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G. Pisier : 46L99, 46M99, 46B99
S. Popa : 46L35, 46L50
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E. Størmer : 46L10, 46L50
L. Vainerman : 81R50, 22A30
D. Voiculescu : 46L50, 60F99, 43A99

Table des Matières

Introduction

This book is the collection of talks given in the conference on operator algebras held in Orléans in July 1992. Orléans has always been a privileged place for operator algebras thanks to François Combes and Claire Delaroche whose kindness and devotion to the subject played a determinant role.

The content of the book describes the recent advances and several major topics of the theory of operator algebras.

First the theory of quantum groups which after the early work of Kac, Takesaki, Enock and Schwartz and Woronowicz is undergoing rapid changes. A very simple definition of these objects was obtained by Baaj and Skandalis, simply as a unitary operator V in the tensor square of a Hilbert space, $H \otimes H$ satisfying suitable multiplicativity conditions. This very fruitful point of view is analysed by Baaj in the special case of the quantum group $E_\mu(2)$ of Woronowicz with special relevance to the modular theory. The equally important deformation aspect of quantum groups ties up (in the paper of Bauval) with continous fields of C^*-algebras. The papers of Boca and Landstad deal with actions of compact quantum groups on C^*-algebras, analysed in the ergodic case by Boca and in the case of single crossed product by Landstad. Finally Vainerman analyses double cosets of compact quantum groups with respect to subgroups and computes corresponding characters in terms of q-orthogonal polynomials.

The second topic widely covered in this book is the analysis of operator algebras associated to free groups, in which the seminal work of Voiculescu on free probability theory plays a dominant role. This work of D. Voiculescu is a major step towards the classification of type II_1 factors beyond the hyperfinite case and the theory has already provided many unexpected results. In his paper Voiculescu gives in particular an explicit way to compute the spectrum of convolution operators on the free group. Radulescu's paper gives a simple presentation of a III_λ factor whose associated II_1 factor is the II_1 factor of a free group with infinitely many generators.

The II_1 factors associated to Fuchsian groups should belong to the class of II_1 factors "next to hyperfinite" and this question is analysed in the paper of de La Harpe and Voiculescu. P. de La Harpe and his collaborators have obtained general results in particular on simplicity for C^*-algebras and von Neumann algebras of discrete groups and a general review is given in de La Harpe's paper.

Finally S. Popa analyses the free analogue of central sequences for II_1 factors and shows that certain universal commuting squares involving amalgamated free products appear asymptotically in any inclusion of finite index, a result of great interest in the theory of subfactors.

The third topic which is quite active at present is the entropy for automorphisms of II_1 factors, a subject reviewed in Størmer's paper. After my initial work with Størmer this subject has evolved slowly due to the difficulty of making explicit computations of entropy. But a number of new results involving, in particular, Narnhofer, Thirring, Størmer and Sauvageot make it quite lively at present.

Thanks to the work of Effros, Haagerup, Pisier and their collaborators, the notion of operator space has found many interesting applications and has become a bridge between operator algebras and Banach spaces. The paper of Pisier develops the notion of exactness in this context, parallel to the well-known work of E. Kirchberg.

Finally, the papers of Blanchard, Brown and Bekka-Vallette deal with questions related to the K-theory of C^*-algebras. The first analyses the notion of tensor product of two C^*-algebras over $C(X)$, the second deals with the subtle nuance between homotopy and equivalence of projections in general C^*-algebras; the third shows that the natural morphism $C^*(H) \to M(C^*(G))$ associated with a group inclusion $H \to G$ is in most cases of interest not injective.

A. Connes

Le colloque "Algèbres d'Opérateurs, Orléans 1992" a été organisé avec le concours

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du Conseil Régional du Centre,
du Conseil Général du Loiret
et de la ville d' Orléans.

Que ces organismes et institutions en soient ici remerciées !

RESUMÉS DES EXPOSÉS

S. BAAJ : *Regular representation of the quantum $E_\mu(2)$ group of Woronowicz*

Let H be a Hilbert space. In this article, under appropriate "regularity" conditions, we associate to every multiplicative unitary $V \in \mathcal{L}(H \otimes H)$, a pair of Hopf C*-algebras in duality. We show that the regular representation of the quantum $E_\mu(2)$ group of Woronowicz is a multiplicative unitary satisfying our conditions and we calculate its covariant representations. We also calculate the Haar measures of $E_\mu(2)$ and its Pontrjagyn dual and we give their modular theory.

A. BAUVAL : *Quantum group - and Poisson - deformation of $SU(2)$*

We endow Woronowicz's family of quantum groups $(SU_\mu(2))_{\mu \in \mathbf{R}^*}$ with a structure of continuous field, and use the underlying continuous field of C^*-algebras to construct a deformation of Poisson-$SU(2)$. We prove that this Poisson-deformation is, in some sense, unique. This enables us to compare it with the one constructed by Sheu.

M.E.B. BEKKA, A. VALETTE : *Lattices in semi-simple Lie groups, and multipliers of group C^*-algebras*

Let Γ be a lattice in a non-compact simple Lie group G. We prove that the canonical map from the full C*-algebra $C^*(\Gamma)$ to the multiplier algebra $M(C^*(G))$ is not injective in general (it is never injective if G has Kazhdan's property (T), and not injective for many lattices either in $SO(n,1)$ or $SU(n,1)$). For a locally compact group G, Fell introduced a property $(WF3)$, stating that for any closed subgroup H of G, the canonical map from $C^*(H)$ to $M(C^*(G))$ is injective. We prove that, for an almost connected G, property $(WF3)$ is equivalent to amenability.

E. BLANCHARD : *Tensor products of $C(X)$-algebras over $C(X)$*

Given a Hausdorff compact space X, we study the C*-(semi)-norms on the algebraic tensor product $A \otimes_{alg,C(X)} B$ of two $C(X)$-algebras A and B over $C(X)$. In particular, if one of the two $C(X)$-algebras defines a continuous field of C*-algebras over X, there exist minimal and maximal C*-norms on $A \otimes_{alg,C(X)} B$, but there does not exist any C*-norm on $A \otimes_{alg,C(X)} B$ in general.

F.P. BOCA : *Ergodic actions of compact matrix pseudogroups on C^*-algebras*

A generalization of the classical finiteness theorem of Høegh-Krohn, Landstad and Størmer for ergodic actions of compact groups on operator algebras is proved for actions of compact matrix pseudogroups on C^*-algebras. This, together with the Takesaki-Takai type duality result of Baaj and Skandalis, show that the reduced C^*-crossed product of a unital C^*-algebra by an ergodic action of a compact matrix pseudogroup is a direct sum of C^*-algebras of compact operators.

L.G. BROWN : *Homotopy of projections in C^*-algebras of stable rank one*

S. Zhang has suggested the study of the following question for a particular projection

p in a C^*-algebra A: Is every projection which is unitarily equivalent to p necessarily homotopic to p? It was shown by Effros, Kaminker and Zhang that the answer is yes if A is a unital or non-unital purely infinite simple C^*-algebra, and by Zhang that the answer is yes if A has real rank zero and (topological) stable rank one. We show that the answer is yes whenever A has stable rank one. We also give an example where A is extremally rich and of real rank zero and the answer is no. A second theorem makes an additional hypothesis which rules out such examples. In addition the paper discusses the concept of extremal richness and its K-theoretic consequences.

P. DE LA HARPE : *Operator algebras, free groups and other groups*

The operator algebras associated to non commutative free groups have received a lot of attention, by F.J. Murray and J. von Neumann, and by later workers. We review some properties of these algebras, both for free groups and for other groups such as lattices in Lie groups and Gromov hyperbolic groups. We have also collected a list of open problems.

P. DE LA HARPE, D. VOICULESCU : *A problem on the II_1-factors of Fuchsian groups*

We discuss a problem concerning the von Neumann algebra $W^*_\lambda(\Gamma)$ of a Fuchsian group Γ which is finitely generated and non elementary. The problem is to find how such an algebra is related to the factors in the Dykema-Radulescu family $(L(F_r))_{1<r\leq\infty}$ interpolating continuously the non abelian free group factors.

M.B. LANDSTAD : *Simplicity of crossed products from ergodic actions of compact matrix pseudogroups*

The result that, for an ergodic covariant system $(\mathcal{M}, \rho, G)$ over a compact group G, the crossed product $\mathcal{M} \times_\rho G$ is a simple C*–algebra iff the multiplicity of each $\pi \in \widehat{G}$ in ρ equals $\dim(\pi)$, is generalised to ergodic actions of the compact matrix pseudogroups defined by S. L. Woronowicz. The crossed product turns out to be simple iff the *quantum* dimension equals the *quantum* multiplicity for each irreducible representation of the pseudogroup. As in the group case, the crossed product is then isomorphic to the algebra of compact operators.

