GLOBAL SOLUTIONS FOR SMALL NONLINEAR LONG RANGE PERTURBATIONS OF TWO DIMENSIONAL SCHRÖDINGER EQUATIONS

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Abstract. — Let Q_1, Q_2 be two quadratic forms, and u a local solution of the two dimensional Schrödinger equation $(i\partial_t + \Delta)u = Q_1(u, \nabla_x u) + Q_2(\overline{u}, \nabla_x \overline{u})$. We prove that if Q_1 and Q_2 do depend on the derivatives of u, and if the Cauchy datum is small enough and decaying enough at infinity, the solution exists for all times. The difficulty of the problem originates in the fact that the nonlinear perturbation is a long range one: by this, we mean that it can be written as the product of (a derivative of) u and of a potential whose L^∞ space-norm is not time integrable at infinity.

Résumé (Solutions globales pour des perturbations nonlinéaires à longue portée de l'équation de Schrödinger en dimension 2)

Soient Q_1, Q_2 deux formes quadratiques et u solution locale de l'équation de Schrödinger en dimension 2 d'espace $(i\partial_t + \Delta)u = Q_1(u, \nabla_x u) + Q_2(\overline{u}, \nabla_x \overline{u})$. Nous prouvons que si Q_1 et Q_2 dépendent effectivement des dérivées de u, et si la donnée de Cauchy est assez petite et assez décroissante à l'infini, la solution existe globalement en temps. La difficulté du problème réside dans le fait que la perturbation nonlinéaire est à longue portée, en ce sens qu'elle s'écrit comme un produit (d'une dérivée) de u par un potentiel dont la norme L^∞ en espace n'est pas intégrable lorsque $t \to +\infty$.

CONTENTS

INTRODUCTION

This paper is devoted to the proof of existence of global solutions for a nonlinear Schrödinger equation in two space dimension with small Cauchy data. Consider the equation

$$(i\partial_t + \Delta)u = F(u, \nabla_x u, \overline{u}, \nabla_x \overline{u})$$
$$u|_{t=0} = \varepsilon u_0$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, F is a polynomial vanishing at least at order 2 at 0 and $\varepsilon > 0$.

The problem of *local existence* for the above equation with a general nonlinearity and for *small Cauchy data* (i.e. small ε) in a convenient Sobolev space has been solved by Kenig, Ponce and Vega [19]. Hayashi and Ozawa [18] obtained local existence in one space dimension for *large Cauchy data*. The case of any space dimension was treated by Chihara [4]. More recently, Kenig, Ponce and Vega [20] proved the similar result for a generalized Schrödinger equation, i.e. an equation in which Δ is replaced by a more general operator.

We are interested in this paper in *global solutions* for *small* enough ε. When the space dimension is larger or equal to 3, and F vanishes *at least at order* 3 at 0, Chihara [5], [6] proved that there is a global solution if the data are small enough in a (weighted) Sobolev space. He also proved the same result in two space dimensions under a convenient restriction on the cubic part of the nonlinearity.

For *quadratic nonlinearities*, and space dimension larger or equal to three, global existence for small data has been obtained under convenient assumptions on the nonlinearity. The most recent results are due to Hayashi and Hirata [10], Hayashi and Kato [11], Hayashi, Miao and Naumkin [12]. We refer the reader to the introduction of [15] for a detailed discussion of these results as well as further references.

The results we have mentioned so far could be qualified of "short range" type ones. By this, we mean the following: the nonlinearity F can be written as a sum of products of a nonlinear potential $V(u, \nabla_x u, \overline{u}, \nabla_x \overline{u})$ times u or $\nabla_x u$ or $\overline{u}$ or $\nabla_x \overline{u}$.

Denote by $k \geqslant 1$ the order of vanishing of V at the origin. Since linear solutions of the Schrödinger operator decay in L^∞ like $t^{-d/2}$ when $t \to +\infty$, we see that V computed on such a solution decays like $t^{-kd/2}$ when $t \to +\infty$. We say that F is a short range perturbation of the linear Schrödinger equation if this quantity is integrable when $t \to +\infty$ i.e. if $kd/2 > 1$. All the results we have indicated above fall into this category.

We are interested in this paper in the long range case, more precisely in the limiting case $kd/2 = 1$. There are only two such possibilities: either the space dimension d is 1 and F is cubic, or $d = 2$ and F is quadratic. The former case has been solved in general by Hayashi and Naumkin [**14**]: they found a sufficient condition on the cubic nonlinearity under which solutions are global for small enough Cauchy data in a weighted Sobolev space. Their method relies on the use of the smoothing property of Doi [**9**]. They could also in [**17**] reduce a particular quadratic nonlinearity to a cubic one, thus obtaining global existence in this case as well.

The case of *quadratic nonlinearities in two space dimensions* is studied by Cohn [**7**] for a very specific nonlinearity, and by Hayashi and Naumkin [**15**], [**16**] in the special case of *real analytic Cauchy data*. Such an assumption allows one to avoid the difficulty of the loss of one derivative in the right hand side of the equation.

Our aim in this paper is to study this quadratic two dimensional problem when the Cauchy datum lies in a weighted Sobolev space. We are thus obliged to cope with the problem of recovering the derivative lost in the right hand side. We state our main theorem of global existence in the first chapter, together with precise assumption on the quadratic nonlinearity we consider. Let us just describe here our general strategy in the special case

$$(D_t + D_x^2)u = u(D_{x_1}u)$$

where $D_t = \frac{1}{i}\partial_t$, $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 1, 2$, and where the datum is given at $t = 1$, the solution being looked for on $\{t \geqslant 1\}$. We first take new coordinates $T = t$, $X = x/t$ and look for u in terms of a new unknown $w(T, X)$ through $u(t,x) = \frac{1}{t}e^{ix^2/4t}w(t, x/t)$. We get for w an equation of form

$$(0.1) \qquad \left(D_T + \frac{D_X^2}{T^2}\right)w = \frac{1}{T}e^{i\theta(T,X)}w\left(\frac{D_{X_1}}{T} + \frac{X_1}{2}\right)w$$

where $\theta = TX^2/4$. Let us take a simplified model forgetting the $X_1/2$ term above:

$$(0.2) \qquad \left(D_T + \frac{D_X^2}{T^2}\right)w = \frac{1}{T}e^{i\theta(T,X)}w\left(\frac{D_{X_1}}{T}w\right).$$

Remark that D_{X_j} is the translation in the new coordinates of the operator $tD_{x_j} - x_j/2$, which is of constant use in the study of global problems for nonlinear Schrödinger equations with small Cauchy data. Consequently, smoothness relatively to D_X will play an essential role. The form of the right hand side of (0.2) shows immediately what are the difficulties we will encounter. First of all, we have a loss of a D_X/T derivative in the nonlinearity. To remove this problem, we shall use the Kato local smoothing

property in the version due to Kenig, Ponce and Vega [**20**], adapted to our long time framework. Secondly, the right hand side of the equation contains the oscillating factor $e^{i\theta}$, which cannot have any D_X-smoothness uniformly as time $T \to +\infty$. To treat these oscillating contributions, we introduce spaces of the following type

$$(0.3) \qquad \{v \in L^2; (D_X/\sqrt{T})^s (D_X/T)^{s'} v \in L^2\}$$

where s and s' are integers. The smoothness relatively to D_X/T corresponds to what is gained by the local smoothing property – and to what is lost in the nonlinearity $w\frac{D_{X_1}}{T}w$. The smoothness relatively to $D_X/\sqrt{T}$ should be understood as a weak version of smoothness relatively to $tD_x - x/2$ for $u(t,x)$. This type of derivative is natural for the problem because of the form of $\theta(T,X) = (\sqrt{T}X)^2/4$. To study products of elements in (0.3), we will need to have s and s' large enough. There will be no problem to ensure that for s', but as $\frac{D_X}{\sqrt{T}}e^{i\theta} = \frac{\sqrt{T}X}{2}e^{i\theta}$ we cannot expect the right hand side of (0.2) to be in a space of type (0.3) with a positive s. Consequently, instead of trying to find directly w in a space of type (0.3), we shall look for w as an expansion $w = v + V_1(v)e^{i\theta}$ where v and $V_1(v)$ will be essentially in a space (0.3) with large enough s, s', and where $V_1(v)$ will moreover decay like $(\sqrt{T}|X|)^{-2}$ when $\sqrt{T}|X| \to +\infty$. When plugging such an expression in $e^{i\theta(T,X)}w(\frac{D_{X_1}}{T}w)$, one gets a first contribution of form $e^{i\theta(T,X)}v(\frac{D_{X_1}}{T}v)$, and remainders decaying like $\langle\sqrt{T}X\rangle^{-2}$. One will choose $V_1(v)$ as a function of v such that $(D_T + \frac{D_X^2}{T^2})(V_1 e^{i\theta})$ equals $\frac{1}{T}v(\frac{D_{X_1}}{T}v)e^{i\theta}$ modulo remainders. This is possible because θ is a non characteristic phase for the operator $D_T + \frac{D_X^2}{T^2}$. In that way, one gets an equation

$$(0.4) \qquad \left(D_T + \frac{D_X^2}{T^2}\right)v = \frac{1}{T}R$$

where R will be a combination of terms $e^{im\theta}$ with coefficients decaying at least like $\langle\sqrt{T}X\rangle^{-2}$. Since $\langle\sqrt{T}X\rangle^{-2}e^{im\theta}$ has some smoothness relatively to $D_X/\sqrt{T}$ uniformly in T (actually, this expression accepts two $D_X/\sqrt{T}$ derivatives), this shows that we have gained some smoothness in comparison with the right hand side of (0.2). Actually one has to repeat such a method once again, to reduce the equation to (0.4) with a right hand side $R = R(v, \frac{D_X}{T}v)$ with values in a space of type (0.3) with $s \sim 4$. This last equation can then be solved globally using the local smoothing property as mentioned before.

Our paper is organized as follows. In the first chapter, we state our main theorem and perform first reductions. The second chapter is devoted to the proof of the local smoothing inequality that will be essential in the rest of the article. We make use of Littlewood-Paley decompositions to define convenient Sobolev spaces, and to prove the smoothing inequality as a consequence of propagation of singularities. Section 3 is devoted to nonlinear estimates. We make extensive use of the ideas of paradifferential calculus of Bony [**1**] to study nonlinear operators acting on the Sobolev spaces defined in chapter 2. We also prove results concerning products or conjugation of an element

of such a Sobolev space with an oscillatory exponential. Section 4 gives the proof of the theorem. We perform the method of elimination of oscillatory exponentials outlined above. The main tool is again paradifferential calculus, which allows us to decompose the right hand side of the equation as a sum of a nice term, and of a really oscillating contribution, that we eliminate using the non charactericity of the phase. Since the true equation is (0.1) rather than (0.2), we use weighted versions of the Sobolev spaces defined in chapter 2 to treat the contribution coming from X in the right hand side of (0.1). When all oscillatory contributions have been cancelled, the proof of the theorem, as well as the description of the asymptotics of the solution, follow from standard arguments.

CHAPTER 1

THE NONLINEAR SCHRÖDINGER EQUATION

1.1. Statement of main results

Denote by $(t, x) = (t, x_1, x_2)$ coordinates on $\mathbb{R} \times \mathbb{R}^2$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D_x = (D_{x_1}, D_{x_2})$. Let us denote by $Q_1(X_0, X_1, X_2)$, $Q_2(X_0, X_1, X_2)$ two polynomials in three indeterminates, with complex coefficients, homogeneous of degree 2, satisfying the following assumption:

(1.1.1) $\qquad\qquad Q_1$ and Q_2 vanish identically along $(X_1, X_2) = (0, 0)$.

We will be interested in solutions to the following Schrödinger equation

(1.1.2) $$(D_t + D_x^2)u = Q_1(u, D_x u) + Q_2(\overline{u}, \overline{D_x u}).$$

We want to prove global existence for small Cauchy data. Since (1.1.2) is invariant under translations relatively to the time variable, we can without any loss of generality take Cauchy data at $t = 1$ instead of $t = 0$, and study solutions for $t \geqslant 1$. Our initial datum will have to be smooth enough and will have to decay rapidly enough at infinity. We thus introduce for $M \in \mathbb{N}$

(1.1.3)
$$\mathcal{H}^M(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2); \forall \alpha, \beta \in \mathbb{N}^2, |\alpha| + |\beta| \leqslant M, x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^2)\}$$
$$= \{u \in H^M(\mathbb{R}^2); \forall \gamma \in \mathbb{N}^2, |\gamma| \leqslant M, x^\gamma u \in H^{M-\gamma}(\mathbb{R}^2)\}$$

where $H^M(\mathbb{R}^2)$ denotes the usual Sobolev space. Remark that the first characterization of $\mathcal{H}^M(\mathbb{R}^2)$ implies that if λ is any real number, $\mathcal{H}^M(\mathbb{R}^2)$ is invariant under the transformation

(1.1.4) $$u \longmapsto e^{i\lambda x^2}u.$$

Let us state our main result:

THEOREM 1.1.1. — *There is $M_0 \in \mathbb{N}$ with the following properties: for any even integer $M \geqslant M_0$, there is $\varepsilon_0 > 0$ such that for any u_0 in the unit ball of $\mathcal{H}^{M+4}(\mathbb{R}^2)$, and any $\varepsilon \in \,]0, \varepsilon_0[$, equation (1.1.2) with Cauchy datum $u|_{t=1} = \varepsilon u_0$ has a unique global solution $u \in C^0([1, +\infty[, \mathcal{H}^M(\mathbb{R}^2))$.*

REMARKS

– The problem of local existence for equation (1.1.2) has been solved by Kenig, Ponce and Vega [**19**] for small Cauchy data. Local existence for arbitrary Cauchy data has been proved by Hayashi and Ozawa [**18**] in one space dimension, by Chihara [**4**] in higher space dimension, and by Kenig, Ponce and Vega [**20**] when Δ is replaced by a more general operator. The uniqueness assertion of our theorem follows from the local uniqueness obtained by these authors (see [**20**], theorem 1.2). Consequently, we will have to prove only global existence.

– The number M_0 in the statement of the theorem has to be taken large enough to apply the aforementioned results of uniqueness. For the proof of existence in the large, we will just have to assume $M_0 \geqslant 6$.

– In the statement of the theorem, there is apparently a loss of smoothness between the assumptions made on the Cauchy data – $u_0 \in \mathcal{H}^{M+4}(\mathbb{R}^2)$ – and the conclusion saying that at time t the solution $u(t, \cdot)$ belongs to $\mathcal{H}^M(\mathbb{R}^2)$. This comes from the fact that the above spaces are not optimal to solve our problem. An optimal statement, without any artificial loss of smoothness, will be given in theorem 4.2.1 after several reductions and the definition of convenient spaces.

The solution of theorem 1.1.1 has the same asymptotic behaviour as linear solutions.

THEOREM 1.1.2. — *Under the assumptions of theorem 1.1.1 there is a L^∞ function $v_\infty(x)$, satisfying $(1 + |x|)^M v_\infty(x) \in L^\infty$, and such that for any $\delta \in\ ]0, 1/2[$ the solution u has the following asymptotic behaviour:*

$$(1.1.5) \quad \left\| \left(1 + \left|\frac{x}{t}\right|\right)^M \left(u(t,x) - \frac{1}{t}e^{ix^2/4t}v_\infty(x/t)\right) \right\|_{L^\infty(dx)} = O(t^{-1-\delta}),\ t \longrightarrow +\infty.$$

1.2. First reductions

We look for a solution to (1.1.2) in new coordinates

$$(1.2.1) \qquad\qquad T = t, \quad X = \frac{x}{t}$$

and take as a new unknown the function $w(T, X)$ related to u through the relation

$$(1.2.2) \qquad\qquad u(t, x) = \frac{1}{t}e^{ix^2/4t}w(t, x/t).$$

Let us set

$$(1.2.3) \qquad \theta(T, X) = \frac{TX^2}{4}, \quad Z_j = \frac{D_{X_j}}{T} + \frac{X_j}{2}, \qquad j = 1, 2,\ Z = (Z_1, Z_2).$$

Then

$$(1.2.4) \qquad D_x u(t,x) = \frac{1}{T} e^{i\theta(T,X)} (Zw)(T,X)|_{T=t,X=x/t}$$

$$(D_t + D_x^2)u(t,x) = \frac{1}{T} e^{i\theta(T,X)} \Big[D_T + \frac{D_X^2}{T^2} \Big] w(T,X)|_{T=t,X=x/t}.$$

We deduce from these relations that u is a solution of (1.1.2) if and only if w solves

$$(1.2.5) \qquad \Big(D_T + \frac{D_X^2}{T^2} \Big) w = \frac{1}{T} e^{i\theta(T,X)} Q_1(w, Zw) + \frac{1}{T} e^{-3i\theta(T,X)} Q_2(\overline{w}, \overline{Zw}).$$

Since the transformation $u \mapsto e^{-ix^2/4} u$ leaves $\mathcal{H}^{M+4}(\mathbb{R}^2)$ invariant, as remarked before, the initial condition for (1.2.5) will be

$$(1.2.6) \qquad\qquad\qquad w|_{T=1} = \varepsilon w_0$$

with w_0 given in a fixed ball of $\mathcal{H}^{M+4}(\mathbb{R}^2)$.

Let us indicate what will be our general strategy to find a global solution to (1.2.5) for small Cauchy data. If, in the right hand side of (1.2.5), we had no oscillating factor like $e^{i\theta}$, $e^{-3i\theta}$, a proof similar to the one of [**8**] for global existence for the nonlinear Klein-Gordon equation would work: namely, we could look for a solution w in a Sobolev space of smooth enough functions of X, endowed with norms depending conveniently on T. One would then have to prove L^2 inequalities for these Sobolev norms, and to supplement them by an L^∞ estimate deduced from the equation. The only new difficulty, in comparison with the Klein-Gordon case, would come from the loss of one derivative in the right hand side, and we would have to compensate for this loss using the smoothing property of the equation.

The oscillating factors $e^{i\theta}$, $e^{-3i\theta}$ prevent us from applying directly such a strategy: actually, taking D_X-derivatives of such exponentials makes appear a TX factor, which has bad behaviour as $T \to +\infty$. In other words, we cannot expect to put the right hand side of (1.2.5) in any space of smooth functions in X uniformly in $T \to +\infty$. The idea will be to use the fact that the phases θ and -3θ are *non-characteristic* for the operator $D_T + D_X^2/T^2$. We shall use this property looking for the solution w as $w = v + V_1 e^{i\theta} + V_{-3} e^{-3i\theta}$ where V_1, V_{-3} will be expressions in terms of the new unknown v. These functions V_1, V_{-3} will have the same smoothness as v, but will enjoy better decay assumptions when $\sqrt{T}|X| \to +\infty$, and will be chosen to cancel the worst contributions to the right hand side of (1.2.5). In that way, we will reduce (1.2.5) to an equation on v, in which the right hand side will no longer contain any harmful oscillating factors, and to which the strategy discussed above can be applied.

REMARK. — Let us comment on the assumptions on the nonlinearity. The fact that we exclude nonlinearities of type $B(u, D_x u; \overline{u}, \overline{D_x u})$, where B is a bilinear form, comes from the fact that they would induce in the right hand side of (1.2.5) terms in $e^{-i\theta}$, where $-\theta$ is a *characteristic phase* for $D_T + D_X^2/T^2$ when $T \to +\infty$. Because of that, we could not get rid of these terms by a normal form method as indicated above. We

believe that this difficulty is not just a technical one: if one tries to find asymptotic solutions through an ansatz inspired by the asymptotics of linear solutions, the same problem appears, preventing one from constructing an asymptotic series as a formal solution.

The second restriction on the nonlinearity coming from (1.1.1), namely the fact that terms like u^2 or $\overline{u}^2$ are excluded, is related to the fact that θ has a critical point at $X = 0$. Because of that θ is a really oscillating factor only when $\sqrt{T}|X| \to +\infty$, and the normal form method alluded to above can be carried out only in a region $\sqrt{T}|X| \geqslant 1$. To be able to treat the contribution of the domain $\sqrt{T}|X| \leqslant 1$, we need some extra-vanishing of the non-linearity at $X = 0$, which is provided by at least one Z-derivative of w. We have no conjecture on what happens when (1.1.1) is not satisfied. We remark anyway that in *three space dimension* Hayashi and Naumkin [**13**] have been able to get global solutions for nonlinearities of type u^2 or $\overline{u}^2$.

CHAPTER 2

LINEAR ESTIMATES

In the rest of that paper, we will only use the coordinates introduced in (1.2.1). We will thus employ the notation (t, x) instead of (T, X) without any risk of confusion. The aim of this chapter is to introduce, using dyadic decompositions, the spaces of distributions in which we will look for the solutions of equation (1.2.5). The first problem will be to prove a linear estimate in these spaces for solutions of $D_t + D_x^2/t^2$, reflecting the Kato smoothing property. We will do so adapting to our long time framework the method used by Kenig, Ponce and Vega [20] in the local in time setting.

2.1. Symbolic calculus and applications

The spaces we shall introduce in the following section will be of uniformly local type. Their definition will rely on the use of cut-off functions relatively to the x-variable. On the other hand, to be able to prove smoothing estimates in those spaces, we shall have to decompose their elements using pseudo-differential partitions of identity invariant under the hamiltonian flow of the Schrödinger operator. To simplify the study of the interaction of these different localizations, we first establish some general results. We shall use throughout the whole paper the notation $\langle x \rangle = (1 + x^2)^{1/2}$.

DEFINITION 2.1.1. — A (C_0, N_0)-temperate weight (for a given positive constant C_0 and a given $N_0 \in \mathbb{N}$) will be a C^∞ function $A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(x, \xi) \mapsto A(x, \xi)$, satisfying for any $(x, \xi) \in \mathbb{R}^{2d}$, any $(y, \eta) \in \mathbb{R}^{2d}$

$$(2.1.1) \qquad A(x, \xi) \geqslant 1, \quad \frac{A(x, \xi)}{A(y, \eta)} + \frac{A(y, \eta)}{A(x, \xi)} \leqslant C_0(\langle x - y \rangle + \langle \xi - \eta \rangle)^{N_0}.$$

DEFINITION 2.1.2. — Let A be a temperate weight. We shall denote by $S(A^{-\infty})$ the space of all smooth functions $(x, \xi) \mapsto a(x, \xi)$, defined on $\mathbb{R}^d \times \mathbb{R}^d$, with values in $\mathbb{C}$,

such that for any $N \in \mathbb{N}, n \in \mathbb{N}$

$$(2.1.2) \qquad |a|_{A,N,n} \overset{\text{def}}{=} \sup_{|\alpha|+|\beta| \leqslant n} \sup_{(x,\xi)} |A(x,\xi)^N \partial_x^\alpha \partial_\xi^\beta a(x,\xi)| < +\infty.$$

We define in the obvious way the notion of a bounded family of symbols.

REMARK. — Since $A \geqslant 1$, it follows that $S(A^{-\infty}) \subset S_{0,0}^0$. Consequently, by the Calderón-Vaillancourt theorem [**2**], any pseudo-differential operator with symbol in $S(A^{-\infty})$ is bounded on $L^2(\mathbb{R}^d)$, its operator norm being controlled in terms of a semi-norm (2.1.2) of the symbol.

We will need a result of symbolic calculus. If A and B are two (C_0, N_0)-temperate weights, if $a \in S(A^{-\infty})$, $b \in S(B^{-\infty})$, we denote by $a\#b$ the symbol of the operator $a(x,D) \circ b(x,D)$, given by the oscillatory integral

$$(2.1.3) \qquad a\#b(x,\eta) = \frac{1}{(2\pi)^d} \int e^{-iy\xi} a(x, \eta - \xi) b(x - y, \eta) \, dy d\xi.$$

In the same way, the adjoint $a(x,D)^*$ of $a(x,D)$ has symbol $a^*(x,\eta)$ given by

$$(2.1.4) \qquad a^*(x,\eta) = \frac{1}{(2\pi)^d} \int e^{-iy\xi} \overline{a}(x - y, \eta - \xi) \, dy d\xi.$$

THEOREM 2.1.3. — *For any $C_0 > 0$, $N_0 \in \mathbb{N}$, $N \in \mathbb{N}$, $n \in \mathbb{N}$, there are $C'_{N,n} > 0$ and $n' \in \mathbb{N}$ such that the following two statements are true:*

(i) For any couple of (C_0, N_0)-temperate weights A, B and for any $a \in S(A^{-\infty})$, $b \in S(B^{-\infty})$, one has $ab \in S((AB)^{-\infty})$, $a\#b - ab \in S((AB)^{-\infty})$ with the following estimates

$$(2.1.5) \qquad \begin{aligned} |ab|_{AB,N,n} &\leqslant C'_{N,n} |a|_{A,N,n} |b|_{B,N,n} \\ |a\#b - ab|_{AB,N,n} &\leqslant C'_{N,n} |\partial_\xi a|_{A,N,n'} |\partial_x b|_{B,N,n'}. \end{aligned}$$

(ii) For any (C_0, N_0)-temperate weight A and for any $a \in S(A^{-\infty})$, one has $a^* \in S(A^{-\infty})$ with the estimate

$$(2.1.6) \qquad |a^* - \overline{a}|_{A,N,n} \leqslant C'_{N,n} |\partial_x \partial_\xi a|_{A,N,n'}.$$

Proof. — (i) The proof of such a result is quite classical. We just have to pay attention to the fact that $C'_{N,n}$ depends on C_0, N_0, N, n but not on the weights themselves. The first inequality (2.1.5) is clear. To prove the second one we write

$$(2.1.7) \quad a\#b(x,\eta) - ab(x,\eta) = \frac{1}{(2\pi)^d} \int e^{-iy\xi} (a(x, \eta - \xi) - a(x,\eta)) b(x - y, \eta) \, dy d\xi$$
$$+ \frac{1}{(2\pi)^d} \int e^{-iy\xi} a(x,\eta) (b(x - y, \eta) - b(x,\eta)) \, dy d\xi$$

where both integrals are oscillatory ones. The last contribution is zero, as the coefficient of the exponential does not depend on ξ and vanishes for $y = 0$. In the first

integral, let us write $a(x, \eta - \xi) - a(x, \eta) = -\xi \widetilde{a}(x, \xi, \eta)$ with

$$\widetilde{a}(x, \xi, \eta) = \int_0^1 (\partial_\eta a)(x, \eta - t\xi) \, dt.$$

We have if $|\alpha| + |\beta| + |\gamma| \leqslant n$

$$(2.1.8) \qquad |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \widetilde{a}(x, \xi, \eta)| \leqslant |\partial_\eta a|_{A,N,n} \int_0^1 A(x, \eta - t\xi)^{-N} \, dt$$
$$\leqslant |\partial_\eta a|_{A,N,n} A(x, \eta)^{-N} C_0^N \langle \xi \rangle^{N_0 N}$$

by (2.1.2), (2.1.1). By integration by parts, the first contribution to the right hand side of (2.1.7) is

$$(2.1.9) \qquad \frac{1}{(2\pi)^d} \int e^{-iy\xi} \widetilde{a}(x, \xi, \eta)(D_x b)(x - y, \eta) \, dy d\xi.$$

Let $L_1(y, D_\xi) = \langle y \rangle^{-2}(1 - y \cdot D_\xi)$, $L_2(\xi, D_y) = \langle \xi \rangle^{-2}(1 - \xi \cdot D_y)$. Integrating by parts, we get that (2.1.9) equals

$$(2.1.10) \quad \frac{1}{(2\pi)^d} \int e^{-iy\xi} ({}^t L_1(y, D_\xi))^k ({}^t L_2(\xi, D_y))^k [\widetilde{a}(x, \xi, \eta)(D_x b)(x - y, \eta)] \, dy d\xi.$$

Using (2.1.8), and the analogous inequality

$$|\partial_x^\alpha \partial_\eta^\beta D_x b(x - y, \eta)| \leqslant |\partial_x b|_{B,N,n} B(x, \eta)^{-N} C_0^N \langle y \rangle^{N_0 N}$$

for $|\alpha| + |\beta| \leqslant n$, we see that the modulus of (2.1.10) is smaller than

$$C_{k,N} \int \langle y \rangle^{-k + N_0 N} \langle \xi \rangle^{-k + N_0 N} \, dy d\xi (A(x, \eta)B(x, \eta))^{-N} |\partial_\eta a|_{A,N,k} |\partial_x b|_{B,N,k}.$$

If k is large enough relatively to N, we get the inequality

$$|a \# b - ab|_{AB,N,0} \leqslant C_{N,k} |\partial_\eta a|_{A,N,k} |\partial_x b|_{B,N,k}$$

where $C_{N,k}$ depends on N, C_0, N_0 but not on A nor B. The derivatives of $a \# b - ab$ being estimated in the same way, the conclusion follows.

(ii) The proof is similar. $\qquad \square$

EXAMPLE. — We will consider in the following families of weights $A_{q,\delta}(x, \xi)$, or $A_{q,\delta}(t, x, \xi)$, depending on parameters $q \in \mathbb{Z}^d$, $\delta \in]0, 1]$, and eventually $t \in [1, +\infty[$, satisfying (2.1.1) with C_0 and N_0 uniform in t, δ, q. We will then consider symbols $a_{q,\delta}(x, \xi)$ or $a_{q,\delta}(t, x, \xi)$ satisfying for any $\alpha, \beta \in \mathbb{N}^d$

$$(2.1.11) \qquad |\partial_x^\alpha \partial_\xi^\beta a_{q,\delta}(t, x, \xi)| \leqslant C_{\alpha,\beta,N} \delta^{|\alpha| + |\beta|} A_q(x, \xi)^{-N}$$

with $C_{\alpha,\beta,N}$ independent of q, δ.

In other words, we will have for any $\alpha, \beta \in \mathbb{N}^d, n \in \mathbb{N}$

$$(2.1.12) \qquad \sup_{q,t} |\partial_x^\alpha \partial_\xi^\beta a_{q,\delta}(t, \cdot, \cdot)|_{A_{q,\delta},N,n} = O(\delta^{|\alpha| + |\beta|}), \quad \delta \longrightarrow 0.$$

This introduction of δ gives us, according to (2.1.12) and (2.1.5), (2.1.6), a class of symbols in which we do have a symbolic calculus when $\delta \to 0$.

As an example, let us take $\phi \in C_0^\infty(]-1,1[^d)$, $0 \leqslant \phi \leqslant 1$, such that if we set $\phi_q(x) = \phi(x-q)$ we have $\sum_{q \in \mathbb{Z}^d} \phi_q(x) \equiv 1$. For $\delta \in]0,1]$ put $\phi_{q,\delta}(x) = \phi_q(\delta x)$. If we set $D_{q,\delta}(x,\xi) = \langle \delta x - q \rangle$, we see that (2.1.1) is satisfied by this weight with a uniform (C_0, N_0). Moreover, $\phi_{q,\delta}(x)$ satisfies (2.1.12) with $A_{q,\delta}$ replaced by $D_{q,\delta}$.

We will need the following proposition:

PROPOSITION 2.1.4. — *Let $A_{q,\delta}(t,x,\xi)$ be a family of weights satisfying (2.1.1) with a uniform (C_0, N_0). Assume moreover that there is $N_1 \in \mathbb{N}$, $C_1 > 0$ with*

$$(2.1.13) \qquad\qquad \sum_q A_{q,\delta}(t,x,\xi)^{-N_1} \leqslant C_1$$

for any $t \geqslant 1$, $\delta \in]0,1]$, $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Let $a_{q,\delta}(t,\cdot) \in S(A_{q,\delta}(t,\cdot)^{-\infty})$ satisfying (2.1.12). Assume that there is a positive constant c such that $\sum_q |a_{q,\delta}(t,x,\xi)|^2 \geqslant c$ for any $t \geqslant 1$, $\delta \in]0,1]$, $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Let $r \in [1,+\infty]$. There is $\delta_0 > 0$, $C > 0$ such that for any $\delta \in]0,\delta_0]$, any interval $I \subset [1,+\infty[$, any $u \in L^2(I \times \mathbb{R}^d)$ one has the estimate

$$
\begin{aligned}
C^{-1} &\left\| \left\| \mathbb{1}_I(t)\phi_{p,\delta}(x)u(t,x) \right\|_{L^2(\frac{dt}{t}dx)} \right\|_{\ell_p^r} \\
(2.1.14) \qquad &\leqslant \left\| \left(\sum_q \left\| \phi_{p,\delta}\mathbb{1}_I(t)a_{q,\delta}(t,x,D)u(t,x) \right\|_{L^2(\frac{dt}{t}dx)}^2 \right)^{1/2} \right\|_{\ell_p^r} \\
&\leqslant C \left\| \left\| \mathbb{1}_I(t)\phi_{p,\delta}(x)u(t,x) \right\|_{L^2(\frac{dt}{t}dx)} \right\|_{\ell_p^r}
\end{aligned}
$$

where ℓ_p^r means the ℓ^r norm relatively to p.
  If $u \in C^0([1,+\infty[, L^2(\mathbb{R}^d))$, one has for any t the inequality

$$
\begin{aligned}
C^{-1} &\left\| \left\| \phi_{p,\delta}(x)u(t,x) \right\|_{L^2(dx)} \right\|_{\ell_p^r} \\
(2.1.15) \qquad &\leqslant \left\| \left(\sum_q \left\| \phi_{p,\delta}a_{q,\delta}(t,x,D)u(t,x) \right\|_{L^2(dx)}^2 \right)^{1/2} \right\|_{\ell_p^r} \\
&\leqslant C \left\| \left\| \phi_{p,\delta}(x)u(t,x) \right\|_{L^2(dx)} \right\|_{\ell_p^r}
\end{aligned}
$$

with a constant C independent of t.
  The last inequality in (2.1.14), (2.1.15) holds without the assumption $\sum_q |a_{q,\delta}|^2 \geqslant c$.

Proof. — Let us write L_I^2 for $L^2(\frac{dt}{t}dx)$ over the interval I. We have

$$(2.1.16) \quad \sum_q \|\phi_{p,\delta}a_{q,\delta}(t,x,D)u\|_{L_I^2}^2 = \int_I \langle \sum_q a_{q,\delta}(t,x,D)^* \phi_{p,\delta}^2 a_{q,\delta}(t,x,D)u, u \rangle \frac{dt}{t}.$$

Let us define

$$(2.1.17) \qquad b_{q,\delta}(t,x,\xi) = |a_{q,\delta}(t,x,\xi)|^2, b_\delta(t,x,\xi) = \sum_q b_{q,\delta}(t,x,\xi).$$

Then $b_{q,\delta} \in S(A_{q,\delta}^{-\infty})$ and satisfies (2.1.12). Moreover, assumption (2.1.13) implies that $b_\delta \in S(1)$ and satisfies $|\partial_x^\alpha \partial_\xi^\beta b_\delta(t,x,\xi)| \leqslant C\delta^{|\alpha|+|\beta|}$. By assumption, we also have $b_\delta \geqslant c$.

Since $a_{q,\delta} \in S(A_{q,\delta}^{-\infty})$, $\phi_{p,\delta} \in S(D_{p,\delta}^{-\infty})$ with $D_{p,\delta} = \langle \delta x - p \rangle$, and since these weights satisfy condition (2.1.1) with uniform (C_0, N_0), properties (2.1.5) and (2.1.6) imply that

$$(2.1.18) \qquad a_{q,\delta}(t,x,D)^* \phi_{p,\delta}^2 a_{q,\delta}(t,x,D) = \phi_{p,\delta} b_{q,\delta}(t,x,D)\phi_{p,\delta} + R_{p,q,\delta}(t,x,D)$$

where $R_{p,q,\delta}(t,x,\xi) \in S((A_{q,\delta}D_{p,\delta})^{-\infty})$. Moreover, since $a_{q,\delta}(t,x,\xi)$ and $\phi_{p,\delta}$ satisfy estimates of type (2.1.12), inequalities (2.1.5), (2.1.6) imply that the semi-norms of $R_{p,q,\delta}$ are $O(\delta^2)$ when $\delta \to 0$ uniformly in p,q. Summing (2.1.18) in q we get

$$(2.1.19) \qquad \sum_q a_{q,\delta}(t,x,D)^* \phi_{p,\delta}^2 a_{q,\delta}(t,x,D) = \phi_{p,\delta} b_\delta(t,x,D)\phi_{p,\delta} + R_{p,\delta}(t,x,D)$$

where, because of (2.1.13), $R_{p,\delta}(t,x,\xi) \in S(D_{p,\delta}^{-\infty})$, the semi-norms of this symbol being $O(\delta^2)$ uniformly in p. Define now

$$C_\delta(t,x,\xi) = \left[b_\delta(t,x,\xi) - \frac{c}{2} \right]^{1/2}.$$

Since $b_\delta \geqslant c$, we get that $C_\delta(t,x,\xi) \in S(1)$ with estimates $|\partial_x^\alpha \partial_\xi^\beta C_\delta| \leqslant C\delta^{|\alpha|+|\beta|}$. Applying again theorem 2.1.3, we see that

$$(2.1.20) \qquad b_\delta(t,x,D) = C_\delta(t,x,D)^* C_\delta(t,x,D) + \frac{c}{2}\mathrm{Id} + R_\delta(t,x,D)$$

where $R_\delta \in S(1)$, with semi-norms in this space $O(\delta^2)$, $\delta \to 0$. Plugging into (2.1.19) we get, changing the notation for $R_{p,\delta}$,

$$(2.1.21) \quad \sum_q a_{q,\delta}(t,x,D)^* \phi_{p,\delta}^2 a_{q,\delta}(t,x,D) = \phi_{p,\delta} C_\delta^* C_\delta \phi_{p,\delta} + \frac{c}{2}\phi_{p,\delta}\phi_{p,\delta} + R_{p,\delta}(t,x,D)$$

where $R_{p,\delta}(t,x,\xi) \in S(D_{p,\delta}^{-\infty})$, with semi-norms $O(\delta^2)$ uniformly in p. The right hand side of (2.1.16) is thus bounded from below by

$$(2.1.22) \qquad \frac{c}{2}\int_I \|\phi_{p,\delta}(x)u(t,x)\|^2_{L^2(dx)} \frac{dt}{t} + \int_I \langle R_{p,\delta}(t,x,D)u, u\rangle \frac{dt}{t}$$

and from above by

$$(2.1.23) \quad \frac{c}{2}\int_I \|\phi_{p,\delta}(x)u(t,x)\|^2_{L^2(dx)} \frac{dt}{t} + \int_I \|C_\delta \phi_{p,\delta} u(t,x)\|^2_{L^2(dx)} \frac{dt}{t}$$
$$+ \int_I \langle R_{p,\delta}(t,x,D)u, u\rangle \frac{dt}{t}.$$

Since $C_\delta(t,x,D)$ is bounded on L^2 uniformly in δ, the conclusion will follow from the following lemma. $\qquad\square$

LEMMA 2.1.5. — *With the above notations,*

$$(2.1.24) \qquad \left\| \left| \int_I \langle R_{p,\delta}(t,x,D)u,u\rangle \frac{dt}{t} \right|^{1/2} \right\|_{\ell_p^r} \leqslant C\delta \left\| \|\mathbb{1}_I(t)\phi_{p,\delta}u\|_{L^2(\frac{dt}{t}dx)} \right\|_{\ell_p^r}$$

for a uniform constant $C > 0$.

Proof. — Write

$$(2.1.25) \qquad \int_I \langle R_{p,\delta}(t,x,D)u,u\rangle \frac{dt}{t} = \sum_\ell \sum_{\ell'} \int_I \langle \widetilde{\phi}_{\ell',\delta}(x)R_{p,\delta}\widetilde{\phi}_{\ell,\delta}(\phi_{\ell,\delta}u),\phi_{\ell',\delta}u\rangle \frac{dt}{t}$$

where $\widetilde{\phi}_{\ell,\delta} = \widetilde{\phi}(\delta x - \ell)$ for a function $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^d)$, $\widetilde{\phi} \equiv 1$ close to $\operatorname{Supp}\phi$. Since $R_{p,\delta}(t,x,\xi) \in S(D_{p,\delta}^{-\infty})$, $\widetilde{\phi}_{\ell,\delta} \in S(D_{\ell,\delta}^{-\infty})$, we deduce from theorem 2.1.3 that the symbol $R_{p,\delta}^{\ell,\ell'}(t,x,\xi)$ of the operator $\widetilde{\phi}_{\ell',\delta} \circ R_{p,\delta}(t,x,D) \circ \widetilde{\phi}_{\ell,\delta}$ lies in $S((D_{\ell,\delta}D_{p,\delta}D_{\ell',\delta})^{-\infty})$. Moreover, its semi-norms are $O(\delta^2)$ when $\delta \to 0$ uniformly in p, ℓ, ℓ' since the same is true for $R_{p,\delta}(t,x,\xi)$ i.e. we have

$$(2.1.26) \qquad |\partial_x^\alpha \partial_\xi^\beta R_{p,\delta}^{\ell,\ell'}(t,x,\xi)| \leqslant C_{\alpha,\beta,N}\delta^2 \langle \delta x - \ell\rangle^{-N} \langle \delta x - \ell'\rangle^{-N} \langle \delta x - p\rangle^{-N}$$

with a constant independent of δ and p, ℓ, ℓ'. We deduce from that the uniform estimates

$$(2.1.27) \qquad |\partial_x^\alpha \partial_\xi^\beta R_{p,\delta}^{\ell,\ell'}(t,x,\xi)| \leqslant C_{\alpha,\beta,N}\delta^2 \langle \ell - p\rangle^{-N} \langle \ell' - p\rangle^{-N}$$

whence a similar estimate for the $\mathcal{L}(L^2)$ operator norm of $R_{p,\delta}(t,x,D)$. We deduce from (2.1.25)

$$\left| \int_I \langle R_{p,\delta}(t,x,D)u,u\rangle \frac{dt}{t} \right|^{1/2} \leqslant C\delta \sum_\ell \langle p - \ell\rangle^{-N} \|\mathbb{1}_I(t)\phi_{\ell,\delta}u\|_{L^2(\frac{dt}{t}dx)}.$$

The conclusion of the lemma follows from this inequality. $\qquad\square$

2.2. Spaces of distributions and linear inequality

Let us introduce the dyadic decompositions that will be used in the rest of that paper. Following for instance [**3**] chapter 2, we choose $\varphi \in C_0^\infty(\mathbb{R}^d - \{0\})$, $\varphi \geqslant 0$, $\operatorname{Supp}\varphi \subset \{\xi; 3/4 < |\xi| < 8/3\}$ and $\chi \in C_0^\infty(\{\xi; |\xi| < 4/3\})$, $\chi \geqslant 0$, $\chi \equiv 1$ close to 0 such that

$$\chi(\xi) + \sum_0^\infty \varphi(2^{-j}\xi) \equiv 1$$

$$(2.2.1) \qquad |j - j'| \geqslant 2 \Longrightarrow \operatorname{Supp}\varphi(2^{-j}\cdot) \cap \operatorname{Supp}\varphi(2^{-j'}\cdot) = \varnothing$$

$$j \geqslant 1 \Longrightarrow \operatorname{Supp}\chi \cap \operatorname{Supp}\varphi(2^{-j}\cdot) = \varnothing.$$

We denote by Δ_j, $j \in \mathbb{N}$ (resp. Δ_{-1}) the Fourier multiplier with symbol $\varphi(2^{-j}\xi)$ (resp. $\chi(\xi)$) so that $\sum_{j=-1}^{+\infty} \Delta_j = \mathrm{Id}$. For $j \in \mathbb{N}$, we set

$$(2.2.2) \qquad S_j = \sum_{k=-1}^{j-1} \Delta_k = \chi(2^{-j}D).$$

For $\ell \in \mathbb{N}$ set $\Gamma_\ell(t) = \mathbb{1}_{\{2^\ell < t < 2^{\ell+1}\}}$.

