The claim follows.

For f a closed immersion the second and third statement can be found in [**17**] before Lemma 2.1. The fourth statement can also be checked on stalks. □

CHAPTER 3

MOTIVIC COMPLEXES I

Let S be the spectrum of a Dedekind domain. For $X \in \mathrm{Smc}_S$ and $r \geq 0$ we denote by $\mathscr{M}^X(r) \in \mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$ Levine's cycle complex. A representative is the complex which has $z^r(\_, 2r-i)$ in cohomological degree i, see [**17**, §3], [**33**].

Here for an equidimensional scheme Y of finite type over a regular one-dimensional scheme, $z^r(Y, i)$ is the free abelian group on closed integral subschemes of $Y \times \Delta^i$ (Δ^i the algebraic i-simplex over $\mathrm{Spec}(\mathbb{Z})$) which intersect all faces properly. If Y' is of finite type over the same regular one-dimensional base we have $z^r(Y \coprod Y', i) \cong z^r(Y, i) \oplus z^r(Y', i)$.

For $r < 0$ we set $\mathscr{M}^X(r) = 0$. When it is clear from the context which X is meant we also write $\mathscr{M}(r)$. We also write $\mathscr{M}^X_{\text{ét}}(r)$ for $\epsilon^* \mathscr{M}^X(r)$ and $\mathscr{M}^X(r)/m$ for $\mathscr{M}^X(r) \otimes^{\mathbb{L}} \mathbb{Z}/m$ (since $\mathscr{M}^X(r)$ is scheme-wise a free abelian group we can also replace here the derived tensor product $\otimes^{\mathbb{L}}$ with the tensor product $\otimes$).

THEOREM 3.1 (Levine). – *Let $i \colon Z \to X$ be a closed inclusion in Smc_S of codimension c and $j \colon U \to X$ the complementary open inclusion. Then there is an exact triangle in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$*

(1) $\mathbb{R}i_* \mathscr{M}^Z(r-c)[-2c] \to \mathscr{M}^X(r) \to \mathbb{R}j_* \mathscr{M}^U(r) \to \mathbb{R}i_* \mathscr{M}^Z(r-c)[-2c+1].$

Proof. – This is [**33**, Theorem 1.7] (note the schemes in Smc_S are of finite type over S, so we can apply this result). □

COROLLARY 3.2. – *Let $i \colon Z \to X$ be a closed inclusion in Smc_S of codimension c. Then there is a canonical isomorphism*

$$\mathbb{R}i^! \mathscr{M}^X(r) \cong \mathscr{M}^Z(r-c)[-2c]$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$.

Proof. – This follows from Theorem 3.1 using Lemma 2.1. □

THEOREM 3.3. – *For $X \in \mathrm{Smc}_S$ we have $\mathscr{H}^k(\mathscr{M}^X(r)) = 0$ for $k > r$.*

Proof. – This is [**17**, Corollary 4.4] (note a connected scheme in Smc_S is either smooth overe S or over a field, so we can apply this result). □

THEOREM 3.4. – *Suppose $X \in \mathrm{Smc}_S$ is of characteristic p. Then there is an isomorphism*

$$\mathscr{M}^X(r)/p^n \cong \nu_n^r[-r] \tag{2}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^n))$.

Proof. – If X is smooth over a perfect field this is [**18**, Theorem 8.3]. The general case follows by a colimit argument (using [**29**, I. (1.10.1)]). □

COROLLARY 3.5. – *Let S be of mixed characteristic. Let p be a prime, $X \in \mathrm{Sm}_S$ and $\pi\colon X \to S$ the structure morphism. Let and $j\colon U := \pi^{-1}(S[\frac{1}{p}]) \to X$ be the open inclusion and $i\colon Z \hookrightarrow X$ the reduced closed complement. Then $\mathscr{H}^k(\mathbb{R}j_* \mathscr{M}^U(r) \otimes^{\mathbb{L}} \mathbb{Z}/p^n) = 0$ for $k > r$ and the natural map*

$$\mathscr{H}^r(\mathbb{R}j_*(\mathscr{M}^U(r)/p^n)) \to i_*\nu_n^{r-1} \tag{3}$$

induced by the triangle (1) *and the isomorphism* (2) *is an epimorphism.*

Proof. – This follows from Theorem 3.3, the exactness of i_* and the long exact sequence of cohomology sheaves induced by the exact triangle (1). □

LEMMA 3.6. – *Suppose $X \in \mathrm{Smc}_S$ is of characteristic p. Then the diagram*

$$\begin{array}{ccc} \mathscr{M}^X(r)/p^{n+1} & \xrightarrow{\cong} & \nu_{n+1}^r[-r] \\ \downarrow & & \downarrow \\ \mathscr{M}^X(r)/p^n & \xrightarrow{\cong} & \nu_n^r[-r] \end{array}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^{n+1}))$ *commutes.*

Suppose m is invertible on $X \in \mathrm{Smc}_S$. Then there is a cycle class map

$$\mathrm{cc}_m^X(r)\colon \mathscr{M}^X(r)/m \to \mathbb{R}\epsilon_*\mathbb{Z}/m(r) \tag{4}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/m))$.

We recall the definition of the cycle class map: For $V \in \mathrm{Smc}_S$ we denote by $c^r(V,n)$ the set of cycles (closed integral subschemes) of codimension r of $V \times \Delta^n$ which intersect all $V \times Y$ with Y a face of Δ^n properly.

Let $\mu_m^{\otimes r} \to \mathscr{G}$ be an injectively fibrant replacement in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{X,\text{ét}}, \mathbb{Z}/m))$.

Let $V \in \mathrm{Sm}_X$. For W a closed subset of V such that each irreducible component has codimension greater or equal to r set $\mathscr{G}^W(V) := \ker(\mathscr{G}(V) \to \mathscr{G}(V \setminus W))$.

As in [**36**, 12.3] there is a canonical isomorphism of $H^{2r}(\mathscr{G}^W(V))$ with the free $\mathbb{Z}/m$-module on the irreducible components of W of codimension r and the map $\tau_{\le 2r}\mathscr{G}^W(V) \to H^{2r}(\mathscr{G}^W(V))[-2r]$ is a quasi-isomorphism.

For $V \in X_{\text{ét}}$ denote by $\mathscr{G}^r(V,n)$ the colimit of the $\mathscr{G}^W(V \times \Delta^n)$ where W runs through the finite unions of elements of $c^r(V,n)$. The simplicial complex of $\mathbb{Z}/m$-modules $\tau_{\le 2r}\mathscr{G}^r(V,\bullet)$ augments to the simplicial abelian group $z^r(V,\bullet)/m[-2r]$. This augmentation is a levelwise quasi-isomorphism. We denote by $\mathcal{G}^r(V)$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r}\mathcal{G}^r(V,\bullet)$. Thus we get a quasi-isomorphism $\mathcal{G}^r(X) \to z^r(X)/m[-2r]$. Here for $V \in \mathrm{Smc}_S$ the complex $z^r(V)$ is defined to be the normalized complex associated to the simplicial abelian group $z^r(V,\bullet)$.

On the other hand for $V \in X_{\text{ét}}$ we have a canonical map $\mathcal{G}^r(V,n) \to \mathcal{G}(V \times \Delta^n)$ compatible with the simplicial structure. We denote by $\mathcal{G}'(V)$ the total complex associated to the double complex which is the normalized complex associated to $\mathcal{G}(V \times \Delta^\bullet)$. We have a canonical quasi-isomorphism $\mathcal{G}(V) \to \mathcal{G}'(V)$ and a canonical map $\mathcal{G}^r(V) \to \mathcal{G}'(V)$. The above groups and maps are functorial in $V \in X_{\text{ét}}$.

Thus we get a map

$$z^r(\_)/m[-2r] \cong \mathcal{G}^r \to \mathcal{G}' \cong \mathcal{G}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\text{ét}}, \mathbb{Z}/m))$. This is (the adjoint of) the cycle class map.

The étale sheafification of the cycle class map is an isomorphism in $\mathsf{D}(\mathrm{Sh}(X_{\text{ét}}, \mathbb{Z}/m))$, see [**17**, Theorem 1.2.4.].

Let $f\colon Y \to X$ be a flat morphism of schemes in Smc_S. Then there is a flat pullback $f^*\mathcal{M}^X(r) \to \mathcal{M}^Y(r)$.

Lemma 3.7. – *Let $f\colon Y \to X$ be a flat morphism of schemes in* Smc_S*. Suppose m is invertible on X. Then the diagram*

$$\begin{array}{ccc} f^*\mathcal{M}^X(r)/m & \xrightarrow{f^*\mathrm{cc}^X_m(r)} & f^*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \\ \downarrow & & \downarrow \\ \mathcal{M}^Y(r)/m & \xrightarrow{\mathrm{cc}^Y_m(r)} & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) \end{array}$$

commutes.

Proof. – This follows from the definition of the étale cycle class map. □

Lemma 3.8. – *Let $X \in \mathrm{Smc}_S$ and suppose m is invertible on X. Let $m'|m$. Then the diagram*

$$\begin{array}{ccc} \mathcal{M}^X(r)/m & \xrightarrow{\mathrm{cc}^X_m(r)} & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) \\ \downarrow & & \downarrow \\ \mathcal{M}^X(r)/m' & \xrightarrow{\mathrm{cc}^X_{m'}(r)} & \mathbb{R}\epsilon_*\mathbb{Z}/m'(r) \end{array}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/m))$ commutes.

Proof. – This follows from the definition of the étale cycle class map. □

THEOREM 3.9. – *Let $X \in \mathrm{Smc}_S$ and suppose m is invertible on X. Then there is an isomorphism*

$$\mathscr{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}\epsilon_* \mathbb{Z}/m(r))$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/m))$ *induced by the cycle class map.*

Proof. – By [**17**, Theorem 1.2.2.] (which we can apply since the Bloch-Kato conjecture is proven, [**58**]) we have

$$\mathscr{M}^X(r) \cong \tau_{\leq (r+1)} \mathbb{R}\epsilon_* \mathscr{M}^X_{\text{ét}}(r).$$

By Theorem 3.3 it follows that $\mathbb{R}^{r+1}\epsilon_* \mathscr{M}^X_{\text{ét}}(r) = 0$. Thus

$$\mathscr{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}\epsilon_* \mathscr{M}^X_{\text{ét}}(r)/m).$$

But by [**17**, Theorem 1.2.4.] we have

$$\mathscr{M}^X_{\text{ét}}(r)/m \cong \mathbb{Z}/m(r)$$

induced by the cycle class map (see the proof of [**17**, Theorem 1.2.4.]). This shows the claim. □

THEOREM 3.10. – *Let $i\colon Z \to X$ be a closed inclusion in Smc_S of codimension c and suppose m is invertible on X. Then there is a canonical isomorphism*

$$\mathbb{R}i^! \mathbb{Z}/m(r) \cong \mathbb{Z}/m(r-c)[-2c]$$

in $\mathsf{D}(\mathrm{Sh}(Z_{\text{ét}}, \mathbb{Z}/m))$.

Proof. – This is [**42**, Théorème 3.1.1.]. □

A consequence is the localization/Gysin exact triangle for étale cohomology.

COROLLARY 3.11. – *Let $i\colon Z \to X$ be a closed inclusion in Smc_S of codimension c and $j\colon U \to X$ the complementary open inclusion. Suppose m is invertible on X. Then there is an exact triangle*

$$i_*\mathbb{Z}/m(r-c)[-2c] \to \mathbb{Z}/m(r) \to \mathbb{R}j_*\mathbb{Z}/m(r) \to i_*\mathbb{Z}/m(r-c)[-2c+1]$$

in $\mathsf{D}(\mathrm{Sh}(X_{\text{ét}}, \mathbb{Z}/m))$ *(note i_* computes the derived push forward in this situation).*

Proof. – This follows from Theorem 3.10 and the corresponding exact triangle involving $\mathbb{R}i^!\mathbb{Z}/m(r)$. □

THEOREM 3.12. – *Let $i\colon Z \to X$ be a closed inclusion in Smc_S of codimension c and suppose m is invertible on X. Then the diagram*

$$\begin{array}{ccc}
\mathbb{R}i^! \mathscr{M}^X(r)/m & \xrightarrow{\cong} & \mathscr{M}^Z(r-c)/m[-2c] \\
\downarrow{\scriptstyle \mathbb{R}i^! \mathrm{cc}^X_m(r)} & & \\
\mathbb{R}i^! \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & & \downarrow{\scriptstyle \mathrm{cc}^Z_m(r-c)[-2c]} \\
\downarrow{\cong} & & \\
\mathbb{R}\epsilon_* \mathbb{R}i^! \mathbb{Z}/m(r) & \xrightarrow{\cong} & \mathbb{R}\epsilon_* \mathbb{Z}/m(r-c)[-2c]
\end{array}$$

in $\mathsf{D}(\mathrm{Sh}(Z_{\mathrm{Zar}}, \mathbb{Z}/m))$ *commutes.*

Proof. – Let $U = X \setminus Z$.

We use the definition of the étale cycle class map for X and the notation $\mathscr{G}$, $\mathscr{G}'$ and $\mathscr{G}^r$ after (4).

Denote by $\tilde{\mathscr{G}}$, $\tilde{\mathscr{G}}'$, $\tilde{\mathscr{G}}^{r-c}$ the analogous objects defined for Z instead for X, so we have a diagram

$$z^{r-c}(\_)/m[-2(r-c)] \xleftarrow{\sim} \tilde{\mathscr{G}}^{r-c} \to \tilde{\mathscr{G}}' \xleftarrow{\sim} \tilde{\mathscr{G}}$$

in $\mathrm{Cpx}(\mathrm{Sh}(Z_{\text{ét}}, \mathbb{Z}/m))$.

For $V \in \mathrm{Sm}_X$ set $\mathscr{G}_Z(V) := \ker(\mathscr{G}(V) \to \mathscr{G}(V|_U))$. Thus $\mathscr{G}_Z \in \mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{X,\text{ét}}, \mathbb{Z}/m))$ computes $i_* \mathbb{R}i^! \mu_m^{\otimes r}$.

There is an absolute purity isomorphism $\mathscr{G}_Z \cong i_* \tilde{\mathscr{G}}[-2c]$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{X,\text{ét}}, \mathbb{Z}/m))$ (see Theorem 3.10). Choose a representative $\varphi\colon \mathscr{G}_Z \to i_* \tilde{\mathscr{G}}[-2c]$ in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{X,\text{ét}}, \mathbb{Z}/m))$ of this isomorphism. This exists since $i_* \tilde{\mathscr{G}}[-2c]$ is injectively fibrant.

For $V \in X_{\text{ét}}$ denote by $\mathscr{G}'_Z(V)$ the total complex associated to the double complex which is the normalized complex associated to $\mathscr{G}_Z(V \times \Delta^\bullet)$. Moreover let $\mathscr{G}^r_Z(V, n)$ be the colimit of the $\mathscr{G}^W(V \times \Delta^n)$ where W runs through the finite unions of elements of $c^{r-c}(V|_Z, n)$. Denote by $\mathscr{G}^r_Z(V)$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r} \mathscr{G}^r_Z(V, \bullet)$. Denote by $z^r_Z(V)$ the complex $z^{r-c}(V|_Z)$.

Set $\mathscr{G}_U(V) := \mathscr{G}(V|_U)$, $\mathscr{G}'_U(V) := \mathscr{G}'(V|_U)$, $\mathscr{G}^r_U(V) := \mathscr{G}^r(V|_U)$ and $z^r_U(V) := z^r(V|_U)$.

We have the diagram

$$\begin{array}{ccccccc}
i_* \tilde{\mathscr{G}}[-2c] & \xleftarrow{\sim} & \mathscr{G}_Z & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{G}_U \\
\downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\
i_* \tilde{\mathscr{G}}'[-2c] & \xleftarrow{\sim} & \mathscr{G}'_Z & \longrightarrow & \mathscr{G}' & \longrightarrow & \mathscr{G}'_U \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
i_* \tilde{\mathscr{G}}^{r-c}[-2c] & \xleftarrow{\sim} & \mathscr{G}^r_Z & \longrightarrow & \mathscr{G}^r & \longrightarrow & \mathscr{G}^r_U \\
\downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\
i_* z^{r-c}(_)[-2r] & \xleftarrow{\cong} & z^r_Z(_)[-2r] & \longrightarrow & z^r(_)[-2r] & \longrightarrow & z^r_U(_)[-2r].
\end{array}$$

The upper three left most horizontal maps are induced by φ. The lower left square commutes by the naturality of the purity maps in étale cohomology. All other squares commute by construction. The last two arrows in each horizontal line compose to 0 and constitute an exact triangle, thus the second vertical line computes $i_*\mathbb{R}i^!$ of the third vertical line. The claim follows. □

Corollary 3.13. – *Let $i\colon Z \to X$ be a closed inclusion in Smc_S of codimension c and $j\colon U \to X$ the complementary open inclusion. Suppose m is invertible on X. Then the diagram*

$$
\begin{array}{ccccccc}
i_*\mathscr{M}^Z(r-c)/m[-2c] & \longrightarrow & \mathscr{M}^X(r)/m & \longrightarrow & \mathbb{R}j_*\mathscr{M}^U(r)/m & \longrightarrow & i_*\mathscr{M}^Z(r-c)/m[-2c+1] \\
\downarrow{\scriptstyle i_*\mathrm{cc}^Z_m(r-c)[-2c]} & & & & \downarrow{\scriptstyle \mathbb{R}j_*\mathrm{cc}^U_m(r)} & & \downarrow{\scriptstyle i_*\mathrm{cc}^Z_m(r-c)[-2c+1]} \\
i_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r-c)[-2c] & & \downarrow{\scriptstyle \mathrm{cc}^X_m(r)} & & \mathbb{R}j_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) & & i_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r-c)[-2c+1] \\
\downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathbb{R}\epsilon_*i_*\mathbb{Z}/m(r-c)[-2c] & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}j_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*i_*\mathbb{Z}/m(r-c)[-2c+1]
\end{array}
$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/m))$ commutes.

Proof. – The diagram

$$
\begin{array}{ccccccc}
i_*\mathbb{R}i^!\mathscr{M}^X(r)/m & \longrightarrow & \mathscr{M}^X(r)/m & \longrightarrow & \mathbb{R}j_*\mathscr{M}^U(r)/m & \longrightarrow & i_*\mathbb{R}i^!\mathscr{M}^X(r)/m[1] \\
\downarrow{\scriptstyle i_*\mathbb{R}i^!\mathrm{cc}^X_m(r)} & & & & & & \downarrow{\scriptstyle i_*\mathbb{R}i^!\mathrm{cc}^X_m(r)[1]} \\
i_*\mathbb{R}i^!\mathbb{R}\epsilon_*\mathbb{Z}/m & & \downarrow{\scriptstyle \mathrm{cc}^X_m(r)} & & \downarrow{\scriptstyle \mathrm{cc}^U_m(r)} & & i_*\mathbb{R}i^!\mathbb{R}\epsilon_*\mathbb{Z}/m[1] \\
\downarrow{\scriptstyle \cong} & & & & & & \downarrow{\scriptstyle \cong} \\
i_*\mathbb{R}\epsilon_*\mathbb{R}i^!\mathbb{Z}/m(r) & & & & \mathbb{R}j_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) & & i_*\mathbb{R}\epsilon_*\mathbb{R}i^!\mathbb{Z}/m(r)[1] \\
\downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathbb{R}\epsilon_*i_*\mathbb{R}i^!\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}j_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*i_*\mathbb{R}i^!\mathbb{Z}/m(r)[1]
\end{array}
$$

commutes. Thus the claim follows from Theorem 3.12. □

Theorem 3.14. – *Let $X \in \mathrm{Smc}_S$. Let $q\colon \mathbb{A}^1_X \to X$ be the projection. Then the canonical map*

$$\mathscr{M}^X(r) \to \mathbb{R}q_*\mathscr{M}^{\mathbb{A}^1_X}(r)$$

is an isomorphism in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$.

Proof. – This is [**17**, Corollary 3.5] (again we can apply this result since any connected component is either smooth over S or over a field). □

CHAPTER 4

THE CONSTRUCTION

For Sections 4 through 7 and Appendix C of the paper we fix a Dedekind domain $\mathscr{D}$ of mixed characteristic and set $S := \mathrm{Spec}(\mathscr{D})$. For a prime p we let $S[\frac{1}{p}] = \mathrm{Spec}(\mathscr{D}[\frac{1}{p}])$ and $Z_p \subset S$ the closed complement of $S[\frac{1}{p}]$ with the reduced scheme structure. Then Z_p is a finite union of spectra of fields of characteristic p.

4.1. The p-parts

4.1.1. Finite coefficients. – We fix a prime p and set $U := S[\frac{1}{p}]$, $Z := Z_p$, $i\colon Z \hookrightarrow S$ the closed and $j\colon U \hookrightarrow S$ the open inclusion.

For a scheme X for which the motivic complexes are defined we set $\mathscr{M}_n^X(r) := \mathscr{M}^X(r)/p^n$.

For $n \geq 1$ and $r \in \mathbb{Z}$ let $L_n(r) := \mu_{p^n}^{\otimes r}$ viewed as sheaf of $\mathbb{Z}/p^n$-modules on $\mathrm{Sm}_{U,\text{ét}}$.

The pullback $j^{-1}\colon \mathrm{Sm}_S \to \mathrm{Sm}_U$, $X \mapsto X \times_S U$, induces a push forward

$$j_*\colon \mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}/p^n) \to \mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}/p^n)$$

(we suppress the dependence on n of the functor j_*). The same is true for étale sheaves.

Similarly, we have the pullback $i^{-1}\colon \mathrm{Sm}_S \to \mathrm{Sm}_Z$, $X \mapsto X \times_S Z$, inducing also a push forward on sheaf categories.

Let $QL_n(1) \to L_n(1)$ be a cofibrant replacement in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^n))$ (the latter category is equipped with the local projective model structure, see Appendix A) and let $QL_n(1) \to RQL_n(1)$ be a fibrant replacement via a cofibration. Thus $\mathscr{T} := RQL_n(1)[1]$ is both fibrant and cofibrant.

Recall the decomposition

$$\underline{\mathbb{R}\mathrm{Hom}}_{\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}},\mathbb{Z}/p^n))}(\mathbb{G}_{m,U}, L_n(1)[1]) = L_n(1)[1] \oplus L_n(0). \tag{5}$$

The first summand splits off because the projection $\mathbb{G}_{m,U} \to U$ has the section $\{1\}$.

To define the isomorphism of the remaining summand with $L_n(0)$ we use the Gysin sequence for the situation

$$\mathbb{G}_{m,U} \hookrightarrow \mathbb{A}^1_U \hookleftarrow \{0\}.$$

Let $\iota\colon \mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}} \to \mathscr{T}$ (for notation see Section 2) be a map which classifies the canonical element $1 \in H^1_{\text{ét}}(\mathbb{G}_{m,U}, L_n(1))$ under the above decomposition (here the source of ι is the chain complex having the indicated object in degree 0). Note that $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}$ is cofibrant.

Remark 4.1. – *The map $\mathbb{Z}/p^n[\mathbb{G}_{m,U}]_{\text{ét}} \to \mathscr{T}$ induced by ι represents the map induced by the last map of the exact triangle*

$$L_n(1) \to \mathbb{G}_{m,U} \xrightarrow{p^n} \mathbb{G}_{m,U} \to L_n(1)[1]$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}},\mathbb{Z}))$. This follows from the construction of the Gysin isomorphism.

We get a map

$$\mathrm{Sym}(\iota)\colon \mathrm{Sym}(\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}) \to \mathrm{Sym}(\mathscr{T})$$

of commutative monoids in symmetric sequences in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}},\mathbb{Z}/p^n))$, in other words $\mathrm{Sym}(\mathscr{T})$ is a commutative monoid in the category of symmetric $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}$-spectra $\mathrm{Sp}^{\Sigma}_{\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}}$ (for symmetric sequences and (symmetric) spectra we refer to **[26]**). In particular it gives rise to an E_∞-object in $\mathrm{Sp}^{\Sigma}_{\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}}$.

Let $Q\mathrm{Sym}(\mathscr{T}) \to \mathrm{Sym}(\mathscr{T})$ be a cofibrant replacement via a trivial fibration and $Q\mathrm{Sym}(\mathscr{T}) \to RQ\mathrm{Sym}(\mathscr{T})$ a fibrant resolution of $Q\mathrm{Sym}(\mathscr{T})$ in $E_\infty(\mathrm{Sp}^{\Sigma}_{\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}})$ (here $E_\infty(\mathrm{Sp}^{\Sigma}_{\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}})$ is equipped with the transferred semi model structure, see Appendix A. In particular $RQ\mathrm{Sym}(\mathscr{T})$ is underlying levelwise fibrant for the local projective model structure and is therefore suitable to compute the derived push forward along $\epsilon\colon \mathrm{Sm}_{U,\text{ét}} \to \mathrm{Sm}_{U,\text{Zar}}$).

Lemma 4.2. – *The map $Q\mathrm{Sym}(\mathscr{T}) \to RQ\mathrm{Sym}(\mathscr{T})$ is a level equivalence, i.e., $\mathrm{Sym}(\mathscr{T})$ is an Ω-spectrum.*

Proof. – This follows from the fact that we have chosen the map ι in such a way that the derived adjoints of the structure maps of $\mathrm{Sym}(\mathscr{T})$ give rise to the isomorphism $\underline{\mathbb{R}\mathrm{Hom}}((\mathbb{G}_{m,U},\{1\}), L_n(r)[r]) \simeq L_n(r-1)[r-1]$. □

Set $A := \epsilon_*(RQ\mathrm{Sym}(\mathscr{T}))$, so the spectrum A is $RQ\mathrm{Sym}(\mathscr{T})$ viewed as E_∞-algebra in $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{Zar}}$-spectra in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{Zar}},\mathbb{Z}/p^n))$.

We denote by A_r the r-th level of A. Thus $A_r \simeq \mathbb{R}\epsilon_* L_n(r)[r]$.

Set $A_r^{\text{tr}} := \tau_{\leq 0}(A_r)$, where $\tau_{\leq 0}$ denotes the good truncation at degree 0, i.e., the complex A_r^{tr} equals A_r in (cohomological) degrees < 0, consists of the cycles in degree 0 and is 0 in positive degree.

Thus by Theorem 3.9 there is for every $X \in \mathrm{Sm}_U$ an isomorphism

$$A_r^{\text{tr}}|_{X_{\text{Zar}}} \cong \mathscr{M}_n^X(r)[r] \qquad (6)$$

in $\mathsf{D}(\mathrm{Sh}(X_{\text{Zar}},\mathbb{Z}/p^n))$, where $A_r^{\text{tr}}|_{X_{\text{Zar}}}$ denotes the restriction of A_r^{tr} to X_{Zar}.

LEMMA 4.3. – *The complexes A^{tr}_r assemble to a $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}$-spectrum A^{tr}. This spectrum is equipped with an E_∞-structure together with a map of E_∞-algebras $A^{\mathrm{tr}} \to A$ which is levelwise the canonical map $A^{\mathrm{tr}}_r \to A_r$.*

Proof. – This follows from the fact that the truncation $\tau_{\leq 0}$ is right adjoint to the symmetric monoidal inclusion of (cohomologically) non-positively graded complexes into all complexes and that $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}$ lies in this subcategory of non-positively graded complexes. □

Let $QA^{\mathrm{tr}} \to A^{\mathrm{tr}}$ be a cofibrant replacement via a trivial fibration in E_∞-algebras in $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}$-spectra in $\mathrm{Cpx}^{\leq 0}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}},\mathbb{Z}/p^n))$ (so QA^{tr} is also cofibrant viewed as an E_∞-algebra in spectra in unbounded complexes) and $QA^{\mathrm{tr}} \to RQA^{\mathrm{tr}}$ be a fibrant resolution (as E_∞-algebras in $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}$-spectra in the model category $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}},\mathbb{Z}/p^n))$, so in particular RQA^{tr} is also underlying fibrant).

PROPOSITION 4.4. – *The map $QA^{\mathrm{tr}} \to RQA^{\mathrm{tr}}$ is a level equivalence, i.e., A^{tr} and QA^{tr} are Ω-spectra.*

Proof. – Set $m := p^n$. Let $X \in \mathrm{Sm}_U$. Let $\tilde{\imath}\colon \{0\} \to \mathbb{A}^1_X$ be the closed, $\tilde{\jmath}\colon \mathbb{G}_{m,X} \to \mathbb{A}^1_X$ the open inclusion and $q\colon \mathbb{A}^1_X \to X$ the projection. By Corollary 3.11 we have an exact triangle

$$\tilde{\imath}_*\mathbb{Z}/m(r-1)[-2] \to \mathbb{Z}/m(r)^{\mathbb{A}^1_X} \to \mathbb{R}\tilde{\jmath}_*\mathbb{Z}/m(r) \to \tilde{\imath}_*\mathbb{Z}/m(r-1)[-1].$$

Note

$$\mathbb{R}q_*\mathbb{R}\tilde{\jmath}_*\mathbb{Z}/m(r) \cong \underline{\mathbb{R}\mathrm{Hom}}_{\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}},\mathbb{Z}/m))}(\mathbb{G}_{m,U},L_n(r))|_{X_{\text{ét}}}$$

and that $\mathbb{R}q_*$ applied to the last map in the triangle gives the projection to the second summand in our decomposition (5). Thus by construction of the map ι this map also gives the inverse of the adjoint of the structure map in $R\mathrm{Sym}(\mathscr{T})$.

By Theorem 3.1 there is an exact triangle

$$\tilde{\imath}_*\mathscr{M}_n(r-1)[-2] \to \mathscr{M}_n^{\mathbb{A}^1_X}(r) \to \mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r) \to \tilde{\imath}_*\mathscr{M}_n(r-1)[-1].$$

Hence by Theorem 3.3 the canonical map

$$\tau_{\leq r}(\mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r)) \to \mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r)$$

is an isomorphism. Thus in view of Theorem 3.9 the same truncation property holds for $\mathbb{R}\tilde{\jmath}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)$. Thus the map

$$\mathbb{R}\tilde{\jmath}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \to \mathbb{R}\epsilon_*\tilde{\imath}_*\mathbb{Z}/m(r-1)[-1]$$

factors through $\tau_{\leq r}(\mathbb{R}\epsilon_*\tilde{\imath}_*\mathbb{Z}/m(r-1)[-1])$.

Moreover the map

$$\mathbb{R}\tilde{\jmath}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \cong \tau_{\leq r}\mathbb{R}\tilde{\jmath}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \to \tau_{\leq r}\mathbb{R}\tilde{\jmath}_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r)$$

is an isomorphism, thus we have a canonical map

$$\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)^{\mathbb{A}^1_X} \to \mathbb{R}\tilde{\jmath}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r).$$

Using Corollary 3.13 these maps fit into the commutative diagram

$$\begin{array}{ccccc}
\mathscr{M}_n^{\mathbb{A}^1_X}(r) & \longrightarrow & \mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r) & \longrightarrow & \tilde{\imath}_*\mathscr{M}_n(r-1)[-1] \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\tau_{\le r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)^{\mathbb{A}^1_X} & \longrightarrow & \mathbb{R}\tilde{\jmath}_*\tau_{\le r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & \tau_{\le r}(\tilde{\imath}_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r-1)[-1]),
\end{array} \tag{7}$$

where the top row is part of the triangle given by Theorem 3.1. The composition

$$\begin{aligned}
A^{\mathrm{tr}}_{r-1}[-r]|_{X_{\mathrm{Zar}}} &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}, A^{\mathrm{tr}}_r[-r])|_{X_{\mathrm{Zar}}} \\
&\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,U}]_{\mathrm{Zar}}, \tau_{\le r}\mathbb{R}\epsilon_* L_n(r))|_{X_{\mathrm{Zar}}} \\
&\cong \mathbb{R}q_*\mathbb{R}\tilde{\jmath}_*\tau_{\le r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \to \tau_{\le r}(\mathbb{R}\epsilon_*\mathbb{Z}/m(r-1)[-1]) \cong A^{\mathrm{tr}}_{r-1}[-r]|_{X_{\mathrm{Zar}}}
\end{aligned}$$

is the identity.

Since

$$\mathbb{Z}/m[\mathbb{G}_{m,U}]_{\mathrm{Zar}} \cong \mathbb{Z}/m \oplus \mathbb{Z}/m[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}} \tag{8}$$

the object $\mathbb{R}q_*\mathbb{R}\tilde{\jmath}_*\tau_{\le r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)$ splits into two summands, the trivial summand corresponding to the left summand in (8) and the non-trivial summand corresponding to the right summand.

By Theorem 3.14 $\mathbb{R}q_*\mathscr{M}_n^{\mathbb{A}^1_X}(r)$ identifies with $\mathscr{M}_n^X(r)$, thus $\mathbb{R}q_*$ applied to the left bottom arrow in (7) is an isomorphism to the trivial summand and $\mathbb{R}q_*$ of the bottom row splits. Thus also $\mathbb{R}q_*$ of the top row splits. This shows that in fact

$$\mathbb{R}q_*\tilde{\imath}_*\mathscr{M}_n(r-1)[-1] \cong \mathscr{M}_n^X(r-1)[-1]$$

via the right vertical isomorphism and the right lower map in the diagram is isomorphic to the non-trivial summand in $\mathbb{R}q_*\mathbb{R}\tilde{\jmath}_*\tau_{\le r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)$. Since this holds over every $X \in \mathrm{Sm}_U$ we are done. □

Thus $B := j_*(RQA^{\mathrm{tr}})$ is a $\mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}$-spectrum and computes also levelwise the derived push forward of A^{tr} along j. (Note that to compute the levelwise push forward we also could have used the levelwise model structure.)

By (6) for every $X \in \mathrm{Sm}_S$ we have

$$B_r|_{X_{\mathrm{Zar}}} \cong \mathbb{R}(j_X)_*(\mathscr{M}_n^{X_U}(r))[r] \tag{9}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}},\mathbb{Z}/p^n))$ (here $X_U = X\times_S U$ and j_X denotes the inclusion $X_U \hookrightarrow X$).

Thus by Corollary 3.5 the map $B^{\mathrm{tr}}_r := \tau_{\le 0}B_r \to B_r$ is a quasi-isomorphism.

As in Lemma 4.3 the B^{tr}_r assemble to an E_∞-algebra B^{tr}, and the natural map $B^{\mathrm{tr}} \to B$ is an equivalence.

By the following lemma we could have used j_*A^{tr} instead of B and B^{tr}.

Lemma 4.5. – *The natural maps $j_*(QA^{\mathrm{tr}}) \to j_*A^{\mathrm{tr}}$ and $j_*(QA^{\mathrm{tr}}) \to B$ are level equivalences.*

Proof. – Note first that each A_r^{tr} is fibrant in $\mathrm{Cpx}^{\leq 0}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}/p^n))$, hence so are the $(QA^{\mathrm{tr}})_r$, thus the $j_*A_r^{\mathrm{tr}}$ and $j_*(QA^{\mathrm{tr}})_r$ are the derived push forwards to the homotopy category of $\mathrm{Cpx}^{\leq 0}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}/p^n))$. But the truncation commutes with derived push forward (both are right adjoints), so the claim follows from the fact that $B^{\mathrm{tr}} \to B$ is an equivalence. □

COROLLARY 4.6. – *There is a natural isomorphism*

$$\mathbb{R}^r j_* L_n(r) \cong \epsilon^* \mathscr{H}^0(B_r^{\mathrm{tr}}) = \mathscr{H}^0(B_r^{\mathrm{tr}})_{\text{ét}}$$

in $\mathrm{Sh}(\mathrm{Sm}_{S,\text{ét}}, \mathbb{Z}/p^n)$.

Proof. – We have $j_*A^{\mathrm{tr}} = \epsilon_*\tau_{\leq 0} j_*(RQ\mathrm{Sym}(\mathscr{T}))$ (here the truncation is understood levelwise), thus

$$\begin{aligned}\mathbb{R}^r j_* L_n(r) &\cong \mathscr{H}^0(j_*((RQ\mathrm{Sym}(\mathscr{T}))_r) = \mathscr{H}^0(\tau_{\leq 0} j_*((RQ\mathrm{Sym}(\mathscr{T}))_r) \\ &= \epsilon^* \mathscr{H}^0(\epsilon_*\tau_{\leq 0} j_*((RQ\mathrm{Sym}(\mathscr{T}))_r) = \epsilon^* \mathscr{H}^0(j_*A_r^{\mathrm{tr}}) = \epsilon^* \mathscr{H}^0(B_r^{\mathrm{tr}}).\end{aligned}$$

At the end we used Lemma 4.5. □

By (9) and Corollary 3.5 we have for every $X \in \mathrm{Sm}_S$ a natural epimorphism

$$s_X \colon \mathscr{H}^0(B_r^{\mathrm{tr}}|_{X_{\mathrm{Zar}}}) \twoheadrightarrow (i_X)_*\nu_n^{r-1}, \tag{10}$$

where i_X is the inclusion $X \times_S Z \hookrightarrow X$.

PROPOSITION 4.7. – *The maps* s_X *assemble to an epimorphism*

$$s \colon \mathscr{H}^0(B_r^{\mathrm{tr}}) \twoheadrightarrow i_*\nu_n^{r-1}$$

in $\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}/p^n)$.

In order to prove this proposition we describe the maps s_X in a way Geisser used to define his version of syntomic cohomology in [**17**, §6].

Let $X \in \mathrm{Sm}_S$. We first give a construction of a map

$$b_X \colon (i_X)^*(\mathbb{R}^r j_* L_n(r)|_{X_{\text{ét}}}) \to \nu_n^{r-1}$$

in $\mathrm{Sh}((X_Z)_{\text{ét}}, \mathbb{Z}/p^n)$. Over a complete discrete valuation ring of mixed characteristic such a map was constructed in [**4**, §(6.6)], see also [**17**, §6]. We fix a point $\mathfrak{p} \in Z$ and let Λ be the completion of the discrete valuation ring $D_{\mathfrak{p}}$. Set $T := \mathrm{Spec}(\Lambda)$. Let η be the generic point of T. Let $X_T := X \times_S T$, and let $X_{\mathfrak{p}}$ be the special fiber and X_η the generic fiber of X_T.

We let $j_{X_T} \colon X_\eta \to X_T$ and $i_{X_T} \colon X_{\mathfrak{p}} \to X_T$ be the canonical inclusions.

Then the map

$$b_{X_T} \colon M^r_{n,X_T} := (i_{X_T})^*\mathbb{R}^r(j_{X_T})_*(\mathbb{Z}/p^n(r)) \to \nu_n^{r-1}$$

in [**4**, §(6.6)] is defined as follows (recall $\mathbb{Z}/p^n(r) = \mu_{p^n}^{\otimes r}$):

By [**4**, Corollary (6.1.1)] the sheaf M^r_{n,X_T} is (étale) locally generated by symbols $\{x_1, \ldots, x_r\}$, $x_i \in (i_{X_T})^*(j_{X_T})_*\mathscr{O}^*_{X_\eta}$ (for the definition of symbol see [**4**, §(1.2)]).

