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ON THE MONODROMY MAP FOR LOGARITHMIC DIFFERENTIAL SYSTEMS

by Marian Aprodu, Indranil Biswas, Sorin Dumitrescu & Sebastian Heller

Abstract. — We study the monodromy map for logarithmic $\mathfrak{g}$-differential systems over an oriented surface S_0 of genus g, with $\mathfrak{g}$ being the Lie algebra of a complex reductive affine algebraic group G. These logarithmic $\mathfrak{g}$-differential systems are triples of the form (X, D, Φ), where $(X, D) \in \mathcal{T}_{g,d}$ is an element of the Teichmüller space of complex structures on S_0 with $d \geq 1$ ordered marked points $D \subset S_0 = X$ and Φ is a logarithmic connection on the trivial holomorphic principal G-bundle $X \times G$ over X, whose polar part is contained in the divisor D. We prove that the monodromy map from the space of logarithmic $\mathfrak{g}$-differential systems to the character variety of

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G-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point in the following two cases:

1. $g \geq 2$, $d \geq 1$, and $\dim_{\mathbb{C}} G \geq d+2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The above monodromy map is nowhere an immersion in the following two cases:

1. $g = 0$ and $d \geq 4$;
2. $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

This extends to the logarithmic case the main results in [5], [2] dealing with nonsingular holomorphic $\mathfrak{g}$-differential systems (which corresponds to the case of $d = 0$).

Résumé (*Sur la monodromie des systèmes différentiels logarithmiques*). — Nous étudions la monodromie des $\mathfrak{g}$-systèmes différentiels logarithmiques au-dessus d'une surface compacte orientée S_0 de genre g, où $\mathfrak{g}$ désigne l'algèbre de Lie d'un groupe de Lie complexe affine réductif G. Ces $\mathfrak{g}$-systèmes différentiels sont des triplets de la forme (X, D, Φ), où $(X, D) \in \mathcal{T}_{g,d}$ est un élément de l'espace de Teichmüller de structures complexes sur S_0, avec $d \geq 1$ points marqués ordonnés $D \subset S_0 = X$ et Φ est une connexion logarithmique sur le G-fibré holomorphe trivial $X \times G$ au-dessus de X et dont la partie polaire est contenue dans le diviseur D.

Nous démontrons que l'application de monodromie définie sur l'espace des $\mathfrak{g}$-systèmes différentiels logarithmiques et à valeurs dans la variété des caractères de G-représentations du groupe fondamental de $S_0 \setminus D$ est une immersion au point générique dans les deux cas suivants :

1. $g \geq 2$, $d \geq 1$, et $\dim_{\mathbb{C}} G \geq d+2$;
2. $g = 1$ et $\dim_{\mathbb{C}} G \geq d$.

L'application de monodromie ci-dessus n'est en aucun point une immersion dans les deux cas suivants :

1. $g = 0$ et $d \geq 4$;
2. $g \geq 1$ et $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

Ceci étend au cas logarithmique les résultats principaux de [5], [2] qui traitent le cas des $\mathfrak{g}$-systèmes différentiels holomorphes non singuliers (qui correspondent ici au cas $d = 0$).

1. Introduction

The study of the Riemann–Hilbert mapping, which associates to a flat (algebraic or holomorphic) connection its monodromy morphism from the fundamental group is a classical topic in algebraic and analytical geometry (see, for instance, [8], [17], and references therein).

We recall the setup and results of [5] and [2], the predecessors of this paper. Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$, with $\dim G > 0$, and let $\mathfrak{g}$ be the Lie algebra of G. A $\mathfrak{g}$-differential system is a pair of the form (X, Φ), where X is a complex structure on a compact oriented smooth surface S_0 of genus g, and Φ is a holomorphic connection on the trivial holomorphic principal G-bundle $X \times G$ over the Riemann surface X. A $\mathfrak{g}$-differential system (X, Φ) is called irreducible if Φ is not induced by a holomorphic connection on $X \times P$ for some proper parabolic subgroup P of G.

Since any holomorphic connection on a Riemann surface is flat, associating the monodromy representation to a holomorphic connection we obtain a map from the space of irreducible $\mathfrak{g}$-differential systems to the irreducible G-character variety $\mathrm{Hom}(\pi_1(S_0), G)^{\mathrm{ir}}/G$. This monodromy map is actually holomorphic.

The main result of [5] says that, if $g = 2$, this Riemann–Hilbert monodromy map is a local diffeomorphism from the space of irreducible $\mathfrak{g}$-differential systems into the irreducible G-character variety, for $G = \mathrm{SL}(2, \mathbb{C})$. Being inspired by [5], in [2] it was shown that, for all $g \geq 2$, the above monodromy map is an immersion on an open dense subset of the space of irreducible $\mathfrak{g}$-differential systems, for all reductive groups G with $\dim_{\mathbb{C}} G \geq 3$.

Our aim here is to study the Riemann–Hilbert monodromy mapping for logarithmic $\mathfrak{g}$-differential systems, where $\mathfrak{g}$ is as above. These logarithmic $\mathfrak{g}$-differential systems are defined by triples of the form (X, D, Φ), where $(X, D) \in \mathcal{T}_{g,d}$ is an element of the Teichmüller space of complex structures on S_0, with d ordered marked points $D \subset S_0 = X$ (see Section 3), and Φ is a logarithmic connection on the trivial holomorphic principal G-bundle $X \times G$ over X, whose polar part is contained in the divisor D.

We prove the following (see Theorem 4.4):

Theorem 1.1. — *Assume that $3g - 3 + d > 0$ and $d \geq 1$. The Riemann–Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$-differential systems to the character variety of irreducible G-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point in the following two cases:*

1. $g \geq 2$ *and* $\dim_{\mathbb{C}} G \geq d + 2$*;*
2. $g = 1$ *and* $\dim_{\mathbb{C}} G \geq d$*.*

The Riemann–Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$-differential systems to the character variety of irreducible G-representations of the fundamental group of $S_0 \setminus D$ is nowhere an immersion in the following two cases:

1. $g = 0$*;*
2. $g \geq 1$ *and* $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$ *(in particular, when* $g = 1$ *and* $\dim_{\mathbb{C}} G < d$*).*

We note that Theorem 1.1 gives a complete answer only when $g = 0$ or $g = 1$. For given $g \geq 2$ and G, there are finitely many cases of d that are not addressed in Theorem 1.1. When $g = 1$ and $d = 0$, from the first part of Theorem 1.1 it follows that the monodromy mapping from the space of irreducible logarithmic $\mathfrak{g}$-differential systems is an immersion at the generic point; see Remark 4.6.

Theorem 1.1, extends to the class of logarithmic $\mathfrak{g}$-differential systems, the main result in [2] which deals with the nonsingular holomorphic $\mathfrak{g}$-differential systems (corresponding to the case $d = 0$). Notice that the hypothesis $3g - 3 + d > 0$ in Theorem 1.1 implies that the above Teichmüller space $\mathcal{T}_{g,d}$ has positive dimension.

Given a reductive complex affine algebraic group G_0, by setting G to be the product group G_0^m, $m \geq 1$, we can make its dimension arbitrarily large.

The proof of Theorem 1.1 is based on a transversality result in the moduli space $\mathcal{B}_G$ of quadruples of the form (X, D, E_G, Φ), where

- $(X, D) \in \mathcal{T}_{g,d}$,
- E_G is a holomorphic principal G-bundle on X such that E_G is topologically trivial, and
- Φ is a logarithmic connection on E_G whose polar part is contained in D.

A key ingredient of this transversality condition is proved in Lemma 4.7, which is an adaptation to the logarithmic case of Theorem 1.1 in [10] (where its proof is attributed to R. Lazarsfeld).

The article is organized as follows. Sections 2 and 3 are preparatory: they introduce the concept of logarithmic connections on holomorphic principal bundles, the above moduli space $\mathcal{B}_G$ of quadruples (X, D, E_G, Φ), and the G-character variety. We describe the infinitesimal deformation space of quadruples (the tangent space of $\mathcal{B}_G$) as the first hypercohomology group of a certain 2-term complex (see Proposition 3.2). Section 4 is devoted to the proof of the main result (Theorem 4.4) and deals with the transversality, in the tangent space of $\mathcal{B}_G$, between the isomonodromy foliation and the subspace of logarithmic $\mathfrak{g}$-differential systems. This transversality condition, which is equivalent to the monodromy map being an immersion on the space logarithmic of $\mathfrak{g}$-differential systems, is proved by combining a criteria given in Lemma 4.3 (also Proposition 4.5), with Lemma 4.7 (dealing with the case $g \geq 3$) and Lemma 4.9 (dealing with the case of $g = 2$).