G. PISIER : *Exact operator spaces*

We study the notion of exactness in the category of operator spaces, in analogy with Kirchberg's work for C^*-algebras. As for C^*-algebras, exactness can be characterized either by the exactness of certain sequences, or by the property that the finite dimensional subspaces embed almost completely isometrically into a nuclear C^*-algebra. Let E be an n-dimensional operator space. We define $d_{SK}(E) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\}$ where the infimum runs over all isomorphims u between E and an arbitrary n-dimensional subspace of the algebra of all compact operators on ℓ_2. An operator space X is exact iff $d_{SK}(E)$ remains bounded when E runs over all possible finite dimensional subspaces of X. In the general case, it can be shown that $d_{SK}(E) \leq \sqrt{n}$ (here again $n = \dim(E)$), and we give examples showing that this cannot be improved at least asymptotically. We show that $d_{SK}(E) \leq C$ iff for all ultraproducts $\widehat{F} = \Pi F_i/\mathcal{U}$ (of operator spaces) the canonical isomorphism (which has norm ≤ 1) v_E: $\Pi(E \otimes_{\min} F_i)/\mathcal{U} \to E \otimes_{\min} (\Pi F_i/\mathcal{U})$ satisfies $\|v_E^{-1}\| \leq C$. Finally, we show that $d_{SK}(E) = d_{SK}(E^*) = 1$ holds iff E is a point of continuity with respect to two natural topologies on the set of all n-dimensional operator spaces.

S. POPA : *Free-independent sequences in type II_1 factors and related problems*

We prove that, unlike central sequences (i.e., commuting-independent sequences) which in general may or may not exist, free-independent sequences exist in any separable type II_1 factor. More generally, we prove that certain universal commuting squares involving amalgamated free products appear asymptotically in any inclusion of finite index.

F. RĂDULESCU *A type III_λ factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$*

We construct a type III factor by using the free product construction introduced by Voiculescu and show that its core is $L(F_\infty) \otimes B(H)$. We prove that $M_2(C) * L^\infty[0,1]$ is a type III_λ factor if $M_2(C)$ is endowed with a nontracial state (depending on λ).

E. STØRMER : *Entropy in operator algebras*

We give a survey of the theory of dynamical entropy in operator algebras as it was by the end of 1992. Since then Problems 4.2 and 6.6 in the article have been solved, the first positively by D.Voiculescu and the second negatively by Narnhofer, Thirring and the author.

L. VAINERMAN : *Hypergroups structures associated with Gel'fand pairs of compact quantum groups*

Double cosets of compact quantum groups with respect to their subgroups are considered and cases of a Gel'fand pair and a strict Gel'fand pair are distinguished. It is shown that every strict Gel'fand pair of compact quantum groups generates a normal commutative hypercomplex system with a compact basis and a commutative discrete hypergroup which are in duality to each other. The examples of strict Gel'fand pairs of compact quantum groups are considered and characters of the corresponding hypergroups are described in terms of q-orthogonal polynomials.

D. VOICULESCU : *Operations on certain non-commutative operator-valued random variables*

In the context of free products with amalgamation over an algebra B, additive and multiplicative free convolution are studied. Analogues of the R- and S- transforms are obtained. Applications to the B-free central limit theorem and to the spectra of convolution operators on free groups are considered.

SAAD BAAJ

Représentation régulière du groupe quantique des déplacements de Woronowicz

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REPRESENTATION REGULIERE DU GROUPE QUANTIQUE DES DEPLACEMENTS DE WORONOWICZ

Saad Baaj

Introduction

Soit H un espace de Hilbert. Un unitaire V qui agit dans $H \otimes H$ est dit multiplicatif s'il vérifie la relation pentagonale $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Un unitaire multiplicatif V est dit régulier [4] si l'adhérence normique $\overline{\mathcal{C}(V)}$ de la sous-algèbre $\mathcal{C}(V) = \{(id \otimes \omega)(\Sigma V) \ / \ \omega \in \mathcal{L}(H)_*\}$ de $\mathcal{L}(H)$ où Σ est la volte, coincide avec la C*-algèbre des opérateurs compacts $\mathcal{K}$ dans H ; il est dit irréductible [4] s'il existe un unitaire $U \in \mathcal{L}(H)$ vérifiant les conditions :

a) $U^2 = 1$ et $(\Sigma(1 \otimes U)V)^3 = 1$

b) l'unitaire $\widehat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ est multiplicatif.

Dans [4], en collaboration avec G.Skandalis, nous avons associé à tout unitaire multiplicatif régulier V, deux C*-algèbres de Hopf $(S_V, \widehat{S_V})$ en dualité, généralisant ainsi le cas des C*-algèbres de Hopf $(C_0(G), C^*_{red}(G))$ associées à un groupe localement compact G. Comme nous l'avons annoncé dans [4], l'hypothèse de régularité, qui correspond en fait à la dualité de Takesaki-Takai pour les produits croisés de C*-algèbres, n'est pas toujours vérifiée. Citons l'exemple suivant qui sera développé ailleurs [5]. Soit G un groupe localement compact, à tout couple (G_1, G_2) de sous-groupes fermés de G, d'intersection triviale et tel que l'ensemble G_1G_2 soit un ouvert dense dans G, on peut associer, comme [4] dans le cas $G = G_1G_2$, un unitaire multiplicatif V qui correspond au biproduit croisé de [17]. Dans ce cas, l'algèbre $\overline{\mathcal{C}(V)}$ est le produit croisé $C_0(G) \rtimes_{red} (G_1 \times G_2)$ où le sous-groupe G_1 (resp. G_2) agit par translation à droite (resp. à gauche) dans G. Remarquons que la C*-algèbre $C_0(G) \rtimes_{red} (G_1 \times G_2)$ contient la C*-algèbre des opérateurs compacts. Cependant, si $G \neq G_1G_2$, cette C*-algèbre admet plus d'une représentation et donc, dans ce cas, l'inclusion $\mathcal{K} \subset \overline{\mathcal{C}(V)}$ est stricte. Notons cependant que l'unitaire multiplicatif V est irréductible.

Dans cet article, nous dégageons deux conditions plus faibles que les conditions de régularité et d'irréductibilité de [4], qui nous permettent de realiser les constructions de [4] et d'obtenir la plupart de ses résultats. La première condition que nous avons appelée "semi-régularité", revient à demander que l'adhérence normique de $\mathcal{C}(V)$ contienne la C*-algèbre des opérateurs compacts. Une conséquence de cette hypothèse est que $\overline{\mathcal{C}(V)}$ est auto-adjointe et donc par la preuve de (cf. [4] 3.5), l'algèbre réduite S_V et l'algèbre réduite duale $\widehat{S_V}$ sont également auto-adjointes. D'autrepart, nous disons que l'unitaire multiplicatif V est "équilibré" s'il existe un unitaire $U \in \mathcal{L}(H)$ tel que $U^2 = 1$ et que l'unitaire $\widehat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ soit multiplicatif.

Si V est un unitaire multiplicatif équilibré et semi-birégulier, i.e V et $\widehat{V}$ semi-réguliers, nous montrons (paragraphe 3) que les C*-algèbres S_V et $\widehat{S_V}$ peuvent être munies de structures de C*-algèbres de Hopf bisimplifiables naturelles. Nous montrons également que les constructions et les résultats de ([4] appendice) restent valables dans ce cadre. En particulier, si W est l'unitaire multiplicatif associé à une représentation covariante (cf. [4] appendice) d'un unitaire multiplicatif V satisfaisant aux conditions précédentes, l'algèbre réduite S_W (resp. l'algèbre réduite duale $\widehat{S_W}$), munie du coproduit $\delta(x) = W(x \otimes 1)W^*$ (resp. $\widehat{\delta}(x) = W^*(1 \otimes x)W$) est une C*-algèbre de Hopf isomorphe à la C*-algèbre de Hopf S_V (resp. $\widehat{S_V}$).

Notons que dans le cas non régulier, on ne peut espérer obtenir la dualité de Takesaki-Takai pour les produits croisés de C*-algèbres. Cependant, comme l'hypothèse de semi-régularité implique la "régularité au sens faible", i.e l'adhérence faible de la sous-algèbre $\mathcal{C}(V)$ coincide avec $\mathcal{L}(H)$, la méthode de [11] s'adapte dans le cadre des unitaires multiplicatifs irréductibles semi-birégulier pour établir la dualité de Takesaki pour les produits croisés d'algèbres de von Neumann.