Then for any $f_1, \dots, f_r \in (i_{X_T})^* \mathcal{O}^*_{X_T}$ the map b_{X_T} sends the symbol $\{f_1, \dots, f_r\}$ to 0 and the symbol $\{f_1, \dots, f_{r-1}, \pi\}$ (π a uniformizer of Λ) to $\mathrm{dlog}\overline{f}_1 \dots \mathrm{dlog}\overline{f}_{r-1}$, where $\overline{f}_i$ is the reduction of f_i to $\mathcal{O}^*_{X_\mathfrak{p}}$.

By multilinearity and the fact that $\{x, -x\} = 0$ for $x \in (i_{X_T})^*(j_{X_T})_* \mathcal{O}^*_{X_\eta}$ this characterizes b_{X_T} uniquely.

The base change morphism for the square

$$\begin{array}{ccc} X_\eta & \xrightarrow{f_{X_U}} & X_U \\ \downarrow{\scriptstyle j_{X_T}} & & \downarrow{\scriptstyle j_X} \\ X_T & \xrightarrow{f_X} & X \end{array}$$

applied to the sheaf $\mathbb{Z}/p^n(r)$ on $(X_U)_{\text{ét}}$ yields

$$(f_X)^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to \mathbb{R}^r (j_{X_T})_* \mathbb{Z}/p^n(r)$$

(note that $(f_{X_U})^* \mathbb{Z}/p^n(r) = \mathbb{Z}/p^n(r)$). Applying $(i_{X_T})^*$ and noting that $(i_{X_T})^*(f_X)^* = (i_\mathfrak{p})^*$ where $i_\mathfrak{p}$ is the inclusion $i_\mathfrak{p} \colon X_\mathfrak{p} \to X$ we get a map

$$(i_\mathfrak{p})^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to M^r_{n, X_T}.$$

Composing with b_{X_T} gives a map

$$(i_\mathfrak{p})^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to \nu_n^{r-1}.$$

Taking the disjoint union over all points in Z we finally get the map

$$b_X \colon (i_X)^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to \nu_n^{r-1},$$

the adjoint of which is a map

$$\flat(b_X) \colon \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to (i_X)_* \nu_n^{r-1}.$$

Together with the isomorphism of Corollary 4.6 we get the composition

$$(11) \qquad s'_X \colon \mathcal{H}^0(B_r^{\mathrm{tr}})|_{X_{\mathrm{Zar}}} \to \epsilon_* \mathcal{H}^0(B_r)_{\text{ét}}|_{X_{\mathrm{Zar}}} \cong \epsilon_* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \to (i_X)_* \nu_n^{r-1}$$

(by our convention ν_n^{r-1} also denotes the logarithmic de Rham-Witt sheaf on $(X_Z)_{\mathrm{Zar}}$).

Proposition 4.8. – *With the notation as above we have $s_X = s'_X$.*

Proof. – We keep the local completed situation at a point $\mathfrak{p}$ of Z from above.

We have a natural map induced by flat pullback $(f_{X_U})^* \mathcal{M}_n^{X_U}(r) \to \mathcal{M}_n^{X_\eta}(r)$, whence we get a base change morphism

$$f_X^* \mathbb{R}^r (j_X)_* \mathcal{M}_n^{X_U}(r) \to \mathbb{R}^r (j_{X_T})_* \mathcal{M}_n^{X_\eta}(r).$$

We get a diagram

$$\begin{array}{ccccc} f_X^*\mathbb{R}^r(j_X)_*\mathscr{M}_n^{X_U}(r) & \longrightarrow & \mathbb{R}^r(j_{X_T})_*\mathscr{M}_n^{X_\eta}(r) & \longrightarrow & \mathscr{H}^{r-1}((i_{X_T})_*\mathscr{M}_n^{X_\mathfrak{p}}(r-1)) \\ \downarrow & & \downarrow & & \downarrow\cong \\ f_X^*\epsilon_*\mathbb{R}^r(j_X)_*\mathbb{Z}/p^n(r) & \longrightarrow & \epsilon_*\mathbb{R}^r(j_{X_T})_*\mathbb{Z}/p^n(r) & \longrightarrow & (i_{X_T})_*\nu_n^{r-1}. \end{array}$$

The left and middle vertical maps are induced by the isomorphism of Corollary 4.6 and (9). The left lower horizontal map is induced by the transformation $f_X^*\epsilon_* \to \epsilon_* f_X^*$. The upper right horizontal arrow is part of the localization sequence for the motivic complexes. The lower right horizontal map is induced by b_{X_T}.

The claim of the proposition follows from the commutativity of the outer square. Indeed, a map from the left upper corner to the right lower corner is adjoint to a map

$$(i_\mathfrak{p})^*\mathbb{R}^r(j_X)_*\mathscr{M}_n^{X_U}(r) = (i_{X_T})^*f^*\mathbb{R}^r(j_X)_*\mathscr{M}_n^{X_U}(r) \to \nu_n^{r-1}.$$

The assertion that the outside compositions are the same implies that the adjoints of s_X and s'_X coincide over the point $\mathfrak{p}$. Since this is true for all points in Z the claim follows.

The left square of the above square commutes by naturality of the cycle class map, Lemma 3.7.

So we are left to prove the commutativity of the right hand square.

Since the right lower corner is an étale sheaf we can also sheafify this square in the étale topology to test commutativity.

The resulting square is adjoint to a square

$$\begin{array}{ccc} (i_{X_T})^*\epsilon^*\mathbb{R}^r(j_{X_T})_*\mathscr{M}_n^{X_\eta}(r) & \longrightarrow & \epsilon^*\mathscr{H}^{r-1}(\mathscr{M}_n^{X_\mathfrak{p}}(r-1)) \\ \downarrow\cong & & \downarrow\cong \\ (i_{X_T})^*\mathbb{R}^r(j_{X_T})_*\mathbb{Z}/p^n(r) & \longrightarrow & \nu_n^{r-1} \end{array}$$

(the left vertical map is an isomorphism by Corollary 4.6 and (9), both understood for the completed situation). This commutativity would follow from the commutativity of the right hand square in the first diagram in the proof of [**17**, Theorem 1.3]. This commutativity is not explicitely stated in loc. cit., but the proof in loc. cit. that the composition $\kappa \circ \alpha \circ c$ (the three maps in this composition are introduced in the proof of [**17**, Theorem 1.3]) is 0 shows the commutativity of our diagram:

As in loc. cit. let R be the strictly henselian local ring of a point in the closed fiber $X_\mathfrak{p}$ of X_T, let L be the field of quotients of R, F the field of quotients of R/π, $V = R_{(\pi)}$, V^h the henselization of V and L^h the quotient field of V^h.

We have to show the commutativity of

$$\begin{array}{ccc} H^r(R[\frac{1}{\pi}], \mathcal{M}_n(r)) & \longrightarrow & H^{r-1}(R/\pi, \mathcal{M}_n(r-1)) \\ \downarrow \cong & & \downarrow \cong \\ H^r_{\text{ét}}(R[\frac{1}{\pi}], \mathbb{Z}/p^n(r)) & \longrightarrow & \nu_n^{r-1}(R/\pi). \end{array}$$

The map $\nu_n^{r-1}(R/\pi) \to \nu_n^{r-1}(F)$ is injective (see the proof of [**17**, Theorem 1.3], where it is attributed to [**20**, Corollary 1.6]).

Thus by the naturality of the localization sequence for motivic complexes and the fact that the b_{X_T} are sheaf maps it is enough to show commutativity of the square which one gets from the last square by replacing $R[\frac{1}{\pi}]$ with L and R/π with F. But this square factors as

$$\begin{array}{ccccc} H^r(L, \mathcal{M}_n(r)) & \longrightarrow & H^r(L^h, \mathcal{M}_n(r)) & \longrightarrow & H^{r-1}(F, \mathcal{M}_n(r-1)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H^r_{\text{ét}}(L, \mathbb{Z}/p^n(r)) & \longrightarrow & H^r_{\text{ét}}(L^h, \mathbb{Z}/p^n(r)) & \longrightarrow & \nu_n^{r-1}(F). \end{array}$$

The right upper horizontal map is induced from the localization sequence of the motivic complexes for V^h, its generic and its closed point.

The left hand square commutes by naturality of the cycle class map, and the commutativity of the right hand square is shown in the proof of [**17**, Theorem 1.3] in the paragraph before the last paragraph. This finishes the proof. □

We next discuss functoriality of the construction of the morphisms s'_X. So let $g\colon Y \to X$ be a morphism in Sm_S. We still keep the local completed situation from above. We let g_Z, g_T, g_η and $g_\mathfrak{p}$ be the base changes of g (over S) to Z, T, η and $\mathfrak{p}$.

Consider the diagram

$$\begin{array}{ccc} Y_\eta & \xrightarrow{g_\eta} & X_\eta \\ \downarrow j_{Y_T} & & \downarrow j_{X_T} \\ Y_T & \xrightarrow{g_T} & X_T \\ \uparrow i_{Y_T} & & \uparrow i_{X_T} \\ Y_\mathfrak{p} & \xrightarrow{g_\mathfrak{p}} & X_\mathfrak{p}. \end{array}$$

A base change morphism gives us

$$(g_T)^*\mathbb{R}^r(j_{X_T})_*(\mathbb{Z}/p^n(r)) \to \mathbb{R}^r(j_{Y_T})_*(\mathbb{Z}/p^n(r)).$$

Applying $(i_{Y_T})^*$ and using $(i_{Y_T})^*(g_T)^* \cong (g_\mathfrak{p})^*(i_{X_T})^*$ gives

$$(g_\mathfrak{p})^* M^r_{n,X_T} \to M^r_{n,Y_T}.$$

Lemma 4.9. – *The diagram*

$$\begin{array}{ccc} (g_{\mathfrak{p}})^* M^r_{n,X_T} & \xrightarrow{(g_{\mathfrak{p}})^*(b_{X_T})} & (g_{\mathfrak{p}})^* \nu_n^{r-1} \\ \downarrow & & \downarrow \\ M^r_{n,Y_T} & \xrightarrow{b_{Y_T}} & \nu_n^{r-1} \end{array}$$

commutes.

Proof. – This follows from the definition of the morphisms b_{X_T} and b_{Y_T} in terms of symbols and the functoriality of the symbols. □

As above for X let f_Y be the map $Y_T \to Y$.

Lemma 4.10. – *The diagram*

$$\begin{array}{ccc} g_T^* f_X^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) & \longrightarrow & g_T^* \mathbb{R}^r (j_{X_T})_* \mathbb{Z}/p^n(r) \\ \downarrow & & \downarrow \\ f_Y^* \mathbb{R}^r (j_Y)_* \mathbb{Z}/p^n(r) & \longrightarrow & \mathbb{R}^r (j_{Y_T})_* \mathbb{Z}/p^n(r), \end{array}$$

where all maps are induced by base change morphisms, commutes.

Proof. – This follows by the naturality of the base change morphisms. □

Corollary 4.11. – *The diagram*

$$\begin{array}{ccc} (g_Z)^*(i_X)^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{(g_Z)^*(b_X)} & (g_Z)^* \nu_n^{r-1} \\ \downarrow & & \downarrow \\ (i_Y)^* \mathbb{R}^r (j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{b_Y} & \nu_n^{r-1}, \end{array}$$

where the left vertical map is induced by a base change morphism, commutes.

Proof. – This follows by combining Lemmas 4.9 and 4.10. □

Corollary 4.12. – *The diagram*

$$\begin{array}{ccc} g^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{g^*(\flat(b_X))} & g^*(i_X)_* \nu_n^{r-1} \\ & & \downarrow \cong \\ \downarrow & & (i_Y)_*(g_Z)^* \nu_n^{r-1} \\ & & \downarrow \\ \mathbb{R}^r (j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{\flat(b_Y)} & (i_Y)_* \nu_n^{r-1} \end{array}$$

commutes.

Proof. – We check that the adjoints with respect to the pair $(i_Y)^*$, $(i_Y)_*$ of the two compositions are the two compositions of Corollary 4.11. For the composition via the left lower corner this is immediate. For the other composition one uses a compatibility between adjoints and pullbacks. □

COROLLARY 4.13. – *The maps s'_X assemble to a map of sheaves $\mathscr{H}^0(B_r^{\mathrm{tr}}) \to i_*\nu_n^{r-1}$.*

Proof. – This follows directly from Corollary 4.12. □

Proof of Proposition 4.7. – The assertion follows by combining Proposition 4.8 and Corollary 4.13. □

Let C_r be the kernel of the composition

$$B_r^{\mathrm{tr}} \twoheadrightarrow \mathscr{H}^0(B_r^{\mathrm{tr}}) \overset{s}{\twoheadrightarrow} i_*\nu_n^{r-1}.$$

Then by construction of the maps s_X we have for any $X \in \mathrm{Sm}_S$ an isomorphism

$$C_r|_{X_{\mathrm{Zar}}} \cong \mathscr{M}_n^X(r)[r] \tag{12}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^n))$ since both objects appear as (shifted) homotopy fibers of the map

$$\mathbb{R}(j_X)_*\mathscr{M}_n^{X_U}(r) \to \mathbb{R}(i_X)_*\mathscr{M}_n^{X_Z}(r-1)[-1].$$

This isomorphism is even uniquely determined since there are no non-trivial maps $\mathscr{M}_n^X(r) \to (i_X)_*\nu_n^{r-1}[-r-1]$ in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^n))$.

LEMMA 4.14. – *Let R be a commutative ring, $T \in \mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, R)$ and E an E_∞-algebra in symmetric T-spectra in $\mathrm{Cpx}^{\leq 0}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, R))$. Let E_r be the levels of E. Let for any $r > 0$ an epimorphism $\mathscr{H}^0(E_r) \twoheadrightarrow e_r$ in $\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, R)[\Sigma_r]$ be given. Let E'_r be the kernel of the induced map $E_r \to e_r$ and set $E'_0 := E_0$. Suppose the canonical map $\varphi\colon T \to E_1$ (which is the composition $T \cong R \otimes_R T \overset{\mathrm{unit}\otimes\mathrm{id}}{\longrightarrow} E_0 \otimes_R T \to E_1$) factors through E'_1 and that for any $r, r' \geq 0$ the composition in $\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, R)$ induced by the E_∞-multiplication on E*

$$\mathscr{H}^0(E'_r) \otimes_R \mathscr{H}^0(E'_{r'}) \to \mathscr{H}^0(E_r) \otimes_R \mathscr{H}^0(E_{r'}) \to \mathscr{H}^0(E_{r+r'}) \to e_{r+r'}$$

is the zero map. Then there is an induced structure of an E_∞-algebra E' in symmetric T-spectra on the collection of the E'_r together with a map of E_∞-algebras $E' \to E$ which is levelwise the canonical map $E'_r \to E_r$.

Proof. – All tensor products are over R. The condition implies that we have natural maps

$$\phi_{r,r'}\colon \mathscr{H}^0(E'_r) \otimes \mathscr{H}^0(E'_{r'}) \to \mathscr{H}^0(E'_{r+r'}).$$

Let $\mathscr{O}$ be our E_∞-operad in $\mathrm{Cpx}^{\leq 0}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, R))$. Note that each E'_r carries an action of Σ_r. The structure maps of the E_∞-algebra in T-spectra E are maps

$$s\colon E_r \otimes T \to E_{r+1}$$

and

$$a\colon \mathscr{O}(k) \otimes E_{r_1} \otimes \cdots \otimes E_{r_k} \to E_r,$$

$r = \sum_{i=1}^{k} r_i$. These are subject to certain conditions. We show that when restricting these maps to the E'_r they factor through E'_r (for the appropriate r). Then it is clear that these new structure maps also satisfy the conditions required.

To show that the composition

$$\mathcal{O}(k) \otimes E'_{r_1} \otimes \cdots \otimes E'_{r_k} \to \mathcal{O}(k) \otimes E_{r_1} \otimes \cdots \otimes E_{r_k} \to E_r$$

factors through E'_r it is sufficient to show that the induced map on $\mathcal{H}^0$ factors through $\mathcal{H}^0(E'_r)$. But since $\mathcal{O}$ is E_∞ the map on $\mathcal{H}^0$ is a map

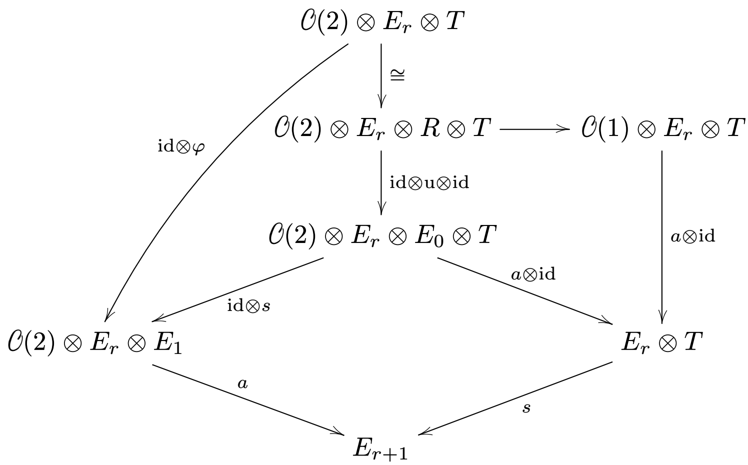
$$\mathcal{H}^0(E'_{r_1}) \otimes \cdots \otimes \mathcal{H}^0(E'_{r_k}) \to \mathcal{H}^0(E_r)$$

and the conditions to be E_∞ imply that this map is an iteration of the maps $\phi_{r',r''}$. Thus we get the factorization.

To handle the case of the T-spectrum structure maps it is again sufficient to show that the composition

$$\psi\colon \mathcal{H}^0(E'_r) \otimes T \to \mathcal{H}^0(E_r) \otimes T \to \mathcal{H}^0(E_{r+1})$$

factors through $\mathcal{H}^0(E'_{r+1})$. But the commutativity of the diagram



(the only horizontal arrow is a structure map of the operad using $R \cong \mathcal{O}(0)$) implies that ψ is the composition

$$\mathcal{H}^0(E'_r) \otimes T \to \mathcal{H}^0(E'_r) \otimes \mathcal{H}^0(E'_1) \to H^0(E_{r+1})$$

which factors through $\mathcal{H}^0(E'_{r+1})$ by assumption. This finishes the proof. □

We want to apply Lemma 4.14 with $T = \mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$, $E = B^{\mathrm{tr}}$ and $e_r = i_* \nu_n^{r-1}$. Then we have $E'_r = C_r$.

LEMMA 4.15. – *The Σ_r-action on $\mathcal{H}^0(B^{\mathrm{tr}}_r)$ is the sign representation.*

Proof. – This follows from the fact that there is a zig zag of Σ_r-equivariant quasi-isomorphisms between $\mathscr{T}^{\otimes r}$ and $(L_n(1)[1])^{\otimes r}$, and on the latter the Σ_r-action is strictly the sign representation, since $L_n(1)[1]$ has as only entry an invertible sheaf of $\mathbb{Z}/p^n$-modules in homological degree 1. □

So if we equip ν_n^{r-1} with the sign representation of Σ_r the map $\mathscr{H}^0(B_r^{\mathrm{tr}}) \to i_*\nu_n^{r-1}$ is Σ_r-equivariant.

The exact sequence

$$0 \to L_n(1) \to \mathbb{G}_{m,U} \xrightarrow{p^n} \mathbb{G}_{m,U} \to 0$$

on $\mathrm{Sm}_{U,\text{ét}}$ induces a boundary homomorphism

$$\beta\colon j_*\mathbb{G}_{m,U} \to \mathbb{R}^1 j_* L_n(1)$$

of sheaves on $\mathrm{Sm}_{S,\text{ét}}$ (note that (after restriction) this is the symbol map defined in [**4**, (1.2)] in degree 1).

Lemma 4.16. – *The diagram*

$$\begin{array}{ccccccccc} \mathbb{G}_{m,S} & \longrightarrow & \mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}} & \longrightarrow & B_1^{\mathrm{tr}} & \longrightarrow & \mathscr{H}^0(B_1^{\mathrm{tr}}) & \longrightarrow & \mathscr{H}^0(B_1^{\mathrm{tr}})_{\text{ét}} \\ \downarrow & & & & & & & & \downarrow \cong \\ j_*\mathbb{G}_{m,U} & & & & \xrightarrow{\quad\beta\quad} & & & & \mathbb{R}^1 j_* L_n(1) \end{array}$$

commutes.

Proof. – This follows from the defining property of the map ι (see Remark 4.1). □

Corollary 4.17. – *The composition*

$$\mathbb{G}_{m,S} \to \mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}} \xrightarrow{\varphi} B_1^{\mathrm{tr}} \to \mathscr{H}^0(B_1^{\mathrm{tr}}) \to i_*\nu_n^{r-1}$$

is the constant map to zero.

Proof. – This follows from Lemma 4.16, the definition of the map b_{X_T} and the definition of symbol: The symbol $\{x\}$ for x an invertible section over a smooth scheme over S is sent to 0 via b_{X_T}. □

Thus the first condition of Lemma 4.14 about the factorization of the map φ is satisfied.

For the second condition we get back to our local completed situation. Let $X \in \mathrm{Sm}_S$, $\mathfrak{p} \in Z$ and let the notation be as above. By [**32**, §3, top of p. 277] there is an exact sequence

$$0 \to U^0 M_n^r \to M_n^r \to \nu_n^{r-1} \to 0 \tag{13}$$

on $(X_{\mathfrak{p}})_{\text{ét}}$, where $U^0 M_n^r$ is the subsheaf of M_n^r generated étale locally by symbols $\{x_1, \ldots, x_r\}$ with $x_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*$. This follows from the exact sequence

$$0 \to U^1 M_n^r \to M_n^r \to \nu_n^r \oplus \nu_n^{r-1} \to 0$$

([**4**, Theorem (1.4)(i)]), where $U^1M_n^r$ is generated étale locally by symbols $\{x_1,\dots,x_r\}$ with $x_1-1\in\pi\cdot(i_{X_T})^*\mathscr{O}_{X_T}$. Indeed, given an element in the kernel of $M_n^r\to\nu_n^{r-1}$ we can first change it by symbols $\{x_1,\dots,x_r\}$ with $x_i\in(i_{X_T})^*\mathscr{O}_{X_T}^*$ to lie also in the kernel of the map $M_n^r\to\nu_n^r$, and then it lies in $U^1M_n^r$ which is also generated by symbols (of the indicated type).

LEMMA 4.18. – *Let $r,r'\geq 0$. The composition*

$$\mathscr{H}^0(C_r)\otimes\mathscr{H}^0(C_{r'})\to\mathscr{H}^0(B_r^{\mathrm{tr}})\otimes\mathscr{H}^0(B_{r'}^{\mathrm{tr}})\to\mathscr{H}^0(B_{r+r'}^{\mathrm{tr}}),$$

where the second map is induced by the E_∞-structure on B^{tr}, factors through $\mathscr{H}^0(C_{r+r'})$.

Proof. – Let y be a local section lying in the kernel of $\mathscr{H}^0(B_r^{\mathrm{tr}})\to\nu_n^{r-1}$, similarly for y'. We may view y and y' as local sections of M_n^r and $M_n^{r'}$. They are mapped to 0 by the maps to ν_n^{r-1} and $\nu_n^{r'-1}$, thus by the exact sequence (13) the sections y and y' can be written locally as linear combinations of symbols of the form $\{x_1,\dots x_r\}$ and $\{x'_1,\dots,x'_{r'}\}$ with $x_i,x'_i\in(i_{X_T})^*\mathscr{O}_{X_T}^*$. But the product of such symbols is just the concatenated symbol $\{x_1,\dots,x_r,x'_1,\dots,x'_{r'}\}$ which thus also lies in the kernel of the map $M_n^{r+r'}\to\nu_n^{r+r'-1}$. This is true over all points $\mathfrak{p}$ of Z, so we see that $y\otimes y'$ is sent to 0 in $i_*\nu_n^{r+r'-1}$. □

COROLLARY 4.19. – *The collection of the C_r forms an E_∞-algebra C in $\mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}$-spectra which comes with a map of E_∞-algebras $C\to B^{\mathrm{tr}}$ which is levelwise the canonical map $C_r\to B_r^{\mathrm{tr}}$.*

Proof. – This follows with Corollary 4.17 and Lemma 4.18 from Lemma 4.14. □

Thus with (12) we have arranged the motivic complexes $\mathscr{M}_n^X(r)[r]$, $r\geq 0$, into an E_∞-algebra in $\mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}$-spectra on $\mathrm{Sm}_{S,\mathrm{Zar}}$.

PROPOSITION 4.20. – *The algebra C is an Ω-spectrum.*

Proof. – Set $m:=p^n$. Let $X\in\mathrm{Sm}_S$. Let $\tilde{\imath}\colon\{0\}\to\mathbb{A}_X^1$ be the closed, $\tilde{\jmath}\colon\mathbb{G}_{m,X}\to\mathbb{A}_X^1$ the open inclusion and $q\colon\mathbb{A}_X^1\to X$ the projection.

Since $\mathbb{Z}/m[\mathbb{G}_{m,S}]_{\mathrm{Zar}}\cong\mathbb{Z}/m\oplus\mathbb{Z}/m[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}$ we have a decomposition

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{G}_{m,S},C_r)\cong C_r\oplus\mathscr{R}.$$

By Theorem 3.1 we have an exact triangle

$$\tilde{\imath}_*\mathscr{M}_n(r-1)[-2]\to\mathscr{M}_n^{\mathbb{A}_X^1}(r)\to\mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r)\to\tilde{\imath}_*\mathscr{M}_n(r-1)[-1].$$

The composition

$$C_r[-r]|_{X_{\mathrm{Zar}}}\cong\mathbb{R}q_*\mathscr{M}_n^{\mathbb{A}_X^1}(r)\to\mathbb{R}q_*\mathbb{R}\tilde{\jmath}_*\mathscr{M}_n(r)$$
$$\cong\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{G}_{m,S},C_r[-r])|_{X_{\mathrm{Zar}}}\cong C_r[-r]|_{X_{\mathrm{Zar}}}\oplus\mathscr{R}[-r]|_{X_{\mathrm{Zar}}}\to C_r[-r]|_{X_{\mathrm{Zar}}}$$

is the identity. Thus when we apply $\mathbb{R}q_*$ to the above triangle we obtain a split triangle. Let $\phi\colon\mathscr{M}_n^X(r-1)[-1]\xrightarrow{\cong}\mathscr{R}[-r]|_{X_{\mathrm{Zar}}}$ be the resulting isomorphism.

We are finished when we prove that the diagram

$$\begin{array}{ccc} C_{r-1}|_{X_{\mathrm{Zar}}} & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}, C_r)|_{X_{\mathrm{Zar}}} \\ \Big\downarrow\cong & & \Big\downarrow\cong \\ \mathscr{M}_n^X(r-1)[r-1] & \xrightarrow[\cong]{\phi} & \mathscr{R}|_{X_{\mathrm{Zar}}}, \end{array}$$

where the upper horizontal map is the derived adjoint of the structure map of the spectrum C, commutes. To see this it is sufficient to show that the post composition of the two compositions with the map $\mathscr{R}|_{X_{\mathrm{Zar}}} \to \mathbb{R}j_*\mathscr{R}'|_{X_{\mathrm{Zar}}}$, where $\mathscr{R}'$ is defined to be the second summand in the decomposition $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{G}_{m,U}, A_r^{\mathrm{tr}}) \cong A_r^{\mathrm{tr}} \oplus \mathscr{R}'$, coincide, since there are no non-trivial maps from $C_{r-1}|_{X_{\mathrm{Zar}}}$ to $(i_X)_*\nu_n^{r-2}[-1]$.

But we have a transformation of diagrams from the above diagram to the diagram

(14)
$$\begin{array}{ccc} B^{\mathrm{tr}}_{r-1}|_{X_{\mathrm{Zar}}} & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}, B^{\mathrm{tr}}_r)|_{X_{\mathrm{Zar}}} \\ \Big\downarrow\cong & & \Big\downarrow\cong \\ \mathbb{R}(j_X)_*\mathscr{M}_n^{X_U}(r-1)[r-1] & \xrightarrow{\cong} & \mathbb{R}j_*\mathscr{R}'|_{X_{\mathrm{Zar}}} \end{array}$$

which commutes by the arguments in the proof of Proposition 4.4. So the two prolonged compositions in question are the two compositions in diagram (14) precomposed with the map $C_{r-1}|_{X_{\mathrm{Zar}}} \to B^{\mathrm{tr}}_{r-1}|_{X_{\mathrm{Zar}}}$, thus they coincide. This finishes the proof. □

4.1.2. The p-completed parts. – In this section we want to arrange (variants of) C for varying n into a compatible family, such that we can then take the (homotopy) limit of this system.

To start with write $\mathbb{Z}/p^\bullet$ for the inverse system comprised by the commutative rings $\mathbb{Z}/p^n$ with the obvious transition maps and $\mathrm{Mod}_{\mathbb{Z}/p^\bullet}$ for the category of modules over this system, i.e., the category whose objects are systems of abelian groups

$$\cdots \to M_n \to \cdots \to M_2 \to M_1$$

where each M_n is annihilated by p^n.

For a site $\mathscr{S}$ write $\mathrm{Sh}(\mathscr{S}, \mathbb{Z}/p^\bullet)$ for $\mathrm{Sh}(\mathscr{S}, \mathrm{Mod}_{\mathbb{Z}/p^\bullet})$.

The system of the $L_n(r)$ comprises a natural object $L_\bullet(r)$ of $\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^\bullet)$.

Let $QL_\bullet(1) \to L_\bullet(1)$ be a cofibrant replacement in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^\bullet))$ (the latter category is equipped with the inverse local projective model structure, i.e., weak equivalences and cofibrations are detected levelwise (note that the indexing category of our systems has a canonical inverse structure; for a definition of the latter notion see [**25**, Definition 5.1.1])) and let $QL_\bullet(1) \to RQL_\bullet(1)$ be a fibrant replacement via a cofibration. Thus $\mathscr{T} := RQL_\bullet(1)[1]$ is both fibrant and cofibrant.

We claim that the maps ι (for varying n) from the beginning of Section 4.1.1 can be arranged to a map

$$\iota\colon \mathbb{Z}/p^{\bullet}[\mathbb{G}_{m,U},\{1\}]_{\text{ét}} \to \mathscr{T}.$$

Indeed, suppose we have already defined ι up to level n in such a way that on each level $k \leq n$ the map represents the canonical element $1 \in H^1_{\text{ét}}(\mathbb{G}_{m,U}, L_k(1))$. We claim that we can extend the system of maps to level $n+1$: Choose a representative

$$\iota'\colon \mathbb{Z}/p^{n+1}[\mathbb{G}_{m,U},\{1\}]_{\text{ét}} \to \mathscr{T}_{n+1}.$$

Then the composition with the fibration $\mathscr{T}_{n+1} \to \mathscr{T}_n$ is homotopic to the map in level n. This homotopy can be lifted giving as second endpoint the required lift.

As in Section 4.1.1 the map of symmetric sequences $\mathrm{Sym}(\iota)$ gives rise to an E_∞-algebra $\mathrm{Sym}(\mathscr{T})$ in Ω-$\mathbb{Z}/p^{\bullet}[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}$-spectra, and we let $Q\mathrm{Sym}(\mathscr{T}) \to \mathrm{Sym}(\mathscr{T})$ be a cofbrant replacement via a trivial fibration and $Q\mathrm{Sym}(\mathscr{T}) \to RQ\mathrm{Sym}(\mathscr{T})$ be a fibrant resolution.

Set $A := \epsilon_*(RQ\mathrm{Sym}(\mathscr{T}))$ and $A^{\mathrm{tr}} := \tau_{\leq 0}(A)$. As in Section 4.1.1 A^{tr} is again an E_∞-algebra. We set $B := j_*A^{\mathrm{tr}}$. By Lemma 4.5 the algebra B computes levelwise in the n-direction the algebra which was denoted B^{tr} in Section 4.1.1.

Thus we have for every n and r the epimorphism

$$s_{r,n}\colon \mathscr{H}^0(B_{r,n}) \to i_*\nu_n^{r-1}$$

of Proposition 4.7.

Lemma 4.21. – *We have a commutative diagram*

$$\begin{array}{ccc} \mathscr{H}^0(B_{r,n+1}) & \xrightarrow{s_{r,n+1}} & i_*\nu_{n+1}^{r-1} \\ \downarrow & & \downarrow \\ \mathscr{H}^0(B_{r,n}) & \xrightarrow{s_{r,n}} & \nu_n^{r-1}. \end{array}$$

Proof. – We only have to verify that a corresponding diagram involving the maps s'_X commutes. This follows by the explicit definition of the maps b_{X_T} (recall the maps $\flat(b_X)$ enter the definition of the maps s'_X (11), and the maps b_{X_T} are used to define the maps b_X). □

We thus get an epimorphism

$$B_r \to i_*\nu_{\bullet}^{r-1}.$$

We denote by C_r the kernel of this epimorphism.

As in Section 4.1.1 we can apply a variant of Lemma 4.14 (or the lemma levelwise in the n-direction and using functoriality) to see that the collection of the C_r gives rise to an E_∞-algebra C together with a map of E_∞-algebras $C \to B$ which is levelwise (for the r-direction) the canonical map $C_r \to B_r$.

Let $X \in \mathrm{Sm}_S$. We want to see that the canonical isomorphisms (12)

$$C_{r,n}|_{X_{\mathrm{Zar}}} \cong \mathscr{M}_n^X(r)[r]$$

are compatible with the reductions $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$.

First by Lemma 3.8 the diagram

$$\begin{array}{ccc} \mathscr{M}_{n+1}^{X_U}(r)[r] & \xrightarrow{\cong} & A'_{r,n+1}|_{(X_U)_{\mathrm{Zar}}} \\ \downarrow & & \downarrow \\ \mathscr{M}_n^{X_U}(r)[r] & \xrightarrow{\cong} & A'_{r,n}|_{(X_U)_{\mathrm{Zar}}} \end{array}$$

commutes.

This shows that if we compose the two compositions in the square

$$\begin{array}{ccc} C_{r,n+1}|_{X_{\mathrm{Zar}}} & \xrightarrow{\cong} & \mathscr{M}_{n+1}^{X}(r)[r] \\ \downarrow & & \downarrow \\ C_{r,n}|_{X_{\mathrm{Zar}}} & \xrightarrow{\cong} & \mathscr{M}_n^{X}(r)[r] \end{array} \tag{15}$$

with the map $\mathscr{M}_n^X(r)[r] \to \mathbb{R}(j_X)_* \mathscr{M}_n^{X_U}(r)[r]$ the resulting two maps coincide. But $\mathrm{Hom}_{\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}},\mathbb{Z}/p^{n+1}))}(C_{r,n+1}|_{X_{\mathrm{Zar}}}, \nu_n^{r-1}[-1]) = 0$, hence (15) commutes.

Let $QC \to C$ be a cofibrant and $QC \to C^{\mathrm{cf}}$ be a fibrant replacement as E_∞-algebras. Then

$$D(p) := \lim_n C^{\mathrm{cf}}_{\bullet,n}$$

is an E_∞-algebra in $\mathbb{Z}_p[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectra in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}_p))$.

Corollary 4.22. – *For $X \in \mathrm{Sm}_S$ there is an isomorphism*

$$D(p)_r|_{X_{\mathrm{Zar}}} \cong (\mathscr{M}^X(r))^{\wedge p}[r]$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}_p))$, where $(\mathscr{M}^X(r))^{\wedge p}$ is the p-completion of $\mathscr{M}^X(r)$.

Proof. – This follows from the commutativity of (15), since the p-completion of $\mathscr{M}^X(r)$ is the homotopy limit over all the $\mathscr{M}_n^X(r)$. □

Next we will equip $D(p)$ with an orientation. Recall that given a ring spectrum E in $\mathsf{SH}(S)$ (i.e., a commutative monoid in $\mathsf{SH}(S)$) an orientation on E is a class in $\mathsf{E}^{2,1}(\mathbb{P}^\infty)$, i.e., a map

$$\Sigma^{-2,-1}\Sigma_+^\infty \mathbb{P}^\infty \to \mathsf{E}$$

in $\mathsf{SH}(S)$ the restriction of which to $\Sigma^{-2,-1}\Sigma_+^\infty\mathbb{P}^1$ is the map

$$\Sigma^{-2,-1}\Sigma_+^\infty\mathbb{P}^1 \to \Sigma^{-2,-1}\Sigma^\infty\mathbb{P}^1 \cong \mathbf{1} \to \mathsf{E},$$

where the first map is the canonical projection to a summand and the third map is the unit.

Denote by $\mathcal{O}^*_{/U}$ the sheaf (in any of the considered topologies) of abelian groups represented by $\mathbb{G}_{m,U}$ over Sm_U, let $\mathcal{O}^*_{/S}$ be defined similarly. For M a sheaf of abelian groups or an object in a triangulated category we set $M/p^n := M \otimes^{\mathbb{L}} \mathbb{Z}/p^n$.

Using the resolution of $\mathcal{O}^*_{/U}$ by the sheaf of meromorphic functions and the sheaf of codimension 1 cycles one sees that $\mathbb{R}^i j_* \mathcal{O}^*_{/U} = 0$ for $i > 0$. Thus we have an exact triangle

$$\mathcal{O}^*_{/S} \to \mathbb{R}j_* \mathcal{O}^*_{/U} \to i_*\mathbb{Z} \to \mathcal{O}^*_{/S}[1]$$

in the Zariski topology, from which we derive an exact triangle

$$\mathcal{O}^*_{/S}/p^n \to \mathbb{R}j_* \mathcal{O}^*_{/U}/p^n \to i_*\mathbb{Z}/p^n \to \mathcal{O}^*_{/S}/p^n[1]. \tag{16}$$

We have a map of exact triangles

$$\begin{array}{ccccccc}
\mathcal{O}^*_{/U} & \xrightarrow{p^n} & \mathcal{O}^*_{/U} & \longrightarrow & \mathcal{O}^*_{/U}/p^n & \longrightarrow & \mathcal{O}^*_{/U}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\epsilon_* \mathcal{O}^*_{/U} & \xrightarrow{p^n} & \mathbb{R}\epsilon_* \mathcal{O}^*_{/U} & \longrightarrow & \mathbb{R}\epsilon_* L_n(1)[1] & \longrightarrow & \mathbb{R}\epsilon_* \mathcal{O}^*_{/U}[1].
\end{array}$$

The third vertical map factors uniquely through a map $\mathcal{O}^*_{/U}/p^n \to \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1])$. Since $\mathbb{R}^1\epsilon_* \mathcal{O}^*_{/U} = 0$ we see by the long exact cohomology sheaf sequences associated to these triangles that this map is an isomorphism. Note we have $A^{\mathrm{tr}}_{1,n} \cong \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1])$ in the derived category, and thus $B_{1,n} \cong \mathbb{R}j_*\tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1]) \cong \mathbb{R}j_* \mathcal{O}^*_{/U}/p^n$.