2. The logarithmic Atiyah bundle

Let X be a compact connected Riemann surface. Let

$$D := \{x_1, \cdots, x_d\} \subset X \tag{1}$$

be d distinct points, with $d \geq 2$. For notational convenience, the divisor $x_1 + \ldots + x_d$ of degree d on X will also be denoted by D. For a holomorphic vector bundle V on X, the holomorphic vector bundles $V \otimes \mathcal{O}_X(D)$ and $V \otimes \mathcal{O}_X(-D)$ will be denoted by $V(D)$ and $V(-D)$, respectively. The holomorphic tangent and cotangent bundles of X will be denoted by TX and K_X, respectively.

Let G be a connected complex affine algebraic group with $\dim G > 0$. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$p : E_G \longrightarrow X \tag{2}$$

be a holomorphic principal G-bundle over X. The action of G on E_G produces an action of G on the holomorphic tangent bundle TE_G of E_G. The quotient

$$\mathrm{At}(E_G) := (TE_G)/G \longrightarrow X \tag{3}$$

is the Atiyah bundle for E_G [1]. Let $dp : TE_G \longrightarrow p^*TX$ be the differential of the map p in (2). Let

$$\mathrm{ad}(E_G) := \mathrm{kernel}(dp)/G \subset (TE_G)/G \tag{4}$$

be the adjoint bundle for E_G. Note that this holomorphic vector bundle, kernel(dp) is identified with the trivial holomorphic vector bundle $E_G \times \mathfrak{g} \longrightarrow E_G$ using the action of G on E_G. Hence $\mathrm{ad}(E_G)$ coincides with the vector bundle $E_G \times^G \mathfrak{g} \longrightarrow X$ associated to E_G for the adjoint action of G on $\mathfrak{g}$.

Thus we have a short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0, \tag{5}$$

where $\mathrm{At}(E_G)$ is defined in (3), and the projection $d'p$ is induced by dp; the sequence in (5) is known as the Atiyah exact sequence. Define

$$\mathrm{At}(E_G)(-\log D) := (d'p)^{-1}(TX(-D)) \subset \mathrm{At}(E_G), \tag{6}$$

where $d'p$ is the homomorphism in (5). So, from (5) we have the *logarithmic Atiyah exact sequence*

$$0 \longrightarrow \mathrm{ad}(E_G) \xrightarrow{\iota_0} \mathrm{At}(E_G)(-\log D) \xrightarrow{\widehat{dp}} TX(-D) \longrightarrow 0, \tag{7}$$

where $\widehat{dp}$ is the restriction of the homomorphism $d'p$ to $\mathrm{At}(E_G)(-\log D)$, and ι_0 is given by the homomorphism $\mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G)$ in (5). We have the following commutative diagram of homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_G) & \xrightarrow{\iota_0} & \mathrm{At}(E_G)(-\log D) & \xrightarrow{\widehat{dp}} & TX(-D) & \longrightarrow & 0, \\ & & \| & & \big\downarrow \iota & & \big\downarrow \iota' & & \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G) & \xrightarrow{d'p} & TX & \longrightarrow & 0, \end{array} \tag{8}$$

where ι and ι' are the natural inclusion maps.

A *logarithmic connection* on E_G with polar part in D is a holomorphic homomorphism

$$\Phi : TX(-D) \longrightarrow \mathrm{At}(E_G)(-\log D)$$

such that

$$\widehat{dp} \circ \Phi = \mathrm{Id}_{TX(-D)}, \tag{9}$$

where $\widehat{dp}$ is the surjective homomorphism in (7).

Since we have $\iota' \circ \widehat{dp} = (d'p) \circ \iota$ (see (8)), and $\iota'(y)(TX(-D)_y) = 0$ for every point $y \in D$ in (1), for a logarithmic connection Φ on E_G, from (9) we have

$$\iota' \circ \widehat{dp} \circ \Phi(TX(-D)_y) = \iota'(y)(TX(-D)_y) = 0,$$

for every $y \in D$. Consequently, from the commutativity of (8) we conclude that $(d'p) \circ \iota \circ \Phi(TX(-D)_y) = 0$. This implies that

$$\iota \circ \Phi(TX(-D)_y) \subset \mathrm{ad}(E_G)_y \subset \mathrm{At}(E_G)_y \tag{10}$$

(see (8)). On the other hand, $TX(-D)_y = \mathbb{C}$ by the Poincaré adjunction formula [13, p. 146]; for any holomorphic coordinate function z on X around y with $z(y) = 0$, the map $\mathbb{C} \longrightarrow TX(-D)_y$ defined by $\lambda \longmapsto (\lambda \frac{dz}{z})(y)$ is actually independent of the choice of the coordinate function z. The element

$$(\iota \circ \Phi)(y)(1) \in \mathrm{ad}(E_G)_y$$

(see (10)) is called the *residue* of Φ at y; see [8].

Fixing X, the infinitesimal deformations of the principal G-bundle E_G are parametrized by $H^1(X, \mathrm{ad}(E_G))$ [9].

We recall that the infinitesimal deformations of the d-pointed Riemann surface (X, D) are parametrized by $H^1(X, TX(-D))$. The infinitesimal deformations of the above triple (X, D, E_G) are parametrized by $H^1(X, \mathrm{At}(E_G)(-\log D))$ [3], [6], [7], [9], [15].

The following lemma is standard (see [3, Section 2.2] and [15]).

Lemma 2.1. — 1. *The homomorphism of cohomologies*

$$\widehat{dp}_* : H^1(X, \mathrm{At}(E_G)(-\log D)) \longrightarrow H^1(X, TX(-D)),$$

induced by the projection $\widehat{dp}$ *in* (7)*, corresponds to the forgetful map from the infinitesimal deformations of the triple* (X, D, E_G) *to the infinitesimal deformations of the pair* (X, D) *obtained by simply forgetting the principal* G*-bundle.*

2. *The homomorphism of cohomologies*

$$\iota_{0*} : H^1(X, \mathrm{ad}(E_G))) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D)),$$

induced by the homomorphism ι_0 *in* (7)*, coincides with the map from the infinitesimal deformations of the principal* G*-bundle* E_G *to the infinitesimal deformations of the triple* (X, D, E_G) *obtained by keeping the pair* (X, D) *fixed.*

3. Logarithmic connections and isomonodromy

3.1. Logarithmic Atiyah bundle. — Since $\mathrm{At}(E_G) := (TE_G)/G$ (see (3)), the subsheaf $\mathrm{At}(E_G)(-\log D) \subset \mathrm{At}(E_G)$ corresponds to a subsheaf of the sheaf of G-invariant holomorphic vector fields on E_G. We will have occasions to use the following description of this subsheaf of the sheaf of G-invariant holomorphic vector fields on E_G.

Let

$$\widetilde{D} := p^{-1}(D) \subset E_G$$

be the divisor, where p is the projection in (2). Let

$$TE_G(-\log \widetilde{D}) \subset TE_G$$

be the corresponding logarithmic tangent bundle. We recall that this subsheaf is characterized by the following property: A holomorphic vector field v, defined on an open subset $U \subset E_G$, is a section of $TE_G(-\log \widetilde{D})$ if and only if for every holomorphic function f on U that vanishes on $\widetilde{D} \cap U$, the function $v(f)$ also vanishes on $\widetilde{D} \cap U$. Since the divisor $\widetilde{D}$ is smooth, it follows that $TE_G(-\log \widetilde{D})$ is a locally free $\mathcal{O}_{E_G}$-submodule of TE_G. Consequently, $TE_G(-\log \widetilde{D})$ is a holomorphic vector bundle on E_G. The above characterizing property of $TE_G(-\log \widetilde{D})$ immediately implies that the Lie bracket operation of locally defined holomorphic vector fields on E_G preserves the subsheaf $TE_G(-\log \widetilde{D})$.

To describe $TE_G(-\log \widetilde{D})$ locally, take a point $x \in \widetilde{D}$. Let $(z_1, z_2, \cdots, z_m)$ be holomorphic coordinate functions on E_G defined around x such that $z_1 = p \circ z$ for some holomorphic coordinate function z on X around $p(x)$, and also $z_i(x) = 0$ for all $1 \leq i \leq m$; here p denotes the projection in (2). Then $TE_G(-\log \widetilde{D})$ around x is generated by the holomorphic vector fields $z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_m}$.