DEFINITION 2.2.1. — Let s, s', α, α' be real numbers, $r \in [1, +\infty]$.

(i) One denotes by $F_{\alpha,\alpha',r}^{s,s'}$ the space of all distributions u belonging to $L_{\mathrm{loc}}^2([1, +\infty[, \mathcal{S}'(\mathbb{R}^d))$ such that there is a constant $C > 0$, a sequence $(c_{jq})_{jq}$ indexed by $j \geqslant -1$, $q \in \mathbb{Z}^d$, satisfying $\sum_j (\sum_q |c_{jq}|^r)^{2/r} \leqslant 1$, such that

$$(2.2.3) \quad \|\phi_q(x)\Delta_j u\Gamma_\ell(t)\|_{L^2(\frac{dt}{t}dx)}$$
$$\leqslant Cc_{jq}2^{(j-\ell/2)\alpha}(1+2^{j-\ell/2})^{-s-\alpha}2^{(j-\ell)\alpha'}(1+2^{j-\ell})^{-s'-\alpha'}$$

for any $j \geqslant -1$, $q \in \mathbb{Z}^d$, $\ell \in \mathbb{N}$.

(ii) One denotes by $H_{\alpha,\alpha',r}^{s,s'}$ the space of all distributions u belonging to $L_{\mathrm{loc}}^\infty([1, +\infty[, \mathcal{S}'(\mathbb{R}^d))$ such that there is a constant $C > 0$, a sequence $(c_{jq})_{jq}$ indexed by $j \geqslant -1$, $q \in \mathbb{Z}^d$, satisfying $\sum_j (\sum_q |c_{jq}|^r)^{2/r} \leqslant 1$, such that

$$(2.2.4) \quad \|\phi_q(x)\Delta_j u\Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}$$
$$\leqslant Cc_{jq}2^{(j-\ell/2)\alpha}(1+2^{j-\ell/2})^{-s-\alpha}2^{(j-\ell)\alpha'}(1+2^{j-\ell})^{-s'-\alpha'}$$

for any $j \geqslant -1$, $q \in \mathbb{Z}^d$, $\ell \in \mathbb{N}$.

(iii) One denotes by $\underline{H}_{\alpha,\alpha'}^{s,s'}$ the space of all distributions u belonging to $L_{\mathrm{loc}}^\infty([1, +\infty[, \mathcal{S}'(\mathbb{R}^d))$ such that there is a constant $C > 0$, and a sequence $(c_j)_j$ in the unit ball of ℓ^2, with

$$(2.2.5) \quad \|\Delta_j u\Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}$$
$$\leqslant Cc_j2^{(j-\ell/2)\alpha}(1+2^{j-\ell/2})^{-s-\alpha}2^{(j-\ell)\alpha'}(1+2^{j-\ell})^{-s'-\alpha'}$$

for any $j \geqslant -1$, $\ell \in \mathbb{N}$.

We define the norms in these spaces as the best constant $C > 0$.

We will use the notation

$$(2.2.6) \qquad \|c_{jq}\|_{\ell_j^2 \ell_q^r} = \left(\sum_j \left(\sum_q |c_{jq}|^r \right)^{2/r} \right)^{1/2}.$$

The definition of the above space is clearer if we write it in the following equivalent way: a distribution u is in $F_{\alpha,\alpha',r}^{s,s'}$ if and only if there is a family of $L_{\mathrm{loc}}^2([1,+\infty[)$-functions $(c_{jq}(t))_{jq}$ and $C > 0$ such that

$$(2.2.7) \quad \|\phi_q(x)\Delta_j u(t,\cdot)\|_{L^2(dx)}$$

$$\leqslant Cc_{jq}(t)\left(\frac{2^j}{\sqrt{t}}\right)^{\alpha}\left(1+\frac{2^j}{\sqrt{t}}\right)^{-s-\alpha}\left(\frac{2^j}{t}\right)^{\alpha'}\left(1+\frac{2^j}{t}\right)^{-s'-\alpha'},$$

$$\sup_{\ell}\|c_{jq}(t)\Gamma_\ell(t)\|_{L^2(dt/t)} \text{ is in the unit ball of } \ell_j^2\ell_q^r.$$

One has a similar characterization of $H_{\alpha,\alpha',r}^{s,s'}$ replacing in the last condition the $L^2(dt/t)$-norm by the L^∞ one, and of $\underline{H}_{\alpha,\alpha'}^{s,s'}$, by

$$\|\Delta_j u(t,\cdot)\|_{L^2(dx)} \leqslant Cc_j\left(\frac{2^j}{\sqrt{t}}\right)^{\alpha}\left(1+\frac{2^j}{\sqrt{t}}\right)^{-s-\alpha}\left(\frac{2^j}{t}\right)^{\alpha'}\left(1+\frac{2^j}{t}\right)^{-s'-\alpha'}$$

with $(c_j)_j$ in the unit ball of ℓ^2. The difference between spaces $\underline{H}_{\alpha,\alpha'}^{s,s'}$ and $H_{\alpha,\alpha',r}^{s,s'}$ is that the latter imposes some x-decay when $|x| \to +\infty$ (for $r < +\infty$). Since $\|\Delta_j u(t,\cdot)\|_{L^2}^2 \leqslant C\sum_q \|\phi_q\Delta_j u(t,\cdot)\|_{L^2}^2$ we have an injection $H_{\alpha,\alpha',2}^{s,s'} \subset \underline{H}_{\alpha,\alpha'}^{s,s'}$.

The meaning of conditions (2.2.7) is as follows: the weight in the right hand side corresponds to two types of derivatives: a stronger one $D_x/\sqrt{t}$ and a weaker one D_x/t. The indices s, s' represent the amount of such derivatives available when $2^j/\sqrt{t}$ or $2^j/t$ go to infinity. The indices α, α' measure the behaviour when $2^j/\sqrt{t} \ll 1$ or $2^j/t \ll 1$. The weaker derivative D_x/t is the one to which the Kato smoothing property will apply, uniformly in t, when solving the linear Schrödinger equation. As a last remark, let us say that the above spaces do not depend on the choice of φ, χ, ϕ.

Let us remark that in definitions (2.2.3), (2.2.4), (2.2.7), we could reverse the orders of ϕ_q and Δ_j:

PROPOSITION 2.2.2. — *A distribution u belongs to $F_{\alpha,\alpha',r}^{s,s'}$ (resp. $H_{\alpha,\alpha',r}^{s,s'}$) if and only if there is a family of $L_{\mathrm{loc}}^2([1,+\infty[)$-functions (resp. a family of $L_{\mathrm{loc}}^\infty([1,+\infty[)$-functions) $(c_{jq}(t))_{jq}$ and $C > 0$ such that*

$$(2.2.8) \quad \|\Delta_j\phi_q u(t,\cdot)\|_{L^2(dx)}$$

$$\leqslant Cc_{jq}(t)\left(\frac{2^j}{\sqrt{t}}\right)^{\alpha}\left(1+\frac{2^j}{\sqrt{t}}\right)^{-s-\alpha}\left(\frac{2^j}{t}\right)^{\alpha'}\left(1+\frac{2^j}{t}\right)^{-s'-\alpha'}$$

with $\sup_\ell \|c_{jq}(t)\Gamma_\ell(t)\|_{L^2(dt/t)}$ (resp. $\sup_\ell \|c_{jq}(t)\Gamma_\ell(t)\|_{L^\infty(dt)} = \|c_{jq}\|_{L^\infty(dt)}$) in the unit ball of $\ell_j^2\ell_q^r$.

Proof. — Assume that (2.2.7) is satisfied and let us prove (2.2.8). Choose $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^d)$, $\widetilde{\phi} \equiv 1$ close to $\mathrm{Supp}\,\phi$, $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^d - \{0\})$, $\widetilde{\varphi} \equiv 1$ close to $\mathrm{Supp}\,\varphi$,

$\widetilde{\chi} \in C_0^\infty(\mathbb{R}^d)$, $\widetilde{\chi} \equiv 1$ close to Supp χ, and define $\widetilde{\phi}_q(x) = \widetilde{\phi}(x - q)$, $\widetilde{\Delta}_j = \widetilde{\varphi}(2^{-j}D)$, $j \in \mathbb{N}$, $\widetilde{\Delta}_{-1} = \widetilde{\chi}(D)$. Write

$$(2.2.9) \qquad \Delta_j \phi_q u = \sum_{q'} \sum_{j' \geqslant -1} (\Delta_j \phi_q(x) \widetilde{\Delta}_{j'} \widetilde{\phi}_{q'}(x))(\phi_{q'}(x) \Delta_{j'} u).$$

It is enough to show that for any $N \in \mathbb{N}$, there is $C_N > 0$ with

$$(2.2.10) \qquad \|\Delta_j \phi_q \widetilde{\Delta}_{j'} \widetilde{\phi}_{q'}\|_{\mathcal{L}(L^2)} \leqslant C_N \langle q - q' \rangle^{-N} 2^{-|j-j'|N}$$

since the assumption (2.2.7) implies that

$$\|\phi_{q'}(x) \Delta_{j'} u(t, x)\|_{L^2(dx)}$$

$$\leqslant C 2^{|j-j'|N_0} c_{j'q'}(t) \left(\frac{2^j}{\sqrt{t}} \right)^\alpha \left(1 + \frac{2^j}{\sqrt{t}} \right)^{-s-\alpha} \left(\frac{2^j}{t} \right)^{\alpha'} \left(1 + \frac{2^j}{t} \right)^{-s'-\alpha'}$$

for some large enough N_0. To prove (2.2.10) we write for any function w

$$(2.2.11) \quad \Delta_j \phi_q(x) \widetilde{\Delta}_{j'} \widetilde{\phi}_{q'}(x) w$$

$$= \sum_p \sum_{p'} \sum_{p''} [\phi_p(D) \Delta_j \phi_q(x) \phi_{p'}(D) \widetilde{\Delta}_{j'} \widetilde{\phi}_{q'}(x) \widetilde{\phi}_{p''}(D)](\phi_{p''}(D) w).$$

Remark that $\phi_p(\xi) \varphi(2^{-j}\xi) \in S(D_p(\xi)^{-\infty})$ where $D_p(\xi) = \langle \xi - p \rangle$, with semi-norms uniformly controlled in p. In the same way $\phi_{p'}(\xi) \widetilde{\varphi}(2^{-j'}\xi) \in S(D_{p'}(\xi)^{-\infty})$, $\widetilde{\phi}_{p''}(\xi) \in S(D_{p''}(\xi)^{-\infty})$, $\phi_q(x) \in S(D_q(x)^{-\infty})$, $\widetilde{\phi}_{q'}(x) \in S(D_{q'}(x)^{-\infty})$, with uniform control of semi-norms. By theorem 2.1.3, the operator inside the bracket in the right hand side of (2.2.11) is given by a symbol in $S(A_{p,p',p'',q,q'}(x,\xi)^{-\infty})$, with

$$(2.2.12) \qquad A_{p,p',p'',q,q'}(x,\xi) = D_p(\xi) D_{p'}(\xi) D_{p''}(\xi) D_q(x) D_{q'}(x),$$

its semi-norms being controlled uniformly in p, p', p'', q, q'. Since (2.2.12) is bounded from below by $c\langle p - p' \rangle^{1/2} \langle p - p'' \rangle^{1/2} \langle q - q' \rangle$, and since in the summation (2.2.11), p (resp. p') stays of magnitude 2^j (resp. $2^{j'}$), the fact that the $\mathcal{L}(L^2)$-norm of an operator is bounded by the semi-norms of its symbol implies that

$$\|\Delta_j \phi_q(x) \widetilde{\Delta}_{j'} \widetilde{\phi}_{q'}(x) w\|_{L^2}^2$$

$$\leqslant C \sum_p \left[\sum_{p'} \sum_{p''} 2^{-N|j-j'|} \langle p - p' \rangle^{-N} \langle p - p'' \rangle^{-N} \langle q - q' \rangle^{-N} \|\phi_{p''}(D) w\|_{L^2} \right]^2$$

$$\leqslant C 2^{-2N|j-j'|} \langle q - q' \rangle^{-2N} \left\| \sum_{p''} \langle p - p'' \rangle^{-N} \|\phi_{p''}(D) w\|_{L^2} \right\|_{\ell_p^2}^2$$

where we used almost orthogonality in p. This inequality implies (2.2.10) and concludes the proof.

The reverse implication (2.2.8) $\Rightarrow$ (2.2.7) is shown in a similar way. □

Our main objective is to prove the following version of the Kato smoothing property in the framework of the spaces introduced in definition 2.2.1.

THEOREM 2.2.3. — *Let s, s', α, α' be real numbers satisfying*

$$(2.2.13) \qquad \frac{\alpha}{2} + \alpha' < 0, \quad s + \alpha \geqslant 0, \quad s' + \alpha' \geqslant 0.$$

Take $v_0 \in H^{s+s'}(\mathbb{R}^d)$, $f \in F^{s,s'-1/2}_{\alpha,\alpha',1}$. Then the solution v of the linear problem

$$(2.2.14) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)v = \frac{1}{t}f$$
$$v|_{t=1} = v_0$$

belongs to $\underline{H}^{s,s'}_{\alpha,\alpha'} \cap F^{s,s'+1/2}_{\alpha,\alpha',\infty}$ and we have the smoothing inequality

$$(2.2.15) \qquad \|v\|_{\underline{H}^{s,s'}_{\alpha,\alpha'}} + \|v\|_{F^{s,s'+1/2}_{\alpha,\alpha',\infty}} \leqslant C[\|v_0\|_{H^{s+s'}} + \|f\|_{F^{s,s'-1/2}_{\alpha,\alpha',1}}].$$

The main step in the proof will be to obtain estimates for dyadic frequency cut-offs of v. For $\delta \in]0,1]$, $q \in \mathbb{Z}^d$, we use the notation $\phi_{q,\delta}(x) = \phi(\delta x - q)$ introduced before proposition 2.1.4 and $\Gamma_\ell(t) = \mathbb{1}_{\{2^\ell < t < 2^{\ell+1}\}}$.

THEOREM 2.2.4. — *There is $\delta_0 \in]0,1]$ and for any $\delta \in]0,\delta_0]$ a positive constant C_δ such that for any v_0, f, v, satisfying the assumptions of theorem 2.2.3, any $j \in \mathbb{Z}$, $j \geqslant -1$, any $\ell \in \mathbb{N}$*

$$(2.2.16) \quad \sup_p \|\Gamma_\ell(t)\phi_{p,\delta}(x)\Delta_j v\|_{L^2(\frac{dt}{t}dx)}$$

$$\leqslant C_\delta(1 + 2^{j-\ell})^{-1/2}\|\Delta_j v_0\|_{L^2}$$

$$+ C_\delta(1 + 2^{j-\ell})^{-1/2} \sum_{m=0}^{\ell} (1 + 2^{j-m})^{-1/2} \left\| \|\Gamma_m(t)\phi_{p,\delta}\Delta_j f\|_{L^2(\frac{dt}{t}dx)} \right\|_{\ell^1_p}.$$

The proof of this theorem will be done in several steps. Define first

$$(2.2.17) \qquad \begin{aligned} a_{q,\delta}(t,x,\xi) &= \phi_{q,\delta}(x + 2\xi(\tfrac{1}{t} - 1)) = \phi(\delta(x + 2\xi(\tfrac{1}{t} - 1)) - q) \\ A_{q,\delta}(t,x,\xi) &= \langle \delta(x + 2\xi(\tfrac{1}{t} - 1)) - q \rangle. \end{aligned}$$

We see that $A_{q,\delta}$ is a family of (C_0, N_0)-temperate weights, with constants C_0, N_0 uniform in $q \in \mathbb{Z}^d$, $\delta \in]0,1]$, and that this family satisfies (2.1.13). Moreover $a_{q,\delta}|_{t=1} = \phi_{q,\delta}(x)$ and $a_{q,\delta} \in S(A_{q,\delta}^{-\infty})$, the semi-norms of these symbols in these spaces being controlled uniformly in q, δ. More precisely, we have for $N \in \mathbb{N}, n \in \mathbb{N}$

$$(2.2.18) \qquad \sup_q |\partial_x^\alpha \partial_\xi^\beta a_{q,\delta}|_{A_{q,\delta},N,n} = O(\delta^{|\alpha|+|\beta|}), \quad \delta \longrightarrow 0.$$

The symbols $a_{q,\delta}$ are essentially constant along the bicharacteristics of $D_t + D_x^2/t^2$, so that they will enjoy nice commutation properties with this operator. More precisely, denote

$$(2.2.19) \qquad b_{q,\delta}(t,x,\xi) = -(\Delta\phi)\left(\delta\left(x + 2\xi\left(\tfrac{1}{t} - 1\right)\right) - q\right).$$

We have

$$(2.2.20) \qquad \left[D_t + \frac{D_x^2}{t^2}, a_{q,\delta}(t,x,D)\right] = \frac{\delta^2}{t^2}b_{q,\delta}(t,x,D)$$

whence the equation

$$(2.2.21) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)a_{q,\delta}(t,x,D)\Delta_j v = \frac{1}{t}a_{q,\delta}(t,x,D)\Delta_j f + \frac{\delta^2}{t^2}b_{q,\delta}(t,x,D)\Delta_j v$$
$$a_{q,\delta}(t,x,D)\Delta_j v|_{t=1} = \phi_{q,\delta}(x)\Delta_j v_0.$$

The first step of the proof is to get an upper bound for the left hand side of (2.2.16) in terms of $a_{q,\delta}(t,x,D)\Delta_j v$ cut-off in convenient time domains.

PROPOSITION 2.2.5. — *Choose* $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \equiv 1$ *on* $[-2,2]^d$, $\psi \geqslant 0$, *and for* $q \in \mathbb{Z}^d$, $\delta \in]0,1]$ *set* $\psi_{q,\delta}(t,\xi) = \psi(2\delta\xi(\frac{1}{t}-1)-q)$. *Denote by* $\widetilde{\Delta}_j$ *the dyadic cut-offs introduced at the beginning of the proof of proposition 2.2.2. There is* $\delta_0 \in]0,1]$ *and* $C > 0$ *such that for any* $\delta \in]0,\delta_0]$, *any* $j \geqslant -1$, *any compact interval* $I \subset [1,+\infty[$, *any* $v \in L^2_{\mathrm{loc}}([1,+\infty[, L^2(dx))$ *one has*

$$(2.2.22) \quad \sup_{p\in\mathbb{Z}^d} \|\phi_{p,\delta}(x)\Delta_j v(t,x)\|_{L_I^2}^2 \leqslant C \sup_{p\in\mathbb{Z}^d} \sum_{q\in\mathbb{Z}^d} \|\psi_{q-p,\delta}(t,D)\widetilde{\Delta}_j a_{q,\delta}(t,x,D)\Delta_j v\|_{L_I^2}^2$$

where $L_I^2 = L^2(I, \frac{dt}{t}dx)$.

To prove this proposition, we shall decompose $\phi_{p,\delta}(x)\Delta_j v(t,x)$ as an almost orthogonal sum using $(a_{q,\delta})_q$. Each term of this decomposition will have microlocal support contained inside $\mathrm{Supp}\,\phi_{p,\delta}(x) \cap \mathrm{Supp}\,a_{q,\delta}(t,x,\xi)$, which will imply a (t,ξ)-localization, given by $\psi_{q-p,\delta}$ in the right hand side of (2.2.22). This can be visualized on the following picture where, at fixed ξ, we represent $u = \frac{1}{t} - 1$ as a function of x:
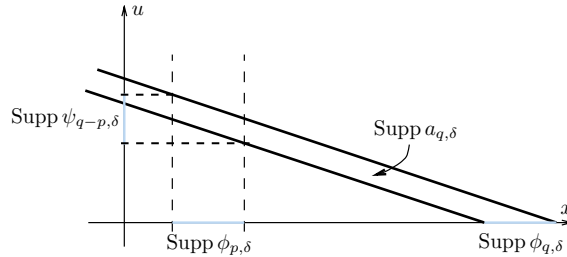
The formal proof will use the following lemma.

LEMMA 2.2.6. — *Let* $A_{q,\delta}(t,x,\xi)$ *be a family of temperate weights with uniform constants* (C_0, N_0), *satisfying for some* $N_1 \in \mathbb{N}$, $C_1 > 0$ *the inequality* $\sum_q A_{q,\delta}(t,x,\xi)^{-N_1} \leqslant C_1$, *uniformly in* t,δ. *Let* $r \in [1,+\infty]$. *Denote* $D_{q',\delta}(x) = \langle \delta x - q' \rangle$.

Let $R_{q,q',\delta}(t,x,\xi)$ *be a family of symbols of* $S(A_{q,\delta}^{-\infty}D_{q',\delta}(x)^{-\infty})$ *with semi-norms in these spaces bounded uniformly in* $\delta \in]0,1]$, $q,q' \in \mathbb{Z}^d$. *There is* $C > 0$ *such that*

for any $w \in L^2_{\mathrm{loc}}([1, +\infty[, L^2(dx))$, any compact interval I,

$$(2.2.23) \qquad \left\| \Big(\sum_q \| R_{q,q',\delta}(t,x,D)w \|^2_{L^2_I} \Big)^{1/2} \right\|_{\ell^r_{q'}} \leqslant C \big\| \| \phi_{q',\delta} w \|_{L^2_I} \big\|_{\ell^r_{q'}}.$$

If $R_{q,\delta}(x,\xi)$ is a family in $S(A^{-\infty}_{q,\delta})$ at fixed t, with uniformly bounded semi-norms, there is $C > 0$, and for any $w \in L^2(\mathbb{R}^2)$

$$(2.2.24) \qquad \sum_q \| R_{q,\delta}(x,D)w \|^2_{L^2} \leqslant C \| w \|^2_{L^2}.$$

Proof. — We decompose $w = \sum_p \phi_{p,\delta}(x)w$ and write

$$(2.2.25) \quad \sum_q \| R_{q,q',\delta}(t,x,D)w \|^2_{L^2_I}$$

$$= \sum_{p,p',q} \int_I \langle [\widetilde{\phi}_{p',\delta}(x) R_{q,q',\delta}(t,x,D)^* R_{q,q',\delta}(t,x,D) \widetilde{\phi}_{p,\delta}] \phi_{p,\delta} w, \phi_{p',\delta} w \rangle \frac{dt}{t}.$$

By theorem 2.1.3, the operator between brackets $L^{p,p'}_{q,q',\delta}(t,x,D)$ has symbol

$$L^{p,p'}_{q,q',\delta}(t,x,\xi) \in S(A^{-\infty}_{q,\delta} D_{q',\delta}(x)^{-\infty} D_{p,\delta}(x)^{-\infty} D_{p',\delta}(x)^{-\infty})$$

with uniformly controlled semi-norms. Since for $N \geqslant N_1$,

$$(2.2.26) \quad \sum_q A_{q,\delta}(t,x,\xi)^{-N} D_{q',\delta}(x)^{-2N} D_{p,\delta}(x)^{-N} D_{p',\delta}(x)^{-N}$$

$$\leqslant C_N \langle q' - p \rangle^{-N} \langle q' - p' \rangle^{-N},$$

$\sum_q L^{p,p'}_{q,q',\delta}(t,x,\xi)$ is in $S^0_{0,0}$ with semi-norms controlled by the right hand side of (2.2.26), uniformly in δ. We deduce from (2.2.25)

$$\sum_q \| R_{q,q',\delta}(t,x,D)w \|^2_{L^2_I} \leqslant \sum_p \sum_{p'} C_N \langle q' - p \rangle^{-N} \langle q' - p' \rangle^{-N} \| \phi_{p,\delta} w \|_{L^2_I} \| \phi_{p',\delta} w \|_{L^2_I}$$

from which (2.2.23) follows. One gets (2.2.24) applying the fixed time version of (2.2.23) to $R_{q,q',\delta}(t,x,\xi) \overset{\text{def}}{=} \phi_{q',\delta}(x) R_{q,\delta}(x,\xi)$ and $r = 2$. $\qquad \square$

Proof of proposition 2.2.5. — By proposition 2.1.4, for small enough δ_0 we have

$$(2.2.27) \qquad \sup_p \| \phi_{p,\delta}(x) \Delta_j v(t,x) \|^2_{L^2_I} \leqslant C \sup_p \sum_q \| \phi_{p,\delta}(x) a_{q,\delta}(t,x,D) \Delta_j v \|^2_{L^2_I}.$$

Remark that if (t,x,ξ) is in the support of $\phi_{p,\delta}(x) a_{q,\delta}(t,x,\xi)$ one has $\delta x - p \in Q$, $\delta(x + 2\xi(\frac{1}{t} - 1)) - q \in Q$, where Q is the cube $]-1,1[^d$, whence $\psi_{q-p,\delta}(t,\xi) = 1$. We thus write

$$(2.2.28) \qquad \phi_{p,\delta}(x) a_{q,\delta}(t,x,D) \Delta_j = \phi_{p,\delta}(x) a_{q,\delta}(t,x,D) (\widetilde{\Delta}_j \psi_{q-p,\delta}(t,D)) \Delta_j.$$

The family $a_{q,\delta}(t,x,\xi)$ has semi-norms in $S(A_{q,\delta}^{-\infty})$ uniformly bounded (where $A_{q,\delta}$ is defined by (2.2.17)) and moreover, for $N \in \mathbb{N}, n \in \mathbb{N}$

$$(2.2.29) \qquad \sup_q |\partial_x^\alpha \partial_\xi^\beta a_{q,\delta}(t,x,\xi)|_{A_{q,\delta},N,n} = O(\delta^{|\alpha|+|\beta|}), \delta \longrightarrow 0.$$

The family $\psi_{q-p,\delta}(t,\xi)\widetilde{\varphi}(2^{-j}\xi)$ has semi-norms uniformly bounded in the space of symbols $S(1) = S_{0,0}^0$. By theorem 2.1.3 we see, taking (2.2.29) into account,

$$\phi_{p,\delta}(x)[a_{q,\delta}(t,x,D)][\widetilde{\Delta}_j \psi_{q-p,\delta}(t,D)]$$
$$= \phi_{p,\delta}(x)[\widetilde{\Delta}_j \psi_{q-p,\delta}(t,D)][a_{q,\delta}(t,x,D)] + \delta R_{q,p,\delta}(t,x,D)$$

where $R_{q,p,\delta}(t,x,\xi)$ is in $S(A_{q,\delta}^{-\infty} D_{p,\delta}(x)^{-\infty})$ with uniformly bounded semi-norms. We deduce from (2.2.27) and from lemma 2.2.6 with $r = \infty$

$$\sup_p \|\phi_{p,\delta}(x)\Delta_j v\|_{L_I^2}^2 \leqslant C \sup_p \sum_q \|\psi_{q-p,\delta}(t,D)\widetilde{\Delta}_j a_{q,\delta}(t,x,D)\Delta_j v\|_{L_I^2}^2$$
$$+ C\delta^2 \sup_p \|\phi_{p,\delta}(x)\Delta_j v\|_{L_I^2}^2.$$

The conclusion follows taking δ small enough. $\square$

COROLLARY 2.2.7. — *With the above assumptions and notations, there is $C > 0$ such that for any $\delta \in]0,\delta_0], \ell \in \mathbb{N}, v \in L_{\mathrm{loc}}^\infty([1,+\infty[,L^2)$*

$$(2.2.30) \quad \sup_p \|\Gamma_\ell(t)\phi_{p,\delta}(x)\Delta_j v(t,x)\|_{L^2(\frac{dt}{t}dx)}$$
$$\leqslant C \inf\left(1, \frac{2^{(\ell-j)/2}}{\sqrt{\delta}}\right)\left(\sum_q \|\sup_t |\Gamma_\ell(t)\mathcal{F}(a_{q,\delta}(t,x,D)\Delta_j v)(t,\xi)|\|_{L^2(d\xi)}^2\right)^{1/2}$$

where we denoted by $\mathcal{F}$ the Fourier transform in x.

Proof. — By proposition 2.2.5, the square of the left hand side is smaller than

$$(2.2.31) \qquad C \sup_{p\in\mathbb{Z}^d} \sum_q \|\Gamma_\ell(t)\psi_{q-p,\delta}(t,D)\widetilde{\Delta}_j a_{q,\delta}(t,x,D)\Delta_j v(t,x)\|_{L^2(\frac{dt}{t}dx)}^2.$$

The general term of this sum is bounded by Plancherel by

$$C \int \sup_t |\Gamma_\ell(t)\mathcal{F}(a_{q,\delta}(t,x,D)\Delta_j v)(t,\xi)|^2 d\xi \times \int \Gamma_\ell(t)\frac{dt}{t}.$$

This gives (2.2.30) when $1 \leqslant 2^{(\ell-j)/2}/\sqrt{\delta}$. In the other case, we can always assume $j \geqslant 0$. Using that on the support of $\psi_{q-p,\delta}(t,\xi)$ we have the inequality

$$|2\delta\xi(\tfrac{1}{t}-1) - (q-p)| \leqslant K,$$

we estimate the general term of (2.2.31) by

$$\int |\Gamma_\ell(t)\psi_{q-p,\delta}(t,\xi)\widetilde{\varphi}(2^{-j}\xi)\mathcal{F}(a_{q,\delta}(t,x,D)\Delta_j v)(t,\xi)|^2\, d\xi \frac{dt}{t}$$

$$\leqslant C2^\ell \int \mathbb{1}_{\{|2\delta\xi(u-1)-(q-p)|\leqslant K\}}\Gamma_\ell(1/u)$$

$$\times \widetilde{\varphi}(2^{-j}\xi)\mathcal{F}(a_{q,\delta}(1/u,x,D)\Delta_j v)(1/u,\xi)|^2\, d\xi\, du$$

$$\leqslant C\frac{2^{\ell-j}}{\delta}\int \sup_t |\Gamma_\ell(t)\mathcal{F}(a_{q,\delta}(t,x,D)\Delta_j v)(t,\xi)|^2\, d\xi$$

where the last inequality follows from the fact that u stays in an interval of length $K/\delta|\xi|$, with K independent of p,q,δ, and $|\xi| \sim 2^j$. This concludes the proof. □

The factor $2^{(\ell-j)/2}$ in the right hand side of (2.2.30), represents the gain coming from the smoothing property of the operator. The next step is to estimate the right hand side of (2.2.30) using the equation. To simplify notations let us set

$$(2.2.32) \qquad v_{jq} = a_{q,\delta}(t,x,D)\Delta_j v,\ \ f_{jq} = a_{q,\delta}(t,x,D)\Delta_j f$$
$$v_{jq}^0 = \phi_{q,\delta}\Delta_j v_0,\ \ r_{jq} = b_{q,\delta}(t,x,D)\Delta_j v.$$

PROPOSITION 2.2.8. — *There are $\delta_0 \in\,]0,1]$, $C > 0$ such that for any $\delta \in\,]0,\delta_0]$, any $j \geqslant -1$, $\ell \in \mathbb{N}$, $v \in L^\infty_{\mathrm{loc}}([1,+\infty[, L^2)$ one has*

$$(2.2.33) \quad \left\|\|\sup_t |\Gamma_\ell(t)\widehat{v}_{jq}(t,\xi)|\|_{L^2(d\xi)}\right\|_{\ell_q^2}$$

$$\leqslant C\left[\left\|\|\widehat{v}_{jq}^0(\xi)\|_{L^2(d\xi)}\right\|_{\ell_q^2} + \left\|\left\|\sum_{m=0}^\ell \int \Gamma_m(t)|\widehat{f}_{jq}(t,\xi)|\frac{dt}{t}\right\|_{L^2(d\xi)}\right\|_{\ell_q^2}\right].$$

Proof. — Equation (2.2.21) may be written

$$(2.2.34) \qquad\qquad \left(D_t + \frac{\xi^2}{t^2}\right)\widehat{v}_{jq}(t,\xi) = \frac{1}{t}\widehat{f}_{jq}(t,\xi) + \frac{\delta^2}{t^2}\widehat{r}_{jq}(t,\xi)$$
$$\widehat{v}_{jq}(1,\xi) = \widehat{v}_{jq}^0(\xi)$$

whence

$$\widehat{v}_{jq}(t,\xi) = \exp(i\frac{\xi^2}{t} - i\xi^2)\widehat{v}_{jq}^0(\xi)$$

$$+ i\int_1^t \exp\left(i\xi^2\left(\frac{1}{t} - \frac{1}{s}\right)\right)\left[\frac{1}{s}\widehat{f}_{jq}(s,\xi) + \frac{\delta^2}{s^2}\widehat{r}_{jq}(s,\xi)\right] ds$$

which implies for any ξ, any $\ell \in \mathbb{N}$,

$$\sup_t |\Gamma_\ell(t)\widehat{v}_{jq}(t,\xi)| \leqslant |\widehat{v}_{jq}^0(\xi)| + \sum_{m=0}^\ell \int \Gamma_m(s)|\widehat{f}_{jq}(s,\xi)|\frac{ds}{s} + \delta^2\sum_{m=0}^\ell \int \Gamma_m(s)|\widehat{r}_{jq}(s,\xi)|\frac{ds}{s^2}.$$

Taking the $L^2(d\xi)$ norm and the ℓ_q^2 norm, and setting

$$\alpha_{\ell j} = \left\| \| \sup_t |\Gamma_\ell(t)\widehat{v}_{jq}(t,\xi)| \|_{L^2(d\xi)} \right\|_{\ell_q^2}$$

$$a_{\ell j} = \left\| \|\widehat{v}_{jq}^0\|_{L^2(d\xi)} \right\|_{\ell_q^2} + \left\| \left\| \sum_{m\leqslant \ell} \int \Gamma_m(s)|\widehat{f}_{jq}(s,\xi)| \frac{ds}{s} \right\|_{L^2(d\xi)} \right\|_{\ell_q^2}$$

$$\beta_j(s) = \left\| \|\widehat{r}_{jq}(s,\xi)\|_{L^2(d\xi)} \right\|_{\ell_q^2}$$

we get the inequality

$$(2.2.35) \qquad \alpha_{\ell j} \leqslant a_{\ell j} + \delta^2 \sum_{m=0}^{\ell} \int \Gamma_m(s)\beta_j(s)\frac{ds}{s^2}.$$

Applying the second and the first inequalities (2.1.15) with $r = 2$, we have by definition of r_{jq}, v_{jq}

$$\beta_j(s) \leqslant C\|\Delta_j v(s,\cdot)\|_{L^2} \leqslant C \left\| \|\widehat{v}_{jq}(s,\xi)\|_{L^2(d\xi)} \right\|_{\ell_q^2}$$

with a constant C independent of s. Plugging into (2.2.35), we get an inequality

$$(2.2.36) \qquad \alpha_{\ell j} \leqslant a_{\ell j} + C\delta^2 \sum_{m=0}^{\ell} \alpha_{mj} 2^{-m}.$$

Using that $a_{\ell j}$ is increasing in ℓ, we deduce from (2.2.36) that if δ is small enough $\alpha_{\ell j} \leqslant 2a_{\ell j}$ which is the conclusion. $\qquad\square$

The last step of the proof is to estimate the right hand side of (2.2.33).

PROPOSITION 2.2.9. — *One has the following estimates, for a uniform constant C:*

$$(2.2.37) \quad \left\| \left\| \int \Gamma_m(t)|\widehat{f}_{jq}(t,\xi)| \frac{dt}{t} \right\|_{L^2(d\xi)} \right\|_{\ell_q^2}$$

$$\leqslant C \inf\left[\frac{2^{(m-j)/2}}{\sqrt{\delta}}, 1 \right] \sum_{p\in\mathbb{Z}^d} \|\Gamma_m(t)\phi_{p,\delta}\Delta_j f\|_{L^2(\frac{dt}{t}dx)}$$

for any $m \in \mathbb{N}$, $j \geqslant -1$.

Proof. — We use the notation $\psi_{r,\delta}(t,\xi)$ introduced in the statement of proposition 2.2.6. We also denote by $\widetilde{\psi}_{r,\delta}(t,\xi)$ the function defined like $\psi_{r,\delta}(t,\xi)$, replacing ψ by a function $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^d)$, $\widetilde{\psi} \geqslant 0$, $\widetilde{\psi} \equiv 1$ on Supp ψ. Write

$$(2.2.38) \quad \int \Gamma_m(t)|\widehat{f}_{jq}(t,\xi)| \frac{dt}{t} \leqslant C \sum_{r\in\mathbb{Z}^d} \sum_{j'} \int \Gamma_m(t)|\psi_{r,\delta}(t,\xi)\varphi(2^{-j'}\xi)\widehat{f}_{jq}(t,\xi)| \frac{dt}{t}$$

where $\varphi(2^{-j'}\xi)$ should be read $\chi(\xi)$ when $j' = -1$. The general term of the above sum can be estimated by

$$(2.2.39) \quad \Big(\int \widetilde{\varphi}(2^{-j'}\xi)\widetilde{\psi}_{r,\delta}(t,\xi)^2 \Gamma_m(t)\frac{dt}{t} \Big)^{1/2}$$
$$\times \Big(\int \Gamma_m(t)\psi_{r,\delta}(t,\xi)^2 \varphi(2^{-j'}\xi)^2 |\widehat{f}_{jq}(t,\xi)|^2 \frac{dt}{t} \Big)^{1/2}.$$

By the same computation as in the proof of corollary 2.2.7, the first factor is smaller than $C\inf[1, 2^{(m-j')/2}/\sqrt{\delta}]$. The left hand side of (2.2.37) will be smaller than

$$(2.2.40) \quad \sum_{j'} C\inf\Big[\frac{2^{(m-j')/2}}{\sqrt{\delta}}, 1\Big]$$
$$\times \Big\| \sum_r \|\Gamma_m(t)\psi_{r,\delta}(t,D)\Delta_{j'}a_{q,\delta}(t,x,D)\Delta_j f\|_{L^2(\frac{dt}{t}dx)} \Big\|_{\ell^2_q}.$$

Let us write

$$(2.2.41) \quad \Gamma_m(t)\psi_{r,\delta}(t,D)\Delta_{j'}a_{q,\delta}(t,x,D)\Delta_j f$$
$$= \sum_{\ell,\ell',\ell''} \sum_p K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)(\Gamma_m(t)\phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f)$$

where

$$(2.2.42) \quad K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)$$
$$= \Delta_{j'}\psi_{r,\delta}(t,D)\phi_{\ell,\delta}(D)a_{q,\delta}(t,x,D)\widetilde{\Delta}_j\phi_{\ell',\delta}(D)\widetilde{\phi}_{p,\delta}(x)\widetilde{\phi}_{\ell'',\delta}(D).$$

We write

$$(2.2.43) \quad \|\Gamma_m(t)\psi_{r,\delta}(t,D)\Delta_{j'}a_{q,\delta}(t,x,D)\Delta_j f\|_{L^2(\frac{dt}{t}dx)}$$
$$\leqslant C\Big\|\Big\| \sum_{\ell',\ell'',p} K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)(\Gamma_m(t)\phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f)\Big\|_{L^2(\frac{dt}{t}dx)}\Big\|_{\ell^2_\ell}$$

since the ℓ-sum is an almost orthogonal one. By theorem 2.1.3 and (2.2.42) the symbol $K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,\xi) \in S(A^{-\infty})$ where

$$A = \langle 2\delta\xi(\tfrac{1}{t}-1) - r\rangle\langle \delta\xi - \ell\rangle\langle \delta(x + 2\xi(\tfrac{1}{t}-1)) - q\rangle\langle \delta\xi - \ell'\rangle\langle \delta x - p\rangle\langle \delta\xi - \ell''\rangle$$

with uniformly controlled semi-norms. This implies that the symbol is in $S(1) = S^0_{0,0}$ with semi-norms controlled by

$$(2.2.44) \quad C_N\langle 2\ell''(\tfrac{1}{t}-1) - r\rangle^{-N}\langle \ell - \ell'\rangle^{-N}\langle \ell - \ell''\rangle^{-N}\langle r - (q-p)\rangle^{-N}$$

uniformly in $\delta \in]0,1], t \geqslant 1$. At fixed t, the $\mathcal{L}(L^2)$ operator norm of $K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)$ is thus smaller than (2.2.44). Moreover, since (2.2.42) implies that ℓ (resp. ℓ') is of

magnitude $2^{j'}$ (resp. 2^j), the $\langle \ell - \ell' \rangle^{-N}$ term of (2.2.44) provides a $2^{-N|j-j'|}$ factor. At fixed t, we thus have

$$(2.2.45) \quad \sum_{\ell''} \Big\| \sum_{\ell',p} K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)(\Gamma_m \phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f) \Big\|_{L^2(dx)}$$

$$\leqslant C_N \sum_{\ell',\ell'',p} \langle 2\ell''(\tfrac{1}{t}-1)-r \rangle^{-N} \langle \ell - \ell' \rangle^{-N} \langle \ell - \ell'' \rangle^{-N} \langle r-(q-p) \rangle^{-N}$$

$$\times\, 2^{-N|j-j'|} \|\Gamma_m(t)\phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f(t,\cdot)\|_{L^2(dx)}.$$

Denote by

$$(2.2.46) \quad \begin{aligned} k^{j,j',r}_{\ell,\ell'',q}(t) &= \Big\| \sum_{\ell',p} K^{j,j',r}_{\ell,\ell',\ell'',p,q}(t,x,D)(\Gamma_m \phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f) \Big\|_{L^2(dx)} \\ \gamma^{j}_{\ell'',p}(t) &= \|\Gamma_m(t)\phi_{\ell'',\delta}(D)\phi_{p,\delta}(x)\Delta_j f(t,\cdot)\|_{L^2(dx)}. \end{aligned}$$

The norm to estimate in (2.2.40) is by (2.2.43) bounded from above by

$$\Big\| \sum_r \Big\| \Big\| \sum_{\ell''} k^{j,j',r}_{\ell,\ell'',q}(t) \Big\|_{L^2(dt/t)} \Big\|_{\ell^2_\ell} \Big\|_{\ell^2_q}.$$

Using (2.2.45), this last term can be controlled by

$$(2.2.47) \quad \Big\| \sum_r \Big\| \sum_{\ell',\ell'',p} \langle \ell-\ell' \rangle^{-N} \langle \ell-\ell'' \rangle^{-N} \langle r-(q-p) \rangle^{-N} 2^{-N|j-j'|}$$

$$\times \Big(\int \langle 2\ell''(\tfrac{1}{t}-1)-r \rangle^{-2N} |\gamma^{j}_{\ell'',p}(t)|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{\ell^2_\ell} \Big\|_{\ell^2_q}.$$

Put

$$(2.2.48) \quad \begin{aligned} \beta^{j}_{\ell'',r,p} &= \Big(\int \langle 2\ell''(\tfrac{1}{t}-1)-r \rangle^{-2N} |\gamma^{j}_{\ell'',p}(t)|^2 \frac{dt}{t} \Big)^{1/2} \\ \widetilde{\beta}^{j}_{r,p} &= \|\beta^{j}_{\ell'',r,p}\|_{\ell^2_{\ell''}}. \end{aligned}$$

We have an upper bound for (2.2.47)

$$2^{-N|j-j'|} \Big\| \sum_r \Big\| \sum_{\ell'',p} \langle \ell-\ell'' \rangle^{-N} \langle r-(q-p) \rangle^{-N} \beta^{j}_{\ell'',r,p} \Big\|_{\ell^2_\ell} \Big\|_{\ell^2_q}$$

$$\leqslant C 2^{-N|j-j'|} \Big\| \sum_r \sum_p \langle r-(q-p) \rangle^{-N} \widetilde{\beta}^{j}_{r,p} \Big\|_{\ell^2_q}$$

$$\leqslant C 2^{-N|j-j'|} \sum_p \Big\| \sum_r \langle r-q \rangle^{-N} \widetilde{\beta}^{j}_{r,p} \Big\|_{\ell^2_q}$$

$$\leqslant C 2^{-N|j-j'|} \sum_p \|\widetilde{\beta}^{j}_{r,p}\|_{\ell^2_r}$$

$$\leqslant C2^{-N|j-j'|}\sum_p \Big(\sum_r \sum_{\ell''}\int \langle 2\ell''(\tfrac{1}{t}-1)-r\rangle^{-2N}|\gamma^j_{\ell'',p}(t)|^2\frac{dt}{t}\Big)^{1/2}$$

$$\leqslant C2^{-N|j-j'|}\sum_p \|\Gamma_m\phi_{p,\delta}(x)\Delta_j f\|_{L^2(\frac{dt}{t}dx)}.$$