We note that the diagram

$$\begin{array}{ccccc}
\mathbb{R}j_* \mathcal{O}^*_{/U}/p^n & & \longrightarrow & & i_*\mathbb{Z}/p^n \\
\downarrow \cong & & & & \downarrow = \\
\mathbb{R}j_*\tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1]) & \longrightarrow & \mathscr{H}^0(B_{1,n}) & \xrightarrow{s_{r,n}} & i_*\mathbb{Z}/p^n
\end{array}$$

commutes (this follows from the definition of the maps $s_{r,n}$, Proposition 4.8 and the definition of the maps s'_X). Thus together with the triangle (16) we derive an isomorphism $C_{1,n} \cong \mathcal{O}^*_{/S}/p^n$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}/p^n))$. This isomorphism is moreover unique since there are no non-trivial homomorphisms from $\mathcal{O}^*_{/S}/p^n$ to $i_*\mathbb{Z}/p^n[-1]$.

We see that there is an isomorphism $D(p)_1 \cong (\mathcal{O}^*_{/S})^{\wedge p}$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}_p))$. We denote any such isomorphism which is compatible with the projections to $C_{1,n}$ and $\mathcal{O}^*_{/S}/p^n$ by φ.

Since $D(p)$ is an Ω-spectrum which satisfies Nisnevich descent and is $\mathbb{A}^1$-local the maps $\Sigma^{-2,-1}\Sigma^\infty_+\mathbb{P}^\infty \to D(p)$ in $\mathsf{SH}(S)$ correspond to maps

$$\mathbb{Z}[\mathbb{P}^\infty]_{\mathrm{Zar}}[-1] \to D(p)_1$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$. We let $o\colon \Sigma^{-2,-1}\Sigma^\infty_+\mathbb{P}^\infty \to D(p)$ correspond to $\mathbb{Z}[\mathbb{P}^\infty]_{\mathrm{Zar}} \to \mathcal{O}^*_{/S}[1] \to (\mathcal{O}^*_{/S})^{\wedge p}[1] \xrightarrow{\varphi^{-1}[1]} D(p)_1[1]$, where the first map classifies the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^\infty$.

The definition of the bonding maps in $D(p)$ implies that the map $\mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}} \to D(p)_1$ corresponding to the unit map $\Sigma^{-1,-1}\Sigma^{\infty}(\mathbb{G}_{m,S},\{1\}) \cong \mathbf{1} \to D(p)$ is the map

$$\mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}} \to \mathscr{O}^*_{/S} \to (\mathscr{O}^*_{/S})^{\wedge p} \xrightarrow{\varphi^{-1}} D(p)_1.$$

Note that this composition is independent of the particular choice of φ since we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))}(\mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}},(\mathscr{O}^*_{/S})^{\wedge p}) &\cong \mathbb{Z}_p \\ &\cong \lim_n \mathrm{Hom}_{\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))}(\mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}},\mathscr{O}^*_{/S}/p^n).\end{aligned}$$

Let $\psi\colon (\mathbb{P}^1,\{\infty\}) \to \mathbb{G}_{m,S} \wedge S^1$ be the canonical isomorphism in $\mathscr{H}_\bullet(S)$ and let $c\colon \mathscr{H}_\bullet(S) \to \mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Nis}},\mathbb{Z}))$ be the canonical map. Then the composition

$$\mathbb{Z}[\mathbb{P}^1,\{\infty\}]_{\mathrm{Zar}} \cong c((\mathbb{P}^1,\{\infty\})) \xrightarrow{c(\psi)} c(\mathbb{G}_{m,S}\wedge S^1) \cong \mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}[1] \to \mathscr{O}^*_{/S}[1]$$

in $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Nis}},\mathbb{Z}))$ classifies the tautological line bundle on $\mathbb{P}^1$. Thus we see that the map

$$\mathbb{Z}[\mathbb{P}^1]_{\mathrm{Zar}} \to \mathbb{Z}[\mathbb{P}^1,\{\infty\}]_{\mathrm{Zar}} \to D(p)_1[1],$$

where the first map is the canonical projection onto a summand and the second map is induced by the unit map of $D(p)$ is the restriction of o to $\mathbb{Z}[\mathbb{P}^1]_{\mathrm{Zar}}$, so o is indeed an orientation.

PROPOSITION 4.23. – *The spectrum in* $\mathsf{SH}(S)$ *associated with* $D(p)$ *is orientable.*

4.2. The completed part

Set $D := \prod_p D(p)$, where the $D(p)$ are the algebras from the last section viewed as E_∞-algebras in spectra in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\hat{\mathbb{Z}}))$ and the product is taken over all primes.

Then for $X \in \mathrm{Sm}_S$ we have

$$D_r|_{X_{\mathrm{Zar}}} \cong (\prod_p (\mathscr{M}^X(r))^{\wedge p})[r]$$

in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}},\hat{\mathbb{Z}}))$.

COROLLARY 4.24. – *The spectrum in* $\mathsf{SH}(S)$ *associated with* D *is orientable.*

Proof. – This follows from Proposition 4.23. □

4.3. The rational parts

We denote by $D_{\mathbb{Q}}$ the rationalization of D as an E_∞-spectrum.

We denote by H_B the Beilinson spectrum over S, see [7, Definition 14.1.2]. It has a natural E_∞-structure ([7, Corollary 14.2.6]) and is orientable ([7, 14.1.5]).

THEOREM 4.25. – *H_B is the initial E_∞-spectrum among rational orientable E_∞-spectra.*

Proof. – This is [7, Corollary 14.2.16 (Rv)]. □

COROLLARY 4.26. – *There is a canonical map of E_∞-spectra $\mathsf{H}_B \to D_{\mathbb{Q}}$.*

Proof. – This follows from Corollary 4.24 and Theorem 4.25. □

4.4. The definition

Before giving the definition of the motivic Eilenberg-MacLane spectrum as an E_∞-object in the category of symmetric $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectra in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ we give a summary of the objects defined on the way.

For each prime p and positive number n we started with the stricly commutative ring spectrum $\mathrm{Sym}(\mathscr{T})$ in étale $\mathbb{Z}/p^n$-sheaves on $\mathrm{Sm}_{S[\frac{1}{p}]}$, took the derived push forward to the Zariski topology (a model of which is A) and then an appropriate truncation (A^{tr}). The derived push forward of that E_∞-spectrum to S is modeled by B and B^{tr} (the latter living in homologically non-negative chain complexes). Appropriate (homotopy) fibers of certain maps to logarithmic de Rham-Witt sheaves define the E_∞-algebra C.

Varying n we obtain in the limit the algebra $D(p)$, whose product over all primes p defines D. The rationalization of the latter object is $D_{\mathbb{Q}}$, which receives an E_∞-map from H_B.

DEFINITION 4.27. – *We denote by $\mathsf{M}\mathbb{Z}$ the homotopy pullback in E_∞-spectra of the diagram*

$$\begin{array}{ccc} & & D \\ & & \downarrow \\ \mathsf{H}_B & \longrightarrow & D_{\mathbb{Q}}, \end{array}$$

where we model the homotopy pullback be replacing all objects fibrantly and all maps by fibrations in the semi model structure on E_∞-algebras in the category of symmetric $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectra in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$.

If we want to emphasize the dependence on S we also write $\mathsf{M}\mathbb{Z}_S$.

The E_∞-structure on the resulting object $\mathsf{M}\mathbb{Z}$ will enable us later to develop for example a motivic functor formalism for triangulated categories of motivic sheaves on base schemes, in the form that we will define these categories to be the homotopy categories of the categories of modules over pullbacks of $\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$.

CHAPTER 5

MOTIVIC COMPLEXES II

5.1. A strictification

In this section we enlarge the motivic complexes from Section 3 to presheaves on all of Sm_S. We need some preparations.

For each $n \in \mathbb{N}$ we define a category $\mathscr{E}_n$ together with a functor $\varphi_n \colon \mathscr{E}_n \to [n]$, where $[n]$ is the category $0 \to 1 \to \cdots \to n$. The objects of $\mathscr{E}_n$ are triples (A, B, i) where $i \in [n]$ and $A \subset B \subset \{i, \dots, n\}$ with $i \in A$. There is exactly one morphism from (A, B, i) to (A', B', j) if $i \leq j$, $B \cap \{j, \dots, n\} \subset B'$ and $A' \subset A$, otherwise there is no such morphism. The functor φ_n is determined by the fact that (A, B, i) is mapped to i. We declare a map f in $\mathscr{E}_n$ to be a weak equivalence if $\varphi_n(f)$ is an identity.

A *category with weak equivalences* is a category $\mathscr{C}$ together with a subcategory $\mathscr{W}$ of $\mathscr{C}$ such that every isomorphism in $\mathscr{C}$ lies in $\mathscr{W}$. A *homotopical category* is a category with weak equivalences satisfying the two out of six property, see [**11**, 8.2].

For a category $\mathscr{C}$ with weak equivalences $\mathscr{W}$ we denote by $L^H_{\mathscr{W}} \mathscr{C}$ its hammock localization, see [**12**]. If it is clear which weak equivalences are meant we also write $L^H \mathscr{C}$. The hammock localization satisfies an ∞-categorical universal property, namely the coherent nerve of (a fibrant replacement of) the hammock localization of $\mathscr{C}$ is up to equivalence the universal ∞-category obtain from the nerve of $\mathscr{C}$ by inverting the maps in $\mathscr{W}$ (in an ∞-categorical sense) see [**22**, 1.2.1 Proposition].

A morphism in $[n]$ is defined to be a weak equivalence if it is an identity. So both $\mathscr{E}_n$ and $[n]$ are homotopical categories. Since $[n]$ is the homotopy category of $L^H([n])$ there is a natural simplicial functor $L^H([n]) \to [n]$ which is an equivalence of simplicial categories (by the universal property given in loc. cit.). Composing with the natural functor $L^H \mathscr{E}_n \to L^H([n])$ gives us the simplicial functor $L^H \mathscr{E}_n \to [n]$.

Proposition 5.1. – *The natural functor $L^H \mathscr{E}_n \to [n]$ is an equivalence of simplicial categories.*

Before giving the proof we need some preparations.

For us a *direct* category is a category with a chosen degree function, see [**25**, Definition 5.1.1].

For a category I a full subcategory $J \subset I$ is called *agreeable* if no arrow in I has a domain which is not in J and a codomain which is in J.

LEMMA 5.2. – *Let I be a direct category and $J \subset I$ an agreeable subcategory. Let $\mathcal{C}$ be a model category and $D\colon I \to \mathcal{C}$ a cofibrant diagram for the projective model structure. Then $D|_J$ is cofibrant in $\mathcal{C}^J$.*

Proof. – The right adjoint r to the restriction functor $\mathcal{C}^I \to \mathcal{C}^J$ is a right Quillen functor since for $i \in I \setminus J$ we have $r(D)(i) = *$. □

LEMMA 5.3. – *Let I be a direct category and $J \subset I$ an agreeable subcategory. Let $\mathcal{C}$ be a model category and $D\colon I \to \mathcal{C}$ a cofibrant diagram for the projective model structure. Then the canonical map $\operatorname{colim}(D|_J) \to \operatorname{colim} D$ is a cofibration.*

Proof. – The object $\operatorname{colim} D$ is obtained from $\operatorname{colim}(D|_J)$ by successively gluing in the $D(i)$ for $i \in I \setminus J$ for increasing degree of i. The domains of the attaching maps are corresponding latching spaces. □

For $i \in [n]$ let $\mathcal{E}_{n,i} := \varphi_n^{-1}(i)$ and $\mathcal{E}_{n,\leq i}$ be the full subcategory of $\mathcal{E}_n$ of objects (A, B, j) with $j \leq i$. It is easily seen that $\mathcal{E}_{n,\leq i}$ can be given the structure of a direct category (choose e.g., the degree functor $d\colon \mathcal{E}_{n,\leq i} \to \mathbb{N}$, $(A, B, j) \mapsto j + |B| - |A| = j + |B \setminus A|$). For $j \leq i \leq n$ let $\mathcal{E}_{n,[j,i]} := \varphi_n^{-1}(\{j, \ldots, i\})$.

LEMMA 5.4. – *Let $\mathcal{C}$ be a model category and $D\colon \mathcal{E}_{n,\leq i} \to \mathcal{C}$ be a projectively cofibrant diagram. Then for $k \leq j \leq i$ the restriction $D|_{\mathcal{E}_{n,[k,j]}}$ is also cofibrant.*

Proof. – Let $F\colon \mathcal{E}_{n,[k,j]} \to \mathcal{E}_{n,\leq i}$ be the inclusion. We claim that the right adjoint r to the restriction functor $\mathcal{C}^{\mathcal{E}_{n,\leq i}} \to \mathcal{C}^{\mathcal{E}_{n,[k,j]}}$ is a right Quillen functor. First note that for $D \in \mathcal{C}^{\mathcal{E}_{n,[k,j]}}$ we have $r(D)(x) = \lim \phi$, where $\phi\colon x/F \to \mathcal{C}$ sends an object $x \to F(y)$ to $D(y)$. Then r is right Quillen since for $l < k$ and an object $(A, B, l) \in \mathcal{E}_{n,\leq i}$ with $A \cap [k, j] \neq \emptyset$ the category $(A, B, l)/F$ has the initial object $(A, B, l) \to (A \cap \{m, \ldots, n\}, B \cap \{m, \ldots, n\}, m)$, where $m = \min(A \cap [k, j])$, and for other objects $x \in \mathcal{E}_{n,\leq i}$ the comma category x/F is empty, so that $r(D)(x) = *$ for D as above. □

LEMMA 5.5. – *Let $\mathcal{C}$ be a model category and $D\colon \mathcal{E}_{n,\leq i} \to \mathcal{C}$ be a projectively cofibrant diagram preserving weak equivalences. Then for any $X \in \varphi_n^{-1}(i)$ the map $D(X) \to \operatorname{colim} D$ is a weak equivalence.*

Proof. – We show by descending induction on j, starting with $j = i$, that for any $X \in \varphi_n^{-1}(i)$ the map $D(X) \to \operatorname{colim} D|_{\mathcal{E}_{n,[j,i]}}$ is a weak equivalence. For $j = i$ this follows from the fact $\mathcal{E}_{n,i}$ has a final object. Let the statement be true for $0 < j+1 \leq i$

and let us show it for j. Let $J \subset \mathcal{E}_{n,j}$ be the full subcategory on objects (A, B, j) such that $A \cap \{j+1, \ldots, i\} \neq \emptyset$. Then we have a pushout diagram

$$\begin{array}{ccc} \operatorname{colim} D|_J & \longrightarrow & \operatorname{colim} D|_{\mathcal{E}_{n,j}} \\ \downarrow & & \downarrow \\ \operatorname{colim} D|_{\mathcal{E}_{n,[j+1,i]}} & \longrightarrow & \operatorname{colim} D|_{\mathcal{E}_{n,[j,i]}}. \end{array}$$

First note that by Lemmas 5.4 and 5.2 all objects in this diagram are cofibrant. Furthermore the upper horizontal map is a cofibration by Lemma 5.3. The full subcategory of J consisting of objects (A, B, j) with $B = \{j, \ldots, n\}$ is homotopy right cofinal in J and contractible (it has an initial object), thus J is contractible. Since the diagram $D|_J$ is weakly equivalent to a constant diagram it follows that $D(X) \to \operatorname{colim} D|_J$ is a weak equivalence for any $X \in J$, thus the upper horizontal map in the above diagram is also a weak equivalence and the induction step follows. □

LEMMA 5.6. – *Let $\mathcal{C}$ be a model category and l the left adjoint to the pull back functor $r\colon \mathcal{C}^{[n]} \to \mathcal{C}^{\mathcal{E}_n}$. Let $D\colon \mathcal{E}_n \to \mathcal{C}$ be (projectively) cofibrant such that for any weak equivalence f in $\mathcal{E}_n$ the map $D(f)$ is a weak equivalence. Then $D \to r(l(D))$ is a weak equivalence.*

Proof. – We have $l(D)(i) = \operatorname{colim} D|_{\mathcal{E}_{n,\leq i}}$. Thus the claim follows from Lemma 5.5. □

LEMMA 5.7. – *Let $F\colon I \to J$ be an essentially surjective functor between small categories and $\mathcal{W} \subset I$ a subcategory making I into a category with weak equivalences. Suppose F sends any map in $\mathcal{W}$ to an isomorphism. Then the natural map $L^H_{\mathcal{W}} I \to J$ is a weak equivalence between simplicial categories if and only if for any projectively cofibrant diagram $D\colon I \to \mathsf{sSet}$ preserving weak equivalences the unit map $D \to r(l(D))$ is a weak equivalence (r is the pullback functor and l its left adjoint).*

Proof. – Let $\mathcal{C}$ be the left Bousfield localization of the model category sSet^I (equipped with the projective model structure) along the maps $\operatorname{Hom}(f, \_)$ where f runs through the maps of $\mathcal{W}$. Then $(L^H_{\mathcal{W}} I)^{\mathrm{op}}$ is weakly equivalent to the full simplicial subcategory of sSet^I consisting of cofibrant fibrant objects which become isomorphic in $\mathsf{Ho}(\mathsf{sSet}^I)$ to objects in the image of the composed functor $I^{\mathrm{op}} \to \mathsf{Ho}(\mathsf{sSet}^I) \to \mathsf{Ho}\,\mathcal{C} \hookrightarrow \mathsf{Ho}(\mathsf{sSet}^I)$. Similarly (but easier) J^{op} is weakly equivalent to a full simplicial subcategory of sSet^J. The functor $(L^H_{\mathcal{W}} I)^{\mathrm{op}} \to J^{\mathrm{op}}$ is described via these equivalences by the restriction of the push forward $\mathsf{sSet}^I \to \mathsf{sSet}^J$ followed by a fibrant replacement functor. The claim follows. □

Proof of Proposition 5.1. – The claim follows from Lemmas 5.6 and 5.7. □

Let $f\colon [n] \to [m]$ be a map in $\triangle$. We define a functor $f_*\colon \mathscr{E}_n \to \mathscr{E}_m$ by setting $f_*((A,B,i)) = (f(A), f(B), f(i))$. One checks that this determines uniquely f_*. Thus we get a *cosimplicial* object $\mathscr{E}\colon [n] \mapsto \mathscr{E}_n$ in the category of small categories with weak equivalences. Applying the hammock localization yields a cosimplicical simplicial category $L^H\mathscr{E}\colon [n] \mapsto L^H\mathscr{E}_n$ together with a map from $L^H\mathscr{E}$ to the standard cosimplicial simplicial category $[\bullet]$ which is levelwise a Dwyer-Kan equivalence.

Let $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$ be the category of triples $(A, n, (a_1, \dots, a_n))$ where A is a $\mathscr{D}$-algebra such that $\mathrm{Spec}(A) \in \mathrm{Sm}_S$ and $a_1, \dots, a_n \in A$ generate A as a $\mathscr{D}$-algebra. Morphisms are morphisms of $\mathscr{D}$-algebras with no compatibility of the generators required. Clearly the functor $(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})^{\mathrm{op}} \to \mathrm{Sm}_S$, $(A, n, (a_1, \dots, a_n)) \mapsto \mathrm{Spec}(A)$, is an equivalence onto the full subcategory of Sm_S of affine schemes.

Let $\Phi'\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathsf{Cat}$ be a strict version (using e.g., **[40]**) of the pseudofunctor which sends X to the category $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Cpx}(\mathsf{Ab}))$ and any map f in Sm_S to f^*. Let

$$\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi \to \mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$$

be the left Grothendieck fibration of the functor $\Phi := \Phi' \circ (\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \to \mathrm{Sm}_S^{\mathrm{op}})$ (see e.g., **[53]** for the Grothendieck construction). For any $X \in \mathrm{Sm}_S$ let I'_X be a set of representatives of open immersions in X_{Nis} and let I''_X be the set of maps $D^n \otimes f$ in $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Cpx}(\mathsf{Ab}))$ (here $D^n \in \mathrm{Cpx}(\mathsf{Ab})$ refers to the n-disk complex), where n runs through $\mathbb{Z}$ and f through I'_X (here $\otimes$ is an exterior tensor product). Let I_X be the union of the set of maps $i_*(I''_Z)$, where $i\colon Z \hookrightarrow X$ runs through a set of representatives of closed immersion in Sm_S with target X. For $F \in \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Cpx}(\mathsf{Ab}))$ let $R_X F$ be obtained from F by applying the small object argument to F for the set I_X of maps and the cardinal ω (here ω is the first infinite cardinal; it suffices to perform the small object argument for that cardinal by the compactness of representables). Then for any $U \in X_{\mathrm{Nis}}$ the complex $(R_X F)(U)$ is a representative of the derived sections of F over U. Moreover the assignment $(X, F) \to (R_X F)(X)$ is functorial in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi$ defining a functor $\Psi\colon \int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi \to \mathrm{Cpx}(\mathsf{Ab})$.

As in Section 3 for $X \in \mathrm{Sm}_S$ we denote by $z^r(X) \in \mathrm{Cpx}(\mathsf{Ab})$ Bloch-Levine's complex of codimension r cycles and by $\hat{z}^r(X)$ the object of $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Cpx}(\mathsf{Ab}))$ given by $U \mapsto z^r(U)$. We let $\tilde{z}^r(X) := \Psi((X, \hat{z}^r(X)))$.

For $X \in \mathrm{Sm}_S$ and $F = \{f_1, \dots f_n\}$ a set of closed immersions $f_i\colon Z_i \hookrightarrow X$ in Sm_S we denote by $z^r_F(X)$ the normalized chain complex associated to the simplicial abelian group $[n] \mapsto z^r_F(X, n)$ which is the subsimplicial abelian group of $z^r(X, \bullet)$ of cycles in good position with respect to the Z_i. Sheafifying we obtain the object $\hat{z}^r_F(X) \in \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Cpx}(\mathsf{Ab}))$, and we let $\tilde{z}^r_F(X) := \Psi((X, \hat{z}^r_F(X)))$.

We also write $z^r_F(A)$ for $z^r_F(\mathrm{Spec}(A))$ and similarly for the other versions. We have the following moving lemma due to Marc Levine.

THEOREM 5.8. – *For $X \in \mathrm{Sm}_S$ the natural map $\tilde{z}^r_F(X) \to \tilde{z}^r(X)$ is a quasi-isomorphism.*

Proof. – This follows from [**34**, Theorem 2.6.2]: We use the notation of loc. cit. and set $B := S$, $q := r$, $\mathcal{C} := F$, $e := 0$ and $E := z^q \in \mathbf{Spt}(B^{(q)})$ (z^q is introduced in loc. cit. before Remark 2.2.1.). Then the local weak equivalence in [**34**, Theorem 2.6.2 (1)] gives us the result. Note that the Axioms (4.1.1), (4.1.2) and (4.1.3) (in the presence of finite residue fields) of loc. cit. are fulfilled for E. □

Let

$$(A_0, k_0, (a_{0,1}, \dots, a_{0,k_0})) \to \cdots \to (A_n, k_n, (a_{n,1}, \dots, a_{n,k_n}))$$

be a chain of maps in $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}}$, i.e., an n-simplex, which we denote by K, in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}}$. Let $i \in [n]$ and $B \subset \{i, \dots, n\}$ with $i \in B$. Set

$$C_{i,B} := \bigotimes_{j \in B \setminus \{i\}} A_i[T_1 \dots, T_{k_j}] \cong \bigotimes_{j \in B \setminus \{i\}} A_i[T_{j,1} \dots, T_{j,k_j}],$$

where the tensor products are over A_i. If $i \leq j \leq n$, $B' \subset \{j, \dots, n\}$ with $j \in B'$ and $B \cap \{j, \dots, n\} \subset B'$ we define a map $g_{i,B,j,B'} \colon C_{i,B} \to C_{j,B'}$ over the map $A_i \to A_j$ by sending a variable $T_{l,m}$ for $l > j$ to the respective variable $T_{l,m}$ and to the image of the element $a_{l,m}$ in A_j for $l \leq j$. If furthermore $j \leq k \leq n$ and $B'' \subset \{k, \dots, n\}$ with $k \in B''$ and $B' \cap \{k, \dots, n\} \subset B''$ then we have

$$g_{j,B',k,B''} \circ g_{i,B,j,B'} = g_{i,B,k,B''}. \tag{17}$$

For $t = (A, B, i) \in \mathcal{E}_n$ we let F_t be the set of closed subschemes of $\mathrm{Spec}(C_{i,B})$ consisting of the $\mathrm{Spec}(g_{i,B,j,B\cap\{j,\dots,n\}})$ for $j \in A \setminus \{i\}$. For such t set $C_t := C_{i,B}$.

These closed subscheme are introduced to obtain a well-defined pullback of cycles: We would like to have a pullback functor along the map $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$. Instead of doing this directly we first pull back (via smooth pullback) to the affine space $\mathrm{Spec}(C_{i,B})$ over $\mathrm{Spec}(A_i)$ and then use the pullback of cycles in good position with respect to these closed subschemes.

Summarizing for $j \in A$ we have a map

$$(\mathrm{Spec}(C_t), \hat{z}^r_{F_t}(C_t)) \to (\mathrm{Spec}(C_{j,B\cap\{j,\dots,n\}}), \hat{z}^r(C_{j,B\cap\{j,\dots,n\}}))$$

in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}}} \Phi$ induced by pullback of cycles (see Appendix B) which for $B' \subset \{j, \dots, n\}$ with $B \cap \{j, \dots, n\} \subset B'$ we can prolong via smooth pullback to a map to $(\mathrm{Spec}(C_{j,B'}), \hat{z}^r(C_{j,B'}))$.

Lemma 5.9. – *Let $t \to s$ be a map in $\mathcal{E}_n$. Then the above map*

$$(\mathrm{Spec}(C_t), \hat{z}^r_{F_t}(C_t)) \to (\mathrm{Spec}(C_s), \hat{z}^r(C_s))$$

in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}}} \Phi$ factors through $(\mathrm{Spec}(C_s), \hat{z}^r_{F_s}(C_s))$.

Proof. – Let $t = (A, B, i)$ and $s = (A', B', j)$. Set $s' := (A \cap \{j, \dots, n\}, B \cap \{j, \dots, n\}, j)$. Without loss of generality we can assume $A' = A \cap \{j, \dots, n\}$. Clearly the map

$$(\mathrm{Spec}(C_t), \hat{z}^r_{F_t}(C_t)) \to (\mathrm{Spec}(C_s), \hat{z}^r(C_{s'}))$$

factors through $(\mathrm{Spec}(C_{s'}), \hat{z}^r_{F_{s'}}(C_{s'}))$. If $A' = \{j\}$ we are done, otherwise fix $k \in A' \setminus \{j\}$. Set $B'' := (B \cap \{j, \dots, n\}) \cup (B' \cap \{j, \dots, k\})$ and $s'' := (A', B'', j)$. Then we have a well defined map

$$(\mathrm{Spec}(C_{s'}), \hat{z}^r_{F_{s'}}(C_{s'})) \to (\mathrm{Spec}(C_{s''}), \hat{z}^r_{\{g_{j,B'',k,B''\cap\{k,\dots,n\}}\}}(C_{s''}))$$

since cycles meet in the correct codimension. Furthermore we have a well defined map

$$(\mathrm{Spec}(C_{s''}), \hat{z}^r_{\{g_{j,B'',k,B''\cap\{k,\dots,n\}}\}}(C_{s''})) \to (\mathrm{Spec}(C_s), \hat{z}^r_{\{g_{j,B',k,B'\cap\{k,\dots,n\}}\}}(C_s))$$

for the same reason. Altogether we see that cycles meet as claimed. □

For $f\colon t \to s$ a map in $\mathscr{E}_n$ we let $\alpha_K(f)\colon \tilde{z}^r_{F_t}(C_t) \to \tilde{z}^r_{F_s}(C_s)$ be the map $\Psi(g)$ for

$$g\colon (\mathrm{Spec}(C_t), \hat{z}^r_{F_t}(C_t)) \to (\mathrm{Spec}(C_s), \hat{z}^r_{F_s}(C_s))$$

the map in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi$ constructed above using Lemma 5.9.

Lemma 5.10. – *For $f\colon t \to s$ and $g\colon s \to r$ two maps in $\mathscr{E}_n$ we have $\alpha_K(g \circ f) = \alpha_K(g) \circ \alpha_K(f)$.*

Proof. – Let $t = (A, B, i)$, $s = (A', B', j)$ and $r = (A'', B'', k)$. Then the map $\alpha_K(f)$ is defined by pulling back cycles via the map $\mathrm{Spec}(g_{i,B,j,B'})$ and the map $\alpha_K(g)$ by pull back via $\mathrm{Spec}(g_{j,B',k,B''})$. Thus the claim follows from (17) and Theorem B.3. □

Setting $\alpha_K(t) := \tilde{z}^r_{F_t}(C_t)$ for t an object of $\mathscr{E}_n$ and using Lemma 5.10 we get a functor $\alpha_K\colon \mathscr{E}_n \to \mathrm{Cpx}(\mathsf{Ab})$.

By restricting everything to opens U in S_{Zar} we get a functor

$$\tilde{\alpha}_K\colon \mathscr{E}_n \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z})).$$

Lemma 5.11. – *The functor $\tilde{\alpha}_K$ sends weak equivalences in $\mathscr{E}_n$ to quasi-isomorphisms.*

Proof. – The weak equivalences in $\mathscr{E}_n$ are sent under $\tilde{\alpha}_K$ to cycle complexes which differ from each other by being defined for different affine spaces over a base and by good intersection properties with respect to different sets of closed subschemes. Thus the statement follows from Theorem 5.8 (which takes care of the different sets of closed subschemes) and Theorem 3.14 (which takes care of the different affine spaces appearing over a base). □

Lemma 5.12. – *Let $f\colon [m] \to [n]$ be a monomorphism in $\triangle$ and K an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$. Then the composition $\mathscr{E}_m \xrightarrow{f_*} \mathscr{E}_n \xrightarrow{\tilde{\alpha}_K} \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ is equal to $\tilde{\alpha}_{f^*K}$.*

Proof. – We use a superscript K or f^*K to distinguish between the objects which are defined above for K respectively f^*K. We have $C_t^{f^*K} = C^K_{f_*t}$ and $F_t^{f^*K} = F^K_{f_*t}$ for t an object of $\mathscr{E}_m$. Thus the claim follows on objects. The definitions of the two functors on morphisms also coincide, thus the claim follows. □

For a category I we let $\mathbb{N}(I)$ be the subcategory of $I \times \mathbb{N}$ (where $\mathbb{N}$ is a category in the usual way) which has all objects and where a map $(A, n) \to (B, m)$ belongs to $\mathbb{N}(I)$ if and only if the map $A \to B$ is the identity or if $m > n$. Note that a composition of non-identity maps is again a non-identity map in $\mathbb{N}(I)$.

We let a map $(A, n) \to (B, m)$ in $\mathbb{N}(I)$ be a weak equivalence if and only if the map $A \to B$ is the identity. We have a canonical projection functor $\mathrm{pr}\colon \mathbb{N}(I) \to I$.

Proposition 5.13. – *For any category I the canonical functor $L^H\mathbb{N}(I) \to I$ is a weak equivalence of simplicial categories.*

Proof. – We use Lemma 5.7. Let $\mathcal{C}$ be a model category and let $\mathcal{C}^{\mathbb{N}(I)}$ be equipped with the projective model structure (which exists since $\mathbb{N}(I)$ has the structure of a direct category). Let $D\colon \mathbb{N}(I) \to \mathcal{C}$ be a cofibrant diagram which preserves weak equivalences. For $i \in I$ the diagram $D|_{\mathrm{pr}/i}$ is also cofibrant by [**49**, Lemma 4.2] (it is not used here that $\mathcal{C}^I$ also should have a model structure). The full subcategory J comprised by the objects $((i, n), p((i, n)) \xrightarrow{\mathrm{id}} i)$ in pr/i is homotopy right cofinal, thus $\mathrm{colim}(D|_{\mathrm{pr}/i}) \simeq \mathrm{hocolim}(D|_J)$ from which it follows that $D \to r(l(D))$, where l is the left adjoint to $r\colon \mathcal{C}^I \to \mathcal{C}^{\mathbb{N}(I)}$, is a weak equivalence. □

Let $\mathcal{N}$ be the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}})$ and $\pi := \mathrm{Nerve}(\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}}) \xrightarrow{\mathrm{pr}} \mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}})$. For any $K \in \mathcal{N}_n$ we let $\sigma_K\colon [n] \to [n']$ be the unique epimorphism in Δ such that $K = \sigma_K^*(K')$ with $K' \in \mathcal{N}_{n'}$ non-degenerate. K' is then also uniquely determined. We let β_K be the composition

$$\mathcal{E}_n \xrightarrow{\sigma_{K,*}} \mathcal{E}_{n'} \xrightarrow{\tilde{\alpha}_{\pi(K')}} \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z})).$$

The $\tilde{\alpha}_K$ are compatible for monomorphisms in Δ (Lemma 5.12). The reason for introducing $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}})$ is that then the following compatibility statement for all maps in Δ holds:

Lemma 5.14. – *Let $h\colon [m] \to [n]$ be a map in Δ and $K \in \mathcal{N}_n$. Then the composition $\mathcal{E}_m \xrightarrow{h_*} \mathcal{E}_n \xrightarrow{\beta_K} \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ is equal to β_{h^*K}.*

Proof. – Since every composition of non-identity maps in $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathcal{D}})$ is a non-identity map we have a commutative diagram

$$\begin{array}{ccc} [m] & \xrightarrow{h} & [n] \\ \downarrow{\scriptstyle f_{h^*K}} & & \downarrow{\scriptstyle f_K} \\ [m'] & \longrightarrow & [n'], \end{array}$$

where the bottom horizontal map is a monomorphism. Thus the claim follows from Lemma 5.12 and the definition of the maps β_K and β_{h^*K}. □

Let $\Gamma\colon \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}},\mathbb{Z})) \to \mathrm{Cpx}(\mathsf{Ab})$ be a fibrant replacement functor followed by the global sections functor. We denote by qi the subcategory of quasi-isomorphisms of $\mathrm{Cpx}(\mathsf{Ab})$. By Lemma 5.11 we get for any $K \in \mathscr{N}_n$ induced functors $L^H(\Gamma \circ \beta_K)\colon L^H \mathscr{E}_n \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})$ which are compatible with maps in $\triangle$ by Lemma 5.14.

Let $q_\bullet\colon Q_\bullet \to L^H \mathscr{E}$ be a map between cosimplicial objects in sCat. By a coend construction we can pair a simplicial set L and any cosimplicial object $P_\bullet$ in sCat to obtain an object of sCat which we denote by $\mathscr{D}^L_{P_\bullet}$ (in the notation of [**23**] this is $P_\bullet \otimes L$, see Definition 16.3.1. of loc. cit.). In the case $L = \mathscr{N}$ we just write $\mathscr{D}_{P_\bullet}$. If L is the nerve of a category C we let $\mathscr{D}^C_{P_\bullet} := \mathscr{D}^L_{P_\bullet}$. We have $\mathscr{D}_{[\bullet]} \cong \mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})$, thus we have a natural map $\mathscr{D}_{Q_\bullet} \to \mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})$.

Lemma 5.15. – *Let $\mathcal{C}$ be a model category and $\phi\colon X \to Y$ a transformation between Reedy cofibrant objects $X, Y \in \mathcal{C}^\triangle$ which is objectwise a weak equivalence. Then $\phi \otimes \mathrm{id}_L\colon X \otimes L \to Y \otimes L$ (where we use the notation of loc. cit.) is a weak equivalence in $\mathcal{C}$ for any simplicial set L.*

Proof. – The object $X \otimes L$ is a colimit over a $\triangle L$-diagram D in $\mathcal{C}$ (here $\triangle L$ is the category of simplices of L), likewise for $Y \otimes L$. By [**23**, Proposition 16.3.12.] these diagrams are Reedy cofibrant, and since $\triangle L$ has fibrant constants ([**23**, Proposition 15.10.4.]) the colimits above are homotopy colimits, whence the claim. □

Lemma 5.16. – *If $Q_\bullet$ is Reedy cofibrant and the map $Q_\bullet \to [\bullet]$ is a weak equivalence then the natural map $\mathscr{D}_{Q_\bullet} \to \mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})$ is a weak equivalence of simplicial categories.*

Proof. – One deduces the result from the analogous statement for the usual adjunction between simplicial sets and simplicial categories involving the coherent nerve functor, [**37**, Theorem 2.2.0.1]: The cosimplicial object X in sCat used to define the coherent nerve functor is Reedy cofibrant (since it is the image of the standard cosimplicial object in sSet, which itself is Reedy cofibrant, with respect to the left adjoint $\mathfrak{C}$ to the coherent nerve functor). If now the simplicial set L is the nerve of a category C then the natural map $\mathfrak{C}(L) = X \otimes L \to C$ in sCat is a weak equivalence. Using a zig zag between X and $Q_\bullet$ over $[\bullet]$ through Reedy cofibrant objects Lemma 5.15 implies that then also $Q_\bullet \otimes L \to C$ is a weak equivalence, whence the claim. □

From now on suppose that $Q_\bullet$ is Reedy cofibrant and that the map $Q_\bullet \to [\bullet]$ is a weak equivalence (which can always be achieved by a cofibrant replacement of $L^H \mathscr{E}$ in $\mathsf{sCat}^\triangle$).

For $K \in \mathscr{N}_n$ let $\gamma_K := L^H(\Gamma \circ \beta_K) \circ q_n$. Then the γ_K are again compatible with maps in $\triangle$.

LEMMA 5.17. – *The compatible maps γ_K give rise to an induced map*

$$\gamma\colon \mathscr{D}_{Q_\bullet} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})$$

of simplicial categories.