The action of G on TE_G, induced by the action of G on E_G, actually preserves the subsheaf $TE_G(-\log \widetilde{D})$. It is now straightforward to check that

$$\mathrm{At}(E_G)(-\log D) = TE_G(-\log \widetilde{D})/G. \tag{11}$$

Let

$$\Phi : TX(-D) \longrightarrow \mathrm{At}(E_G)(-\log D) \tag{12}$$

be a logarithmic connection on E_G. Let

$$\widetilde{\Phi} : \mathrm{At}(E_G)(-\log D) \longrightarrow \mathrm{ad}(E_G) \tag{13}$$

be the holomorphic homomorphism uniquely determined by the following conditions:

1. $\widetilde{\Phi} \circ \iota_0 = \mathrm{Id}_{\mathrm{ad}(E_G)}$, where ι_0 is the injective homomorphism in (7), and
2. $\mathrm{kernel}(\widetilde{\Phi}) = \Phi(TX(-D))$.

In view of (4) and (11), the homomorphism $\widetilde{\Phi}$ in (13) produces a G-invariant surjective holomorphic homomorphism

$$\Phi'_0 \; : TE_G(-\log \widetilde{D}) \longrightarrow \mathrm{kernel}(dp), \tag{14}$$

where p is the projection in (2).

Let w be a holomorphic vector field on an open subset $U \subset X$ that vanishes on $U \cap D$. In view of (11), the section $\Phi(w)$ of $\mathrm{At}(E_G)(-\log D)\big|_U$ corresponds to a unique G-invariant holomorphic section of $TE_G(-\log \widetilde{D})\big|_{p^{-1}(U)}$ satisfying the condition that

$$dp(\Phi(w)) = p^* w$$

(as sections of p^*TX)); let

$$\Phi(w)' \in H^0(p^{-1}(U), TE_G(-\log \widetilde{D})\big|_{p^{-1}(U)}) \tag{15}$$

denote this section constructed from w.

Lemma 3.1. — *Let v be a G-invariant holomorphic section of the logarithmic tangent bundle $TE_G(-\log \widetilde{D})\big|_{p^{-1}(U)}$. Then the following three statements hold:*

1. *The holomorphic section $\Phi'_0([\Phi(w)', v])$ of* kernel(dp) *is G-invariant, where Φ'_0 is the homomorphism in* (14), *and $\Phi(w)'$ is the section of $TE_G(-\log \widetilde{D})\big|_{p^{-1}(U)}$ constructed above from w.*
2. *For every holomorphic function h on U,*
$$\Phi'_0([\Phi(h \cdot w)', v]) = (h \circ p) \cdot \Phi'_0([\Phi(w)', v]).$$
3. *If $v = \Phi(v_1)'$ for some holomorphic section v_1 of $T(-D)\big|_U$, then*
$$\Phi'_0([\Phi(w)', v]) = 0.$$

Proof. — As noted before, the Lie bracket operation of locally defined holomorphic vector fields on E_G preserves the subsheaf $TE_G(-\log \widetilde{D})$. Since the homomorphism Φ'_0 in (14) is G-invariant, and $\Phi(w)'$ is G-invariant, while v is given to be G-invariant, it follows that $\Phi'_0([\Phi(w)', v])$ is also G-invariant.

To prove the second statement, consider the identity

$$[\Phi(h \cdot w)', v] = (h \circ p) \cdot [\Phi(w)', v] - v(h \circ p) \cdot \Phi(w)'.$$

Since $\Phi'_0(\Phi(w)') = 0$, where Φ'_0 is the homomorphism in (14), the second statement follows from this identity.

The third statement follows from the fact that any holomorphic one-dimensional distribution is integrable. ☐

In view of (4) and (11), from Lemma 3.1(2) we get a homomorphism

$$\widetilde{\Phi} : TX(-D) \otimes \mathrm{At}(E_G)(-\log D) \longrightarrow \mathrm{ad}(E_G).$$

Then the homomorphism

$$\widetilde{\Phi} \otimes \mathrm{Id}_{TX(-D)^*} : \mathrm{At}(E_G)(-\log D) TX(-D) \otimes TX(-D)^* \longrightarrow \mathrm{ad}(E_G) \otimes TX(-D)^*$$

produces a homomorphism

$$\widehat{\Phi} : \mathrm{At}(E_G)(-\log D) \longrightarrow \mathrm{ad}(E_G) \otimes TX(-D)^* = \mathrm{ad}(E_G) \otimes K_X(D) \tag{16}$$

using the duality pairing $TX(-D) \otimes TX(-D)^* \longrightarrow \mathcal{O}_X$.

Let $\mathcal{C}_\bullet$ be the two-term complex of sheaves on X

$$\mathcal{C}_\bullet : \mathcal{C}_0 := \mathrm{At}(E_G)(-\log D) \xrightarrow{\widehat{\Phi}} \mathcal{C}_1 := \mathrm{ad}(E_G) \otimes K_X(D), \tag{17}$$

where $\widehat{\Phi}$ is the homomorphism constructed in (16), and $\mathcal{C}_i$ is at the i-th position.

From Lemma 3.1(3), we know that

$$\widehat{\Phi} \circ \Phi = 0.$$

Consequently, the logarithmic connection Φ in (12) produces a homomorphism of complexes

$$\Phi^C : TX(-D) \longrightarrow \mathcal{C}_\bullet, \tag{18}$$

where $TX(-D)$ is the one-term complex concentrated at the 0-th position, and $\mathcal{C}_\bullet$ is the complex in (17). In other words, we have the commutative diagram

$$\begin{array}{ccccc} TX(-D) : & TX(-D) & \longrightarrow & 0 \\ \Phi^C \Big\downarrow & \Big\downarrow \Phi & & \Big\downarrow \\ \mathcal{C}_\bullet \quad : & \mathcal{C}_0 & \xrightarrow{\widehat{\Phi}} & \mathcal{C}_1 \end{array} \tag{19}$$

3.2. Character variety. — Let $\mathcal{T}_{g,d}$ denote the Teichmüller space of compact connected Riemann surfaces of genus g with d ordered marked points, where $d \geq 1$. We will always assume that $3g - 3 + d > 0$. This $\mathcal{T}_{g,d}$ is a complex manifold of dimension $3g - 3 + d$. We recall a description of $\mathcal{T}_{g,d}$, which will be used here. Let S_0 be an oriented C^∞ surface of genus g and let $D_0 \subset S_0$ be d ordered distinct points. Let $\mathbf{C}(S_0)$ denote the space of all C^∞ complex structures on S_0 compatible with the given orientation of S_0. Let $\mathrm{Diff}^0_{D_0}(S_0)$ denote the group of all orientation preserving diffeomorphisms

$$\beta : S_0 \longrightarrow S_0$$

such that

- $\beta(x) = x$ for every $x \in D_0$, and
- β is homotopic to the identity map of S_0 through a continuous family of diffeomorphisms β_t of S_0, $0 \leq t \leq 1$, such that $\beta_t(x) = x$ for all t and all $x \in D_0$.

The group $\mathrm{Diff}^0_{D_0}(S_0)$ acts on $\mathbf{C}(S_0)$ by pushing forward complex structures using diffeomorphisms. Then we have

$$\mathcal{T}_{g,d} = \mathbf{C}(S_0)/\mathrm{Diff}^0_{D_0}(S_0).$$

We now assume the complex connected affine algebraic group G to be reductive. The complement $S_0 \setminus D_0$ will be denoted by S'_0. Let

$$\mathcal{R}_G(S'_0) := \mathrm{Hom}^{\mathrm{ir}}(\pi_1(S'_0), G)/G \tag{20}$$

be the irreducible G-character variety for S'_0; the space $\mathrm{Hom}^{\mathrm{ir}}(\pi_1(S'_0), G)$ consists of all homomorphisms $\gamma : \pi_1(S'_0) \longrightarrow G$ such that $\gamma(\pi_1(S'_0))$ is not contained in any proper parabolic subgroup of G. We note that $\mathcal{R}_G(S'_0)$ does not depend on the choice of the base point needed to define the fundamental group of S'_0. Since $\pi_1(S'_0)$ is finitely presented, the complex algebraic structure of G

produces a complex algebraic structure on $\mathcal{R}_G(S'_0)$, so $\mathcal{R}_G(S'_0)$ is a complex affine variety. It is, in fact, a smooth complex orbifold. We have

$$\dim \mathcal{R}_G(S'_0) = (2g+d-1)\cdot \dim_{\mathbb{C}} G - \dim_{\mathbb{C}}[G,G]. \tag{21}$$

For more details of the above dimension count the reader is referred to [11] and [20, Proposition 49] (to which the monodromy around the poles should be added).

3.3. Monodromy of logarithmic connections. — A logarithmic connection Φ on a holomorphic principal G-bundle $E_G \longrightarrow X$ is called *irreducible*, if there is no holomorphic reduction of structure group $E_P \subset E_G\big|_{X\setminus D}$ to some proper parabolic subgroup $P \subset G$, over the open subset $X \setminus D \subset X$, such that Φ is induced by a holomorphic connection on E_P.