Dans le paragraphe 4, nous étudions un exemple important d'unitaire multiplicatif irréductible, semi-birégulier mais non régulier : la représentation régulière du groupe quantique [25,26] des déplacements $E_\mu(2)$ de Woronowicz. Rappelons qu'étant donné un nombre réel $\mu > 1$, la C*-algèbre de Hopf (A, δ) des "fonctions continues sur $E_\mu(2)$ tendant vers 0 à l'infini" est [25] le produit croisé $A = C_0(\mathbf{C}_\mu) \rtimes_\alpha \mathbf{Z}$ où $\mathbf{C}_\mu = \{z \in \mathbf{C} \, / \, |z| \in \mu^{\mathbf{Z}}\} \cup \{0\}$, pour l'action définie par $\alpha(f)(\zeta) = f(\mu^{-1}\zeta)$. Il est facile de deviner [3] une mesure positive ν sur l'espace $\mathbf{C}_\mu$ telle que le poids dual ([15], [22]) correspondant Φ soit une mesure de Haar pour $E_\mu(2)$. La preuve de l'invariance à gauche et à droite de cette mesure de Haar est alors basée sur l'expression de ce poids Φ comme une somme de formes positives sur A et sur le calcul du produit de convolution de ces formes.

Comme nous le montrons dans un cadre assez général au paragraphe 2, à toute C*-algèbre de Hopf munie d'une mesure de Haar, nous associons une isométrie pentagonale qui correspond dans le cas des groupes à la représentation régulière. Dans le cas du groupe quantique $E_\mu(2)$, l'isométrie V obtenue est un unitaire multiplicatif semi-régulier mais non régulier. Comme on peut s'y attendre dans une "situation avec mesure de Haar", la représentation régulière V est irréductible. Nous montrons que la C*-algèbre de Hopf réduite (S_V, δ_V) coincide avec (A, δ) et que la C*-algèbre de Hopf réduite duale $\widehat{S_V}$ munie du coproduit opposé est isomorphe à la C*-algèbre de Hopf [26] des "fonctions continues sur le dual de Pontrjagyn $\widehat{E_\mu(2)}$ tendant vers 0 à l'infini" au sens de Woronowicz.

Par les résultats du paragraphe 3 et la moyennabilité de $E_\mu(2)$ et de $\widehat{E_\mu(2)}$, nous savons que les représentations (resp. coreprésentations) de V sont les représentations de la C*-algèbre $\widehat{S_V}$ (resp. S_V). S'appuyant sur la description (théorème 4.10) de l'unitaire multiplicatif V comme multiplicateur de la C*-algèbre $\widehat{S_V} \otimes S_V$, on peut déduire la description [26] des représentations du groupe quantique $E_\mu(2)$ donné par Woronowicz.

Une autre conséquence de la moyennabilité de $E_\mu(2)$ est que le produit croisé réduit $S_V \rtimes \widehat{S_V}$ coincide avec le produit croisé "max". Il s'ensuit que les représentations de la C*-algèbre $B = S_V \rtimes \widehat{S_V}$ coincident avec les représentatons covariantes [4] de l'unitaire multiplicatif V. Nous montrons que la C*-algèbre B est une extension des opérateurs compacts par les compacts. Il en résulte que V n'admet que deux représentations covariantes : la représentation régulière et une deuxième que nous décrivons.

Nous terminons le paragraphe 4 par le calcul des mesures de Haar duales et de leur théorie modulaire. Pour déduire les mesures de Haar duales à partir de la mesure de Haar Φ, on peut procéder comme dans le cas classique des algèbres de Kac ([12], [13], [16], [24]), i.e construire des algèbres hilbertiennes à gauche et montrer que les poids correspondants ([8], [22]) vérifient les propriétés d'invariance voulues. Pour garder à cet article une longueur raisonnable, nous avons préféré procéder directement en donnant les formes positives sur $\widehat{S_V}$ qui permettent d'exprimer les mesures de Haar duales comme somme de formes positives ; le calcul de leur produit de convolution permet alors comme dans le cas de Φ, de montrer les propriétés d'invariance . Nous montrons ensuite que les théories modulaires de Φ et des mesures de Haar duales $\widehat{\Phi}$ et $\widehat{\Psi}$ vérifient la conjecture de ([21] paragraphe 6.).

S'appuyant sur une conséquence du formulaire de [21], nous montrons que le poids $\Phi \otimes \widehat{\Phi}$ est une mesure de Haar invariante à gauche et à droite sur le double quantique ([4], [28]) de $E_\mu(2)$. Procédant comme dans le cas classique, on peut déduire dans ce cas les mesures de Haar duales et montrer que leur théorie modulaire satisfait le formulaire de [21].

Enfin, dans une première appendice, nous rassemblons les propriétés que nous avons utilisées dans le paragraphe 4, des coefficients de Fourier $(A(m,n))_{(m,n)\in \mathbf{Z}^2}$ de la suite de fonctions notée $(U(m,.))_{m\in\mathbf{Z}}$ introduite par Woronowicz dans [25]. Dans une seconde appendice, nous complétons la preuve du théorème 4.2 et nous montrons l'unicité de la mesure de Haar de $E_\mu(2)$.

Durant l'élaboration de cet article, j'ai bénéficié de nombreuses et fructueuses discussions avec G.Skandalis sur ce sujet ; je l'en remercie très sincèrement.

1. Préliminaires

Dans ce paragraphe, nous fixons les notations constamment utilisées dans la suite et nous rappelons quelques définitions.

Soit E un espace de Banach et $X \subset E$ un sous-ensemble de E. Nous notons $\overline{X}$ l'adhérence de X dans E et nous désignons par $\overline{lin}\ X$ l'espace vectoriel fermé engendré par X dans E.

Tous les produits tensoriels de C*-algèbres, sauf mention expresse du contraire, sont supposés munis de la norme spatiale (produits tensoriels "min").

Soit A une C*-algèbre, nous notons $\widetilde{A}$ la C*-algèbre obtenue à partir de A par adjonction d'un élément unité et $M(A)$ la C*-algèbre des multiplicateurs [18] de A. Si J est un idéal bilatère fermé de A, on pose $M(A,J) = \{m \in M(A) \ / \ mA + Am \subset J\}$.

Un homomorphisme de C*-algèbres $\pi : A \to M(B)$ est dit non dégénéré si, pour une unité approchée (e_i) de A, $\pi(e_i) \to 1$ pour la topologie stricte.

1.1. DÉFINITION. — *(cf. [4]) Une C*-algèbre de Hopf est un couple (A,δ) où A est une C*-algèbre et $\delta : A \to M(\tilde{A} \otimes A + A \otimes \tilde{A}; A \otimes A)$ est un homomorphisme non dégénéré, appelé le coproduit de A, vérifiant $(id \otimes \delta)\delta = (\delta \otimes id)\delta$. Une C*-algèbre de Hopf est dite bisimplifiable si on a $A \otimes A = \overline{lin}\ \delta(A)(1 \otimes A) = \overline{lin}\ \delta(A)(A \otimes 1)$.*

Soit H un espace de Hibert, si $T \in \mathcal{L}(H \otimes H)$, on définit $T_{12}, T_{13}, T_{23} \in \mathcal{L}(H \otimes H \otimes H)$ comme dans [4]. On note $\Sigma \in \mathcal{L}(H \otimes H)$ la volte donnée par $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ et on pose $T_{21} = \Sigma T \Sigma$.

Pour $T \in \mathcal{L}(H \otimes H)$ et $\omega \in \mathcal{L}(H)_*$, on définit les opérateurs $(id \otimes \omega)(T)$ et $(\omega \otimes id)(T)$ par les formules :

$$(\xi \mid (id \otimes \omega)(T)\eta) = \omega(\theta_\xi^* T \theta_\eta) \quad , \quad (\xi \mid (\omega \otimes id)(T)\eta) = \omega(\theta'^*_\xi T \theta'_\eta)$$

où $\theta_\xi, \theta'_\eta \in \mathcal{L}(H, H \otimes H)$ sont définies par $\theta_\xi(\eta) = \theta'_\eta(\xi) = \xi \otimes \eta$.