Proof. – The simplicial category $\mathscr{D}_{Q_\bullet}$ arises in such a way that for each $K \in \mathscr{N}_n$, $n \in \mathbb{N}$, one takes a copy of Q_n and glues these copies according to the simplical structure in $\mathscr{N}$ and the cosimplicial structure in $Q_\bullet$. The compatibility of the maps γ_K ensures that these maps taken together (which will be a map from the coproduct of the Q_n indexed by all $n \in \mathbb{N}$, $K \in \mathscr{N}_n$, to $L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})$) factors through the glued simplicial category. □

LEMMA 5.18. – *The map γ gives rise to a diagram $\mathsf{ho}(\gamma) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})})$ which is well-defined up to canonical isomorphism.*

Proof. – This follows from a strictification result, see [**37**, Proposition 4.2.4.4]. □

REMARK 5.19. – *The above procedure to obtain from a family of maps γ_K (or even from the family of maps α_K or $\tilde{\alpha}_K$) an object in the homotopy category of a diagram category with values in* $\mathrm{Cpx}(\mathsf{Ab})$ *will be used in variations several times below.*

We define the motivic complex $\mathscr{M}(r)$ to be the push forward of $\mathsf{ho}(\gamma)[-2r]$ with respect to the composition

$$\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})}) \to \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}}) \to \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z})),$$

where the first map is the (left) adjoint to the pullback along $p\colon \mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \to \mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$ (in fact by the definition of the maps β_K and (the proof of) Proposition 5.13 the object $\mathsf{ho}(\gamma)[-2r]$ lies in the full subcategory $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}}) \hookrightarrow \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})})$, so we could have used also the right adjoint) and the second map is the Zariski localization map (note the canonical map from the derived category of Zariski sheaves on affines in Sm_S to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ is an equivalence).

5.2. Properties of the motivic complexes

Let $\mathcal{C}$, $\mathscr{S}$ be categories and I a small category. Let $\mathcal{E}'$ be a cosimplicial object in Cat over $[\bullet]$. Let for any n-simplex K of the nerve of $\mathscr{S}$ be a functor $\alpha_K\colon \mathcal{E}'_n \times I \to \mathcal{C}$ be given. Suppose these functors are compatible for monomorphisms in $\triangle$, i.e., that for $f\colon [m] \to [n]$ a monomorphism we have $\alpha_K \circ (f_* \times \mathrm{id}) = \alpha_{f^*K}$. Then for $\tilde{K}$ an n-simplex of the nerve of $\mathscr{S} \times I$ we let $T(\alpha)_{\tilde{K}}$ be the composition $\mathcal{E}'_n \to \mathcal{E}'_n \times I \xrightarrow{\alpha_K} \mathcal{C}$, where the second component of the first map is the composition $\mathcal{E}'_n \to [n] \to I$ (the second map being the second component of $\tilde{K}$) and where K is the first component of $\tilde{K}$. The $T(\alpha)_{\tilde{K}}$ are then again compatible for monomorphisms in $\triangle$.

Let $\rho\colon \mathscr{S}' \to \mathscr{S} \times I$ be a functor and suppose that the composition in $\mathscr{S}'$ of two non-identity maps is a non-identity map. Let K be an n-simplex of the nerve of $\mathscr{S}'$. Let $\sigma\colon [n] \to [n']$ be the unique epimorphism in $\triangle$ such that $K = \sigma^*(K')$ for a non-degenerate n'-simplex K'. Let $T^{\rho}(\alpha)_K$ be the composition $\mathscr{E}'_n \xrightarrow{\sigma_*} \mathscr{E}'_{n'} \xrightarrow{T(\alpha)_{\tilde{K}}} \mathcal{C}$, where $\tilde{K}$ is the image of K' in the nerve of $\mathscr{S} \times I$. Then the $T^{\rho}(\alpha)_K$ are compatible for all maps in $\triangle$.

In our applications $\mathscr{S}'$ will be $\mathbb{N}(\mathscr{S}) \times I$.

5.2.1. Comparison to flat maps. – Let the notation be as in the last section. We denote by $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}$ the subcategory of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ consisting of flat maps.

Let K be an n-simplex in the nerve of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}$. In particular we have a chain $A_0 \to \cdots \to A_n$ of smooth $\mathscr{D}$-algebras where each map is flat. We associate to this the functor $\alpha_K^{\mathrm{flat}} \colon [n] \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ which sends i to $(U \mapsto z^r(\mathrm{Spec}(A_i) \times_S U))$ and where the maps are induced by flat pullback of cycles. We denote by $\tilde{\alpha}_K^{\mathrm{flat}}$ the composition $\mathscr{E}_n \xrightarrow{\varphi_n} [n] \xrightarrow{\alpha_K^{\mathrm{flat}}} \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$.

Recall the maps $\tilde{\alpha}_K$. We have a natural transformation $\tilde{\alpha}_K^{\mathrm{flat}} \to \tilde{\alpha}_K$ which is induced by the maps $A_i \to C_t$ for $t = (A, B, i) \in \mathscr{E}_n$ (recall C_t is a polynomial algebra over A_i). We note that the cycle conditions given by the F_t are fulfilled since for a map $t \to s$ in $\mathscr{E}_n$ with $s = (A', B', j)$ the diagram

$$\begin{array}{ccc} C_t & \longrightarrow & C_s \\ \uparrow & & \uparrow \\ A_i & \longrightarrow & A_j \end{array}$$

commutes.

We denote by $\alpha_K \colon \mathscr{E}_n \times [1] \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ the functor corresponding to this natural transformation.

Thus as in the beginning of Section 5.2 we get a compatible family of maps $T^{\rho}(\alpha)_K$, where ρ is the functor $\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times [1] \to \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}} \times [1]$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times [1]$ let $\tilde{\gamma}_K := L^H(\Gamma \circ T^p(\alpha)_K) \circ q_n$.

The $\tilde{\gamma}_K$ glue to give a map

$$\tilde{\gamma} \colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times [1]} \to L_{qi}^H \mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\tilde{\gamma}) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times [1]})$ the diagram canonically associated to $\tilde{\gamma}$ (see Remark 5.19 for the last two steps).

LEMMA 5.20. – *$\mathsf{ho}(\gamma)|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}})}$ and $\mathsf{ho}(\tilde{\gamma})|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times \{1\}}$ are canonically isomorphic.*

Proof. – This follows by construction of $\mathsf{ho}(\gamma)$ and $\mathsf{ho}(\tilde{\gamma})$. □

LEMMA 5.21. – *$\mathsf{ho}(\tilde{\gamma})|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}}) \times \{0\}}$ is canonically isomorphic to the diagram on $\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}})$ which associates to an $(A, n, (a_1, \ldots, a_n), m)$ the cycle complex $z^r(A)$.*

Proof. – This follows by construction of $\mathsf{ho}(\tilde{\gamma})$. □

Let $\mathrm{Sm}_S^{\mathrm{fl}}$ be the subcategory of Sm_S of flat maps.

COROLLARY 5.22. – *The complex $\mathscr{M}(r)|_{\mathrm{Sm}_S^{\mathrm{fl}}}$ is canonically isomorphic to the diagram $X \mapsto z^r(X)[-2r]$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}^{\mathrm{fl}}_{S,\mathrm{Zar}}, \mathbb{Z}))$.*

Proof. – This follows from Lemmas 5.20 and 5.21, the fact that the map in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen,fl}})})$ associated to $\mathsf{ho}(\tilde{\gamma})$ is an isomorphism, the fact (which follows from these Lemmas) that the push forward of $\mathsf{ho}(\gamma)$ with respect to $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})}) \to \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}})$ has Zariski descent and from the proof of Proposition 5.13. □

COROLLARY 5.23. – *For $X \in \mathrm{Sm}_S$ there is a canonical isomorphism $\mathscr{M}^X(r) \cong \mathscr{M}(r)|_{X_{\mathrm{Zar}}}$ in $\mathsf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$.*

Proof. – This follows from Corollary 5.22. □

5.2.2. Some localization triangles. – We still keep the notation of Section 5.1. Let A be a smooth $\mathscr{D}$-algebra. Let K be an n-simplex in the nerve of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$. For $t \in \mathscr{E}_n$ set $C_t^A := A \otimes_D C_t$ and $F_t^A := \{\mathrm{Spec}(A) \times_S a \mid a \in F_t\}$. Then as in Section 5.1 we get functors

$$\alpha_K^A \colon \mathscr{E}_n \to \mathrm{Cpx}(\mathsf{Ab}), t \mapsto \tilde{z}^r_{F_t^A}(C_t^A),$$

and

$$\tilde{\alpha}_K^A \colon \mathscr{E}_n \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z})).$$

Now let $a_1, \ldots, a_k \in A$ be generators of A. Set $\underline{A} := (A, k, (a_1, \ldots, a_k)) \in \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$. For $(A', k', (a'_1, \ldots, a'_{k'})) \in \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ let

$$\underline{A} \otimes (A', k', (a'_1, \ldots, a'_{k'})) := \\ (A \otimes_D A', k + k', (a_1 \otimes 1, \ldots, a_k \otimes 1, 1 \otimes a'_1, \ldots, 1 \otimes a'_{k'})) \in \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}.$$

Similarly for an n-simplex K in the nerve of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ the n-simplex $\underline{A} \otimes K$ is defined (we assume the tensor products involved are chosen from now on).

For K an n-simplex in the nerve of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ we have a natural transformation $\tilde{\alpha}_K^A \to \tilde{\alpha}_{\underline{A} \otimes K}$ induced by applying Ψ to obvious maps in $\int_{\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}} \Phi$. We denote by

$$\overline{\alpha}_K^A \colon \mathscr{E}_n \times [1] \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$$

the functor corresponding to this natural transformation.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}) \times [1]$ we let

$$\gamma_K^A := L^H(\Gamma \circ T^\rho(\overline{\alpha}^A)_K) \circ q_n,$$

where ρ is the functor $\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}) \times [1] \to \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}} \times [1]$.

The γ_K^A glue to give a map

$$\gamma^A\colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times[1]} \to L_{qi}^H\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^A) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times[1]})$ the diagram canonically associated to γ^A (see Remark 5.19 for the last two steps).

Lemma 5.24. – *The push forward of* $\mathsf{ho}(\gamma^A)[-2r]|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times\{1\}}$ *to* $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$ *is canonically isomorphic to* $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{\mathrm{Zar}}, \mathscr{M}(r))$.

Proof. – Let $j\colon \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}} \to \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ be given by $X \mapsto \underline{A}\otimes X$ and $\mathbb{N}(j)\colon \mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}) \to \mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})$ be the induced functor. Let $i_1\colon \mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}) \to \mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times[1]$ be the map given by $X\mapsto X\times\{1\}$. Let i_1' be the map induced by i_1 on nerves. Then for $K\in\mathscr{N}_n$ we have $T^\rho(\overline{\alpha}^A)_{i_1'(K)} = \beta_{\underline{A}\otimes K}$. It follows that the composition

$$\mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})} \xrightarrow{\mathscr{D}_{Q_\bullet}^{i_1}} \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times[1]} \xrightarrow{\gamma^A} L_{qi}^H\mathrm{Cpx}(\mathsf{Ab}),$$

which gives rise to $\mathsf{ho}(\gamma^A)[-2r]|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times\{1\}}$, is identical to the composition

$$\mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})} \xrightarrow{\mathscr{D}_{Q_\bullet}^{\hat{j}}} \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})} \xrightarrow{\gamma} L_{qi}^H\mathrm{Cpx}(\mathsf{Ab}),$$

which gives rise to $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{\mathrm{Zar}}, \mathscr{M}(r))$, whence the claim. □

Corollary 5.25. – *The push forward of* $\mathsf{ho}(\gamma^A)[-2r]|_{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})\times\{0\}}$ *to* $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$ *is canonically isomorphic to* $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{\mathrm{Zar}}, \mathscr{M}(r))$.

Proof. – This follows from the fact that the map in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}})})$ associated to $\mathsf{ho}(\gamma^A)$ is an isomorphism. □

Now let $f\colon A\to A'$ be a flat map to a smooth $\mathscr{D}$-algebra A', let $a_1',\dots,a_{k'}'\in A'$ be generators and $\underline{A}' := (A',k',(a_1',\dots,a_{k'}'))\in\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$.

We have functors

$$\begin{aligned} a\colon \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}} &\to \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}, (B,l,(b_1,\dots,b_l)) \\ &\mapsto (A\otimes_D B, k+l, (a_1\otimes 1,\dots,a_k\otimes 1, 1\otimes b_1,\dots,1\otimes b_l)) \end{aligned}$$

and

$$\begin{aligned} b\colon \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}} &\to \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}, (B,l,(b_1,\dots,b_l)) \\ &\mapsto (A'\otimes_D B, k'+l, (a_1'\otimes 1,\dots,a_{k'}'\otimes 1, 1\otimes b_1,\dots,1\otimes b_l)), \end{aligned}$$

and a natural transformation $a\to b$ induced by f. We let $G\colon \mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}\times[1]\to\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}$ be the corresponding functor.

Let K be an n-simplex in the nerve of $\mathrm{SmAlg}_{\mathscr{D}}^{\mathrm{gen}}\times[1]$. Let

$$\alpha_{2,K}^f := \tilde{\alpha}_{G(K)}\colon \mathscr{E}_n \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}},\mathbb{Z})).$$

Let K be an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$. We have a natural transformation $\tilde{\alpha}^{A}_{K} \to \tilde{\alpha}^{A'}_{K}$ induced by f. Let $\alpha^{f}_{K}\colon \mathscr{E}_n \times [1] \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ be the corresponding functor.

For K an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \times [1]$ let

$$\alpha^{f}_{1,K} := T(\alpha^{f})_K\colon \mathscr{E}_n \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$$

(for notation see the beginning of Section 5.2).

We have a natural transformation $\alpha^{f}_{1,K} \to \alpha^{f}_{2,K}$ induced by applying Ψ to obvious maps in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi$. We denote by $\overline{\alpha}^{f}_{K}\colon \mathscr{E}_n \times [1] \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$ the corresponding functor.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2$ we let $\gamma^{f}_{K} := L^{H}(\Gamma \circ T^{\rho}(\overline{\alpha}^{f})_K) \circ q_n$, where ρ is the composition $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2 \to \mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \times [1]^2 \cong (\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \times [1]) \times [1]$ (the $\mathscr{S}$ from the beginning of Section 5.2 is now $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \times [1]$).

The γ^{f}_{K} glue to give a map

$$\gamma^{f}\colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2} \to L^{H}_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^{f}) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2})$ the diagram canonically associated to γ^{f} (see Remark 5.19 for the last two steps).

LEMMA 5.26. – *The push forward to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ of the map in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})})$ associated to $\mathsf{ho}(\gamma^{f})[-2r]|_{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1] \times \{1\}}$ is canonically isomorphic to the map*

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{\mathrm{Zar}}, \mathscr{M}(r)) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A')]_{\mathrm{Zar}}, \mathscr{M}(r)).$$

Proof. – This follows from the definition of $\mathsf{ho}(\gamma^{f})$. □

COROLLARY 5.27. – *The push forward to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ of the map in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})})$ associated to $\mathsf{ho}(\gamma^{f})[-2r]|_{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1] \times \{0\}}$ is canonically isomorphic to the map*

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{\mathrm{Zar}}, \mathscr{M}(r)) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathrm{Spec}(A')]_{\mathrm{Zar}}, \mathscr{M}(r)).$$

PROPOSITION 5.28. – *Let $i\colon Z \to X$ be a closed immersion of affine schemes in Sm_S of codimension 1 with open affine complement U. Then there is an exact triangle*

$$\begin{aligned}\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathscr{M}(r-1))[-2] &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathscr{M}(r)) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[U]_{\mathrm{Zar}}, \mathscr{M}(r)) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathscr{M}(r-1))[-1]\end{aligned}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$, where the second map is induced by the morphism $U \to X$.

Proof. – Let $A \to A''$ be the map of function algebras corresponding to i and $A \to A'$ the map corresponding to the open inclusion $U \subset X$.

For K an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$ we define a functor $\alpha^{\cdot}_{K}{}^{: \mathscr{E}_n \times [1]^2 \to \mathrm{Cpx}(\mathsf{Ab})}$ by sending $(t,0,0)$ to $\tilde{z}^{r-1}_{F^{A''}_t}(C^{A''}_t)$, $(t,1,0)$ to $\tilde{z}^{r}_{F^{A}_t}(C^{A}_t)$, $(t,1,1)$ to $\tilde{z}^{r}_{F^{A'}_t}(C^{A'}_t)$ and $(t,0,1)$ to 0. Sheafification on S yields a functor $\tilde{\alpha}^{\cdot}_{K}{}^{: \mathscr{E}_n \times [1]^2 \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))}$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2$ let $\gamma^{\cdot}_K = L^H(\Gamma \circ T^{\rho}(\tilde{\alpha}^{\cdot}_K) \circ q_n$.

The γ_K glue to give a map

$$\gamma^{: \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})}.$$

We denote by $\mathsf{ho}(\gamma^{)} \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2})$ the diagram canonically associated to γ (see again Remark 5.19).

The square in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ associated to the push forward of $\mathsf{ho}(\gamma^{)}[-2r]$ is exact by [**33**, Theorem 1.7]. Moreover by Corollary 5.25 the entries in this square in the places $(0,0)$, $(1,0)$ and $(1,1)$ are $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathscr{M}(r-1))[-2]$, $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathscr{M}(r))$ and $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[U]_{\mathrm{Zar}}, \mathscr{M}(r))$, and the map from entry $(1,0)$ to $(1,1)$ is the one induced by the map $U \subset X$ by Corollary 5.27. Thus by [**38**, Definition 1.1.2.11] we get the exact triangle as required. □

5.2.3. The étale cycle class map. – For $X \in \mathrm{Sm}_S$ and F a finite set of closed immersions in Sm_S with target X we denote by $c^r_F(X,n)$ the set of cycles (closed integral subschemes) of codimension r of $X \times \Delta^n$ which intersect all $Z \times Y$ with $Z \in F \cup \{X\}$ and Y a face of Δ^n properly.

Let $U \subset S$ open. Let m be an integer which is invertible on U. Let $\mu_m^{\otimes r} \to \mathscr{G}$ be an injectively fibrant replacement in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/m))$.

Let $X \in \mathrm{Sm}_U$. For W a closed subset of X such that each irreducible component has codimension greater or equal to r set $\mathscr{G}^W(X) := \ker(\mathscr{G}(X) \to \mathscr{G}(X \setminus W))$.

As in [**36**, 12.3] there is a canonical isomorphism of $H^{2r}(\mathscr{G}^W(X))$ with the free $\mathbb{Z}/m$-module on the irreducible components of W of codimension r and the map $\tau_{\leq 2r}\mathscr{G}^W(X) \to H^{2r}(\mathscr{G}^W(X))[-2r]$ is a quasi-isomorphism.

For F a finite set of closed immersions in Sm_U with target X denote by $\mathscr{G}^r_F(X,n)$ the colimit of the $\mathscr{G}^W(X \times \Delta^n)$ where W runs through the finite unions of elements of $c^r_F(X,n)$. The simplicial complex of $\mathbb{Z}/m$-modules $\tau_{\leq 2r}\mathscr{G}^r_F(X,\bullet)$ augments to the simplicial abelian group $z^r_F(X,\bullet)/m[-2r]$. This augmentation is a levelwise quasi-isomorphism. We denote by $\mathscr{G}^r_F(X)$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r}\mathscr{G}^r_F(X,\bullet)$. Thus we get a quasi-isomorphism $\mathscr{G}^r_F(X) \to z^r_F(X)/m[-2r]$.

On the other hand we have a canonical map $\mathscr{G}^r_F(X,n) \to \mathscr{G}(X \times \Delta^n)$ compatible with the simplicial structure. We denote by $\mathscr{G}'(X)$ the total complex associated to the double complex which is the normalized complex associated to $\mathscr{G}(X \times \Delta^\bullet)$. We have a canonical quasi-isomorphism $\mathscr{G}(X) \to \mathscr{G}'(X)$ and a canonical map $\mathscr{G}^r_F(X) \to \mathscr{G}'(X)$.

(Thus in $\mathsf{D}(\mathbb{Z}/m)$ we get a map

$$z^r_F(X)/m[-2r] \cong \mathscr{G}^r_F(X) \to \mathscr{G}'(X) \cong \mathscr{G}(X).)$$

Let $\widehat{\mathscr{G}}^r_F(X)$ be the object $(X, U \mapsto \mathscr{G}^r_F(U))$ of $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi$ and similarly for $\widehat{\mathscr{G}}'(X)$. Let $\widetilde{\mathscr{G}}^r_F(X) := \Psi(\widehat{\mathscr{G}}^r_F(X))$ and similarly for $\widetilde{\mathscr{G}}'(X)$.

Our next aim is to make this assignment functorial in X for all maps in Sm_U. In the following we sometimes insert into the above definitions A instead of $\mathrm{Spec}(A)$. Let I be the category $0 \leftarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 4$ (we will obtain the cycle class map as an I-diagram in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}))$ where all except the map indexed by $1 \to 2$ are isomorphisms). We denote by $(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m$ the full subcategory of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$ such that m is invertible in the algebras belonging to the objects. We use the notation of Section 5.1. Let K be an n-simplex in the nerve of $(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m$. We assign to K the following functor $\alpha'_K \colon \mathscr{E}_n \times I \to \mathrm{Cpx}(\mathsf{Ab})$: $(t,0) \mapsto \alpha_K(t)/m[-2r]$, $(t,1) \mapsto \widetilde{\mathscr{G}}^r_{F_t}(C_t)$, $(t,2) \mapsto \widetilde{\mathscr{G}}'(C_t)$, $(t,3) \mapsto \mathscr{G}(C_t)$, $(t,4) \mapsto \mathscr{G}(A_{\varphi_n(t)})$. Sheafifying on U_{Zar} yields a functor $\tilde{\alpha}'_K \colon \mathscr{E}_n \times I \to \mathrm{Cpx}(\mathrm{Sh}(U_{\mathrm{Zar}}, \mathbb{Z}))$. These functors are compatible for monomorphisms in $\triangle$.

For K an n-simplex of the nerve of $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I$ let $\gamma^I_K := L^H(\Gamma \circ T^\rho(\tilde{\alpha}')_K) \circ q_n$, where ρ is the functor $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I \to (\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m \times I$. The γ^I_K glue to give a map

$$\gamma^I \colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^I) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I})$ the diagram canonically associated to γ^I (see Remark 5.19).

The push forward of the I-diagram in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)})$ corresponding to the diagram $\mathsf{ho}(\gamma^I)$ to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}))$ is an I-diagram of the form $(\mathscr{M}(r)/m)|_U \cong \bullet \to \bullet \cong \bullet \cong \mathbb{R}\epsilon_* \mu_m^{\otimes r}$ which yields the cycle class map.

Next we wish to show the compatibility of this cycle class map with the original cycle class map defined for flat morphisms.

If in the following notation a collection F of closed subschemes is missing we assume that this F is empty. For K an n-simplex in the nerve of $(\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m$ (with the obvious notation) we define a functor $\alpha''_K \colon \mathscr{E}_n \times I \to \mathrm{Cpx}(\mathsf{Ab})$ in the following way: $(t,0) \mapsto z^r(A_{\varphi_n(t)})[-2r]$, $(t,1) \mapsto \mathscr{G}^r(A_{\varphi_n(t)})$, $(t,2) \mapsto \mathscr{G}'(A_{\varphi_n(t)})$, $(t,3), (t,4) \mapsto \mathscr{G}(A_{\varphi_n(t)})$. Sheafifying on U yields a functor $\tilde{\alpha}''_K \colon \mathscr{E}_n \times I \to \mathrm{Cpx}(\mathrm{Sh}(U_{\mathrm{Zar}}, \mathbb{Z}))$. There is an obvious natural transformation $\tilde{\alpha}''_K \to \tilde{\alpha}'_K$. We denote by $\overline{\alpha}_K \colon \mathscr{E}_n \times I \times [1] \to \mathrm{Cpx}(\mathrm{Sh}(U_{\mathrm{Zar}}, \mathbb{Z}))$ the corresponding functor.

For K an n-simplex of the nerve of $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m) \times I \times [1]$ let $\gamma^{I \times [1]}_K := L^H(\Gamma \circ T^\rho(\overline{\alpha})_K) \circ q_n$, where ρ is the functor $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m) \times I \times [1] \to (\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m \times I \times [1]$. The $\gamma^{I \times [1]}_K$ glue to give a map

$$\gamma^{I \times [1]} \colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m) \times I \times [1]} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^{I\times[1]}) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m)\times I\times[1]})$ the diagram canonically associated to $\gamma^{I\times[1]}$ (see Remark 5.19).

The push forward of the $I\times[1]$-diagram in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen,fl}}_{\mathscr{D}})_m)})$ corresponding to the diagram $\mathsf{ho}(\gamma^{I\times[1]})$ to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}^{\mathrm{fl}}_{U,\mathrm{Zar}},\mathbb{Z}))$ is an $I\times[1]$-diagram where the subdiagram indexed on $I\times\{0\}$ gives the old cycle class map and the subdiagram indexed on $I\times\{1\}$ the new cycle class map restricted to flat maps. Thus the two cycle class maps are canonically isomorphic (over flat maps).

Corollary 5.29. – *For $X\in\mathrm{Sm}_U$ the cycle class map $\mathscr{M}^X(r)/m \to \mathbb{R}\epsilon_*\mathbb{Z}/m(r)$ from Section 3 is canonically isomorphic to the cycle class map $(\mathscr{M}(r)/m)_{|_U} \to \mathbb{R}\epsilon_*\mu_m^{\otimes r}$ restricted to X_{Zar}.*

Now we also use the notation of Section 5.2.2. We assume m is invertible in A. For an n-simplex K of the nerve of $(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m$ we define a functor $(\alpha')^A_K\colon \mathscr{E}_n\times I\to \mathrm{Cpx}(\mathsf{Ab})$ in the following way: $(t,0)\mapsto \alpha^A_K(t)/m[-2r]$, $(t,1)\mapsto \widetilde{\mathscr{G}}^r_{F^A_t}(C^A_t)$, $(t,2)\mapsto \widetilde{\mathscr{G}}'(C^A_t)$, $(t,3)\mapsto \mathscr{G}(C^A_t)$, $(t,4)\mapsto \mathscr{G}(A\otimes_D A_{\varphi_n(t)})$. Sheafifying on U_{Zar} yields a functor $(\tilde{\alpha}')^A_K\colon \mathscr{E}_n\times I\to\mathrm{Cpx}(\mathrm{Sh}(U_{\mathrm{Zar}},\mathbb{Z}))$. These functors are compatible for monomorphisms in $\triangle$.

We have a natural transformation $(\tilde{\alpha}')^A_K \to \tilde{\alpha}'_{\underline{A\otimes K}}$ induced by applying Ψ to obvious maps in $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}}\Phi$. We denote by $\overline{\alpha}^A_K\colon \mathscr{E}_n\times I\times[1]\to \mathrm{Cpx}(\mathrm{Sh}(U_{\mathrm{Zar}},\mathbb{Z}))$ the corresponding functor.

For K an n-simplex of the nerve of $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)\times I\times[1]$ let $\gamma^{A,I\times[1]}_K := L^H(\Gamma\circ T^\rho(\overline{\alpha}^A)_K)\circ q_n$, where ρ is the functor $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)\times I\times[1]\to(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m\times I\times[1]$. The $\gamma^{A,I\times[1]}_K$ glue to give a map

$$\gamma^{A,I\times[1]}\colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m\times I\times[1]} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^{A,I\times[1]}) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)\times I\times[1]})$ the diagram canonically associated to $\gamma^{A,I\times[1]}$ (see Remark 5.19).

The push forward of the $I\times[1]$-diagram in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)})$ corresponding to the diagram $\mathsf{ho}(\gamma^{A,I\times[1]})$ to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}},\mathbb{Z}))$ is an $I\times[1]$-diagram where the subdiagram indexed on $I\times\{1\}$ gives the functor $\underline{\mathbb{R}\mathrm{Hom}}(\mathrm{Spec}(A),\_)$ applied to the cycle class map $(\mathscr{M}(r)/m)_{|_U}\to\mathbb{R}\epsilon_*\mu_m^{\otimes r}$.

Corollary 5.30. – *The subdiagram indexed on $I\times\{0\}$ of the above $I\times[1]$ diagram in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}},\mathbb{Z}))$ yields a map canonically isomorphic to the map*

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathrm{Spec}(A),(\mathscr{M}(r)/m)_{|_U}) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathrm{Spec}(A),\mathbb{R}\epsilon_*\mu_m^{\otimes r})$$

induced by the cycle class map.

Let $i\colon Z \to X$ be a closed immersion of affine schemes in Sm_S of codimension 1 with open affine complement V. The exact triangle

$$\begin{aligned}\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathscr{M}(r-1))[-2] &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathscr{M}(r)) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V]_{\mathrm{Zar}}, \mathscr{M}(r)) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathscr{M}(r-1))[-1]\end{aligned}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ from Proposition 5.28 yields an exact triangle

$$\begin{aligned}\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathscr{M}(r-1)/m)_{|_U})[-2] &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X_U]_{\mathrm{Zar}}, (\mathscr{M}(r)/m)_{|_U}) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V_U]_{\mathrm{Zar}}, (\mathscr{M}(r)/m)_{|_U}) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathscr{M}(r-1)/m)_{|_U})[-1]\end{aligned}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}))$.

Proposition 5.31. – *Let the notation be as above. Then the diagram*

$$\begin{array}{ccc}\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathscr{M}(r-1)/m)_{|_U})[-2] & \longrightarrow & \mathbb{R}\epsilon_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{ét}}, \mu_m^{\otimes(r-1)})[-2] \\ \downarrow & & \downarrow \\ \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X_U]_{\mathrm{Zar}}, (\mathscr{M}(r)/m)_{|_U}) & \longrightarrow & \mathbb{R}\epsilon_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X_U]_{\mathrm{ét}}, \mu_m^{\otimes r}) \\ \downarrow & & \downarrow \\ \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V_U]_{\mathrm{Zar}}, (\mathscr{M}(r)/m)_{|_U}) & \longrightarrow & \mathbb{R}\epsilon_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V_U]_{\mathrm{ét}}, \mu_m^{\otimes r}) \\ \downarrow & & \downarrow \\ \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathscr{M}(r-1)/m)_{|_U})[-1] & \longrightarrow & \mathbb{R}\epsilon_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{ét}}, \mu_m^{\otimes(r-1)})[-1],\end{array}$$

where the first vertical row is the exact triangle from above, the second vertical row is the corresponding exact triangle for étale sheaves and where the horizontal maps are induced by the cycle class maps, commutes.

Proof. – Let $A \to A''$ be the map of function algebras corresponding to i and $A \to A'$ the map corresponding to the open inclusion $V \to X$. We let J be the category which is defined by gluing the object $(0,0)$ of $[1]^2$ to the object 0 of $[1]$. We call c the object 1 of $[1]$ viewed as object of J, the other objects are numbered (k,l), $k,l \in \{0,1\}$. Let $\mu_m^{\otimes(r-1)} \to \tilde{\mathscr{G}}$ be an injectively fibrant replacement in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{ét}}, \mathbb{Z}/m))$. We let $\mathscr{H}_t(n)$ be the colimit of the $\mathscr{G}^W((\mathrm{Spec} C_t^A) \times \Delta^n)$, where W runs through the finite unions of elements of $c^{r-1}(\mathrm{Spec} C_t^{A''}, n)$. We denote by $\mathscr{H}_t$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r}\mathscr{H}_t(\bullet)$. We have an absolute purity isomorphism φ from the sheaf

$$\mathrm{Sm}_U \ni Y \mapsto \ker(\mathscr{G}(Y \times_S X) \to \mathscr{G}(Y \times_X V))$$

to

$$\mathrm{Sm}_U \ni Y \mapsto \tilde{\mathscr{G}}(Y \times_S Z)[-2]$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/m))$. This can be lifted to a map of (complexes of) sheaves since the target of the map is injectively fibrant. We denote any such lift also by φ. We let $\tilde{\mathscr{G}}^{r-1}$, $\tilde{\mathscr{G}}'$, $\tilde{\widetilde{\mathscr{G}}}^{r-1}$ and $\tilde{\widetilde{\mathscr{G}}}'$ be the analogs of $\mathscr{G}^r$, $\mathscr{G}'$, $\widetilde{\mathscr{G}}^r$ and $\widetilde{\mathscr{G}}'$ (and in the first and third cases also with the cycle conditions).

For K an n-simplex in the nerve of $(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m$ we define, using φ, a functor

$$\alpha^{\heartsuit}_K \colon \mathscr{E}_n \times I \times J \to \mathrm{Cpx}(\mathsf{Ab})$$

by sending

$$\begin{aligned}
&(t,0,c) \text{ to } \tilde{z}^{r-1}_{F_t^{A''}}(C_t^{A''})[-2r], && (t,1,c) \text{ to } \tilde{\widetilde{\mathscr{G}}}^{r-1}_{F_t^{A''}}(C_t^{A''})[-2],\\
&(t,0,(0,0)) \text{ to } \tilde{z}^{r-1}_{F_t^{A''}}(C_t^{A''})[-2r], && (t,1,(0,0)) \text{ to } \mathscr{H}_t,\\
&(t,0,(1,0)) \text{ to } \tilde{z}^{r}_{F_t^{A}}(C_t^{A})[-2r], && (t,1,(1,0)) \text{ to } \widetilde{\mathscr{G}}^{r}_{F_t^{A}}(C_t^{A}),\\
&(t,0,(1,1)) \text{ to } \tilde{z}^{r}_{F_t^{A'}}(C_t^{A'})[-2r], && (t,1,(1,1)) \text{ to } \widetilde{\mathscr{G}}^{r}_{F_t^{A'}}(C_t^{A'}),\\
&(t,0,(0,1)) \text{ to } 0, && (t,1,(0,1)) \text{ to } 0,\\
&(t,2,c) \text{ to } \tilde{\widetilde{\mathscr{G}}}'(C_t^{A''})[-2], && (t,3,c) \text{ to } \tilde{\mathscr{G}}(C_t^{A''})[-2],\\
&(t,2,(0,0)) \text{ to } \ker(\widetilde{\mathscr{G}}'(C_t^{A}) \to \widetilde{\mathscr{G}}'(C_t^{A'})), && (t,3,(0,0)) \text{ to } \ker(\mathscr{G}(C_t^{A}) \to \mathscr{G}(C_t^{A'})),\\
&(t,2,(1,0)) \text{ to } \widetilde{\mathscr{G}}'(C_t^{A}), && (t,3,(1,0)) \text{ to } \mathscr{G}(C_t^{A}),\\
&(t,2,(1,1)) \text{ to } \widetilde{\mathscr{G}}'(C_t^{A'}), && (t,3,(1,1)) \text{ to } \mathscr{G}(C_t^{A'}),\\
&(t,2,(0,1)) \text{ to } 0, && (t,3,(0,1)) \text{ to } 0,\\
&(t,4,c) \text{ to } \tilde{\mathscr{G}}(A'' \otimes_D A_{\varphi_n(t)})[-2],\\
&(t,4,(0,0)) \text{ to } \ker(\mathscr{G}(A \otimes_D A_{\varphi_n(t)}) \to \mathscr{G}(A' \otimes_D A_{\varphi_n(t)})),\\
&(t,4,(1,0)) \text{ to } \mathscr{G}(A \otimes_D A_{\varphi_n(t)}),\\
&(t,4,(1,1)) \text{ to } \mathscr{G}(A' \otimes_D A_{\varphi_n(t)})
\end{aligned}$$

and $(t,4,(0,1))$ to 0

(the functoriality of this assignment uses a combination of the functoriality used to define the cycle class map and the functoriality used to obtain the localization triangle).

Sheafifying we obtain a functor $\tilde{\alpha}^{\heartsuit}_K \colon \mathscr{E}_n \times I \times J \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$. For K an n-simplex in the nerve of $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I \times J$ we let $\gamma^{\heartsuit}_K := L^H(\Gamma \circ T^{\rho}(\tilde{\alpha}^{\heartsuit})_K) \circ q_n$, where ρ is the functor $\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I \times J \to (\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m \times I \times J$.

The $\gamma^{\heartsuit}_K$ glue to give a map

$$\gamma^{\heartsuit} \colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m) \times I \times J} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^{\heartsuit}) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}((\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m)\times I\times J})$ the diagram canonically associated to $\gamma^{\heartsuit}$ (see Remark 5.19).

The commutativity of the push forward of the corresponding $I \times J$-diagram in $\mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}})_m})$ to $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Zar}}, \mathbb{Z}))$ shows the claim, using Corollary 5.30. □

5.3. The naive $\mathbb{G}_m$-spectrum

PROPOSITION 5.32. – *There exists a canonical isomorphism*

$$\mathscr{M}(r-1)[-1] \cong \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}, \mathscr{M}(r))$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$.

Proof. – By Proposition 5.28 there is an exact triangle

$$\begin{aligned}\mathscr{M}(r-1)[-2] &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{A}^1_S]_{\mathrm{Zar}}, \mathscr{M}(r)) \\ &\to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}]_{\mathrm{Zar}}, \mathscr{M}(r)) \to \mathscr{M}(r-1)[-1].\end{aligned}$$

There is a split $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}]_{\mathrm{Zar}}, \mathscr{M}(r)) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{A}^1_S]_{\mathrm{Zar}}, \mathscr{M}(r))$ induced by $\{1\} \subset \mathbb{G}_{m,S}$ and the $\mathbb{A}^1$-invariance of $\mathscr{M}(r)$. This induces the required isomorphism. □

In analogy with [**41**, I 6, Définition I.124] we define a naive $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectrum E in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ to consist of a sequence of objects $\mathsf{E}_n \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$, $n \in \mathbb{N}$, together with bonding maps $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}} \otimes \mathsf{E}_n \to \mathsf{E}_{n+1}$ such that the adjoints $\mathsf{E}_n \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}, \mathsf{E}_{n+1})$ are isomorphisms. A map of naive spectra is defined in the obvious way. The theory developed in loc. cit. is also valid in our case, in particular there is always a lift of a naive spectrum to an object in the homotopy category of spectra and the functor from the homotopy category of spectra to naive spectra is conservative and full ([**41**, I 6, Proposition I.126]).

We thus get from Proposition 5.32 a naive $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectrum $\mathscr{M}$ in the category $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ with entry $\mathscr{M}(r)[r]$ in level r. We also denote a lift of $\mathscr{M}$ to the homotopy category of $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$-spectra by $\mathscr{M}$. If we want to emphasize the dependence of $\mathscr{M}$ on S we write $\mathscr{M}_S$ instead of $\mathscr{M}$.

CHAPTER 6

MOTIVIC COMPLEXES OVER A FIELD

We first note that the material from Section 5 carries over verbatim to the case of smooth schemes over a field k, except that we do not have to use the constructions involving $\int_{\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}} \Phi$ and Ψ and the functor Γ in the constructions (this is because over a field the Bloch cycle complex computes motivic cohomology over each open (so there is no need to derive the global sections functor)). We denote the resulting motivic complexes in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$ by $\mathscr{M}(r)_k$. The resulting naive $\mathbb{G}_m$-spectrum is denoted by $\mathscr{M}_k$, the same notation is used for a lift to a spectrum. In this section we will use the notation of Section 5 (like C_t^A etc.) carried over to the field case.