The above definition of irreducibility needs clarification, because in the special case where E_G is the trivial holomorphic principal G-bundle, and D is the zero divisor — so the logarithmic connection Φ is holomorphic, meaning it has no poles — this definition of irreducibility is, a priori, weaker than the definition, given in the Introduction, of irreducible holomorphic $\mathfrak{g}$-differential systems. More precisely, in the definition, given in the Introduction, of irreducible holomorphic $\mathfrak{g}$-differential systems, the principal P-bundle is required to be the trivial bundle $X \times P \longrightarrow X$, while the above definition does not impose any other condition on E_P apart from the condition that the logarithmic connection Φ is induced by a logarithmic connection on E_P. We will show the following:

Let Φ be a holomorphic connection on the trivial principal G-bundle

$$\mathcal{E}^0_G := X \times G \longrightarrow X,$$

and let $E_P \subset \mathcal{E}^0_G$ be a holomorphic reduction of structure group to P over X, such that Φ is induced by a holomorphic connection on E_P. Then E_P is the trivial principal P-bundle $X \times P \longrightarrow X$.

To prove the above statement, first note that a holomorphic reduction of structure group $E_P \subset \mathcal{E}^0_G$ to P is given by a holomorphic map $\phi : X \longrightarrow G/P$. For this map ϕ, we have

$$\phi^* T(G/P) = \mathrm{ad}(\mathcal{E}^0_G)/\mathrm{ad}(E_P). \tag{22}$$

If E_P admits a holomorphic connection Φ_P, then Φ_P induces holomorphic connections on both $\mathrm{ad}(\mathcal{E}^0_G)$ and $\mathrm{ad}(E_P)$. This implies that

$$\mathrm{degree}(\mathrm{ad}(\mathcal{E}^0_G)) = 0 = \mathrm{degree}(\mathrm{ad}(E_P)),$$

and hence from (22) it follows that

$$\mathrm{degree}(\phi^* T(G/P)) = 0. \tag{23}$$

Since the anticanonical line bundle $K_{G/P}^{-1}$ on G/P is ample, from (23) we conclude that ϕ is a constant map. Consequently, P is the trivial principal P-bundle $X \times P \longrightarrow X$.

Let

$$\varphi : \mathcal{B}_G \longrightarrow \mathcal{T}_{g,d} \tag{24}$$

be the moduli space of irreducible logarithmic connections on topologically trivializable holomorphic principal G-bundles. So, $\mathcal{B}_G$ is the moduli space of quadruples of the form (X, D, E_G, Φ), where

- $(X, D) \in \mathcal{T}_{g,d}$,
- E_G is a holomorphic principal G-bundle on X such that E_G is topologically trivial, and
- Φ is an irreducible logarithmic connection on E_G whose polar part is contained in D.

The map φ in (24) sends any (X, D, E_G, Φ) to the pair (X, D). The moduli space $\mathcal{B}_G$ is a smooth complex orbifold.

Any logarithmic connection on a Riemann surface X is flat because $\bigwedge^2 (TX)^* = 0$ (consequently, its curvature two-form vanishes identically). So, considering monodromy representation of logarithmic connections, we get a holomorphic map

$$\theta : \mathcal{B}_G \longrightarrow \mathcal{R}_G(S_0'), \tag{25}$$

where $\mathcal{R}_G(S_0')$ is constructed in (20).

We will prove that a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

First, let E_G be a holomorphic principal G-bundle on X equipped with a logarithmic connection Φ, such that Φ is *not* irreducible. So, there is a proper parabolic subgroup $P \subset G$, a holomorphic reduction of the structure group of $E_G\big|_{X\setminus D}$ to P, given by a subbundle $E_P \subset E_G\big|_{X\setminus D}$, and a holomorphic connection Φ_P on E_P, such that the logarithmic connection on E_G induced by Φ_P coincide with Φ. Since the monodromy of Φ_P coincides with the monodromy of Φ, the monodromy of Φ is contained in P, and hence the monodromy representation for Φ is not irreducible. To prove the converse let Φ be an irreducible logarithmic connection on a holomorphic principal G-bundle E_G on X. Take a point $x_0 \in X \setminus D$ and fix a point $z_0 \in (E_G)_{x_0}$ in the fiber of E_G over x_0. Taking parallel translations of z_0 along all possible homotopy classes of loops based at x_0 we get the monodromy representation

$$H_\Phi : \pi_1(X \setminus D, x_0) \longrightarrow G$$

of Φ. Assume that the image of H_Φ is contained in a parabolic subgroup $P \subsetneq G$. Let $\mathcal{S} \subset E_G\big|_{X\setminus D}$ be the subset obtained by taking parallel translations of z_0

along all possible homotopy classes of paths starting at x_0. Then

$$E_P := \mathcal{S}P \subset E_G\big|_{X\setminus D}$$

(recall that G acts on E_G) is a holomorphic reduction of the structure group of $E_G\big|_{X\setminus D}$ to P over $X \setminus D$. The logarithmic connection Φ produces a holomorphic connection on the holomorphic principal P-bundle E_P, which, in turn, induces Φ. Consequently, the logarithmic connection Φ is not irreducible. Thus, a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

3.4. Isomonodromy. — Let

$$d\theta : T\mathcal{B}_G \longrightarrow \theta^* T\mathcal{R}_G(S'_0)$$

be the differential of the map θ in (25). The map θ is a holomorphic submersion, meaning that $d\theta$ is surjective. The kernel of $d\theta$

$$\mathcal{I} := \mathrm{kernel}(d\theta) \subset T\mathcal{B}_G \tag{26}$$

is a holomorphic foliation on $\mathcal{B}_G$; it is known as the *isomonodromy foliation.*

For any point $(X, D) \in \mathcal{T}_{g,d}$, the restriction of θ to $\varphi^{-1}((X, D))$, where φ is the projection in (24), is a holomorphic local diffeomorphism. Consequently, for any point $z \in \mathcal{B}_G$, the differential of φ

$$d\varphi(z) : T_z\mathcal{B}_G \longrightarrow T_{\varphi(z)}\mathcal{T}_{g,d},$$

when restricted to the subspace $\mathcal{I}_z \subset T_z\mathcal{B}_G$ in (26) produces an isomorphism

$$\mathcal{I}_z \xrightarrow{\sim} T_{\varphi(z)}\mathcal{T}_{g,d}.$$

Therefore, there is a unique holomorphic homomorphism

$$\mathbb{L} : \varphi^* T\mathcal{T}_{g,d} \longrightarrow T\mathcal{B}_G \tag{27}$$

such that

- $d\varphi \circ \mathbb{L} = \mathrm{Id}_{\varphi^* T\mathcal{T}_{g,d}}$, and
- $\mathbb{L}(\varphi^* T\mathcal{T}_{g,d}) \subset \mathcal{I}$, where $\mathcal{I}$ is constructed in (26).

Since for any point $(X, D) \in \mathcal{T}_{g,d}$, the restriction of θ to $\varphi^{-1}((X, D))$ is a holomorphic local diffeomorphism, it follows that $\mathbb{L}$ actually satisfies the condition that

$$\mathbb{L}(\varphi^* T\mathcal{T}_{g,d}) = \mathcal{I}. \tag{28}$$

Proposition 3.2. — *Take any point $z = (X, D, E_G, \Phi) \in \mathcal{B}_G$.*

1. *The tangent space to $\mathcal{B}_G$ at z is the first hypercohomology*

$$T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet),$$

where $\mathcal{C}_\bullet$ is the complex in (17).

2. *The homomorphism*

$$\mathbb{L}(z) : T_{\varphi(z)}\mathcal{T}_{g,d} = H^1(X, TX(-D)) \longrightarrow T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$$

in (27) *coincides with the homomorphism of hypercohomologies*

$$\Phi^C_* : H^1(X, TX(-D)) \longrightarrow \mathbb{H}^1(X, \mathcal{C}_\bullet)$$

induced by the homomorphism Φ^C *in* (18).

For the proof of Proposition 3.2, the reader is referred to [18, Proposition 3.8] (Proposition 3.4 of the arxiv version of [18]), [7, p. 1417, Proposition 5.1], [6], [16], and [4].