Going back to (2.2.40) we see that we proved that the left hand side of (2.2.37) is smaller than

$$C\sum_{j'}\inf[2^{(m-j')/2}/\sqrt\delta,1]2^{-N|j-j'|}\sum_p \|\Gamma_m(t)\phi_{p,\delta}(x)\Delta_j f\|_{L^2(\frac{dt}{t}dx)}.$$

The conclusion of the proposition follows.   □

Proof of theorem 2.2.4. — One has just to apply corollary 2.2.7 followed by proposition 2.2.8 and proposition 2.2.9, remarking that we have a bound for $\|\,\|\widehat{v}^0_{jq}(\xi)\|_{L^2(d\xi)}\|_{\ell^2_q}$ in terms of $\|\Delta_j v_0\|_{L^2}$.   □

Proof of theorem 2.2.3. — From now on, we shall fix δ small enough so that inequality (2.2.16) of theorem 2.2.4 holds true. We can then, in the left and right hand sides of (2.2.16), replace $\phi_{p,\delta}$ by ϕ_p, loosing a bigger constant in the right hand side. Let us prove first that $\|v\|_{F^{s,s'+1/2}_{\alpha,\alpha',\infty}}$ is smaller than the right hand side of (2.2.15). We deduce from (2.2.16) that for any q

$$(2.2.49)\quad 2^{-(j-\frac{\ell}{2})\alpha}(1+2^{j-\frac{\ell}{2}})^{s+\alpha}2^{-(j-\ell)\alpha'}(1+2^{j-\ell})^{s'+\frac{1}{2}+\alpha'}\|\Gamma_\ell(t)\phi_q\Delta_j v\|_{L^2(\frac{dt}{t}dx)}$$

$$\leqslant Ca_{j\ell}\|\Delta_j v_0\|_{L^2}2^{j(s+s')}$$

$$+C\sum_{m=0}^{\ell}b_{j\ell m}2^{-(j-m/2)\alpha}(1+2^{j-m/2})^{s+\alpha}2^{-(j-m)\alpha'}(1+2^{j-m})^{s'-1/2+\alpha'}$$

$$\times\Big\|\|\Gamma_m\phi_p(x)\Delta_j f\|_{L^2(\frac{dt}{t}dx)}\Big\|_{\ell^1_p}$$

with

$$a_{j\ell}=2^{-j(s+s'+\alpha+\alpha')+\ell(\alpha+2\alpha')/2}(1+2^{j-\ell/2})^{s+\alpha}(1+2^{j-\ell})^{s'+\alpha'}$$

$$b_{j\ell m}=2^{(\ell-m)(\alpha+2\alpha')/2}(1+2^{j-\ell/2})^{s+\alpha}(1+2^{j-m/2})^{-s-\alpha}$$

$$\times(1+2^{j-\ell})^{s'+\alpha'}(1+2^{j-m})^{-s'-\alpha'}.$$

Since $s+\alpha\geqslant 0$, $s'+\alpha'\geqslant 0$, we get $a_{j\ell}\leqslant C2^{\ell(\alpha+2\alpha')/2}$ and $b_{j\ell m}\leqslant 2^{(\ell-m)(\alpha+2\alpha')/2}$. Since $\frac{\alpha}{2}+\alpha'<0$, we get taking the $\ell^2_j\ell^\infty_q$ norm of (2.2.49)

$$\|v\|_{F^{s,s'+1/2}_{\alpha,\alpha',\infty}}\leqslant C\|v_0\|_{H^{s+s'}}+C\sum_{m=0}^{\ell}2^{(\ell-m)(\alpha+2\alpha')/2}\|f\|_{F^{s,s'-1/2}_{\alpha,\alpha',1}}$$

$$\leqslant C(\|v_0\|_{H^{s+s'}}+\|f\|_{F^{s,s'-1/2}_{\alpha,\alpha',1}}).$$

We still have to prove that $v \in \underline{H}_{\alpha,\alpha'}^{s,s'}$. Using inequality (2.1.15) with $r = 2$ we get for any fixed t

$$\|\Delta_j v(t,\cdot)\|_{L^2(dx)} \leqslant C \left\| \|\phi_p \Delta_j v(t,\cdot)\|_{L^2(dx)} \right\|_{\ell_p^2}$$

$$(2.2.50) \qquad \leqslant C \left\| \left\| \|\phi_p a_q(t,x,D)\Delta_j v(t,\cdot)\|_{L^2(dx)} \right\|_{\ell_q^2} \right\|_{\ell_p^2}$$

$$\leqslant C \left\| \|\mathcal{F}(a_q(t,x,D)\Delta_j v)(t,\xi)\|_{L^2(d\xi)} \right\|_{\ell_q^2}.$$

Using notations (2.2.32), for t in the support of $\Gamma_\ell(t)$, we can bound $\|\Delta_j v(t,\cdot)\|_{L^2}$ by (2.2.33), whence using estimate (2.2.37) the inequality

$$\|\Gamma_\ell(t)\Delta_j v(t,\cdot)\|_{L^2} \leqslant C \left[\|\Delta_j v_0\|_{L^2} + \sum_{m=0}^{\ell} (1+2^{j-m})^{-1/2} \left\| \|\Gamma_m(t)\phi_p \Delta_j f\|_{L^2(\frac{dt}{t}dx)} \right\|_{\ell_p^1} \right].$$

Arguing as for inequality (2.2.49), we deduce from that

$$\|v\|_{\underline{H}_{\alpha,\alpha'}^{s,s'}} \leqslant C \left[\|v_0\|_{H^{s+s'}} + \|f\|_{F_{\alpha,\alpha',1}^{s,s'-1/2}} \right].$$

This concludes the proof of theorem 2.2.3. $\qquad\qquad\square$

CHAPTER 3

NONLINEAR ESTIMATES

3.1. Estimates for products

The aim of this section is to study regularity of products in the framework of spaces $F_{\alpha,\alpha',r}^{s,s'}$ and $H_{\alpha,\alpha',r}^{s,s'}$ introduced in definition 2.2.1. Let us define a few more notations: let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \equiv 1$ close to 0.

DEFINITION 3.1.1. — One denotes by $\widetilde{H}_{\alpha,\alpha',r}^{s,s'}$ (resp. $\underline{\widetilde{H}}_{\alpha,\alpha'}^{s,s'}$) the space of those $v \in H_{\alpha,\alpha',r}^{s,s'}$ (resp. in $\underline{H}_{\alpha,\alpha'}^{s,s'}$) such that

$$(3.1.1) \qquad (\|\phi_q(x)\chi(D/\sqrt{t})v(t,x)\|_{L^\infty(dtdx)})_q \in \ell_q^r,$$

(resp.

$$(3.1.2) \qquad \chi(D/\sqrt{t})v(t,x) \in L^\infty(dtdx).$$

The above definition depends on the choice of χ except in the case $\alpha' > d/2$, which is the only one we will encounter:

LEMMA 3.1.2. — *Assume $\alpha' > d/2$. Then definition 3.1.1 is independent of the choice of χ. More generally, if $\psi(t,\xi)$ is a smooth function satisfying $\mathrm{Supp}\,\psi \subset \{(t,\xi); |\xi| < C\sqrt{t}\}$ and $|\partial_\xi^\beta \psi(t,\xi)| \leqslant C_\beta t^{-|\beta|/2}$ for any $\beta \in \mathbb{N}^d$, we have for $v \in \widetilde{H}_{\alpha,\alpha',r}^{s,s'}$ (resp. $v \in \underline{\widetilde{H}}_{\alpha,\alpha'}^{s,s'}$)*

$$(3.1.3) \qquad (\|\phi_q(x)\psi(t,D)v\|_{L^\infty(dtdx)})_q \in \ell_q^r.$$

(resp.

$$(3.1.4) \qquad \psi(t,D)v \in L^\infty(dtdx).)$$

If, moreover, $\alpha + 2\alpha' > 0$, then $\widetilde{H}_{\alpha,\alpha',r}^{s,s'} = H_{\alpha,\alpha',r}^{s,s'}$ (resp. $\underline{\widetilde{H}}_{\alpha,\alpha'}^{s,s'} = \underline{H}_{\alpha,\alpha'}^{s,s'}$) and one has more precisely

$$(3.1.5) \qquad (\|\phi_q(x)t^\delta \chi(D/\sqrt{t})v\|_{L^\infty(dtdx)})_q \in \ell_q^r$$

(resp.

(3.1.6) $$t^\delta \chi(D/\sqrt{t})v \in L^\infty(dtdx))$$

for any $\delta \in [0, \min(\frac{\alpha}{2} + \alpha', \frac{\alpha'}{2} - \frac{d}{4})[.$

Proof. — For v satisfying (3.1.1), write

(3.1.7) $$\phi_q \psi(t, D)v = \phi_q \psi(t, D)\chi(D/\sqrt{t})v + \phi_q \psi(t, D)(1 - \chi)(D/\sqrt{t})v.$$

We shall again denote by $\widetilde{\phi}_{q'}$ the function $\widetilde{\phi}_{q'}(x) = \widetilde{\phi}(x - q')$ for $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^d)$, $\widetilde{\phi} \equiv 1$ close to $\operatorname{Supp} \phi$. Let us remark that an operator of form $K_{q,q'}^t = \phi_q(x) \circ \psi(t, D) \circ \widetilde{\phi}_{q'}(x)$ has a kernel whose modulus is bounded from above by

(3.1.8) $$Ct^{d/2}(1 + \sqrt{t}|x - y|)^{-2N}|\phi_q(x)||\widetilde{\phi}_{q'}(y)| \leqslant Ct^{d/2}(1 + \sqrt{t}|x - y|)^{-N}\langle q - q'\rangle^{-N}$$

whence

(3.1.9) $$\begin{aligned} \|K_{qq'}^t\|_{\mathcal{L}(L^\infty)} &\leqslant C_N \langle q - q'\rangle^{-N} \\ \|K_{qq'}^t\|_{\mathcal{L}(L^2, L^\infty)} &\leqslant C_N t^{d/4}\langle q - q'\rangle^{-N}. \end{aligned}$$

We estimate the first contribution to (3.1.7) by

(3.1.10) $$\begin{aligned} \|\phi_q \psi(t, D)\chi(D/\sqrt{t})v\|_{L^\infty} &\leqslant \sum_{q'} \|(\phi_q \psi(t, D)\widetilde{\phi}_{q'})(\phi_{q'}\chi(D/\sqrt{t})v)\|_{L^\infty(dtdx)} \\ &\leqslant \sum_{q'} C_N \langle q - q'\rangle^{-N} \|\phi_{q'}\chi(D/\sqrt{t})v\|_{L^\infty(dtdx)} \end{aligned}$$

which shows that this sequence is in ℓ_q^r because of assumption (3.1.1). The second contribution to the right hand side of (3.1.7) is estimated at fixed t by

$$\begin{aligned} \|\phi_q \psi(t, D)&(1 - \chi)(D/\sqrt{t})v\|_{L^\infty} \\ &\leqslant \sum_{q'} \sum_j \|(\phi_q \psi(t, D)(1 - \chi)(D/\sqrt{t})\widetilde{\phi}_{q'})(\phi_{q'}\Delta_j v)(t, \cdot)\|_{L^\infty(dx)} \end{aligned}$$

where the j-sum is restricted to indices satisfying $C_1\sqrt{t} < 2^j < C_2\sqrt{t}$. By (3.1.9), we get the upper bound

(3.1.11) $$\sum_{q'} \sum_{j; 2^j \sim \sqrt{t}} 2^{jd/2}\langle q - q'\rangle^{-N}\|\phi_{q'}\Delta_j v(t, \cdot)\|_{L^2(dx)}$$

$$\leqslant \sum_{q'} \sum_{j; 2^j \sim \sqrt{t}} 2^{jd/2}\langle q - q'\rangle^{-N}c_{jq'}(2^j/\sqrt{t})^\alpha (2^j/t)^{\alpha'}$$

for a sequence $(c_{jq'})_{jq'}$ in $\ell_j^2 \ell_{q'}^r$ using the definition of $H_{\alpha,\alpha',r}^{s,s'}$. The supremum in t of this last quantity is smaller than Cc_q' with $(c_q')_q \in \ell_q^r$ since $\alpha' > d/2$.

Finally, when $\alpha + 2\alpha' > 0$, $\|\phi_q \chi(D/\sqrt{t})v\|_{L^\infty(dx)}$ is bounded from above by the sum in the right hand side of (3.1.11), extended to $2^j < C\sqrt{t}$ i.e by

$$\sum_{2^j < C\sqrt{t}} 2^{j(\frac{d}{2} + \alpha + \alpha')}t^{-\alpha/2 - \alpha'}c_{jq} \leqslant \frac{C}{t^\delta}c_q'$$

for $0 \leqslant \delta < \min[\frac{\alpha}{2} + \alpha', \frac{\alpha'}{2} - \frac{d}{4}]$ and $(c'_q)_q \in \ell^r$.

The proof for $v \in \underline{\widetilde{H}}^{s,s'}_{\alpha,\alpha'}$ is similar and simpler. $\square$

We will study products of elements of $F^{s,s'}_{\alpha,\alpha',r}, H^{s,s'}_{\alpha,\alpha',r}$ using Bony's paraproducts [**1**]. Define for $u, v \in \mathcal{S}'(\mathbb{R}^d)$

$$(3.1.12) \qquad T_u v = \sum_{j \geqslant 1} S_{j-1} u \Delta_j v.$$

The series converge in $\mathcal{S}'(\mathbb{R}^d)$ and each of its terms has Fourier transform supported inside a ring of size $C2^j$. Moreover, we set

$$(3.1.13) \qquad R(u, v) = \sum_{j,j';|j-j'| \leqslant 1} \Delta_j u \Delta_{j'} v$$

when the above series converges. We will use the same notations when u, v depend on the parameter $t \in [1, +\infty[$. When (3.1.13) has a meaning, one gets Bony's decomposition of the product uv:

$$(3.1.14) \qquad uv = T_u v + T_v u + R(u, v).$$

We will also use notations similar to (3.1.12), (3.1.13) in a slightly more general framework. Assume we are given constants $C_1 > 0$, $C_2 > 0$ and families of C^∞ functions $(\varphi_j(\xi))_{j \in \mathbb{N}}, (\chi_j(\xi))_{j \in \mathbb{N}}$ satisfying

$$(3.1.15) \qquad \begin{aligned} &\operatorname{Supp} \varphi_j(\xi) \subset \{\xi; C_1^{-1} 2^j < |\xi| < C_1 2^j\}, \quad |\partial^\beta_\xi \varphi_j(\xi)| \leqslant C_\beta 2^{-j|\beta|} \\ &\operatorname{Supp} \chi_j(\xi) \subset \{\xi; |\xi| < C_2 2^j\}, \qquad |\partial^\beta_\xi \chi_j(\xi)| \leqslant C_\beta 2^{-j|\beta|}. \end{aligned}$$

We will use the generic notation $\widetilde{T}_u v$ and $\widetilde{R}(u, v)$ for operators

$$(3.1.16) \qquad \begin{aligned} \widetilde{T}_u v &= \sum_{j \geqslant N_1} (\chi_{j-N_1}(D)u)(\varphi_j(D)v) \\ \widetilde{R}(u, v) &= \sum_{j,j';|j-j'| \leqslant N_2} (\varphi_j(D)u)(\varphi_{j'}(D)v) \end{aligned}$$

with $N_1, N_2 \in \mathbb{N}$, with N_1 large enough so that the Fourier transform of each contribution to $\widetilde{T}_u v$ be supported inside a ring of size 2^j. In the definition of $\widetilde{R}$, we also eventually allow low frequency contributions of type $(\chi_0(D)u)(\varphi_{j'}(D)v)$, $|j'| \leqslant N_2$, $(\varphi_j(D)u)(\chi_0(D)v)$, $|j| \leqslant N_2$.

PROPOSITION 3.1.3
  (i) Let $u \in L^\infty(\frac{dt}{t}dx)$ and $v \in F^{s,s'}_{\alpha,\alpha',r}$ (resp. $v \in H^{s,s'}_{\alpha,\alpha',r}$). Then $T_u v$ and $\widetilde{T}_u v$ are in $F^{s,s'}_{\alpha,\alpha',r}$ (resp. $H^{s,s'}_{\alpha,\alpha',r}$).
  (ii) Let $s_j, s'_j, \alpha_j, \alpha'_j, r_j, j = 1, 2$ be indices with $r_j \in [1, +\infty]$. Assume $\alpha'_1 > d/2, s_1 \neq d$. Let $u \in \widetilde{H}^{s_1,s'_1}_{\alpha_1,\alpha'_1,r_1}, v \in F^{s_2,s'_2}_{\alpha_2,\alpha'_2,r_2}$ (resp. $v \in \underline{H}^{s_2,s'_2}_{\alpha_2,\alpha'_2}$). Then $T_u v$ and

$\widetilde{T}_u v$ *belong to* $F_{\alpha,\alpha',r}^{s,s'}$ *(resp.* $H_{\alpha,\alpha',r_1}^{s,s'}$*) where*

$$\alpha = \alpha_2 + (\alpha_1 + 2\alpha_1')_+, \quad \alpha' = \alpha_2',$$

(3.1.17) $$s = s_2 - (d - s_1)_+, \quad s' = s_2' - [-s_1' - \frac{d}{2}]_+,$$

$$\frac{1}{r} = \min[1, \frac{1}{r_1} + \frac{1}{r_2}].$$

Proof

(i) For fixed t we write

(3.1.18) $$\|\phi_q(x)\Delta_j(T_u v)(t,\cdot)\|_{L^2(dx)} \leqslant \sum_{j';|j-j'|\leqslant N_0} \|\phi_q(x)\Delta_j(S_{j'-1}u\Delta_{j'}v)(t,\cdot)\|_{L^2(dx)}$$

for a large enough integer N_0. The general term of the above sum is smaller than

$$\sum_{q'} \|(\phi_q\Delta_j\widetilde{\phi}_{q'})(S_{j'-1}u\phi_{q'}(x)\Delta_{j'}v)(t,\cdot)\|_{L^2(dx)}.$$

Using that the $\mathcal{L}(L^2)$ operator norm of the composition $\phi_q\Delta_j\widetilde{\phi}_{q'}$ is smaller than $C_N\langle q - q'\rangle^{-N}$ we get for (3.1.18) the upper bound

$$\sum_{q'}\sum_{j';|j-j'|\leqslant N_0} C_N\langle q - q'\rangle^{-N}\|S_{j'-1}u\|_{L^\infty}\|\phi_{q'}(x)\Delta_{j'}v(t,\cdot)\|_{L^2(dx)}.$$

Since $\|S_{j'-1}u\|_{L^\infty} \leqslant C\|u\|_{L^\infty}$ the result follows multiplying by a cut-off $\Gamma_\ell(t)$ and computing the $L^2(dt/t)$ (resp. $L^\infty(dt)$) norm.

(ii) To argue in the same way, we need an estimate for quantities like

$$\|\phi_q S_j u(t,\cdot)\|_{L^\infty(dx)}.$$

LEMMA 3.1.4. — *Let* $u \in \widetilde{H}_{\alpha,\alpha',r}^{s,s'}$ *with* $\alpha' > d/2$, $s \neq d$. *We have for any j and any t the estimate*

(3.1.19) $$\|\phi_q(x)S_j u(t,\cdot)\|_{L^\infty(dx)} \leqslant c_{jq}(2^j/\sqrt{t})^{(\alpha+2\alpha')_+}(1 + 2^j/\sqrt{t})^{-(\alpha+2\alpha')_++(d-s)_+}$$
$$\times (1 + 2^j/t)^{[-\frac{d}{2}-s']_+}\|u\|_{\widetilde{H}_{\alpha,\alpha',r}^{s,s'}}$$

where $(c_{jq})_{jq}$ *is a sequence in* $\ell_j^\infty \ell_q^r$.

Proof. — Consider first j, t with $2^j < \sqrt{t}$. When $\alpha + 2\alpha' \leqslant 0$, we argue as in (3.1.10) to obtain

$$\|\phi_q S_j u(t,\cdot)\|_{L^\infty(dx)} = \|\phi_q S_j \chi(D/\sqrt{t})u(t,\cdot)\|_{L^\infty(dx)}$$
$$\leqslant \sum_{q'} \|(\phi_q S_j\widetilde{\phi}_{q'})(\phi_{q'}\chi(D/\sqrt{t})u(t,\cdot))\|_{L^\infty(dx)}$$
$$\leqslant C_N \sum_{q'}\langle q - q'\rangle^{-N}\|\phi_{q'}\chi(D/\sqrt{t})u(t,\cdot)\|_{L^\infty(dx)}$$

and we use condition (3.1.1) of the definition of $\widetilde{H}_{\alpha,\alpha',r}^{s,s'}$ to get an upper bound in terms of a ℓ_q^r sequence. When $\alpha + 2\alpha' > 0$, we write instead

$$
\begin{aligned}
\|\phi_q S_j u(t,\cdot)\|_{L^\infty(dx)} &\leqslant \sum_{k\leqslant j-1} \|\phi_q \Delta_k u(t,\cdot)\|_{L^\infty(dx)} \\
&\leqslant \sum_{k\leqslant j-1}\sum_{q'} \|(\phi_q \widetilde{\Delta}_k \widetilde{\phi}_{q'})(\phi_{q'}\Delta_k u)\|_{L^\infty(dx)}
\end{aligned}
\tag{3.1.20}
$$

where $\widetilde{\Delta}_k$ is such that $\widetilde{\Delta}_k \Delta_k = \Delta_k$. Since the $\mathcal{L}(L^2, L^\infty)$ operator norm of $\phi_q \widetilde{\Delta}_k \widetilde{\phi}_{q'}$ is smaller than $C 2^{kd/2}\langle q - q'\rangle^{-N}$ we get the upper bound

$$
\begin{aligned}
(3.1.21) \quad C\sum_{k\leqslant j-1}\sum_{q'} 2^{kd/2}\langle q-q'\rangle^{-N} &\|\phi_{q'}\Delta_k u(t,\cdot)\|_{L^2} \\
&\leqslant C\sum_{k\leqslant j-1}\sum_{q'} \langle q-q'\rangle^{-N} 2^{kd/2}(2^k/\sqrt{t})^{\alpha}(2^k/t)^{\alpha'} c_{kq'}
\end{aligned}
$$

with $(c_{kq'})_{kq'} \in \ell_k^2 \ell_{q'}^r$. We write (3.1.21) as $(2^j/\sqrt{t})^{\alpha+2\alpha'} d_{jq}$ with

$$
d_{jq} = \sum_k \mathbb{1}_{\{k\leqslant j-1\}} 2^{k\frac{d}{2}+k(\alpha+\alpha')-j(\alpha+2\alpha')} c'_{kq}
$$

with $(c'_{kq})_{kq}$ in $\ell_k^2 \ell_q^r$. Since $\alpha + 2\alpha' > 0, \alpha' > d/2$, $(d_{jq})_{jq}$ is a $\ell_j^\infty \ell_q^r$ sequence.

Consider next the case $C\sqrt{t} < 2^j < Ct$. Arguing as in (3.1.20), (3.1.21) we get the upper bound

$$
\begin{aligned}
\|\phi_q S_j u(t,\cdot)\|_{L^\infty} &\leqslant \|\phi_q \chi(D/\sqrt{t})u(t,\cdot)\|_{L^\infty(dx)} \\
&\quad + C_N \sum_{C\sqrt{t}<2^k\leqslant 2^{j-1}}\sum_{q'} 2^{kd/2}\langle q-q'\rangle^{-N}\|\phi_{q'}\Delta_k u(t,\cdot)\|_{L^2}.
\end{aligned}
$$

The first term in the right hand side is smaller than a ℓ_q^r sequence, uniformly in t, by definition of $\widetilde{H}_{\alpha,\alpha',r}^{s,s'}$, and the second one is smaller than

$$
\sum_{q'}\sum_k \mathbb{1}_{\{C\sqrt{t}<2^k<2^j<Ct\}}\langle q-q'\rangle^{-N} 2^{kd/2}(2^k/\sqrt{t})^{-s}(2^k/t)^{\alpha'} c_{kq'}.
\tag{3.1.22}
$$

When $\frac{s}{2}-\alpha' \geqslant 0$, the supremum in t of (3.1.22) is smaller than the sum $\sum_k 2^{k(\frac{d}{2}-\alpha')} c'_{kq}$ for a $\ell_k^2 \ell_q^r$ sequence $(c'_{kq})_{kq}$, and so smaller than a ℓ_q^r sequence since $\alpha' > d/2$.

When $\frac{s}{2} - \alpha' < 0$ and $d < s$, the supremum in t of (3.1.22) is smaller than

$$
\sum_k \mathbb{1}_{\{k\leqslant j-1\}} 2^{k\frac{d}{2}-(k-\frac{j}{2})s+(k-j)\alpha'} c'_{kq} \leqslant \sum_k 2^{\frac{k}{2}(d-s)} c'_{kq}
$$

which is a ℓ_q^r sequence. Assume now $\frac{s}{2} - \alpha' < 0$ and $d > s$. We write (3.1.22) as $(2^j/\sqrt{t})^{d-s} d_{jq}$ with

$$
d_{jq} = \sum_k \mathbb{1}_{\{C\sqrt{t}<2^k<2^j<Ct\}} 2^{(k-j)(d-s)}(2^k/t)^{\alpha'-\frac{d}{2}} c'_{kq}.
$$

The supremum in t is smaller than $C \sum_k \mathbb{1}_{\{k \leqslant j-1\}} 2^{(k-j)(\frac{d}{2}-s+\alpha')} c'_{kq}$ which is a $\ell_j^2 \ell_q^r$ sequence. This gives again (3.1.19).

Finally, consider the case $Ct \leqslant 2^j$. We estimate

$$\|\phi_q S_j u(t, \cdot)\|_{L^\infty} \leqslant \|\phi_q \chi(D/t) u(t, \cdot)\|_{L^\infty(dx)}$$
$$+ C_N \sum_{Ct \leqslant 2^k < 2^j} \sum_{q'} 2^{kd/2} \langle q - q' \rangle^{-N} \|\phi_{q'} \Delta_k u(t, \cdot)\|_{L^2}.$$

The first term is estimated by the right hand side of (3.1.19), as easily seen reproducing the computations of the two preceding cases. We are left with estimating

$$\sum_{Ct < 2^k < 2^j} 2^{kd/2} (2^k/\sqrt{t})^{-s} (2^k/t)^{-s'} c'_{kq} \leqslant (2^j/t)^{[-\frac{d}{2}-s']_+} \sum_{Ct < 2^k < 2^j} (2^k/\sqrt{t})^{d-s} c'_{kq}$$

for a $\ell_k^2 \ell_{q'}^r$ sequence $(c'_{kq})_{kq}$. If $d - s > 0$, the last sum is smaller than $(2^j/\sqrt{t})^{d-s} c''_{jq}$ for a $\ell_j^2 \ell_q^r$ sequence $(c''_{jq})_{jq}$. If $d - s < 0$, the supremum in t of that sum is smaller than $\sum_k 2^{k(d-s)/2} c'_{kq}$ which is a ℓ_q^r sequence. In both cases, this gives the upper bound of (3.1.19). $\qquad\square$

End of the proof of proposition 3.1.3. — When $v \in F_{\alpha_2, \alpha_2', r_2}^{s_2, s_2'}$, we estimate the general term of the right hand side of (3.1.18) by

$$\sum_{q'} \|(\phi_q \Delta_j \widetilde{\phi}_{q'})(\widetilde{\phi}_{q'} S_{j'-1} u)(\phi_{q'} \Delta_{j'} v)(t, \cdot)\|_{L^2(dx)}$$
$$\leqslant C_N \sum_{q'} \langle q - q' \rangle^{-N} \|\widetilde{\phi}_{q'} S_{j'-1} u(t, \cdot)\|_{L^\infty} \|\phi_{q'} \Delta_{j'} v(t, \cdot)\|_{L^2(dx)}.$$

We just have to apply lemma 3.1.4 to get the conclusion.

When $v \in \underline{H}_{\alpha_2, \alpha_2'}^{s_2, s_2'}$, we bound the general term of the right hand side of (3.1.18) by

$$\sum_{q'} \|(\phi_q \Delta_j \widetilde{\phi}_{q'})((\phi_{q'} S_{j'-1} u)(\Delta_{j'} v)(t, \cdot))\|_{L^2(dx)}$$
$$\leqslant C_N \sum_{q'} \langle q - q' \rangle^{-N} \|\phi_{q'} S_{j'-1} u(t, \cdot)\|_{L^\infty} \|\Delta_{j'} v(t, \cdot)\|_{L^2(dx)}.$$

and we apply again lemma 3.1.4. $\qquad\square$

We will now prove an estimate for the remainder.

PROPOSITION 3.1.5

   (i) *Let* $(s_1, s_1', \alpha_1, \alpha_1', r_1)$, $(s_2, s_2', \alpha_2, \alpha_2', r_2)$ *be real numbers satisfying*

(3.1.23)
$$s_1 + s_2 + s_1' + s_2' > 0, \quad s_1 + s_2 > \alpha_1' + \alpha_2'$$
$$\alpha_1 + \alpha_2 + \alpha_1' + \alpha_2' < 0, \quad r_j \geqslant 1, \ j = 1, 2.$$

Let $u \in F_{\alpha_1,\alpha_1',r_1}^{s_1,s_1'}$ (resp. $u \in \underline{H}_{\alpha_1,\alpha_1'}^{s_1,s_1'}$) and $v \in H_{\alpha_2,\alpha_2',r_2}^{s_2,s_2'}$. Then $R(u,v)$ belongs to $F_{\alpha,\alpha',r}^{s,s'}$ (resp. to $H_{\alpha,\alpha',r_2}^{s,s'}$) with

$$s = s_1 + s_2 - d, \quad s' = s_1' + s_2' + \frac{d}{2}$$

(3.1.24)
$$\alpha = \alpha_1 + \alpha_2 + d, \quad \alpha' = \alpha_1' + \alpha_2' - \frac{d}{2}$$

$$\frac{1}{r} = \min[1, \frac{1}{r_1} + \frac{1}{r_2}].$$

(ii) Let us assume instead of (3.1.23)

$$s_1 + s_2 + s_1' + s_2' > 0, \quad s_1 + s_2 > \alpha_1' + \alpha_2'$$

(3.1.25)
$$\alpha_1 + \alpha_1' < 0, \quad \alpha_2' > \frac{d}{2}, \quad r_j \geqslant 1, \ j = 1, 2$$

and suppose that $v \in \widetilde{H}_{\alpha_2,\alpha_2',r_2}^{s_2,s_2'}$. *We then have* $R(u,v) \in F_{\alpha_1,\alpha_1',r}^{s,s'}$ *(resp.* $R(u,v) \in H_{\alpha_1,\alpha_1',r_2}^{s,s'}$) *for the values of* s, s', r *given by* (3.1.24).

Proof. — We will estimate for $u \in F_{\alpha_1,\alpha_1',r_1}^{s_1,s_1'}$

(3.1.26)
$$\phi_q \Delta_j R(u,v) = \sum_{\substack{k,k' \\ |k-k'| \leqslant 1 \\ k \geqslant j - N_0}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi}_{q'})[(\widetilde{\phi}_{q'} \Delta_k u)(\phi_{q'} \Delta_{k'} v)]$$

where N_0 is a large enough integer. At fixed t, the $L^2(dx)$ norm of the general term of (3.1.26) is bounded from above by

(3.1.27)
$$C_N \langle q - q' \rangle^{-N} 2^{jd/2} \|\widetilde{\phi}_{q'} \Delta_k u(t,\cdot)\|_{L^2(dx)} \|\phi_{q'} \Delta_{k'} v(t,\cdot)\|_{L^2(dx)}.$$

We multiply this expression by $\Gamma_\ell(t)$ and compute the $L^2(dt/t)$ norm in function of $\|\widetilde{\phi}_{q'} \Delta_k u \Gamma_\ell(t)\|_{L^2(\frac{dt}{t}dx)}$ and of $\|\phi_{q'} \Delta_{k'} v \Gamma_\ell(t)\|_{L^\infty(dt,L^2(dx))}$. By definition of the spaces, we get an upper bound given by an expression $C_N \langle q - q' \rangle^{-N} d_{jk\ell q'}$ with

(3.1.28) $\quad d_{jk\ell q'} = 2^{jd/2} 2^{(k-\ell/2)(\alpha_1+\alpha_2)} (1 + 2^{k-\ell/2})^{-s_1-s_2-\alpha_1-\alpha_2}$
$$\times 2^{(k-\ell)(\alpha_1'+\alpha_2')} (1 + 2^{k-\ell})^{-s_1'-s_2'-\alpha_1'-\alpha_2'} c_{kq'}$$

with $(c_{kq'})_{kq'} \in \ell_k^1 \ell_{q'}^r$, where we used that $|k - k'| \leqslant 1$. We must show that

(3.1.29)
$$\sum_{k \geqslant j - N_0} d_{jk\ell q'} \leqslant C 2^{(j-\ell/2)\widetilde{\alpha}} (1 + 2^{j-\ell/2})^{-s-\widetilde{\alpha}} 2^{(j-\ell)\widetilde{\alpha}'} (1 + 2^{j-\ell})^{-s'-\widetilde{\alpha}'} c_{jq'}'$$

with $(c_{jq'}')_{jq'}$ in the unit ball of $\ell_j^1 \ell_{q'}^r \subset \ell_j^2 \ell_{q'}^r$, and where $(\widetilde{\alpha}, \widetilde{\alpha}') = (\alpha, \alpha')$ in case (i), and $(\widetilde{\alpha}, \widetilde{\alpha}') = (\alpha_1, \alpha_1')$ in case (ii).

Case $j \geqslant \ell$. — We have

$$(3.1.30) \qquad \sum_{k \geqslant j-N_0} d_{jk\ell q'} \leqslant C \sum_{k \geqslant j-N_0} 2^{jd/2} 2^{-k(s_1+s_2+s_1'+s_2')} 2^{\ell(\frac{s_1+s_2}{2}+s_1'+s_2')} c_{kq'}$$

$$\leqslant C 2^{-(j-\ell/2)s-(j-\ell)s'} c_{jq'}'$$

with $(c_{jq'}')_{jq'} \in \ell_j^1 \ell_{q'}^r$ because of assumptions (3.1.23) or (3.1.25), s and s' being given by (3.1.24). This gives (3.1.29) both in cases (i) and (ii).

Case $\ell > j \geqslant \ell/2$. — We have

$$(3.1.31) \qquad \sum_{k \geqslant j-N_0} d_{jk\ell q'} \leqslant C 2^{j\frac{d}{2}-\frac{\ell}{2}(s_1+s_2)} c_{\ell q'}'$$

$$+ \sum_{k; \ell > k > j-N_0} 2^{jd/2} 2^{-k(s_1+s_2-\alpha_1'-\alpha_2')} 2^{\ell(\frac{s_1+s_2}{2}-\alpha_1'-\alpha_2')} c_{kq'}$$

where the first contribution is given by the sum (3.1.30) taken for $k \geqslant \ell$ instead of $k \geqslant j-N_0$, and where $(c_{\ell q'}')_{\ell q'} \in \ell_\ell^1 \ell_{q'}^r$. The right hand side of (3.1.31) is smaller than

$$C 2^{-(j-\ell/2)(s_1+s_2-d)} 2^{(j-\ell)(\alpha_1'+\alpha_2'-d/2)} \times [2^{(j-\ell)(s_1+s_2-\alpha_1'-\alpha_2')} c_{\ell q'}' + c_{jq'}]$$

because of (3.1.23), (3.1.25). The sequence between brackets is smaller than a $\ell_j^1 \ell_{q'}^r$ sequence independent of ℓ. This gives the estimate (3.1.29) in cases (i) and (ii), remembering for the last one that $\alpha_2' > d/2$.

Case $\ell/2 > j$. — We first treat case (i). We have

$$(3.1.32) \qquad \sum_{k \geqslant j-N_0} d_{jk\ell q'} \leqslant C [2^{j\frac{d}{2}-\frac{\ell}{2}(s_1+s_2)} c_{\ell q'}' + 2^{j\frac{d}{2}-\frac{\ell}{2}(\alpha_1'+\alpha_2')} c_{\ell q'}'']$$

$$+ \sum_{k; \ell/2 > k > j-N_0} 2^{jd/2} 2^{k(\alpha_1+\alpha_2+\alpha_1'+\alpha_2')} 2^{-\frac{\ell}{2}(\alpha_1+\alpha_2+2(\alpha_1'+\alpha_2'))} c_{kq'}$$

where the first two terms are given by the sum (3.1.31) taken for $k \geqslant \ell/2$ instead of $k \geqslant j-N_0$, and where $(c_{\ell q'}')_{\ell q'}$, $(c_{\ell q'}'')_{\ell q'}$ are in $\ell_\ell^1 \ell_{q'}^r$. Using (3.1.23) we get for the right hand side of (3.1.32) the estimate

$$C 2^{(j-\ell/2)(\alpha_1+\alpha_2+d)} 2^{(j-\ell)(\alpha_1'+\alpha_2'-d/2)}$$

$$(3.1.33) \qquad \times \big[2^{-(j-\ell/2)(\alpha_1+\alpha_2+\alpha_1'+\alpha_2')} 2^{-\frac{\ell}{2}(s_1+s_2-\alpha_1'-\alpha_2')} c_{\ell q'}'$$

$$+ 2^{-(j-\ell/2)(\alpha_1+\alpha_2+\alpha_1'+\alpha_2')} c_{\ell q'}'' + \widetilde{c}_{jq'} \big]$$

with $(\widetilde{c}_{jq'})_{jq'} \in \ell_j^1 \ell_{q'}^r$. The term between brackets is smaller than a sequence indexed by (j, q'), independent of ℓ, and belonging to $\ell_j^1 \ell_{q'}^r$. This gives conclusion (3.1.29) in case (i). In case (ii), we estimate the $L^2(dx)$ norm of the general term of (3.1.26)

using (3.1.27) when $k \geqslant \ell/2$, and writing

$$(3.1.34) \quad \|(\phi_q \Delta_j \widetilde{\phi}_{q'})[(\widetilde{\phi}_{q'} \Delta_k u)(\phi_{q'} \Delta_{k'} v)](t, \cdot)\|_{L^2}$$
$$\leqslant C\langle q - q'\rangle^{-N} \|\widetilde{\phi}_{q'} \Delta_k u(t, \cdot)\|_{L^2} \|\phi_{q'} \Delta_{k'} v(t, \cdot)\|_{L^\infty}$$

when $\ell/2 > k \geqslant j - N_0$. The assumption $(\|\phi_q \chi(D/\sqrt{t})v\|_{L^\infty})_q \in \ell_q^{r_2}$ implies that there is $(c_{q'})_{q'} \in \ell_{q'}^{r_2}$ such that for any k' with $k' \leqslant \ell/2 + N_1$, $\|\phi_{q'} \Delta_{k'} v\|_{L^\infty} \leqslant c_{q'}$. Consequently, if we multiply (3.1.34) by $\Gamma_\ell(t)$ and compute the $L^2(dt/t)$ norm, we get an upper bound of form $C_N \langle q - q'\rangle^{-N} e_{k\ell q'}$ with

$$e_{k\ell q'} = 2^{(k-\ell/2)\alpha_1} 2^{(k-\ell)\alpha_1'} c_{kq'}$$

with $(c_{kq'})_{kq'} \in \ell_k^2 \ell_{q'}^r$. To finish the proof, we must show that

$$(3.1.35) \qquad \sum_{k; \ell/2 > k \geqslant j-N_0} e_{k\ell q'} + \sum_{k \geqslant \ell/2} d_{jk\ell q'}$$

is smaller than the right hand side of (3.1.29) with $\widetilde{\alpha} = \alpha_1$, $\widetilde{\alpha}' = \alpha_1'$. The contribution of the second sum in (3.1.35) is smaller than the first two contributions in the right hand side of (3.1.32) i.e. to

$$C 2^{(j-\ell/2)\alpha_1} 2^{(j-\ell)\alpha_1'} \left[2^{j(d/2-\alpha_2')} 2^{(j-\ell/2)(\alpha_2'-\alpha_1-\alpha_1')} \widetilde{c}_{\ell q'} \right]$$

with $(\widetilde{c}_{q'})_{\ell q'} \in \ell_\ell^1 \ell_{q'}^r$, and the term between brackets is smaller than a sequence in $\ell_j^1 \ell_{q'}^r$, because of the assumption (3.1.25) $\alpha_2' > d/2, \alpha_1 + \alpha_1' < 0$. The first sum in (3.1.35) is smaller than

$$C 2^{(j-\ell/2)\alpha_1} 2^{(j-\ell)\alpha_1'} \left[\sum_{k; \ell/2 > k \geqslant j-N_0} 2^{(k-j)(\alpha_1+\alpha_1')} c_{kq'} \right]$$

and the term between brackets is a $\ell_j^2 \ell_{q'}^r$ sequence since $\alpha_1 + \alpha_1' < 0$. This concludes the proof in the case $u \in F_{\alpha_1, \alpha_1', r_1}^{s_1, s_1'}$.

To prove (i) and (ii) for $u \in \underline{H}_{\alpha_1, \alpha_1'}^{s_1, s_1'}$ we write instead of (3.1.26)

$$\phi_q \Delta_j R(u, v) = \sum_{\substack{k, k' \\ |k-k'| \leqslant 1 \\ k \geqslant j-N_0}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi}_{q'})[(\Delta_k u)(\phi_{q'} \Delta_{k'} v)].$$

The $L^2(dx)$ norm of the general term of the above sum is smaller than

$$C_N \langle q - q'\rangle^{-N} 2^{jd/2} \|\Delta_k u(t, \cdot)\|_{L^2(dx)} \|\phi_{q'} \Delta_{k'} v(t, \cdot)\|_{L^2(dx)}.$$

We multiply this expression by $\Gamma_\ell(t)$ and compute the $L^\infty(dt)$ norm in function of $\|\Delta_k u(t, \cdot)\|_{L^\infty(dt, L^2(dx))}$ and of $\|\phi_{q'} \Delta_{k'} v(t, \cdot)\Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}$. We get an upper bound given by (3.1.28) with a sequence $(c_{kq'})_{kq'}$ in $\ell_k^1 \ell_{q'}^{r_2}$ instead of $\ell_k^1 \ell_{q'}^r$. The rest of the proof is then identical as above. $\qquad\square$

We will also need:

PROPOSITION 3.1.6

(i) Let $\kappa_1 \in \mathbb{N}$, $\kappa_2 \in \mathbb{N}^d$, s, s', α, α' be real numbers satisfying

(3.1.36)
$$s + s' > 2\kappa_1 + |\kappa_2|, \quad s > 2\kappa_1 + \alpha'$$
$$s > 2\kappa_1 + \alpha + 2\alpha', \quad \alpha + \alpha' < d/2.$$

Let $m \in \mathbb{Z}^*, r \in [1, +\infty]$, and $w \in F_{\alpha,\alpha',r}^{s,s'}$ (resp. $w \in H_{\alpha,\alpha',r}^{s,s'}$). Then

(3.1.37)
$$R(w, (tD_t)^{\kappa_1}(D_x/t)^{\kappa_2}e^{im\theta}) \in F_{\alpha,\alpha',r}^{s-2\kappa_1,s'-\kappa_2}$$

(resp.