We let $z_e^r(X) = C_*(z_{\mathrm{equi}}(\mathbb{A}^r, 0))(X)$ (for notation see e.g., [**39**]) be the complex introduced by Friedlander and Suslin ([**16**]), so $z_e^r \in \mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$ and the Zariski hypercohomology of z_e^r computes Bloch's higher Chow groups (see loc. cit.).

Let $A = k[T_1, \ldots, T_r]$. For K an n-simplex in the nerve of $SmAlg_k^{\mathrm{gen}}$ (with corresponding chain $A_0 \to \cdots \to A_n$ of k-algebras) we define a functor $\alpha_K^e \colon \mathscr{E}_n \times [1] \to \mathrm{Cpx}(\mathsf{Ab})$ by sending $(t, 0)$ to $z_e^r(A_{\varphi_n(t)})$ and $(t, 1)$ to $z^r_{F_t^A}(C_t^A)$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [1]$ let $\gamma_K^e := L^H(T^\rho(\alpha^e)_K) \circ q_n$. The γ_K^e glue to give a map

$$\gamma^e \colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [1]} \to L_{qi}^H \mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^e) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [1]})$ the diagram canonically associated to γ^e.

The map in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$ associated to the push forward of $\mathsf{ho}(\gamma^e)[-2r]$ is an isomorphism. Moreover the target is canonically isomorphic to $\mathscr{M}(r)_k$. We get the

Proposition 6.1. – *The complexes $z_e^r[-2r]$ and $\mathscr{M}(r)_k$ are canonically isomorphic in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$.*

Pairing of cycles gives us pairings $z_e^r \otimes z_e^s \to z_e^{r+s}$ involving the Eilenberg-Zilber map). Using Proposition 6.1 this gives us pairings

$$\mathscr{M}(r)_k \otimes^{\mathbb{L}} \mathscr{M}(s)_k \to \mathscr{M}(r+s)_k \tag{18}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}},\mathbb{Z}))$ which are unital, associative and commutative. Using the diagonal we obtain for any $X \in \mathrm{Sm}_k$ pairings

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(r)_k) \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(s)_k) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(r+s)_k)$$

such that the map

$$\mathcal{M}(\bullet)_k \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(\bullet)_k)$$

is a map of $\mathbb{N}$-graded algebras.

We derive an action

$$\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(s)_k) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(r+s)_k). \tag{19}$$

In order to achieve a compatibility between the localization triangle (Proposition 5.28) and this action we study an action of z^r_e directly on Bloch's complexes which appear in the definition of the $\mathcal{M}(s)_k$.

Let A be a smooth k-algebra and let $A' := A[T_1, \ldots, T_r]$. For K an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_k$ (with corresponding chain $A_0 \to \cdots \to A_n$ of k-algebras) we define a functor $\alpha^a_K \colon \mathcal{E}_n \times [1] \to \mathrm{Cpx}(\mathsf{Ab})$ by sending $(t,0)$ to $z^r_e(A_{\varphi_n(t)}) \otimes z^s_{F^A_t}(C^A_t)$ and $(t,1)$ to $z^{r+s}_{F^{A'}_t}(C^{A'}_t)$ (the transition maps from 0 to 1 are induced by pairing of cycles).

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k) \times [1]$ let $\gamma^a_K := L^H(T^\rho(\alpha^a)_K) \circ q_n$. The γ^a_K glue to give a map

$$\gamma^a \colon \mathcal{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k)\times[1]} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^a) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k)\times[1]})$ the diagram canonically associated to γ^a.

The map in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}},\mathbb{Z}))$ associated to the push forward of $\mathsf{ho}(\gamma^a)[-2r-2s]$ yields an action map

$$\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(s)_k) \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(r+s)_k), \tag{20}$$

where $X = \mathrm{Spec}(A)$. We will show in Lemma 6.3 that the action maps (19) and (20) coincide.

Lemma 6.2. – *The pairing* (20) *for* $A = k$ *coincides with the pairing* (18).

Proof. – Let $A := k[T_1, \ldots, T_s]$ and $A' := A[T_1, \ldots, T_r]$. For K an n-simplex in the nerve of $\mathcal{S}$ (with corresponding chain $A_0 \to \cdots \to A_n$ of k-algebras) we define a functor $\alpha^c_K \colon \mathcal{E}_n \times [1]^2 \to \mathrm{Cpx}(\mathsf{Ab})$ by sending $(t,0,0)$ to $z^r_e(A_{\varphi_n(t)}) \otimes z^s_e(A_{\varphi_n(t)})$, $(t,0,1)$ to $z^{r+s}_e(A_{\varphi_n(t)})$, $(t,1,0)$ to $z^r_e(A_{\varphi_n(t)}) \otimes z^s_{F^A_t}(C^A_t)$ and $(t,1,1)$ to $z^{r+s}_{F^{A'}_t}(C^{A'}_t)$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k) \times [1]^2$ let $\gamma^c_K := L^H(T^\rho(\alpha^c)_K) \circ q_n$. The γ^c_K glue to give a map

$$\gamma^c \colon \mathcal{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k)\times[1]^2} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^c) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_k)\times[1]^2})$ the diagram canonically associated to γ^c.

The commutativity of the square in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$ associated to the push forward of $\mathsf{ho}(\gamma^c)[-2r-2s]$ shows the claim for the A under consideration. The claim for $A = k$ is shown in a similar manner. □

LEMMA 6.3. – *For affine X the action maps* (19) *and* (20) *coincide.*

Proof. – Let $X = \mathrm{Spec}(A)$ and let $A' := A[T_1, \dots, T_r]$. For K an n-simplex in the nerve of $\mathcal{S}$ (with corresponding chain $A_0 \to \cdots \to A_n$ of k-algebras) we define a functor $\alpha_K^g \colon \mathcal{E}_n \times [2] \to \mathrm{Cpx}(\mathsf{Ab})$ by sending $(t, 0)$ to $z_e^r(A_{\varphi_n(t)}) \otimes z^s_{F_t^A}(C_t^A)$, $(t, 1)$ to $z_e^r(A \otimes_k A_{\varphi_n(t)}) \otimes z^s_{F_t^A}(C_t^A)$ and $(t, 2)$ to $z^{r+s}_{F_t^{A'}}(C_t^{A'})$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [2]$ let $\gamma_K^g := L^H(T^\rho(\alpha^g)_K) \circ q_n$. The γ_K^g glue to give a map

$$\gamma^g \colon \mathcal{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [2]} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab}).$$

We denote by $\mathsf{ho}(\gamma^g) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [2]})$ the diagram canonically associated to γ^g.

Using methods as in the beginning of Section 5.2.2 one shows that the second map in the diagram $[2] \to \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathcal{S}})$ associated to $\mathsf{ho}(\gamma^g)$ is $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X], f)$, where f is the map induced by $\mathsf{ho}(\gamma^a)$ with A the k-algebra k. The composite map associated to $\mathsf{ho}(\gamma^g)$ gives the action map (20), thus the claim follows from Lemmma 6.2. □

PROPOSITION 6.4. – *Let $i \colon Z \to X$ be a closed immersion of affine schemes in Sm_k of codimension 1 with open affine complement U. Then the diagram*

$$\begin{array}{ccc}
\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathcal{M}(s-1)_k)[-2] & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathcal{M}(r+s-1)_k)[-2] \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(s)_k) & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X]_{\mathrm{Zar}}, \mathcal{M}(r+s)_k) \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[U]_{\mathrm{Zar}}, \mathcal{M}(s)_k) & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[U]_{\mathrm{Zar}}, \mathcal{M}(r+s)_k) \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathcal{M}(s-1)_k)[-1] & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z]_{\mathrm{Zar}}, \mathcal{M}(r+s-1)_k)[-1]
\end{array}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$, where the horizontal maps are the above action maps and the columns are the triangles from Proposition 5.28, commutes.

Proof. – For K an n-simplex in the nerve of $\mathcal{S}$ one defines a functor $\mathcal{E}_n \times [1] \times [1]^2 \to \mathrm{Cpx}(\mathsf{Ab})$ combining the action maps from above and the functors used in the proof of Proposition 5.28. The commutativity of the diagram associated to the resulting functor $\mathcal{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}_k^{\mathrm{gen}}) \times [1] \times [1]^2} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})$ shows the claim. □

The isomorphism

$$\mathbb{Z}[-1] \cong \mathscr{M}(0)_k[-1] \cong \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}, \mathscr{M}(1)_k)$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}},\mathbb{Z}))$ from Proposition 5.32 induces a map

$$\iota_1 \colon \mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}[-1] \to \mathscr{M}(1)_k.$$

LEMMA 6.5. – *The composition*

$$\mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}[-1] \otimes^{\mathbb{L}} \mathscr{M}(r-1)_k \to \mathscr{M}(1)_k \otimes^{\mathbb{L}} \mathscr{M}(r-1)_k \to \mathscr{M}(r)_k,$$

where the first map is induced by ι_1 and the second map is the above multiplication, is adjoint to the isomorphism from Proposition 5.32.

Proof. – We consider the diagram

$$\begin{array}{ccc}
\mathscr{M}(r-1)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}, \mathscr{M}(1)_k) & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}, \mathscr{M}(r)_k) \\
\downarrow & & \downarrow \\
\mathscr{M}(r-1)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k}]_{\mathrm{Zar}}, \mathscr{M}(1)_k) & \longrightarrow & \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k}]_{\mathrm{Zar}}, \mathscr{M}(r)_k) \\
\downarrow & & \downarrow \\
\mathscr{M}(r-1)_k \otimes^{\mathbb{L}} \mathscr{M}(0)_k[-1] & \longrightarrow & \mathscr{M}(r-1)_k[-1]
\end{array}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}},\mathbb{Z}))$, where the upper vertical maps are inclusions of direct summands and the lower square is the lower square from Proposition 6.4. Thus the diagram commutes, the compositions of the left vertical and the right vertical maps are isomorphisms and also the upper and lower horizontal maps are isomorphisms.

The adjoint of the upper horizontal map is the composition

$$\begin{aligned}
\mathscr{M}(r-1)_k \otimes^{\mathbb{L}} \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}, \mathscr{M}(1)_k) \otimes^{\mathbb{L}} \mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}} \\
\to \mathscr{M}(r-1)_k \otimes^{\mathbb{L}} \mathscr{M}(1)_k \to \mathscr{M}(r)_k,
\end{aligned}$$

where the first map is the tensor product of the identity on $\mathscr{M}(r-1)_k$ and the adjoint of an identity, and the second map is the multiplication. Thus the adjoint of the map which starts at the lower left corner of the above diagram, goes via the inverse isomorphism to the upper left corner and then to the upper right corner is the composition stated in the lemma.

The adjoint of the map from the lower right corner to the upper right corner (again via the inverse isomorphism) is the second map in question, thus the maps indeed coincide. □

Recall the isomorphisms

$$z_e^r[-2r] \cong C_*(\mathbb{Z}_{\mathrm{tr}}((\mathbb{G}_{m,k},\{1\})^{\wedge r}))[-r] \tag{21}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}},\mathbb{Z}))$ constructed in [**55**]. We get a natural map

$$\iota_2 \colon \mathbb{Z}[\mathbb{G}_{m,k},\{1\}]_{\mathrm{Zar}}[-1] \to C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_{m,k},\{1\}))[-1] \cong z_e^1[-2] \cong \mathscr{M}(1)_k.$$

LEMMA 6.6. – *The maps ι_1 and ι_2 agree.*

Proof. – Let $i_1 : \mathbb{G}_{m,k} \to \mathbb{A}^1_k$ be the natural inclusion and $i_2 : \mathbb{G}_{m,k} \to \mathbb{A}^1_k$ the inversion followed by the natural inclusion. Let Q be the sheaf cokernel of the map

$$C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_{m,k}, \{1\})) \xrightarrow{i_1 \ominus i_2} C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\})) \oplus C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\})).$$

Since this map is injective and the target is acyclic Q is a representative of the shifted complex $C_*(\mathbb{Z}_{\mathrm{tr}}((\mathbb{G}_{m,k}, \{1\})))[1]$.

For $X \in \mathrm{Sm}_k$ and maps $f, g\colon X \to \mathbb{A}^1_k$ let $h(f, g)$ be the map $X \times \Delta^1 \to \mathbb{A}^1_k$ given by $sf + (1 - s)g$, where s is the standard coordinate on the algebraic 1-simplex Δ^1.

Let $c\colon \mathbb{G}_{m,k} \to \mathbb{A}^1_k$ be the constant map to 1. Let $\varphi \in C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\}))(\mathbb{G}_{m,k})$ be given by $h(i_1, c)$, in a similar manner let ψ be given by $-h(i_2, c)$. Then

$$\partial(\varphi, \psi) \in C_0(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\}))(\mathbb{G}_{m,k}) \oplus C_0(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\}))(\mathbb{G}_{m,k})$$

is the image of $\overline{\mathrm{id}}_{\mathbb{G}_{m,k}} \in C_0(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_{m,k}, \{1\}))(\mathbb{G}_{m,k})$ with respect to the map $i_1 \ominus i_2$. Thus the canonical map $\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\mathrm{Zar}}[1] \to Q$ is represented by the image of (φ, ψ) in $Q_1(\mathbb{G}_{m,k})$.

Note there is a canonical map $Q \to C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\}))$ which is induced by the two canonical covering maps $\mathbb{A}^1_k \to \mathbb{P}^1_k$. We denote the image of (φ, ψ) in the group $C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\}))(\mathbb{G}_{m,k})$ by η. Thus η induces a map

$$\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\mathrm{Zar}}[1] \to C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\})). \tag{22}$$

The comparison isomorphism (21) is constructed using the natural map

$$C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\})) \to C_*(z_{\mathrm{equi}}(\mathbb{P}^1_k \setminus \{1\}, 0)) \cong C_*(z_{\mathrm{equi}}(\mathbb{A}^1_k, 0)),$$

and precomposition with (22) gives the map ι_2 (modulo the identification $z^1_e[-2] \cong \mathscr{M}(1)_k$ and a shift). Let us denote the image of η in $z^1(\mathbb{G}_{m,k} \times_k \mathbb{A}^1_k, 1)$ by η'. The cycle (in the sense of homological algebra) η' is a sum $\varphi' + \psi'$ of chains (where each summand is a chain constituted by an algebraic cycle or the negative thereof). Here φ' (resp. ψ') is the image of φ (resp. ψ). We want to compute the boundary of η' for the triangle defined by the sequence

$$z^0(\{0\} \times_k \mathbb{A}^1_k) \to z^1(\mathbb{A}^1_k \times_k \mathbb{A}^1_k) \to z^1(\mathbb{G}_{m,k} \times_k \mathbb{A}^1_k). \tag{23}$$

Therefore we lift η' to the middle complex, take the boundary and view it as an element of the left complex. We first give a lift of φ'.

Let $c'\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the constant map to $\{1\}$. We let $\tilde{\varphi} \in C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\}))(\mathbb{A}^1_k)$ be given by $h(\mathrm{id}_{\mathbb{A}^1_k}, c')$ and $\tilde{\varphi}'$ be the image of $\tilde{\varphi}$ with respect to the composition

$$C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1_k, \{1\}))(\mathbb{A}^1_k) \to C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\}))(\mathbb{A}^1_k) \to z^1_e(\mathbb{A}^1_k, 1) \to z^1(\mathbb{A}^1_k \times_k \mathbb{A}^1_k, 1).$$

Then $\tilde{\varphi}'$ is a lift of φ'. The boundary of the image of $\tilde{\varphi}$ in $C_1(\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_k, \{1\}))(\mathbb{A}^1_k)$ is the graph of the canonical embedding $\mathbb{A}^1_k \to \mathbb{P}^1_k$.

We let t be the standard coordinate on $\mathbb{G}_{m,k}$, s the standard coordinate on Δ^1 and $[x_0 : x_1]$ homogeneous coordinates on $\mathbb{P}^1_k$. Then the effective cycle corresponding to the image of $-\psi$ in $C_1(\mathbb{Z}_{\mathrm{tr}}((\mathbb{P}^1_k, \{1\})))(\mathbb{G}_{m,k})$ is given by the homogeneous equation

$$sx_0 + t(1-s)x_0 = tx_1.$$

The closure Z in $\mathbb{A}^1_k \times \Delta^1 \times_k \mathbb{P}^1_k$ is given by the same equation. Intersecting with $s = 0$ (resp. $s = 1$) gives the closed subscheme with equation $t(x_0 - x_1) = 0$ (resp. $x_0 = tx_1$). This shows that the intersections of this closure with the faces of Δ^1 are proper. We view the restriction of Z to $\mathbb{P}^1_k \setminus \{1\}$ as a cycle in $z^1(\mathbb{A}^1_k \times_k \mathbb{A}^1_k, 1)$ and denote its negative by $\tilde{\psi}'$. Thus $\tilde{\psi}'$ is a lift of ψ'.

We see that the boundary of $\tilde{\varphi}'$ cancels with the contribution of the boundary of $\tilde{\psi}'$ for $s = 1$. Thus the boundary of $\tilde{\varphi}' + \tilde{\psi}'$ is given by the equation $t = 0$. It follows that the boundary of η' for the triangle defined by (23) corresponds to 1. The claim follows. □

Theorem 6.7. – *The spectrum $\mathcal{M}_k$ is isomorphic to the motivic Eilenberg-MacLane spectrum $\mathsf{M}\mathbb{Z}_k$ over k.*

Proof. – The isomorphisms (21) are compatible with the product structures, see [**30**, Proposition 3.3] for the case of a perfect ground field and [**31**] for the general case. Thus the claim follows from Lemmas 6.5 and 6.6. □

CHAPTER 7

COMPARISONS

7.1. The exceptional inverse image of $\mathscr{M}$

We let x be a closed point of S, k its residue field and $i\colon \mathrm{Spec}(k) \hookrightarrow S$ the corresponding closed inclusion. Set $U := S \setminus \{x\}$, $U = \mathrm{Spec}(\mathscr{D}')$, and let j be the open inclusion $U \to S$. We view k as a $\mathscr{D}$-algebra in the canonical way.

We use the notation of Section 5. Let K be an n-simplex in the nerve of $\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}$. For $t \in \mathscr{E}_n$ set $C'_t := \mathscr{D}' \otimes_{\mathscr{D}} C_t$ and $F'_t := \{U \times_S a \mid a \in F_t\}$. Also set $C''_t := k \otimes_{\mathscr{D}} C_t$ and $F''_t := \{\{x\} \times_S a \mid a \in F_t\}$. Define a functor $\alpha^!_K\colon \mathscr{E}_n \times [1]^2 \to \mathrm{Cpx}(\mathsf{Ab})$ by sending $(t,0,0)$ to $z^{r-1}_{F''_t}(C''_t)$, $(t,1,0)$ to $\tilde{z}^r_{F_t}(C_t)$, $(t,1,1)$ to $\tilde{z}^r_{F'_t}(C'_t)$ and $(t,0,1)$ to 0. Sheafification on S yields a functor $\tilde{\alpha}^!_K\colon \mathscr{E}_n \times [1]^2 \to \mathrm{Cpx}(\mathrm{Sh}(S_{\mathrm{Zar}}, \mathbb{Z}))$.

For K an n-simplex in the nerve of $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2$ let $\gamma^!_K := L^H(\Gamma \circ T^\rho(\tilde{\alpha}^!)_K) \circ q_n$, where ρ is the functor $\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2 \to \mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}} \times [1]^2$.

The $\gamma^!_K$ glue to give a map $\gamma^!\colon \mathscr{D}_{Q_\bullet}^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2} \to L^H_{qi}\mathrm{Cpx}(\mathsf{Ab})$.

We denote by $\mathsf{ho}(\gamma^!) \in \mathsf{Ho}(\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{N}(\mathrm{SmAlg}^{\mathrm{gen}}_{\mathscr{D}}) \times [1]^2})$ the diagram canonically associated to $\gamma^!$.

The square in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ associated to the push forward of $\mathsf{ho}(\gamma^!)[-2r]$ is exact. We thus obtain the

Proposition 7.1. – *There is an exact triangle*

$$i_* \mathscr{M}(r-1)_k[-2] \to \mathscr{M}(r) \to j_* j^* \mathscr{M}(r) \to i_* \mathscr{M}(r-1)_k[-1]$$

in $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Nis}}, \mathbb{Z}))$.

Corollary 7.2. – *There is a canonical isomorphism*

$$i^! \mathscr{M}(r) \cong \mathscr{M}(r-1)_k[-2]$$

in $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Nis}}, \mathbb{Z}))$.

Corollary 7.3. – *There is a canonical isomorphism of naive $\mathbb{G}_m$-spectra*

$$i^! \mathscr{M} \cong \mathscr{M}_k(-1)[-2]$$

and also such an isomorphism of spectra.

Proof. – The bonding maps are the same. □

Theorem 7.4. – *There is an isomorphism of spectra*

$$i^! \mathscr{M} \cong \mathsf{M}\mathbb{Z}_k(-1)[-2].$$

Proof. – This follows from Corollary 7.3 and Theorem 6.7. □

7.2. Pullback to the generic point

Let K be the fraction field of D and $f\colon \mathrm{Spec}(K) \to S$ the canonical morphism.

Lemma 7.5. – *There is an isomorphism $f^* \mathscr{M}(r) \cong \mathscr{M}(r)_K$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{K,\mathrm{Zar}}, \mathbb{Z}))$.*

Proof. – Let $X \in \mathrm{Sm}_K$ and x a point of X. Locally around x the scheme X has a model $\widetilde{X} \in \mathrm{Sm}_S$ with corresponding point $\tilde{x}$, such that we have an isomorphism $\mathcal{O}_{X,x} \cong \mathcal{O}_{\widetilde{X},\tilde{x}}$, and locally around $\tilde{x}$ such models are canonically isomorphic. It follows that the fiber of $f^* \mathscr{M}(r)$ at x is canonically equivalent to the fiber of $\mathscr{M}(r)$ at $\tilde{x}$. By the continuity of the cycle complexes (say for cofiltered systems where all transition maps are open immersions) it follows that both fibers are equivalent to (a shift of) Bloch's cycle complex attached to $\mathrm{Spec}(\mathcal{O}_{X,x})$. □

Theorem 7.6. – *There is an isomorphism $f^* \mathscr{M} \cong \mathsf{M}\mathbb{Z}_K$ in $\mathsf{SH}(K)$.*

Proof. – There is an isomorphism $f^* \mathscr{M} \cong \mathscr{M}_K$. The result now follows from Theorem 6.7. □

7.3. Weight 1 motivic complexes

We keep the notation of the last section.

Proposition 7.7. – *Let k be a field, $\mathbb{Z}(1) = C_*(\mathbb{Z}_{\mathrm{tr}}((\mathbb{G}_{m,k}, \{1\})))[-1]$ be the motivic complex of weight 1 (in the notation of* [**55**] *or* [**39**]*). Then there is an isomorphism $\mathbb{Z}(1) \cong \mathcal{O}^*[-1]$ in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Zar}}, \mathbb{Z}))$. Moreover the map $\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\mathrm{Zar}} \to \mathcal{O}^*$ induced by this map is the canonical one.*

Proof. – The first part is [**39**, Theorem 4.1], the second part is contained in the proof of [**39**, Lemma 4.4]. □

We denote by $S^{(1)}$ the set of codimension 1 points of S, and for each $\mathfrak{p} \in S^{(1)}$ we let $\kappa(\mathfrak{p})$ be the residue field of $\mathfrak{p}$ and $i_{\mathfrak{p}}$ the corresponding inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to S$.

LEMMA 7.8. – *There is an exact triangle*

$$\bigoplus_{\mathfrak{p}\in S^{(1)}} i_{\mathfrak{p},*}\mathscr{M}(r-1)_{\kappa(\mathfrak{p})}[-2] \to \mathscr{M}(r) \to f_*\mathscr{M}(r)_K \to \bigoplus_{\mathfrak{p}\in S^{(1)}} i_{\mathfrak{p},*}\mathscr{M}(r-1)_{\kappa(\mathfrak{p})}[-1]$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$.

Proof. – This follows from Corollary 7.2 and Lemma 7.5. □

COROLLARY 7.9. – *We have* $\mathscr{H}^i(\mathscr{M}(1)) \cong 0$ *for* $i \neq 1$ *and there is an exact sequence*

$$0 \to \mathscr{H}^1(\mathscr{M}(1)) \to f_*\mathscr{O}^*_{/K} \to \bigoplus_{\mathfrak{p}\in S^{(1)}} i_{\mathfrak{p},*}\mathbb{Z} \to 0$$

in $\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z})$.

Proof. – This follows from Lemma 7.8 and Proposition 7.7. □

THEOREM 7.10. – *There is a canonical isomorphism*

$$\mathscr{M}(1) \cong \mathscr{O}^*_{/S}[-1]$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$.

Proof. – We have a canonical map $\mathbb{Z}[\mathbb{G}_{m,S}]_{\mathrm{Zar}} \to \mathscr{H}^1(\mathscr{M}(1))$ whose composition with the map $\mathscr{H}^1(\mathscr{M}(1)) \to f_*\mathscr{O}^*_{/K}$ is the canonical map by Proposition 7.7. The image of this canonical map $\mathbb{Z}[\mathbb{G}_{m,S}]_{\mathrm{Zar}} \to f_*\mathscr{O}^*_{/K}$ is $\mathscr{O}^*_{/S}$. The claim follows now from Corollary 7.9. □

Let $\mathsf{H}_{B,1} \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$ be the first $\mathbb{A}^1$-and Nisnevich-local space in a Ω-$\mathbb{G}_{m,S}$-spectrum model of H_B.

THEOREM 7.11. – *There is a canonical isomorphism*

$$\mathsf{H}_{B,1} \cong (\mathscr{O}^*_{/S})_{\mathbb{Q}}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}},\mathbb{Z}))$.

Proof. – The proof is similar to the proof of Theorem 7.10. □

7.4. Rational spectra

We keep the notation of the last sections.

COROLLARY 7.12. – *There is an exact triangle*

$$\bigoplus_{\mathfrak{p}\in S^{(1)}} i_{\mathfrak{p},*}\mathsf{M}\mathbb{Z}_{\kappa(\mathfrak{p})}(-1)[-2] \to \mathscr{M} \to f_*\mathsf{M}\mathbb{Z}_K \to \bigoplus_{\mathfrak{p}\in S^{(1)}} i_{\mathfrak{p},*}\mathsf{M}\mathbb{Z}_{\kappa(\mathfrak{p})}(-1)[-1]$$

in $\mathsf{SH}(S)$.

Proof. – This follows from Theorems 7.4 and 7.6. □

We call a spectrum $E \in \mathsf{SH}(S)$ a Beilinson motive if it is H_B-local (compare with [7, Definition 14.2.1]). This is the case if and only if the canonical map $E \to \mathsf{H}_B \wedge E$ is an isomorphism ([7, Corollary 14.2.16]).

COROLLARY 7.13. – *The rationalization* $\mathscr{M}_{\mathbb{Q}}$ *is a Beilinson motive.*

Proof. – The rational motivic Eilenberg-MacLane spectra $\mathsf{M}\mathbb{Q}_{\kappa(\mathfrak{p})}$ for $\mathfrak{p} \in S^{(1)}$ and $\mathsf{M}\mathbb{Q}_K$ are orientable, thus their push forwards to S are Beilinson motives ([7, Corollary 14.2.16 (Ri)]). Now the claim follows from Corollary 7.12. □

By construction we have $\mathscr{M}_{0,0} = \mathbb{Z}$. By Corollary 7.13 the elements of $(\mathscr{M}_{\mathbb{Q}})_{0,0}$ correspond bijectively to maps $\mathsf{H}_B \to \mathscr{M}_{\mathbb{Q}}$, and we let $u\colon \mathsf{H}_B \to \mathscr{M}_{\mathbb{Q}}$ be the map corresponding to $1 \in \mathbb{Q} = (\mathscr{M}_{\mathbb{Q}})_{0,0}$.

THEOREM 7.14. – *The map u is an isomorphism.*

LEMMA 7.15. – *For* $\mathfrak{p} \in S^{(1)}$ *the map*

$$i_{\mathfrak{p}}^{!}\mathsf{H}_B \xrightarrow{i_{\mathfrak{p}}^{!}u} i_{\mathfrak{p}}^{!}\mathscr{M}_{\mathbb{Q}}$$

is an isomorphism.

Proof. – The definition of u is in such a way that the map $u_1\colon \mathsf{H}_{B,1} \to \mathscr{M}(1)_{\mathbb{Q}}[1]$ induced by u is compatible with the natural maps from $\mathbb{Q}[\mathbb{G}_{m,S}]_{\mathrm{Zar}}$ to $\mathsf{H}_{B,1}$ and $\mathscr{M}(1)_{\mathbb{Q}}[1]$. Since these latter maps are surjections it follows that the composition

$$(\mathscr{O}^*_{/S})_{\mathbb{Q}} \cong \mathsf{H}_{B,1} \xrightarrow{u_1} \mathscr{M}(1)_{\mathbb{Q}}[1] \cong (\mathscr{O}^*_{/S})_{\mathbb{Q}},$$

where the first resp. third map is the identification from Theorem 7.11 resp. Theorem 7.10, is the identity. It follows that the map

$$\varphi\colon \mathsf{H}_{B,\kappa(\mathfrak{p})}(-1)[-2] \cong i_{\mathfrak{p}}^{!}\mathsf{H}_B \xrightarrow{i_{\mathfrak{p}}^{!}u} i_{\mathfrak{p}}^{!}\mathscr{M}_{\mathbb{Q}} \cong \mathsf{M}\mathbb{Q}_{\kappa(\mathfrak{p})}(-1)[-2],$$

where the first isomorphism is from [7, Theorem 14.4.1], induces an isomorphism on zeroth spaces of Ω-$\mathbb{G}_{m,\kappa(\mathfrak{p})}$-spectra, thus $\varphi(1)[2]$ corresponds to a nonzero element in $\mathbb{Q} = \mathrm{Hom}_{\mathsf{SH}(\kappa(\mathfrak{p}))}(\mathsf{H}_{B,\kappa(\mathfrak{p})}, \mathsf{M}\mathbb{Q}_{\kappa(\mathfrak{p})})$. The claim follows. □

Proof of Theorem 7.14. – The map f^*u is an isomorphism. Now the claim follows from Lemma 7.15 and the map between triangles of the form as in Corollary 7.12 induced by u. □

7.5. The isomorphism between $\mathsf{M}\mathbb{Z}$ and $\mathscr{M}$

First we recast the definition of $\mathsf{M}\mathbb{Z}$ working purely in triangulated categories. We use the notation of Section 4.1.1.

The canonical triangle

$$L_n(r-1)[-2] \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{A}^1_U]_{\text{ét}}, L_n(r)) \\ \to \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,U}]_{\text{ét}}, L_n(r)) \to L_n(r-1)[-1]$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^n))$ induces a canonical isomorphism

$$L_n(r-1)[-1] \cong \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}, L_n(r)).$$

We thus get a naive $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}$-spectrum $\mathscr{L}_n$ with entry $L_n(r)[r]$ in level r. By the choice of the map ι the underlying naive $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}$-spectrum of the spectrum $\mathrm{Sym}(\mathscr{T})$ is $\mathscr{L}_n$.

Since we have canonical isomorphisms

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}, \mathbb{R}\epsilon_*\mathscr{L}_{n,r}) \cong \mathbb{R}\epsilon_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}, \mathscr{L}_{n,r})$$

we get a naive $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}$-spectrum $\mathbb{R}\epsilon_*\mathscr{L}_n$. The naive prespectrum (with the obvious definition of naive prespectrum) $\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_n$ with entries $\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r}$ is a naive spectrum by Proposition 4.4.

We also have canonical isomorphisms

$$\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}, \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r}) \cong \mathbb{R}j_*\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Zar}}, \tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r}),$$

thus $H_n := \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_n$ is a naive $T_n := \mathbb{Z}/p^n[\mathbb{G}_{m,S},\{1\}]_{\mathrm{Zar}}$-spectrum.

By construction the underlying naive T_n-spectrum of B' (see Section 4.1.1) equals H_n.

Proposition 4.7 furnishes canonical maps

$$H_{n,r} \to i_*\nu_n^{r-1}$$

whose homotopy fibers we denote by $F_{n,r}$.

As after equation (12) it follows that the $F_{n,r}$ are determined up to canonical isomorphisms.

Using the structure maps of H_n we get maps

$$T_n \otimes F_{n,r} \to T_n \otimes H_{n,r} \to H_{n,r+1} \to i_*\nu_n^r$$

which are 0 since we know already from Section 4.1.1 that the $F_{n,r}$ organize themselves into a naive T-spectrum such that the maps $F_{n,r} \to H_{n,r}$ form a map of naive T_n-spectra.

Thus in turn the maps $T_n \otimes F_{n,r} \to H_{n,r+1}$ factorize through $F_{n,r+1}$. These factorizations are unique since there are no non-trivial maps $T_n \otimes F_{n,r} \to i_*\nu_n^r[-1]$.

Thus we see that the $F_{n,r}$ assemble in a unique way into a naive T_n-spectrum F_n together with a map of naive T_n-spectra $F_n \to H_n$.

Of course the underlying naive T_n-spectrum of C (see Section 4.1.1) is F_n.

Set $\mathscr{M}_n(r) := \mathscr{M}(r)/p^n$ and $\mathscr{M}_n := \mathscr{M}/p^n$. We have étale cycle class maps

$$\mathscr{M}_n(r)|_U \to \mathbb{R}\epsilon_* L_n(r)$$

(see Section 5.2.3).

Proposition 5.31 (applied with $X = \mathbb{A}^1_S$ and $Z = \{1\}$) implies that these cycle class maps combine to give a map of naive $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{\mathrm{Zar}}$-spectra $j^* \mathscr{M}_n \to \mathbb{R}\epsilon_* \mathscr{L}_n$.

By Theorem 3.3 this map factors uniquely through $\tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_n$ (by an isomorphism, see Theorem 3.9) and by adjointness we get a map of naive T_n-spectra $\mathscr{M}_n \to H_n$ which factors as

$$\mathscr{M}_n \to \mathbb{R}j_* j^* \mathscr{M}_n \xrightarrow{\cong} H_n.$$

We have commutative diagrams

$$\begin{array}{ccccccc}
\mathscr{M}_n(r)[r] & \longrightarrow & \mathbb{R}j_* j^* \mathscr{M}_n(r)[r] & \longrightarrow & i_* \mathscr{M}_{n,Z}(r-1)[r-1] & \longrightarrow & \mathscr{M}_n(r)[r+1] \\
 & & \downarrow{\cong} & & \downarrow{\cong} & & \\
 & & H_{n,r} & \longrightarrow & i_* \nu_n^{r-1}. & &
\end{array}$$

(the vertical maps being isomorphisms) with an exact triangle as upper row: The exact triangle is induced by a variant of Proposition 7.1. The right vertical isomorphism is a global version of Theorem 3.4. The diagram commutes by a global version of Proposition 3.4.

Thus we get a unique factorization $\mathscr{M}_n \to F_n$ which is an isomorphism.

Set $T := \mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}}$. By adjointness we get a map of naive T-spectra $\mathscr{M} \to F_n$.

For $F, G \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}/p^n))$ we denote by $\mathrm{map}_{\mathbb{Z}/p^n}(F, G) \in \mathsf{Ho}(\mathsf{sSet})$ the mapping space between F and G. We use the same notation for étale sheaves and also for spectra. If the coefficients are $\mathbb{Z}$ we use the notation map.

Lemma 7.16. – *For any* $r \geq 0$ *we have* $\mathrm{map}_{\mathbb{Z}/p^n}(F_{n,r}, F_{n,r}) \cong \mathbb{Z}/p^n$.

Proof. – We have

$$\begin{aligned}
\mathrm{map}_{\mathbb{Z}/p^n}(F_{n,r}, \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}) &\cong \mathrm{map}_{\mathbb{Z}/p^n}(j^* F_{n,r}, \tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}) \\
&\cong \mathrm{map}_{\mathbb{Z}/p^n}(\tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}, \tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}) \\
&\cong \mathrm{map}_{\mathbb{Z}/p^n}(\tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}, \mathbb{R}\epsilon_* \mathscr{L}_{n,r}) \\
&\cong \mathrm{map}_{\mathbb{Z}/p^n}(\epsilon^*\tau_{\leq 0}\mathbb{R}\epsilon_* \mathscr{L}_{n,r}, \mathscr{L}_{n,r}) \\
&\cong \mathrm{map}_{\mathbb{Z}/p^n}(\mathscr{L}_{n,r}, \mathscr{L}_{n,r}) \cong \mathbb{Z}/p^n.
\end{aligned}$$

We have a long exact sequence

$$\cdots \to \mathrm{Hom}(F_{n,r}[i+1], i_*\nu_n^{r-1}) \to \mathrm{Hom}(F_{n,r}[i], F_{n,r})$$
$$\to \mathrm{Hom}(F_{n,r}[i], \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r}) \to \mathrm{Hom}(F_{n,r}[i], i_*\nu_n^{r-1}) \to \cdots$$

Thus the maps

$$\mathrm{Hom}(F_{n,r}[i], F_{n,r}) \to \mathrm{Hom}(F_{n,r}[i], \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r})$$

are isomorphisms for $i > 0$ and injective for $i = 0$. But since

$$\mathrm{Hom}(F_{n,r}, \mathbb{R}j_*\tau_{\leq 0}\mathbb{R}\epsilon_*\mathscr{L}_{n,r}) \cong \mathbb{Z}/p^n$$

the map for $i = 0$ is also surjective, so the claim follows. □

Corollary 7.17. – *The naive T-spectra F_n have lifts to T-spectra which are unique up to canonical isomorphism (in the homotopy category). Denoting these lifts also by F_n we have* $\mathrm{map}_{\mathbb{Z}/p^n}(F_n, F_n) \cong \mathbb{Z}/p^n$*. Moreover we have* $\mathrm{map}(\mathscr{M}, F_n) \cong \mathbb{Z}/p^n$ *(the latter mapping space is computed in T-spectra).*

Proof. – The mapping space from a lift of F_n to spectra to itself is computed as

$$\mathrm{holim}_r \mathrm{map}_{\mathbb{Z}/p^n}(F_{n,r}, F_{n,r}) \cong \mathbb{Z}/p^n,$$

from this the result follows. □

Clearly we have $F_{n+1} \otimes^{\mathbb{L}}_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \cong F_n$.

Thus we get compatible maps $\mathscr{M} \to F_n$ for all n in the homotopy category of T-spectra. This furnishes a map $\mathscr{M} \to \mathrm{holim}_n F_n$ which is the p-completion map. Note that the homotopy limit is uniquely determined up to canonical isomorphism.