1.2. DÉFINITION. — *(cf. [4]) Un unitaire $V \in \mathcal{L}(H \otimes H)$ est dit multiplicatif s'il vérifie la relation pentagonale :*

$$V_{12} V_{13} V_{23} = V_{23} V_{12}$$

Si $V \in \mathcal{L}(H \otimes H)$ est un unitaire multiplicatif et $\omega \in \mathcal{L}(H)_*$, on pose $L(\omega) = (\omega \otimes id)(V)$ et $\rho(\omega) = (id \otimes \omega)(V)$. L'algèbre réduite [4] (resp. l'algèbre réduite duale) S_V (resp. $\widehat{S_V}$) de V est par définition l'adhérence normique dans $\mathcal{L}(H)$ de la sous-algèbre $A(V) = \{L(\omega) \ / \ \omega \in \mathcal{L}(H)_*\}$ (resp. $\widehat{A}(V) = \{\rho(\omega) \ / \ \omega \in \mathcal{L}(H)_*\}$). Quand aucune confusion n'est possible, on note simplement S et $\widehat{S}$ ces algèbres.

Une représentation [4] (resp. coreprésentation) de V dans l'espace de Hilbert K est un unitaire $X \in \mathcal{L}(K \otimes H)$ (resp. $X \in \mathcal{L}(H \otimes K)$) vérifiant la relation $X_{12} X_{13} V_{23} = V_{23} X_{12}$ (resp. $V_{12} X_{13} X_{23} = X_{23} V_{12}$). Dans ce cas, pour tout $\omega \in \mathcal{L}(H)_*$, on pose $\rho_X(\omega) = (id \otimes \omega)(X)$ (resp. $L_X(\omega) = (\omega \otimes id)(X)$) ; l'espace vectoriel $\widehat{A_X} = \{\rho_X(\omega) \ / \ \omega \in \mathcal{L}(H)_*\}$ (resp. $A_X = \{L_X(\omega) \ / \ \omega \in \mathcal{L}(H)_*\}$) est une sous-algèbre ([4] A.3) de $\mathcal{L}(K)$; on note alors $\widehat{S_X}$ (resp. S_X) son adhérence normique dans $\mathcal{L}(K)$.

Une représentation covariante ([4] appendice) de V dans un espace de Hilbert K est un couple (X, Y) où X est une représentation et Y est une coreprésentation de V dans le même espace de Hilbert K vérifiant la relation de covariance $Y_{12} V_{13} X_{23} = X_{23} Y_{12}$.

2. Mesure de Haar sur une C*-algèbre de Hopf

Dans ce paragraphe, nous associons à toute C*-algèbre de Hopf munie d'une mesure de Haar, une isométrie pentagonale qui correspond dans le cas des groupes à la représentation régulière. Notons que notre définition d'une mesure de Haar est moins restrictive que celle de [14]. Auparavant, nous rappelons quelques notations et résultats de la théorie [7] des poids sur une C*-algèbre.

Un poids [7] Φ sur une C*-algèbre A est une application de A_+ dans $[0,\infty]$ telle que $\Phi(x+y) = \Phi(x) + \Phi(y)$ pour $x, y \in A_+$ et $\Phi(\lambda x) = \lambda\Phi(x)$ pour $\lambda \geq 0$ et $x \in A_+$. Un poids Φ est dit semi-fini (normiquement) si le sous -espace vectoriel :

$$\mathfrak{M}_\Phi = \{x_1 - x_2 + i(x_3 - x_4)\ ,\ x_j \in A_+\ ,\ \Phi(x_j) < \infty\} = \mathfrak{N}_\Phi^* \mathfrak{N}_\Phi$$

où $\mathfrak{N}_\Phi = \{x \in A / \Phi(x^* x) < \infty\}$, est dense dans A.

Si Φ est un poids sur une C*-algèbre A, sa représentation [9] GNS $(\Lambda_\Phi, H_\Phi, \pi_\Phi)$ est définie de la façon suivante. L'idéal à gauche $\mathfrak{N}_\Phi$ de A, muni du produit scalaire $(x \mid y) = \Phi(x^* y)$, est un espace préhilbertien. Soient H_Φ le séparé complété de cet espace préhilbertien et Λ_Φ l'application canonique de $\mathfrak{N}_\Phi$ dans H_Φ ; posant $\pi_\Phi(a)(\Lambda_\Phi(x)) = \Lambda_\Phi\,(ax)$ pour $a \in A$ et $x \in \mathfrak{N}_\Phi$, on obtient une représentation π_Φ de la C*-algèbre A dans l'espace de Hilbert H_Φ.

Si Φ est un poids normal semi-fini fidèle sur une algèbre de von Neumann M, on note [22] σ_t^Φ le groupe d'automorphismes modulaires de M associé.

Soit Φ est un poids s.c.i et semi-fini sur une C*-algèbre A, alors [9] la représentation π_Φ est non dégénérée et [7,9] Φ admet un prolongement canonique noté $\bar{\Phi}$, à l'algèbre de von Neumann $M = \pi_\Phi(A)''$, donné par :

$$\bar{\Phi}(x) = \sup\{f(x)\ ,\ f \in M_*^+\ ,\ f \circ \pi_\Phi \leq \Phi\}$$

$\bar{\Phi}$ est donc un poids normal semi-fini (pour la topologie ultrafaible) sur M vérifiant $\bar{\Phi} \circ \pi_\Phi = \Phi$. D'aprés [1], on a également pour tout $x \in M_+$:

$$\bar{\Phi}(x) = \inf\{l \ / \text{ il existe } a_i \in A \ \text{ t.q } \pi_\Phi(a_i) \to x \text{ (ultrafort) et } \Phi(a_i) \to l\}$$

Soient Φ, Ψ des poids s.c.i et semi-finis sur une C*-algèbre A, alors le produit tensoriel $\Phi \otimes \Psi$ des poids Φ et Ψ est le poids s.c.i et semi-fini sùr $A \otimes A$ défini par $\Phi \otimes \Psi = (\bar{\Phi} \otimes \bar{\Psi}) \circ (\pi_\Phi \otimes \pi_\Psi)$, où $\bar{\Phi} \otimes \bar{\Psi}$ désigne le produit tensoriel des poids normaux [22] sur le produit tensoriel d'algèbres de von Neumann.

2.1. LEMME. — *(cf. [9], [14]) Soient M une algèbre de von Neumann, B une sous-C*-algèbre de M faiblement dense dans M et Φ un poids normal, semi-fini ultrafaiblement et fidèle sur M. Posons $\phi = \Phi | B^+$ et supposons que ϕ soit normiquement semi-fini. Notons H_Φ (resp. H_ϕ) l'espace de la représentation GNS du poids Φ (resp. ϕ). Si le groupe à un paramètre d'automorphismes modulaires (σ_t^Φ) laisse B invariante et* *définit par restriction, un groupe à un paramètre d'automorphismes normiquement continu de la C^*-algèbre B, alors l'isométrie $U : H_\phi \to H_\Phi$ définie par $U\,\Lambda_\phi x = \Lambda_\Phi x$ est un unitaire qui entrelace les représentations π_ϕ et $\pi_\Phi|B$ de B.*

Démonstration. Soit (u_j) une suite généralisée dans $\mathfrak{M}_\phi^+$ normiquement bornée telle que $u_j \to 1$ fortement. Posons $v_j = \dfrac{1}{\sqrt{\pi}} \displaystyle\int e^{-t^2}\, \sigma_t^\Phi(u_j)\, dt$. On a $v_j \in \mathfrak{M}_\phi^+$ et [22] pour tout $z \in \mathbf{C}$, on a $\sigma_z^\Phi(v_j) \to 1$ pour la topologie *-forte. Pour tout $y \in \mathfrak{N}_\Phi$, on a :

$$\Lambda_\Phi\, y v_j = J_\Phi \pi_\Phi(\sigma^\Phi_{-\frac{i}{2}}(v_j)) J_\Phi\, \Lambda_\Phi\, y \to \Lambda_\Phi\, y$$

Soit alors (b_k) une suite généralisée dans B qui converge fortement vers y, comme $b_k v_j \in \mathfrak{N}_\phi$ pour tout k et tout j, le vecteur $\Lambda_\Phi\, y$ est dans l'image de l'isométrie U. ∎

Soit A une C*-algèbre et soit Ψ un poids s.c.i et semi-fini sur A, supposons que le prolongement canonique $\bar{\Psi}$ de Ψ à l'algèbre de von Neumann $M = \pi_\Psi(A)''$ soit fidèle et que le groupe d'automorphismes modulaires $(\sigma_t^{\bar{\Psi}})$ laisse la C*-algèbre $B = \pi_\Psi(A)$ invariante et définit par restriction, un groupe d'automorphismes de B normiquement continu, alors d'aprés le lemme précédent, l'isométrie $U : H_\Psi \to H_{\bar{\Psi}}$ définie par $U\Lambda_\Psi\, y = \Lambda_{\bar{\Psi}}\, \pi_\Psi(y)$ est un unitaire qui entrelace les représentations π_Ψ et $\pi_{\bar{\Psi}} \circ \pi_\Psi$. De même, il résulte de (2.1) que l'isométrie naturelle $U : H_\Psi \otimes H_\Psi \to H_{\Psi\otimes\Psi}$ définie par $U(\Lambda_\Psi\, x \otimes \Lambda_\Psi\, y) = \Lambda_{\Psi\otimes\Psi}\,(x \otimes y)$ est un unitaire. On identifie également l'espace de Hilbert $H_{\Psi\otimes\Psi\otimes\Psi}$ au produit tensoriel $H_\Psi \otimes H_\Psi \otimes H_\Psi$...