4. Monodromy map on logarithmic differential systems

4.1. Logarithmic differential systems. — As before, G is a connected complex reductive affine algebraic group with $\dim G > 0$, and

$$d_s := \dim_{\mathbb{C}}[G, G]. \tag{29}$$

Consider the moduli space $\mathcal{B}_G$ in (24). Let

$$\mathbb{T}(G) \subset \mathcal{B}_G \tag{30}$$

be the locus of all (X, D, E_G, Φ) such that the holomorphic principal G-bundle E_G on X is holomorphically trivial. Note that E_G is topologically trivial by the definition of $\mathcal{B}_G$; also by the definition of $\mathcal{B}_G$ the logarithmic connection Φ is irreducible. The subset $\mathbb{T}(G)$ in (30) is a complex subspace.

Proposition 4.1. — *The complex space* $\mathbb{T}(G)$ *in* (30) *is a complex orbifold of dimension* $(g + d - 1) \cdot \dim_{\mathbb{C}} G - d_s + 3g - 3 + d$*, where* d_s *is defined in* (29).

Proof. — Let $\varpi : C_{g,d} \longrightarrow \mathcal{T}_{g,d}$ be the universal Riemann surface equipped with the universal divisor $\mathcal{D} \subset C_{g,d}$ of relative degree d over $\mathcal{T}_{g,d}$. Let $\mathcal{K} \longrightarrow C_{g,d}$ be the relative holomorphic cotangent bundle for the projection ϖ. Let $\varphi' : \mathbb{T}(G) \longrightarrow \mathcal{T}_{g,d}$ be the restriction of the map φ in (24).

For any Riemann surface X, the space of all logarithmic connections on the trivial holomorphic principal G-bundle $X \times G \longrightarrow X$ with a polar part contained in $D \subset X$ is the vector space $H^0(X, K_X(D)) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Consequently, $\mathbb{T}(G)$ is the quotient of an open dense subset of the total space of $\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D})) \otimes \mathfrak{g}$ by the adjoint action of G; the group G acts trivially on $\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D}))$, and it has the adjoint action on $\mathfrak{g}$.

Take a point

$$\mathbf{w} \in \varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D})) \otimes \mathfrak{g}$$

that defines an irreducible logarithmic connection on the trivial principal G-bundle. The adjoint action of the center of G on $\mathfrak{g}$ is trivial. The isotropy subgroup of $[G,G]$ for the action of $[G,G]$ on $\mathbf{w}$ is a finite subgroup of $[G,G]$. The proposition follows from these. ☐

Let

$$\widehat{\theta} : \mathbb{T}(G) \longrightarrow \mathcal{R}_G(S_0') \tag{31}$$

be the restriction to $\mathbb{T}(G) \subset \mathcal{B}_G$ of the monodromy map θ in (25). We are interested in the following question: When is the map $\widehat{\theta}$ an immersion over an open dense subset of $\mathbb{T}(G)$?

Remark 4.2. — In [2], it was proved that $\widehat{\theta}$ is an immersion over an open dense subset of $\mathbb{T}(G)$, if $g \geq 2$, $d = 0$, and $\dim_{\mathbb{C}} G \geq 3$. From this, it can be deduced that $\widehat{\theta}$ is an immersion over an open dense subset of $\mathbb{T}(G)$, if $g \geq 2$, $d = 1$ and $\dim_{\mathbb{C}} G \geq 3$. To see this, first note that there is no logarithmic one-form on a compact Riemann surface X with exactly one pole, because the residue has to be zero. So the space $\mathbb{T}(G)$ in (31) for $d = 1$ coincides with $\mathbb{T}(G)$ for $d = 0$. On the other hand, the natural map from the space $\mathcal{R}_G(S_0')$ in (31) for $d = 0$ to the space $\mathcal{R}_G(S_0')$ for $d = 1$ is an embedding; this natural map corresponds to restricting any flat G-connection on S_0 to the open subset $S_0' = S_0 \setminus D_0$ of it. Therefore, we conclude that the map $\widehat{\theta}$ is an immersion over an open dense subset of $\mathbb{T}(G)$, if $g \geq 2$, $d = 1$ and $\dim_{\mathbb{C}} G \geq 3$.

4.2. The main theorem. — We first state a lemma of linear algebra that will be used in the proof of Theorem 4.4.

Lemma 4.3. — *Let $\beta : V \longrightarrow W$ be a linear map between two finite dimensional complex vector spaces. Let S_1 and S_2 be two subspaces of V such that*

1. $\mathrm{kernel}(\beta) \subset S_1$, *and*
2. *the homomorphism $\beta\big|_{S_2} : S_2 \longrightarrow W$ is injective.*

Then $\dim S_1 \cap S_2 = \dim \beta(S_1) \cap \beta(S_2)$.

Proof. — Since $\beta\big|_{S_2}$ is injective, the restriction $\beta\big|_{S_1 \cap S_2}$ is injective. For any $v \in S_2$ with $\beta(v) \in \beta(S_1)$, there is an element $w \in S_1$ such that $\beta(v) = \beta(w)$. But then $v - w \in \mathrm{kernel}(\beta) \subset S_1$, and hence $v \in S_1$. Consequently, the restriction $\beta\big|_{S_1 \cap S_2}$ is realized as an isomorphism between $S_1 \cap S_2$ and $\beta(S_1) \cap \beta(S_2)$. ☐

Theorem 4.4. — *Assume that $3g - 3 + d > 0$ and $d \geq 1$. The map $\widehat{\theta}$ in* (31) *is an immersion over a nonempty open dense subset of $\mathbb{T}(G)$ in the following two cases:*

1. $g \geq 2$ *and* $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ *and* $\dim_{\mathbb{C}} G \geq d$.

The map $\widehat{\theta}$ in (31) *is nowhere an immersion in the following two cases:*

1. $g = 0$;
2. $g \geq 1$ *and* $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

Proof. — First assume that $g = 0$. The trivial holomorphic principal G-bundle over $\mathbb{CP}^1$ is rigid [19], [14]. In other words, in any holomorphic family of a holomorphic principal G-bundle over $\mathbb{CP}^1$, parametrized by a complex manifold Z, the locus of points of Z over which the principal G-bundle on $\mathbb{CP}^1$ is holomorphically trivial is an open subset of Z. Therefore, the map $\widehat{\theta}$ in (31) is nowhere an immersion. We note that this also follows from the fact that

$$\dim \mathbb{T}(G) - \mathcal{R}_G(S'_0) = d - 3 > 0$$

if $g = 0$ (see (21) and Proposition 4.1).

So, we assume that $g \geq 1$.

If $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$, then from (21) and Proposition 4.1, we have

$$\dim \mathbb{T}(G) - \mathcal{R}_G(S'_0) = 3g - 3 + d - g \cdot \dim_{\mathbb{C}} G > 0.$$

Hence the map $\widehat{\theta}$ in (31) is nowhere an immersion in this case also.

So, we assume that at least one of the following two holds:

1. $g \geq 2$ and $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The map $\widehat{\theta}$ in (31) is an immersion over the subset of $\mathbb{T}(G)$ over which the homomorphism

$$\bigwedge^e d\widehat{\theta} : \bigwedge^e T\mathbb{T}(G) \longrightarrow \widehat{\theta}^* \bigwedge^e T\mathcal{R}_G(S'_0)$$

is fiber-wise nonzero, where $e = \dim_{\mathbb{C}} \mathbb{T}(G)$ and $d\widehat{\theta}$ is the differential of the map $\widehat{\theta}$. Therefore, to prove the theorem, it suffices to show that there is a point $z \in \mathbb{T}(G)$ such that the differential at z

$$d\widehat{\theta}(z) : T_z\mathbb{T}(G) \longrightarrow T_{\widehat{\theta}(z)}\mathcal{R}_G(S'_0) \tag{32}$$

is injective; recall that $\mathbb{T}(G)$ is irreducible.

Take a point

$$z = (X, D, E_G, \Phi) \ \in \mathbb{T}(G). \tag{33}$$

We recall that $\mathcal{I}(z) = \text{kernel}((d\theta)(z))$ (see (26)). The homomorphism $d\widehat{\theta}(z)$ (see (32)) is injective if and only if

$$\mathcal{I}(z) \cap T_z\mathbb{T}(G) = 0; \tag{34}$$

note that both $\mathcal{I}(z)$ and $T_z\mathbb{T}(G)$ are subspaces of the tangent space $T_z\mathcal{B}_G$.

We will use Lemma 4.3 to prove that (34) holds when z is chosen suitably.

We recall from Proposition 3.2(1) that $T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$, where $\mathcal{C}_\bullet$ is the complex in (17). We also recall that the infinitesimal deformations of the triple (X, D, E_G) are parametrized by $H^1(X, \text{At}(E_G)(-\log D))$. Let

$$\rho : \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow H^1(X, \text{At}(E_G)(-\log D)) \tag{35}$$

be the forgetful map that sends any infinitesimal deformation of the quadruple

$$z = (X, D, E_G, \Phi)$$

in (33) to the infinitesimal deformation of the triple (X, D, E_G) obtained from it by simply forgetting the logarithmic connection. We shall describe ρ explicitly.