We have a canonical isomorphism $\mathrm{holim}_n F_n \cong D(p)$, where we use the notation of Section 4.1.2. Thus we get a canonical map $\mathscr{M} \to D$ (notation from Section 4.2) in the homotopy category of T-spectra.

Moreover the diagram

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & D \\ \Big\downarrow{\scriptstyle u^{-1}\circ f} & & \Big\downarrow \\ \mathsf{H}_B & \longrightarrow & D_{\mathbb{Q}}, \end{array}$$

where $f\colon \mathscr{M} \to \mathscr{M}_{\mathbb{Q}}$ is the rationalization map and and u is from Section 7.4, commutes (maps out of $\mathscr{M}_{\mathbb{Q}} \cong \mathsf{H}_B$ into Beilinson motives correspond to elements in $\pi_{0,0}$). Thus we obtain a map $\mathscr{M} \to \mathsf{M}\mathbb{Z}$ (for the latter see Definition 4.27). This map is an isomorphism since u is an isomorphism (Theorem 7.14) and each map $\mathscr{M} \to D(p)$ is the p-completion map.

We have shown

Theorem 7.18. – *There is a canonical isomorphism* $\mathscr{M} \cong \mathsf{M}\mathbb{Z}$ *in the homotopy category of T-spectra. We have* $\mathrm{map}(\mathscr{M}, \mathscr{M}) \cong \mathbb{Z}$*, where* map *can denote the mapping space in T-spectra or in* $\mathsf{SH}(S)$*.*

We leave the last assertion as an exercise to the reader.

COROLLARY 7.19. – *For $X \in \mathrm{Sm}_S$ there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathsf{SH}(S)}(\Sigma^\infty X_+, \mathsf{M}\mathbb{Z}(n)[i]) \cong H^i_{\mathrm{mot}}(X,n),$$

where the latter group denotes Levine's motivic cohomology.

For $X \in \mathrm{Sm}_S$ we denote by $\mathrm{DM}(X)$ the homotopy category of the category of $f^*\mathsf{M}\mathbb{Z}$-modules, where f is the structural morphism of X.

COROLLARY 7.20. – *For $X \in \mathrm{Sm}_S$ there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}, \mathbb{Z}(n)[i]) \cong H^i_{\mathrm{mot}}(X,n).$$

CHAPTER 8

BASE CHANGE

The main result of this section is Theorem 8.22 which states that the pullback of our motivic Eilenberg-MacLane spectrum over a Dedekind domain of mixed characteristic to the spectrum of a residue field of positive characteristic is canonically isomorphic to Voevodsky's motivic Eilenberg-MacLane spectrum. Proposition 8.7 handles the case where the characteristic p of the residue field is invertible in the coefficients, whereas Theorem 8.18 treats the main part for the case of $\mathbb{Z}/p$-coefficients.

For the next result note that for a base scheme S, commutative ring R and topology $t \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$ there is for any $X \in \mathrm{Sm}_S$ a restriction functor $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, R)) \to \mathsf{D}(\mathrm{Sh}(X_t, R))$, $F \mapsto F|_X$, to the small site. Also for a map $f\colon V \to U$ between base schemes there are (derived) pullback functors

$$f^*\colon \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,t}, R)) \to \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{V,t}, R))$$

and

$$f^*\colon \mathsf{D}(\mathrm{Sh}(U_t, R)) \to \mathsf{D}(\mathrm{Sh}(V_t, R)).$$

We will need the following proposition to carry over base change results for the small sites to the big sites.

PROPOSITION 8.1. – *Let $f\colon T \to S$ be a morphism of base schemes, R a commutative ring and $t \in \{\mathrm{Zar}, \mathrm{Nis}, \text{ét}\}$. Let $F \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, R))$. For each $X \in \mathrm{Sm}_S$ let f_X be the map $X_T := T \times_S X \to X$. Suppose that for each $X \in \mathrm{Sm}_S$ the object $f_X^*(F|_{X_t}) \in \mathsf{D}(\mathrm{Sh}(X_{T,t}, R))$ is zero. Then $f^*F \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{T,t}, R))$ is zero.*

Proof. – We use the language of ∞-categories. For any base scheme U let $\theta(U)$ be the functor on $\mathrm{Sm}_U^{\mathrm{op}}$ which associates to any $X \in \mathrm{Sm}_U$ the ∞-category associated to the model category $\mathrm{Cpx}(\mathrm{Sh}(X_t, R))$ (see Appendix A for the model structure). This functor is associated to the left Quillen presheaf

$$\begin{array}{rccc} \tilde{\theta}(U)\colon & \mathrm{Sm}_U^{\mathrm{op}} & \to & \mathrm{ModCat} \\ & X & \mapsto & \mathrm{Cpx}(\mathrm{Sh}(X_t, R)). \end{array}$$

We denote by $\mathrm{Sect}(\tilde{\theta}(U))$ the ∞-category which is associated to the model category of sections of $\tilde{\theta}(U)$. The objects of the category of sections of $\tilde{\theta}(U)$ are collections of objects $G_X \in \tilde{\theta}(U)(X)$ for any $X \in \mathrm{Sm}_U$ together with transition maps $g^*G_X \to G_Y$ for any map $g\colon Y \to X$ in Sm_U which have to satisfy a cocycle condition. We note that colimits and limits are computed sectionwise in $\mathrm{Sect}(\tilde{\theta}(U))$.

We let $\mathrm{Sect}(\tilde{\theta}(U))_{\text{ét-cart}}$ be the full subcategory of $\mathrm{Sect}(\tilde{\theta}(U))$ which consists of objects which are cartesian for étale morphisms in Sm_U. Then $\mathrm{Sect}(\tilde{\theta}(U))_{\text{ét-cart}}$ is canonically equivalent to the ∞-category associated to $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,t}, R))$. Note that the inclusion

$$\mathrm{Sect}(\tilde{\theta}(U))_{\text{ét-cart}} \hookrightarrow \mathrm{Sect}(\tilde{\theta}(U))$$

preserves colimits (since pullback functors are left adjoints) and also limits (since pullbacks with respect to étale maps are also right adjoints), so it has both a left and a right adjoint.

Let $g\colon V \to U$ be a morphism of base schemes. Then there is an induced left adjoint $g^*\colon \mathrm{Sect}(\tilde{\theta}(U)) \to \mathrm{Sect}(\tilde{\theta}(V))$ which can be described as follows:

We denote by $\tilde{\theta}(V/U)$ the left Quillen presheaf on $\mathrm{Sm}_U^{\mathrm{op}}$ which assigns to $X \in \mathrm{Sm}_U$ the model category $\mathrm{Cpx}(\mathrm{Sh}(X_{V,t}, R))$ and by $\theta(V/U)$ the associated functor with values in ∞-categories. Pulling back along the maps $X_V \to X$ for $X \in \mathrm{Sm}_U$ defines a natural transformation of functors

$$\tilde{\theta}(U) \to \tilde{\theta}(V/U)$$

which induces an adjunction

$$\mathrm{Sect}(\tilde{\theta}(U)) \rightleftarrows \mathrm{Sect}(\tilde{\theta}(V/U))$$

which is given objectwise by the adjunctions

$$\theta(U)(X) \rightleftarrows \theta(V/U)(X),$$

$X \in \mathrm{Sm}_U$.

Moreover $\tilde{\theta}(V/U)$ is given as the composition $\tilde{\theta}(V) \circ (f^{-1})^{\mathrm{op}}$, where $f^{-1}\colon \mathrm{Sm}_U \to \mathrm{Sm}_V$ is the pullback functor. Thus we get an induced adjunction

$$\mathrm{Sect}(\tilde{\theta}(V/U)) \rightleftarrows \mathrm{Sect}(\tilde{\theta}(V))$$

whose right adjoint is given by pulling back a section in $\mathrm{Sect}(\tilde{\theta}(V))$ along f^{-1} to a section in $\mathrm{Sect}(\tilde{\theta}(V/U))$.

The functor g^* is then given as the composition of left adjoints

$$\mathrm{Sect}(\tilde{\theta}(U)) \to \mathrm{Sect}(\tilde{\theta}(V/U)) \to \mathrm{Sect}(\tilde{\theta}(V)).$$

The base change $\mathbb{L}g^*\colon \mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,t}, R)) \to \mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{V,t}, R))$ is modeled by the composition $\tilde{g}$

$$\mathrm{Sect}(\tilde{\theta}(U))_{\text{ét-cart}} \hookrightarrow \mathrm{Sect}(\tilde{\theta}(U)) \xrightarrow{g^*} \mathrm{Sect}(\tilde{\theta}(V)) \to \mathrm{Sect}(\tilde{\theta}(V))_{\text{ét-cart}},$$

where the last morphism is the left adjoint to the inclusion, since the right adjoint ψ of $\tilde{g}$ is canonically equivalent to the right adjoint of $\mathbb{L}g^*$: Note that the right adjoint

$\mathrm{Sect}(\tilde{\theta}(V)) \to \mathrm{Sect}(\tilde{\theta}(U))$ preserves the étale-cartesian objects, since for an étale map $e\colon Y \to X$ in Sm_U and $G \in \theta(V)(X_V)$ the canonical map

$$e^*(g_X)_*G \to (g_Y)_*(e_V)^*G$$

associated to the diagram

$$\begin{array}{ccc} Y_V & \xrightarrow{e_V} & X_V \\ \downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle g_X} \\ Y & \xrightarrow{e} & X \end{array}$$

is an equivalence in $\theta(U)(Y)$. Thus for $G \in \mathrm{Sect}(\tilde{\theta}(V))_{\text{ét-cart}}$ we have

$$(\psi(G)(X))(X) = (G(X_V))(X_V)$$

for any $X \in \mathrm{Sm}_U$, showing that ψ is equivalent to the right adjoint of $\mathbb{L}g^*$.

The assumption on F implies that the composition ψ'

$$\mathrm{Sect}(\tilde{\theta}(S))_{\text{ét-cart}} \hookrightarrow \mathrm{Sect}(\tilde{\theta}(S)) \to \mathrm{Sect}(\tilde{\theta}(T/S))$$

applied to F already gives zero, since for $X \in \mathrm{Sm}_S$ we have

$$\psi'(F)(X) \simeq (\mathbb{L}f_X^*)(F|_{X_t}) \simeq 0.$$

Since by the above considerations $\mathbb{L}f^*$ is equivalent to the composition of ψ' and the functor

$$\mathrm{Sect}(\tilde{\theta}(T/S)) \to \mathrm{Sect}(\tilde{\theta}(T)) \to \mathrm{Sect}(\tilde{\theta}(T))_{\text{ét-cart}}$$

the claim follows. □

REMARK 8.2. – *As explained by one of the referees one can model the ∞-categories $\mathrm{Sect}(\tilde{\theta}(U))$ by sheaves on an enlarged site $\mathrm{SM}_{U,t}$. The objects in SM_U are factorizations $Y \to X \to U$, where the first morphism is étale and the second morphism smooth. Morphisms are commutative diagrams*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \\ & \searrow & \downarrow \\ & & U \end{array}$$

and covers are sets of morphisms of the form

$$\begin{array}{ccc} Y_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & = & X \\ & \searrow & \downarrow \\ & & U \end{array}$$

such that $\{Y_i \to Y\}$ is a cover in the topology t. Then the inclusion

$$\mathrm{Sect}(\tilde{\theta}(U))_{\text{ét-cart}} \hookrightarrow \mathrm{Sect}(\tilde{\theta}(U))$$

is induced by the map of sites

$$\mathrm{SM}_{U,t} \to \mathrm{Sm}_{U,t}$$

which sends $Y \to X \to U$ to $Y \to U$.

PROPOSITION 8.3. – *Let R be a commutative ring, X a connected base scheme, x a geometric point of X, $G := \pi_1^{\text{ét}}(X,x)$ the étale fundamental group of X at x and $p\colon \widetilde{X} \to X$ the universal cover of X determined by x. Then for any $F \in \mathsf{D}(\mathrm{Sh}(X_{\text{ét}},R))$ the object $\mathbb{R}\Gamma(\widetilde{X},p^*F)$ is naturally an object of $\mathsf{D}_G(R)$, the derived category of R-modules equipped with a continuous G-action, and there is a natural isomorphism $\mathbb{R}\Gamma(X,F) \cong \mathbb{R}\Gamma(\widetilde{X},p^*F)^{hG}$ in $\mathsf{D}(R)$.*

Proof. – The left adjoint to the global sections functor $\Gamma\colon \mathrm{Cpx}(\mathrm{Sh}(X_{\text{ét}},R)) \to \mathrm{Cpx}(R)$ can be factored as

$$\mathrm{Cpx}(R) \to \mathrm{Cpx}(R)[G]^{\mathrm{cts}} \to \mathrm{Cpx}(\mathrm{Sh}(X_{\text{ét}},R)),$$

where $\mathrm{Cpx}(R)[G]^{\mathrm{cts}}$ is the category of complexes of R-modules equipped with a continuous G-action. The second functor of this composition is fully faithful with essential image the complexes of those sheaves which become constant after pullback to $\widetilde{X}$. When we equip the three categories appearing in this composition with the injective model structures the two functors in the composition become left Quillen functors. Moreover the right adjoint of the second functor is given by $\mathbb{R}\Gamma(\widetilde{X},p^*(-))$ (the latter objects carry natural continuous G-actions). Applying the derived right adjoints to both functors yields the result. □

LEMMA 8.4. – *Let R be a commutative ring, X a base scheme, K a field, $x\colon \mathrm{Spec}(K) \to X$ a point inducing an isomorphism on the residue fields and $\varepsilon\colon X_{\text{ét}} \to X_{\mathrm{Nis}}$ the map of sites. Then for any $F \in \mathsf{D}(\mathrm{Sh}(X_{\text{ét}},R))$ the canonical map*

$$(x^*\mathbb{R}\varepsilon_*F)(\mathrm{Spec}(K)) \to \mathbb{R}\Gamma(\mathrm{Spec}(K),x^*F)$$

in $\mathsf{D}(R)$ is an isomorphism.

Proof. – Let $Y := \mathrm{Spec}(\mathcal{O}_{X,x}^h)$ (the spectrum of the henselization of the local ring of X at x). Let y be the geometric point of Y corresponding to an algebraic closure of K and $\widetilde{Y}$ the universal cover of Y with respect to y ($\widetilde{Y}$ is then the spectrum of the strict henselization of $\mathcal{O}_{X,x}$). Let $F_Y \in \mathsf{D}(\mathrm{Sh}(Y_{\text{ét}},R))$ be the pullback of F to Y, similarly for $F_{\widetilde{Y}}$. Then the left hand side of the map in question is naturally isomorphic to $\mathbb{R}\Gamma(Y,F_Y)$. By Proposition 8.3 the latter object can be naturally identified with $\mathbb{R}\Gamma(\widetilde{Y},F_{\widetilde{Y}})^{hG}$, where G is the absolute Galois group of K (with respect to the chosen algebraic closure). But $\mathbb{R}\Gamma(\widetilde{Y},F_{\widetilde{Y}})$ is canonically isomorphic to y^*F_Y (since $\widetilde{Y}$ is the spectrum of a strictly henselian local ring) equipped with the natural G-action. The canonical isomorphism $(y^*F_Y)^{hG} \cong \mathbb{R}\Gamma(\mathrm{Spec}(K),x^*F)$ yields the result. □

Proposition 8.5. – *Let $f\colon Y \to X$ be a morphism of base schemes which induces isomorphisms on residue fields and R a commutative ring. Let ε denote the maps of sites $X_{\text{ét}} \to X_{\text{Nis}}$ and $Y_{\text{ét}} \to Y_{\text{Nis}}$. Let $F \in \mathsf{D}(\mathrm{Sh}(X_{\text{ét}}, R))$. Then the canonical map $f^*\mathbb{R}\varepsilon_* F \to \mathbb{R}\varepsilon_* f^* F$ is an isomorphism in $\mathsf{D}(\mathrm{Sh}(Y_{\mathrm{Nis}}, R))$.*

Proof. – Let K be a field and $x\colon \mathrm{Spec}(K) \to Y$ a map inducing an isomorphism on residue fields. Applying Lemma 8.4 twice we get

$$(x^*(\mathbb{R}\varepsilon_*(f^*F)))(\mathrm{Spec}(K)) \cong \mathbb{R}\Gamma(\mathrm{Spec}(K), x^*f^*F) \cong (x^*f^*\mathbb{R}\varepsilon_*F)(\mathrm{Spec}(K)).$$

This shows the claim. □

Proposition 8.6. – *Let $i\colon Z \hookrightarrow X$ be a closed immersion between base schemes and R a commutative ring. Let $F \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{X,\mathrm{Nis}}, R))$, $G \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{Z,\mathrm{Nis}}, R))$ and $\varphi\colon \mathbb{L}i^*F \to G$ be a map. Suppose that for any $Y \in \mathrm{Sm}_X$ the map $i_Y^*(F|_{Y_{\mathrm{Nis}}}) \to G|_{Y_{Z,\mathrm{Nis}}}$ (i_Y the induced closed immersion $Z_Y \to Y$) induced by φ is an isomorphism in $\mathsf{D}(\mathrm{Sh}(Y_{Z,\mathrm{Nis}}, R))$. Then φ is an $\mathbb{A}^1$-weak equivalence.*

Proof. – By the morphism induced by φ we mean the composition

$$i_Y^*(F|_{Y_{\mathrm{Nis}}}) \to i_Y^*((\mathbb{R}i_*G)|_{Y_{\mathrm{Nis}}}) \cong i_Y^*(\mathbb{R}i_{Y,*}(G|_{Y_{Z,\mathrm{Nis}}})) \cong G|_{Y_{Z,\mathrm{Nis}}},$$

where the first morphism comes from the adjoint $F \to \mathbb{R}i_*G$ of φ.

The cofiber of the adjoint $F \to \mathbb{R}i_*G$ of φ satisfies the assumption of Proposition 8.1, thus the map $\mathbb{L}i^*F \to \mathbb{L}i^*\mathbb{R}i_*G$ is an isomorphism. But the map $i^*\mathbb{R}i_*G \to G$ is an $\mathbb{A}^1$-weak equivalence, because $\mathbb{R}i_*$ preserves $\mathbb{A}^1$-weak equivalences (since i is finite) and the composition

$$\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{Z,\mathrm{Nis}}, R)) \to \mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{X,\mathrm{Nis}}, R)) \to \mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{Z,\mathrm{Nis}}, R))$$

is naturally equivalent to the identity. Therefore φ is the composition of an isomorphism and an $\mathbb{A}^1$-weak equivalence. □

We will now prove base change for finite coefficients which are invertible on the base. Let U be the spectrum of a Dedekind domain of mixed characteristic and p a prime which is invertible on U. Let $x \in U$ be a closed point of positive residue characteristic and $\kappa := \kappa(x)$. We denote by i the closed inclusion $\{x\} \hookrightarrow U$. We let $L_{U,n}(r) = \mu_{p^n}^{\otimes r}$ viewed as object of $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^n))$, similarly $L_{\kappa,n}(r) = \mu_{p^n}^{\otimes r}$ viewed as object of $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\text{ét}}, \mathbb{Z}/p^n))$. We have a natural map $\varphi\colon L_{U,n}(r) \to \mathbb{R}i_*L_{\kappa,n}(r)$. Let ε denote the maps of sites $\mathrm{Sm}_{U,\text{ét}} \to \mathrm{Sm}_{U,\mathrm{Nis}}$ and $\mathrm{Sm}_{\kappa,\text{ét}} \to \mathrm{Sm}_{\kappa,\mathrm{Nis}}$. The adjoint of φ induces the second map in the composition

$$\mathbb{L}i^*\mathbb{R}\varepsilon_*L_{U,n}(r) \to \mathbb{R}\varepsilon_*\mathbb{L}i^*L_{U,n}(r) \to \mathbb{R}\varepsilon_*L_{\kappa,n}(r).$$

Applying $\tau_{\leq r}$ to this composition yields the second map in the composition

$$g_{n,r}\colon \mathbb{L}i^*\tau_{\leq r}\mathbb{R}\varepsilon_*L_{U,n}(r) \to \tau_{\leq r}\mathbb{L}i^*\mathbb{R}\varepsilon_*L_{U,n}(r) \to \tau_{\leq r}\mathbb{R}\varepsilon_*L_{\kappa,n}(r),$$

whereas the first map canonically exists since $\mathbb{L}i^*$ preserves $(-r-1)$-connected objects (being generated by homotopy colimits and extensions by suitable shifts of free sheaves of $\mathbb{Z}/p^n$-modules on representables).

The sequence $((\tau_{\leq r}\mathbb{R}\varepsilon_* L_{U,n}(r))[r])_{r\in\mathbb{N}}$ assembles into a naive $\mathbb{Z}/p^n[\mathbb{G}_{m,U},\{1\}]_{\mathrm{Nis}}$-spectrum F_n in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Nis}},\mathbb{Z}/p^n))$, and the sequence $((\tau_{\leq r}\mathbb{R}\varepsilon_* L_{\kappa,n}(r))[r])_{r\in\mathbb{N}}$ into a naive $\mathbb{Z}/p^n[\mathbb{G}_{m,\kappa},\{1\}]_{\mathrm{Nis}}$-spectrum G_n in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\mathrm{Nis}},\mathbb{Z}/p^n))$, and the $g_{n,r}$ give a map of naive prespectra

$$g_n\colon \mathbb{L}i^*F_n \to G_n.$$

Proposition 8.7. – *The maps g_n are levelwise $\mathbb{A}^1$-weak equivalences.*

Proof. – Let $X \in \mathrm{Sm}_U$, X_κ the inverse image of x and $i_X\colon X_\kappa \hookrightarrow X$ the closed inclusion. Proposition 8.5 implies that the canonical map

$$i_X^*\mathbb{R}\varepsilon_*\mu_{p^n}^{\otimes r} \to \mathbb{R}\varepsilon_*\mu_{p^n}^{\otimes r}$$

is an isomorphism in $\mathsf{D}(\mathrm{Sh}(X_{\kappa,\mathrm{Nis}},\mathbb{Z}/p^n))$ (here the first $\mu_{p^n}^{\otimes r}$ denotes an object in $\mathsf{D}(\mathrm{Sh}(X_{\text{ét}},\mathbb{Z}/p^n))$, whereas the second an object in $\mathsf{D}(\mathrm{Sh}(X_{\kappa,\text{ét}},\mathbb{Z}/p^n)))$. By the exactness of i_X^* the canonical map

$$i_X^*\tau_{\leq r}\mathbb{R}\varepsilon_*\mu_{p^n}^{\otimes r} \to \tau_{\leq r}\mathbb{R}\varepsilon_*\mu_{p^n}^{\otimes r}$$

is thus also an isomorphism (in the same category). Thus the claim follows from Proposition 8.6. □

This establishes our first goal (base change for coefficients which are invertible on the base).

Note that $\mathrm{map}_{\mathbb{Z}/p^n}(G_{n,r},G_{n,r}) \cong \mathbb{Z}/p^n$ (compare to Lemma 7.16) (here $G_{n,r}$ denotes the r-th level of the naive spectrum G_n), thus the G_n lift to spectra G_n which are unique up to canonical isomorphism (compare to Corollary 7.17) (these spectra are $\mathbb{A}^1$-local, so can be considered as spectra in the $\mathbb{A}^1$-local categories).

Using the techniques of Section 5 one constructs as in Section 7.5 a map of naive spectra $\mathcal{M}_\kappa \to G_n$ ($\mathcal{M}_\kappa$ as in Section 6) which is induced by étale cycle class maps. The induced map $\mathcal{M}_\kappa/p^n \to G_n$ is an isomorphism by Theorem 3.9. The object $D(p) := \mathrm{holim}_n G_n$ is canonically defined and the canonically induced map $\mathcal{M}_\kappa \to D(p)$ is the p-completion map.

Let $p := \mathrm{char}(\kappa)$. For each $n \in \mathbb{N}$ define a naive $\mathbb{Z}/p^n[\mathbb{G}_{m,\kappa},\{1\}]_{\mathrm{Nis}}$-spectrum E_n by $E_{n,r} := \nu_n^r \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\mathrm{Nis}},\mathbb{Z}/p^n))$. The bonding maps are the compositions

$$E_{n,r}\otimes^{\mathbb{L}}\mathbb{Z}/p^n[\mathbb{G}_{m,\kappa},\{1\}]_{\mathrm{Nis}} \to E_{n,r}\otimes^{\mathbb{L}}\mathcal{O}^*_{/\kappa} \to E_{n,r}\otimes^{\mathbb{L}}\nu_n^1 \to \nu_n^{r+1} \cong E_{n,r+1}.$$

There is a map of naive spectra $\mathcal{M}_\kappa \to E_n$ such that the induced map $\mathcal{M}_\kappa/p^n \to E_n$ is an isomorphism.

Note that $\mathrm{map}_{\mathbb{Z}/p^n}(E_{n,r},E_{n,r})$ is (homotopy) discrete, so that E_n has a canonical model as spectrum which we also denote by E_n (which is again $\mathbb{A}^1$-local). Moreover $E := \mathrm{holim}_n E_n$ is well-defined up to canonical isomorphism, and the canonical map $\mathcal{M}_\kappa \to E$ is the p-completion map.

We denote by $\mathsf{H}_{B,\kappa}$ the Beilinson spectrum over κ. There is a canonical map

$$\mathsf{H}_{B,\kappa} \to (E \times \prod_{p \neq \mathrm{char}(\kappa)} D(p))_{\mathbb{Q}},$$

and the canonical diagram

$$\begin{array}{ccc} \mathscr{M}_\kappa & \longrightarrow & E \times \prod_{p\neq \mathrm{char}(\kappa)} D(p) \\ \downarrow & & \downarrow \\ \mathsf{H}_{B,\kappa} & \longrightarrow & (E \times \prod_{p \neq \mathrm{char}(\kappa)} D(p))_{\mathbb{Q}} \end{array} \tag{24}$$

is homotopy cartesian.

Suppose now that the U from above is the spectrum of a complete discrete valuation ring Λ and x is the closed point of U. We have $p = \mathrm{char}(\kappa)$. Above we have for any prime $l \neq p$ constructed maps of naive spectra

$$\mathbb{L}i^* \mathsf{M}\mathbb{Z}_U/l^n \cong \mathbb{L}i^* F_n \to G_n$$

(here the dependence of G_n and F_n on l is suppressed) which are isomorphisms by Proposition 8.7 (here we view the naive spectra taking values in the $\mathbb{A}^1$-local categories). Thus these maps lift uniquely to isomorphisms between the corresponding spectra. We get canonical maps $\mathbb{L}i^*\mathsf{M}\mathbb{Z}_U \to D_l$ for all primes $l \neq p$.

Let η be the complement of $\{x\}$ in U and $j\colon \eta \hookrightarrow U$ the open inclusion. We again have the objects $L_{\eta,n}(r) := \mu_{p^n}^{\otimes r} \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\eta,\text{ét}}, \mathbb{Z}/p^n))$ and the map of sites $\varepsilon\colon \mathrm{Sm}_{\eta,\text{ét}} \to \mathrm{Sm}_{\eta,\mathrm{Nis}}$. By the proof of Lemma 4.5 we get the first isomorphism in the chain of isomorphisms

$$\mathbb{R}j_*\tau_{\leq r}\mathbb{R}\varepsilon_* L_{\eta,n}(r) \cong \tau_{\leq r}\mathbb{R}j_*\mathbb{R}\varepsilon_* L_{\eta,n}(r) \cong \tau_{\leq r}\mathbb{R}\varepsilon_*\mathbb{R}j_* L_{\eta,n}(r) \tag{25}$$

in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Nis}}, \mathbb{Z}/p^n))$ (the second isomorphism follows from functoriality of sites).

Mapping to $\mathscr{H}^r$ we get a map

$$\begin{aligned} \mathbb{R}j_*\tau_{\leq r}\mathbb{R}\varepsilon_* L_{\eta,n}(r) &\to \mathscr{H}^r(\mathbb{R}\varepsilon_*\mathbb{R}j_*L_{\eta,n}(r))[-r] \\ &\to \varepsilon_*\varepsilon^*\,\mathscr{H}^r(\mathbb{R}\varepsilon_*\mathbb{R}j_*L_{\eta,n}(r))[-r] \cong \varepsilon_*\mathbb{R}^r j_* L_{\eta,n}(r)[-r]. \end{aligned}$$

For any $X \in \mathrm{Sm}_U$ we have a map

$$(i_X)^*\mathbb{R}^r j_{X,*}(L_{\eta,n}(r)|_{X_{\eta,\text{ét}}}) \to \nu_n^r \oplus \nu_n^{r-1}$$

in $\mathrm{Sh}(X_{\kappa,\text{ét}}, \mathbb{Z}/p^n)$ constructed in [**4**, §(6.6)]. The second projection of this map was already described in Section 4.1.1, the first projection is similar: it sends a symbol $\{f_1,\dots,f_r\}$, $f_1,\dots,f_r \in (i_X)^*\mathscr{O}_X^*$, to $\mathrm{dlog}\overline{f}_1 \dots \mathrm{dlog}\overline{f}_r$ and a symbol $\{f_1,\dots,f_{r-1},\pi\}$, π a fixed uniformizer of Λ, to 0.

As in Section 4.1.1 these maps glue to give a map

$$\mathbb{R}^r j_* L_{\eta,n}(r) \xrightarrow{\varphi\oplus\psi} i_*\nu_n^r \oplus i_*\nu_n^{r-1}.$$

As in Section 7.5 we denote by $F_{n,r}$ the homotopy fiber of the composition

$$\mathbb{R}j_*\tau_{\leq r}\mathbb{R}\varepsilon_* L_{\eta,n}(r)[r] \to \varepsilon_*\mathbb{R}^r j_* L_{\eta,n}(r) \to i_*\nu_n^r \oplus i_*\nu_n^{r-1} \to i_*\nu_n^{r-1},$$

and the sequence $(F_{n,r})_{r\in\mathbb{N}}$ assembles into a naive spectrum F_n (we do not need the objects denoted by the same symbol from above any longer).

Note that the maps φ give rise to maps $F_{n,r} \to i_*\nu_n^r$, thus to maps

$$\alpha_{n,r}\colon \mathbb{L}i^*F_{n,r} \to \nu_n^r = E_{n,r}.$$

LEMMA 8.8. – *The maps $\alpha_{n,r}$ assemble to a map $\alpha_n\colon \mathbb{L}i^*F_n \to E_n$ of naive prespectra.*

Proof. – We leave the verification to the reader. □

Our next goal is to show that the $\alpha_{n,r}$ are $\mathbb{A}^1$-weak equivalences (this is achieved with Theorem 8.18). The main tools are a filtration of Bloch and Kato on p-adic étale cohomology, a descent construction for this filtration, a gluing procedure for the graded pieces of this filtration from small to big sites and vanishing statements for higher push forwards from the étale to the Nisnevich topology of certain sheaves.

PROPOSITION 8.9. – *Let X be a base scheme of characteristic p and $F \in \mathrm{Sh}(X_{\text{ét}}, \mathbb{Z})$ a p-torsion sheaf. Then $\mathbb{R}^i\varepsilon_*F \in \mathrm{Sh}(X_{\text{Nis}}, \mathbb{Z})$ is zero for $i > 1$.*

Proof. – Let K be a field and $x\colon \mathrm{Spec}(K) \to X$ a map inducing an isomorphism on residue fields. Then

$$\Gamma(\mathrm{Spec}(K), x^*\mathbb{R}\varepsilon_*F) \cong \mathbb{R}\Gamma(\mathrm{Spec}(K), x^*F)$$

in $\mathsf{D}(\mathsf{Ab})$ (Lemma 8.4). The latter complex (which is Galois cohomology) vanishes in cohomological degrees > 1 by [**46**, II, Prop. 3]. □

PROPOSITION 8.10. – *Let X be a base scheme and F a quasi coherent sheaf on $X_{\text{ét}}$. Then $\mathbb{R}^i\varepsilon_*F \in \mathrm{Sh}(X_{\text{Nis}}, \mathbb{Z})$ is zero for $i > 0$.*

Proof. – Let K be a field and $x\colon \mathrm{Spec}(K) \to X$ a map inducing an isomorphism on residue fields. Then

$$\Gamma(\mathrm{Spec}(K), x^*\mathbb{R}\varepsilon_*F) \cong \mathbb{R}\Gamma(\mathrm{Spec}(K), x^*F)$$

in $\mathsf{D}(\mathsf{Ab})$ (Lemma 8.4). But for a finite Galois extension L/K the object $(x^*F)(\mathrm{Spec}(L))$ is an induced $\mathrm{Gal}(L/K)$-module, thus its cohomology vanishes in degrees > 0. □

For $X \in \mathrm{Sm}_\kappa$ let Ω_X^1 be the sheaf on $X_{\text{ét}}$ (and thus also on X_{Nis} and X_{Zar}) of absolute Kähler differentials on X. It is quasi coherent. Let $\Omega_X^\bullet$ be the exterior algebra over $\mathcal{O}_X$ of Ω_X^1. Define subsheaves

$$B_X^i := \mathrm{im}(d\colon \Omega_X^{i-1} \to \Omega_X^i)$$
$$\text{and } Z_X^i := \ker(d\colon \Omega_X^i \to \Omega_X^{i+1})$$

of Ω_X^i on $X_{\text{ét}}$.

LEMMA 8.11. – *For $X \in \mathrm{Sm}_\kappa$ we have $\mathbb{R}^i\varepsilon_*\Omega_X^j = \mathbb{R}^i\varepsilon_*B_X^j = \mathbb{R}^i\varepsilon_*Z_X^j = 0$ in $\mathrm{Sh}(X_{\text{Nis}}, \mathbb{F}_p)$ for $j \geq 0$ and $i > 0$.*

Proof. – We have $\mathbb{R}^i\varepsilon_*\Omega^j_X = 0$ for $j \geq 0$ and $i > 0$ by Proposition 8.10. We have isomorphisms

$$\Omega^j_X \cong Z^j_X/B^j_X$$

given by the inverse Cartier operator, see [**4**, top of p. 112], thus the claim follows for $j = 0$. Suppose by induction the claim for j. The exact sequence

$$0 \to Z^j_X \to \Omega^j_X \to B^{j+1}_X \to 0$$

shows the claim for B^{j+1}_X, the above isomorphism for $j+1$ shows the claim for Z^{j+1}_X. This finishes the proof. □

Let $L_\eta(r) := L_{\eta,1}(r)$. For any $X \in \mathrm{Sm}_U$ we have the following isomorphisms

$$\begin{aligned} i^*_X(\mathbb{R}j_*\tau_{\leq r}\mathbb{R}\varepsilon_* L_\eta(r))|_{X_{\mathrm{Nis}}} &\cong i^*_X(\tau_{\leq r}\mathbb{R}\varepsilon_*\mathbb{R}j_* L_\eta(r)|_{X_{\mathrm{Nis}}}) \\ &\cong \tau_{\leq r} i^*_X(\mathbb{R}\varepsilon_*\mathbb{R}j_* L_\eta(r)|_{X_{\mathrm{Nis}}}) \\ &\cong \tau_{\leq r}\mathbb{R}\varepsilon_* i^*_X \tau_{\leq r}(\mathbb{R}j_* L_\eta(r)|_{X_{\text{ét}}}) \end{aligned}$$

in $\mathsf{D}(\mathrm{Sh}(X_{\kappa,\mathrm{Nis}}, \mathbb{F}_p))$.

The first isomorphism uses (25), the second the exactness of i^*_X and the third also Proposition 8.5 (strictly speaking we do not need the third isomorphism, but we give it for motivation).

Set

$$K_{X,0} := i^*_X\tau_{\leq r}(\mathbb{R}j_* L_\eta(r)|_{X_{\text{ét}}}) \in \mathsf{D}(\mathrm{Sh}(X_{\kappa,\text{ét}}, \mathbb{F}_p)). \tag{26}$$

We will define descending filtrations on the sheaves $\mathscr{H}^k(K_{X,0}) \in \mathrm{Sh}(X_{\kappa,\text{ét}}, \mathbb{F}_p)$, $0 \leq k \leq r$ (compare with [**4**]). We start with $k = r$.

For $m \geq 1$ let $U^m\mathscr{H}^r(K_{X,0})$ be the subsheaf of $\mathscr{H}^r(K_{X,0})$ generated étale locally by sections of the form $\{x_1, \ldots, x_r\}$, $x_i \in i^*_X j_{X,*}\mathscr{O}^*_{X_\eta}$, such that $x_1 - 1 \in \pi^m i^*_X\mathscr{O}_X$, see [**4**, p. 111]. We define $U^0\mathscr{H}^r(K_{X,0}) := \mathscr{H}^r(K_{X,0})$.

Let e be the absolute ramification index of Λ and $e' := \frac{ep}{p-1}$.

We denote by $\mathrm{gr}^m U^\bullet\mathscr{H}^r(K_{X,0})$ the m-th graded piece of the filtration $U^\bullet\mathscr{H}^r(K_{X,0})$.

Then we have

$$\mathrm{gr}^m U^\bullet\mathscr{H}^r(K_{X,0}) \cong \begin{cases} \nu^r_1 \oplus \nu^{r-1}_1, & 0 = m, \\ \Omega^{r-1}_{X_\kappa}, & 1 \leq m < e', \quad p \nmid m, \\ B^r_{X_\kappa} \oplus B^{r-1}_{X_\kappa}, & 1 \leq m < e', \quad p \mid m, \\ 0, & e' \leq m, \end{cases}$$

see [**4**, Cor. (1.4.1)]. We denote these graded pieces by Q^m_X. These sheaves Q^m_X glue to sheaves Q^m on $\mathrm{Sm}_{\kappa,\text{ét}}$.

To define the filtrations for $k < r$ we have to adjoin a p-th root of unity and descend a filtration upstairs.

Let $\widetilde{\Lambda}$ be the integral closure of Λ in $\widetilde{K} := K(\zeta_p)$, where K is the quotient field of Λ and ζ_p is a primitive p-th root of unity. Let d be the degree of $\widetilde{K}$ over K and

$G := \mathrm{Gal}(\widetilde{K}/K)$. We have $d \mid p-1$. There is a canonical injective group homomorphism $\psi_1 \colon G \hookrightarrow \mathbb{F}_p^*$ characterized by $g(\zeta_p) = \zeta_p^{\psi_1(g)}$ for $g \in G$.