2.2. DÉFINITION. — *On appelle mesure de Haar à gauche (resp. à droite) sur une C^*-algèbre de Hopf (A, δ), un poids $\Phi : A_+ \to [0, \infty]$ s.c.i et semi-fini vérifiant :*
a) $\bar{\Phi}$ est fidèle sur l'algèbre de von Neumann $M = \pi_\Phi(A)''$.
b) $\sigma_t^{\bar{\Phi}}(\pi_\Phi(A)) = \pi_\Phi(A)$ et pour tout $a \in A$ la fonction $t \to \sigma_t^{\bar{\Phi}}(\pi_\Phi(a))$ est continue normiquement.
c) Pour tout $a \in A_+$ et toute forme positive f sur A, on a $\Phi(a * f) = \|f\|\Phi(a)$ (resp. $\Phi(f * a) = \|f\|\Phi(a)$).

Rappelons que si (A, δ) est une C*-algèbre de Hopf, $a \in A$, $f, f' \in A^*$, on pose :

$$a * f = (f \otimes id)(\delta(a))\ ,\ f * a = (id \otimes f)(\delta(a))\ ,\ f * f' = (f \otimes f') \circ \delta$$

2.3. PROPOSITION. — *Soit (A, δ) une C^*-algèbre de Hopf munie d'une mesure de Haar à droite Ψ.*
a) Pour tout $x, y \in \mathfrak{N}_\Psi$, on a $(\Psi \otimes \Psi)((1 \otimes y^*)\delta(x^*x)(1 \otimes y)) = \Psi(x^*x)\Psi(y^*y)$.
b) Pour tout $a \in \mathfrak{N}_{\Psi\otimes\Psi}$ et tout $y \in \mathfrak{N}_\Psi$, on a $(\Psi \otimes \Psi \otimes \Psi)((1 \otimes 1 \otimes y^*)(id \otimes \delta)(a^*a)(1 \otimes 1 \otimes y)) = (\Psi \otimes \Psi)(a^*a)\Psi(y^*y)$.

Démonstration. Posons $f = y\,\Psi\,y^*$, f est une forme positive sur A et on a $\|f\| = \Psi(y^*y)$. Par [19], il existe une famille de formes normales positives $(\omega_i)_{i\in I}$ sur M telle que $\bar{\Psi} = \sum_i \omega_i$. On en déduit ([22] 8.3) que $(\bar{\Psi} \otimes \bar{\Psi}) = \sum_i (\omega_i \otimes \bar{\Psi})$, d'où :

$$\begin{aligned}(\Psi\otimes\Psi)((1\otimes y^*)\delta(x^*x)(1\otimes x)) &= \sum_i(\omega_i\otimes\bar{\Psi})\circ(\pi_\Psi\otimes\pi_\Psi)((1\otimes y^*)\delta(x^*x)(1\otimes x))\\ &= \sum_i\Psi(y^*(x^*x*\omega_i\circ\pi_\Psi)y)\\ &= \sum_i f(x^*x*\omega_i\circ\pi_\Psi))\\ &= \sum_i(\omega_i\circ\pi_\Psi)(f*x^*x)\\ &= \Psi(f*x^*x)=\Psi(y^*y)\Psi(x^*x).<\infty\end{aligned}$$

d'où le a).

Démonstration analogue pour le b). ∎

Avec les notations ci-dessus, notons V l'isométrie dans l'espace de Hilbert $H_\Psi\otimes H_\Psi = H_{\Psi\otimes\Psi}$ définie par :

$$V(\Lambda_\Psi\, x\otimes\Lambda_\Psi\, y)=\Lambda_{\Psi\otimes\Psi}(\delta(x)(1\otimes y))$$

où $x,y\in\mathfrak{N}_\Psi$. Nous avons :

2.4. THÉORÈME. — *L'isométrie V vérifie la relation pentagonale $V_{12}V_{13}V_{23}=V_{23}V_{12}$.*

Pour la preuve de ce résultat, on a besoin de :

2.5. LEMME. — *Pour tout $a\in\mathfrak{N}_{\Psi\otimes\Psi}$ et tout $z\in\mathfrak{N}_\Psi$, on a $V_{23}\Lambda_{\Psi\otimes\Psi\otimes\Psi}(a\otimes z)=\Lambda_{\Psi\otimes\Psi\otimes\Psi}((id\otimes\delta)(a)(1\otimes 1\otimes z))$.*

Démonstration. Il est clair que si a appartient au produit tensoriel algébrique $\mathfrak{N}_\Psi\odot\mathfrak{N}_\Psi$, on a par définition de V :

$$V_{23}\,\Lambda_{\Psi\otimes\Psi\otimes\Psi}(a\otimes z)=\Lambda_{\Psi\otimes\Psi\otimes\Psi}((id\otimes\delta)(a)(1\otimes 1\otimes z))$$

Dans le cas général, soit (a_n) une suite dans le produit tensoriel algébrique $\mathfrak{N}_\Psi\odot\mathfrak{N}_\Psi$ telle que (2.1) $\Lambda_{\Psi\otimes\Psi}\,a_n\to\Lambda_{\Psi\otimes\Psi}\,a$. Par (2.3), on a :

$$\Lambda_{\Psi\otimes\Psi\otimes\Psi}((id\otimes\delta)(a_n)(1\otimes 1\otimes z))\to\Lambda_{\Psi\otimes\Psi\otimes\Psi}((id\otimes\delta)(a)(1\otimes 1\otimes z))$$

∎

démonstration du théorème. Pour $x,y,z\in\mathfrak{N}_\Psi$, on a par le lemme précédent :

$$\begin{aligned}V_{23}V_{12}\,(\Lambda_\Psi\,x\otimes\Lambda_\Psi\,y\otimes\Lambda_\Psi\,z) &= V_{23}\Lambda_{\Psi\otimes\Psi\otimes\Psi}\,((\delta(x)(1\otimes y))\otimes z)\\ &= \Lambda_{\Psi\otimes\Psi\otimes\Psi}\,(\delta^2(x)(1\otimes(\delta(y)(1\otimes z)))\end{aligned}$$

D'autrepart, soit (b_n) une suite dans le produit tensoriel algébrique $\mathfrak{N}_\Psi \odot \mathfrak{N}_\Psi$ telle que $\Lambda_{\Psi\otimes\Psi}\, b_n \to \Lambda_{\Psi\otimes\Psi}\,(\delta(y)(1\otimes z))$, on a :

$$\begin{aligned} V_{12}V_{13}V_{23}\,(\Lambda_\Psi\, x\otimes\Lambda_\Psi\, y\otimes\Lambda_\Psi\, z) &= V_{12}V_{13}\,\Lambda_\Psi\, x\otimes\Lambda_{\Psi\otimes\Psi}\,(\delta(y)(1\otimes z)) \\ &= \lim_{n\to\infty} V_{12}V_{13}\,\Lambda_{\Psi\otimes\Psi\otimes\Psi}\,(x\otimes b_n) \\ &= \lim_{n\to\infty} V_{12}\,\Lambda_{\Psi\otimes\Psi\otimes\Psi}\delta(x)_{13}(1\otimes b_n) \\ &= \lim_{n\to\infty} \Lambda_{\Psi\otimes\Psi\otimes\Psi}\delta^2(x)(1\otimes b_n)\ \ (\text{par}\quad(2.5)) \end{aligned}$$