(3.1.38)
$$R(w, (tD_t)^{\kappa_1}(D_x/t)^{\kappa_2}e^{im\theta}) \in H_{\alpha,\alpha',r}^{s-2\kappa_1,s'-\kappa_2}).$$

(ii) Let $\kappa_1 \in \{0,1\}$, $|\kappa_2| = 1, s, s', \alpha, \alpha'$ be real numbers satisfying

(3.1.39)
$$s + s' > 2\kappa_1 + 1, \quad s > 2\kappa_1 + \alpha', \quad s > 2\kappa_1 - 2 + 2(\alpha + 2\alpha' + 2)$$
$$-\frac{d}{2} - 1 < \alpha + \alpha', \quad 2\alpha + 3\alpha' + 1 < \frac{d}{2}.$$

Then if $\beta = 2(\alpha + \alpha' + 1)$, $w \in F_{\alpha',\alpha',r}^{s,s'}$ (resp. $w \in H_{\alpha',\alpha',r}^{s,s'}$) one has

(3.1.40)
$$R(w, (tD_t)^{\kappa_1}(D_x/t)^{\kappa_2}e^{im\theta}) \in F_{\beta,\alpha',r}^{s-2\kappa_1,s'-1}$$

(resp.

(3.1.41)
$$R(w, (tD_t)^{\kappa_1}(D_x/t)^{\kappa_2}e^{im\theta}) \in H_{\beta,\alpha',r}^{s-2\kappa_1,s'-1}).$$

The same results hold true for $\widetilde{R}$ instead of R.

Proof. — We write with $f = (tD_t)^{\kappa_1}(D_x/t)^{\kappa_2}e^{im\theta}$

(3.1.42)
$$\phi_q \Delta_j R(w, f)(t, \cdot) = \sum_{\substack{k,k' \\ |k-k'| \leqslant 1}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi}_{q'})((\phi_{q'} \Delta_k w)(\Delta_{k'} f)).$$

We first treat case (i). Remark that since

$$tD_t(e^{im\theta}) = m\frac{tx^2}{4}e^{im\theta} = \frac{1}{m}\Big[\frac{D_x^2}{t} + im\Big]e^{im\theta}$$

we have

(3.1.43)
$$\Big\|\Delta_k((tD_t)^{\kappa_1}\Big(\frac{D_x}{t}\Big)^{\kappa_2}e^{im\theta})\Big\|_{L^\infty(dx)} \leqslant C\Big(1 + \frac{2^k}{\sqrt{t}}\Big)^{2\kappa_1}\Big(\frac{2^k}{t}\Big)^{|\kappa_2|}$$
$$\Big\|\Delta_k(tD_t)^{\kappa_1}\Big(\frac{D_x}{t}\Big)^{\kappa_2}e^{im\theta}\Big\|_{L^2(dx)} \leqslant C\Big(\frac{2^k}{t}\Big)^{d/2}\Big(1 + \frac{2^k}{\sqrt{t}}\Big)^{2\kappa_1}\Big(\frac{2^k}{t}\Big)^{|\kappa_2|}.$$

This shows in particular that it is enough to prove (3.1.37), (3.1.38) in the special case $\kappa_1 = \kappa_2 = 0$, to which one can reduce changing the values of s, s'. We deduce from (3.1.42) that for $p = 2$ (resp. $p = \infty$)

(3.1.44)
$$\|\phi_q \Delta_j R(w, f)\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))} \leqslant C\sum_{q'} \langle q - q'\rangle^{-N} d_{j\ell q'}$$

with

$$(3.1.45) \quad d_{j\ell q'} = \sum_{\substack{k,k' \\ j-N_0 \leqslant k < \ell-j \\ |k-k'| \leqslant 1}} 2^{j\frac{d}{2}} \|\phi_{q'}\Gamma_\ell(t)\Delta_k w\|_{L^p(\frac{dt}{t},L^2(dx))} \|\Delta_{k'} f\Gamma_\ell(t)\|_{L^\infty(L^2)}$$

$$+ \sum_{\substack{k,k' \\ k \geqslant j-N_0 \\ k \geqslant \ell-j \\ |k-k'| \leqslant 1}} \|\phi_{q'}\Gamma_\ell(t)\Delta_k w\|_{L^p(\frac{dt}{t},L^2(dx))} \|\Delta_{k'} f\Gamma_\ell(t)\|_{L^\infty(dtdx)}.$$

We estimate (3.1.45) in the following cases:

Case $j \geqslant \ell$. — In this case, only the second sum in (3.1.45) remains. It is bounded from above by

$$\sum_{k \geqslant j-N_0} 2^{-(k-\ell/2)s-(k-\ell)s'} c_{kq'} \leqslant C 2^{-(j-\ell/2)s-(j-\ell)s'} c'_{jq'}$$

for $\ell^2\ell^r$ sequences $(c_{kq'})_{kq'}, (c'_{jq'})_{jq'}$, since $s+s' > 0$. This gives the needed conclusion.

Case $\ell > j \geqslant \ell/2$. — Again, only the second sum of (3.1.45) contributes. We get the upper bound

$$\sum_{k=j-N_0}^{\ell} 2^{-(k-\ell/2)s+(k-\ell)\alpha'} c_{kq'} + 2^{-\ell s/2} c'_{\ell q'}$$

$$= 2^{-(j-\ell/2)s+(j-\ell)\alpha'} \left[\sum_{k=j-N_0}^{\ell} 2^{-(k-j)(s-\alpha')} c_{kq'} + 2^{(j-\ell)(s-\alpha')} c'_{\ell q'} \right]$$

for $\ell^2\ell^r$ sequences $(c_{kq'})_{kq'}, (c'_{\ell q'})_{\ell q'}$, and since $s > \alpha'$, the expression between brackets is a $\ell_j^2\ell_{q'}^r$ sequence, which gives the result.

Case $\ell/2 > j$. — The first sum in (3.1.45) is bounded from above, using (3.1.43), by

$$\sum_{k=j-N_0}^{\ell/2} 2^{jd/2} 2^{(k-\ell/2)\alpha+(k-\ell)(\alpha'+d/2)} c_{kq'} + \sum_{k=\ell/2}^{\ell-j} 2^{jd/2} 2^{-(k-\ell/2)s+(k-\ell)(\alpha'+d/2)} c_{kq'}$$

$$= 2^{(j-\ell/2)\alpha+(j-\ell)\alpha'} \left[\sum_{k=j-N_0}^{\ell/2} 2^{(j-k)(d/2-\alpha-\alpha')+(k-\ell/2)d} c_{kq'} \right.$$

$$\left. + \sum_{k=\ell/2}^{\ell-j} 2^{(j+k-\ell)(d/2-\alpha-\alpha')+(k-\ell/2)(\alpha+2\alpha'-s)} c_{kq'} \right]$$

and since $d/2 > \alpha+\alpha'$, $s > \alpha+2\alpha'$, the term between brackets is smaller than a $\ell_j^2\ell_{q'}^r$ sequence independent of ℓ. This gives the wanted estimate. We still have to study

the contribution to the second sum in (3.1.45). We get an upper bound

$$C2^{(j-\ell/2)\alpha+(j-\ell)\alpha'}\Big[\sum_{k=\ell-j}^{\ell}2^{-(j+k-\ell)(s-\alpha')+(j-\ell/2)(s-\alpha-2\alpha')}c_{kq'}$$
$$+\sum_{k>\ell}2^{-(k-\ell)(s+s')+(j-\ell/2)(s-\alpha-2\alpha')-j(s-\alpha')}c_{kq'}\Big].$$

Since $s>\alpha', s>\alpha+2\alpha', s+s'>0$, the term between brackets is in $\ell_j^2\ell_{q'}^r$. This concludes the proof of the proposition in case (i).

To treat case (ii), let us remark first that assumption (3.1.39) implies the first two relations (3.1.36). Moreover, conclusions (3.1.40), (3.1.41) coincide with (3.1.37), (3.1.38) when the frequency j in the left hand side of (3.1.44) satisfies $j\geqslant\ell/2$. Since, in case (i), the estimates of (3.1.44) when $j\geqslant\ell/2$ used only the first two relations (3.1.36), we see that we just need to study the case $j<\ell/2$. We can also reduce to $\kappa_1=0$, replacing s by $s-2\kappa_1$. Using (3.1.43) with $|\kappa_2|=1$, we estimate the first sum in (3.1.45) by

$$2^{(j-\ell/2)\beta+(j-\ell)\alpha'}\Big[\sum_{k=j}^{\ell/2}2^{(k-\ell/2)(\alpha+\alpha'+\frac{d}{2}+1)}2^{-(j-\ell/2)(2\alpha+3\alpha'+1-\frac{d}{2})}2^{-j}c_{kq'}$$
$$+\sum_{k=\ell/2}^{\ell-j}2^{(\alpha'+\frac{d}{2}+1-s)(k+j-\ell)}2^{(j-\ell/2)(s+2-2(\alpha+2\alpha'+2))}2^{-j}c_{kq'}\Big].$$

Since by (3.1.39), $\alpha+\alpha'+\frac{d}{2}+1>0$ and $2\alpha+3\alpha'+1-\frac{d}{2}<0$, the first sum inside the bracket is a $\ell_j^2\ell_{q'}^r$ sequence. When the coefficient of $(k+j-\ell)$ in the second sum is strictly positive, we also get that this sum is a $\ell_j^2\ell_{q'}^r$ sequence since, by (3.1.39), $s>-2+2(\alpha+2\alpha'+2)$. When the coefficient of $(k+j-\ell)$ is non-positive, we get for this sum an upper bound

$$(3.1.46)\qquad\qquad\qquad C2^{(j-\ell/2)(\frac{d}{2}-1-2\alpha-3\alpha'-\delta)}2^{-j}c'_{\ell q'}$$

with $(c'_{\ell q'})_{\ell q'}\in\ell_\ell^2\ell_{q'}^r$, $\delta>0$ when the coefficient of $k+j-\ell$ is zero, $\delta=0$ otherwise. Since $\frac{d}{2}-1-2\alpha-3\alpha'>0$, we see that the supremum in $\ell>2j$ of (3.1.46) is in $\ell_j^2\ell_{q'}^r$.

Let us estimate now the second sum in (3.1.45): using (3.1.43), we get the estimate

$$2^{(j-\ell/2)\beta+(j-\ell)\alpha'}\Big[\sum_{k=\ell-j}^{\ell}2^{-(k-\ell)(s-1-\alpha')}2^{-(j-\ell/2)(2(\alpha+2\alpha'+2)-s-2)}2^{-j(s-\alpha')}c_{kq'}$$
$$+\sum_{k=\ell}^{+\infty}2^{-(k-\ell)(s+s'-1)}2^{(j-\ell/2)(s+2-2(\alpha+2\alpha'+2))}2^{-j(s-\alpha')}c_{kq'}\Big].$$

When $s-1-\alpha'>0$, the first sum is smaller than

$$2^{-(j-\ell/2)(2(\alpha+2\alpha'+2)-s-2)}2^{-j}c'_{\ell-j,q'}$$

with $(c'_{\ell q'})_{\ell q'} \in \ell^2_\ell \ell^r_{q'}$, so smaller than a $\ell^2_j \ell^r_{q'}$ sequence using assumption (3.1.39) and $j \leqslant \ell/2$. When $s - 1 - \alpha' \leqslant 0$, we get an upper bound

$$2^{(j-\ell/2)(s+2-2(\alpha+2\alpha'+2))} 2^{-j(s-\alpha')+j\delta} c_{\ell q'},$$

with $\delta > 0$ arbitrarily small, which gives again a $\ell^2_j \ell^r_{q'}$ upper bound using $s > \alpha'$. For the second sum, we have $s + s' > 1$, whence again an upper bound in terms of a $\ell^2_j \ell^r_{q'}$ sequence. $\qquad\square$

We want to deduce from the preceding results a statement concerning products. Let us set the following definition.

DEFINITION 3.1.7. — Let s, s', α, α' be real numbers, $r \in [1, +\infty]$, $M \in \mathbb{N}$. Denote by $\frac{1}{\tilde{r}} = \min[1, \frac{1}{r} + \frac{1}{2}]$. We set

(3.1.47)
$$E^{s,s'}_{\alpha,\alpha',r} = H^{s,s'}_{\alpha,\alpha',\tilde{r}} \cap F^{s,s'+1/2}_{\alpha,\alpha',r}, \quad \underline{E}^{s,s'}_{\alpha,\alpha',r} = \underline{H}^{s,s'}_{\alpha,\alpha'} \cap F^{s,s'+1/2}_{\alpha,\alpha',r}$$

$$\widetilde{E}^{s,s'}_{\alpha,\alpha',r} = \widetilde{H}^{s,s'}_{\alpha,\alpha',\tilde{r}} \cap F^{s,s'+1/2}_{\alpha,\alpha',r}, \quad \underline{\widetilde{E}}^{s,s'}_{\alpha,\alpha',r} = \underline{\widetilde{H}}^{s,s'}_{\alpha,\alpha'} \cap F^{s,s'+1/2}_{\alpha,\alpha',r}$$

(3.1.48)
$$\mathcal{E}^{s,s'}_{\alpha,\alpha',r}(M) = \{u \in E^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, x^\gamma u \in E^{s,s'-|\gamma|}_{\alpha,\alpha',r}\}$$

$$\underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M) = \{u \in \underline{E}^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, x^\gamma u \in \underline{E}^{s,s'-|\gamma|}_{\alpha,\alpha',r}\}$$

(3.1.49)
$$\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M) = \{u \in \widetilde{E}^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, x^\gamma u \in \widetilde{E}^{s,s'-|\gamma|}_{\alpha,\alpha',r}\}$$

$$\underline{\widetilde{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',r}(M) = \{u \in \underline{\widetilde{E}}^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, x^\gamma u \in \underline{\widetilde{E}}^{s,s'-|\gamma|}_{\alpha,\alpha',r}\}.$$

COROLLARY 3.1.8. — Let $s, s', \alpha, \alpha', \sigma_1, \sigma_2, \sigma'_1, \sigma'_2$ be real numbers, $M \in \mathbb{N}$, $\lambda \in \mathbb{R}_+$ a parameter, satisfying

(3.1.50)
$$\alpha' > d/2, \quad s > \alpha' + \frac{\sigma_1 + \sigma_2}{2}, \quad M > s' - \lambda + \frac{d}{2}$$

$$\sigma_j \geqslant 0, \quad s - \sigma_j \neq d, \quad \sigma'_j \in [0,1], \ j = 1,2, \ s \geqslant d.$$

(i) Assume moreover

(3.1.51)
$$-(2\alpha' + 1) < \alpha < -(\alpha' + 1).$$

Then if $\gamma \in \mathbb{N}^d$, $|\gamma| \leqslant 2M - d - 1$, $u_1 \in \underline{\widetilde{\mathcal{E}}}^{s-\sigma_1, s'-\sigma'_1}_{\alpha+2,\alpha',\infty}(M)$, $u_2 \in \underline{\widetilde{\mathcal{E}}}^{s-\sigma_2, s'-\sigma'_2}_{\alpha,\alpha',\infty}(M)$ *and if*

(3.1.52)
$$s + s' > \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma'_1 + \sigma'_2}{2} + \frac{|\gamma|}{2} + \frac{d}{2}$$

we have that $x^\gamma u_1 u_2 \in \widetilde{E}^{\widehat{s}, \widehat{s}'(\gamma)}_{\beta,\alpha',1}$ *where*

(3.1.53)
$$\widehat{s} = s - \sigma_1 - \sigma_2, \ \beta = 2(\alpha + \alpha' + 1)$$

$$\widehat{s}'(\gamma) = s' - \max(\sigma'_1, \sigma'_2) - [1 + \lambda - s' + |\gamma| + \delta + \frac{d}{2}]_+$$

where $\delta > 0$ *is arbitrarily small.*

(ii) Assume (3.1.50), (3.1.52) *and*

(3.1.54)
$$\alpha + \alpha' < 0.$$

Then if $u_1 \in \underline{\widetilde{\mathcal{E}}}^{s-\sigma_1, s'-\sigma'_1}_{\alpha,\alpha',\infty}(M)$, $u_2 \in \underline{\widetilde{\mathcal{E}}}^{s-\sigma_2, s'-\sigma'_2}_{\alpha,\alpha',\infty}(M)$, *one has* $x^\gamma u_1 u_2 \in \widetilde{E}^{\widehat{s}, \widehat{s}'(\gamma)}_{\alpha,\alpha',1}$.

We prove first:

LEMMA 3.1.9. — *Let $u \in \underline{\widetilde{E}}_{\alpha,\alpha',r}^{s,s'}$ with $x^\gamma u \in \underline{\widetilde{E}}_{\alpha,\alpha',r}^{s,s'-|\gamma|}$ for $|\gamma| \leqslant d+1$. Then for any $\delta > 0$, $u \in \widetilde{E}_{\alpha,\alpha',1}^{s,s'-d-\delta}$.*

Proof. — Write for $q \in \mathbb{Z}^d$, $\ell \in \mathbb{N}$

$$
\|\phi_q \Delta_j u \Gamma_\ell(t)\|_{L^\infty(dt,L^2(dx))} \leqslant C \langle q \rangle^{-d-1} \sum_{|\gamma| \leqslant d+1} \|x^\gamma \phi_q \Delta_j u \Gamma_\ell(t)\|_{L^\infty(L^2)}
$$

(3.1.55)

$$
\leqslant C \langle q \rangle^{-d-1} \sum_{|\gamma| \leqslant d+1} \|\widetilde{\Delta}_{j,\gamma}(x^\gamma u)\Gamma_\ell(t)\|_{L^\infty(L^2)}
$$

where $\widetilde{\Delta}_{j,\gamma}$ is a family of Fourier multipliers defined by a cut-off satisfying conditions (3.1.15). If we use (2.2.5) for u and $x^\gamma u$ (replacing s' by $s' - |\gamma|$) in the right hand side of (3.1.55), and then interpolate with (2.2.5), we obtain that $u \in H_{\alpha,\alpha',1}^{s,s'-d-\delta}$. The fact that $u \in F_{\alpha,\alpha',1}^{s,s'+\frac{1}{2}-d-\delta}$ is proved in the same way. One proves similarly that $\|\phi_q \chi(D/\sqrt{t})u\|_{L^\infty} \leqslant C \langle q \rangle^{-d-1}$. This concludes the proof. $\qquad\square$

Proof of corollary 3.1.8. — We will prove at the same time assertions (i) and (ii). We set $\widetilde{\alpha} = \alpha + 2$ in case (i) and $\widetilde{\alpha} = \alpha$ in case (ii). We decompose

$$
(3.1.56) \qquad\qquad u_1 u_2 = T_{u_1} u_2 + T_{u_2} u_1 + R(u_1, u_2).
$$

Let $\gamma \in \mathbb{N}^d$, $|\gamma| \leqslant 2M - d - 1$. If $|\gamma| \leqslant M - d - 1$, we remark using notation (3.1.16) that $x^\gamma T_{u_1} u_2$ may be written as a linear combination of expressions of type $\widetilde{T}_{(x^{\gamma'} u_1)} u_2$ with $|\gamma'| \leqslant |\gamma|$. Actually, arguing by induction, only the case $|\gamma| = 1$ has to be considered. One then writes

$$
x T_{u_1} u_2 = \sum_{j \geqslant 1}([x, S_{j-1}]u_1)(\Delta_j u_2) + T_{(x u_1)} u_2
$$

and remarks that $[x, S_{j-1}] = \chi_j(D)$ for a function $\chi_j(\xi)$ satisfying the second condition (3.1.15).

By assumption, we have

$$
(3.1.57) \qquad\qquad x^{\gamma'} x^{\gamma''} u_1 \in \underline{\widetilde{E}}_{\widetilde{\alpha},\alpha',\infty}^{s-\sigma_1,s'-\sigma_1'-|\gamma'|-|\gamma''|}
$$

for $|\gamma''| \leqslant d+1$. By lemma 3.1.9, we deduce $x^{\gamma'} u_1 \in \widetilde{E}_{\widetilde{\alpha},\alpha',1}^{s-\sigma_1,s'-\sigma_1'-|\gamma'|-d-\delta} \subset \widetilde{H}_{\widetilde{\alpha},\alpha',1}^{s-\sigma_1,s'-\sigma_1'-|\gamma'|-d-\delta}$ for any $\delta > 0$. We apply proposition 3.1.3 (ii) and get

$$
(3.1.58) \qquad x^\gamma T_{u_1} u_2 \in E_{\alpha+(\widetilde{\alpha}+2\alpha')_+,\alpha',1}^{s-\sigma_2-(d-s+\sigma_1)_+,s'-\sigma_2'-[s'+\sigma_1'+|\gamma|+\delta+\frac{d}{2}]_+}.
$$

Since $s \geqslant d$, $\sigma_1' \in [0,1]$, $\sigma_1 \geqslant 0$ and, in case (i), $\alpha + (\widetilde{\alpha} + 2\alpha')_+ = \alpha + (\alpha + 2\alpha' + 2)_+ \geqslant \beta$, we see that this last space is contained inside $E_{\beta,\alpha',1}^{\widehat{s},\widehat{s}'(\gamma)}$ in case (i), and in $E_{\alpha,\alpha',1}^{\widehat{s},\widehat{s}'(\gamma)}$ in case (ii). When $|\gamma| \geqslant M - d$, we decompose $\gamma = \gamma_1 + \gamma_2$ with $|\gamma_1| = M - d - 1$, $|\gamma_2| \leqslant M$

and write $x^\gamma T_{u_1} u_2$ as a linear combination of terms $\widetilde{T}_{(x^{\gamma'_1} u_1)}(x^{\gamma'_2} u_2)$ with $|\gamma'_1| \leqslant |\gamma_1|$, $|\gamma'_2| \leqslant |\gamma_2|$. We get that $x^\gamma T_{u_1} u_2$ is then inside

$$(3.1.59) \qquad E^{s-\sigma_2-(d-s+\sigma_1)_+,\, s'-\sigma'_2-|\gamma_2|-[-s'+\sigma'_1+|\gamma_1|+\delta+\frac{d}{2}]_+}_{\alpha+(\widetilde{\alpha}+2\alpha')_+,\, \alpha',1}.$$

Since

$$|\gamma_2| + [-s' + \sigma'_1 + |\gamma_1| + \delta + \frac{d}{2}]_+ \leqslant [-s' + \lambda + 1 + |\gamma| + \delta + \frac{d}{2}]_+$$

as $-s' + \lambda + 1 + |\gamma_1| + \frac{d}{2} > 0$ by (3.1.50), we get that (3.1.59) is contained in $E^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1}$ in case (i) and in $E^{\widehat{s},\widehat{s}'(\gamma)}_{\alpha,\alpha',1}$ in case (ii).

The proof that $x^\gamma T_{u_2} u_1$ belongs to $E^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1}$ (resp. $E^{\widehat{s},\widehat{s}'(\gamma)}_{\alpha,\alpha',1}$) in case (i) (resp. in case (ii)) is similar: one has just to replace in (3.1.58), (3.1.59) the first lower index by $\widetilde{\alpha} + (\alpha + 2\alpha')_+$, which is larger than β (resp. α) in case (i) (resp. case (ii)).

Let us study $R(u_1, u_2)$. For $|\gamma| \leqslant 2M - d - 1$, we write $x^\gamma R(u_1, u_2)$ as a linear combination of terms of type $\widetilde{R}(x^{\gamma_1} u_1, x^{\gamma_2} u_2)$ with $|\gamma_1| + |\gamma_2| \leqslant |\gamma|$, $|\gamma_1| \leqslant M - d - 1$, $|\gamma_2| \leqslant M$. We will thus have $x^{\gamma_2} u_2 \in \underline{\widetilde{E}}^{s-\sigma_2,\, s'-\sigma'_2-|\gamma_2|}_{\alpha,\alpha',\infty}$ and by lemma 3.1.9 $x^{\gamma_1} u_1 \in \widetilde{E}^{s-\sigma_1,\, s'-\sigma'_1-|\gamma_1|-d-\delta}_{\widetilde{\alpha},\alpha',1} \subset \widetilde{H}^{s-\sigma_1,\, s'-\sigma'_1-|\gamma_1|-d-\delta}_{\widetilde{\alpha},\alpha',1}$. In case (i), we apply (i) of proposition 3.1.5. We check that for $\delta > 0$ small enough,

$$(3.1.60) \qquad \begin{aligned} 2s + 2s' - (\sigma_1 + \sigma_2) - (\sigma'_1 + \sigma'_2) - d - \delta - |\gamma| &> 0 \\ 2s - (\sigma_1 + \sigma_2) &> 2\alpha' \\ 2(\alpha + \alpha' + 1) &< 0 \end{aligned}$$

because of assumptions (3.1.50), (3.1.51), (3.1.52), i.e. all conditions (3.1.23) are satisfied. Consequently $\widetilde{R}(x^{\gamma_1} u_1, x^{\gamma_2} u_2)$ belongs to

$$E^{2s-(\sigma_1+\sigma_2)-d,\, 2s'-(\sigma'_1+\sigma'_2)-|\gamma|-\delta-\frac{d}{2}}_{2\alpha+2+d,\, 2\alpha'-d/2,1}$$

which is contained, since $s \geqslant d$ and $\alpha' > d/2$, inside $E^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1}$. This concludes the proof of the fact that $x^\gamma u_1 u_2 \in \widetilde{E}^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1}$ in case (i) since (3.1.51) implies $\frac{\beta}{2} + \alpha' > 0$ whence $\widetilde{E}^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1} = E^{\widehat{s},\widehat{s}'(\gamma)}_{\beta,\alpha',1}$ by lemma 3.1.2.

In case (ii), one has to check (3.1.25) to apply proposition 3.1.5 i.e. to verify conditions (3.1.60) with the last inequality replaced by $\alpha + \alpha' < 0$, $\alpha' > d/2$. This follows again from (3.1.50), (3.1.52), (3.1.54), and implies that $\widetilde{R}(x^{\gamma_1} u_1, x^{\gamma_2} u_2) \in E^{\widehat{s},\widehat{s}'(\gamma)}_{\alpha,\alpha',1}$.

To conclude the proof of the corollary, we still have to get the L^∞ estimate involved in the definition of $\widetilde{E}^{\widehat{s},\widehat{s}'(\gamma)}_{\alpha,\alpha',1}$ i.e. we must show that if $|\gamma| \leqslant 2M-d-1$ and $\chi \in C_0^\infty(\mathbb{R}^d)$, $\mathrm{Supp}\, \chi \subset B(0,1)$,

$$(3.1.61) \qquad (\|\phi_q(x)\chi(D/\sqrt{t})(x^\gamma u_1 u_2)\|_{L^\infty(dtdx)})_q \in \ell_q^1.$$

Consider $A(t, D) = \sum_{k; 2^k > 10\sqrt{t}} \Delta_k$ and split $\gamma = \gamma_1 + \gamma_2$, with the inequalities $|\gamma_1| \leqslant M - d - 1$, $|\gamma_2| \leqslant M$. We have

$$(3.1.62) \quad \phi_q(x)\chi(D/\sqrt{t})((x^{\gamma_1} u_1)A(t, D)(x^{\gamma_2} u_2))$$
$$= \sum_{\substack{k, k' \\ |k-k'| \leqslant N_0 \\ 2^k > 10\sqrt{t}}} \phi_q(x)\chi(D/\sqrt{t})[\Delta_{k'}(x^{\gamma_1} u_1)\Delta_k(x^{\gamma_2} u_2))].$$

for a large enough integer N_0. The general term in the sum (3.1.62) may be written

$$(3.1.63) \qquad \sum_{q'} \phi_q(x)\chi(D/\sqrt{t})\widetilde{\phi}_{q'}[\phi_{q'}\Delta_{k'}(x^{\gamma_1} u_1)\Delta_k(x^{\gamma_2} u_2)].$$

For fixed t, the $L^\infty(dx)$ norm of (3.1.63) is smaller than

$$(3.1.64) \qquad C_N \sum_{q'} \langle q - q' \rangle^{-N} t^{d/2} \|\phi_{q'}\Delta_{k'}(x^{\gamma_1} u_1)(t, \cdot)\|_{L^2} \|\Delta_k(x^{\gamma_2} u_2)(t, \cdot)\|_{L^2}.$$

The L^∞ norm of (3.1.62) is thus smaller than the q' sum of $C_N \langle q - q' \rangle^{-N}$ times

$$\sum_{k; \sqrt{t} < 2^k < t} t^{d/2} \left(\frac{2^k}{\sqrt{t}}\right)^{-2s + (\sigma_1 + \sigma_2)} \left(\frac{2^k}{t}\right)^{2\alpha'} c_{kq'}$$
$$+ \sum_{k; t < 2^k} t^{d/2} \left(\frac{2^k}{\sqrt{t}}\right)^{-2s + (\sigma_1 + \sigma_2)} \left(\frac{2^k}{t}\right)^{-2s' + (\sigma_1' + \sigma_2') + |\gamma| + d + \delta} c_{kq'}$$
$$\leqslant \sum_{k; \sqrt{t} < 2^k < t} t^{d/2 - \alpha'} \left(\frac{2^k}{\sqrt{t}}\right)^{-2s + (\sigma_1 + \sigma_2) + 2\alpha'} c_{kq'}$$
$$+ \sum_{k; t < 2^k} t^{\frac{d}{2} + \frac{\sigma_1 + \sigma_2}{2} - s} \left(\frac{2^k}{t}\right)^{-2(s+s') + (\sigma_1 + \sigma_2 + \sigma_1' + \sigma_2') + |\gamma| + d + \delta} c_{kq'}$$

for a positive δ and a $\ell_k^1 \ell_{q'}^1$ sequence $(c_{kq'})_{kq'}$. Since, by (3.1.50), $\alpha' > d/2$, $s > \alpha' + \frac{\sigma_1 + \sigma_2}{2} > \frac{d}{2} + \frac{\sigma_1 + \sigma_2}{2}$, and because of (3.1.52), we see that the last expression is smaller than $C \sum_k 2^{-k\delta'} c_{kq'}$, for some positive δ', i.e. the L^∞ norm of (3.1.62) is smaller than a ℓ_q^1 sequence. We are thus reduced to the study of

$$\|\phi_q[\chi(D/\sqrt{t})((x^{\gamma_1} u_1)(\mathrm{Id} - A(t, D))(x^{\gamma_2} u_2))]\|_{L^\infty}.$$

Repeating the preceding reasoning, we further reduce ourselves to

$$\|\phi_q[\chi(D/\sqrt{t})((\mathrm{Id} - A(t, D))(x^{\gamma_1} u_1)(\mathrm{Id} - A(t, D))(x^{\gamma_2} u_2))]\|_{L^\infty}.$$

We get that this sequence is in ℓ_q^1 applying lemma 3.1.2 with $\psi(t, \xi) = 1 - A(t, \xi)$. This concludes the proof. $\qquad \square$

3.2. Conjugation by an oscillatory exponential

Our aim in this section is to study expressions of type

$$(3.2.1) \qquad e^{-im\theta} T_w((1-\chi)(D/\sqrt{t})e^{im\theta})$$

where $m \in \mathbb{Z}^*$ and $w \in \widetilde{E}^{s,s'}_{\alpha,\alpha',r}$. We want to show that (3.2.1) has essentially the same smoothness as w. This will follow from the fact that making act a $D_x/\sqrt{t}$ or a D_x/t derivative on (3.2.1) gives either a term like (3.2.1) – with w replaced by $\frac{D_x}{\sqrt{t}}w$ or $\frac{D_x}{t}w$ – or two contributions coming from the action of the derivative on the exponentials, that will compensate each other modulo remainders.

Let us introduce some notations:

DEFINITION 3.2.1. — For $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}_+$ one denotes by Σ^μ_ν the space of smooth functions $(t,\xi) \mapsto a(t,\xi)$ defined on $[1,+\infty[\times \mathbb{R}^d$ with values in $\mathbb{C}$, satisfying the following inequalities

$$(3.2.2) \qquad |(t\partial_t)^k(\sqrt{t}\partial_\xi)^\gamma a(t,\xi)| \leqslant C_{k\gamma}\Big(\frac{\langle\xi\rangle}{t}\Big(1+\frac{\langle\xi\rangle}{t}\Big)^{-1}\Big)^\nu\Big(1+\frac{\langle\xi\rangle}{\sqrt{t}}\Big)^{\mu-|\gamma|},$$

for any γ when $|\xi| > c\sqrt{t}$, and for $\gamma = 0$ when $|\xi| \leqslant c\sqrt{t}$.

We shall also denote by S^μ the space of usual symbols of order μ i.e. smooth functions b satisfying for any $\gamma \in \mathbb{N}^d$

$$(3.2.3) \qquad |\partial^\gamma_x b(x)| \leqslant C_\gamma \langle x\rangle^{-|\gamma|+\mu}.$$

PROPOSITION 3.2.2. — *Let* $s, s', \alpha, \alpha', r, \mu, \rho, \nu$ *be real numbers satisfying*

$$(3.2.4) \qquad \begin{aligned} r \in [1,+\infty], \quad \alpha+\alpha' < d/2, \quad \alpha' > d/2 \\ \rho > \mu \geqslant 0, \quad \nu \geqslant 0. \end{aligned}$$

Let $a(t,\xi) \in \Sigma^{-\rho}_\nu$ *and* $b(x) \in S^\mu$. *Assume moreover that for some* $c > 0$,

$$(3.2.5) \qquad \operatorname{Supp} a \subset \{(t,\xi); |\xi| > c\sqrt{t}\}.$$

and that

$$(3.2.6) \qquad \Big(\frac{\alpha}{2}+\alpha' > 0,\ \alpha+\alpha'+\frac{d}{2} > 0\Big) \text{ or } \Big(\frac{\alpha}{2}+\alpha' < 0,\ \alpha+\alpha'+\frac{d}{2}+\nu > 0\Big).$$

Then if $w \in \widetilde{E}^{s,s'}_{\alpha,\alpha',r}$, *we have for any* $m \in \mathbb{Z}^*$

$$(3.2.7) \qquad e^{-im\theta}b(\sqrt{t}x)T_w(a(t,D)e^{im\theta}) \in \widetilde{E}^{s+\rho-\mu,s'}_{\alpha+\zeta,\alpha',r} \subset \widetilde{E}^{s+\rho-\mu,s'}_{\alpha,\alpha',r}$$

where ζ *is any number in* $]0,\min(\rho-\mu,d,\frac{d}{2}-\alpha-\alpha')[$.

We write by definition of paraproduct

$$(3.2.8) \qquad e^{-im\theta}b(\sqrt{t}x)T_w[a(t,D)e^{im\theta}] = \sum_k (S_{k-1}w)g_k$$

$$(3.2.9) \qquad g_k = e^{-im\theta}b(\sqrt{t}x)a(t,D)\Delta_k(e^{im\theta}).$$

The assumption (3.2.5) implies that in the right hand side of (3.2.8) the summation is taken for k satisfying $2^k > c\sqrt{t}$. We begin by the following lemma:

LEMMA 3.2.3. — *For any $N \in \mathbb{N}$, one has the following estimates, uniformly in $t \geqslant 1$, k with $2^k > \sqrt{t}$:*

$$(3.2.10) \quad \|\Delta_j g_k(t, \cdot)\|_{L^2}$$

$$\leqslant C_N 2^{-j\frac{d}{2}} \left(\frac{2^{j+k}}{t}\right)^d \left(1 + \frac{2^{j+k}}{t}\right)^{-N} \left(\frac{2^k}{\sqrt{t}}\right)^{-\rho+\mu} \left[\frac{2^k}{t}\left(1 + \frac{2^k}{t}\right)^{-1}\right]^\nu$$

$$(3.2.11) \quad \|S_j g_k(t, \cdot)\|_{L^\infty}$$

$$\leqslant C_N \left(\frac{2^{j+k}}{t}\right)^d \left(1 + \frac{2^{j+k}}{t}\right)^{-d} \left(\frac{2^k}{\sqrt{t}}\right)^{-\rho+\mu} \left[\frac{2^k}{t}\left(1 + \frac{2^k}{t}\right)^{-1}\right]^\nu.$$

Proof. — Compute the Fourier transform of $b(\sqrt{t}x)e^{-im\theta}$ as the oscillatory integral

$$(3.2.12) \qquad \int e^{-ix\xi - imtx^2/4} b(\sqrt{t}x)\, dx = \frac{1}{t^{d/2}} e^{i\xi^2/mt} B(\xi/\sqrt{t})$$

where

$$(3.2.13) \qquad B(\eta) = \int e^{-imx^2/4} b(x - 2\eta/m)\, dx.$$

If $L = (1 + m^2 x^2)^{-1}(1 - 2mx \cdot D_x)$, so that $L(e^{-imx^2/4}) = e^{-imx^2/4}$, we have

$$|({}^t L)^k [b(x - 2\eta/m)]| \leqslant C_k \langle x \rangle^{-k} (1 + |x - 2\eta/m|)^\mu$$

whence, integrating by parts in (3.2.13), the estimate $B(\eta) \leqslant C\langle \eta \rangle^\mu$. Treating the derivatives in the same way, we get that $B \in S^\mu$. We now write the Fourier transform of g_k as a multiple of

$$\mathcal{F}(e^{-im\theta} b(\sqrt{t}x)) * \left[a(t, \eta)\varphi(2^{-k}\eta)\frac{e^{-i\eta^2/mt}}{t^{d/2}}\right]$$

that is, using (3.2.12), (3.2.13),

$$(3.2.14) \qquad e^{i\xi^2/mt} \left(\frac{2^k}{t}\right)^d \int e^{-2i\frac{\xi\eta}{mt}2^k} B\left(\frac{\xi - 2^k\eta}{\sqrt{t}}\right) \varphi(\eta) a(t, 2^k\eta)\, d\eta.$$

Integrating by parts in η we obtain an estimate of the modulus of (3.2.14) by

$$C\left(\frac{2^k}{t}\right)^d \left(1 + \frac{2^k|\xi|}{t}\right)^{-N} \left[\frac{2^k}{t}\left(1 + \frac{2^k}{t}\right)^{-1}\right]^\nu \left(\frac{2^k}{\sqrt{t}}\right)^{-\rho}$$

$$\times \sum_{N'=0}^{N} \int \left(1 + \frac{|\xi - 2^k\eta|}{\sqrt{t}}\right)^{\mu-N'} \left(\frac{2^k}{\sqrt{t}}\right)^{N'} \varphi_1(\eta)\, d\eta,$$

where we used (3.2.2), $B \in S^\mu$, $2^k > c\sqrt{t}$, and where φ_1 has the same support as φ. Consider first N' with $\mu - N' \leqslant 0$. When the integration is made over a domain

$|\xi - 2^k\eta| > c2^k$ for a small $c > 0$, we get an upper bound

$$(3.2.15) \qquad C\left(\frac{2^k}{t}\right)^d \left(1 + \frac{2^k|\xi|}{t}\right)^{-N} \left[\frac{2^k}{t}\left(1 + \frac{2^k}{t}\right)^{-1}\right]^\nu \left(\frac{2^k}{\sqrt{t}}\right)^{\mu-\rho}.$$

On the other hand, the contribution of the integration for $|\xi - 2^k\eta| < c2^k$ gives, since this implies $|\xi| \sim 2^k$, and since $-2N + N' \leqslant -N$,

$$C\left(\frac{2^k}{t}\right)^d \left[\frac{2^k}{t}\left(1 + \frac{2^k}{t}\right)^{-1}\right]^\nu \left(\frac{2^k}{\sqrt{t}}\right)^{-\rho-N}$$

which is still of form (3.2.15) changing the value of N.

On the other hand, for N' such that $\mu - N' > 0$, we get immediately an upper bound of type (3.2.15), changing the value of N.

Inequality (3.2.10) (resp. (3.2.11)) follows from estimate (3.2.15) computing the L^2 (resp. L^1) norm of $\widehat{\Delta_j g_k}(\xi)$ over a ring (resp. a ball) of radius 2^j. $\qquad\square$

Proof of proposition 3.2.2. — We decompose (3.2.8) as

$$\sum_{k;2^k > c\sqrt{t}} T_{g_k}(S_{k-1}w) + \sum_{k;2^k > c\sqrt{t}} R(g_k, S_{k-1}w) + \sum_{k;2^k > c\sqrt{t}} T_{S_{k-1}w}g_k$$

and we shall estimate successively for a fixed j

$$(3.2.16) \qquad\qquad \sum_{k;2^k > c\sqrt{t}} \Delta_j(T_{g_k}(S_{k-1}w))$$

$$(3.2.17) \qquad\qquad \sum_{k;2^k > c\sqrt{t}} \Delta_j(R(g_k, S_{k-1}w))$$

$$(3.2.18) \qquad\qquad \sum_{k;2^k > c\sqrt{t}} \Delta_j(T_{S_{k-1}w}g_k).$$

The assumptions on a, b of proposition 3.2.2 will enter the proof only through the estimates (3.2.10), (3.2.11). Since the parameters ρ and μ appear in the right hand sides of these inequalities only in the difference $\rho - \mu$, we can without reducing the generality assume $\mu = 0$, $\rho > 0$.

Estimate of (3.2.16). — We write the product of (3.2.16) with a function ϕ_q as

$$(3.2.19) \qquad \sum_{\substack{j' \\ |j-j'| \leqslant N_0}} \sum_{\substack{k \\ 2^k > c\sqrt{t} \\ k \geqslant j-N_0}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi}_{q'})((S_{j'-1}g_k)\phi_{q'}\Delta_{j'}S_{k-1}w)$$

for a large enough integer N_0. We multiply by $\Gamma_\ell(t)$ and compute the $L^p(\frac{dt}{t}, L^2(dx))$ norm for $p = 2$ or $p = \infty$ of (3.2.19). We get an upper bound

$$(3.2.20) \qquad \sum_{\substack{j' \\ |j-j'| \leqslant N_0}} \sum_{\substack{k \\ 2^k > c2^{\ell/2} \\ k \geqslant j-N_0}} \sum_{q'} C_N \langle q - q' \rangle^{-N} \|\Gamma_\ell(t)(S_{j'-1}g_k)\|_{L^\infty(dtdx)}$$

$$\times \|\Gamma_\ell(t)\phi_{q'}\Delta_{j'}S_{k-1}w\|_{L^p(\frac{dt}{t}, L^2(dx))}.$$

The last norm in the above formula is bounded from above by

$$(3.2.21) \qquad C2^{(j-\ell/2)\alpha+(j-\ell)\alpha'}(1+2^{j-\ell/2})^{-s-\alpha}(1+2^{j-\ell})^{-\tilde{s}'-\alpha'}c_{jq'}$$

with $\tilde{s}' = s'$ if $p = \infty$ and $\tilde{s}' = s' + 1/2$ if $p = 2$, $(c_{jq'})_{jq'}$ in the unit ball of $\ell_j^2\ell_{q'}^r$. On the other hand, we have by (3.2.11)

$$(3.2.22) \qquad \sum_{\substack{k \\ k \geqslant \frac{\ell}{2}-N_0 \\ k \geqslant j-N_0}} \|\Gamma_\ell(t)S_{j'-1}g_k\|_{L^\infty(dtdx)}$$

$$\leqslant \sum_{\substack{k \\ k \geqslant \frac{\ell}{2}-N_0 \\ k \geqslant j-N_0}} 2^{(j+k-\ell)d}(1+2^{j+k-\ell})^{-d}2^{-(k-\ell/2)\rho}(2^{k-\ell}(1+2^{k-\ell})^{-1})^\nu.$$

Denote by ζ' any positive number with $\zeta' < \min[\rho, d]$. When $\ell/2 \geqslant j$, this last sum is smaller than

$$\sum_{k > \ell-j} 2^{-(k-\ell/2)\rho} + \sum_{k;\ell-j \geqslant k \geqslant \ell/2-N_0} 2^{(j+k-\ell)d-(k-\ell/2)\rho} \leqslant C2^{(j-\ell/2)\zeta'}.$$

When $j > \ell/2$, we have the estimate

$$\sum_{k \geqslant j-N_0} 2^{-(k-\ell/2)\rho} \leqslant C2^{-(j-\ell/2)\rho}.$$

We see that (3.2.22) is smaller than

$$(3.2.23) \qquad 2^{(j-\ell/2)\zeta'}(1+2^{j-\ell/2})^{-\rho-\zeta'}.$$

Consequently, (3.2.20), (3.2.21), (3.2.22), (3.2.23) show that (3.2.16) belongs to $E^{s+\rho,s'}_{\alpha+\zeta',\alpha',r}$.