LEMMA 8.12. – *There exists a uniformizer $\tilde{\pi}$ of $\widetilde{\Lambda}$ such that $\tilde{\pi}^d \in \Lambda$.*

Proof. – Since Λ is complete K contains all $(p-1)$-st roots of unity, in particular all d'th roots of unity. Thus by Kummer theory there is an $a \in K$ such that $\widetilde{K} = K(\sqrt[d]{a})$. Let $a = s\pi^k$ for a unit $s \in \Lambda$ and $k \in \mathbb{Z}$ (π the uniformizer of Λ). Let M be the subgroup of $\mathbb{Q}$ generated by 1 and $\frac{k}{d}$ and let $e = |M/\mathbb{Z}|$, so $M = \frac{1}{e}\mathbb{Z}$. Since $K(\sqrt[d]{a^e}) = K(\sqrt[d]{s^e})$ (note $d \mid (ek)$) is unramified over K the ramification index of L over K is e. Let $\alpha, \beta \in \mathbb{Z}$ with $\frac{1}{e} = \alpha + \beta\frac{k}{d}$, then $\tilde{\pi} := \pi^\alpha \cdot (\sqrt[d]{a})^\beta$ is as desired. □

From now on $\tilde{\pi}$ denotes a uniformizer of $\widetilde{\Lambda}$ as in Lemma 8.12. We have a natural map $\psi \colon G \to \mu_{p-1} \subset K^*$ characterized by $g(\tilde{\pi}) = \psi(g) \cdot \tilde{\pi}$ for $g \in G$. We let ψ_2 be the composition $G \xrightarrow{\psi} \mu_{p-1} \xrightarrow{\cong} \mathbb{F}_p^*$.

EXAMPLE 8.13. – *In the case $\Lambda = \mathbb{Z}_p$ we can choose $\tilde{\pi} = \sqrt[p-1]{-p}$ (see* [**8**, Theorem 4.3.18]*). In this case we have $\psi_1 = \psi_2$.*

Let $\widetilde{U} := \mathrm{Spec}\widetilde{\Lambda}$, $\tilde{\eta}$ the generic point of $\widetilde{U}$, $\tilde{\kappa}$ the residue field of $\widetilde{\Lambda}$.

For $X \in \mathrm{Sm}_U$ denote by $\widetilde{X}$ the base change to $\widetilde{U}$.

The notations $\widetilde{X}_{\tilde{\eta}}$, $\widetilde{X}_{\tilde{\kappa}}$, $j_{\widetilde{X}}$ and $i_{\widetilde{X}}$ explain themselves.

We fix now $X \in \mathrm{Sm}_U$.

We have the commutative diagram

$$\begin{array}{ccccc} \widetilde{X}_{\tilde{\kappa}} & \xhookrightarrow{i_{\widetilde{X}}} & \widetilde{X} & \xhookleftarrow{j_{\widetilde{X}}} & \widetilde{X}_{\tilde{\eta}} \\ \downarrow{\scriptstyle f^\kappa} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^\eta} \\ X_\kappa & \xhookrightarrow{i_X} & X & \xhookleftarrow{j_X} & X_\eta. \end{array}$$

This is naturally a diagram of schemes equipped with G-actions. We will need to consider G-equivariant étale sheaves on these schemes. To fix terminology let Y be a base scheme equipped with an action of a finite group H and R a commutative ring. We denote by $\mathrm{Sh}_H(Y_{\text{ét}}, R)$ the abelian category of H-equivariant étale sheaves on Y. An object $F \in \mathrm{Sh}_H(Y_{\text{ét}}, R)$ in particular gives rise to natural isomorphisms $F(U) \to F(Y \times_{h,Y} U)$ for any $U \in Y_{\text{ét}}$ and $h \in H$ which have to satisfy various compatibilities. Equivalently F can be viewed as an étale sheaf on the stack $[Y/H]$. We let $\mathsf{D}(\mathrm{Sh}_H(Y_{\text{ét}}, R))$ be the derived category of $\mathrm{Sh}_H(Y_{\text{ét}}, R)$.

This triangulated category can equivalently be defined as (homotopy) cartesian objects in the derived category of étale sheaves of R-modules on the standard simplicial scheme the H-action on Y gives rise to.

For H-equivariant maps between H-base schemes there are the usual adjunctions between these categories. Moreover there are various change of groups functors, of which we only will need the functor $\mathsf{D}(\mathrm{Sh}(Y_{\text{ét}}, R)) \to \mathsf{D}(\mathrm{Sh}_H(Y_{\text{ét}}, R))$ which equips an object with the trivial H-action in case H acts trivially on Y and its right adjoint, taking H-homotopy fixed points, denoted $(\_)^{hH}$.

LEMMA 8.14. – *Let $W \to Y$ be an H-Galois cover of base schemes. Then*

$$\mathsf{D}(\mathrm{Sh}_H(W_{\text{ét}}, R)) \simeq \mathsf{D}(\mathrm{Sh}(Y_{\text{ét}}, R)).$$

This equivalence can be obtained as the composition

$$\mathsf{D}(\mathrm{Sh}_H(W_{\text{ét}}, R)) \to \mathsf{D}(\mathrm{Sh}_H(Y_{\text{ét}}, R)) \xrightarrow{(\_)^{hH}} \mathsf{D}(\mathrm{Sh}(Y_{\text{ét}}, R)).$$

Proof. – The first statement follows from the fact that $[W/H] \simeq Y$, the second statement from considering the corresponding left adjoints. □

Now suppose we are given a map of groups $\psi\colon H \to R^*$ and $n \in \mathbb{Z}$. Then we can twist on object $F \in \mathrm{Sh}_H(Y_{\text{ét}}, R)$ by multiplying for any $U \in Y_{\text{ét}}$ and $h \in H$ the structure map $F(U) \to F(Y \times_{h,Y} U)$ with $\psi(h)^n$. We denote the resulting equivariant sheaf by $F\{n\}^\psi$. We use the same notation for the induced functor on the derived categories. We note that the operation $(\_)\{n\}^\psi$ commutes with (derived) pull backs and push forwards and with taking cohomology sheaves. Also for $F, G \in \mathsf{D}(\mathrm{Sh}_H(Y_{\text{ét}}, R))$ we have

$$F\{n\}^\psi \otimes_R G\{m\}^\psi \cong (F \otimes_R G)\{n+m\}^\psi.$$

For any $n \in \mathbb{Z}$ we denote by $\widetilde{L}(n)$ the sheaf $\mu_p^{\otimes n}$ which we view as an object of $\mathsf{D}_G(\mathrm{Sh}(\widetilde{X}_{\tilde{\eta},\text{ét}}, \mathbb{F}_p))$ (since it pulls back from X_η it is equipped with a natural G-action). We also let $L(n)$ be the sheaf $\mu_p^{\otimes n}$ viewed as object of $\mathsf{D}(\mathrm{Sh}(X_{\eta,\text{ét}}, \mathbb{F}_p))$. By Lemma 8.14 we have a canonical isomorphism $L(r) \cong (\mathbb{R}f_*^\eta \widetilde{L}(r))^{hG}$. Since the order of G is prime to p the cohomology sheaves of the homotopy fixed points are the fixed points of the cohomology sheaves (in fact these fixed points split off as direct summands defined by canonical projectors). Therefore we will write from now on $(\_)^G$ instead of $(\_)^{hG}$. Taking homotopy fixed points commutes with $\mathbb{R}j_{X,*}$ and by the above also with i_X^*, thus we have

$$i_X^*\mathbb{R}j_{X,*}L(r) \cong (i_X^*\mathbb{R}j_{X,*}\mathbb{R}f_*^\eta\widetilde{L}(r))^G \cong (i_X^*\mathbb{R}f_*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(r))^G \cong (\mathbb{R}f_*^\kappa i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(r))^G.$$

For the last isomorphism we have used the proper base change formula (note the left hand square in the above diagram is not necessarily cartesian, but the map from the left upper corner to the pullback is the closed inclusion of the pullback equipped with its reduced structure, so induces equivalences of étale sheaf categories).

So we have

$$\mathscr{H}^k(i_X^*\mathbb{R}j_{X,*}L(r)) \cong (f_*^\kappa\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(r)))^G$$

(we use that f_*^κ commutes with taking cohomology sheaves since f^κ is finite).

We have an isomorphism $\widetilde{L}(0)\{1\}^{\psi_1} \cong \widetilde{L}(1)$ in $\mathsf{D}_G(\mathrm{Sh}(\widetilde{X}_{\tilde{\eta},\text{ét}}, \mathbb{F}_p))$ sending 1 to ζ_p. This gives us an isomorphism $\widetilde{L}(0)\{n\}^{\psi_1} \cong \widetilde{L}(n)$ for any $n \in \mathbb{Z}$. We get the isomorphism

$$\widetilde{L}(r) \cong \widetilde{L}(k) \otimes \widetilde{L}(r-k) \cong \widetilde{L}(k) \otimes \widetilde{L}(0)\{r-k\}^{\psi_1} \cong \widetilde{L}(k)\{r-k\}^{\psi_1}.$$

We thus have

$$f_*^{\kappa}\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(r)) \cong (f_*^{\kappa}\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k)))\{r-k\}^{\psi_1}$$

in $\mathrm{Sh}_G(X_{\kappa,\text{ét}},\mathbb{F}_p)$.

We have a filtration on $\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k))$ by the $U^m\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k))$, where the latter subsheaf is generated as above by sections of the form $\{x_1,\dots,x_r\}$ such that $x_1-1\in\tilde{\pi}^m i_{\widetilde{X}}^*\mathscr{O}_{\widetilde{X}}$ ($m\geq 1$, for $m=0$ we again take the whole sheaf). The $U^m\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k))$ are invariant under the G-action, thus the

$$U_X^{k,m} := ((f_*^{\kappa}U^m\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k)))\{r-k\}^{\psi_1})^G$$

filter $\mathscr{H}^k(i_X^*\mathbb{R}j_{X,*}L(r))\cong\mathscr{H}^k(K_{X,0})$.

Let $\tilde{e}$ be the absolute ramification index of $\widetilde{\Lambda}$ and $\tilde{e}' := \frac{\tilde{e}p}{p-1}$.

We denote by

$$\widetilde{Q}_X^{k,m} := \mathrm{gr}^m U^{\bullet}\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k)) \in \mathrm{Sh}_G(\widetilde{X}_{\tilde{\kappa},\text{ét}},\mathbb{F}_p)$$

the m-th graded piece of the descending filtration $U^{\bullet}\,\mathscr{H}^k(i_{\widetilde{X}}^*\mathbb{R}j_{\widetilde{X},*}\widetilde{L}(k))$.

Then we have

$$\mathrm{forg}(\widetilde{Q}_X^{k,m}) \cong \begin{cases} \nu_1^k\oplus\nu_1^{k-1}, & 0=m,\\ \Omega^{k-1}_{\widetilde{X}_{\tilde{\kappa}}}, & 1\leq m<\tilde{e}',\quad p\nmid m,\\ B^k_{\widetilde{X}_{\tilde{\kappa}}}\oplus B^{k-1}_{\widetilde{X}_{\tilde{\kappa}}}, & 1\leq m<\tilde{e}',\quad p\mid m,\\ 0, & \tilde{e}'\leq m,\end{cases}$$

where forg(−) denotes forgetting the G-action, see [**4**, Cor. (1.4.1)].

Let $P_X^{k,m} := f_*^{\kappa}\widetilde{Q}_X^{k,m}\in\mathrm{Sh}_G(X_{\kappa,\text{ét}},\mathbb{F}_p)$.

The sheaves $\nu_1^k\oplus\nu_1^{k-1}$, $\Omega^{k-1}_{\widetilde{X}_{\tilde{\kappa}}}$ and $B^k_{\widetilde{X}_{\tilde{\kappa}}}\oplus B^{k-1}_{\widetilde{X}_{\tilde{\kappa}}}$ are equivariant with respect to the $\mathrm{Gal}(\tilde{\kappa}/\kappa)$-action on $\widetilde{X}_{\tilde{\kappa}}$, thus they are also equivariant with respect to the G-action. This defines a G-action on $\mathrm{forg}(P_X^{k,m})$, denoted $R_X^{k,m}\in\mathrm{Sh}_G(X_{\kappa,\text{ét}},\mathbb{F}_p)$, which possibly differs from $P_X^{k,m}$.

The formulas in [**4**, (4.3)] (the maps ρ_m defined there are used in Paragraphs 5 and 6 in loc. cit. to define the isomorphisms above) and the definition of the map to the 0-graded part show that there are isomorphisms

$$P_X^{k,m}\cong R_X^{k,m}\{m\}^{\psi_2}.$$

We let $Q_X^{k,m} := \mathrm{gr}^m(U_X^{k,\bullet})\cong(P_X^{k,m}\{r-k\}^{\psi_1})^G$.

The considerations made show the following

Proposition 8.15. – *The sheaves $Q_X^{k,m}$ on $X_{\kappa,\text{ét}}$ only depend on X_κ and glue to a sheaf $Q^{k,m}$ on $\mathrm{Sm}_{\kappa,\text{ét}}$.*

Define inductively objects $K_{X,m} \in \mathsf{D}(\mathrm{Sh}(X_{\kappa,\text{ét}}, \mathbb{F}_p))$ in the following way. For $m = 0$ we have already defined the object (see (26)). We define $K_{X,1}$ to be the homotopy fiber of the composition

$$K_{X,0} \to \mathscr{H}^r(K_{X,0})[-r] \to Q_X^0[-r].$$

There is a canoncial map $K_{X,1} \to Q_X^1[-r]$. Suppose $K_{X,m}$ together with a map

$$K_{X,m} \to Q_X^m[-r]$$

is already defined for $m < e'$. Define then $K_{X,m+1}$ to be the homotopy fiber of this last map. If $m+1 < e'$ there is a map $K_{X,m+1} \to Q_X^{m+1}[-r]$. If $m+1 \geq e'$ we have $K_{X,m+1} \cong \tau_{\leq (r-1)} K_{X,0}$ and there is a map

$$K_{X,m+1} \to Q_X^{r-1,0}[-r+1].$$

Keep going this way splitting off successively the

$$Q_X^{r-1,0}[-r+1], Q_X^{r-1,1}[-r+1], \dots, Q_X^{r-k,m}[-k], \dots, Q_X^{0,0}$$

(where for this m we require $0 \leq m < \tilde{e}'$) obtaining the $K_{X,m+2}, \dots, K_{X,N} = 0$.

By construction we have triangles

$$K_{X,m+1} \to K_{X,m} \to Q_X^{k(m),m'(m)}[-k(m)] \to K_{X,m+1}[1],$$

where $k(m)$ and $m'(m)$ depend in a way on m which we do not make explicit.

Set $H_0 := \tau_{\leq r} \mathbb{R} j_* L_\eta(r)$. The maps $H_0|_{X_{\text{ét}}} \to i_{X,*} Q_X^0[-r]$ glue to a map of sheaves $H_0 \to i_* Q^0[-r]$. We let H_1 be the homotopy fiber of this last map. We have a map $i_X^* H_1|_{X_{\text{ét}}} \to K_{X,0}$ which factors uniquely through $K_{X,1}$, thus we get a map $i_X^* H_1|_{X_{\text{ét}}} \to Q_X^1$ with adjoint $H_1|_{X_{\text{ét}}} \to i_{X,*} Q_X^1$. These maps glue to a map $H_1 \to i_* Q^1$ whose homotopy fiber we denote by H_2. Inductively one constructs objects $H_m \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{F}_p))$, $0 \leq m \leq N$, with maps

$$i_X^* H_m|_{X_{\text{ét}}} \to K_{X,m} \to Q_X^{k(m),m'(m)}[-k(m)]$$

(here we suppose $m > e'$, the other case is similar) whose adjoints glue to a map

$$H_m \to \mathscr{H}^{k(m)}(H_m)[-k(m)] \to i_* Q^{k(m),m'(m)}[-k(m)].$$

H_{m+1} is then defined to be the homotopy fiber of this map. By construction we have triangles

$$H_{m+1} \to H_m \to i_* Q^{k(m),m'(m)}[-k(m)] \to H_{m+1}[1].$$

Moreover for $X \in \mathrm{Sm}_U$ we have

$$i_X^*(H_m|_{X_{\text{ét}}}) \cong K_{X,m}.$$

Note that we have $\mathbb{R}^j \varepsilon_* i_* Q^m \cong i_* \mathbb{R}^j \varepsilon_* Q^m = 0$ for $j > 0$ and $m \geq 1$ by Lemma 8.11, similarly we have $\mathbb{R}^j \varepsilon_* i_* Q^{k,m} = 0$ for $j > 1$ by Proposition 8.9. Thus the canonical maps

$$\tau_{\leq r} \mathbb{R} \varepsilon_* H_m \to \mathbb{R} \varepsilon_* H_m$$

are isomorphisms for $m \geq 1$. Since $\mathscr{H}^{r+1}(\mathbb{R}\varepsilon_* H_1) = 0$ it also follows that the map

$$\mathscr{H}^r(\mathbb{R}\varepsilon_* H_0) \to \varepsilon_* i_* Q^0 \cong i_* \nu_1^r \oplus i_* \nu_1^{r-1}$$

is an epimorphism. It follows that we have an exact triangle

$$\mathbb{R}\varepsilon_* H_1 \to \tau_{\leq r}\mathbb{R}\varepsilon_* H_0 \to \varepsilon_* i_* Q^0 \to \mathbb{R}\varepsilon_* H_1[1]. \tag{27}$$

LEMMA 8.16. – *For any $F \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\text{ét}}, \mathbb{F}_p))$ we have $\mathbb{R}\varepsilon_* F \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\mathrm{Nis}}, \mathbb{F}_p))$ is $\mathbb{A}^1$-weakly contractible (i.e., becomes 0 in the $\mathbb{A}^1$-localization $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{\kappa,\mathrm{Nis}}, \mathbb{F}_p))$).*

Proof. – Tensoring F with the Artin-Schreier exact triangle

$$\mathbb{F}_p \to \mathbb{G}_a \to \mathbb{G}_a \to \mathbb{F}_p[1]$$

shows that it is sufficient to show that $\mathbb{R}\varepsilon_*(F \otimes \mathbb{G}_a)$ is $\mathbb{A}^1$-contractible. The standard $\mathbb{A}^1$-contraction of $\mathbb{G}_a$ does the job. □

PROPOSITION 8.17. – *The homotopy cofiber of the map*

$$\mathbb{R}\varepsilon_* H_N \to \mathbb{R}\varepsilon_* H_1$$

is $\mathbb{A}^1$-weakly contractible in $\mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Nis}}, \mathbb{F}_p))$ (i.e., becomes 0 in the $\mathbb{A}^1$-localization $\mathsf{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{U,\mathrm{Nis}}, \mathbb{F}_p))$).

Proof. – This homotopy cofiber is filtered with graded pieces the $\mathbb{R}\varepsilon_* i_* Q^m \cong i_* \mathbb{R}\varepsilon_* Q^m$, $m \geq 1$, and the $\mathbb{R}\varepsilon_* i_* Q^{k,m} \cong i_* \mathbb{R}\varepsilon_* Q^{k,m}$, so it is sufficient to show that these are $\mathbb{A}^1$-weakly contractible. But i_* preserves $\mathbb{A}^1$-weak equivalences since it is finite, so the claim follows from Lemma 8.16. □

By Proposition 8.5 for any $X \in \mathrm{Sm}_U$ we have $i_X^*(\mathbb{R}\varepsilon_* H_N)|_{X_{\mathrm{Nis}}} \cong \mathbb{R}\varepsilon_* i_X^*(H_N|_{X_{\text{ét}}}) \cong \mathbb{R}\varepsilon_* K_{X,N} \cong 0$, thus due to Proposition 8.1 we get $\mathbb{L}i^* \mathbb{R}\varepsilon_* H_N \cong 0$. Proposition 8.17 implies that $\mathbb{L}i^* \mathbb{R}\varepsilon_* H_1$ is $\mathbb{A}^1$-weakly contractible. By (27) it follows that

$$\mathbb{L}i^* \tau_{\leq r} \mathbb{R}\varepsilon_* H_0 \to \mathbb{L}i^* \varepsilon_* i_* Q^0$$

is an $\mathbb{A}^1$-weak equivalence.

Altogether we obtain

THEOREM 8.18. – *The maps $\alpha_{n,r}$ defined before Lemma 8.8 are $\mathbb{A}^1$-weak equivalences.*

So we have isomorphisms of naive spectra

$$\mathbb{L}i^* \mathsf{M}\mathbb{Z}_U/p^n \cong \mathbb{L}i^* F_n \cong E_n$$

in the $\mathbb{A}^1$-local categories which lift uniquely to isomorphisms of spectra. The induced map

$$\mathbb{L}i^* \mathsf{M}\mathbb{Z} \to E$$

is the p-completion map.

We get that $\mathbb{L}i^* \mathsf{M}\mathbb{Z}_U$ sits in the same homotopy cartesian square as $\mathscr{M}_\kappa$ (see diagram (24)), whence (using Theorem 6.7)

COROLLARY 8.19. – *There are canonical isomorphisms $\mathbb{L}i^* \mathsf{M}\mathbb{Z}_U \cong \mathscr{M}_\kappa \cong \mathsf{M}\mathbb{Z}_\kappa$.*

We are next going to construct natural comparison maps for our spectra for morphisms between Dedekind domains of mixed characteristic.

So let $\mathscr{D} \to \widetilde{\mathscr{D}}$ be a map of Dedekind domains of mixed characteristic. We use the notation of Section 4.1.2, without tildas for the situation over $S = \mathrm{Spec}(\mathscr{D})$ and with tildas for the situation over $\widetilde{S} = \mathrm{Spec}(\widetilde{\mathscr{D}})$. Note that for the various categories of complexes of sheaves we have Quillen adjunctions between the categories attached to S and $\widetilde{S}$. We let f denote the various maps from the situation with tildes to the situation without, e.g., $f\colon \widetilde{S} \to S$ or $f\colon \widetilde{U} \to U$.

We have an isomorphism $\varphi\colon f^*L_\bullet(1) \cong \widetilde{L}_\bullet(1)$. Choose a map $\psi\colon f^*\mathscr{T} \to \widetilde{\mathscr{T}}$ lifting the image of φ in the homotopy category. Thus we get a map

$$f^*\mathrm{Sym}(\mathscr{T}) \to \mathrm{Sym}(\widetilde{\mathscr{T}})$$

of commutative monoids in symmetric $\mathbb{Z}/p^\bullet[\mathbb{G}_{m,\widetilde{U}}, \{1\}]_{\text{ét}}$-spectra. Using lifting arguments one gets a map

$$f^*RQ\mathrm{Sym}(\mathscr{T}) \to RQ\mathrm{Sym}(\widetilde{\mathscr{T}}).$$

One gets induced maps $f^*A \to \widetilde{A}$, $f^*A' \to \widetilde{A}'$, $f^*B \to \widetilde{B}$, $f^*C \to \widetilde{C}$, $f^*C' \to \widetilde{C}'$ and $f^*D(p) \to \widetilde{D}(p)$.

By the definition of $\mathsf{M}\mathbb{Z}_S$ and $\mathsf{M}\mathbb{Z}_{\widetilde{S}}$ it is then clear that we get the comparison map

$$\Phi_f\colon \mathbb{L}f^*\mathsf{M}\mathbb{Z}_S \to \mathsf{M}\mathbb{Z}_{\widetilde{S}}$$

which is a map of E_∞-spectra.

Lemma 8.20. – *If $\widetilde{\mathscr{D}}$ is a filtered colimit of smooth $\mathscr{D}$-algebras then the comparison map Φ_f is an isomorphism.*

Proof. – Let $\widetilde{\mathscr{D}} = \mathrm{colim}_\alpha \mathscr{D}_\alpha$, where each $\mathscr{D}_\alpha$ is a smooth $\mathscr{D}$-algebra and set $S_\alpha := \mathrm{Spec}(\mathscr{D}_\alpha)$. Let $f_\alpha\colon S_\alpha \to S$ be the canonical maps. We show that for any $X \in \mathrm{Sm}_{\widetilde{S}}$ and integers p, q the induced map

$$\mathrm{Hom}_{\mathsf{SH}(\widetilde{S})}(\Sigma^{p,q}\Sigma^\infty_+ X, f^*\mathsf{M}\mathbb{Z}_S) \to \mathrm{Hom}_{\mathsf{SH}(\widetilde{S})}(\Sigma^{p,q}\Sigma^\infty_+ X, \mathsf{M}\mathbb{Z}_{\widetilde{S}}) \tag{28}$$

is an isomorphism. By the remarks after [**27**, Definition A.1.] we can write $X = \lim_\alpha X_\alpha$, where each X_α is a smooth and separated S_α-scheme of finite type. By [**27**, Lemma A.7.(1)] the left side of (28) can be written as

$$\mathrm{colim}_\alpha \mathrm{Hom}_{\mathsf{SH}(S_\alpha)}(\Sigma^{p,q}\Sigma^\infty_+ X_\alpha, f_\alpha^*\mathsf{M}\mathbb{Z}_S).$$

A similar formula holds for the right hand side, using the continuity of the constructions used to define $\mathsf{M}\mathbb{Z}$ (more precisely we use continuity of étale cohomology, of the logarithmic de Rham-Witt sheaves and of Beilinson motives). □

Corollary 8.21. – *Suppose $\widetilde{\mathscr{D}}$ is the completion of a local ring of S at a closed point of positive residue characteristic. Then Φ_f is an isomorphism.*

Proof. – In this case $\widetilde{\mathscr{D}}$ is a filtered colimit of smooth $\mathscr{D}$-algebras: We an apply Popescu's theorem ([**9**, Theorem 1.3]), since the map $\mathscr{D} \to \widetilde{\mathscr{D}}$ is regular (in the sense of [**21**, Définition (6.8.1)]), because $\mathscr{D}$ is of mixed characteristic. So Lemma 8.20 applies. □

THEOREM 8.22. – *Let $S = \operatorname{Spec}(D)$, D a Dedekind domain of mixed characteristic, $x \in S$ a closed point of positive residue characteristic and $i\colon \{x\} \to S$ the inclusion. Then there is a canonical isomorphism $\mathbb{L}i^*\mathsf{M}\mathbb{Z}_S \cong \mathsf{M}\mathbb{Z}_{\kappa(x)}$ which respects the E_∞-structures.*

Proof. – The isomorphism as spectra follows now from Corollary 8.19 and Corollary 8.21. The isomorphism can be made to respect the E_∞-structures by the uniqueness of E_∞-structures on $\mathsf{M}\mathbb{Z}_{\kappa(x)}$, which holds since this spectrum is the zero-slice of the sphere spectrum (see [**35**]). □

LEMMA 8.23. – *Let $g\colon k \to l$ be a field extension. Then the natural map $g^*\mathsf{M}\mathbb{Z}_k \to \mathsf{M}\mathbb{Z}_l$ is an isomorphism.*

Proof. – This follows from [**27**, Theorem 4.18] (taking U to be the spectrum of the prime field contained in k). □

LEMMA 8.24. – *Let $\widetilde{S} = \operatorname{Spec}(\widetilde{\mathscr{D}})$, $\widetilde{\mathscr{D}}$ a Dedekind domain, and let $\varphi\colon E \to F$ be any map in $\mathsf{SH}(\widetilde{S})$. Suppose for any $x \in \widetilde{S}$ that $\mathbb{L}i_x^*\varphi$ is an isomorphism, where i_x denotes the inclusion $\{x\} \hookrightarrow \widetilde{S}$. Then φ is an isomorphism.*

Proof. – This follows from [**7**, Proposition 4.3.9] and localization. □

THEOREM 8.25. – *For any map f between spectra of Dedekind domains of mixed characteristic the comparison map Φ_f is an isomorphism.*

Proof. – This follows from Theorem 8.22, Lemma 8.23 and Lemma 8.24. □

CHAPTER 9

THE MOTIVIC FUNCTOR FORMALISM

For any base scheme X we let $\mathsf{M}\mathbb{Z}_X := f^*\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$, where $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ is the structure morphism. We let $\mathsf{M}\mathbb{Z}_X - \mathrm{Mod}$ be the model category of highly structured $\mathsf{M}\mathbb{Z}_X$-module spectra and set $\mathrm{DM}(X) := \mathsf{Ho}(\mathsf{M}\mathbb{Z}_X - \mathrm{Mod})$. This is done e.g., along the lines of [**44**]. For any map of base schemes $f\colon X \to Y$ we get an adjunction

$$f^*\colon \mathrm{DM}(Y) \rightleftarrows \mathrm{DM}(X)\colon f_*.$$

The categories $\mathrm{DM}(X)$ are closed tensor triangulated and the functors f^* are symmetric monoidal.

If f is smooth the functor f^* has a left adjoint $f_\sharp$.

Note that all these functors commute with the forgetful functors

$$\mathrm{DM}(X) \to \mathsf{SH}(X).$$

For the functors f^* and f_* this follows since these functors are the lifts of the respective functors on the categories $\mathsf{SH}(-)$ to the module categories, and for f smooth this follows for the functors $f_\sharp$ since the functor $f_\sharp\colon \mathsf{SH}(X) \to \mathsf{SH}(Y)$ is a $\mathsf{SH}(Y)$-module functor (i.e., the projection formula holds).

It follows that the assignment

$$X \mapsto \mathrm{DM}(X)$$

has the structure of a stable homotopy functor in the sense of [**1**].

Thus the main results of loc. cit. are valid for this assignment, and the extensions of these results from [**7**] (e.g., removing the (quasi-)projectivity assumptions) are available.

In particular for a morphism of finite type $f\colon X \to Y$ we have an adjoint pair

$$f_!\colon \mathrm{DM}(X) \rightleftarrows \mathrm{DM}(Y)\colon f^!.$$

Moreover the proper base change theorem holds.

We also have the

THEOREM 9.1. – *Let* $i\colon Z \hookrightarrow X$ *be a closed inclusion of base schemes and* $j\colon U \hookrightarrow X$ *the open complement. Then for any* $F \in \mathrm{DM}(X)$ *there is an exact triangle*

$$j_!j^*F \to F \to i_*i^*F \to j_!j^*F[1]$$

in $\mathrm{DM}(X)$.

CHAPTER 10

FURTHER APPLICATIONS

10.1. The Hopkins-Morel isomorphism

We first equip $\mathsf{M}\mathbb{Z}$ with an orientation.

PROPOSITION 10.1. – *Let X be a smooth scheme over Dedekind domain of mixed characteristic. Then there is a unique orientation on $\mathsf{M}\mathbb{Z}_X$. The corresponding formal group law is the additive one.*

Proof. – Let S be the spectrum of a Dedekind domain of mixed characteristic. Let $P \in \mathsf{D}(\mathrm{Sh}(\mathrm{Sm}_{S,\mathrm{Zar}}, \mathbb{Z}))$ be the first $\mathbb{A}^1$-and Nisnevich-local space of an Ω-$\mathbb{P}^1$-spectrum model of $\mathsf{M}\mathbb{Z}_S$. Then by Theorem 7.10 there is a canonical isomorphism $P \cong \mathcal{O}^*_{/S}[1]$. Moreover by the proof of this theorem the canonical map $\mathbb{Z}[\mathbb{P}^1, \{\infty\}]_{\mathrm{Zar}} \to P$ induced by the first bonding map is induced by the suspension of the canonical map $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{\mathrm{Zar}} \to \mathcal{O}^*_{/S}$, using the canonical isomorphism $(\mathbb{P}^1, \{\infty\}) \cong \mathbb{G}_{m,S} \wedge S^1$ in $\mathscr{H}_\bullet(S)$. Thus our map $\mathbb{Z}[\mathbb{P}^1, \{\infty\}]_{\mathrm{Zar}} \to \mathcal{O}^*_{/S}[1]$ classifies the line bundle $\mathcal{O}(-1)$. So the map $\Sigma^{-2,-1}\Sigma^\infty_+ \mathbb{P}^\infty \to \mathsf{M}\mathbb{Z}_S$ corresponding to the map $\mathbb{Z}[\mathbb{P}^\infty]_{\mathrm{Zar}} \to \mathcal{O}^*_{/S}[1]$ which classifies the tautological line bundle is an orientation of $\mathsf{M}\mathbb{Z}_S$. Pulling back to any smooth scheme X over S gives an orientation of $\mathsf{M}\mathbb{Z}_X$. Since motivic cohomology of X with negative weight vanishes this orientation is unique and the corresponding formal group law is the additive one. □

By pulling back the unique orientation of $\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ to any base scheme X we see that $\mathsf{M}\mathbb{Z}_X$ has a canonical additive orientation.

REMARK 10.2. – *We note that over smooth schemes X over Dedekind domains of mixed characteristic or over fields the orientation map $\mathsf{MGL}_X \to \mathsf{M}\mathbb{Z}_X$ has a unique structure of an E_∞-map. This E_∞-map is achieved as the composition of the E_∞-maps*

$$\mathsf{MGL}_X \to s_0\mathsf{MGL}_X \cong s_0\mathbf{1} \to \mathsf{s}_0\mathsf{M}\mathbb{Z}_X \to \mathsf{M}\mathbb{Z}_X,$$

where the second isomorphism is [48, Corollary 3.3] *and where the last map exists since the map $f_0\mathsf{M}\mathbb{Z}_X \to s_0\mathsf{M}\mathbb{Z}_X$ is an isomorphism since $\mathsf{M}\mathbb{Z}_X$ is 0-truncated.*

Thus for any base scheme X the orientation $\mathsf{MGL}_X \to \mathsf{M}\mathbb{Z}_X$ has a canonical E_∞-structure. Since MGL_X has a strong periodization this gives an alternative proof that $\mathsf{M}\mathbb{Z}_X$ is strongly periodizable.

We see that we can factor the orientation map $\mathsf{MGL} \to \mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ through the quotient $\mathsf{MGL}/(x_1, x_2, \ldots)\mathsf{MGL}$, where the x_i are images of generators of MU_* with respect to the natural map $\mathsf{MU}_* \to \mathsf{MGL}_{2*,*}$. Pulling back this factorization to any base scheme X we get a map $\Phi_X \colon \mathsf{MGL}_X/(x_1, x_2, \ldots)\mathsf{MGL}_X \to \mathsf{M}\mathbb{Z}_X$.

THEOREM 10.3. – *Let R be a commutative ring and X a base scheme whose positive residue characteristics are all invertible in R. Then the map $\Phi_X \wedge M_R$, where M_R denotes the Moore spectrum on R, is an isomorphism.*

Proof. – We only have to show this statement for X being equal to the spectrum of a localization of $\mathbb{Z}$. Then it follows by pullback to the points of X and [**27**, Theorem 7.12] using Theorem 8.22 and Lemma 8.24. □

COROLLARY 10.4. – *Let R be a commutative ring and X a base scheme whose positive residue characteristics are all invertible in R. Then $\mathsf{M}R_X$ (which denotes $\mathsf{M}\mathbb{Z}_X$ with R-coefficients) is (stably) cellular, i.e., is contained in the smallest localizing triangulated subcategory of $\mathsf{SH}(X)$ which contains the spheres $S^{0,q}$ for all $q \in \mathbb{Z}$.*

10.2. The dual motivic Steenrod algebra

In the whole section we fix a prime l.

For a base scheme S we denote by $\underline{\mathrm{Pic}}_S$ a strictification of the 2-functor which assigns to any $X \in \mathrm{Sm}_S$ the Picard groupoid of line bundles on X. We denote by $\nu\underline{\mathrm{Pic}}_S$ the motivic space which assigns to any $X \in \mathrm{Sm}_S$ the nerve of $\underline{\mathrm{Pic}}_S(X)$.

PROPOSITION 10.5. – *Let S be a regular base scheme and let $f \colon \mathbb{P}^\infty_S \to \nu\underline{\mathrm{Pic}}_S$ be a map classifying a $\mathbb{G}_m$-torsor P on $\mathbb{P}^\infty_S$. Then there is an $\mathbb{A}^1$-fiber sequence*

$$P \to \mathbb{P}^\infty_S \to \nu\underline{\mathrm{Pic}}_S$$

of motivic spaces.

Proof. – The sequence is a fiber sequence in simplicial presheaves equipped with a model structure with objectwise weak equivalences. Thus the claim follows from the $\mathbb{A}^1$-and Nisnevich-locality of $\nu\underline{\mathrm{Pic}}_S$ (and e.g., right properness of motivic model structures). □

For a base scheme S we let $W_{S,n,k}$ be the $\mathbb{G}_m$-torsor on $\mathbb{P}^k_S$ corresponding to the line bundle $\mathcal{O}_{\mathbb{P}^k}(-n)$. We let $W_{S,n} := \mathrm{colim}_k W_{S,n,k}$ be the corresponding $\mathbb{G}_m$-torsor on $\mathbb{P}^\infty_S$.

We are going to compute the motivic cohomology of $W_{S,n}$ with $\mathbb{Z}/m$-coefficients for $m|n$ relative to the motivic cohomology of the base S. We orient ourselves along the lines of [**57**, §6].

We have a cofibration sequence

$$W_{S,n,+} \to \mathcal{O}_{\mathbb{P}^\infty_S}(-n)_+ \to \mathrm{Th}(\mathcal{O}_{\mathbb{P}^\infty_S}(-n)). \tag{29}$$

For a motivic space $\mathfrak{X}$ over S let

$$H^{p,q}(\mathfrak{X}) := \mathrm{Hom}_{\mathsf{SH}(S)}(\Sigma^\infty \mathfrak{X}_+, \Sigma^{p,q}\mathsf{M}\mathbb{Z})$$

be the motivic cohomology of $\mathfrak{X}$. More generally for an abelian group A we set

$$H^{p,q}(\mathfrak{X}, A) := \mathrm{Hom}_{\mathsf{SH}(S)}(\Sigma^\infty \mathfrak{X}_+, \Sigma^{p,q}\mathsf{M}A).$$

We denote the respective reduced motivic cohomology groups of pointed motivic spaces by $\widetilde{H}$.

Then (29) gives a long exact sequence

$$\cdots \to H^{*-2,*-1}(S)\sigma \xrightarrow{n\sigma} H^{*,*}(S)\sigma \to H^{*,*}(W_{S,n}) \to H^{*-1,*-1}(S)\sigma \to \cdots. \tag{30}$$

Here σ is the class of $\mathcal{O}_{\mathbb{P}^\infty_S}(-1)$ in $H^{2,1}(\mathbb{P}^\infty_S)$.

For any $m > 0$ let $\beta_m\colon H^{*,*}(\_, \mathbb{Z}/m) \to H^{*+1,*}(\_)$ be the Bockstein homomorphism.

Let $v_n \in H^{2,1}(W_{S,n})$ be the pullback of σ under the canonical map $W_{S,n} \to \mathbb{P}^\infty_S$.

LEMMA 10.6. – *For any $m > 0$ there is a $u \in H^{1,1}(W_{S,n}, \mathbb{Z}/m)$ such that the restriction of u to $*$ is 0 and such that $\beta_m(u) = \frac{n}{\gcd(m,n)} \cdot v_n$. If S is smooth over a Dedekind domain of mixed characteristic or over a field then this u is unique with these properties.*

Proof. – Let $\tilde{v} \in H^{2,1}(W_{S,n})$ be any m-torsion class which restricts to 0 on $*$. Note that $\frac{n}{\gcd(m,n)} \cdot v_n$ is such a class. We will prove that then there is a unique $\tilde{u} \in H^{1,1}(W_{S,n}, \mathbb{Z}/m)$ which restricts to 0 on $*$ such that $\beta_m(\tilde{u}) = \tilde{v}$, assuming S is smooth over a Dedekind domain of mixed characteristic or over a field. The general statement about existence follows then by base change (e.g., from $\mathrm{Spec}(\mathbb{Z})$ to S).