Pour tout $u\in\mathfrak{M}_\Psi^+$, on a avec les mêmes notations :

$$\begin{aligned} \lim_{n\to\infty} \Lambda_{\Psi\otimes\Psi\otimes\Psi}\delta^2(x)(u\otimes b_n) &= \lim_{n\to\infty} (\pi_\Psi\otimes\pi_\Psi\otimes\pi_\Psi)(\delta^2(x))\,\Lambda_{\Psi\otimes\Psi\otimes\Psi}(u\otimes b_n) \\ &= (\pi_\Psi\otimes\pi_\Psi\otimes\pi_\Psi)(\delta^2(x))\,\Lambda_{\Psi\otimes\Psi\otimes\Psi}(u\otimes(\delta(y)(1\otimes z)) \end{aligned}$$

Par densité normique dans A^+ des éléments u de $\mathfrak{M}_\Psi^+$ tels que $\pi_\Psi(u)$ soient analytiques pour le groupe d'automorphismes modulaires (σ_t^Ψ), on déduit :

$$\lim_{n\to\infty} \Lambda_{\Psi\otimes\Psi\otimes\Psi}\delta^2(x)(1\otimes b_n) = \Lambda_{\Psi\otimes\Psi\otimes\Psi}\,(\delta^2(x)(1\otimes(\delta(y)(1\otimes z)))$$

■

2.6. Remarque. — Soit (A,δ) une C*-algèbre de Hopf.

1) Si Ψ est une mesure de Haar à droite sur (A,δ), la condition a) de (2.2) entraine que l'espace des vecteurs bornés à droite relativement à la représentation GNS du [9] système hilbertien à gauche $(A,\mathfrak{N}_\Psi,s_\Psi)$ où $s_\Psi(x,y)=\Psi(x^*y)$, est [9] dense dans H_Ψ ; procédant comme par exemple dans [12], on montre facilement que $V\in\mathcal{L}(H_\Psi)\otimes M$ où M est l'algèbre de von Neumann engendrée par $\pi_\Psi(A)$.

2) Si Φ est une mesure de Haar à gauche sur (A,δ), alors on montre de même que la formule $V^*(\Lambda_\Phi\, x\otimes\Lambda_\Phi\, y)=\Lambda_{\Phi\otimes\Phi}\,(\delta(y)(x\otimes 1))$ définit une co-isométrie V de l'espace de Hilbert $H_\Phi\otimes H_\Phi$ qui vérifie la relation pentagonale ; on a evidemment que $V\in M\otimes\mathcal{L}(H)$ où M est l'algèbre de von Neumann engendrée par $\pi_\Phi(A)$.

3) Si Ψ est une mesure de Haar à droite sur (A,δ), alors pour tout $x\in\mathfrak{N}_\Psi$ et toute forme $f\in A^*$, on a $(f*x)\in\mathfrak{N}_\Psi$ et $\Psi((f*x)^*(f*x))\le\|f\|^2\,\Psi(x^*x)$; voir par exemple ([23],[13]).

3. Unitaires multiplicatifs semi-réguliers

Le but de cette partie est de montrer qu'avec des hypothèses de régularité et d'irréductibilité plus faibles que celles de [4], les algèbres de Banach S_V et $\widehat{S_V}$ canoniquement associées à un unitaire multiplicatif V, restent munies d'une structure de C*-algèbre de Hopf bisimplifiable (cf.[4] 3.). Sous les mêmes hypothèses, nous étudions également les représentations covariantes de ces unitaires multiplicatifs.

Commençons par la définition suivante :

3.1. DÉFINITION. — *On dit que l'unitaire multiplicatif V dans H est semi-régulier si l'adhérence normique de l'algèbre $\mathcal{C}(V)$ contient la C^*-algèbre des opérateurs compacts $\mathcal{K}$. On dit que V est semi-birégulier si V est semi-régulier et que l'adhérence normique de l'espace vectoriel $\{(\omega \otimes id)(\Sigma V) \ / \ \omega \in \mathcal{L}(H)_*\}$ contient $\mathcal{K}$.*

Si V est semi-régulier, il est clair que $\Sigma V^* \Sigma$ est semi-régulier. Si deux unitaires multiplicatifs sont équivalents et que l'un est semi-régulier, l'autre l'est.

Nous verrons au paragraphe 4 que la représentation régulière du groupe quantique $E_\mu(2)$ de Woronowicz est unitaire multiplicatif semi-régulier mais non régulier. Cependant, dans le cas unifère (i.e $1 \in A(V)$) , nous allons montrer que la semi-régularité d'un unitaire multiplicatif V entraine sa régularité .

On a :

3.2. PROPOSITION. — *Soit V un unitaire multiplicatif dans H semi- régulier, S et $\widehat{S}$ les algèbres de Banach associées.*

a) L'adhérence normique de la sous-algèbre $\mathcal{C}(V)$ de $\mathcal{L}(H)$ est auto-adjointe.

b) S et $\widehat{S}$ sont des sous-C^*-algèbres de $\mathcal{L}(H)$.

Notons d'abord que si l'adhérence normique de $\mathcal{C}(V)$ est stable par l'involution, S et $\widehat{S}$ le sont également par la même preuve que ([4] 3.5) ; d'où le b) . Pour démontrer le a), on a besoin de :

3.3. LEMME. — *Soient H un espace de Hilbert, B une sous-algèbre normiquement fermée de $\mathcal{L}(H)$ et B_0 une sous-algèbre de B normiquement dense. Supposons que les ensembles $B_0^* B_0$ et $B_0 B_0^*$ soient contenus dans B. Alors B est une sous-C^*-algèbre de $\mathcal{L}(H)$.*

Démonstration. Il suffit de montrer que $B_0^* \subset B$. Or, pour tout $x \in B_0$, il existe une suite de polynômes (P_n) telle que $P_n(0) = 0$ pour tout n et que x soit limite normique de $(xP_n(x^*x))$. Comme $\overline{P_n}(x^*x)x^* \in B$, on a $x^* \in B$.

démonstration de la proposition

Posons $B_0 = \mathcal{C}(V)$ et notons B son adhérence normique dans $\mathcal{L}(H)$. Nous allons montrer les ensembles $B_0^* B_0$ et $B_0 B_0^*$ sont contenus dans B. Pour $\omega, \omega' \in \mathcal{L}(H)_*$, on a :

$$\begin{aligned}(id \otimes \omega)(\Sigma V)^*(id \otimes \omega')(\Sigma V) &= (id \otimes \omega' \otimes \omega^*)(V_{13}^* \Sigma_{13} \Sigma_{12} V_{12}) \\ &= (id \otimes \omega' \otimes \omega^*)(\Sigma_{12} V_{23}^* V_{13} \Sigma_{23})\end{aligned}$$

Il résulte du calcul précédent que si $x, y \in B_0$, alors $x^* y$ appartient à l'adhérence normique de l'espace vectoriel engendré par $\{(id \otimes \alpha \otimes \beta)(\Sigma_{12} V_{23}^* (a \otimes b \otimes 1) V_{13}) \ / \ \alpha, \beta \in \mathcal{L}(H)_* \, ; \, a, b \in \mathcal{K}\}$. Or il résulte de la semi- régularité de V et de ([4] 3.1) que l'adhérence normique de l'espace vectoriel engendré par $\underline{\{(a \otimes 1)V(1 \otimes b) \ / \ a, b \in \mathcal{K}\}}$ contient la C*-algèbre $\mathcal{K} \otimes \mathcal{K}$. On en déduit que $x^* y \in \overline{lin}\, \{(id \otimes \alpha \otimes \beta)(\Sigma_{12} V_{23}^* V_{12} V_{13}) \ / \ \alpha, \beta \in$

$\mathcal{L}(H)_*\}$. Comme $V^*_{23}V_{12}V_{13} = V_{12}V^*_{23}$, on a $x^*y \in B$. Remplaçant V par $\Sigma V^*\Sigma$, on obtient également que $xy^* \in B$.

■

3.4. Remarque. — Si X est une coreprésentation (resp. représentation) de l'unitaire multiplicatif semi-régulier V dans un espace de Hilbert K, alors la sous-algèbre ([4] A.3) S_X (resp. $\widehat{S}_X$) est une sous-C*-algèbre de $\mathcal{L}(K)$; la preuve est la même que dans ([4] A.3).