Estimate of (3.2.17). — We write the product of (3.2.17) with a function ϕ_q as

$$(3.2.24) \qquad \sum_{\substack{j',j'' \\ |j'-j''|\leqslant N_0 \\ j' \geqslant j-N_0}} \sum_{\substack{k \\ 2^k > c\sqrt{t} \\ k \geqslant j''-N_0}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi}_{q'})((\Delta_{j'}g_k)(\phi_{q'}\Delta_{j''}S_{k-1}w))$$

for a large enough integer N_0. We multiply by $\Gamma_\ell(t)$ and compute the $L^p(\frac{dt}{t}, L^2(dx))$ norm for $p = 2$ or $p = \infty$ of (3.2.24). We get an upper bound

$$(3.2.25) \qquad \sum_{\substack{j',j'' \\ |j'-j''|\leqslant N_0 \\ j' \geqslant j-N_0}} \sum_{\substack{k \\ k \geqslant \ell/2-N_0 \\ k \geqslant j''-N_0}} \sum_{q'} C_N \langle q-q' \rangle^{-N}$$

$$\times 2^{jd/2}\|\Delta_{j'}g_k\|_{L^\infty(dt,L^2(dx))}\|\Gamma_\ell(t)\phi_{q'}\Delta_{j''}S_{k-1}w\|_{L^p(\frac{dt}{t},L^2(dx))}.$$

for a large enough N_0. We use (3.2.10) to estimate the first norm in (3.2.25) and remind that the second norm is bounded from above by (3.2.21) with j replaced by

$j'' \sim j'$. Consequently, (3.2.25) is smaller than

$$(3.2.26) \quad \sum_{\substack{j' \\ j' \geqslant j - N_0}} \sum_{\substack{k \\ k \geqslant \ell/2 - N_0 \\ k \geqslant j' - N_0}} \sum_{q'} C_N \langle q - q' \rangle^{-N} 2^{(j'+k-\ell)d + (j-j')d/2} (1 + 2^{j'+k-\ell})^{-N}$$

$$\times 2^{-(k-\ell/2)\rho} [2^{k-\ell}(1 + 2^{k-\ell})^{-1}]^{\nu} 2^{(j'-\ell/2)\alpha} 2^{(j'-\ell)\alpha'} c_{j'q'}$$

since the terms in $s, \tilde{s}'$ of (3.2.21) can always be compensated for by $(1 + 2^{j'+k-\ell})^{-N}$ as $k \geqslant \frac{\ell}{2} - N_0$. This expression is smaller than

$$\sum_{\substack{j' \\ j' \geqslant j - N_0}} \sum_{q'} C_N \langle q - q' \rangle^{-N} c_{j'q'} 2^{(j-j')d/2 + (j'-\ell/2)\alpha + (j'-\ell)\alpha'} (1 + 2^{j'-\ell/2})^{-N+d}$$

$$\times \sum_{\substack{k \\ k \geqslant \ell/2 - N_0 \\ k \geqslant j' - N_0}} 2^{(j'+k-\ell)d} (1 + 2^{j'+k-\ell})^{-d} 2^{-(k-\ell/2)\rho} [2^{k-\ell}(1 + 2^{k-\ell})^{-1}]^{\nu}$$

$$\leqslant \sum_{\substack{j' \\ j' \geqslant j - N_0}} \sum_{q'} C_N \langle q - q' \rangle^{-N} c_{j'q'} 2^{(j-j')d/2 + (j'-\ell/2)\alpha + (j'-\ell)\alpha'}$$

$$\times 2^{(j'-\ell/2)\zeta'} (1 + 2^{j'-\ell/2})^{-\rho - \zeta' - N + d}$$

by the estimate (3.2.23) of (3.2.22). If we take $\zeta < \min[\zeta', \frac{d}{2} - \alpha - \alpha']$, we obtain an upper bound of type

$$C c'_{jq} 2^{(j-\ell/2)(\alpha+\zeta) + (j-\ell)\alpha'} (1 + 2^{j-\ell/2})^{-N'}$$

for any $N' \in \mathbb{N}$, with $(c'_{jq})_{jq}$ in $\ell_j^2 \ell_q^r$. This shows that (3.2.17) is in $E_{\alpha+\zeta,\alpha',r}^{+\infty,+\infty}$.

Estimate of (3.2.18). — We write the product of (3.2.18) with a function ϕ_q as

$$(3.2.27) \quad \sum_{\substack{j' \\ |j-j'| \leqslant N_0}} \sum_{\substack{k \\ 2^k > c\sqrt{t}}} \sum_{q'} (\phi_q \Delta_j \widetilde{\phi_{q'}})((\phi_{q'} S_{j'-1} S_{k-1} w)(\Delta_{j'} g_k))$$

for a large enough integer N_0. We multiply by $\Gamma_\ell(t)$ and compute the $L^p(\frac{dt}{t}, L^2(dx))$ norm for $p = 2$ or $p = \infty$. We get the upper bound

$$(3.2.28) \quad \sum_{\substack{j' \\ |j-j'| \leqslant N_0}} \sum_{\substack{k \\ k \geqslant \ell/2 - N_0}} \sum_{q'} C_N \langle q - q' \rangle^{-N} \|\phi_{q'} S_{j'-1} S_{k-1} w \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty(dx))}$$

$$\times \|\Delta_{j'} g_k \Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}.$$

We consider first the case $\alpha/2 + \alpha' > 0$, $\alpha + \alpha' + d/2 > 0$. Let us show that for some $\mu_0 \in \mathbb{N}$ and some sequence $(c_{jq'})_{jq'}$ in $\ell_j^2 \ell_{q'}^r$ we have the estimate

$$(3.2.29) \quad \|\phi_{q'} S_{j'-1} S_{k-1} w \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty(dx))}$$

$$\leqslant C c_{jq'} 2^{j'd/2 + (j'-\ell/2)\alpha + (j'-\ell)\alpha'} (1 + 2^{j'-\ell/2})^{\mu_0}.$$

We write

$$\|\phi_{q'}S_{j'-1}S_{k-1}w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^\infty(dx))}$$
$$\leqslant C_N \sum_{q''}\sum_{k'\leqslant j'-2}\langle q'-q''\rangle^{-N}2^{k'd/2}\|\phi_{q''}\Delta_{k'}w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2(dx))}.$$

When $j' < \ell/2$ we get an estimate of type

$$\sum_{k'\leqslant j'-2}2^{k'd/2+(k'-\ell/2)\alpha+(k'-\ell)\alpha'}c'_{k'q'}$$

for some $\ell^2_{k'}\ell^r_{q'}$ sequence $(c'_{k'q'})_{k'q'}$. Since $\frac{d}{2}+\alpha+\alpha' > 0$, (3.2.29) follows. When $j' > \ell/2$, we estimate the left hand side of (3.2.29) by

$$\sum_{k'<\ell/2}2^{k'd/2+(k'-\ell/2)\alpha+(k'-\ell)\alpha'}c'_{k'q'} + \sum_{k';\ell/2\leqslant k'}2^{k'd/2+(k'-\ell)\alpha'-(k'-\ell/2)\mu_1}c'_{k'q'}2^{(j'-\ell/2)\mu_2}$$

for large enough μ_1 and μ_2. This is smaller than

$$2^{\frac{\ell}{2}\frac{d}{2}-\frac{\ell}{2}\alpha'+(j'-\frac{\ell}{2})\mu_2}c''_{\ell q'} = 2^{j'\frac{d}{2}+(j'-\ell)\alpha'}c''_{\ell q'}2^{(j'-\frac{\ell}{2})(\mu_2-\alpha'-\frac{d}{2})}$$

for some $\ell^2_\ell\ell^r_{q'}$ sequence $(c''_{\ell q'})_{\ell q'}$, whence (3.2.29).

Next, we compute from (3.2.10)

$$\sum_{k\geqslant\ell/2-N_0}\|\Delta_{j'}g_k\Gamma_\ell(t)\|_{L^\infty(dt,L^2)}$$

(3.2.30)
$$\leqslant C2^{-j'd/2}\sum_{k\geqslant\ell/2-N_0}2^{(j'+k-\ell)d}(1+2^{j'+k-\ell})^{-N}2^{-(k-\ell/2)\rho}$$
$$\leqslant C2^{-j'd/2}2^{(j'-\ell/2)\zeta'}(1+2^{j'-\ell/2})^{-N+d-\zeta'}$$

where ζ' is any number $\zeta' < \min[\rho, d]$. Then (3.2.29) and (3.2.30) show that (3.2.28) is smaller than

$$Cc_{jq}2^{(j-\ell/2)(\alpha+\zeta')}2^{(j-\ell)\alpha'}(1+2^{j-\ell/2})^{-N'}$$

for any N' i.e. the contribution of (3.2.18) is in $E^{+\infty,+\infty}_{\alpha+\zeta',\alpha',r}$.

Let us now study the case $\frac{\alpha}{2}+\alpha' < 0$, $\alpha+\alpha'+\frac{d}{2}+\nu > 0$. We have

(3.2.31)
$$\sum_{k\geqslant\ell/2-N_0}\|\Delta_{j'}g_k\Gamma_\ell(t)\|_{L^\infty(dt,L^2)}$$
$$\leqslant C2^{-j'\frac{d}{2}}\sum_{k\geqslant\ell/2-N_0}2^{(j'+k-\ell)d}(1+2^{j'+k-\ell})^{-N}2^{-(k-\frac{\ell}{2})\rho}[2^{k-\ell}(1+2^{k-\ell})^{-1}]^\nu.$$

When $j' < \ell/2$, we estimate the sum by

$$\sum_{k;\ell/2-N_0\leqslant k<\ell-j'}2^{(j'+k-\ell)d}2^{-(k-\ell/2)\rho-j'\nu} + \sum_{k;k\geqslant\ell-j'}2^{-(j'+k-\ell)(N-d)-(k-\ell/2)\rho+(k-\ell)\nu}$$
$$\leqslant C2^{(j'-\ell/2)\zeta'-j'\nu}$$

with $\zeta' < \min(\rho, d)$. When $j' > \ell/2$, the sum is smaller than

$$\sum_{k;k\geqslant\ell/2-N_0} 2^{-(j'+k-\ell)(N-d)-(k-\ell/2)\rho+(k-\ell)\nu} \leqslant C2^{-(j'-\ell/2)N'-j'\nu}$$

for a convenient N'. Consequently, (3.2.31) is smaller than

$$(3.2.32) \qquad C_{N'}2^{(j'-\ell/2)\zeta'}\left(1+2^{j'-\ell/2}\right)^{-N}2^{-j'(\nu+d/2)}$$

for any N. If we use (3.1.19), we get also

$$(3.2.33) \qquad \|\phi_{q'}S_{j'-1}S_k w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^\infty(dx))} \leqslant C(1+2^{j'-\ell/2})^{\mu_0}c_{jq}$$

with $(c_{jq})_{jq}$ in $\ell_j^\infty\ell_q^r$. Combining with (3.2.32), we obtain that (3.2.28) is smaller than

$$C_N 2^{(j-\ell/2)(\alpha+\zeta')+(j-\ell)\alpha'}\left(1+2^{j-\ell/2}\right)^{-N}c_{jq\ell}$$

with $c_{jq\ell} = 2^{\ell(\frac{\alpha}{2}+\alpha')}2^{-j(\nu+\frac{d}{2}+\alpha+\alpha')}c_{jq}$. Since $\frac{\alpha}{2}+\alpha' < 0$, $\nu+\frac{d}{2}+\alpha+\alpha' > 0$, this sequence is smaller than a $\ell_j^2\ell_q^r$ sequence independent of ℓ. This shows that contribution (3.2.18) is in $E_{\alpha+\zeta',\alpha',r}^{+\infty,+\infty}$.

We have proved up to now that (3.2.7) belongs to $E_{\alpha+\zeta,\alpha',r}^{s+\rho,s'} \subset E_{\alpha,\alpha',r}^{s+\rho,s'}$. We still have to show that we can replace E by $\widetilde{E}$.

When $\frac{\alpha}{2}+\alpha' > 0$, lemma 3.1.2 together with the assumption $\alpha' > d/2$, shows that these two spaces are identical. Let us assume now that $\frac{\alpha}{2}+\alpha' < 0$. By (3.2.8) we have to estimate the quantity

$$(3.2.34) \quad \phi_q\chi(D/\sqrt{t})\Big(\sum_{k;2^k>c\sqrt{t}} S_{k-1}wg_k\Big)$$
$$= \sum_{k;2^k>c\sqrt{t}} \phi_q\chi(D/\sqrt{t})[(\chi(D/10\sqrt{t})S_{k-1}w)g_k]$$
$$+ \sum_{k;2^k>c\sqrt{t}} \phi_q\chi(D/\sqrt{t})[((1-\chi(D/10\sqrt{t}))S_{k-1}w)g_k]$$

with $\chi \in C_0^\infty(\mathbb{R}^d)$, Supp $\chi \subset B(0,1)$, $\chi \equiv 1$ on $B(0,1/2)$. The first term in the right hand side may be written

$$(3.2.35) \qquad \sum_k\sum_{q'}(\phi_q\chi(D/\sqrt{t})\widetilde{\phi}_{q'})((\phi_{q'}\chi(D/10\sqrt{t})S_{k-1}w)g_k).$$

Since $w \in \widetilde{E}_{\alpha,\alpha',r}^{s,s'}$, we have, by lemma 3.1.2

$$(3.2.36) \qquad \|\phi_{q'}\chi(D/10\sqrt{t})S_{k-1}w\|_{L^\infty(dtdx)} \leqslant Cc_{q'}$$

for a constant C and a $\ell_{q'}^r$ sequence $(c_{q'})_{q'}$ independent of k. Moreover, by (3.2.11) and the assumption $\rho-\mu > 0$, $\sum_{k;2^k>c\sqrt{t}}\|g_k\|_{L^\infty(dtdx)} \leqslant C$. We thus deduce from

(3.2.35), (3.2.36) that the first term in the right hand side of (3.2.34) has L^∞ norm in ℓ^r_q. The last sum in (3.2.34) may be written

$$(3.2.37) \qquad \sum_k \sum_{q'} (\phi_q \chi(D/\sqrt{t})\widetilde{\phi}_{q'})[(\phi_{q'}(1-\chi)(D/10\sqrt{t})S_{k-1}w)(1-\chi)(D/\sqrt{t})g_k].$$

as follows from inspection of the supports of the Fourier transforms. We estimate

$$(3.2.38) \quad \|(\phi_{q'}(1-\chi)(D/10\sqrt{t}))S_{k-1}w(t,\cdot)\|_{L^\infty(dx)}$$
$$\leqslant \sum_{q''} \sum_{k';c\sqrt{t}<2^{k'}<2^{k-1}} C_N \langle q'-q''\rangle^{-N} 2^{k'd/2}\|\phi_{q''}\Delta_{k'}w(t,\cdot)\|_{L^2}.$$

When $2^k < t$ we get an upper bound

$$\sum_{k';c\sqrt{t}<2^{k'}<2^{k-1}} 2^{k'd/2}\left(\frac{2^{k'}}{\sqrt{t}}\right)^{-s}\left(\frac{2^{k'}}{t}\right)^{\alpha'} c_{k'q'} \leqslant \sum_{k';c\sqrt{t}<2^{k'}<2^{k-1}} 2^{k'(d/2-\alpha')}\left(\frac{2^{k'}}{\sqrt{t}}\right)^{\mu_0} c_{k'q'}$$

where $(c_{k'q'})_{k'q'}$ is in $\ell^2_{k'}\ell^r_{q'}$ and μ_0 is chosen large enough. If $\mu_0 + \frac{d}{2} - \alpha' > 0$, we get an upper bound of type

$$C 2^{k(d/2-\alpha')} c'_{kq'} \left(\frac{2^k}{\sqrt{t}}\right)^{\mu_0} \leqslant C\widetilde{c}_{q'}\left(\frac{2^k}{\sqrt{t}}\right)^{\mu_0}$$

where $(c'_{kq'})_{kq'} \in \ell^2_k\ell^r_{q'}, (\widetilde{c}_{q'})_{q'} \in \ell^r$ independent of k,t.

When $2^k > t$, we control (3.2.38) by

$$\sum_{k';c\sqrt{t}<2^{k'}<t} 2^{k'(d/2-\alpha')}\left(\frac{2^{k'}}{\sqrt{t}}\right)^{\mu_0} c_{k'q'} + \sum_{k';t<2^{k'}<2^{k-1}} 2^{k'd/2}\left(\frac{2^{k'}}{\sqrt{t}}\right)^{-s}\left(\frac{2^{k'}}{t}\right)^{-s'} c_{k'q'}$$

which is also controlled by $C\widetilde{c}_{q'}\left(\frac{2^k}{\sqrt{t}}\right)^{\mu_0}$. We thus have

$$\|\phi_{q'}(1-\chi)(D/10\sqrt{t})S_{k-1}w(t,\cdot)\|_{L^\infty(dx)} \leqslant C\widetilde{c}_{q'}\left(\frac{2^k}{\sqrt{t}}\right)^{\mu_0}$$

for some μ_0 and some $\ell^r_{q'}$ sequence $(\widetilde{c}_{q'})_{q'}$. Moreover we deduce from (3.2.10)

$$\|(1-\chi)(D/\sqrt{t})g_k\|_{L^\infty(dx)} \leqslant C \sum_{2^j>c\sqrt{t}} 2^{jd/2}\|\Delta_j g_k\|_{L^2(dx)} \leqslant C_N(2^k/\sqrt{t})^{-N}.$$

We deduce from these two inequalities that the $L^\infty(dxdt)$ norm of (3.2.37) is smaller than a ℓ^r_q sequence. This shows that the $L^\infty(dxdt)$ norm of (3.2.34) is in ℓ^r_q and concludes the proof of proposition 3.2.2. $\qquad\square$

Our next objective is to estimate quantities of type

$$(3.2.39) \qquad\qquad T_{a(t,x)w}b(t,D)e^{im\theta}$$

where $m \in \mathbb{Z}^*$, $a(t,x)$ is a smooth function of x satisfying for any $\gamma \in \mathbb{N}^d$

$$(3.2.40) \qquad |(D_x/\sqrt{t})^\gamma a(t,x)| \leqslant C_\gamma(1+\sqrt{t}|x|)^{m_1-|\gamma|}(1+|x|)^{m'_1}$$

and where $b(t, \xi)$ is in $\Sigma_\nu^{m_2}$, using the notations of definition 3.2.1.

PROPOSITION 3.2.4. — *Let m_1, m_1', m_2, ν be real numbers satisfying the inequalities $m_1 \leqslant 0, m_1' \leqslant 0$, and let a be a symbol verifying (3.2.40) and $b \in \Sigma_\nu^{m_2}$.*

(i) Take $s, s', \alpha, \alpha', r$ such that $s, s', \alpha, \alpha' \in \mathbb{R}$, $r \in [1, +\infty]$,

$$(3.2.41) \qquad -d < \alpha + \alpha' < -\frac{d}{2}, \quad s < \alpha + 2\alpha', \quad s + s' < -1/2, \quad \nu = \frac{d}{2}.$$

Let $w \in E_{\alpha,\alpha',r}^{s,s'}$ be such that $a(t,x)w \in E_{\alpha,\alpha',r}^{s,s'}$. Then (3.2.39) belongs to $E_{\alpha+2d,\alpha',r}^{s-m_1-m_2,s'-m_1'}$.

(ii) Assume instead of (3.2.41),

$$(3.2.42) \qquad -\frac{d}{2} < \alpha + \alpha', \quad s < \alpha' - \frac{d}{2}, \quad s + s' < -\frac{1}{2}, \quad \nu = 0.$$

Then if $w \in E_{\alpha,\alpha',r}^{s,s'}$ and $a(t,x)w \in E_{\alpha,\alpha',r}^{s,s'}$, (3.2.39) belongs to the space $E_{\alpha+d,\alpha',r}^{s-m_1-m_2,s'-m_1'}$.

We can in the above statements replace operator T by operator $\widetilde{T}$ of (3.1.16).

We first prove the following lemmas:

LEMMA 3.2.5. — *Let a be a symbol satisfying (3.2.40) with $m_1 \leqslant 0$, $m_1' \leqslant 0$, $b \in \Sigma_\nu^{m_2}, m \in \mathbb{Z}^*$. We have the following estimates*

$$(3.2.43)$$
$$\|a(t,x)\Delta_j b(t,D)e^{im\theta}\|_{L^2(dx)} \leqslant C\left(\frac{2^j}{t}\right)^{\nu+d/2}\left(1 + \frac{2^j}{t}\right)^{-\nu+m_1'}\left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_1+m_2}$$

$$(3.2.44)$$
$$\|a(t,x)\Delta_j b(t,D)e^{im\theta}\|_{L^\infty(dx)} \leqslant C\left(\frac{2^j}{t}\right)^{\nu}\left(1 + \frac{2^j}{t}\right)^{-\nu+m_1'}\left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_1+m_2}.$$

Proof. — Remark first that since $m_1 \leqslant 0$, $m_1' \leqslant 0$, (3.2.40) implies that

$$(3.2.45) \qquad \|\mathbb{1}_{\{|x| \geqslant c2^j/t\}}a(t,x)\|_{L^\infty} \leqslant C\left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_1}\left(1 + \frac{2^j}{t}\right)^{m_1'}.$$

Moreover, a direct computation and the assumption $b(t,\xi) \in \Sigma_\nu^{m_2}$ shows that

$$(3.2.46) \quad \begin{aligned} \|\Delta_j b(t,D)e^{im\theta}\|_{L^2(dx)} &\leqslant C\left(\frac{2^j}{t}\right)^{\nu+d/2}\left(1 + \frac{2^j}{t}\right)^{-\nu}\left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_2} \\ \|\Delta_j b(t,D)e^{im\theta}\|_{L^\infty(dx)} &\leqslant C\left(\frac{2^j}{t}\right)^{\nu}\left(1 + \frac{2^j}{t}\right)^{-\nu}\left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_2}. \end{aligned}$$

Together with (3.2.45), these inequalities show that (3.2.43), (3.2.44) hold true if we replace in the left hand side $a(t,x)$ by $\mathbb{1}_{\{|x| \geqslant c2^j/t\}}a(t,x)$. Consequently, we are left with estimating in L^2 or L^∞ the quantity

$$(3.2.47) \qquad |\mathbb{1}_{\{|x| < c2^j/t\}}a(t,x)\Delta_j b(t,D)e^{im\theta}| \leqslant C|\mathbb{1}_{\{|x| < c2^j/t\}}\Delta_j b(t,D)e^{im\theta}|,$$

where we used that a is bounded since $m_1 \leqslant 0$, $m_1' \leqslant 0$. When $2^j < \sqrt{t}$, the wanted upper bound follows from the above estimate of the L^2 or L^∞ norms of $\Delta_j b(t, D)e^{im\theta}$. When $2^j \geqslant \sqrt{t}$, we write $\Delta_j b(t, D)e^{im\theta}$ as a constant multiple of

$$(3.2.48) \qquad \int e^{ix\xi - i\xi^2/mt}\varphi(2^{-j}\xi)b(t,\xi)\frac{d\xi}{t^{d/2}} = \left(\frac{2^j}{\sqrt{t}}\right)^d \int e^{i2^j[x\xi - \frac{2^j\xi^2}{mt}]}\widetilde{b}_j(t,\xi)\,d\xi$$

where $\widetilde{b}_j(t,\xi) = \varphi(\xi)b(t, 2^j\xi)$ is supported in a fixed ring of size 1 and satisfies by (3.2.2) for any $\gamma \in \mathbb{N}^d$

$$(3.2.49) \qquad |\partial_\xi^\gamma \widetilde{b}_j(t,\xi)| \leqslant C_\gamma \left(\frac{2^j}{t}\right)^\nu \left(1 + \frac{2^j}{t}\right)^{-\nu} \left(1 + \frac{2^j}{\sqrt{t}}\right)^{m_2}.$$

Define the operator

$$(3.2.50) \qquad L_j(t, x, \xi, D_\xi) = [1 + 2^{2j}(x - 2^{j+1}\tfrac{\xi}{mt})^2]^{-1}[1 + 2^j(x - 2^{j+1}\tfrac{\xi}{mt}) \cdot D_\xi].$$

Remark that when $\xi \in \mathrm{Supp}\,\varphi$ and x stays in the domain $|x| \leqslant c2^j/t$ with c small enough, this operator is of form $L_j = \alpha_j^0(t, x, \xi) + \alpha_j^1(t, x, \xi) \cdot D_\xi$ where α_j^0, α_j^1 are smooth coefficients satisfying for $\ell = 0, 1$ and $\gamma \in \mathbb{N}^d$

$$(3.2.51) \qquad |\partial_\xi^\gamma \alpha_j^\ell(t, x, \xi)| \leqslant C_\gamma(1 + 2^j/\sqrt{t})^{-2}.$$

Consequently, ${}^tL_j$ is of the same type, and integrating by parts, we write (3.2.48) as

$$(3.2.52) \qquad \left(\frac{2^j}{\sqrt{t}}\right)^d \int e^{i2^j[x\xi - \frac{2^j\xi^2}{mt}]}({}^tL_j)^M \widetilde{b}_j(t, \xi)\,d\xi$$

when x stays in the domain $\{|x| < c2^j/t\}$, for any $M \in \mathbb{N}$. It follows then from (3.2.49), (3.2.51) that (3.2.47) is smaller than

$$C_N \mathbb{1}_{\{|x| < c2^j/t\}} \left(\frac{2^j}{t}\right)^\nu \left(1 + \frac{2^j}{\sqrt{t}}\right)^{-N}$$

for any N. This implies immediately an upper bound of the L^∞ (resp. L^2) norm of this quantity by the right hand side of (3.2.44) (resp. (3.2.43)) and concludes the proof. $\qquad\square$

LEMMA 3.2.6. — *If s' is a real number, define $s_2' = s' + \frac{1}{2}, s_\infty' = s'$. For $r \in [1, +\infty]$, set $r_2 = r$, $r_\infty = \widetilde{r}$, with $\widetilde{r}$ defined in definition 3.1.7.*

(i) Let $s, s', \alpha, \alpha', r$ be real numbers, with $r \in [1, +\infty]$, and assume that

$$(3.2.53) \qquad -d < \alpha + \alpha' < -d/2, \quad s < \alpha + 2\alpha', \quad s + s' < -1/2.$$

If $w \in E_{\alpha,\alpha',r}^{s,s'}$, one has the following estimates for $p = 2$ or $p = \infty$:

$$(3.2.54) \qquad \|\phi_q(x)\chi(D/\sqrt{t})w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty)} \leqslant C2^{-s(j-\ell/2)+(j-\ell)(\alpha'-d)}c_{jq}$$

when $\ell/2 \leqslant j \leqslant \ell$,

$$(3.2.55) \qquad \|\phi_q(x)S_j w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty)} \leqslant C2^{-\frac{\ell}{2}\alpha - \ell\alpha'}c_q$$

when $j \leqslant \ell/2$,

(3.2.56) $$\|\phi_q(x)\chi(D/\sqrt{t})w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)} \leqslant C2^{-s(j-\ell/2)-s'_p(j-\ell)}c_{jq}$$

when $\ell \leqslant j$,

(3.2.57) $$\|\phi_q(x)(1-\chi(D/\sqrt{t}))S_j w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}$$
$$\leqslant C2^{-s(j-\ell/2)+(j-\ell)\alpha'}(1+2^{j-\ell})^{-\alpha'-s'_p}c_{jq}$$

when $\ell/2 \leqslant j$, *where* $(c_{jq})_{jq}$ *(resp.* $(c_q)_q$*) is a sequence in the unit ball of* $\ell_j^2\ell_q^{r_p}$ *(resp.* $\ell_q^{r_p}$*).*

   (ii) Assume instead of (3.2.53)

(3.2.58) $$-d/2 < \alpha + \alpha', \quad s < \alpha' - d/2, \quad s + s' < -1/2.$$

If $w \in E_{\alpha,\alpha',r}^{s,s'}$, *one has for* $p = 2$ *or* $p = \infty$,

(3.2.59) $$\|\phi_q(x)\chi(D/\sqrt{t})w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^\infty)} \leqslant C2^{-s(j-\ell/2)+(j-\ell)(\alpha'-d/2)}c_{jq}$$

when $\ell/2 \leqslant j \leqslant \ell$,

(3.2.60) $$\|\phi_q(x)S_j w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^\infty)} \leqslant C2^{(j-\frac{\ell}{2})\alpha+(j-\ell)\alpha'}c_{jq}2^{j\frac{d}{2}}$$

when $j \leqslant \ell/2$, *and* (3.2.56), (3.2.57) *when respectively* $\ell \leqslant j$ *and* $\ell/2 \leqslant j$, *for some sequence* $(c_{jq})_{jq}$ *in the unit ball of* $\ell_j^2\ell_q^r$.

Proof. — Let us prove first (3.2.54) and (3.2.55). We estimate, with some $N_1 \in \mathbb{N}$,

$$\|\phi_q(x)\chi(D/\sqrt{t})w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^\infty)}$$

(3.2.61)
$$\leqslant \sum_{q'}\sum_{j';j'\leqslant\ell/2+N_1}\|(\phi_q(x)\chi(D/\sqrt{t})\widetilde{\phi}_{q'})(\phi_{q'}\Delta_{j'}w\Gamma_\ell(t))\|_{L^p(\frac{dt}{t},L^\infty)}$$
$$\leqslant \sum_{q'}\sum_{j';j'\leqslant\ell/2+N_1}C_N 2^{j'd/2}\langle q-q'\rangle^{-N}\|\phi_{q'}\Delta_{j'}w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}.$$

Using the definition of $E_{\alpha,\alpha',r}^{s,s'}$, we get the upper bound

(3.2.62) $$C_N \sum_{q'}\sum_{j';j'\leqslant\ell/2+N_1}2^{j'd/2}\langle q-q'\rangle^{-N}2^{\alpha(j'-\ell/2)+\alpha'(j'-\ell)}c_{j'q'}$$

for a $\ell_{j'}^2\ell_{q'}^{r_p}$ sequence $(c_{j'q'})_{j'q'}$. In case of assertion (i) of the lemma, under assumption (3.2.53), (3.2.62) is smaller than $C2^{-\frac{\ell}{2}\alpha-\ell\alpha'}c_q'$ for a $\ell_q^{r_p}$ sequence $(c_q')_q$. This last expression may be written

$$C2^{-s(j-\ell/2)+(j-\ell)(\alpha'-d)}c_{jq\ell}$$

with

$$c_{jq\ell} = 2^{j(s-\alpha'+d)}2^{\ell(-s/2-d-\alpha/2)}c_q'.$$

This quantity is smaller than $2^{-j(\alpha+\alpha'+d)}c_q'$ when $s/2+d+\alpha/2 \leqslant 0, \ell/2 \leqslant j \leqslant \ell$, and than $2^{j(s/2-\alpha'-\alpha/2)}c_q'$ when $s/2+d+\alpha/2 \geqslant 0, \ell/2 \leqslant j \leqslant \ell$. In both cases, because

of (3.2.53), we get an upper bound by a $\ell_j^2 \ell_q^{r_p}$ sequence, whence (3.2.54). One proves (3.2.55) in the same way.

Let us now prove inequality (3.2.59) of assertion (ii) of the lemma. Because of (3.2.58), the quantity (3.2.62) is smaller than $C2^{\frac{\ell}{2}(\frac{d}{2}-\alpha')}c_{\ell q}$ with $(c_{\ell q})_{\ell q} \in \ell_\ell^2 \ell_q^r$. We write this upper bound $C2^{-s(j-\ell/2)+(j-\ell)(\alpha'-d/2)}c_{jq\ell}$ with

$$c_{jq\ell} = 2^{(j-\ell/2)(s-\alpha'+d/2)}c_{\ell q}.$$

Since $s - \alpha' + d/2 < 0$, the supremum for $\ell \leqslant 2j$ of this sequence is in $\ell_j^2 \ell_q^r$. This gives the conclusion (3.2.59). One gets (3.2.60) by a similar computation.

Let us prove now (3.2.56). Arguing as in (3.2.61) we write

$$(3.2.63) \quad \|\phi_q(x)\chi(D/\sqrt{t})w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}$$
$$\leqslant C_N \sum_{q'} \sum_{j';j' \leqslant \ell/2+N_1} \langle q-q'\rangle^{-N} 2^{\alpha(j'-\ell/2)+\alpha'(j'-\ell)}c_{j'q'}.$$

Let us consider first the case $\alpha + \alpha' \leqslant 0$. Then the right hand side of (3.2.63) is smaller than $C2^{-\frac{\ell}{2}\alpha - \ell\alpha' + \ell\varepsilon}c_q'$ where $(c_q')_q \in \ell_q^r$, $\varepsilon = 0$ if $\alpha + \alpha' < 0$, $\varepsilon > 0$ arbitrarily small if $\alpha + \alpha' = 0$. We write this quantity $C2^{-(j-\ell/2)s-(j-\ell)s_p'}c_{jq\ell}$ where

$$(3.2.64) \qquad c_{jq\ell} = 2^{j(s+s_p')}2^{-\frac{\ell}{2}(\alpha+2\alpha'+s+2s_p'-2\varepsilon)}c_q' \leqslant 2^{j(\varepsilon+\delta)}c_q'$$

if $\ell \leqslant j$ and $\delta = \max[s+s_p', \frac{s}{2} - \frac{\alpha}{2} - \alpha']$. The assumptions (3.2.53) or (3.2.58) imply that $\delta < 0$, so the right hand side of (3.2.64) is in $\ell_j^2 \ell_q^r$ if $\varepsilon < -\delta$, whence (3.2.56) in this case. When $\alpha + \alpha' > 0$ we are necessarily in case (ii) and the right hand side of (3.2.63) is smaller than $c2^{-\frac{\ell}{2}\alpha'}c_{\ell q}'$ for a $\ell_\ell^2 \ell_q^r$ sequence $(c_{\ell q}')_{\ell q}$. We write this quantity $2^{-s(j-\ell/2)-s_p'(j-\ell)}c_{jq\ell}$, with for $\ell \leqslant j$

$$c_{jq\ell} = 2^{j(s+s_p')}2^{-\frac{\ell}{2}[s+2s_p'+\alpha']}c_{\ell q}' \leqslant 2^{j\max[s+s_p', \frac{s}{2} - \frac{\alpha'}{2}]}c_{\ell q}'.$$

Since by (3.2.58) $s + s_p' < 0$, $s < \alpha'$, the supremum in $\ell \leqslant j$ of this last sequence belongs to $\ell_j^2 \ell_q^r$. This concludes the proof of (3.2.56).

Finally, let us prove (3.2.57) in cases (i) and (ii). We write

$$(3.2.65) \quad \|\phi_q(x)(1-\chi)(D/\sqrt{t})S_j w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}$$
$$\leqslant \sum_{q'} \sum_{j';\ell/2+N_1 \leqslant j' \leqslant j-1} \|(\phi_q(1-\chi)(D/\sqrt{t})\widetilde{\phi}_{q'})(\phi_{q'}\Delta_{j'}w\Gamma_\ell(t))\|_{L^p(\frac{dt}{t},L^2)}$$
$$\leqslant C_N \sum_{q'} \Bigg(\sum_{j';\ell/2+N_1 \leqslant j' \leqslant \inf(\ell,j-1)} \langle q-q'\rangle^{-N} 2^{(j'-\ell)\alpha'-s(j'-\ell/2)}c_{j'q'}$$
$$+ \sum_{j';\ell<j'<j} \langle q-q'\rangle^{-N} 2^{-s(j'-\ell/2)-s_p'(j'-\ell)}c_{j'q'} \Bigg)$$

where $(c_{j'q'})_{j'q'} \in \ell^2_{j'}\ell^r_{q'}$ and where we used the definition of $E^{s,s'}_{\alpha,\alpha',r}$. When $j - 1 \leqslant \ell$, since in both cases (i) and (ii) $\alpha' - s > 0$, we get for (3.2.65) the upper bound

$$C 2^{(j-\ell)\alpha'} 2^{-s(j-\ell/2)} c'_{jq}$$

for a $\ell^2_j \ell^r_q$ sequence $(c'_{jq})_{jq}$. When $\ell < j - 1$, using that $s + s'_p < 0$, $\alpha' - s > 0$ we obtain the upper bound

$$C 2^{-s(j-\ell/2)-s'_p(j-\ell)}[c'_{jq} + c''_{\ell q} 2^{(s+s'_p)(j-\ell)}]$$

for a $\ell^2_j \ell^r_q$ (resp. $\ell^2_\ell \ell^r_q$) sequence $(c'_{jq})_{jq}$ (resp. $(c''_{\ell q})_{\ell q}$). The conclusion follows from these estimates. $\qquad\square$

Proof of proposition 3.2.4. — We want to estimate

$$(3.2.66) \qquad \phi_q \Delta_j [T_{a(t,x)w} b(t,D) e^{im\theta}] \Gamma_\ell(t)$$

in $L^2(\frac{dt}{t}, L^2(dx))$ and $L^\infty(dt, L^2(dx))$. We study successively the following cases.

Estimate for $j \leqslant \ell/2$. — We write for $|j - j'| \leqslant N_0$, N_0 a fixed integer, and $p = 2$ or $p = \infty$

$$(3.2.67) \quad \|\phi_q \Delta_j [S_{j'-1}(a(t,x)w) \Gamma_\ell(t) \Delta_{j'} b(t,D) e^{im\theta}]\|_{L^p(\frac{dt}{t}, L^2)}$$

$$\leqslant \sum_{q'} C_N \langle q - q' \rangle^{-N} \|\phi_{q'} S_{j'-1}[a(t,x)w] \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty)}$$

$$\times \|\Delta_{j'}[b(t,D) e^{im\theta}]\|_{L^\infty(L^2)}.$$

In case (i), we use inequality (3.2.55) (for aw instead of w) together with (3.2.43) with $\nu = d/2$. We get the upper bound

$$(3.2.68) \qquad C 2^{-\frac{\ell}{2}\alpha-\ell\alpha'} c_q 2^{(j-\ell)d} = C 2^{(j-\ell/2)(\alpha+2d)} 2^{(j-\ell)\alpha'} c_{jq}$$

with $(c_q)_q \in \ell^{r_p}_q$ and $c_{jq} = 2^{-j(\alpha+\alpha'+d)} c_q$, a sequence in $\ell^2_j \ell^{r_p}_q$, as we have $\alpha+\alpha'+d > 0$ by (3.2.41). In case (ii), we apply (3.2.60) together with (3.2.43) with $\nu = 0$. We get the upper bound

$$(3.2.69) \qquad C 2^{(j-\ell/2)\alpha+(j-\ell)\alpha'} c_{jq} 2^{jd/2} 2^{(j-\ell)d/2} = C 2^{(j-\ell/2)(\alpha+d)+(j-\ell)\alpha'} c_{jq}$$

for a $\ell^2_j \ell^{r_p}_q$ sequence $(c_{jq})_{jq}$. The expressions (3.2.68), (3.2.69) show that (3.2.66) belongs to the spaces indicated in the statement of the proposition.