Let $\mathcal{A}_\bullet$ be the one-term complex with $\mathrm{At}(E_G)(-\log D)$ at the 0-th position.

Consider the homomorphism $\mathcal{H}$ of complexes

(36)
$$\begin{array}{rcl}
\mathcal{C}_\bullet \ : \mathrm{At}(E_G)(-\log D) & \xrightarrow{\widehat{\Phi}} & \mathrm{ad}(E_G) \otimes K_X(D) \\
\mathcal{H}\downarrow \qquad\qquad \Vert \qquad\qquad & & \downarrow \\
\mathcal{A}_\bullet \ : \mathrm{At}(E_G)(-\log D) & \longrightarrow & 0
\end{array}$$

where $\mathcal{C}_\bullet$ is the complex in (17). Let

(37)
$$\mathcal{H}_* : \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow \mathbb{H}^1(X, \mathcal{A}_\bullet) = H^1(X, \mathrm{At}(E_G)(-\log D))$$

be the homomorphism of hypercohomologies induced by this homomorphism of complexes. Then the homomorphism ρ in (35) coincides with $\mathcal{H}_*$ in (37).

In Lemma 4.3, set $V = \mathbb{H}^1(X, \mathcal{C}_\bullet)$, $W = H^1(X, \mathrm{At}(E_G)(-\log D))$, $\beta = \mathcal{H}_*$ (see (37)), $S_1 = T_z\mathbb{T}(G)$ (see (30)), and $S_2 = \mathcal{I}(z)$ (see (26)).

We will show that the hypotheses in Lemma 4.3 are satisfied.

Proposition 4.5. — *For the above data, the two conditions in Lemma 4.3 hold.*

Proof of Proposition 4.5. — The first condition in Lemma 4.3 says that

(38)
$$\mathrm{kernel}(\mathcal{H}_*) \subset T_z\mathbb{T}(G).$$

To prove (38) we will identify the kernel of $\mathcal{H}_*$. For this, observe that the homomorphism of complexes $\mathcal{H}$ in (36) fits in the following short exact sequence of complexes:

$$\begin{array}{ccccc}
 & 0 & & & 0 \\
 & \downarrow & & & \downarrow \\
 & 0 & & & \\
 & \downarrow & & & \downarrow \\
\mathcal{A}'_\bullet : & 0 & \longrightarrow & & \mathrm{ad}(E_G) \otimes K_X(D) \\
\downarrow & \downarrow & & & \Vert \\
\mathcal{C}_\bullet : & \mathrm{At}(E_G)(-\log D) & \xrightarrow{\widehat{\Phi}} & & \mathrm{ad}(E_G) \otimes K_X(D) \\
\mathcal{H}\downarrow & \Vert & & & \downarrow \\
\mathcal{A}_\bullet : & \mathrm{At}(E_G)(-\log D) & \longrightarrow & & 0 \\
\downarrow & \downarrow & & & \downarrow \\
0 & 0 & & & 0
\end{array}$$

This short exact sequence of complexes yields the following long exact sequence of hypercohomologies:

$$\longrightarrow \mathbb{H}^1(X,\mathcal{A}'_\bullet) = H^0(X,\mathrm{ad}(E_G)\otimes K_X(D)) \xrightarrow{\nu} \mathbb{H}^1(X,\mathcal{C}_\bullet)$$
$$\xrightarrow{\mathcal{H}_*} \mathbb{H}^1(X,\mathcal{A}_\bullet) = H^1(X,\mathrm{At}(E_G)(-\log D)) \longrightarrow \cdots .$$

The above homomorphism ν corresponds to moving the holomorphic connection on the trivializable holomorphic principal G-bundle E_G, keeping the triple (X,D,E_G) fixed. This immediately implies that (38) holds.

The second condition in Lemma 4.3 says that the restriction of the homomorphism $\mathcal{H}_*$ to $\mathcal{I}(z)$

$$\mathcal{H}_*\big|_{\mathcal{I}(z)} : \mathcal{I}(z) \longrightarrow H^1(X,\mathrm{At}(E_G)(-\log D)) \tag{39}$$

is injective.

To prove that the homomorphism in (39) is injective, from (28) we conclude that $\mathcal{H}_*\big|_{\mathcal{I}(z)}$ is injective if the composition of homomorphisms

$$H^1(X,TX(-D)) \xrightarrow{\mathbb{L}(z)} \mathbb{H}^1(X,\mathcal{C}_\bullet) \xrightarrow{\mathcal{H}_*} H^1(X,\mathrm{At}(E_G)(-\log D)) \tag{40}$$

is injective, where $\mathbb{L}(z)$ is the homomorphism in (27). From Proposition 3.2(2) we know that $\mathbb{L}(z) = \Phi^C_*$. Therefore, the composition of homomorphisms in (40) coincides with the homomorphism of cohomologies

$$\Phi_* : H^1(X,TX(-D)) \longrightarrow H^1(X,\mathrm{At}(E_G)(-\log D))$$

induced by the logarithmic connection $\Phi : TX(-D) \longrightarrow \mathrm{At}(E_G)(-\log D)$ in (33). But from the definition of a logarithmic connection we know that

- the homomorphism Φ is fiber-wise injective, and
- $\Phi(TX(-D))$ is a direct summand of $\mathrm{At}(E_G)(-\log D)$.

Consequently, the above homomorphism Φ_* is injective. Hence the composition of homomorphisms in (40) is injective. This implies that the homomorphism in (39) is injective. This completes the proof of Proposition 4.5. □

Continuing with the proof of Theorem 4.4, in view of Proposition 4.5, from Lemma 4.3 we conclude that the statement in (34) is equivalent to the following statement:

$$\mathcal{H}_*(\mathcal{I}(z)) \cap \mathcal{H}_*(T_z\mathbb{T}(G)) = 0, \tag{41}$$

where $\mathcal{H}_*$ is the homomorphism in (37).

Fix a holomorphic trivialization of the principal G-bundle E_G in (33). Using it we will identify E_G with the trivial holomorphic principal G-bundle $X \times G \longrightarrow X$. So $\mathrm{ad}(E_G)$ is the trivial holomorphic vector bundle $X\times\mathfrak{g} \longrightarrow X$, where $\mathfrak{g}$ is the Lie algebra of G, and also

$$\mathrm{At}(E_G)(-\log D) = \mathrm{ad}(E_G)\oplus TX(-D) = X\times\mathfrak{g}\oplus TX(-D). \tag{42}$$

Let Φ_0 be the trivial logarithmic (in fact, it is holomorphic) connection on the trivial holomorphic principal G-bundle $X \times G \longrightarrow X$. Note that the trivial holomorphic connection on E_G does not depend on the choice of the trivialization of E_G. The homomorphism

$$TX(-D) \longrightarrow \mathrm{At}(E_G)(-\log D)$$

that defines Φ_0 coincides with the inclusion map

$$TX(-D) \hookrightarrow \mathrm{ad}(E_G) \oplus TX(-D) = \mathrm{At}(E_G)(-\log D)$$

(see (42)). So we have

$$\Phi = \Phi_0 + \delta, \tag{43}$$

where

$$\delta \in H^0(X, K_X(D) \otimes \mathfrak{g}) = H^0(X, K_X(D)) \otimes \mathfrak{g};$$

recall that $\mathrm{ad}(E_G) = X \times \mathfrak{g}$.

Consider the infinitesimal deformations of the triple (X, D, E_G) in (33) such that the principal G-bundle remains trivial, but the pair (X, D) moves. These correspond to the image of the homomorphism

$$\begin{aligned} H^1(X, TX(-D)) \longrightarrow\ & H^1(X, \mathrm{At}(E_G)(-\log D)) \\ &= H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)) \end{aligned}$$

(see (42) for the decomposition) given by the identity map of $H^1(X, TX(-D))$ and the zero map of $H^1(X, TX(-D))$ to $H^1(X, \mathrm{ad}(E_G))$. In other words, these correspond to the image of the homomorphism of cohomologies

$$H^1(X, TX(-D)) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D))$$

induced by the inclusion map $TX(-D) \hookrightarrow \mathrm{ad}(E_G) \oplus TX(-D)$, which is defined using (42).