Nous ne savons pas si la semi-régularité d'un unitaire multiplicatif V suffit pour munir la C*-algèbre S (resp. $\widehat{S}$) , via le morphisme $\delta(x) = V(x \otimes 1)V^*)$ (resp. $\delta(x) = V^*(1 \otimes x)V)$, d'une structure de C*-algèbre de Hopf. Cependant, si S ou $\widehat{S}$ est unifère, c'est le cas car alors l'unitaire multiplicatif est régulier. Pour la preuve, montrons le résultat suivant :

3.5. LEMME. — *Soient V un unitaire multiplicatif, X une coreprésentation (resp. représentation) de V dans un espace de Hilbert K. On a :*

$$\overline{lin}\,(\mathcal{C}(V)^* \odot 1)X(\mathcal{K} \odot 1) = X(\overline{\mathcal{K} \odot A_X})$$

$$(resp.\,\overline{lin}\,(1 \odot \mathcal{K})X(1 \odot \mathcal{C}(V)^*) = (\overline{\widehat{A}_X \odot \mathcal{K})X})$$

Démonstration. Pour $\omega \in \mathcal{L}(H)_*, k \in \mathcal{K}$, on a :

$$\begin{aligned}(id \otimes \omega \otimes id)(V^*_{12}\Sigma_{12}X_{13}(k \otimes 1 \otimes 1)) =& (id \otimes \omega \otimes id)(V^*_{12}X_{23}\Sigma_{12}(k \otimes 1 \otimes 1)) \\ =& (id \otimes \omega \otimes id)(X_{13}X_{23}V^*_{12}\Sigma_{12}(k \otimes 1 \otimes 1))\end{aligned}$$

On conclut grâce à l'égalité $V^*\Sigma(\overline{\mathcal{K} \odot \mathcal{K}}) = \mathcal{K} \otimes \mathcal{K}$.

L'assertion resp. résulte de celle-ci en remplaçant V par $\Sigma V^*\Sigma$ et X par $\Sigma X^*\Sigma$.

■

3.6. PROPOSITION. — *Soit V un unitaire multiplicatif dans H de type compact (resp. discret). Si V est semi-régulier, alors V est régulier.*

Démonstration. Notons d'abord qu' on a par ([4] 3.1), $\overline{lin}\,(\mathcal{K}\odot 1)V(\mathcal{K}\odot 1) = \overline{\mathcal{K} \odot A(V)}$. Si V est semi-régulier, (3.5) entraine que $\mathcal{K} \otimes S \subset V(\mathcal{K} \otimes S)$. Si V est de plus de type compact, alors pour tout $k \in \mathcal{K}$ on a $V^*(k \otimes 1) = \lim_{n \to \infty} \sum_j^{p_n} (k_j \otimes s_j)$ avec $k_j \in \mathcal{K}$ et $s_j \in S$.

Pour tout $\xi, \eta \in H$, posons alors $t = (id \otimes \omega_{k\eta,\xi})(\Sigma V)$ et $t_n = \sum_j^{p_n} \theta_{s_j^*\xi, k_j\eta}$ pour tout entier n. Pour tout $\alpha, \beta \in H$, nous avons :

$$\begin{aligned}(\alpha \mid (t - t_n)\beta) &=(\alpha \otimes k\eta \mid (\Sigma V)(\beta \otimes \xi)) - \sum_j^{p_n} (\alpha \mid s_j^* \xi)(k_j \eta \mid \beta) \\ &=((V^*(k \otimes 1) - \sum_j^{p_n} (k_j \otimes s_j))(\eta \otimes \alpha) \mid \beta \otimes \xi)\end{aligned}$$

donc $\|t - t_n\| \to 0$ quand $n \to \infty$ et V est régulier.

L'assertion resp. résulte de celle-ci en remplaçant V par $\Sigma V^* \Sigma$. ∎

Par les résultats de [4], V est en fait (à la multiplicité prés), irréductible.

3.7. DÉFINITION. — *On dit qu'un unitaire multiplicatif V dans H est équilibré s'il existe un unitaire $U \in \mathcal{L}(H)$ tel que :*

a) $U^2 = 1$.

b) L' unitaire $\widehat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ est multiplicatif.

Par abus de langage (et de notation), nous dirons que le couple (V, U) est un unitaire multiplicatif équilibré s'il vérifie les conditions de la définition précédente.

3.8. Remarque. — On déduit immédiatement de la définition de $\widehat{V}$ que si (V, U) est un unitaire multiplicatif équilibré, V est semi-birégulier ssi V et $\widehat{V}$ sont semi-réguliers. Notons aussi que l'unitaire $\widetilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma = (U \otimes U)\widehat{V}(U \otimes U)$ est également multiplicatif.

Dans la proposition suivante, nous donnons des propriétés algébriques des co-représentations et représentations d'un unitaire multiplicatif équilibré (V, U). Pour cela, si X est une coreprésentation (resp. représentation) de V , nous posons $\widehat{X} = (1 \otimes U)X_{21}(1 \otimes U)$ (resp. $\widetilde{X} = (U \otimes 1)X_{21}(U \otimes 1)$). Nous avons :

3.9. PROPOSITION. — *Soient (V, U) un unitaire multiplicatif équilibré dans H et X une co-représentation (resp. représentation) de V dans un espace de Hilbert K.*

a) $\widetilde{V}_{12} X_{13} \widetilde{V}_{12}^* = X_{13} X_{23}$ *(resp.* $\widehat{V}_{23}^* X_{13} \widehat{V}_{23} = X_{12} X_{13}$*)*

b) $\widehat{X}_{23}^* V_{13} \widehat{X}_{23} = X_{12} V_{13}$ *(resp.* $\widetilde{X}_{12} V_{13} \widetilde{X}_{12}^* = V_{13} X_{23}$*)*

c) $\overline{lin}\,(\mathcal{K} \odot 1)X(\mathcal{C}(\widetilde{V})^* \odot 1) = \overline{lin}\,(\mathcal{K} \odot A_X)X$

(resp. $\overline{lin}\,(1 \odot \mathcal{C}(\widehat{V})^*)X(1 \odot \mathcal{K}) = \overline{lin}\,X(\widehat{A}_X \odot \mathcal{K})$)

Démonstration. a) Cf. [4] A.7 b).

b) En conjuguant l'égalité $\widetilde{V}_{12} X_{13} \widetilde{V}_{12}^* = X_{13} X_{23}$ par l'unitaire $(1 \otimes U \otimes 1)\Sigma_{12}$, on obtient $V_{12} \widehat{X}_{32} V_{12}^* = \widehat{X}_{32} X_{13}$, d'où $V_{13} \widehat{X}_{23} V_{13}^* = \widehat{X}_{23} X_{12}$. On démontre de même l'assertion (resp.).

c) Pour $\omega \in \mathcal{L}(H)_*$, on a :

$$\begin{aligned}(id\otimes\omega\otimes id)((k\otimes 1\otimes 1)X_{13}\widetilde{V}^*_{12}\Sigma_{12}) =&(id\otimes\omega\otimes id)((k\otimes 1\otimes 1)\widetilde{V}^*_{12}X_{13}X_{23}\Sigma_{12})\\ =&(id\otimes\omega\otimes id)((k\otimes 1\otimes 1)\widetilde{V}^*_{12}\Sigma_{12}X_{23}X_{13})\end{aligned}$$

On conclut grâce à l'égalité $\overline{(\mathcal{K}\odot\mathcal{K})}\widetilde{V}^*\Sigma = \mathcal{K}\otimes\mathcal{K}$.

L'assertion resp. résulte de celle-ci en remplaçant V par $\Sigma V^*\Sigma$ et X par $\Sigma X^*\Sigma$. ∎

3.10. PROPOSITION. — *Soient (V,U) un unitaire multiplicatif équilibré et semi-birégulier, X une coreprésentation (resp. représentation) de V. Nous avons :*

a) $X\in M(\mathcal{K}\otimes S_X)$ (resp. $X\in M(\widehat{S}_X\otimes\mathcal{K})$).

b) $X\in M(\widehat{S}\otimes S_X)$ (resp. $X\in M(\widehat{S}_X\otimes S)$).

Démonstration. a) Remarquons d'abord qu' on a par ([4] 3.1), $\mathcal{K}\otimes S_X = \overline{lin}\ (\mathcal{K}\otimes 1)X(\mathcal{K}\otimes 1)$. Comme V est semi- régulier, grâce à (3.5), on a donc $\mathcal{K}\otimes S_X\subset X(\mathcal{K}\otimes S_X)$. L'unitaire multiplicatif $\widehat{V}$ étant également semi- régulier , on a aussi par (3.9 c)) , $\mathcal{K}\otimes S_X\subset(\mathcal{K}\otimes S_X)X$. Donc X est un multiplicateur de $\mathcal{K}\otimes S_X$.