Estimate for $\ell/2 < j \leqslant \ell$. — For j' with $|j - j'|$ smaller than some given integer N_1, we write

$$(3.2.70) \qquad S_{j'-1}(aw) \Delta_{j'}(b(t,D) e^{im\theta}) = U_{j'} \cdot V_{j'}$$

where, if $c(t,x) = (1 + tx^2)^{-m_1/2}(1 + x^2)^{-m'_1/2}$, we set

$$(3.2.71) \qquad \begin{aligned} U_{j'} &= c(t,x) S_{j'-1}(aw) \\ V_{j'} &= c(t,x)^{-1} \Delta_{j'}[b(t,D) e^{im\theta}]. \end{aligned}$$

We further decompose, choosing N_2 such that $j'' \geqslant j' + N_2$ and $j \geqslant \ell/2$ imply $(1 - \chi)(D/\sqrt{t})\Delta_{j''}\Gamma_\ell = \Delta_{j''}\Gamma_\ell,$

$$(3.2.72) \qquad U_{j'} = U'_{j'} + U''_{j'} + \sum_{j'' \geqslant j' + N_2} U_{j'}^{j''}$$

$$U'_{j'} = c(t,x)S_{j'-1}[a\chi(D/\sqrt{t})w]$$

$$U''_{j'} = c(t,x)S_{j'-1}[a(1 - \chi)(D/\sqrt{t})S_{j'+N_2}w]$$

$$U_{j'}^{j''} = c(t,x)S_{j'-1}[a\Delta_{j''}w]$$

when t is in the support of $\Gamma_\ell(t)$. We have the estimate

$$(3.2.73) \quad \|\phi_q(x)U_{j'}V_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))}$$

$$\leqslant \|\phi_q(x)U'_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty(dx))}\|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}$$

$$+ \|\phi_q(x)U''_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))}\|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dtdx)}$$

$$+ \sum_{j'' \geqslant j'} \|\phi_q(x)U_{j'}^{j''}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))}\|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dtdx)}.$$

We write

$$\phi_q U'_{j'} = \sum_{q'} K_{j'qq'}[\phi_{q'}\chi(D/\sqrt{t})w]$$

$$(3.2.74) \qquad \phi_q U''_{j'} = \sum_{q'} K_{j'qq'}[\phi_{q'}(1 - \chi)(D/\sqrt{t})S_{j'+N_2}w]$$

$$\phi_q U_{j'}^{j''} = \sum_{q'} K_{j'qq'}^{j''}[\phi_{q'}\Delta_{j''}w]$$

where the operator $K_{j'qq'}$, (resp. $K_{j'qq'}^{j''}$) is given by

$$(3.2.75) \qquad K_{j'qq'}w(x) = \phi_q(x)c(t,x)2^{j'd}\int h(2^{j'}(x-y))a(t,y)\widetilde{\phi}_{q'}(y)w(y)\,dy$$

(resp.

$$(3.2.76) \quad K_{j'qq'}^{j''}w(x) = \phi_q(x)c(t,x)2^{j'd}\int h(2^{j'}(x-y))a(t,y)\widetilde{\Delta}_{j''}(\widetilde{\phi}_{q'}(y)w(y))\,dy)$$

for $h \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{h} \in C_0^\infty(\mathbb{R}^d)$. Let us remark that the kernel $k_{j'qq'}(x,y)$ of $K_{j'qq'}$ satisfies for any $N \in \mathbb{N}$

$$(3.2.77) \qquad \begin{aligned} \sup_x \int |k_{j'qq'}(x,y)|\,dy &\leqslant C_N\langle q - q'\rangle^{-N}, \\ \sup_y \int |k_{j'qq'}(x,y)|\,dx &\leqslant C_N\langle q - q'\rangle^{-N}, \end{aligned}$$

since, on the domain $|y| \geqslant \varepsilon|x|$ $(\varepsilon > 0)$ $|c(t,x)a(t,y)| \leqslant C$ as $m_1 \leqslant 0, m'_1 \leqslant 0$, and on the domain $|y| < \varepsilon|x|$, the decay of h compensates the growth of $c(t,x)$, since

$2^{j'} \geqslant c2^{\ell/2} \geqslant c\sqrt{t}$ on the support of $\Gamma_\ell(t)$. Consequently, for $t \sim 2^\ell$, $\sigma \in [1, +\infty]$

$$(3.2.78) \qquad \|K_{j'qq'}\|_{\mathcal{L}(L^\sigma, L^\sigma)} \leqslant C_N \langle q - q' \rangle^{-N}$$

for any N, uniformly in j', q, q'. By integration by parts, we can write (3.2.76)

$$(3.2.79) \quad \phi_q(x) c(t, x) 2^{j'd} \int (2^{-j''} D_y)^M [h(2^{j'}(x - y)) a(t, y)]$$
$$\times (2^{-j''} D_y)^{-M} \widetilde{\Delta}_{j''} [\widetilde{\phi}_{q'}(y) w(y)] \, dy$$

for any $M \in \mathbb{N}$. Since, by (3.2.40),

$$|D_y^\gamma a(t, y)| \leqslant C t^{|\gamma|/2} (1 + \sqrt{t}|y|)^{m_1} (1 + |y|)^{m_1'}$$

and, for $t \sim 2^\ell$, $\sqrt{t} \leqslant C 2^{\ell/2} \leqslant C 2^j$, we deduce from (3.2.79) that the kernel $k_{j'qq'}^{j''}(x, y)$ of $K_{j'qq'}^{j''}$ satisfies

$$(3.2.80) \qquad \begin{aligned} \sup_x \int |k_{j'qq'}^{j''}(x, y)| \, dy &\leqslant C_N \langle q - q' \rangle^{-N} 2^{-N(j'' - j')}, \\ \sup_y \int |k_{j'qq'}^{j''}(x, y)| \, dx &\leqslant C_N \langle q - q' \rangle^{-N} 2^{-N(j'' - j')}, \end{aligned}$$

whence, for $t \sim 2^\ell$, $\sigma \in [1, +\infty]$

$$(3.2.81) \qquad \|K_{j'qq'}^{j''}\|_{\mathcal{L}(L^\sigma, L^\sigma)} \leqslant C_N \langle q - q' \rangle^{-N} 2^{-N(j'' - j')}.$$

Let us estimate the first contribution to the right hand side of (3.2.73). Using (3.2.74), (3.2.78) we see that

$$\|\phi_q(x) U_{j'}' \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty(dx))} \|V_{j'} \Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}$$
$$\leqslant C_N \sum_{q'} \langle q - q' \rangle^{-N} \|\phi_{q'}(x) \chi(D/\sqrt{t}) w \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^\infty(dx))}$$
$$\times \|V_{j'} \Gamma_\ell(t)\|_{L^\infty(dt, L^2(dx))}.$$

Applying (3.2.54) and (3.2.43) with $a = c^{-1}$, $\nu = d/2$ in case (i) (resp. (3.2.59) and (3.2.43) with $a = c^{-1}, \nu = 0$ in case (ii)), we get the upper bound

$$(3.2.82) \qquad C 2^{-(s - m_1 - m_2)(j - \ell/2)} 2^{(j - \ell)\alpha'} c_{jq}$$

in both cases, for a $\ell_j^2 \ell_q^{r_p}$ sequence $(c_{jq})_{jq}$. This is the wanted conclusion. To study the second contribution to the right hand side of (3.2.73) we write using again (3.2.74), (3.2.78)

$$\|\phi_q(x) U_{j'}'' \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))} \|V_{j'} \Gamma_\ell(t)\|_{L^\infty(dtdx)}$$
$$\leqslant C_N \sum_{q'} \langle q - q' \rangle^{-N} \|\phi_{q'}(x)(1 - \chi)(D/\sqrt{t}) S_{j' + N_2} w \Gamma_\ell(t)\|_{L^p(\frac{dt}{t}, L^2(dx))}$$
$$\times \|V_{j'} \Gamma_\ell(t)\|_{L^\infty(dtdx)}.$$

Since (3.2.44) with $a = c^{-1}$ implies in any case $\|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dtdx)} \leqslant C2^{(j-\ell/2)(m_1+m_2)}$ we obtain from (3.2.57) the upper bound

$$C2^{-(s-m_1-m_2)(j-\ell/2)}2^{(j-\ell)\alpha'}c'_{jq}$$

for a $\ell_j^2 \ell_q^{r_p}$ sequence $(c'_{jq})_{jq}$. This is again the wanted conclusion.

To study the third contribution to the right hand side of (3.2.73), we write using (3.2.74), (3.2.81)

$$(3.2.83) \quad \sum_{j'' \geqslant j'} \|\phi_q(x)U_{j'}^{j''}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2(dx))}\|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dtdx)}$$

$$\leqslant C_N \sum_{q'} \sum_{j'' \geqslant j'} \langle q-q'\rangle^{-N}2^{-N(j''-j')}\|\phi_{q'}(x)\Delta_{j''}w\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2(dx))}$$
$$\times \|V_{j'}\Gamma_\ell(t)\|_{L^\infty(dtdx)}.$$

Using (3.2.44) and the definition of $E_{\alpha,\alpha',r}^{s,s'}$, we estimate (3.2.83) by

$$C_N \sum_{q'} \sum_{j'' \geqslant j'} \langle q-q'\rangle^{-N}2^{-N(j''-j')}2^{-(j''-\ell/2)s}2^{(j''-\ell)\alpha'}(1+2^{j''-\ell})^{-s'_p-\alpha'}c_{j''q'}$$
$$\times 2^{(j'-\ell/2)(m_1+m_2)}$$

$$\leqslant C2^{-(j-\ell/2)(s-m_1-m_2)}2^{(j-\ell)\alpha'}c'_{jq}$$

with a $\ell_{j''}^2 \ell_{q'}^{r_p}$ (resp. $\ell_j^2 \ell_q^{r_p}$) sequence $(c_{j''q'})_{j''q'}$ (resp. $(c'_{jq})_{jq}$). This gives the wanted conclusion.

Estimate for $\ell \leqslant j$. — Using decomposition (3.2.72), we estimate

$$(3.2.84) \quad \|\phi_q U_{j'}V_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)} \leqslant \|\phi_q U'_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}\|\Gamma_\ell(t)V_{j'}\|_{L^\infty(dtdx)}$$
$$+ \|\phi_q U''_{j'}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}\|\Gamma_\ell(t)V_{j'}\|_{L^\infty(dtdx)}$$
$$+ \sum_{j'' \geqslant j'}\|\phi_q U_{j'}^{j''}\Gamma_\ell(t)\|_{L^p(\frac{dt}{t},L^2)}\|\Gamma_\ell(t)V_{j'}\|_{L^\infty(dtdx)}.$$

We estimate the first term in the right hand side using (3.2.74), (3.2.78), (3.2.56) and (3.2.44) by

$$(3.2.85) \qquad\qquad C2^{-(s-m_1-m_2)(j-\ell/2)-(s'_p-m'_1)(j-\ell)}c_{jq}$$

for a $\ell_j^2 \ell_q^{r_p}$ sequence $(c_{jq})_{jq}$. This is the wanted conclusion. The second contribution to the right hand side of (3.2.84) is estimated in the same way, using (3.2.57) instead of (3.2.56). Finally, the third contribution is smaller, by (3.2.81), (3.2.44) and the definition of $E_{\alpha,\alpha',r}^{s,s'}$, than

$$C_N \sum_{j'' \geqslant j'} \sum_{q'} \langle q-q'\rangle^{-N}2^{-N(j''-j')}2^{(-j''-\ell/2)s-(j''-\ell)s'_p}c_{j''q'}2^{(j-\ell)m'_1}2^{(j-\ell/2)(m_1+m_2)}$$

for a $\ell_{j''}^2 \ell_{q'}^{r_p}$ sequence $(c_{j''q'})_{j''q'}$. This is again smaller than (3.2.85), which concludes the proof of the proposition. $\qquad\square$

We close this chapter with a corollary of proposition 3.2.4 which will be used in the rest of this paper. Remind that we defined in definition 3.1.7 spaces $\mathcal{E}^{s,s'}_{\alpha,\alpha',r}(M)$, $\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M)$.

COROLLARY 3.2.7. — *Take $d = 2$ and let $b(t,\xi) \in \Sigma^{m_2}_\nu$. Let $m \in \mathbb{Z}^*$, $M \in 2\mathbb{N}$. Let w be an element of $\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M)$ such that for a given $\mu \in [0,4]$, $\langle\sqrt{t}x\rangle^\mu w \in \widetilde{\mathcal{E}}^{s-\mu,s'}_{\alpha,\alpha',r}(M)$. Assume*

$$(3.2.86) \quad -2 < \alpha + \alpha' < -1, \quad s - \mu < \alpha + 2\alpha', \quad s + s' - \mu - M < -1/2, \quad \nu = 1$$

or

$$(3.2.87) \qquad -1 < \alpha + \alpha', \quad s - \mu < \alpha' - 1, \quad s + s' - \mu - M < -1/2, \quad \nu = 0.$$

Then under assumptions (3.2.86), (resp. (3.2.87)) $T_w b(t,D)(e^{im\theta})$ belongs to $\mathcal{E}^{s-m_2,s'}_{\alpha+4,\alpha',r}(M)$ (resp. $\mathcal{E}^{s-m_2,s'}_{\alpha+2,\alpha',r}(M)$).

Proof. — Take $\gamma \in \mathbb{N}^d$ with $|\gamma| \leqslant M$. We have to show that the expression $x^\gamma T_w b(t,D)(e^{im\theta})$ belongs to $E^{s-m_2,s'-|\gamma|}_{\alpha+4,\alpha',r}$ (resp. $E^{s-m_2,s'-|\gamma|}_{\alpha+2,\alpha',r}$). Using the notations introduced in (3.1.16), we can write this expression as a linear combination of quantities $\widetilde{T}_{x^{\gamma'}w} b(t,D)(e^{im\theta})$ for $|\gamma'| \leqslant |\gamma|$. Write $x^{\gamma'} w = a\widetilde{w}$ with

$$(3.2.88) \qquad \begin{aligned} a(t,x) &= x^{\gamma'}(1+x^2)^{-M/2}\langle\sqrt{t}x\rangle^{-\mu} \\ \widetilde{w} &= \langle\sqrt{t}x\rangle^\mu(1+x^2)^{M/2}w. \end{aligned}$$

We see that a satisfies (3.2.40) with $m_1 = -\mu \leqslant 0$, $m'_1 = -M + |\gamma'| \leqslant -M + |\gamma| \leqslant 0$. Moreover, by assumption, $\widetilde{w} \in \widetilde{E}^{s-\mu,s'-M}_{\alpha,\alpha',r}$. When (3.2.86), (resp. (3.2.87)) is satisfied, we deduce from (i) (resp. from (ii)) of proposition 3.2.4 that $T_{a\widetilde{w}} b(t,D)(e^{im\theta})$ belongs to $E^{s-m_2,s'-|\gamma|}_{\alpha+4,\alpha',r}$ (resp. $E^{s-m_2,s'-|\gamma|}_{\alpha+2,\alpha',r}$) which is the wanted conclusion. $\qquad\square$

CHAPTER 4

PROOF OF THE MAIN THEOREM

4.1. Main reductions

Let us recall that we have seen in section 1.2 that theorem 1.1.1 will be proved if we succeed to construct a solution to the equation

$$(4.1.1) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)w = \frac{1}{t}e^{i\theta(t,x)}Q_1(w, Zw) + \frac{1}{t}e^{-3i\theta(t,x)}Q_2(\overline{w}, \overline{Zw}).$$
$$w|_{t=1} = \varepsilon w_0$$

defined for $t \geqslant 1$, when ε is small enough and w_0 stays in the unit ball of $\mathcal{H}^{M+4}(\mathbb{R}^2)$.

Using the definition of $Z = \frac{D_x}{t} + \frac{x}{2}$ and the assumptions on Q_1, Q_2, we can rewrite the first equation (4.1.1) as

$$(4.1.2) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)w = \frac{1}{t}\sum_{|\gamma|\leqslant 2} q_1^{\gamma}(w, \tfrac{D_x}{t}w)x^{\gamma}e^{i\theta} + \frac{1}{t}\sum_{|\gamma|\leqslant 2} q_2^{\gamma}(\overline{w}, \tfrac{D_x}{t}\overline{w})x^{\gamma}e^{-3i\theta}$$

where $\gamma \in \mathbb{N}^d$, $|\gamma| \leqslant 2$ in the summation, and $q_1^{\gamma}(Y_0, Y_1)$, $q_2^{\gamma}(Y_0, Y_1)$ are homogeneous of degree 2 in (Y_0, Y_1), their degree in Y_1 $\deg_{Y_1} q_j^{\gamma}(Y_0, Y_1)$ satisfying the relation

$$(4.1.3) \qquad 0 < \deg_{Y_1} q_j^{\gamma}(Y_0, Y_1) + |\gamma| \leqslant 2, \quad j = 1, 2.$$

The spaces we will use to look for a solution are the spaces $\widetilde{\mathcal{E}}_{\alpha,\alpha',r}^{s,s'}(M)$ and $\widetilde{\underline{\mathcal{E}}}_{\alpha,\alpha',r}^{s,s'}(M)$ introduced in (3.1.49). Let us recall their definition

$$(4.1.4) \qquad \begin{aligned} \widetilde{\mathcal{E}}_{\alpha,\alpha',r}^{s,s'}(M) &= \{v \in \widetilde{E}_{\alpha,\alpha',r}^{s,s'}; \, \forall \gamma \in \mathbb{N}^d, \, |\gamma| \leqslant M, x^{\gamma}v \in \widetilde{E}_{\alpha,\alpha',r}^{s,s'-|\gamma|}\} \\ \widetilde{\underline{\mathcal{E}}}_{\alpha,\alpha',r}^{s,s'}(M) &= \{v \in \widetilde{\underline{E}}_{\alpha,\alpha',r}^{s,s'}; \, \forall \gamma \in \mathbb{N}^d, \, |\gamma| \leqslant M, x^{\gamma}v \in \widetilde{\underline{E}}_{\alpha,\alpha',r}^{s,s'-|\gamma|}\}. \end{aligned}$$

Since we have relations

$$Z^{\gamma} = \sum_{\gamma'+\gamma''\leqslant\gamma} a_{\gamma',\gamma''}(D_x/t)^{\gamma'} x^{\gamma''}$$

$$x^{\gamma} = \sum_{\gamma'+\gamma''\leqslant\gamma} b_{\gamma',\gamma''}(D_x/t)^{\gamma'} Z^{\gamma''}$$

with coefficients $a_{\gamma',\gamma''}, b_{\gamma',\gamma''}$ uniformly bounded for $t \geqslant 1$, we see that these spaces are also characterized by

$$(4.1.5) \quad \begin{aligned} \widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M) &= \{v \in \widetilde{E}^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, Z^\gamma v \in \widetilde{E}^{s,s'-|\gamma|}_{\alpha,\alpha',r}\} \\ \underline{\widetilde{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',r}(M) &= \{v \in \underline{\widetilde{E}}^{s,s'}_{\alpha,\alpha',r}; \forall \gamma \in \mathbb{N}^d, |\gamma| \leqslant M, Z^\gamma v \in \underline{\widetilde{E}}^{s,s'-|\gamma|}_{\alpha,\alpha',r}\}. \end{aligned}$$

We also have an injection $\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M) \subset \underline{\widetilde{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',\infty}(M)$ because of the corresponding property $\widetilde{H}^{s,s'}_{\alpha,\alpha',2} \subset \underline{\widetilde{H}}^{s,s'}_{\alpha,\alpha'}$.
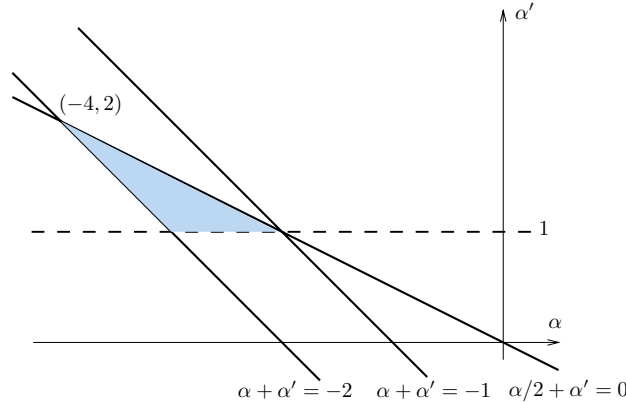
The indices $s, s', \alpha, \alpha', M$ used in (4.1.4), (4.1.5) will be fixed satisfying the following conditions

$$(4.1.6) \quad \begin{aligned} \alpha + 2\alpha' + 4 > s > \alpha' + 2 > 3, \quad \alpha + \alpha' < -1, \quad M = s + s' - 4 \geqslant 6 \\ \alpha + 2\alpha' + 4 > s > 2(\alpha + 2\alpha' + 2). \end{aligned}$$

Remark that the conditions concerning (α, α') read

$$(4.1.7) \quad -2 < \alpha + \alpha' < -1, \quad \alpha' > 1, \quad \frac{\alpha}{2} + \alpha' < 0$$

i.e. (α, α') has to stay in the shaded area in the following figure and s must stay in the

two intervals defined in (4.1.6). One should think of (α, α') as being close to $(-4, 2)$, and in the shaded area, and s being close to $4 - 0$, in the intervals (4.1.6). The fact that s has to be pretty large is imposed by the nonlinear estimates we will need to study the right hand side of (4.1.2). We cannot hope to put this right hand side in a space $\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',r}(M)$ for such a large s: in fact, this would impose to be able to make act on it $s \frac{D_x}{\sqrt{t}}$ derivatives, which is incompatible (if one wants uniform time estimates) with the presence of the exponentials $e^{i\theta}, e^{-3i\theta}$. To get around this difficulty, the main idea is to get rid of these exponentials exploiting the fact that their phases are non-characteristic for the operator $D_t + \frac{D_x^2}{t^2}$.

We fix now $s, s', \alpha, \alpha', M$ satisfying (4.1.6), we take $\mu \in]0, 2[$, close enough to 2, and introduce the following notations:

$$
\begin{aligned}
\mathcal{G} &= \{v \in \underline{\widetilde{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',\infty}(M); \, (tD_t)^k v \in \underline{\widetilde{\mathcal{E}}}^{s-2k,s'}_{\alpha,\alpha',\infty}(M), \, k = 0, 1, 2\} \\
\mathcal{G}' &= \{f \in \mathcal{E}^{s,s'-1}_{\beta,\alpha',1}(M); \, tD_t f \in \mathcal{E}^{s-2,s'-1}_{\beta,\alpha',1}(M)\} \\
\mathcal{V} &= \{V \in \widetilde{\mathcal{E}}^{s+2,s'-1}_{\beta,\alpha',1}(M); \, tD_t V \in \widetilde{\mathcal{E}}^{s,s'-1}_{\beta,\alpha',1}(M), \, \forall c \in S^\mu, \, \forall \gamma \in \mathbb{N}^2, |\gamma| \leqslant 1, \\
&\qquad c(\sqrt{t}x)(D/t)^\gamma (tD_t)^k V \in \widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\alpha+2|\gamma|,\alpha',1}(M), \, k = 0, 1\}
\end{aligned}
$$

(4.1.8)

where the index β is defined in terms of α by

$$(4.1.9) \qquad\qquad \beta = 2(\alpha + \alpha' + 1).$$

We will look for the solution w of (4.1.2) as

$$(4.1.10) \qquad\qquad w = v + V_1 e^{i\theta} + V_{-3} e^{-3i\theta}$$

with $v \in \mathcal{G}$, $V_1, V_{-3} \in \mathcal{V}$. This expression should be understood as the beginning of an expansion of w, since the definition of $\mathcal{V}$ in (4.1.8) shows that V_1, V_{-3} will decay like $\langle \sqrt{t}x \rangle^{-\mu}$. The two functions V_1, V_{-3} will be determined as functions of v in order to cancel the worst oscillating terms in the right hand side of (4.1.2). Let us compute the expression we get when substituting (4.1.10) in the right hand side of (4.1.2). Set the following notations for sets of indices

$$(4.1.11) \qquad\qquad I_1 = \{-4, -2, 2, 0\}, \quad I_2 = \{-5, -1, 3\}.$$

Denote by V the pair (V_1, V_{-3}). We have the following lemma:

LEMMA 4.1.1. — *There are*

– quadratic polynomials $P^1_\gamma(Y_0, Y_1), P^{-3}_\gamma(\overline{Y}_0, \overline{Y}_1)$ in two indeterminates, indexed by $\gamma \in \mathbb{N}^2$, satisfying

$$(4.1.12) \qquad 0 < \deg_{Y_1} P^1_\gamma + |\gamma| \leqslant 2, \; 0 < \deg_{\overline{Y}_1} P^{-3}_\gamma + |\gamma| \leqslant 2,$$

– quadratic polynomials $P^m_\gamma(Y_0, W_0, \overline{Y}_0, \overline{W}_0; Y_1, W_1, \overline{Y}_1, \overline{W}_1)$ in indeterminates $Y_0, \overline{Y}_0 \in \mathbb{C}$, $Y_1, \overline{Y}_1, W_0, \overline{W}_0 \in \mathbb{C}^2$, $W_1, \overline{W}_1 \in \mathbb{C}^4$, linear in the variables $(Y_0, \overline{Y}_0, Y_1, \overline{Y}_1)$ and $(W_0, \overline{W}_0, W_1, \overline{W}_1)$, indexed by $m \in I_1$ and $\gamma \in \mathbb{N}^2$, satisfying

$$(4.1.13) \qquad 0 < \deg_{(Y_1, W_1, \overline{Y}_1, \overline{W}_1)} P^m_\gamma + |\gamma| \leqslant 2,$$

– quadratic polynomials $P^m_\gamma(W_0, \overline{W}_0; W_1, \overline{W}_1)$ indexed by $m \in I_2$ and $\gamma \in \mathbb{N}^2$ satisfying

$$(4.1.14) \qquad 0 < \deg_{(W_1, \overline{W}_1)} P^m_\gamma + |\gamma| \leqslant 2,$$

such that, if w is given by (4.1.10) we have

$$\sum_{|\gamma|\leqslant 2} q_1^\gamma(w,\tfrac{D}{t}w)x^\gamma e^{i\theta} + \sum_{|\gamma|\leqslant 2} q_2^\gamma(\overline{w},\tfrac{D}{t}\overline{w})x^\gamma e^{-3i\theta} = S_1 + S_2 + S_3$$

(4.1.15)
$$S_1 = \sum_{|\gamma|\leqslant 2} P_\gamma^1(v,\tfrac{D}{t}v)x^\gamma e^{i\theta} + \sum_{|\gamma|\leqslant 2} P_\gamma^{-3}(\overline{v},\tfrac{D}{t}\overline{v})x^\gamma e^{-3i\theta}$$

$$S_2 = \sum_{m\in I_1}\sum_{|\gamma|\leqslant 2} P_\gamma^m(v,V,\overline{v},\overline{V};\tfrac{D}{t}v,\tfrac{D}{t}V,\tfrac{D}{t}\overline{v},\tfrac{D}{t}\overline{V})x^\gamma e^{im\theta}$$

$$S_3 = \sum_{m\in I_2}\sum_{|\gamma|\leqslant 2} P_\gamma^m(V,\overline{V};\tfrac{D}{t}V,\tfrac{D}{t}\overline{V})x^\gamma e^{im\theta}.$$

The lemma is proved substituting (4.1.10) into the left hand side of (4.1.15) and sorting terms according to the degree of homogeneity in $(v,\overline{v},\tfrac{D}{t}v,\tfrac{D}{t}\overline{v})$. Our first task will be to write S_1, S_2, S_3 in (4.1.15) making appear the main contributions and remainders. We have to introduce some notations.

DEFINITION 4.1.2. — A quadratic remainder will be a continuous function $v\mapsto R(v)$ (resp. $(v,V)\mapsto R(v,V)$, resp. $V\mapsto R(V)$) defined on the open unit ball of $\mathcal{G}$ (resp. $\mathcal{G}\times\mathcal{V}$, resp. $\mathcal{V}$), with values in $\mathcal{G}'$, vanishing at the origin, and such that one has the following estimates for v,v' in the unit ball of $\mathcal{G}$ (resp. $(v,V),(v',V')$ in the unit ball of $\mathcal{G}\times\mathcal{V}$, resp. V,V' in the unit ball of $\mathcal{V}$)

(4.1.16)
$$\|R(v)-R(v')\|_{\mathcal{G}'} \leqslant C(\|v\|_{\mathcal{G}}+\|v'\|_{\mathcal{G}})\|v-v'\|_{\mathcal{G}}$$
$$\|R(v,V)-R(v',V')\|_{\mathcal{G}'} \leqslant C(\|(v,V)\|_{\mathcal{G}\times\mathcal{V}}+\|(v',V')\|_{\mathcal{G}\times\mathcal{V}})$$
$$\times \|(v-v',V-V')\|_{\mathcal{G}\times\mathcal{V}}$$
$$\|R(V)-R(V')\|_{\mathcal{G}'} \leqslant C(\|V\|_{\mathcal{V}}+\|V'\|_{\mathcal{V}})\|V-V'\|_{\mathcal{V}}.$$

(Actually $\mathcal{V}$ is not endowed with a norm, but with a family of semi-norms. We should thus speak not of the unit ball of $\mathcal{V}$, but of a bounded set, and not use a notation like $\|\cdot\|_{\mathcal{V}}$. Anyway, to avoid introducing some more notations, we shall use such an abuse of notations).

Remind that we defined in definition 3.2.1 the class Σ_ν^μ of symbols $b(t,\xi)$. Let us denote by $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ three generic spaces. We set the following definition:

DEFINITION 4.1.3. — A quadratic symbol $A(v,t,\xi)$ with values in $\mathcal{E}\otimes\Sigma_\nu^\mu$, such that $tD_tA\in\mathcal{E}'\otimes\Sigma_\nu^\mu$, and $(tD_t)^2A\in\mathcal{E}''\otimes\Sigma_\nu^\mu$, will be a linear combination of functions $v\mapsto a(v)b(t,\xi)$ with $b\in\Sigma_\nu^\mu$ and where a is a function defined on the unit ball of $\mathcal{G}$, with values in $\mathcal{E}$, vanishing identically at $v=0$, and such that

(4.1.17)
$$\|a(v)-a(v')\|_{\mathcal{E}} \leqslant C(\|v\|_{\mathcal{G}}+\|v'\|_{\mathcal{G}})\|v-v'\|_{\mathcal{G}}$$
$$\|tD_t(a(v)-a(v'))\|_{\mathcal{E}'} \leqslant C(\|v\|_{\mathcal{G}}+\|v'\|_{\mathcal{G}})\|v-v'\|_{\mathcal{G}}$$
$$\|(tD_t)^2(a(v)-a(v'))\|_{\mathcal{E}''} \leqslant C(\|v\|_{\mathcal{G}}+\|v'\|_{\mathcal{G}})\|v-v'\|_{\mathcal{G}}$$

for any v, v' in the unit ball of $\mathcal{G}$.

We use a similar terminology for symbols $A(v, V, t, \xi)$ defined on the unit ball of $\mathcal{G} \times \mathcal{V}$.

Before beginning the study of equation (4.1.2) let us introduce a last notation. If $A = a \otimes b$ with a a function of (t, x) belonging to a space $\mathcal{E}$ and b a symbol in Σ_ν^μ, we shall denote by $\mathrm{Op}^{\mathcal{B}}(A)$ the paradifferential operator of symbol A, defined using (3.1.12) by

$$(4.1.18) \qquad \mathrm{Op}^{\mathcal{B}}(A)u = T_{a(t,x)}(b(t, D)u).$$

We shall also use the generic notation $\widetilde{\mathrm{Op}}^{\mathcal{B}}(A)$ for the operator

$$(4.1.19) \qquad \widetilde{\mathrm{Op}}^{\mathcal{B}}(A)u = \widetilde{T}_{a(t,x)}(b(t, D)u).$$

The study of equation (4.1.2) will be done in several steps. For each of them, we will present the underlying idea on the example used in the introduction, namely the equation

$$(4.1.20) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)w = \frac{1}{t}e^{i\theta}w\left(\frac{D_x}{t}w\right)$$

where D_x denotes generically one of the derivatives D_{x_j}. The right hand side in the above equation is one of the contributions to the right hand side of (4.1.2).

Step 1: Expression of the equation in terms of v, V

Let us indicate the idea of this first step, when w solves (4.1.20), and when we look for it as $w = v + V_1 e^{i\theta}$ with some V_1 to be determined in function of v in a foregoing step. The assumptions $v \in \mathcal{G}, V_1 \in \mathcal{V}$ mean essentially that v, V_1 are smooth, and moreover that V_1 decays essentially like $\langle \sqrt{t}x \rangle^{-2}$. We plug the expression for w in (4.1.20) and get

$$
\begin{aligned}
\left(D_t + \frac{D_x^2}{t^2}\right)v &= \frac{1}{t}e^{i\theta}v\frac{D_x}{t}v + \frac{1}{t}e^{2i\theta}\left[v\frac{D_x}{t}V_1 + V_1\frac{D_x}{t}v\right] \\
&\quad + \frac{1}{t}e^{3i\theta}V_1\frac{D_x}{t}V_1 - \left(D_t + \frac{D_x^2}{t^2}\right)(V_1(v)e^{i\theta}) \\
&\quad + \text{other terms.}
\end{aligned}
$$
$$(4.1.21)$$

We look at the different terms in the right hand side of this equation and explain how they appear in (4.1.26) – which is the expression we shall obtain in general. First of all write

$$(4.1.22) \qquad V_1\frac{D_x}{t}V_1 e^{i\theta} = T_{e^{i\theta}}\left(V_1\frac{D_x}{t}V_1\right) + R\left(e^{i\theta}, V_1\frac{D_x}{t}V_1\right) + T_{V_1\frac{D_x}{t}V_1}e^{i\theta}.$$

Since V_1 is smooth, so is $V_1\frac{D_x}{t}V_1$ (with a loss of one D_x/t-derivative). Consequently, the first two terms in the right hand side of (4.1.22) are smooth, i.e. will contribute to the R-term in (4.1.26). The third term in the right hand side of (4.1.22) is also

smooth: this will be the contains of lemma 4.1.9. Actually, modulo remainders, it can be written

$$T_{(\langle\sqrt{t}x\rangle^2 V_1)(\langle\sqrt{t}x\rangle^2 D_x V_1/t)}(\langle\sqrt{t}x\rangle^{-4} e^{i\theta}).$$

Since $\langle\sqrt{t}x\rangle^{-4} e^{i\theta}$ accepts four $D_x/\sqrt{t}$-derivatives, and since V_1 decays essentially like $\langle\sqrt{t}x\rangle^{-2}$, this expression will belong to $\mathcal{G}'$ (this uses the fact that the index s satisfies $s < 4$). Consequently we get again a R-type contribution to (4.1.26).

Let us look now at

$$(4.1.23) \qquad v\frac{D_x}{t}v e^{i\theta} = T_{e^{i\theta}}\left(v\frac{D_x}{t}v\right) + R\left(e^{i\theta}, v\frac{D_x}{t}v\right) + T_{v\frac{D_x}{t}v}e^{i\theta}.$$

The first two terms in the right hand side can be handled as above, and give a R-contribution to (4.1.26). The last term $T_{v\frac{D_x}{t}v}e^{i\theta}$ has no chance at all of being smooth, as v has no decay (unlike V_1). We thus keep this term in the right hand side of (4.1.26), writing it as $\mathrm{Op}^{\mathcal{B}}(A_1(v))e^{i\theta}$ where $A_1(v) = v\frac{D_x}{t}v$ is a symbol independent of ξ. Finally, arguing in the same way, we decompose the $e^{2i\theta}$-term in (4.1.21) as a term of type $\mathrm{Op}^{\mathcal{B}}(A_2(v,V))e^{2i\theta}$ and a $R(v,V)$-like term.

Let us turn now to the general case. Our main goal in this first step will be the proof of the following proposition.

PROPOSITION 4.1.4. — *Set $M' = 2M - 5$. There are*

  – *a real number $\mu \in\,]0,2[$,*
  – *quadratic symbols $A_m(v,t,\xi)$, $m = 1, -3$, with values in the space*

$$\widetilde{\mathcal{E}}^{s,s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma^0_0 + \widetilde{\mathcal{E}}^{s,s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma^0_1,$$

such that, for $k = 1,2$, $(tD_t)^k A_m(v,t,\xi)$ belongs to

$$\widetilde{\mathcal{E}}^{s-2k,s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma^0_0 + \widetilde{\mathcal{E}}^{s-2k,s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma^0_1,$$

  – *quadratic symbols $A_m(v,V,t,\xi)$, $m \in I_1 - \{0\}$, with values in the space $\widetilde{\mathcal{E}}^{s,s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma^0_0 + \widetilde{\mathcal{E}}^{s,s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma^0_1$, such that for any $c \in S^{\mu'}$, $\mu' \in [0,\mu]$ and $k = 0,1$*
(4.1.24)
$$c(\sqrt{t}x)(tD_t)^k A_m(v,V,t,\xi) \in \widetilde{\mathcal{E}}^{s-2k-\mu',s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma^0_0 + \widetilde{\mathcal{E}}^{s-2k-\mu',s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma^0_1,$$

  – *two bilinear forms $(v,V) \mapsto H_j(v,V)$, $j = 1,2$, defined on $\mathcal{G} \times \mathcal{V}$ with values in $\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',1}(M+1)$, satisfying, for $k = 0,1$, $c \in S^{\mu'}$, $\mu' \in [0,\mu]$*

$$(4.1.25) \qquad \|c(\sqrt{t}x)(tD_t)^k H_j(v,V)\|_{\widetilde{\mathcal{E}}^{s-\mu'-2k,s'}_{\alpha,\alpha',1}(M+1)} \leqslant C\|v\|_{\mathcal{G}}\|V\|_{\mathcal{V}},$$

  – *quadratic remainders $v \mapsto R(v)$, $(v,V) \mapsto R(v,V)$ defined on the unit ball of $\mathcal{G}$ and $\mathcal{G} \times \mathcal{V}$ respectively, satisfying the conditions of definition 4.1.2,*

such that if (v, V) satisfies the equation

(4.1.26)
$$\left(D_t + \frac{D_x^2}{t^2}\right)v = \frac{1}{t}\sum_{m\in\{1,-3\}}\left[\mathrm{Op}^{\mathcal{B}}(A_m(v,t,\xi))(e^{im\theta}) - \left(D_t + \frac{D_x^2}{t^2}\right)(V_m e^{im\theta})\right]$$
$$+ \frac{1}{t}\sum_{m\in I_1-\{0\}}\mathrm{Op}^{\mathcal{B}}(A_m(v,V,t,\xi))(e^{im\theta})$$
$$+ \frac{1}{t}\sum_{j=1}^{2}x_j H_j(v,V) + \frac{1}{t}R(v) + \frac{1}{t}R(v,V),$$

then w given by (4.1.10) is a solution to (4.1.2).

The proof will be made of several lemmas showing that S_1, S_2, S_3 in (4.1.15) can be reduced to terms in the right hand side of (4.1.26).

LEMMA 4.1.5. — *We fix $\lambda > 2$ close to 2. Let $\gamma_1, \gamma_2, \gamma' \in \mathbb{N}^2$ with $|\gamma_1| \leqslant 1$, $|\gamma_2| \leqslant 1$, $|\gamma'| \leqslant 2M - 3$.*
(i) Let v_1, v_2 be two elements of $\mathcal{G}$. Then for $k = 0, 1, 2$

(4.1.27)
$$(tD_t)^k[x^{\gamma'}((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

belongs to the space $\widetilde{E}_{\beta,\alpha',1}^{s-2k,s'-(\lambda+|\gamma'|-s'+2+\delta)_+-1}$ when $|\gamma_1| + |\gamma_2| > 0$, and to the space $\widetilde{E}_{\alpha,\alpha',1}^{s-2k,s'-(\lambda+|\gamma'|-s'+2+\delta)_+}$ when $\gamma_1 = \gamma_2 = 0$ for any $\delta > 0$. Moreover the norm of (4.1.27) in these spaces is estimated by $C\|v_1\|_{\mathcal{G}}\|v_2\|_{\mathcal{G}}$.
(ii) Let $v_1 \in \mathcal{G}$, $v_2 \in \mathcal{V}$. Then for $k = 0, 1$, $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

(4.1.28)
$$c(\sqrt{t}x)(tD_t)^k[x^{\gamma'}((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

belongs to the space $\widetilde{E}_{\beta,\alpha',1}^{s-2k-\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+-1}$ when $|\gamma_1| + |\gamma_2| > 0$ and to the space $\widetilde{E}_{\alpha,\alpha',1}^{s-2k-\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+}$ when $\gamma_1 = \gamma_2 = 0$ for any $\delta > 0$. Moreover the norm of (4.1.28) in these spaces is estimated by $C\|v_1\|_{\mathcal{G}}\|v_2\|_{\mathcal{V}}$.
(iii) Let $v_1 \in \mathcal{V}$, $v_2 \in \mathcal{V}$. Then for $k = 0, 1$, $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

(4.1.29)
$$c(\sqrt{t}x)(tD_t)^k[x^{\gamma'}((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

belongs to the space $\widetilde{E}_{\beta,\alpha',1}^{s-2k-\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+-1}$ when $|\gamma_1| + |\gamma_2| > 0$ and to the space $\widetilde{E}_{\alpha,\alpha',1}^{s-2k-\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+}$ when $\gamma_1 = \gamma_2 = 0$, and for $c \in S^{2\mu'}$, $\mu' \in [0, \mu]$

(4.1.30)
$$c(\sqrt{t}x)[x^{\gamma'}((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

belongs to the space $\widetilde{E}_{\beta,\alpha',1}^{s-2\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+-1}$ when $|\gamma_1|+|\gamma_2| > 0$ and to the space $\widetilde{E}_{\alpha,\alpha',1}^{s-2\mu',s'-(\lambda+|\gamma'|-s'+2+\delta)_+}$ when $\gamma_1 = \gamma_2 = 0$. Moreover the norm of (4.1.28) in these spaces is estimated by $C\|v_1\|_{\mathcal{V}}\|v_2\|_{\mathcal{V}}$.

Proof

(i) The quantity to study may be written as a combination of expressions $x^{\gamma'} w_1 w_2$ with

$$w_1 = (tD_t)^{k_1}(\tfrac{D}{t})^{\gamma_1} v_1 \in \widetilde{\underline{\mathcal{E}}}_{\alpha,\alpha'+|\gamma_1|,\infty}^{s-2k_1,s'-|\gamma_1|}(M) \subset \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_1|,\alpha',\infty}^{s-2k_1,s'-|\gamma_1|}(M)$$

$$w_2 = (tD_t)^{k_2}(\tfrac{D}{t})^{\gamma_2} v_2 \in \widetilde{\underline{\mathcal{E}}}_{\alpha,\alpha'+|\gamma_2|,\infty}^{s-2k_2,s'-|\gamma_2|}(M) \subset \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_2|,\alpha',\infty}^{s-2k_2,s'-|\gamma_2|}(M)$$

for $k_1 + k_2 = k$. When $|\gamma_1| + |\gamma_2| > 0$, one of the first lower indices of the above spaces equals $\alpha + 2$, so we can apply (i) of corollary 3.1.8 with $\sigma_j = 2k_j$, $\sigma_j' = |\gamma_j|$. One checks that assumptions (3.1.50), (3.1.51) are satisfied because of (4.1.6) and of $\lambda > 2$. Moreover, condition (3.1.52) is satisfied as long as

$$s + s' > 2 + \frac{1}{2}(|\gamma'| + |\gamma_1| + |\gamma_2|) + 1$$

(since $\sigma_1 + \sigma_2 \leqslant 4$), which is true since $s + s' = M + 4$, $|\gamma'| \leqslant 2M - 3$. By (3.1.53) we get that $x^{\gamma'} w_1 w_2$ is in the wanted space. When $\gamma_1 = \gamma_2 = 0$, we use (ii) of corollary 3.1.8, and get also the conclusion of the lemma, since in this case $\max(\sigma_1', \sigma_2') = 0$.

(ii) We write expression (4.1.28) in terms of $x^{\gamma'} w_1 w_2$ with

$$w_1 = (tD_t)^{k_1}(\tfrac{D}{t})^{\gamma_1} v_1 \in \widetilde{\underline{\mathcal{E}}}_{\alpha,\alpha'+|\gamma_1|,\infty}^{s-2k_1,s'-|\gamma_1|}(M) \subset \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_1|,\alpha',\infty}^{s-2k_1,s'-|\gamma_1|}(M)$$

$$w_2 = c(\sqrt{t}x)(tD_t)^{k_2}(\tfrac{D}{t})^{\gamma_2} v_2 \in \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_2|,\alpha',\infty}^{s-\mu'-2k_2,s'-|\gamma_2|}(M) \subset \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_2|,\alpha',\infty}^{s-\mu'-2k_2,s'-|\gamma_2|}(M)$$

for $k_1 + k_2 = k$, where the last inclusion in the first line comes from the definition of the spaces. We then apply corollary 3.1.8 as in the proof of (i), taking here $\sigma_1 = 2k_1$, $\sigma_2 = 2k_2 + \mu'$ which still satisfies $\sigma_1 + \sigma_2 \leqslant 4$ since $k \leqslant 1$, $\mu' \leqslant 2$.

(iii) In this case, (4.1.29) is estimated as (4.1.28). For (4.1.30), we write this expression in terms of $x^{\gamma'} w_1 w_2$ where, if $c_j \in S^{\mu'}$ with $c_1 c_2 = c$,

$$w_j = c_j(\sqrt{t}x)(\tfrac{D}{t})^{\gamma_j} v_j \in \widetilde{\underline{\mathcal{E}}}_{\alpha+2|\gamma_j|,\alpha',\infty}^{s-\mu',s'-|\gamma_j|}(M)$$

and we apply again corollary 3.1.8. $\square$

Let us study now term S_1 of (4.1.15).