Consequently, the subspace in (41)

$$\mathcal{H}_*(T_z\mathbb{T}(G)) \subset H^1(X, \mathrm{At}(E_G)(-\log D)) = H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$\begin{aligned} 0 \oplus H^1(X, TX(-D)) = H^1(X, TX(-D)) &\subset H^1(X, \mathrm{At}(E_G)(-\log D)) \\ &= H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)). \end{aligned}$$

Consider the section δ in (43). Using the natural duality pairing

$$TX(-D) \otimes K_X(D) \longrightarrow \mathcal{O}_X$$

it produces a homomorphism

$$\widehat{\delta} : TX(-D) \longrightarrow \mathcal{O}_X \otimes \mathfrak{g} = \mathrm{ad}(E_G). \tag{44}$$

Let

$$\widehat{\delta}_* : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes \mathfrak{g} = H^1(X, \mathrm{ad}(E_G)) \tag{45}$$

be the homomorphism of cohomologies induced by $\widehat{\delta}$ in (44).

We will now show that the subspace in (41)

$$\mathcal{H}_*(\mathcal{I}(z)) \subset H^1(X, \mathrm{At}(E_G)(-\log D)) = H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$\{(\widehat{\delta}_*(v), v) \mid v \in H^1(X, TX(-D))\} \subset H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)),$$

where $\widehat{\delta}_*$ is the homomorphism in (45).

To prove this, let

$$\iota_{0*} : H^1(X, \mathrm{ad}(E_G)) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D)) \tag{46}$$

be the homomorphism of cohomologies induced by the homomorphism ι_0 of sheaves in (7). We note that ι_{0*} coincides with the natural map that sends any infinitesimal deformation of E_G (keeping (X, D) fixed) to the corresponding infinitesimal deformation of (X, D, E_G), where only E_G is moving.

Consider the homomorphism Φ^C in (18) constructed from the connection Φ. Let

$$\Phi^{0,C} : TX(-D) \longrightarrow \mathcal{C}^0_\bullet \tag{47}$$

be the homomorphism as in (18) constructed for the trivial connection Φ_0 on E_G; here $\mathcal{C}^0_\bullet$ is the complex as in (17) for the trivial connection Φ_0. From (43), it follows immediately that

$$\mathcal{H} \circ \Phi^C - \mathcal{H}^0 \circ \Phi^{0,C} = \iota_0 \circ \widehat{\delta}, \tag{48}$$

where $\widehat{\delta}$, $\mathcal{H}$, and ι_0 are the homomorphisms in (44), (36), and (7) respectively, while $\mathcal{H}^0$ is the homomorphism for the trivial holomorphic connection Φ_0 constructed as in (36) (by substituting Φ_0 in place of Φ in the construction of $\mathcal{H}$). As in Proposition 3.2(2), let

$$\Phi^{0,C}_* : H^1(X, TX(-D)) \longrightarrow \mathbb{H}^1(X, \mathcal{C}^0_\bullet) \tag{49}$$

be the homomorphism of hypercohomologies induced by $\Phi^{0,C}$ in (47). From (48) we conclude that

$$\mathcal{H}_* \circ \Phi^C_* - \mathcal{H}^0_* \circ \Phi^{0,C}_* = \iota_{0*} \circ \widehat{\delta}_*, \tag{50}$$

where $\widehat{\delta}_*$, $\mathcal{H}_*$, $\Phi^{0,C}_*$, ι_{0*} and Φ^C_* are the homomorphisms in (45), (37), (49), (46), and Proposition 3.2(2), respectively, and

$$\mathcal{H}^0_* : \mathbb{H}^1(X, \mathcal{C}^0_\bullet) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D)) \tag{51}$$

is the homomorphism of hypercohomologies induced by the homomorphism $\mathcal{H}^0$ in (48). Note that both sides of (50) are actually homomorphisms from

$H^1(X, TX(-D))$ to $H^1(X, \mathrm{At}(E_G)(-\log D))$. Also, note that from the decomposition in (42), it follows immediately that the homomorphism ι_{0*} in (50) is injective. In fact, the decomposition in (42) realizes $H^1(X, \mathrm{ad}(E_G))$ as a direct summand of $H^1(X, \mathrm{At}(E_G)(-\log D))$.

Now consider the homomorphism

$$\mathbb{L}(z) : T_{(X,D)}\mathcal{T}_{g,d} \longrightarrow T_z\mathcal{B}_G$$

in (27) constructed for the connection Φ in the expression of z in (33). Let

$$\mathbb{L}^0 : T_{(X,D)}\mathcal{T}_{g,d} \longrightarrow T_{(X,D,E_G,\Phi_0)}\mathcal{B}_G$$

be the homomorphism as in (27) constructed for the trivial connection Φ_0. From Proposition 3.2(2), we know that

$$\mathcal{H}_* \circ \mathbb{L} - \mathcal{H}^0_* \circ \mathbb{L}^0 = \mathcal{H}_* \circ \Phi^C_* - \mathcal{H}^0_* \circ \Phi^{0,C}_*, \tag{52}$$

where $\mathcal{H}_*$ and $\mathcal{H}^0_*$ are the homomorphisms in (37) and (51) respectively; recall that $T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$ and $T_{(X,D,E_G,\Phi_0)}\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}^0_\bullet)$.

Combining (50) and (52) it follows that

$$\mathcal{H}_* \circ \mathbb{L} - \mathcal{H}^0_* \circ \mathbb{L}^0 = \iota_{0*} \circ \widehat{\delta}_*, \tag{53}$$

where $\widehat{\delta}_*$ is the homomorphism in (45).

It was noted earlier that the decomposition in (42) realizes $H^1(X, \mathrm{ad}(E_G))$ as a direct summand of $H^1(X, \mathrm{At}(E_G)(-\log D))$.

Therefore, from (53) and (28) we conclude that the subspace in (41)

$$\mathcal{H}_*(\mathcal{I}(z)) \subset H^1(X, \mathrm{At}(E_G)(-\log D)) = H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D))$$

(see (42) for the above decomposition) coincides with the subspace

$$\{(\widehat{\delta}_*(v), v) \mid v \in H^1(X, TX(-D))\} \subset H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)).$$

On the other hand, it was shown earlier that the subspace in (41)

$$\mathcal{H}_*(T_z\mathbb{T}(G)) \subset H^1(X, \mathrm{At}(E_G)(-\log D)) = H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$0 \oplus H^1(X, TX(-D)) = H^1(X, TX(-D)) \subset H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)).$$

Combining these two we obtain an isomorphism

$$\eta : \mathrm{kernel}(\widehat{\delta}_*) \xrightarrow{\sim} \mathcal{H}_*(\mathcal{I}(z)) \cap \mathcal{H}_*(T_z\mathbb{T}(G)) \tag{54}$$

that sends any $v \in \mathrm{kernel}(\widehat{\delta}_*) \subset H^1(X, TX(-D))$ to

$$(0, v) \in H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX(-D)) = H^1(X, \mathrm{At}(E_G)(-\log D)).$$

Consequently, (41) holds if and only if we have

$$\mathrm{kernel}(\widehat{\delta}_*) = 0, \tag{55}$$

where $\widehat{\delta}_*$ is the homomorphism constructed in (45).

Take any subspace

$$V \subset H^0(X, K_X(D)).$$

Let $H^1(X, TX(-D)) \otimes V \longrightarrow H^1(X, \mathcal{O}_X)$ be the homomorphism constructed using the duality pairing $TX(-D) \otimes K_X(D) \longrightarrow \mathcal{O}_X$. Let

$$F_V : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes V^* \tag{56}$$

be the homomorphism given by it. From the construction of $\widehat{\delta}_*$ in (45) we see that

$$\mathrm{kernel}(\widehat{\delta}_*) = \mathrm{kernel}(F_V),$$

where $V \subset H^0(X, K_X(D))$ is the image of the homomorphism

$$H_\delta : \mathfrak{g}^* \longrightarrow H^0(X, K_X(D)) \tag{57}$$

given by δ in (43); note that since $\delta \in H^0(X, K_X(D)) \otimes \mathfrak{g}$, it produces a homomorphism H_δ as in (57) by sending any $w \in \mathfrak{g}^*$ to $w(\delta) \in H^0(X, K_X(D))$. Consequently, (55) holds if and only if

$$\mathrm{kernel}(F_{H_\delta(\mathfrak{g}^*)}) = 0, \tag{58}$$

where H_δ and $F_{H_\delta(\mathfrak{g}^*)}$ are the homomorphisms constructed in (57) and (56), respectively.