Consider the commutative diagram

$$\begin{array}{ccccccc}
H^{1,1}(W_{S,n}) & \longrightarrow & H^{1,1}(W_{S,n}, \mathbb{Z}/m) & \longrightarrow & H^{2,1}(W_{S,n}) & \xrightarrow{\cdot m} & H^{2,1}(W_{S,n}) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{1,1}(S) & \longrightarrow & H^{1,1}(S, \mathbb{Z}/m) & \longrightarrow & H^{2,1}(S), & &
\end{array}$$

with exact rows and where the vertical maps are restriction to $*$ which split the maps on cohomology induced by the structure map $W_{S,n} \to S$. The exact sequence (30) around $H^{1,1}(W_{S,n})$ shows that the first vertical map is an isomorphism. A diagram chase then shows existence and uniqueness of $\tilde{u}$ with the required properties. □

We denote the canonical class in $\widetilde{H}^{1,1}(W_{S,n}, \mathbb{Z}/m)$ obtained this way by $u_{n,m}$ (by demanding that these classes are compatible with base change). We set $u_n := u_{n,n}$.

We let

$$K(\mathbb{Z}/n(1), 1)_S, K(\mathbb{Z}(1), 2)_S \in \mathscr{H}_\bullet(S)$$

be the motivic Eilenberg-MacLane spaces which represent the functors $\widetilde{H}^{1,1}(\_, \mathbb{Z}/n)$ and $\widetilde{H}^{2,1}(\_)$ on $\mathscr{H}_\bullet(S)$ respectively.

Proposition 10.7. – *If S is smooth over a Dedekind domain of mixed characteristic or over a field then we have $K(\mathbb{Z}(1), 2)_S \cong \nu\underline{\mathrm{Pic}}_S \cong B\mathbb{G}_{m,S} \cong \mathbb{P}^\infty_S$ in $\mathscr{H}_\bullet(S)$.*

Proof. – This follows from the fact the motivic sheaf of weight 1 is in this case $\mathcal{O}^*_{/S}[-1]$. □

Proposition 10.8. – *If S is smooth over a Dedekind domain of mixed characteristic or over a field then we have $K(\mathbb{Z}/n(1), 1)_S \cong W_{S,n}$ in $\mathscr{H}_\bullet(S)$. The isomorphism is given by the class u_n.*

Proof. – Let $f\colon \mathbb{P}^\infty_S \to \nu\underline{\mathrm{Pic}}_S$ be the map classifying the line bundle $\mathcal{O}_{\mathbb{P}^\infty_S}(-n)$. $W_{S,n}$ is the corresponding $\mathbb{G}_m$-torsor over $\mathbb{P}^\infty_S$. Then the diagram

$$\begin{array}{ccccc} W_{S,n} & \longrightarrow & \mathbb{P}^\infty_S & \xrightarrow{f} & \nu\underline{\mathrm{Pic}}_S \\ & & \downarrow & & \downarrow \\ K(\mathbb{Z}/n(1),1)_S & \longrightarrow & K(\mathbb{Z}(1),2)_S & \xrightarrow{\cdot n} & K(\mathbb{Z}(1),2)_S \end{array}$$

in $\mathscr{H}_\bullet(S)$, where the vertical maps are the canonical identifications, commutes. Moreover the rows are fiber sequences: the first one by Proposition 10.5, the second one by definition. It follows that there is a vertical isomorphism $u'\colon W_{S,n} \to K(\mathbb{Z}/n(1), 1)_S$ in $\mathscr{H}_\bullet(S)$ making the whole diagram commutative. The uniqueness clause of Lemma 10.6 shows that $u' = u_n$ finishing the proof. □

For any $X \in \mathrm{Sm}_S$ consider the functor $T_n\colon \underline{\mathrm{Pic}}(X) \to \underline{\mathrm{Pic}}(X)$, $\mathscr{L} \mapsto \mathscr{L}^{\otimes n}$. Its homotopy fiber is the Picard groupoid $\underline{G}_n(X)$ whose objects are pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is a line bundle on X and $\varphi\colon \mathscr{L}^{\otimes n} \to \mathcal{O}_X$ is an isomorphism, and whose morphisms are isomorphisms of line bundles compatible with the trivializations. Note that we have a fiber sequences

$$\nu\underline{G}_n(X) \to \nu\underline{\mathrm{Pic}}(X) \xrightarrow{\nu T_n} \nu\underline{\mathrm{Pic}}(X)$$

functorial in X and that these fiber sequences also make sense for $X \in \mathsf{Set}^{\mathrm{Sm}_S^{\mathrm{op}}}$.

As in the proof of Proposition 10.8 it follows that we have a canonical equivalence $K(\mathbb{Z}/n(1), 1)_S \cong \nu\underline{G}_n$ in $\mathscr{H}_\bullet(S)$, provided that S is smooth over a Dedekind ring of mixed characteristic or over a field.

Since $\nu\underline{G}_n$ is Nisnevich- and $\mathbb{A}^1$-local it follows

PROPOSITION 10.9. – *Suppose S is smooth over a Dedekind ring of mixed characteristic or over a field and let X be in Sm_S or $\mathsf{Set}^{\mathrm{Sm}_S^{\mathrm{op}}}$. There is a canonical group isomorphism between $H^{1,1}(X,\mathbb{Z}/n)$ and the group of isomorphism classes of $\underline{G}_n(X)$. The boundary map $H^{1,1}(X,\mathbb{Z}/n) \to H^{2,1}(X)$ corresponds to the map on groupoids which forgets the trivialization. Moreover the map*

$$\mathcal{O}(X)^* \cong H^{1,1}(X,\mathbb{Z}) \to H^{1,1}(X,\mathbb{Z}/n) \cong \underline{G}_n(X)$$

sends a unit x to the trivial line bundle with n-th tensor power trivialized by x.

LEMMA 10.10. – *Suppose S is smooth over a Dedekind ring of mixed characteristic or over a field. The class $u_{n,m}$ corresponds under the isomorphism of Proposition 10.9 to the isomorphism class of the object*

$$(p^*\mathcal{O}_{\mathbb{P}^\infty_S}(-\frac{n}{\gcd(m,n)}), (p^*\mathcal{O}_{\mathbb{P}^\infty_S}(-\frac{n}{\gcd(m,n)}))^{\otimes m} \cong p^*\mathcal{O}_{\mathbb{P}^\infty_S}(-\mathrm{lcm}(m,n)) \cong \mathcal{O}_{W_{S,n}}),$$

where p is the map $W_{S,n} \to \mathbb{P}^\infty_S$ and the last isomorphism is the $\frac{m}{\gcd(m,n)}$-th tensor power of the canonical isomorphism $p^*\mathcal{O}_{\mathbb{P}^\infty_S}(-n) \cong \mathcal{O}_{W_{S,n}}$.

Proof. – This element clearly satisfies the requirements of Lemma 10.6. □

LEMMA 10.11. – *The image of the constant function on 1 under the isomorphisms*

$$(\mathbb{Z}/m)^{\pi_0(S)} \cong H^{0,0}(S,\mathbb{Z}/m) \cong \widetilde{H}^{1,1}(\mathbb{G}_{m,S},\mathbb{Z}/m)$$

corresponds under the isomorphism of Proposition 10.9 to the object $(\mathcal{O}_{\mathbb{G}_{m,S}}, \varphi)$, where φ is given by multiplication with the canonical unit in $\mathcal{O}(\mathbb{G}_{m,S})$.

Proof. – This unit corresponds to 1 under the map

$$\mathcal{O}^*(\mathbb{G}_{m,S}) \cong H^{1,1}(\mathbb{G}_{m,S}) \to \widetilde{H}^{1,1}(\mathbb{G}_{m,S}) \cong H^{0,0}(S) \cong \mathbb{Z}^{\pi_0(S)}. \qquad \square$$

COROLLARY 10.12. – *Suppose S is smooth over a Dedekind ring of mixed characteristic or over a field. Then the image of $u_{n,m}|_{W_{S,n,0}}$ under the isomorphisms*

$$\widetilde{H}^{1,1}(W_{S,n,0},\mathbb{Z}/m) \cong \widetilde{H}^{1,1}(\mathbb{G}_{m,S},\mathbb{Z}/m) \cong H^{0,0}(S,\mathbb{Z}/m) \cong (\mathbb{Z}/m)^{\pi_0(S)}$$

is the constant function on the class of $\frac{m}{\gcd(m,n)}$.

Proof. – This follows from Lemmas 10.10 and 10.11. □

REMARK 10.13. – *This shows that the constant c introduced in the proof of* [**57**, Proposition 6.6] *is in fact* 1.

DEFINITION 10.14. – *Let E be a motivic ring spectrum (i.e., a commutative monoid in $\mathsf{SH}(S)$) such that $E_{0,0}$ is a $\mathbb{Z}/n$-algebra. A* mod-n orientation *on E consists of an orientation $c \in E^{2,1}(\mathbb{P}^\infty_S)$ and a class $u \in \widetilde{E}^{1,1}(W_{S,n})$ which restricts to* 1 *under the map*

$$\widetilde{E}^{1,1}(W_{S,n}) \to \widetilde{E}^{1,1}(W_{S,n,0}) \cong \widetilde{E}^{1,1}(\mathbb{G}_{m,S}) \cong E_{0,0}.$$

REMARK 10.15. – *This notion has its origin in the notion of mod-p-orientation used in* [**51**].

It follows from Corollary 10.12 that the usual orientation of $\mathsf{M}\mathbb{Z}/m$ together with the class $u_{n,m}$ defines a mod-n orientation on $\mathsf{M}\mathbb{Z}/m$ provided $m|n$. We call this orientation the canonical mod-n orientation of $\mathsf{M}\mathbb{Z}/m$.

Note also that any mod-n orientation gives rise to a mod-n' orientation for $n \mid n'$.

THEOREM 10.16. – *Let E be a motivic ring spectrum such that $E_{0,0}$ is a $\mathbb{Z}/n$-algebra with a mod-n orientation given by classes $c \in E^{2,1}(\mathbb{P}^\infty_S)$ and $u \in E^{1,1}(W_{S,n})$. Let $v \in E^{2,1}(W_{S,n})$ be the pullback of c under the canonical projection $W_{S,n} \to \mathbb{P}^\infty_S$. Let $\mathfrak{X}$ be a motivic space. Denote by u and v also the pullbacks of u and v to the E-cohomology of $\mathfrak{X} \times W_{S,n}$. Then the elements v^i, uv^i, $i \geq 0$ form a topological basis of $E^{*,*}(\mathfrak{X} \times W_{S,n})$ over $E^{*,*}(\mathfrak{X})$. More precisely, the elements v^i, uv^i, $0 \leq i \leq k$, form a basis of $E^{*,*}(\mathfrak{X} \times W_{S,n,k})$ over $E^{*,*}(\mathfrak{X})$, v^{k+1} is zero in $E^{*,*}(\mathfrak{X} \times W_{S,n,k})$ and the canonical map*

$$E^{*,*}(\mathfrak{X} \times W_{S,n}) \to \lim_k E^{*,*}(\mathfrak{X} \times W_{S,n,k}),$$

where the transition maps are surjective, is an isomorphism.

Proof. – By writing $\mathfrak{X}$ has the homotopy colomit over Δ^{op} of a diagram with entries disjoint unions of objects from Sm_S and replacing cohomology groups by mapping spaces we reduce to the case where $\mathfrak{X} = X \in \mathrm{Sm}_S$. The induced long exact sequences in E-cohomology from the cofiber sequence

$$W_{X,n,k,+} \to \mathcal{O}_{\mathbb{P}^k_X}(-n)_+ \to \mathrm{Th}(\mathcal{O}_{\mathbb{P}^k_X}(-n))$$

split into short exact sequences

$$0 \to E^{*,*}(X)[\sigma]/(\sigma^{k+1}) \to E^{*,*}(X \times W_{S,n,k}) \to E^{*-1,*-1}(X)[\sigma]/(\sigma^{k+1}) \to 0$$

since $E_{0,0}$ is a $\mathbb{Z}/n$-algebra. The image of u in the right group is of the form $1 + \sigma \cdot r$. Using the fact that these sequences are $E^{*,*}(X)[\sigma]/(\sigma^{k+1})$-module sequences the claim follows. □

It follows that every element in $E^{*,*}(\mathfrak{X} \times W_{S,n})$ can be uniquely written as a power series

$$\sum_{i \geq 0} (a_i v^i + b_i u v^i)$$

with $a_i, b_i \in E^{*,*}(\mathfrak{X})$. Similar statements are valid for elements in $E^{*,*}(\mathfrak{X} \times W^j_{S,n})$. The latter group can be written as the j-fold completed tensor product over $E^{*,*}(\mathfrak{X})$ of copies of $E^{*,*}(\mathfrak{X} \times W_{S,n})$.

Note that if n is odd we have

$$E^{*,*}(\mathfrak{X} \times W^j_{S,n}) \cong E^{*,*}(\mathfrak{X}) v_1, \ldots, v_j(u_1, \ldots, u_j),$$

but if n is even there can be more complicated relations for the u_i^2.

The object $W_{S,n} \in \mathcal{H}(S)$ is naturally a commutative group object (it represents motivic cohomology over certain S, in particular $S = \mathrm{Spec}(\mathbb{Z})$, and pulls back). Moreover it has exponent n. This gives $E^{*,*}(\mathfrak{X} \times W_{S,n})$ the structure of a cocommutative

Hopf algebra object in a category whose tensor structure is the completed tensor product.

The comultiplication

$$E^{*,*}(\mathfrak{X} \times W_{S,n}) \to E^{*,*}(\mathfrak{X} \times W^2_{S,n})$$

is uniquely determined be the images of u and v which can be written as power series in u_1, u_2, v_1, v_2. These power series obey laws which are similar to the familiar formal group laws. We won't spell out these properties, suffices it to say that they are grouped into unitality, associativity, commutativity, exponent n and independence of the image of v of u_1, u_2.

For $E = \mathsf{M}\mathbb{Z}/m$, $m|n$, we have the additive law: $u \mapsto u_1 + u_2$, $v \mapsto v_1 + v_2$. This follows from weight reasons for $S = \mathrm{Spec}(\mathbb{Z})$ and thus is true in general.

There is the notion of a strict isomorphism of such laws (power series), again given by two power series (in u and v) in the target complete ring which start with u respectively v. Moreover the second power series is independent of u. Caution is required in the case n is even since then the complete rings in question might not have standard form.

Two mod-n orientations on a motivic ring spectrum give rise to such a strict isomorphism.

PROPOSITION 10.17. – *Let E be a motivic ring spectrum such that $E_{0,0}$ is an $\mathbb{F}_l$-algebra equipped with two additive mod-l orientations. Then the corresponding strict isomorphism has the form*

$$u \mapsto u + a_0 v + a_1 v^l + \cdots + a_i v^{l^i} + \cdots ,$$
$$v \mapsto v + b_1 v^l + \cdots + b_i v^{l^i} + \cdots .$$

Proof. – The proof is similar to the case of the additive formal group law over an $\mathbb{F}_l$-algebra. □

LEMMA 10.18. – *The suspension spectrum $\Sigma^\infty_+ W_{S,n,k}$ is finite cellular, in particular dualizable. The suspension spectrum $\Sigma^\infty_+ W_{S,n}$ is cellular.*

Proof. – This is a standard argument. □

In the following let $T_k := \Sigma^\infty_+ W_{S,n,k}$.

LEMMA 10.19. – *Let E be a mod-n oriented motivic ring spectrum. For any $0 \leq i \leq 1$, $0 \leq j \leq k$ let $\Sigma^{-2j-i,-j-i} E \to E \wedge T_k^\vee$ be the E-module map corresponding to the element $u^i v^j$ in the homotopy of the target spectrum. Then the induced map*

$$\bigoplus_{i,j} \Sigma^{-2j-i,-j-i} E \longrightarrow E \wedge T_k^\vee$$

is an isomorphism.

Proof. – Applying the functors $\mathrm{Hom}(\Sigma^{p,q}\Sigma^\infty_+ X, -)$ for $X \in \mathrm{Sm}_S$, $p, q \in \mathbb{Z}$, shows the claim. □

LEMMA 10.20. – *Let E be a motivic ring spectrum and U be a dualizable spectrum such that $E \wedge U$ is a finite sum of shifts of E as a E-module. Then $E \wedge U^\vee$ is the sum over the corresponding negative shifts of E.*

Proof. – For F a motivic spectrum we have

$$\begin{aligned}\mathrm{Hom}(F, E \wedge U^\vee) &= \mathrm{Hom}(F \wedge U, E) = \mathrm{Hom}_E(E \wedge U \wedge F, E) \\ &= \mathrm{Hom}_E(\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} E \wedge F, E) = \mathrm{Hom}(F, \bigoplus_\alpha \Sigma^{-p_\alpha, -q_\alpha} E). \qquad \square\end{aligned}$$

LEMMA 10.21. – *Let E be a mod-n oriented motivic ring spectrum. We have E-module isomorphisms*

$$\bigoplus_{i, 0 \le j \le k} \Sigma^{2j+i, j+i} E \cong E \wedge T_k$$

and

$$\bigoplus_{i, 0 \le j} \Sigma^{2j+i, j+i} E \cong E \wedge \Sigma^\infty_+ W_{S,n},$$

where the corresponding generators are the duals of the v^j, uv^j.

Proof. – Use Lemmas 10.19 and 10.20 (the latter is applied with $U = T_k^\vee$). $\square$

Let E, F be mod-n oriented motivic ring spectra.

LEMMA 10.22. – *The natural map*

$$E_{**}F \otimes_{F_{**}} F_{**}T_k^\vee \to (E \wedge F \wedge T_k^\vee)_{**}$$

is an isomorphism.

Proof. – This follows from Lemma 10.19. $\square$

From the above lemma we derive a coaction map

$$E^{**}(W_{S,n,k}) \cong E_{-*,-*}T_k^\vee \to (E \wedge F \wedge T_k^\vee)_{-*,-*} \cong E_{-*,-*}F \otimes_{F_{-*,-*}} F^{**}(W_{S,n,k}),$$

where for the second map we use the unit of F. These are compatible for different values of k, yielding in the limit a coaction map

$$E^{**}(W_{S,n}) \to E_{-*,-*}F \hat{\otimes}_{F_{-*,-*}} F^{**}(W_{S,n}).$$

We write the image of u as

$$\sum_{j \ge 0} (\alpha_j \otimes v^j + \beta_j \otimes uv^j),$$

similarly we write the image of v as

$$\sum_{j \ge 0} \gamma_i \otimes v^j.$$

(The latter sum is independent of u since the relation comes already from the projective space. Note also that the u's and v's on both sides are lying in different groups.)

PROPOSITION 10.23. – *The strict isomorphism relating the two mod-n orientations on $E \wedge F$ has the form*

$$u_E = \sum_{j \geq 0} (\alpha_j v_F^j + \beta_j u_F v_F^j),$$
$$v_E = \sum_{j \geq 0} \gamma_i v_F^j.$$

Here the u_E, v_E are those generators coming from the orientation on E, similarly for u_F, v_F.

Proof. – We leave the verification to the reader. □

REMARK 10.24. – *The coefficients $\alpha_i, \beta_i, \gamma_i$ can also be described by images of canonical homology generators with respect to the maps on F-homology of the orientation maps*

$$\Sigma^{-2,-1}\Sigma_+^\infty \mathbb{P}_S^\infty \to E$$

and

$$\Sigma^{-1,-1}\Sigma_+^\infty W_{S,n} \to E.$$

We specialize now to the case $E = F = \mathsf{M}\mathbb{F}_l$. The structure result below is analogous to results from [**57**, §12].

COROLLARY 10.25. – *The coaction map*

$$H^{**}(W_{S,l}, \mathbb{F}_l) \to (\mathsf{M}\mathbb{F}_l \wedge \mathsf{M}\mathbb{F}_l)_{-*,-*} \hat{\otimes}_{(\mathsf{M}\mathbb{F}_l)_{-*,-*}} H^{**}(W_{S,l}, \mathbb{F}_l)$$

is given by

$$u \mapsto u + \sum_{i \geq 0} \tau_i \otimes v^{l^i},$$
$$v \mapsto v + \sum_{i \geq 1} \xi_i \otimes v^{l^i}$$

with $\tau_i \in (\mathsf{M}\mathbb{F}_l \wedge \mathsf{M}\mathbb{F}_l)_{2l^i-1, l^i-1}$ and $\xi_i \in (\mathsf{M}\mathbb{F}_l \wedge \mathsf{M}\mathbb{F}_l)_{2(l^i-1), l^i-1}$.

Proof. – This follows from Propositions 10.23 and 10.17. □

Set $\mathscr{A}_{**} := (\mathsf{M}\mathbb{F}_l \wedge \mathsf{M}\mathbb{F}_l)_{**}$. We denote by B the set of sequences $(\epsilon_0, r_1, \epsilon_1, r_2, \ldots)$ with $\epsilon_i \in \{0, 1\}$ and $r_i \geq 0$ with only finitely many non-zero terms. For any $I \in B$ let

$$\omega(I) := \tau_0^{\epsilon_0} \xi_1^{r_1} \tau_1^{\epsilon_1} \xi_2^{r_2} \cdots \in \mathscr{A}_{p(I), q(I)}.$$

THEOREM 10.26. – *Suppose l is invertible on S. Then the map*

$$\bigoplus_{I \in B} \Sigma^{p(I),q(I)} \mathsf{M}\mathbb{F}_l \to \mathsf{M}\mathbb{F}_l \wedge \mathsf{M}\mathbb{F}_l,$$

where the map on the summand indexed by I is the $\mathsf{M}\mathbb{F}_l$-module map (where we use the right module structure on the target) corresponding to the element $\omega(I)$, is an isomorphism.

Proof. – It is sufficient to show the statement for $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{l}])$. This follows from [**28**, Theorem 1.1] using Theorem 8.22 and Lemma 8.24. □

REMARK 10.27. – *In the situation of the theorem the pair $(H^{-*,-*}(S, \mathbb{F}_l), \mathscr{A}_{**})$ has the structure of a Hopf algebroid. The operations of $\mathsf{M}\mathbb{F}_l$, i.e., $\mathrm{Hom}^{**}(\mathsf{M}\mathbb{F}_l, \mathsf{M}\mathbb{F}_l)$, are the dual of $\mathscr{A}_{**}$.*

APPENDIX A

(SEMI) MODEL STRUCTURES

Proposition A.1. – *Let $\mathcal{C}$ be a symmetric monoidal cofibrantly generated model category and I an (essentially) small category with 2-fold coproducts. Then the projective model structure on $\mathcal{C}^I$ is symmetric monoidal (with the objectwise symmetric monoidal structure). If I has an initial object and the tensor unit in $\mathcal{C}$ is cofibrant then the tensor unit in $\mathcal{C}^I$ is also cofibrant.*

Proof. – The assertions follow from the formula

$$(\mathrm{Hom}(i, \_) \cdot f) \square (\mathrm{Hom}(j, \_) \cdot g) \cong \mathrm{Hom}(i \sqcup j, \_) \cdot (f \square g)$$

for maps f and g in $\mathcal{C}$ and objectwise considerations. □

Proposition A.2. – *Let $\mathcal{C}$ be a left proper combinatorial model category and $\mathcal{S}$ be an (essentially) small site. Then the projective model structure on $\mathcal{C}^{\mathcal{S}^{\mathrm{op}}}$ can be localized to a local projective model structure where the local objects are presheaves satisfying descent for all hypercovers of $\mathcal{S}$.*

Proof. – We localize at the set of maps

$$\mathrm{hocolim}_{n \in \triangle^{\mathrm{op}}}(U_n \times QA) \to QA,$$

where $U_\bullet \to X$ runs through a set of dense hypercovers (see **[10]**) of $\mathcal{S}$ and A through the set of domains and codomains of a set of generating cofibrations of $\mathcal{C}$ (QA denotes a cofibrant replacement of A). □

Proposition A.3. – *Let R be a commutative ring and $\mathcal{C} = \mathrm{Cpx}_{(\geq 0)}(R)$ be the category of (non-negative) chain complexes of R-modules equipped with its standard projective model structure. Let $\mathcal{S}$ be an (essentially) small site with 2-fold products and enough points. Then the local projective model structure on $\mathcal{C}^{\mathcal{S}^{\mathrm{op}}}$ is symmetric monoidal.*

Proof. – The projective model structure is symmetric monoidal by Proposition A.1. It remains to see that the pushout product of a generating cofibration with a trivial cofibration is a weak equivalence. Checking this on stalks does the job (here use the injective model structure on $\mathcal{C}$). □

Remark A.4. – *This result is also contained in* [**15**].

Theorem A.5. – *Let R be a commutative ring and $\mathscr{S}$ an (essentially) small site with 2-fold products and enough points. Then $\mathrm{Cpx}_{(\geq 0)}(\mathrm{Sh}(\mathscr{S}, R))$ carries a local projective symmetric monoidal cofibrantly generated model structure transferred from the local model structure on presheaves. The weak equivalences are the quasi-isomorphisms.*

Proof. – One applies the transfer principle (see e.g., [**3**, §2.5]): One has to check that transfinite compositions of pushouts by images of generating trivial cofibrations are weak equivalences. This follows from the existence of the injective model structure and the fact that the sheafification functor preserves all weak equivalences. The same applies to prove that the model structure is symmetric monoidal. □

Let R and $\mathscr{S}$ be as in the theorem above. Then the canonical generating cofibrations of $\mathrm{Cpx}_{(\geq 0)}(\mathrm{Sh}(\mathscr{S}, R))$ have cofibrant domain. Thus for a cofibrant $T \in \mathrm{Cpx}_{(\geq 0)}(\mathrm{Sh}(\mathscr{S}, R))$ by [**26**, Theorem 8.11] there is a stable symmetric monoidal model structure on the category Sp_T^Σ of symmetric T-spectra in $\mathrm{Cpx}_{(\geq 0)}(\mathrm{Sh}(\mathscr{S}, R))$.

It follows from [**47**, Theorem 4.7] that for a Σ-cofibrant operad $\mathscr{O}$ in Sp_T^Σ the category of $\mathscr{O}$-algebras inherits a semi model structure. In particular for the image of the linear isometries operad in Sp_T^Σ we obtain a semi model category $E_\infty(\mathrm{Sp}_T^\Sigma)$ of E_∞-spectra.

APPENDIX B

PULLBACK OF CYCLES

For a regular separated Noetherian scheme X of finite Krull dimension we let $X^{(p)}$ be the set of codimension p points on X and $Z^p(X)$ the free abelian group on $X^{(p)}$. Let $C \in X^{(p)}$, $D \in X^{(q)}$. We say that C and D intersect properly if the scheme theoretic intersection Z of the closures of C and D in X has codimension everywhere $\geq p+q$. If C and D intersect properly then for a point W of Z of codimension $p+q$ in X we set

$$m(W;C,D) := \sum_{i\geq 0}(-1)^i \mathrm{length}_{\mathcal{O}_{X,W}}(\mathrm{Tor}_i^{\mathcal{O}_{X,W}}(\mathcal{O}_{\overline{C},W}, \mathcal{O}_{\overline{D},W})),$$

known as Serre's intersection multiplicity.

We extend the notion of proper intersection and the intersection multiplicity at an arbitrary $W \in X^{(p+q)}$ in the canonical way to elements of $Z^p(X)$ and $Z^q(X)$.

For $C \in Z^p(X)$ and $D \in Z^q(X)$ which intersect properly we let

$$C \cdot D := \sum_{W\in X^{(p+q)}} m(W;C,D) \cdot W.$$

For a coherent sheaf $\mathcal{F}$ on X whose support has everywhere codimension $\geq p$ we let $Z_p(\mathcal{F}) \in Z^p(X)$ be given by

$$Z_p(\mathcal{F}) := \sum_{W\in X^{(p)}} \mathrm{length}_{\mathcal{O}_{X,W}}(\mathcal{F}_W) \cdot W.$$

Proposition B.1. – *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X. Suppose that the supports of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ have everywhere at least codimension p, q and $p+q$ respectively. Then*

$$Z_p(\mathcal{F}) \cdot Z_q(\mathcal{G}) = \sum_{i\geq 0}(-1)^i Z_{p+q}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Of course this proposition is a special case of a statement valid for perfect complexes on X.

Proof. – Since the question is local on X we can assume X is local and the support of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the closed point of X. Then the proof proceeds as the proof of [**45**, V.C. Proposition 1], using [**43**, Theorem 1] or [**19**] and a filtration argument (using e.g., [**13**, Proposition 3.7]). $\square$

PROPOSITION B.2. – *Let $C \in Z^p(X)$, $D \in Z^q(X)$ and $E \in Z^r(X)$ such that $C \cdot D$, $(C \cdot D) \cdot E$ and $D \cdot E$ are well defined. Then we have*

$$(C \cdot D) \cdot E = C \cdot (D \cdot E)$$

in $Z^{p+q+r}(X)$.

Proof. – The proof proceeds as the proof of [**45**, V.C.3.b) Associativity], using a spectral sequence argument and Proposition B.1. $\square$

For a flat map $X \to Y$ between regular separated Noetherian schemes of finite Krull dimension there is a flat pullback $f^*\colon Z^p(Y) \to Z^p(X)$. (Note that in contrast for the relative cycles considered in [**52**] there is a pullback map for arbitrary morphisms of the base schemes.)

Let now S be a regular separated Noetherian scheme of finite Krull dimension. Let $f\colon X \to Y$ be a morphism in Sm_S and $C \in Z^p(Y)$. We say that f and C are in good position if for every $W \in X^{(p)}$ with a non-zero coefficient in C the scheme theoretic inverse image $f^{-1}(\overline{W})$ has everywhere codimension $\geq p$. If this is the case we define $f^*(C) \in Z^p(X)$ by

$$f^*(C) := \Gamma_f \cdot \mathrm{pr}_Y^*(C).$$

(We view this intersection, which takes place on $X \times_S Y$, in a canonical way as an element of $Z^p(X)$. Note also that the graph is not in general of a well defined codimension, but for the definition we can e.g., assume X and Y to be connected.)

THEOREM B.3. – *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in Sm_S. Let $C \in Z^p(Z)$ and assume g and C are in good position and f and $g^*(C)$ are in good position. Then $g \circ f$ and C are in good position and*

$$(g \circ f)^*(C) = f^*(g^*(C))$$

in $Z^p(X)$.

Proof. – Let $U := \Gamma_f \times_S Z \subset X \times_S Y \times_S Z$ and $V := X \times_S \Gamma_g \subset X \times_S Y \times_S Z$. Let $\mathrm{pr}_Z\colon X \times_S Y \times_S Z \to Z$ be the projection. The assertion follows from the associativity

$$(U \cdot V) \cdot \mathrm{pr}_Z^*(C) = U \cdot (V \cdot \mathrm{pr}_Z^*(C))$$

which holds by Proposition B.2. $\square$

APPENDIX C

AN EXPLICIT PERIODIZATION OF $\mathsf{M}\mathbb{Z}$

In this section we show in an explicit way that $\mathsf{M}\mathbb{Z}$ is strongly periodizable in the sense of [**50**, Definition 4.1]. This also follows from the existence of an E_∞-orientation on $\mathsf{M}\mathbb{Z}$ and the fact that MGL is strongly periodizable ([**50**, Theorem 6.1]), see Remark 10.2.

Recall that a strong periodization of an E_∞-ring spectrum E in $\mathsf{SH}(S)$ in the following: Assuming the self-map $\mathrm{id} \wedge \tau$ of $\mathsf{E} \wedge S^{2,1} \wedge S^{2,1}$, where τ is the twist, is the identity (which is for example the case if E is orientable) the object

$$\bigvee_{i \in \mathbb{Z}} S^{2i,i} \wedge \mathsf{E}$$

is canonically a commutative monoid under E in $\mathsf{SH}(S)^{\mathbb{Z}}$. The E_∞-ring E is then called *strongly periodizable* if this structure can be made into an E_∞-structure, i.e., if there exists an E_∞-object in graded motivic symmetric spectra under a model of E giving rise to the non-highly structured version above. Such a structure on $\mathsf{M}\mathbb{Z}$ has for example the consequence that there exists an E_∞-version of motivic cochains, i.e., a natural graded E_∞-algebra in complexes of abelian groups computing the motivic cohomology of the base, see Corollary C.3.

We set ourselves in the situation of Section 4.1.2 before the definition of A. Since $L_\bullet(r) = L_\bullet(1)^{\otimes r}$ for any $r \in \mathbb{Z}$ the collection of the $L_\bullet(r)[2r]$ gives rise to a strictly commutative algebra $L_\bullet(*)[2*]$ in $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^\bullet))^{\mathbb{Z}}$. We denote by e the embedding

$$\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^\bullet)) \to \mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\text{ét}}, \mathbb{Z}/p^\bullet))^{\mathbb{Z}}$$

which sets everything into outer degree 0 and by the same symbol the induced embedding of $\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\text{ét}}$-spectra into $e(\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\text{ét}})$-spectra. The tensor product in $e(\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\text{ét}})$-spectra of $e(\mathrm{Sym}(\mathscr{T}))$ with the suspension spectrum of $L_\bullet(*)[2*]$ can be written as the outer tensor product $\mathrm{Sym}(\mathscr{T}) \otimes L_\bullet(*)[2*]$. We let

$$\mathrm{Sym}(\mathscr{T}) \otimes L_\bullet(*)[2*] \to R(\mathrm{Sym}(\mathscr{T}) \otimes L_\bullet(*)[2*])$$

be a fibrant replacement in E_∞-algebras in $e(\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\text{ét}})$-spectra (i.e., in the semi model category $E_\infty((\mathrm{Sp}^{\Sigma}_{\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U},\{1\}]_{\text{ét}}})^{\mathbb{Z}})$). Set $A := \epsilon_*(R(\mathrm{Sym}(\mathscr{T}) \otimes L_\bullet(*)[2*]))$.

For $k \in \mathbb{Z}$ we denote by A_k the contribution of A in outer $\mathbb{Z}$-degree k, so A_k is a $\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\text{ét}}$-spectrum. We set $A_k^{\text{tr}} := \tau_{\leq(-k)}(A_k)$. The A_k^{tr} assemble to an E_∞-algebra $A^{\text{tr}} \in E_\infty((\mathrm{Sp}^\Sigma_{\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U},\{1\}]_{\text{Zar}}})^{\mathbb{Z}})$. We set $B := j_* A^{\text{tr}}$.

As in Section 4.1.2 we have canonical epimorphisms

$$B_{k,r} \to i_* \nu_\bullet^{k+r-1}[k].$$

We denote by $C_{k,r}$ the kernels of these epimorphisms. A variant of Lemma 4.14 implies that the collection of the $C_{k,r}$ gives rise to an E_∞-algebra $C \in E_\infty((\mathrm{Sp}^\Sigma_{\mathbb{Z}/p^\bullet[\mathbb{G}_{m,S},\{1\}]_{\text{Zar}}})^{\mathbb{Z}})$. Let $C^c \to C$ be a cofibrant and $C^c \to C^{\text{cf}}$ be a fibrant replacement via a cofibration in the latter semi model category and set $D(p) := \lim_n C^{\text{cf}}_{*,\bullet,n} \in E_\infty((\mathrm{Sp}^\Sigma_{\mathbb{Z}_p[\mathbb{G}_{m,S},\{1\}]_{\text{Zar}}})^{\mathbb{Z}})$.

Set $D := \prod_p D(p) \in E_\infty((\mathrm{Sp}^\Sigma_{\hat{\mathbb{Z}}[\mathbb{G}_{m,S},\{1\}]_{\text{Zar}}})^{\mathbb{Z}})$. We let PH_B be the periodic version of H_B, then there is a canonical E_∞-map $\mathsf{PH}_B \to D_{\mathbb{Q}}$.

Definition C.1. – *We let* $\mathsf{PM}\mathbb{Z}$ *denote the homotopy pullback in* E_∞*-spectra of the diagram*

$$\begin{array}{ccc} & & D \\ & & \downarrow \\ \mathsf{PH}_B & \longrightarrow & D_{\mathbb{Q}}. \end{array}$$

Clearly we have

Theorem C.2. – *The* E_∞*-spectrum* $\mathsf{PM}\mathbb{Z}$ *is a strong periodization of* $\mathsf{M}\mathbb{Z}$ *in the sense of* [**50**].

For a base scheme X let $\mathrm{DMT}(X)$ be the full localizing triangulated subcategory of $\mathrm{DM}(X)$ spanned by the $\mathbb{Z}(n)$, $n \in \mathbb{Z}$. We denote by $\mathrm{DMT}_{\text{gm}}(X)$ the full subcategory of $\mathrm{DMT}(X)$ of compact objects.

Corollary C.3. – *For a base scheme* X *there is a* E_∞*-algebra* $\mathscr{A}_X$ *in* $\mathrm{Cpx}(\mathsf{Ab})^{\mathbb{Z}}$ *and a tensor triangulated equivalence* $\mathrm{DMT}(X) \simeq \mathsf{D}(\mathscr{A}_X)$.

Proof. – This follows now from [**50**, Theorem 4.3]. □

Corollary C.4. – *Let* $X \in \mathrm{Sm}_S$ *be connected such that for any* n *we have* $H^i_{\text{mot}}(X, n)_{\mathbb{Q}} = 0$ *for* $i \ll 0$ *(for example* $X = \mathrm{Spec}(R)$, R *the localization of a number ring, or* $X = \mathbb{P}^1_R \setminus \{0, 1, \infty\}$*). Then there is an affine derived group scheme* G *over* $\mathbb{Z}$ *such that* $\mathrm{Perf}(G)$*, the (derived) category of perfect representations of* G*, is tensor triangulated equivalent to* $\mathrm{DMT}_{\text{gm}}(X)$.

Proof. – This follows from [**49**, Theorem 6.21]. □

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We construct a motivic Eilenberg-MacLane spectrum with a highly structured multiplication over general base schemes which represents Levine's motivic cohomology, defined via Bloch's cycle complexes, over smooth schemes over Dedekind domains. Our method is by gluing p-completed and rational parts along an arithmetic square. Hereby the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser's version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As an application the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of motives with integral coefficients.

Our spectrum is compatible with base change giving rise to a formalism of six functors for triangulated categories of motivic sheaves over general base schemes including the localization triangle.

Further applications are a generalization of the Hopkins-Morel isomorphism and a structure result for the dual motivic Steenrod algebra in the case where the coefficient characteristic is invertible on the base scheme.