LEMMA 4.1.6. — *Set $M' = 2M - 5$. There are quadratic symbols $A_m(v,t,\xi)$, $m = 1, -3$, with values in $\widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s,s'-1}(M') \otimes \Sigma_0^0 + \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'-1}(M') \otimes \Sigma_1^0$ such that for $k = 1, 2$, $(tD_t)^k A_m \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2k,s'-1}(M') \otimes \Sigma_0^0 + \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-2k,s'-1}(M') \otimes \Sigma_1^0$, and a quadratic remainder $v \mapsto R(v)$ such that*

$$(4.1.31) \qquad S_1 = \sum_{m \in \{1,-3\}} \mathrm{Op}^{\mathcal{B}}(A_m(v,t,\xi))(e^{im\theta}) + R(v).$$

Proof. — We will study the case of contribution $P_\gamma^1(v,(D/t)v)e^{i\theta}x^\gamma$ to S_1, the other contribution being treated in the same way. Because of (4.1.12) such an expression is a linear combination of quantities of type $((D/t)^{\gamma_1}v)((D/t)^{\gamma_2}v)x^{\gamma_3}e^{i\theta}$ for $\gamma_1, \gamma_2, \gamma_3$

satisfying $|\gamma_1| \leqslant 1$, $|\gamma_2| \leqslant 1$, $0 < |\gamma_1| + |\gamma_2| + |\gamma_3| \leqslant 2$. We decompose such an expression as

$$(4.1.32) \quad \mathrm{Op}^{\mathcal{B}}(x^{\gamma_3}e^{i\theta})[((D/t)^{\gamma_1}v)((D/t)^{\gamma_2}v)] + R(x^{\gamma_3}e^{i\theta}, ((D/t)^{\gamma_1}v)((D/t)^{\gamma_2}v))$$
$$+ \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1}v(D/t)^{\gamma_2}v)(x^{\gamma_3}e^{i\theta}) = W_1 + W_2 + W_3.$$

To show that W_1 is a quadratic remainder, we just have to prove by definition of $\mathcal{G}'$ that for any $\gamma \in \mathbb{N}^2$ with $|\gamma| \leqslant M$, any $k = 0, 1$, any $v_1, v_2 \in \mathcal{G}$,

$$(4.1.33) \qquad (tD_t)^k x^{\gamma} \mathrm{Op}^{\mathcal{B}}(x^{\gamma_3}e^{i\theta})[((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

belongs to $E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}$, and that its norm in this space is estimated by $C\|v_1\|_{\mathcal{G}}\|v_2\|_{\mathcal{G}}$. Remark that it follows from formula (3.1.12) that when we multiply a paraproduct $T_a b$ by x, $xT_a b$ may be written, commuting x with the Δ_j's, as a linear combination of terms of type $\widetilde{T}_a b$, $T_a(xb)$. In the same way, an expression $T_{(xa)}b$ is a linear combination of expressions $\widetilde{T}_a b$, $T_a(xb)$. Applying this to (4.1.33), we see that this term may be written as a linear combination of expressions

$$(4.1.34) \qquad \mathrm{Op}^{\mathcal{B}}(x^{\gamma_3'}(tD_t)^{k'}(e^{i\theta}))[(tD_t)^{k''}x^{\gamma'}((D/t)^{\gamma_1}v_1)((D/t)^{\gamma_2}v_2)]$$

with $k' + k'' = k$, $|\gamma'| \leqslant |\gamma| + 1 \leqslant M + 1 \leqslant 2M - 3$, $\gamma_3' = 0$ when $|\gamma_1| + |\gamma_2| > 0$, $|\gamma_3'| = 1$ when $(\gamma_1, \gamma_2) = (0,0)$. We apply (4.1.27) to the term inside the bracket of (4.1.34). Since $s' = M + 4 - s > 5$ as $M > 5$, assertion (i) of lemma 4.1.5 shows that this term belongs to $E_{\beta,\alpha',1}^{s-2k'',s'-|\gamma|-1}$ when $|\gamma_1| + |\gamma_2| > 0$, and to $E_{\alpha,\alpha',1}^{s-2k'',s'-|\gamma|}$ when $(\gamma_1, \gamma_2) = (0,0)$. Moreover, we have the estimate

$$\|S_j(x^{\gamma_3'}(tD_t)^{k'}e^{i\theta})\|_{L^\infty} = \left\| S_j\left(\left(\frac{2D_x}{t}\right)^{\gamma_3'}\left(\frac{D_x^2}{t} + \frac{i}{2}\right)^{k'}e^{i\theta}\right) \right\|_{L^\infty}$$
$$\leqslant C\left(\frac{2^j}{t}\right)^{|\gamma_3'|}\left(1 + \frac{2^j}{\sqrt{t}}\right)^{2k'}.$$

We conclude that when $|\gamma_1| + |\gamma_2| > 0$, $\gamma_3' = 0$, (4.1.34) is in $E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}$ and when $(\gamma_1, \gamma_2) = (0,0)$, $|\gamma_3'| = 1$, (4.1.34) is in

$$E_{\alpha,\alpha'+1,1}^{s-2k,s'-|\gamma|-1} \subset E_{\alpha+2,\alpha',1}^{s-2k,s'-|\gamma|-1} \subset E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}.$$

This is the wanted conclusion, which shows that W_1 is a quadratic remainder.

Let us show that W_2 is also a quadratic remainder, proving that for any $\gamma \in \mathbb{N}^2$, $|\gamma| \leqslant M$, any $k = 0, 1$, any $v_1, v_2 \in \mathcal{G}$

$$(4.1.35) \qquad (tD_t)^k x^{\gamma} R(x^{\gamma_3}e^{i\theta}, (D/t)^{\gamma_1}v_1(D/t)^{\gamma_2}v_2)$$

belongs to $E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}$, with estimates by $\|v_1\|_{\mathcal{G}}\|v_2\|_{\mathcal{G}}$. As for (4.1.33), we can write (4.1.35) as a linear combination of quantities

$$(4.1.36) \qquad \widetilde{R}((tD_t)^{k'}x^{\gamma_3'}e^{i\theta}, (tD_t)^{k''}x^{\gamma'}(D/t)^{\gamma_1}v_1(D/t)^{\gamma_2}v_2)$$

with $k' + k'' = k$, $|\gamma'| \leqslant |\gamma| + 1 \leqslant M + 1 \leqslant 2M - 3$, $\gamma_3' = 0$ if $|\gamma_1| + |\gamma_2| > 0$, $|\gamma_3'| = 1$ if $(\gamma_1, \gamma_2) = (0,0)$. We have seen above that $w = (tD_t)^{k''}x^{\gamma'}(D/t)^{\gamma_1}v_1(D/t)^{\gamma_2}v_2$ belongs

to $E_{\beta,\alpha',1}^{s-2k'',s'-|\gamma|-1}$ when $|\gamma_1| + |\gamma_2| > 0$ and to $E_{\alpha,\alpha',1}^{s-2k'',s'-|\gamma|}$ when $(\gamma_1, \gamma_2) = (0,0)$. In the first case, we apply (i) of proposition 3.1.6 with $\kappa_1 = k'$, $\kappa_2 = 0$ and α, (resp. s, resp. s') replaced by β (resp. $s - 2k''$, resp. $s' - |\gamma| - 1$). By (4.1.6), conditions (3.1.36) are satisfied, so (4.1.36) belongs to $E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}$. When $(\gamma_1, \gamma_2) = (0,0)$, we use (ii) of proposition 3.1.6 with $\kappa_1 = k' \leqslant 1$, $\kappa_2 = \gamma_3'$ and s (resp. s') replaced by $s - 2k''$ (resp. $s' - |\gamma|$). Since (4.1.6) is satisfied, assumptions (3.1.39) are realized, so (4.1.36) again belongs to $E_{\beta,\alpha',1}^{s-2k,s'-|\gamma|-1}$. This gives the conclusion.

Finally, let us show that W_3 gives a contribution to the sum in the right hand side of (4.1.31). This term is the quadratic form associated to the bilinear form

$$(4.1.37) \qquad \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2)(x^{\gamma_3} e^{i\theta}).$$

Consider first the case $\gamma_3 = 0$ whence $|\gamma_1| + |\gamma_2| > 0$. We write then (4.1.37) as $\mathrm{Op}^{\mathcal{B}}(a(t,x) \otimes b(t,\xi))(e^{i\theta})$ where $b \equiv 1 \in \Sigma_0^0$, and where $a = (D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2$ satisfies by lemma 4.1.5 (i) for $k = 0, 1, 2$ $(tD_t)^k a \in \mathcal{E}_{\beta,\alpha',1}^{s-2k,s'-1}(M')$. In other words, W_3 is of form $\mathrm{Op}^{\mathcal{B}}(A_1)(e^{i\theta})$ for a quadratic symbol A_1. When $\gamma_3 \neq 0$, we take $\chi \in C_0^\infty(\mathbb{R}^2)$, $\chi \equiv 1$ close to 0 and decompose (4.1.37) as

$$(4.1.38) \quad \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2 \chi(\xi/t))((2D/t)^{\gamma_3} e^{i\theta})$$
$$+ \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2 (1-\chi)(\xi/t))((2D/t)^{\gamma_3} e^{i\theta}).$$

The first term in (4.1.38) may be written $\mathrm{Op}^{\mathcal{B}}(a \otimes b)(e^{i\theta})$, with $b = \chi(\xi/t)(2\xi/t)^{\gamma_3}$, element of Σ_1^0, and $a = (D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2$. We just have to use lemma 4.1.5 (i), to conclude that $(tD_t)^k a \in \mathcal{E}_{\alpha,\alpha',1}^{s-2k,s'-1}(M')$, i.e. that $a \otimes b$ is a quadratic symbol of wanted type.

The second contribution to (4.1.38) may be written as a linear combination of terms

$$(4.1.39) \qquad \widetilde{\mathrm{Op}}^{\mathcal{B}}(x^{\gamma_3'} (D/t)^{\gamma_1} v_1 (D/t)^{\gamma_2} v_2 (1-\chi)(\xi/t))(e^{i\theta})$$

with $|\gamma_3'| \leqslant 1$ if $|\gamma_1| + |\gamma_2| > 0$, $|\gamma_3'| \leqslant 2$ if $\gamma_1 = \gamma_2 = 0$. For $\gamma \in \mathbb{N}^2$ with $|\gamma| \leqslant M'$, we have $|\gamma| + |\gamma_3'| \leqslant M' + 2 \leqslant 2M - 3$. Again, lemma 4.1.5 (i) shows that (4.1.39) may be written $\mathrm{Op}^{\mathcal{B}}(a \otimes b)(e^{i\theta})$ with $b = (1-\chi)(\xi/t) \in \Sigma_1^0$, $(tD_t)^k a \in \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-2k,s'-1}(M')$ for $k = 0, 1, 2$. This concludes the proof of the lemma. $\qquad \square$

We study now the contribution to S_2 in (4.1.15) indexed by $m \in I_1 - \{0\}$.

LEMMA 4.1.7. — *Set $M' = 2M - 5$. Denote by S_2' the contribution to S_2 given by the second sum (4.1.15) limited to $m \in I_1 - \{0\}$. There are quadratic symbols $A_m(v, V, t, \xi)$ $m \in I_1 - \{0\}$, with values in $\widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s,s'-1}(M') \otimes \Sigma_0^0 + \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'-1}(M') \otimes \Sigma_1^0$, satisfying (4.1.24), and there is a quadratic remainder $R(v, V)$ satisfying the second condition (4.1.16), such that*

$$(4.1.40) \qquad S_2' = \sum_{m \in I_1 - \{0\}} \mathrm{Op}^{\mathcal{B}}(A_m(v, V, t, \xi))(e^{im\theta}) + R(v, V).$$

Proof. — The general term of S_2' may be written

$$(4.1.41) \qquad ((D/t)^{\gamma_1} v)((D/t)^{\gamma_2} V)(x^{\gamma_3} e^{im\theta})$$

where $|\gamma_1| \leqslant 1$, $|\gamma_2| \leqslant 1$, $0 < |\gamma_1| + |\gamma_2| + |\gamma_3| \leqslant 2$, and where v (resp. V) should be read indifferently v or $\overline{v}$ (resp. V or $\overline{V}$). We decompose (4.1.41) as

$$(4.1.42) \qquad \begin{aligned} \mathrm{Op}^{\mathcal{B}}(x^{\gamma_3} e^{im\theta})[((\tfrac{D}{t})^{\gamma_1} v)((\tfrac{D}{t})^{\gamma_2} V)] &+ R(x^{\gamma_3} e^{im\theta}, ((\tfrac{D}{t})^{\gamma_1} v)((\tfrac{D}{t})^{\gamma_2} V)) \\ &+ \mathrm{Op}^{\mathcal{B}}((\tfrac{D}{t})^{\gamma_1} v(\tfrac{D}{t})^{\gamma_2} V)(x^{\gamma_3} e^{im\theta}). \end{aligned}$$

Remark that the assumption $V \in \mathcal{V}$ implies

$$(tD_t)^k V \in \widetilde{\mathcal{E}}^{s-2k,s'}_{\alpha,\alpha',\infty}(M) \subset \underline{\widetilde{\mathcal{E}}}^{s-2k,s'}_{\alpha,\alpha',\infty}(M), \ k = 0,1,$$

i.e. V satisfies the same assumptions as v. Consequently, the proof of the preceding lemma implies that the first two terms in (4.1.42) provide a contribution to the quadratic remainder $R(v,V)$ (the fact that v admits two tD_t derivatives by definition of $\mathcal{G}$, instead of only one for V, does not play any role, since this is not used in the study of the first two terms in (4.1.32)). Let us study the last term in (4.1.42). When $\gamma_3 = 0$, so $|\gamma_1| + |\gamma_2| > 0$, (ii) of lemma 4.1.5 implies that for $k = 0,1$, $c \in S^{\mu'}$, $\mu' \in [0,\mu]$, $a = (D/t)^{\gamma_1} v(D/t)^{\gamma_2} V$ satisfies

$$c(\sqrt{t}x)(tD_t)^k a \in \widetilde{\mathcal{E}}^{s-2k-\mu',s'-1}_{\beta,\alpha',1}(M').$$

Consequently, the last term of (4.1.42) is of type $\mathrm{Op}^{\mathcal{B}}(a \otimes b)(e^{im\theta})$ with $b \in \Sigma_0^0$, i.e. satisfies (4.1.24). When $\gamma_3 \neq 0$, write the last term in (4.1.42) as

$$(4.1.43) \quad \begin{aligned} \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1} v(D/t)^{\gamma_2} V \chi(\xi/t))((2D/mt)^{\gamma_3} e^{im\theta}) \\ + \mathrm{Op}^{\mathcal{B}}((D/t)^{\gamma_1} v(D/t)^{\gamma_2} V(1-\chi)(\xi/t))(x^{\gamma_3} e^{im\theta}). \end{aligned}$$

The first term may be written $\mathrm{Op}^{\mathcal{B}}(a \otimes b)(e^{im\theta})$ with a symbol

$$b(t,\xi) = \chi(\xi/t)(2\xi/tm)^{\gamma_3} \in \Sigma_1^0,$$

since $|\gamma_3| \geqslant 1$, and with a satisfying

$$(4.1.44) \qquad c(\sqrt{t}x)(tD_t)^k a \in \widetilde{\mathcal{E}}^{s-2k-\mu',s'-1}_{\alpha,\alpha',1}(M')$$

for $k = 0,1$, $c \in S^{\mu'}$, $\mu' \in [0,\mu]$ by (ii) of lemma 4.1.5. The second contribution to (4.1.43) may be written as linear combination of terms

$$\widetilde{\mathrm{Op}}^{\mathcal{B}}(x^{\gamma_3'}(D/t)^{\gamma_1} v(D/t)^{\gamma_2} V \chi_1(\xi/t))(e^{im\theta})$$

where $\chi_1 \in C^\infty$ is bounded as well as its derivatives, $\chi_1 \equiv 0$ close to 0 and $|\gamma_3'| \leqslant |\gamma_3|$. This may be written as $\mathrm{Op}^{\mathcal{B}}(a \otimes b)(e^{im\theta})$ with $b = \chi_1(\xi/t) \in \Sigma_1^0$ and a satisfying (4.1.44) because of (ii) of lemma 4.1.5 and of the fact that we have $s' > 5$, since $s' = M + 4 - s$ and $M > 5$. The conclusion of the lemma follows. $\qquad \square$

Let us examine now the non-oscillating contribution to S_2.

LEMMA 4.1.8. — *The sum* $\sum_{|\gamma|\leqslant 2} P_\gamma^0(v, V, \overline{v}, \overline{V}; \frac{D}{t}v, \frac{D}{t}V, \frac{D}{t}\overline{v}, \frac{D}{t}\overline{V})$ *may be written*

$$(4.1.45) \qquad \sum_{j=1}^2 x_j H_j(v, V) + R(v, V)$$

where H_j satisfies (4.1.25) and $R(v, V)$ is a quadratic remainder.

Proof. — The general term of the expression to be studied is of type

$$(4.1.46) \qquad ((D/t)^{\gamma_1} v)((D/t)^{\gamma_2} V) x^{\gamma_3}$$

for $|\gamma_1| \leqslant 1$, $|\gamma_2| \leqslant 1$, $0 < |\gamma_1| + |\gamma_2| + |\gamma_3| \leqslant 2$, and where v (resp. V) should be read indifferently v or $\overline{v}$ (resp. V or $\overline{V}$). When $|\gamma_1| + |\gamma_2| > 0$, and so $|\gamma_3| \leqslant 1$, (ii) of lemma 4.1.5 shows that this expression belongs to $\mathcal{G}'$, and so provides a contribution to the quadratic remainder $R(v, V)$. When $\gamma_1 = \gamma_2 = 0$, we have $1 \leqslant |\gamma_3| \leqslant 2$, so we can write (4.1.46) $\sum_{j=1}^2 x_j H_j(v, V)$ where H_j is of form $x^{\gamma_3'} v V$ for γ_3' satisfying $|\gamma_3'| \leqslant |\gamma_3| - 1 \leqslant 1$. By (ii) of lemma 4.1.5, and the assumption $s' > 5$, which follows from $M > 5$, we obtain

$$c(\sqrt{t}x)(tD_t)^k H_j \in \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-2k-\mu',s'}(M+1)$$

for $c \in S^{\mu'}$, $\mu' \in [0, \mu]$, which is the wanted conclusion. $\square$

Finally, let us study contribution S_3 to (4.1.15).

LEMMA 4.1.9. — *If μ is in $]0, 2[$ close enough to 2, the contribution S_3 to (4.1.15) is a quadratic remainder $R(V)$ satisfying the last condition (4.1.16).*

Proof. — We write S_3 as a linear combination of expressions

$$(4.1.47) \qquad x^{\gamma_3}((D/t)^{\gamma_1} V)((D/t)^{\gamma_2} V) e^{im\theta}$$

with $m \in I_2$, $0 < |\gamma_1| + |\gamma_2| + |\gamma_3| \leqslant 2$, and where again one should read indifferently V or $\overline{V}$ for V. We decompose (4.1.47) as

$$(4.1.48) \qquad \begin{aligned} &\mathrm{Op}^{\mathcal{B}}(x^{\gamma_3} e^{im\theta})[((\tfrac{D}{t})^{\gamma_1} V)((\tfrac{D}{t})^{\gamma_2} V)] + R(x^{\gamma_3} e^{im\theta}, ((\tfrac{D}{t})^{\gamma_1} V)((\tfrac{D}{t})^{\gamma_2} V)) \\ &\qquad\qquad + \mathrm{Op}^{\mathcal{B}}((\tfrac{D}{t})^{\gamma_1} V(\tfrac{D}{t})^{\gamma_2} V)(x^{\gamma_3} e^{im\theta}). \end{aligned}$$

We have seen in the proof of lemma 4.1.7 that V satisfies the same assumptions as v, and so that the first two terms are quadratic remainders of type $R(V)$. We are thus left with the study of the bilinear term in $(V', V'') \in \mathcal{V} \times \mathcal{V}$

$$(4.1.49) \qquad \mathrm{Op}^{\mathcal{B}}(((D/t)^{\gamma_1} V')((D/t)^{\gamma_2} V''))(x^{\gamma_3} e^{im\theta}).$$

When $\gamma_3 = 0$, so $|\gamma_1| + |\gamma_2| > 0$, lemma 4.1.5 (iii) shows that $w = (D/t)^{\gamma_1} V'(D/t)^{\gamma_2} V''$ satisfies

$$\langle \sqrt{t}x \rangle^{2\mu} w \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2\mu,s'-1}(M), \quad \langle \sqrt{t}x \rangle^{\mu} tD_t w \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-\mu,s'-1}(M).$$

We can exploit these decay assumptions on w to prove that (4.1.49) is actually a remainder: one checks immediately using (4.1.6) that if μ is chosen in $]0, 2[$ close enough to 2, assumption (3.2.87) is satisfied for w (replacing in (3.2.87) (α, μ, s') by

$(\beta, 2\mu, s' - 1))$ and for $tD_t w$ (replacing in (3.2.87) (α, s, s') by $(\beta, s - 2, s' - 1)$). We thus deduce from corollary 3.2.7 that

$$\mathrm{Op}^{\mathcal{B}}(w)(e^{im\theta}) \in \mathcal{E}^{s,s'-1}_{\beta+2,\alpha',1}(M) \subset \mathcal{E}^{s,s'-1}_{\beta,\alpha',1}(M)$$

$$\mathrm{Op}^{\mathcal{B}}(tD_t w)(e^{im\theta}) \in \mathcal{E}^{s-2,s'-1}_{\beta+2,\alpha',1}(M) \subset \mathcal{E}^{s-2,s'-1}_{\beta,\alpha',1}(M).$$

Since $\mathrm{Op}^{\mathcal{B}}(w)(tD_t e^{im\theta})$ belongs also to this last space, and since the norms in these spaces are estimated by $C\|V'\|_{\mathcal{V}}\|V''\|_{\mathcal{V}}$, this shows that (4.1.49) is a quadratic remainder.

When $\gamma_3 \neq 0$, we write (4.1.49) as a combination of terms

(4.1.50)
$$\mathrm{Op}^{\mathcal{B}}(((D/t)^{\gamma_1} V')((D/t)^{\gamma_2} V'')\chi(\xi/t))[(2D/mt)^{\gamma_3} e^{im\theta}],$$
$$\mathrm{Op}^{\mathcal{B}}(x^{\gamma_3'}((D/t)^{\gamma_1} V')((D/t)^{\gamma_2} V'')\chi_1(\xi/t))[e^{im\theta}]$$

with $\chi_1 \in C^{\infty}(\mathbb{R}^d)$, bounded as well as all its derivatives, $\chi_1 \equiv 0$ close to 0, and $|\gamma_3'| \leqslant |\gamma_3|$. All these expressions can be written as quantities of type $\widetilde{\mathrm{Op}}^{\mathcal{B}}(w(t,x)b(t,\xi))(e^{im\theta})$ with $b \in \Sigma_1^0$, and w satisfying, because of (iii) of lemma 4.1.5,

$$\langle \sqrt{t}x \rangle^{2\mu} w \in \widetilde{\mathcal{E}}^{s-2\mu,s'-1}_{\alpha,\alpha',1}(M), \langle \sqrt{t}x \rangle^{\mu}(tD_t)w \in \widetilde{\mathcal{E}}^{s-2-\mu,s'-1}_{\alpha,\alpha',1}(M).$$

Since (3.2.86) (with s' replaced by $s' - 1$) is satisfied by (4.1.6), we get from corollary 3.2.7 that (4.1.50) belongs to $\mathcal{E}^{s,s'-1}_{\alpha+4,\alpha',1}(M) \subset \mathcal{E}^{s,s'-1}_{\beta,\alpha',1}(M)$, and its image by tD_t to $\mathcal{E}^{s-2,s'-1}_{\beta,\alpha',1}(M)$. $\square$

Proposition 4.1.4 now follows from (4.1.2), (4.1.10), (4.1.15) and lemmas 4.1.6 to 4.1.9.

Step 2: Cancelation of main oscillating terms

We want now to choose V_1, V_{-3} to eliminate in the right hand side of (4.1.26) the main oscillating contributions, namely $\mathrm{Op}^{\mathcal{B}}(A_m)(e^{im\theta})$, $m = 1, -3$. In the case of example (4.1.20), the oscillating term to get rid of is $T_{(v\frac{D_x}{t}v)}e^{i\theta}$. To do so, we remark that an explicit computation shows that

(4.1.51)
$$\left(D_t + \frac{D_x^2}{t^2}\right)e^{i\theta} = \frac{1}{t}\Lambda(t,D)e^{i\theta}$$

where $\Lambda(t,\xi) = 2\frac{\xi^2}{t} + i$ is an elliptic symbol of order 2 in $\xi/\sqrt{t}$. We thus set

(4.1.52)
$$V_1(v) = e^{-i\theta}T_{(v\frac{D_x}{t}v)}(\Lambda(t,D)^{-1}e^{i\theta}).$$

The equality (4.1.51) allows one to prove that modulo remainders

$$\left(D_t + \frac{D_x^2}{t^2}\right)[V_1(v)e^{i\theta}] = \frac{1}{t}T_{(v\frac{D_x}{t}v)}e^{i\theta}$$

i.e. this choice of V_1 eliminates the $\mathrm{Op}^{\mathcal{B}}(A_1)(e^{i\theta})$ contribution to (4.1.26) (this is the contains of lemma 4.1.11). Moreover V_1 will satisfy the wanted decay and smoothness requirements: this is established in general in proposition 4.1.10. The idea is that $\Lambda(t,D)^{-1}e^{i\theta}$ in (4.1.52) admits two $D_x/\sqrt{t}$-derivatives. But when acting on an

oscillatory exponential $e^{i\theta}$, a gain of $(D_x/\sqrt{t})^{-2}$ in smoothness is equivalent to a gain of $\langle\sqrt{t}x\rangle^{-2}$ in decay. Let us now proceed to the general case.

PROPOSITION 4.1.10. — *There exists for $m = 1, -3$ Lipschitz functions $v \mapsto V_m(v)$ defined on the unit ball of $\mathcal{G}$, with values in $\mathcal{V}$, vanishing at 0, and quadratic symbols $v \mapsto A'_m(v, t, \xi)$ with values in*

$$(4.1.53) \qquad \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-\ell,s'-1}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-2-\ell,s'-1}(M') \otimes \Sigma_1^{-2-\ell}$$

satisfying

$$(4.1.54) \qquad tD_t A'_m \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-4-\ell,s'-1}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-4-\ell,s'-1}(M') \otimes \Sigma_1^{-2-\ell}$$

such that for $m = 1, -3$

$$(4.1.55) \quad \left(D_t + \frac{D_x^2}{t^2}\right)(V_m e^{im\theta}) = \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(A_m(v, t, \xi))(e^{im\theta}) - \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(A'_m(v, t, \xi))(e^{im\theta}).$$

We first prove the following lemma.

LEMMA 4.1.11. — *Let $\ell \in \mathbb{N}$, $m \in \mathbb{Z} - \{0, -1\}$, $k \in \{1, 2\}$, $M' = 2M - 5$.*
    (i) Let $A(t, x, \xi)$ be a symbol satisfying for $k' = 0, \cdots, k$

$$(4.1.56) \qquad\qquad (tD_t)^{k'} A \in \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\ell-2k',s'-1}(M') \otimes \Sigma_1^{-\ell}$$

(resp.

$$(4.1.57) \qquad\qquad (tD_t)^{k'} A \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\ell-2k',s'-1}(M') \otimes \Sigma_0^{-\ell}).$$

There are $B(t, x, \xi), C(t, x, \xi)$ satisfying

$$(4.1.58) \qquad
\begin{aligned}
(tD_t)^{k'} B &\in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\ell-\ell'-2k',s'-1}(M') \otimes \Sigma_1^{-\ell-\ell'-2}, \quad k' = 0, \cdots, k \\
(tD_t)^{k'} C &\in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\ell-\ell'-2k'-2,s'-1}(M') \otimes \Sigma_1^{-\ell-\ell'-2}, \quad k' = 0, \cdots, k-1
\end{aligned}$$

(resp.

$$(4.1.59) \qquad
\begin{aligned}
(tD_t)^{k'} B &\in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\ell-\ell'-2k',s'-1}(M') \otimes \Sigma_0^{-\ell-\ell'-2}, \quad k' = 0, \cdots, k \\
(tD_t)^{k'} C &\in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\ell-\ell'-2k'-2,s'-1}(M') \otimes \Sigma_0^{-\ell-\ell'-2}, \quad k' = 0, \cdots, k-1)
\end{aligned}$$

with B supported in the domain $\{\xi; |\xi| > c\sqrt{t}\}$, and such that

$$(4.1.60) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)[\mathrm{Op}^{\mathcal{B}}(B)(e^{im\theta})] = \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(A)(e^{im\theta}) + \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(C)(e^{im\theta}).$$

Moreover, the semi-norms of B and C are controlled linearly by the semi-norms of A.

(ii) Assume that A satisfies for $k' = 0, 1$, and any $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

$$(4.1.61) \qquad c(\sqrt{t}x)(tD_t)^{k'}A \in \widetilde{\mathcal{E}}^{s-\ell-\mu'-2k',s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma_1^{-\ell}$$

(resp.

$$(4.1.62) \qquad c(\sqrt{t}x)(tD_t)^{k'}A \in \widetilde{\mathcal{E}}^{s-\ell-\mu'-2k',s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma_0^{-\ell}).$$

Then the symbols B and C satisfy for any $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

$$(4.1.63) \qquad \begin{aligned} & c(\sqrt{t}x)(tD_t)^{k'}B \in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}^{s-\ell-\ell'-\mu'-2k',s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma_1^{-\ell-\ell'-2}, \quad k' = 0, 1 \\ & c(\sqrt{t}x)C \in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}^{s-\ell-\ell'-\mu'-2,s'-1}_{\alpha,\alpha',1}(M') \otimes \Sigma_1^{-\ell-\ell'-2}, \end{aligned}$$

(resp.

$$(4.1.64) \qquad \begin{aligned} & c(\sqrt{t}x)(tD_t)^{k'}B \in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}^{s-\ell-\ell'-\mu'-2k',s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma_0^{-\ell-\ell'-2}, \quad k' = 0, 1 \\ & c(\sqrt{t}x)C \in \sum_{\ell'=0}^{1} \widetilde{\mathcal{E}}^{s-\ell-\ell'-\mu'-2,s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma_0^{-\ell-\ell'-2}). \end{aligned}$$

Proof. — The lemma asserts that if A is given, we can find a B, which has the same smoothness and a better $\xi/\sqrt{t}$-decay than A, so that (4.1.60) is true, with a remainder C that is two degree less regular than A but has a better $\xi/\sqrt{t}$ decay.

Assume that A satisfies (4.1.56). Take $\chi \in C_0^\infty(\mathbb{R}^2)$, $\chi \equiv 1$ close to 0 and write

$$A(t, x, \xi) = A(t, x, \xi)(1 - \chi(\xi/\sqrt{t})) + A(t, x, \xi)\chi(\xi/\sqrt{t}).$$

We incorporate the last term in C i.e. we assume that A vanishes for $|\xi| < c\sqrt{t}$. For $m \in \mathbb{Z} - \{0, -1\}$, set

$$(4.1.65) \qquad \Lambda_m(t, \xi) = \left(1 + \frac{1}{m}\right)\frac{\xi^2}{t} + i$$

so that $\left(D_t + \frac{D_x^2}{t^2}\right)e^{im\theta} = \frac{1}{t}\Lambda_m(t, D)e^{im\theta}$. We define

$$(4.1.66) \qquad B(t, x, \xi) = A(t, x, \xi)\Lambda_m(t, \xi)^{-1} - 2\frac{D_x}{\sqrt{t}}A(t, x, \xi) \cdot \frac{\xi}{\sqrt{t}}\Lambda_m(t, \xi)^{-2}.$$

Then B satisfies (4.1.58), and a direct computation shows that

$$\left(D_t + \frac{D_x^2}{t^2}\right)[\mathrm{Op}^{\mathcal{B}}(B)(e^{im\theta})] - \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(A)(e^{im\theta}) = \frac{1}{t}\mathrm{Op}^{\mathcal{B}}(C)(e^{im\theta})$$

with

$$C(t, x, \xi) = \left(tD_t + \frac{D_x^2}{t}\right)(B(t, x, \xi)) - 4\frac{D_x^2}{t}A(t, x, \xi)\left(\frac{\xi}{\sqrt{t}}, \frac{\xi}{\sqrt{t}}\right)\Lambda_m^{-2}$$

which satisfies (4.1.58). The proof under assumption (4.1.57) is identical. Assertion (ii) of the lemma follows from the above expressions for B and C. □

REMARK. — We use in the above proof the assumption $m \neq -1$ in an essential way. If $m = -1$, $\Lambda_m(t, \xi)$ is no longer of order 2, and we can no longer make the asymptotic construction in the proof of the above lemma. In other words we *are not able* to eliminate oscillatory terms of type $\mathrm{Op}^{\mathcal{B}}(A)(e^{-i\theta})$. These terms do not appear in the right hand side of our equation (4.1.26) because of the restrictive assumption on the form of the nonlinearity in the right hand side of (1.1.2).

Proof of proposition 4.1.10. — We apply lemma 4.1.11 (i) to the symbol $A = A_m$, $m = 1, -3$, with $k = 2$, $\ell = 0$. We denote by $-A'_m$ the symbols C of lemma 4.1.11: they satisfy conditions (4.1.53), (4.1.54). Consequently, if we set

$$V_m = e^{-im\theta} \mathrm{Op}^{\mathcal{B}}(B_m)(e^{im\theta})$$

where B_m satisfies (4.1.58), or (4.1.59), we just have to see that V_m is a Lipschitz function of v on the unit ball of $\mathcal{G}$, with values in $\mathcal{V}$. The Lipschitz dependence will follow from the fact that A_m, and thus B_m, are Lipschitz in v by construction. The main fact to prove is that V_m fulfills the conditions (4.1.8) defining $\mathcal{V}$. We will make use of proposition 3.2.2. Consider first the case corresponding to (4.1.58) i.e.

$$(4.1.67) \qquad (tD_t)^{k'} B_m \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\ell-2k',s'-1}(M') \otimes \Sigma_1^{-\ell-2}.$$

We apply proposition 3.2.2 with $\rho = 2$ or $\rho = 3$, $\nu = 1$, $b \equiv 1$, $\mu = 0$. We check that because of (4.1.7), condition (3.2.4) and the second condition (3.2.6) are satisfied. We deduce from (3.2.7) that

$$V_m \in \widetilde{E}_{\alpha+2-0,\alpha',1}^{s+2,s'-1} \subset \widetilde{E}_{\beta,\alpha',1}^{s+2,s'-1}.$$

Writing for $\gamma \in \mathbb{N}^2$, $|\gamma| \leqslant M$, $x^\gamma V_m$ as a linear combination of expressions $e^{-im\theta} \mathrm{Op}^{\mathcal{B}}(x^{\gamma'} B_m)(e^{im\theta})$ for $|\gamma'| \leqslant |\gamma|$, and using (4.1.67), we get in the same way that $V_m \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s+2,s'-1}(M)$.

Let us study now the derivative

$$(4.1.68) \qquad tD_t V_m = e^{-im\theta} \mathrm{Op}^{\mathcal{B}}(tD_t B_m)(e^{im\theta}) - e^{-im\theta}[mtx^2/4, \mathrm{Op}^{\mathcal{B}}(B_m)](e^{im\theta}).$$

Using (4.1.67) and proposition 3.2.2, we get that the first term in the right hand side belongs to $\widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s,s'-1}(M)$. Going back to the definition of $\mathrm{Op}^{\mathcal{B}}(\cdot)$ we see that the last term in (4.1.68) is the opposite of

$$(4.1.69) \qquad e^{-im\theta} \sum_j S_{j-1} B_m(t, x, D)[mtx^2/4, \Delta_j](e^{im\theta}).$$

If for $\Delta_j = \varphi(2^{-j}D)$, we set $\Delta_j^{(1)} = (\nabla\varphi)(2^{-j}D)$, $\Delta_j^{(2)} = (\Delta\varphi)(2^{-j}D)$, we have that

$$(4.1.70) \qquad \left[\frac{mtx^2}{4}, \Delta_j\right](e^{im\theta}) = \left[-\frac{m}{4}(2^{-j}\sqrt{t})^2 \Delta_j^{(2)} + i\Delta_j^{(1)} \cdot (2^{-j}D)\right](e^{im\theta}).$$

Since B_m is supported in $\{|\xi| > c\sqrt{t}\}$, the sum in (4.1.69) is restricted to $2^j > c\sqrt{t}$, so (4.1.70) shows that (4.1.69) is of type $e^{-im\theta}\widetilde{\mathrm{Op}}^{\mathcal{B}}(B_m)(e^{im\theta})$, and so belongs to $\widetilde{\mathcal{E}}^{s+2,s'-1}_{\beta,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s,s'-1}_{\beta,\alpha',1}(M)$.

To study

$$(4.1.71) \qquad c(\sqrt{t}x)(D/t)^\gamma(tD_t)^k V_m, \quad k = 0,1, \ |\gamma| \leqslant 1$$

with $c \in S^\mu$, we argue as above when $\gamma = 0$: we apply proposition 3.2.2 remarking that in (3.2.7) we can take $\zeta > 0$ since $\mu < 2$. We get that (4.1.71) belongs to $\widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1}_{\alpha,\alpha',1}(M)$. When $|\gamma| = 1$ in (4.1.71), we write this expression as

$$(4.1.72) \qquad (D/t)^\gamma[c(\sqrt{t}x)(tD_t)^k V_m] + \frac{1}{\sqrt{t}}c_1(\sqrt{t}x)(tD_t)^k V_m$$

where $c_1 \in S^{\mu-1}$. The first contribution belongs to

$$\widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\alpha,\alpha'+|\gamma|,1}(M) \subset \widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\alpha+2|\gamma|,\alpha',1}(M)$$

which is the wanted conclusion. To study the second contribution, we apply again proposition 3.2.2 with μ replaced by $\mu - 1$. In this case, in (3.2.7), we can take $\zeta = 1$ since $\mu - 1 < 1$ whence $c_1(\sqrt{t}x)(tD_t)^k V_m \in \widetilde{\mathcal{E}}^{s+2-(\mu-1)-2k,s'-1}_{\alpha+1,\alpha',1}(M)$. Coming back to the definitions of these spaces, we deduce that the second term in (4.1.72) belongs to the same space as the first one.

When instead of (4.1.67) one assumes that

$$(tD_t)^{k'}B_m \in \sum_{\ell=0}^1 \widetilde{\mathcal{E}}^{s-\ell-2k',s'-1}_{\beta,\alpha',1}(M') \otimes \Sigma_0^{-\ell-2}$$

(i.e. when we study the case corresponding to (4.1.59)), we use proposition 3.2.2 with $\nu = 0$, and α replaced by β. We remark that the first condition (3.2.6), which reads $\frac{\beta}{2} + \alpha' > 0$, $\beta + \alpha' + 1 > 0$ is satisfied by (4.1.6). We deduce from (3.2.7) with $\zeta = 0+$ (which is possible as $\frac{d}{2} - \beta - \alpha' > 0$) that

$$(4.1.73) \qquad (tD_t)^k V_m \in \widetilde{\mathcal{E}}^{s+2-2k,s'-1}_{\beta,\alpha',1}(M)$$

which is the wanted conclusion. The study of (4.1.71) when $\gamma = 0$ is similar, and shows that this term belongs to $\widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1}_{\beta,\alpha',1}(M)$. When $|\gamma| = 1$, we use decomposition (4.1.72). The first term will belong to

$$\widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\beta,\alpha'+1,1}(M) \subset \widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\beta+2,\alpha',1}(M)$$

which gives the conclusion as $\beta + 2 > \alpha + 2$. To study the second contribution to (4.1.72), we remark first that

$$c_1(\sqrt{t}x)(tD_t)^k V_m \in \widetilde{\mathcal{E}}^{s+2-(\mu-1)-2k,s'-1}_{\beta,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1}_{\alpha+1,\alpha',1}(M)$$

again by (3.2.7) and since (4.1.6) implies $\beta > \alpha + 1$. Exploiting the $1/\sqrt{t}$ factor in the second term of (4.1.72), we see that this expression belongs to $\widetilde{\mathcal{E}}^{s+2-\mu-2k,s'-1-|\gamma|}_{\alpha+2,\alpha',1}(M)$. This concludes the proof. $\qquad \square$

COROLLARY 4.1.12. — *There exists:*

  – for $m = 1, -3$ Lipschitz functions $v \mapsto V_m(v)$ defined on the unit ball of $\mathcal{G}$, with values in $\mathcal{V}$, vanishing at $v = 0$,

  – for $m = 1, -3$ quadratic symbols $v \mapsto A'_m(v, t, \xi)$ with values in (4.1.53), satisfying (4.1.54),

  – for $m \in I_1 - \{0\}$, quadratic symbols $v \mapsto A'_m(v, t, \xi)$ satisfying for $k = 0, 1$, $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

$$(4.1.74) \quad c(\sqrt{t}x)(tD_t)^k A'_m(v, t, \xi)$$
$$\in \widetilde{\mathcal{E}}^{s-2k-\mu', s'-1}_{\beta, \alpha', 1}(M') \otimes \Sigma^0_0 + \widetilde{\mathcal{E}}^{s-2k-\mu', s'-1}_{\alpha, \alpha', 1}(M') \otimes \Sigma^0_1,$$

  – A function $v \mapsto H(v) = \sum_{j=1}^2 x_j H_j(v)$ where $H_j(v)$ are defined on the unit ball of $\mathcal{G}$, vanish at 0, and satisfy for $k = 0, 1$, $c \in S^{\mu'}$, $\mu' \in [0, \mu]$, v, v' in this unit ball

$$(4.1.75) \quad \|c(\sqrt{t}x)(tD_t)^k[H_j(v) - H_j(v')]\|_{\widetilde{\mathcal{E}}^{s-\mu'-2k, s'}_{\alpha, \alpha', 1}(M+1)}$$
$$\leqslant C(\|v\|_{\mathcal{G}} + \|v'\|_{\mathcal{G}})\|v - v'\|_{\mathcal{G}},$$

  – A quadratic remainder $v \mapsto R(v)$ defined on the unit ball of $\mathcal{G}$, with values in $\mathcal{G}'$, such that, if we set $I = \{1, -3\} \cup (I_1 - \{0\}) = \{-4, -3, -2, 2, 1\}$ and

$$(4.1.76) \qquad F(v) = \sum_{m \in I} \mathrm{Op}^{\mathcal{B}}(A'_m(v, t, \xi))(e^{im\theta}) + H(v) + R(v),$$

for any element v of the unit ball of $\mathcal{G}$ which is a solution of

$$(4.1.77) \qquad\qquad \left(D_t + \frac{D_x^2}{t^2}\right)v = \frac{1}{t}F(v),$$

the function w given by (4.1.10) is a solution to (4.1.2).

Proof. — We define $V_m(v)$ by proposition 4.1.10 and denote $V(v) = (V_1(v), V_{-3}(v))$. We just have to see that if v satisfies (4.1.77) with a convenient definition of A'_m, H, R in (4.1.76), then (v, V) is a solution to (4.1.26). By (4.1.55), the first sum in the right hand side of (4.1.26) is $\frac{1}{t}\sum_{m \in \{1, -3\}} \mathrm{Op}^{\mathcal{B}}(A'_m)(e^{im\theta})$. For $m \in I_1 - \{0\}$ we define $A'_m(v, t, \xi) = A_m(v, V, t, \xi)$. Since $v \mapsto V(v)$ is Lipschitz with values in $\mathcal{V}$, (4.1.24) implies that A'_m satisfies (4.1.74) for $m \in I_1 - \{0\}$. We define also the function $H_j(v)$ satisfying (4.1.75) from the bilinear forms of (4.1.25) by the formula $H_j(v) = H_j(v, V(v))$. Since $v \mapsto V(v)$ is Lipschitz, (4.1.75) follows from (4.1.25). In the same way, if we substitute $V(v)$ into the last term $R(v, V)$ of (4.1.26), we get a quadratic remainder. A solution v to (4.1.77) thus allows us to construct a solution (v, V) to (4.1.26), and so a solution w to (4.1.2). $\qquad\square$

Our next task is to find a global solution to (4.1.77) for small enough Cauchy data. We cannot do that directly on equation (4.1.77): actually we want to find a solution v, which belongs in particular to $\underline{E}^{s, s'}_{\alpha, \alpha', \infty}$. In the right hand side, $R(v)$ belongs to

$E_{\beta,\alpha',1}^{s,s'-1} \subset E_{\alpha,\alpha',1}^{s,s'-1} \subset F_{\alpha,\alpha',1}^{s,s'-1/2}$. The smoothing property of theorem 2.2.3 tells us that the corresponding linear solution belongs to $\underline{E}_{\alpha,\alpha',\infty}^{s,s'}$, which is nice. The trouble comes from the $\mathrm{Op}^{\mathcal{B}}(A_m')(e^{im\theta})$ terms in (4.1.76). We cannot hope to put these terms in $F_{\alpha,\alpha',1}^{s,s'-1/2}$. In fact, when $m = 1, -3$, A_m' belongs to Σ_0^{-2}: this means that the gain of smoothness we can get when applying $\mathrm{Op}^{\mathcal{B}}(A_m')$ to $e^{im\theta}$ is at most 2 i.e. these terms cannot be in any $F_{\alpha,\alpha',1}^{s,s'-1/2}$ space for $s > 2$. In the same way, for $m \in I_1 - \{0\}$, A_m' is a symbol of order 0, which decays like $\langle \sqrt{t}x \rangle^{-2+0}$ relatively to space. Since the action of $D_x/\sqrt{t}$ over $e^{im\theta}$ results in a loss of $\sqrt{t}x$, this shows that the best that can be expected for $\mathrm{Op}^{\mathcal{B}}(A_m')(e^{im\theta})$ ($m \in I_1$) is to lie in a $F_{\alpha,\alpha',1}^{s,s'-1/2}$ space with $s < 2$. On the other hand, because of conditions (4.1.6), we do need to be able to solve our problem in a space of that type with $s > 2$ (actually s close to 4). Because of that we will exploit again the fact that the phases of oscillations in (4.1.76) are non-characteristic ones to further reduce the equation.