It is evident that there is an element $z = (X, D, E_G, \Phi) \in \mathbb{T}(G)$ such that (58) holds if and only if there is a subspace $V \subset H^0(X, K_X(D))$, with $\dim V \leq \dim \mathfrak{g}$, satisfying the condition that the homomorphism F_V in (56) is injective. Indeed, choosing a homomorphism

$$\delta' : \mathfrak{g}^* \longrightarrow H^0(X, K_X(D))$$

for which $V \subset \delta'(\mathfrak{g}^*)$, consider the element $\delta \in H^0(X, K_X(D)) \otimes \mathfrak{g}$ given by δ'. Then the logarithmic connection $(X, D, X \times G, \Phi_0 + \delta)$ satisfies (58), where Φ_0 is the trivial holomorphic connection on $X \times G \longrightarrow X$.

First assume that $g = 1$ (hence, by hypothesis, $d \geq 1$) and $\dim_{\mathbb{C}} G \geq d$. This implies that

$$\dim H^0(X, K_X(D)) = d \leq \dim_{\mathbb{C}} G.$$

So, in this case, there is a subspace $V \subset H^0(X, K_X(D))$, with $\dim V \leq \dim \mathfrak{g}$, for which the homomorphism F_V in (56) is injective, if the homomorphism

$$F_{H^0(X, K_X(D))} : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes H^0(X, K_X(D))^* \tag{59}$$

is injective; if the homomorphism in (59) is injective, then we may take V to be $H^0(X, K_X(D))$ itself, and the homomorphism F_V is injective.

The homomorphism in (59) is injective if the dual homomorphism

$$F^*_{H^0(X, K_X(D))} : H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^{\otimes 2}(D)) \tag{60}$$

is surjective. Now, since $\dim H^0(X, K_X) = 1$, the homomorphism in (60) is injective. On the other hand, we have

$$\dim H^0(X, K_X(D)) = d = \dim H^0(X, K_X^{\otimes 2}(D)),$$

so the homomorphism in (60) is an isomorphism; in particular, it is surjective. This proves the theorem when $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

Now assume that $g \geq 2$ and $\dim_{\mathbb{C}} G \geq d + 2$.

Since $\dim \mathfrak{g} \geq d{+}2$, we conclude that there is an element $z = (X, D, E_G, \Phi) \in \mathbb{T}(G)$ such that (58) holds if there is a subspace $V \subset H^0(X, K_X(D))$, with $\dim V \leq d + 2$, for which the homomorphism F_V in (56) is injective. From Lemmas 4.7 and 4.9 (see also Remark 4.10), it follows that such a subspace V exists. This completes the proof of the theorem. □

Remark 4.6. — From Theorem 4.4, it follows that when $g = 1$ and $d = 0$, the map $\widehat{\theta}$ in (31) is an immersion over a nonempty open dense subset of $\mathbb{T}(G)$. Indeed, from Remark 4.2, we know that $\mathbb{T}(G)$ for $d = 0$ coincides with $\mathbb{T}(G)$ for $d = 1$. On the other hand, $\mathcal{R}_G(S'_0)$ for $d = 0$ is embedded into $\mathcal{R}_G(S'_0)$ for $d = 1$. From Theorem 4.4, we know that the map $\widehat{\theta}$ in (31) is an immersion over a nonempty open dense subset of $\mathbb{T}(G)$ if $g = 1$ and $d = 1$. Therefore, the same holds when $g = 1$ and $d = 0$. Recall that $\dim G > 0$.

In view of Remark 4.2, we assume that $d > 1$ when $g > 1$.

Lemma 4.7. — *Take integers $g > 1$ and $d > 1$. Then for any compact connected nonhyperelliptic Riemann surface X of genus $g \geq 3$, and any effective divisor D on X of degree d, there exists a subspace $W \subset H^0(X, K_X(D))$, with* $\dim W = d + 2$, *such that the homomorphism constructed in* (56)

$$F_W : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes W^*$$

is injective.

Proof. — For a compact Riemann surface X of genus g and an effective divisor D on X of degree d, denote the holomorphic line bundle $K_X^{\otimes 2} \otimes \mathcal{O}_X(D))$ by $K_X^2(D)$. For any subspace $V \subset H^0(X, K_X(D))$, let

$$F_V^* : H^0(X, K_X) \otimes V \longrightarrow K^0(X, K_X^2(D))$$

be the dual of the homomorphism F_V in (56).

We need to show that there is a W with $\dim W = d + 2$ such that the above homomorphism

$$F_W^* : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D)) \tag{61}$$

is surjective.

Consider the natural homomorphism

$$J : H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D)). \tag{62}$$

We will now show that under our assumptions, the homomorphism J in (62) is surjective. To this end, we apply [12, Theorem (4.e.1)] and see that it suffices to prove that

$$h^1(X, \mathcal{O}_X(D)) \leq g - 2. \tag{63}$$

When D is nonspecial, (63) evidently holds. So we suppose that D is special. In order to prove (63), first assume that $d \geq 4$. Then Clifford's theorem (see [13, p. 251]) says that $h^0(X, \mathcal{O}_X(D)) \leq d/2+1$. Now using the Riemann–Roch theorem we get that

$$d + 1 - g = h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) \leq \frac{d}{2} + 1 - h^1(X, \mathcal{O}_X(D)).$$

This implies that (63) holds, and hence J is surjective in this case by [12, Theorem (4.e.1)].

Assume now that $d = 2$ or $d = 3$. Since X is not hyperelliptic, if $d = 2$, then we have $h^0(X, \mathcal{O}_X(D)) = 1$. If $d = 3$, Clifford's theorem implies that $h^0(X, \mathcal{O}_X(D)) \leq 2$. Then the Riemann–Roch theorem implies that (63) holds in both these cases. Applying [12, Theorem (4.e.1)], we infer that J is surjective in these cases as well.

Consequently, we have obtained the surjectivity of the map J in (62) for any pair (X, D), as in the lemma.

From the commutative diagram

$$\begin{array}{ccc} H^0(X, K_X) \otimes H^0(X, K_X(D)) & \xrightarrow{\quad J \quad} & H^0(X, K_X^2(D)) \\ \downarrow & & \downarrow \\ H^0(X, K_X) \otimes H^0(X, K_X(D))/H^0(X, K_X) & \longrightarrow & H^0(X, K_X^2(D))/H^0(X, K_X^2) \end{array}$$

we notice that the surjectivity of J implies the surjectivity of the map

$$H^0(X, K_X) \otimes (H^0(X, K_X(D))/H^0(X, K_X)) \longrightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2).$$

Consider $U \subset H^0(X, K_X(D))$ of dimension $(d-1)$ such that $U \cap H^0(X, K_X) = \{0\}$ inside $H^0(X, K_X(D))$. Then the map

$$U \longrightarrow H^0(X, K_X(D))/H^0(X, K_X)$$

is an isomorphism, and hence the induced map

$$H^0(X, K_X) \otimes U \longrightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2) \tag{64}$$

is surjective.

On the other hand, since X is nonhyperelliptic, [10, Theorem 1.1] (whose proof is attributed to Lazarsfeld) shows that for a general subspace $W_0 \subset H^0(X, K_X)$ of dimension 3, the multiplication map

$$H^0(X, K_X) \otimes W_0 \longrightarrow H^0(X, K_X^2) \tag{65}$$

is surjective. Set

$$W = W_0 \oplus U \subset H^0(X, K_X(D)).$$

The surjectivity of the maps in (64) and (65) implies the surjectivity of

$$F^*W : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D)),$$

which concludes the proof. □

Remark 4.8. — Lemma 4.7 is optimal in the following sense. If $W \subset H^0(X, K_X(D))$ is a subspace such that the intersection $W \cap H^0(X, K_X)$ inside $H^0(X, K_X(D))$ is at least three-dimensional, and $H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D))$ is surjective, then $\dim W \geq d+2$. This claim is easily obtained by reverting the argument in the proof of Lemma 4.7.

Lemma 4.7 excluded the case of $g = 2$. This is dealt with separately below.

Lemma 4.9. — *Let X be a compact connected Riemann surface of genus 2 and let D be an effective divisor of degree $d > 1$ such that $D \notin |K_X|$. Then the multiplication map*

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D))$$

is surjective.

Proof. — We start with the short exact sequence

$$0 \longrightarrow TX \longrightarrow H^0(X, K_X) \otimes \mathcal{O}_X \longrightarrow K_X \longrightarrow 0,$$

twist it by $K_X(D)$, and take the corresponding long exact sequence of cohomologies

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D)) \longrightarrow H^1(X, \mathcal{O}_X(D)) \longrightarrow .$$

By the hypothesis, we have $H^1(X, \mathcal{O}_X(D)) = 0$, and hence from this exact sequence of cohomologies, it follows that the multiplication map is surjective. □

Remark 4.10. — Note that, under the hypotheses of Lemma 4.9, the Riemann–Roch theorem implies that $h^0(X, K_X(D)) = d + 1 < d + 2$.

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