Step 3: Last reduction of the equation

Up to now, we have eliminated by an appropriate choice of V the first line in the right hand side of (4.1.26). We still have to eliminate the $\mathrm{Op}^{\mathcal{B}}(A_m(v, V, t, \xi))(e^{im\theta})$ terms (or equivalently, the terms of type $\mathrm{Op}^{\mathcal{B}}(A_m'(v, t, \xi))(e^{im\theta})$ in (4.1.76)). In the case of example (4.1.20) these are coming from the rectangular terms in (4.1.21) i.e. are expressions of type $\mathrm{Op}^{\mathcal{B}}(A_2(v, V))(e^{2i\theta})$, where $A_2(v, V)$ is bilinear. In particular, this A_2 decays like $\langle \sqrt{t}x \rangle^{-2}$, which means that the corresponding term admits two $D_x/\sqrt{t}$-derivatives. This is not enough to incorporate it to the remainders in (4.1.26), since the latter have to accept about four of these derivatives. In this third step, we thus repeat the procedure of step 2, to get rid of these last annoying terms.

PROPOSITION 4.1.13. — *There exists a function $v \mapsto K(v)$ (resp. $v \mapsto \widetilde{R}(v)$) defined on the unit ball of $\mathcal{G}$, with values in $\widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'}(M)$ (resp. $\mathcal{E}_{\beta,\alpha',1}^{s,s'}(M)$), such that for any v, v' in the unit ball of $\mathcal{G}$*

(4.1.78)
$$\|K(v) - K(v')\|_{\widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'}(M)} \leqslant C\|v - v'\|_{\mathcal{G}}(\|v\|_{\mathcal{G}} + \|v'\|_{\mathcal{G}})$$
$$\|\widetilde{R}(v) - \widetilde{R}(v')\|_{\mathcal{E}_{\beta,\alpha',1}^{s,s'}(M)} \leqslant C\|v - v'\|_{\mathcal{G}}(\|v\|_{\mathcal{G}} + \|v'\|_{\mathcal{G}}),$$

and satisfying the following: if v and v' are two elements of the unit ball of $\mathcal{G}$, solutions to

(4.1.79)
$$\left(D_t + \frac{D_x^2}{t^2}\right)(v - K(v')) = \frac{1}{t}H(v') + \frac{1}{t}\widetilde{R}(v')$$

then (v, v') solves

(4.1.80)
$$\left(D_t + \frac{D_x^2}{t^2}\right)v = \frac{1}{t}F(v')$$

where F and H are defined in (4.1.76).

Proof. — We have to define $K(v')$ in such a way that the contribution

$$\sum_{m \in I} \mathrm{Op}^{\mathcal{B}}(A_m')(e^{im\theta})$$

in (4.1.76) is canceled. Apply first lemma 4.1.11 (i) to A_m', $m = 1, -3$. Since then A_m' belongs to (4.1.53), and $t D_t A_m'$ to (4.1.54), we get from this lemma symbols B_m' and C_m' satisfying for $m = 1, -3$

(4.1.81)
$$B_m' \in \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-\ell,s'-1}(M') \otimes \Sigma_0^{-4-\ell} + \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-2-\ell,s'-1}(M') \otimes \Sigma_1^{-4-\ell}$$

$$C_m' \in \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-4-\ell,s'-1}(M') \otimes \Sigma_0^{-4-\ell} + \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-4-\ell,s'-1}(M') \otimes \Sigma_1^{-4-\ell}.$$

When $m \in I_1 - \{0\}$, A_m' satisfies (4.1.74) and lemma 4.1.11 (ii) provides symbols B_m', C_m' such that for $c \in S^{\mu'}$, $\mu' \in [0, \mu]$

(4.1.82)

$$c(\sqrt{t}x)B_m' \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\mu'-\ell,s'-1}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\mu'-\ell,s'-1}(M') \otimes \Sigma_1^{-2-\ell}$$

$$c(\sqrt{t}x)C_m' \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\mu'-2-\ell,s'-1}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\mu'-2-\ell,s'-1}(M') \otimes \Sigma_1^{-2-\ell}.$$

With these notations, we set

(4.1.83)
$$K(v') = \sum_{m \in I} \mathrm{Op}^{\mathcal{B}}(B_m')(e^{im\theta})$$

$$R_1(v') = \sum_{m \in I} \mathrm{Op}^{\mathcal{B}}(C_m')(e^{im\theta})$$

and (4.1.60) shows that (4.1.80) implies (4.1.79) if we can prove that $K(v')$ belongs to $\widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'}(M)$ and $R_1(v')$ to $\mathcal{E}_{\beta,\alpha',1}^{s,s'-1}(M)$ (the quadratic estimates (4.1.78) follow then from the linear dependence of (4.1.83) in B_m', C_m', and the quadratic nature of these terms).

Consider first the contribution to $K(v')$ coming from terms of type $\mathrm{Op}^{\mathcal{B}}(B_m')(e^{im\theta})$, $m = 1, -3$. Condition (4.1.81) implies

$$B_m' \in \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-4-\ell,s'}(M') \otimes \Sigma_0^{-4-\ell} + \sum_{\ell=0}^{2} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-4-\ell,s'}(M') \otimes \Sigma_1^{-4-\ell}.$$

We apply corollary 3.2.7 with $b \in \Sigma_0^{-4-\ell}$ (resp. $b \in \Sigma_1^{-4-\ell}$), $\mu = 0$, s replaced by $s-4-\ell$, M replaced by $M' = 2M-5$, and we check that conditions (3.2.87), (3.2.86) which read respectively

$$-1 < \beta + \alpha', \quad s-4-\ell < \alpha'-1, \quad s+s'-4-\ell-M' < -1/2$$

$$-2 < \alpha + \alpha' < -1, \quad s-4-\ell < \alpha+2\alpha', \quad s+s'-4-\ell-M' < -1/2$$

are satisfied because of (4.1.6). We deduce that the corresponding contribution to $K(v')$ belongs to $\mathcal{E}^{s,s'}_{\alpha+2,\alpha',1}(M)$. Since $\frac{1}{2}(\alpha+2)+\alpha' > 0$, the last assertion of lemma 3.1.2 implies that this space equals $\widetilde{\mathcal{E}}^{s,s'}_{\alpha+2,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',1}(M)$. Let us examine now the contribution to $K(v')$ coming from $\mathrm{Op}^{\mathcal{B}}(B'_m)(e^{im\theta}), m \in I_1 - \{0\}$. We deduce from (4.1.82) that for $c \in S^\mu$

$$c(\sqrt{t}x)B'_m \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}^{s-\mu-2-\ell,s'}_{\beta,\alpha',1}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}^{s-\mu-2-\ell,s'}_{\alpha,\alpha',1}(M') \otimes \Sigma_1^{-2-\ell}.$$

We apply corollary 3.2.7. We check that conditions (3.2.87), (3.2.86) are satisfied if $\mu \in \,]0,2[$ is taken close enough to 2, because of (4.1.6). We obtain that the corresponding contribution to $K(v')$ is in $\mathcal{E}^{s,s'}_{\alpha+2,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',1}(M)$.

Next, let us consider $R_1(v')$. To study the contribution coming from $\mathrm{Op}^{\mathcal{B}}(C'_m)(e^{im\theta})$, $m = 1, -3$, we use assumption (4.1.81) together with corollary 3.2.7. We get that these terms belong to $\mathcal{E}^{s,s'-1}_{\beta,\alpha',1}(M)$. For the contribution of $\mathrm{Op}^{\mathcal{B}}(C'_m)(e^{im\theta})$, $m \in I_1 - \{0\}$, we use (4.1.82) and corollary 3.2.7 and get the same conclusion. The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

To close this chapter, we prove a lemma concerning the quantity $F(v)$ defined in (4.1.76).

LEMMA 4.1.14. — *Denote by $S(v)$ the difference $S(v) = F(v) - H(v)$. For v, v' in the unit ball of $\mathcal{G}$, we have the following inequalities:*

$$(4.1.84) \quad \begin{aligned} &\sum_{k=0}^{1} \|(tD_t)^k(S(v) - S(v'))\|_{\widetilde{\mathcal{E}}^{s-2-2k,s'}_{\beta,\alpha',1}(M)} \leqslant C\|v-v'\|_{\mathcal{G}}(\|v\|_{\mathcal{G}} + \|v'\|_{\mathcal{G}}) \\ &\sum_{k=0}^{1} \|(tD_t)^k(F(v) - F(v'))\|_{\widetilde{\mathcal{E}}^{s-2-2k,s'}_{\alpha,\alpha',1}(M)} \leqslant C\|v-v'\|_{\mathcal{G}}(\|v\|_{\mathcal{G}} + \|v'\|_{\mathcal{G}}). \end{aligned}$$

Proof. — In (4.1.76) we have, because of the definition (4.1.8) of $\mathcal{G}'$,

$$R(v) \in \mathcal{E}^{s,s'-1}_{\beta,\alpha',1}(M) = \widetilde{\mathcal{E}}^{s,s'-1}_{\beta,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s-2,s'}_{\beta,\alpha',1}(M)$$
$$tD_tR(v) \in \mathcal{E}^{s-2,s'-1}_{\beta,\alpha',1}(M) = \widetilde{\mathcal{E}}^{s-2,s'-1}_{\beta,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s-4,s'}_{\beta,\alpha',1}(M)$$

where the equality between $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ spaces follows from $\frac{\beta}{2}+\alpha' > 0$ and lemma 3.1.2. Consequently, the contributions of $R(v)$ to the left hand sides of (4.1.84) satisfy the wanted conclusion.

Since $H(v) \in \widetilde{\mathcal{E}}^{s,s'-1}_{\alpha,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s-2,s'}_{\alpha,\alpha',1}(M)$, $tD_tH(v) \in \widetilde{\mathcal{E}}^{s-2,s'-1}_{\alpha,\alpha',1}(M) \subset \widetilde{\mathcal{E}}^{s-4,s'}_{\alpha,\alpha',1}(M)$ by (4.1.75), the contribution of $H(v)$ to the left hand side of the second inequality (4.1.84) satisfies also the inequality. We are thus reduced to the study of

$\sum_{m \in I} \mathrm{Op}^{\mathcal{B}}(A'_m(v,t,\xi))(e^{im\theta})$. For $m = 1, -3$, (4.1.53) and (4.1.54) imply

$$(4.1.85) \quad (tD_t)^k A'_m \in \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-4-2k-\ell,s'}(M') \otimes \Sigma_0^{-2-\ell} + \sum_{\ell=0}^{1} \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-4-2k-\ell,s'}(M') \otimes \Sigma_1^{-2-\ell}.$$

We apply corollary 3.2.7 with $\mu = 0$, s replaced by $s - 4 - 2k - \ell$, M replaced by M', and α replaced by β for the contributions of the first sum in (4.1.85). One checks that (3.2.86) and (3.2.87) (with α replaced by β) are satisfied because of (4.1.6), whence the conclusion

$$\mathrm{Op}^{\mathcal{B}}((tD_t)^k A'_m)(e^{im\theta}) \in \mathcal{E}_{\beta+2,\alpha',1}^{s-2-2k,s'}(M) + \mathcal{E}_{\alpha+4,\alpha',1}^{s-2-2k,s'}(M)$$
$$\subset \mathcal{E}_{\beta,\alpha',1}^{s-2-2k,s'}(M) = \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-2k,s'}(M).$$

Since $\mathrm{Op}^{\mathcal{B}}(A'_m)(tD_t e^{im\theta})$ belongs also to $\widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-4,s'}(M)$, as one sees writing $tD_t(e^{im\theta}) = \frac{1}{m}\left(im + \frac{D_x^2}{t}\right)e^{im\theta}$ and applying again corollary 3.2.7, we see that the contributions of $(tD_t)^k \mathrm{Op}^{\mathcal{B}}(A'_m)(e^{im\theta})$, $m = 1, -3$ to $(tD_t)^k S(v)$, $k = 0, 1$ belong to $\widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-2k,s'}(M)$. The fact that these contributions satisfy the first inequality (4.1.84) follows from the quadratic nature of A'_m.

Let us consider now $\mathrm{Op}^{\mathcal{B}}(A'_m)(e^{im\theta})$ for $m \in I_1 - \{0\}$. By (4.1.74), for $k = 0, 1$, $\mu' \in [0, \mu]$

$$(4.1.86) \quad \langle\sqrt{t}x\rangle^{\mu'}(tD_t)^k A'_m \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-\mu'-2k-2,s'}(M') \otimes \Sigma_0^0 + \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s-\mu'-2k-2,s'}(M') \otimes \Sigma_1^0.$$

We apply again corollary 3.2.7 with s replaced by $s - 2k - 2$, $m_2 = 0$, α replaced by β for the contribution of the first term in the right hand side of (4.1.86). If $\mu \in]0, 2[$ is close enough to 2, conditions (3.2.86), (3.2.87) are satisfied by (4.1.6), and we obtain

$$\mathrm{Op}^{\mathcal{B}}((tD_t)^k A'_m)(e^{im\theta}) \in \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s-2-2k,s'}(M)$$

which again gives the wanted conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Existence of the global solution

The main remaining step in the proof of theorem 1.1.1 is to prove the following result: denote by $\mathcal{G}_0$ the space

$$\mathcal{G}_0 = \{v \in H^{M+4}(\mathbb{R}^2); \forall \gamma, |\gamma| \leqslant M, (D + x/2)^\gamma v \in H^{M+4-|\gamma|}\}$$
$$= \{v \in H^{M+4}(\mathbb{R}^2); \forall \gamma, |\gamma| \leqslant M, x^\gamma v \in H^{M+4-|\gamma|}\}$$

THEOREM 4.2.1. — *There is $\varepsilon_0 > 0$ such that for any $v_0 \in \mathcal{G}_0$ with $\|v_0\|_{\mathcal{G}_0} < \varepsilon_0$, the problem*

$$(4.2.1) \qquad\qquad \left(D_t + \frac{D_x^2}{t^2}\right)(v - K(v)) = \frac{1}{t}H(v) + \frac{1}{t}\widetilde{R}(v)$$
$$v|_{t=1} = v_0$$

has a global solution $v \in \mathcal{G}$.

We shall construct v using the standard iterative scheme

$$(4.2.2) \qquad \left(D_t + \frac{D_x^2}{t^2}\right)(v^{\nu+1} - K(v^\nu)) = \frac{1}{t}H(v^\nu) + \frac{1}{t}\widetilde{R}(v^\nu)$$
$$v^{\nu+1}|_{t=1} = v_0$$

for $\nu \geqslant 0$, v^0 being defined as the solution to $\left(D_t + \frac{D_x^2}{t^2}\right)v^0 = 0$, $v^0|_{t=1} = v_0$. The main step is to prove:

PROPOSITION 4.2.2. — *There is a constant $C > 0$ such that if v^ν belongs to the unit ball of $\mathcal{G}$, $v^{\nu+1}$ satisfies*

$$(4.2.3) \qquad \|v^{\nu+1}\|_{\mathcal{G}} \leqslant C[\|v_0\|_{\mathcal{G}_0} + \|v^\nu\|_{\mathcal{G}}^2] \quad (\nu \geqslant 0)$$
$$\|v^{\nu+1} - v^\nu\|_{\mathcal{G}} \leqslant C\|v^\nu - v^{\nu-1}\|_{\mathcal{G}}(\|v^{\nu-1}\|_{\mathcal{G}} + \|v^\nu\|_{\mathcal{G}}) \quad (\nu \geqslant 1).$$

The above inequalities imply in a standard way the existence of a global solution $v \in \mathcal{G}$ to (4.2.2) if the Cauchy data are taken small enough. To prove the proposition, we will study successively the different components of the norms of $\mathcal{G}$.

LEMMA 4.2.3. — *The norm $\|v^{\nu+1}\|_{\underline{\mathcal{E}}_{\alpha,\alpha',\infty}^{s,s'}(M)}$ (resp. the norm $\|v^{\nu+1} - v^\nu\|_{\underline{\mathcal{E}}_{\alpha,\alpha',\infty}^{s,s'}(M)}$) is estimated by the right hand side of the first (resp. second) inequality (4.2.3).*

Proof. — Since Z commutes to $D_t + \frac{D_x^2}{t^2}$, we deduce from (4.2.2) that for any $\gamma \in \mathbb{N}^2, |\gamma| \leqslant M$

$$\left(D_t + \frac{D_x^2}{t^2}\right)(Z^\gamma v^{\nu+1} - Z^\gamma K(v^\nu)) = \frac{1}{t}Z^\gamma H(v^\nu) + \frac{1}{t}Z^\gamma \widetilde{R}(v^\nu).$$

Since by corollary 4.1.12 $H(v^\nu) \in \widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'-1}(M)$ and $\widetilde{R}(v^\nu) \in \mathcal{E}_{\beta,\alpha',1}^{s,s'-1}(M)$, the characterization (4.1.5) of these spaces shows that $Z^\gamma H(v^\nu)$ and $Z^\gamma \widetilde{R}(v^\nu)$ belong to $E_{\alpha,\alpha',1}^{s,s'-|\gamma|-1} \subset F_{\alpha,\alpha',1}^{s,s'-|\gamma|-\frac{1}{2}}$, their norms in these spaces being controlled by $C\|v^\nu\|_{\mathcal{G}}^2$. The smoothing inequality (2.2.15) of theorem 2.2.3 implies

$$(4.2.4) \quad \|Z^\gamma(v^{\nu+1} - K(v^\nu))\|_{\underline{E}_{\alpha,\alpha',\infty}^{s,s'-|\gamma|}} \leqslant C\Big[\|Z^\gamma(v^{\nu+1} - K(v^\nu))|_{t=1}\|_{H^{s+s'-|\gamma|}}$$
$$+ \|Z^\gamma H(v^\nu)\|_{F_{\alpha,\alpha',1}^{s,s'-|\gamma|-\frac{1}{2}}} + \|Z^\gamma \widetilde{R}(v^\nu)\|_{F_{\alpha,\alpha',1}^{s,s'-|\gamma|-\frac{1}{2}}}\Big]$$

if the inequalities (2.2.13) are satisfied i.e. $\frac{\alpha}{2} + \alpha' < 0$, $s + \alpha \geqslant 0$, $s' - |\gamma| + \alpha' \geqslant 0$. The first two of these inequalities follow from (4.1.6). Since $|\gamma| \leqslant M = s + s' - 4$ the last one is satisfied if $s \leqslant \alpha' + 4$, which follows also from (4.1.6). The last two norms in the right hand side of (4.2.4) are controlled by $C(\|H(v^\nu)\|_{\widetilde{\mathcal{E}}_{\alpha,\alpha',1}^{s,s'-1}(M)} + \|\widetilde{R}(v^\nu)\|_{\mathcal{E}_{\alpha,\alpha',1}^{s,s'-1}(M)})$ and so by $C\|v^\nu\|_{\mathcal{G}}^2$ because of the properties of H and $\widetilde{R}$ listed in (4.1.75) and (4.1.78).

In the first term of the right hand side we have

$$\|Z^\gamma v^{\nu+1}|_{t=1}\|_{H^{s+s'-|\gamma|}} = \left\|\left(D_x + \frac{x}{2}\right)^\gamma v_0\right\|_{H^{s+s'-|\gamma|}} \leqslant \|v_0\|_{\mathcal{G}_0}$$

$$\|Z^\gamma K(v^\nu)|_{t=1}\|_{H^{s+s'-|\gamma|}} \leqslant \|Z^\gamma K(v^\nu)\|_{E^{s,s'-|\gamma|}_{\alpha,\alpha',\infty}}$$

$$\leqslant \|K(v^\nu)\|_{\widetilde{\mathcal{E}}^{s,s'}_{\alpha,\alpha',1}(M)} \leqslant C\|v^\nu\|^2_{\mathcal{G}}.$$

Consequently, (4.2.4) implies

$$\|Z^\gamma v^{\nu+1}\|_{\underline{E}^{s,s'-|\gamma|}_{\alpha,\alpha',\infty}} \leqslant \|Z^\gamma K(v^\nu)\|_{\underline{E}^{s,s'-|\gamma|}_{\alpha,\alpha',\infty}} + \|v_0\|_{\mathcal{G}_0} + C\|v^\nu\|^2_{\mathcal{G}}$$

which implies the wanted estimate for $\|v^{\nu+1}\|_{\mathcal{E}^{s,s'}_{\alpha,\alpha',\infty}(M)}$ using again inequality (4.1.78) and the injection $\mathcal{E}^{s,s'}_{\alpha,\alpha',\infty}(M) \subset \underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M)$. The estimate for $v^{\nu+1} - v^\nu$ is obtained in the same way using the equation

$$\left(D_t + \frac{D_x^2}{t^2}\right)(v^{\nu+1} - v^\nu - (K(v^\nu) - K(v^{\nu-1})))$$
$$= \frac{1}{t}(H(v^\nu) - H(v^{\nu-1})) + \frac{1}{t}(\widetilde{R}(v^\nu) - \widetilde{R}(v^{\nu-1})).$$

$\square$

LEMMA 4.2.4. — *The norm* $\sum_{k=1}^2 \|(tD_t)^k v^{\nu+1}\|_{\underline{\widetilde{\mathcal{E}}}^{s-2k,s'}_{\alpha,\alpha',\infty}(M)}$ *(resp. the norm* $\sum_{k=1}^2 \|(tD_t)^k(v^{\nu+1} - v^\nu)\|_{\underline{\widetilde{\mathcal{E}}}^{s-2k,s'}_{\alpha,\alpha',\infty}(M)}$*) is estimated by the right hand side of the first (resp. second) inequality* (4.2.3).

Proof. — By proposition 4.1.13, a solution $(v^\nu, v^{\nu+1})$ of (4.2.2) solves also

$$(4.2.5) \qquad\qquad \left(D_t + \frac{D_x^2}{t^2}\right)v^{\nu+1} = \frac{1}{t}F(v^\nu)$$

whence commuting $Z^\gamma, |\gamma| \leqslant M$ to the equation

$$(4.2.6) \qquad tD_t(Z^\gamma v^{\nu+1}) = -\frac{D_x^2}{t}(Z^\gamma v^{\nu+1}) + Z^\gamma F(v^\nu)$$

$$(tD_t)^2(Z^\gamma v^{\nu+1}) = -(tD_t)\frac{D_x^2}{t}(Z^\gamma v^{\nu+1}) + tD_t(Z^\gamma F(v^\nu)).$$

We have

$$\frac{D_x^2}{t}(Z^\gamma v^{\nu+1}) \in \underline{E}^{s-2,s'-|\gamma|}_{\alpha+2,\alpha',\infty} = \underline{\widetilde{E}}^{s-2,s'-|\gamma|}_{\alpha+2,\alpha',\infty} \subset \underline{\widetilde{E}}^{s-2,s'-|\gamma|}_{\alpha,\alpha',\infty}$$

when $v^{\nu+1} \in \underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M)$. Consequently, $\|\frac{D_x^2}{t}(Z^\gamma v^{\nu+1})\|_{\underline{\widetilde{E}}^{s-2,s'-|\gamma|}_{\alpha,\alpha',\infty}}$ is controlled in terms of $\|v^{\nu+1}\|_{\underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M)}$, and so, by lemma 4.2.3, by the right hand side of the first inequality (4.2.3). By lemma 4.1.14, the quantity $\|Z^\gamma F(v^\nu)\|_{\widetilde{E}^{s-2,s'-|\gamma|}_{\alpha,\alpha',\infty}}$ is also controlled by $C\|v^\nu\|^2_{\mathcal{G}}$. The first equality (4.2.6) thus shows that $\|tD_t v^{\nu+1}\|_{\underline{\widetilde{\mathcal{E}}}^{s-2,s'}_{\alpha,\alpha',\infty}(M)}$ is estimated by the wanted quantity. One studies in the same way $(tD_t)^2 v^{\nu+1}$ and $(tD_t)^k(v^{\nu+1} - v^\nu), k = 1, 2$. $\square$

The preceding lemma shows that to estimate the left hand side of (4.2.3) we just need to bound $\|v^{\nu+1}\|_{\widetilde{\underline{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',\infty}(M)}$ and $\|v^{\nu+1}-v^\nu\|_{\widetilde{\underline{\mathcal{E}}}^{s,s'}_{\alpha,\alpha',\infty}(M)}$ instead of the $\underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M)$ norms which have been yet studied in lemma 4.2.3. In other words, we must consider for $|\gamma| \leqslant M$

$$(4.2.7) \qquad \begin{aligned} &\|\chi(D/\sqrt{t})(Z^\gamma v^{\nu+1})\|_{L^\infty(dtdx)} \\ &\|\chi(D/\sqrt{t})(Z^\gamma(v^{\nu+1}-v^\nu))\|_{L^\infty(dtdx)}. \end{aligned}$$

LEMMA 4.2.5. — *The first (resp. second) norm (4.2.7) is estimated by the right hand side of the first (resp. second) inequality (4.2.3).*

Proof. — We shall prove that there is $\delta > 0$, and for any $\gamma \in \mathbb{N}^2$ with $|\gamma| \leqslant M$, a constant $C > 0$, such that for any $t \geqslant 1$

$$(4.2.8) \qquad \begin{aligned} &|D_t[\chi(D/\sqrt{t})Z^\gamma v^{\nu+1}]| \leqslant \frac{C}{t^{1+\delta}}[\|v_0\|_{\mathcal{G}_0} + \|v^\nu\|^2_{\mathcal{G}}], \\ &|D_t[\chi(D/\sqrt{t})Z^\gamma(v^{\nu+1}-v^\nu)]| \leqslant \frac{C}{t^{1+\delta}}\|v^\nu-v^{\nu-1}\|_{\mathcal{G}}(\|v^\nu\|_{\mathcal{G}} + \|v^{\nu-1}\|_{\mathcal{G}}). \end{aligned}$$

The control of (4.2.7) by the right hand side of (4.2.3) will follow from the time integration of the above inequalities (and the fact that at $t = 1$ we have a trivial estimate for $\chi(D)(D_x + \frac{x}{2})^\gamma v_0$).

To prove (4.2.8), we deduce from (4.2.5) the equalities

$$(4.2.9) \quad D_t[\chi(D/\sqrt{t})Z^\gamma v^{\nu+1}] = \frac{i}{2t}\frac{D}{\sqrt{t}} \cdot \chi'\left(\frac{D}{\sqrt{t}}\right) Z^\gamma v^{\nu+1}$$
$$- \frac{1}{t}\chi\left(\frac{D}{\sqrt{t}}\right)\frac{D_x^2}{t}Z^\gamma v^{\nu+1} + \frac{1}{t}\chi\left(\frac{D}{\sqrt{t}}\right)Z^\gamma F(v^\nu)$$

$$(4.2.10) \quad D_t[\chi(D/\sqrt{t})Z^\gamma(v^{\nu+1}-v^\nu)] = \frac{i}{2t}\frac{D}{\sqrt{t}} \cdot \chi'\left(\frac{D}{\sqrt{t}}\right) Z^\gamma(v^{\nu+1}-v^\nu)$$
$$- \frac{1}{t}\chi\left(\frac{D}{\sqrt{t}}\right)\frac{D_x^2}{t}Z^\gamma(v^{\nu+1}-v^\nu) + \frac{1}{t}\chi\left(\frac{D}{\sqrt{t}}\right)Z^\gamma(F(v^\nu)-F(v^{\nu-1})).$$

We have that $\frac{D}{\sqrt{t}} \cdot \chi'\left(\frac{D}{\sqrt{t}}\right)Z^\gamma v^{\nu+1}$ and $\chi\left(\frac{D}{\sqrt{t}}\right)\frac{D_x^2}{t}Z^\gamma v^{\nu+1}$ belong to the space $\underline{E}^{+\infty,+\infty}_{\alpha+2,\alpha',\infty}$. Since $\frac{\alpha}{2} + 1 + \alpha' > 0$, property (3.1.6) of lemma 3.1.2 implies that the modulus of the first two terms in the right hand side of (4.2.9) is bounded by $Ct^{-1-\delta}\|v^{\nu+1}\|_{\underline{\mathcal{E}}^{s,s'}_{\alpha,\alpha',\infty}(M)}$, and so by the right hand side of the first inequality (4.2.8), using lemma 4.2.3. One has a similar statement for the first two contributions to the right hand side of (4.2.10). Let us study now the contribution of $F(v^\nu)$, $F(v^\nu) - F(v^{\nu-1})$. We decomposed in lemma 4.1.14 $F(v^\nu) = H(v^\nu) + S(v^\nu)$ with $S(v^\nu) \in \widetilde{\mathcal{E}}^{s-2,s'}_{\beta,\alpha',1}(M)$ whence $\chi(D/\sqrt{t})Z^\gamma S(v^\nu) \in E^{+\infty,+\infty}_{\beta,\alpha',\infty}$. Using $\beta/2 + \alpha' > 0$ and property (3.1.6) we get for $\frac{1}{t}|\chi\left(\frac{D}{\sqrt{t}}\right)Z^\gamma S(v^\nu)|$ an estimate by the right hand side

of the first equation (4.2.8). A similar statement holds true for the contributions of $S(v^\nu) - S(v^{\nu-1})$ to (4.2.10). To conclude the proof, we need to estimate

$$(4.2.11) \qquad\qquad \frac{1}{t}|\chi(D/\sqrt{t})Z^\gamma H(v^\nu)|$$

$$(4.2.12) \qquad\qquad \frac{1}{t}|\chi(D/\sqrt{t})Z^\gamma(H(v^\nu) - H(v^{\nu-1}))|.$$

Using the definition of Z, we can write

$$(4.2.13) \qquad Z^\gamma H(v^\nu) = x^\gamma H(v^\nu) + \sum_{\substack{\gamma',\gamma'',\ell \\ |\gamma'|+|\gamma''|+\ell \leqslant |\gamma| \\ |\gamma''|+\ell > 0}} t^{-\ell}a_{\gamma',\gamma'',\ell}(D_x/t)^{\gamma''}(x^{\gamma'}H(v^\nu))$$

for convenient coefficients $a_{\gamma',\gamma'',\ell}$. By corollary 4.1.12, we know that $H(v^\nu) \in \widetilde{\mathcal{E}}^{s,s'-1}_{\alpha,\alpha',1}(M)$, whence

$$t^{-\ell}(D_x/t)^{\gamma''}(x^{\gamma'}H(v^\nu)) \in \widetilde{E}^{s,s'-|\gamma|-1}_{\alpha,\alpha'+|\gamma'|+\ell,1} \subset \widetilde{E}^{s,s'-|\gamma|-1}_{\alpha+2,\alpha',1}$$

if $|\gamma''| + \ell \geqslant 1$. We then deduce from (3.1.6) the inequality, for $|\gamma'| + |\gamma''| + \ell \leqslant |\gamma|$, $|\gamma''| + \ell > 0$

$$|\chi(D/\sqrt{t})t^{-\ell}(D/t)^{\gamma''}(x^{\gamma'}H(v^\nu))| \leqslant \frac{C}{t^\delta}\|H(v^\nu)\|_{\widetilde{\mathcal{E}}^{s,s'-1}_{\alpha,\alpha',1}(M)} \leqslant \frac{C}{t^\delta}\|v^\nu\|^2_{\mathcal{G}}$$

for some $\delta > 0$, where the last inequality comes from (4.1.75). To estimate (4.2.11) by the right hand side of the first inequality (4.2.8), we just have to study the first term in the right hand side of (4.2.13). Write $H(v) = \sum_{j=1}^2 x_j H_j(v)$. We know by (4.1.75) that $\langle\sqrt{t}x\rangle H_j(v) \in \widetilde{\mathcal{E}}^{s-1,s'}_{\alpha,\alpha',1}(M+1)$. We shall deduce below from that the inequality

$$(4.2.14) \qquad\qquad |x^\gamma H_j(v^\nu)| \leqslant C\langle\sqrt{t}x\rangle^{-1}\|v^\nu\|^2_{\mathcal{G}}.$$

This will imply

$$|\chi(D/\sqrt{t})x_j x^\gamma H_j(v^\nu)| \leqslant \|x_j\langle\sqrt{t}x\rangle^{-1}\|_{L^\infty(dx)}\|v^\nu\|^2_{\mathcal{G}}$$
$$\leqslant Ct^{-1/2}\|v^\nu\|^2_{\mathcal{G}}.$$

This shows that the contribution of $x^\gamma H(v^\nu)$ to the left hand side of (4.2.11) is controlled by the right hand side of the first equation (4.2.8).

To prove (4.2.14), write that $w = \langle\sqrt{t}x\rangle x^\gamma H_j(v) \in \widetilde{E}^{s-1,s'-|\gamma|}_{\alpha,\alpha',1}$ and decompose

$$\phi_q(x)w = \phi_q(x)\chi(D/\sqrt{t})w + \sum_{q'}\sum_{k;2^k > c\sqrt{t}}(\phi_q(x)\widetilde{\Delta}_k\widetilde{\phi}_{q'})(\phi_{q'}\Delta_k w).$$

By definition of $\widetilde{E}^{s-1,s'-|\gamma|}_{\alpha,\alpha',1}$, the first term is smaller than $Cc_q\|v^\nu\|^2_{\mathcal{G}}$ for a ℓ^1 sequence $(c_q)_q$. The modulus of the second contribution is bounded, using the definition of

$H^{s-1,s'-|\gamma|}_{\alpha,\alpha',1}$ by

$$(4.2.15) \quad C \sum_{q'} \langle q - q' \rangle^{-N} \sum_{k;2^k > c\sqrt{t}} 2^k \|\phi_{q'} \Delta_k w(t, \cdot)\|_{L^2(dx)}$$

$$\leqslant C \|v^\nu\|^2_{\mathcal{G}} \sum_{q'} \langle q - q' \rangle^{-N} \Bigg(\sum_{k;c\sqrt{t} < 2^k < t} 2^k c_{kq'} \Big(\frac{2^k}{\sqrt{t}} \Big)^{-s+1} \Big(\frac{2^k}{t} \Big)^{\alpha'}$$

$$+ \sum_{k;t < 2^k} 2^k c_{kq'} \Big(\frac{2^k}{\sqrt{t}} \Big)^{-s+1} \Big(\frac{2^k}{t} \Big)^{-s'+|\gamma|} \Bigg)$$

for a $\ell^2_k \ell^1_{q'}$ sequence $(c_{kq'})_{kq'}$. Since by (4.1.6) $s > \alpha' + 2 > 3$, $s + s' - |\gamma| - 2 \geqslant s + s' - M - 2 \geqslant 2$, we bound (4.2.15) by a ℓ^1_q sequence times $\|v^\nu\|^2_{\mathcal{G}}$, whence (4.2.14).

The corresponding estimate for (4.2.12) being obtained in the same way, this concludes the proof of lemma 4.2.5. □

Lemmas 4.2.3, 4.2.4, and 4.2.5 imply immediately proposition 4.2.2, by definition of the norm on $\mathcal{G}$. As remarked earlier, theorem 4.2.1 is then an immediate consequence of proposition 4.2.2.

Proof of theorem 1.1.1. — If we are given $v_0 \in \mathcal{G}_0$ with small enough norm in that space, theorem 4.2.1 provides a solution $v \in \mathcal{G}$ defined for $t \geqslant 1$ to (4.2.1). By proposition 4.1.13, v is then a solution to

$$\Big(D_t + \frac{D^2_x}{t^2} \Big) v = \frac{1}{t} F(v)$$

and by corollary 4.1.12, the function $w = v + V_1(v)e^{i\theta} + V_{-3}(v)e^{-3i\theta}$ is a solution to (4.1.2), hence to (4.1.1), defined for $t \geqslant 1$. Since this in turn implies theorem 1.1.1, the only thing we still have to prove is that for any $w_0 \in \mathcal{H}^{M+4}$ in a small enough neighborhood of 0, there is v_0 in a small enough neighborhood of 0 in $\mathcal{H}^{M+4} \subset \mathcal{G}_0$ such that

$$(4.2.16) \qquad w_0 = v_0 + V_1(v_0)e^{ix^2/4} + V_{-3}(v_0)e^{-3ix^2/4}.$$

□

LEMMA 4.2.6. — *The map $v_0 \mapsto w_0$ given by (4.2.16) is a local diffeomorphism from a neighborhood of 0 in $\mathcal{H}^{M+4}$ to a neighborhood of 0 in $\mathcal{H}^{M+4}$.*

Proof. — Since multiplication by $e^{i\lambda x^2}$ leaves $\mathcal{H}^{M+4}$ unchanged, we just have to see that $v_0 \mapsto V_m(v_0)$, $m = 1, -3$ is bounded from $\mathcal{H}^{M+4}$ to $\mathcal{H}^{M+4}$ and satisfies $\|V_m(v_0)\|_{\mathcal{H}^{M+4}} \leqslant C \|v_0\|^2_{\mathcal{H}^{M+4}}$ when v_0 stays in the unit ball of $\mathcal{H}^{M+4}$. The conclusion will then follow from the local inversion theorem.

Remind the construction of V_1, V_{-3} at fixed time $t = 1$: in lemma 4.1.6, we defined symbols A_m $m = 1, -3$, which are at $t = 1$ linear combinations of expressions $a \otimes b$ where $b(\xi)$ is a symbol of order 0, and a is a quadratic expression of type $a = x^{\gamma_3}(D^{\gamma_1}_x v_0)(D^{\gamma_2}_x v_0)$ with $|\gamma_1| \leqslant 1$, $|\gamma_2| \leqslant 1$, $0 < |\gamma_1| + |\gamma_2| + |\gamma_3| \leqslant 2$. One gets

immediately that $a \in \mathcal{H}^{M+3}$. One then constructs from A_m symbols B_m, through formula (4.1.66), which at $t = 1$ gives

$$B_m \in \mathcal{H}^{M+3} \otimes S^{-2} + \mathcal{H}^{M+2} \otimes S^{-3}$$

and one define $V_m(v_0)$ as

$$V_m(v_0) = e^{-imx^2/4} \mathrm{Op}^{\mathcal{B}}(B_m)(e^{imx^2/4}).$$

Then proposition 3.2.2, applied at $t = 1$, shows that $V_m \in \mathcal{H}^{M+4}$. Moreover, since A_m is quadratic in v_0, we have the wanted quadratic estimate. This concludes the proof.   □

Proof of theorem 1.1.2. — Using the change of unknown (1.2.2), we have to find v_∞ such that for any $\gamma \in \mathbb{N}^2$ with $|\gamma| \leqslant M$

$$(4.2.17) \qquad \|x^\gamma(w(t,x) - v_\infty(x))\|_{L^\infty(dx)} = O(t^{-\delta}), \ t \longrightarrow +\infty.$$

By (4.1.10), $w(t,x) = v(t,x) + V_1 e^{i\theta} + V_{-3} e^{-3i\theta}$, where V_1, V_{-3} lie in $\mathcal{V} \subset \widetilde{\mathcal{E}}_{\beta,\alpha',1}^{s+2,s'-1}(M)$. We shall choose the indices in (4.1.6) as optimal as possible i.e. (α, α') close to $(-4, 2)$, in the shaded area of the figure of page 64. Then s will be close to 4 and $\beta = 2(\alpha + \alpha' + 1)$ will be close to -2. If $|\gamma| \leqslant M$, $m = 1, -3$, we thus have $x^\gamma V_m \in \widetilde{E}_{\beta,\alpha',1}^{s+2,s'-1-|\gamma|}$ whence by (3.1.5)

$$\|\chi(D/\sqrt{t})x^\gamma V_m\|_{L^\infty(dx)} = O(t^{-\delta}), \ t \longrightarrow +\infty$$

for any $\delta \in]0, 1/2[$. Moreover

$$\|(1-\chi)(D/\sqrt{t})x^\gamma V_m\|_{L^\infty(dx)} \leqslant C \sum_{k;c\sqrt{t}<2^k<t} 2^k \left(\frac{2^k}{\sqrt{t}}\right)^{-s-2} \left(\frac{2^k}{t}\right)^{\alpha'}$$
$$+ C \sum_{k;t<2^k} 2^k \left(\frac{2^k}{\sqrt{t}}\right)^{-s-2} \left(\frac{2^k}{t}\right)^{-s'+1+|\gamma|}$$

using Sobolev injections and the definition of $E_{\beta,\alpha',1}^{s+2,s'-1-|\gamma|}$. Since $s > \alpha' - 1$, $s + s' > M \geqslant |\gamma|$, we get an $O(t^{-\delta})$ upper bound. Consequently, the asymptotic behaviour of $x^\gamma w$ will be given by the one of $x^\gamma v$. Since $v \in \widetilde{\mathcal{E}}_{\alpha,\alpha',\infty}^{s,s'}(M)$, we have for $|\gamma| \leqslant M$

$$\|(1-\chi)(D/\sqrt{t})x^\gamma v\|_{L^\infty(dx)} \leqslant C \sum_{k;c\sqrt{t}<2^k<t} 2^k \left(\frac{2^k}{\sqrt{t}}\right)^{-s} \left(\frac{2^k}{t}\right)^{\alpha'}$$
$$+ C \sum_{k;t<2^k} 2^k \left(\frac{2^k}{\sqrt{t}}\right)^{-s} \left(\frac{2^k}{t}\right)^{-s'+|\gamma|}$$

and since $s > \alpha' + 1$, $s + s' > M + 1 \geqslant |\gamma| + 1$, we get a $O(t^{-\delta})$ estimate for this term also. We thus have to study $\chi(D/\sqrt{t})(x^\gamma v)$. Since any term of type $\chi(D/\sqrt{t})((D/t)x^{\gamma'}v)$,

$|\gamma'| \leqslant |\gamma| - 1$ will be also $O(t^{-\delta})$ in L^{∞} (still because $\frac{D}{t}(x^{\gamma'}v) \in \underline{E}_{\alpha+2,\alpha',\infty}^{s,s'-|\gamma'|-1}$), we reduce ourselves to the study of $\chi(D/\sqrt{t})(Z^{\gamma}v)$. But it follows from (4.2.8) that

$$(4.2.18) \qquad\qquad |D_t(\chi(D/\sqrt{t})(Z^{\gamma}v))| \leqslant Ct^{-1-\delta}$$

for any $\delta \in]0, 1/2[$ (the fact that δ can be taken as close as we want to $1/2$ follows by inspection of the proof of lemma 4.2.5, using that we took (α, α') near $(-4, 2)$). We deduce from (4.2.18) that $\chi(D/\sqrt{t})(Z^{\gamma}v)$ has a limit in L^{∞} when $t \to +\infty$, and that the distance between the function and its limit is $O(t^{-\delta})$. This concludes the proof. $\qquad\qquad\square$

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