L'assertion resp. résulte de celle-ci en remplaçant V par $\Sigma V^*\Sigma$ et X par $\Sigma X^*\Sigma$.

b) Il résulte clairement du a) qu'on a $V\in M(\mathcal{K}\otimes S)$. Remplaçant V par $\widetilde{V}$, on en déduit que $\widetilde{V}\in M(\mathcal{K}\otimes\widehat{S})$. Par (3.9 a)), on a $X_{23} = X^*_{13}\widetilde{V}_{12}X_{13}\widetilde{V}^*_{12}$. Comme $\widetilde{V}_{12}$ et X_{13} sont des multiplicateurs de $\mathcal{K}\otimes\widehat{S}\otimes S_X$, il en va de même pour X_{23}.

L'assertion resp. résulte de celle-ci en remplaçant V par $\Sigma V^*\Sigma$ et X par $\Sigma X^*\Sigma$. ∎

3.11. COROLLAIRE. — *Soit V un unitaire multiplicatif équilibré et semi-birégulier, S et $\widehat{S}$ les C^*-algèbres associées.*

a) $V\in M(\widehat{S}\otimes S)$.

b) Les adhérences des espaces vectoriels engendrés par $\{V(x\otimes 1)V^*(1\otimes y)\ /\ x,y\in S\}$ et $\{V(x\otimes 1)V^*(y\otimes 1)\ /\ x,y\in S\}$ sont toutes deux égales à $S\otimes S$.

c) Les adhérences des espaces vectoriels engendrés par $\{V^*(1\otimes x)V(1\otimes y)\ /\ x,y\in\widehat{S}\}$ et $\{V^*(1\otimes x)V(y\otimes 1)\ /\ x,y\in\widehat{S}\}$ sont toutes deux égales à $\widehat{S}\otimes\widehat{S}$.

Démonstration. a) résulte clairement de (3.10 a)).

b) Pour $a\in\mathcal{K}$, $\omega\in\mathcal{L}(H)_*$ et $y\in S$, on a :

$$V(L(a\omega)\otimes 1)V^*(1\otimes y) = (\omega\otimes id\otimes id)(V_{12}V_{13}(a\otimes 1\otimes y))$$

Par (3.10 a)) l'espace vectoriel engendré par $\{V(a\otimes y)\ /\ a\in\mathcal{K}, y\in S\}$ est dense dans $\mathcal{K}\otimes S$.

Pour $a \in \mathcal{K}$, $\omega \in \mathcal{L}(H)_*$ et $y \in S$, on a :

$$(y \otimes 1)V(L(\omega a) \otimes 1)V^* = (\omega \otimes id \otimes id)((a \otimes y \otimes 1)V_{12}V_{13})$$

Par (3.10 a)) l'espace vectoriel engendré par $\{(a \otimes y)V \ / \ a \in \mathcal{K}, y \in S\}$ est dense dans $\mathcal{K} \otimes S$.

L'assertion c) résulte de b) en remplaçant V par $\Sigma V^* \Sigma$. ∎

Maintenant on a exactement comme dans ([4] 3.8) :

3.12. THÉORÈME. — *Soit V un unitaire multiplicatif équilibré et semi-birégulier. Munie du coproduit δ donné par $\delta(x) = V(x \otimes 1)V^*$, l'algèbre réduite S de V est une C^*-algèbre de Hopf (1.1) bisimplifiable. Munie du coproduit $\widehat{\delta}$ donné par $\widehat{\delta}(x) = V^*(1 \otimes x)V$, l'algèbre réduite duale $\widehat{S}$ de V est une C^*-algèbre de Hopf bisimplifiable.*

Si V est un unitaire multiplicatif semi-régulier, alors on peut construire comme dans [4] les C*-algèbres pleines S_p et $\widehat{S}_p$. Si de plus, V est équilibré et semi-birégulier, on a :

3.13. THÉORÈME. — *(cf. [4] A.6) Soit V un unitaire multiplicatif équilibré et semi-birégulier, alors les C^*-algèbres pleines S_p et $\widehat{S}_p$ sont munies naturellement de structures de C^*-algèbres de Hopf bisimplifiables. De plus, les coreprésentations unitaires de la C^*-algèbre de Hopf S (resp. $\widehat{S}$) correspondent exactement aux représentations de la C^*-algèbre $\widehat{S}_p$ (resp. S_p).*

Grace à (3.10), la preuve est la même que celle de ([4] A.6).

Enfin, pour affirmer avec ces hypothèses que V est en fait un multiplicateur de $\widehat{S}_p \otimes_{max} S_p$ et que les relations $V_{12}V_{13}V_{23} = V_{23}V_{12}$ et $\widehat{V}_{23}V_{12}V_{13} = V_{13}\widehat{V}_{23}$ ont lieu respectivement dans $\mathcal{L}(\widehat{S}_p \otimes_{max} H \otimes_{max} \otimes S_p)$ et $\mathcal{L}(\widehat{S}_p \otimes_{max} S_p \otimes H)$, il suffit de montrer le résultat suivant :

3.14. LEMME. — *Soit V un unitaire multiplicatif dans l'espace de Hilbert H.*

a) Si (K, X) est une représentation et (K, Y) une coreprésentation de V dans le même espace de Hilbert K vérifiant $[Y_{12}, X_{23}] = 0$ dans $\mathcal{L}(H \otimes K \otimes H)$, alors nous avons $[X^*_{12}Y_{21}X_{12}Y^*_{21}, \Sigma_{23}V_{23}] = 0$ dans $\mathcal{L}(K \otimes H \otimes H)$.

b) Si (K, X) est une représentation et (K, Y) une coreprésentation de V dans le même espace de Hilbert K, alors nous avons $[X^*_{12}Y_{23}X_{12}Y^*_{23}, \Sigma_{24}V_{24}] = 0$ dans $\mathcal{L}(K \otimes H \otimes K \otimes H)$.

Démonstration. a) Nous avons :

$$\begin{aligned}
X_{12}^*Y_{21}X_{12}Y_{21}^*\Sigma_{23}V_{23} &= \Sigma_{23}X_{13}^*Y_{31}X_{13}Y_{31}^*V_{23} \\
&= \Sigma_{23}X_{13}^*Y_{31}X_{13}V_{23}Y_{31}^*Y_{21}^* \\
&= \Sigma_{23}X_{13}^*Y_{31}X_{12}^*V_{23}X_{12}Y_{31}^*Y_{21}^* \\
&= \Sigma_{23}X_{13}^*X_{12}^*Y_{31}V_{23}Y_{31}^*X_{12}Y_{21}^* \\
&= \Sigma_{23}X_{13}^*X_{12}^*V_{23}Y_{21}X_{12}Y_{21}^* = \Sigma_{23}V_{23}X_{12}^*Y_{21}X_{12}Y_{21}^*
\end{aligned}$$

b) Posons $K' = K \otimes K$ et $X' = X_{13}$ (resp. $Y' = Y_{13}$) dans $\mathcal{L}(K' \otimes H)$ (resp. $\mathcal{L}(H \otimes K')$. Il est clair qu'on a $[Y'_{12}, X'_{23}] = 0$ dans $\mathcal{L}(H \otimes K' \otimes H)$; le b) résulte alors du a) appliqué à X' et Y'.

∎

En s'appuyant sur le lemme précédent, il est facile de voir que les assertions b), c) et d) du lemme A.7 et la proposition A.8 de [4] sont vraies pour un unitaire multiplicatif équilibré et semi-birégulier.

Passons mantenant aux représentations covariantes de tels unitaires multiplicatifs.

Notons d'abord qu'à toute représentation covariante (X, Y) dans un espace de Hilbert K d'un unitaire multiplicatif semi-régulier, on peut associer (3.14 b)) un unitaire $W \in \mathcal{L}(K \otimes K)$ donné par $W_{13} = X_{12}^*Y_{23}X_{12}Y_{23}^*$ dans $\mathcal{L}(K \otimes H \otimes K)$. Comme dans ([4] A.10), on montre que W est multiplicatif et que (Y, X) est une représentation covariante de W dans l'espace de Hilbert H.

3.15. LEMME. — *Soient V un unitaire multiplicatif dans H et (X, Y) une représentation covariante de V dans l'espace de Hilbert K. Nous avons :*

$$\overline{\mathcal{C}(V) \odot \mathcal{K}(K)} = \overline{lin}\,(1 \odot \mathcal{K}(K))X_{21}(\mathcal{K} \odot 1)Y(1 \odot \mathcal{K}(K))$$

Si de plus V est semi-régulier, l'unitaire multiplicatif W dans K associé à (X, Y) est semi-régulier.

Démonstration. Pour $k \in \mathcal{K}(K)$ et $\omega \in \mathcal{L}(H)_*$, nous avons $(id \otimes id \otimes \omega)((1 \otimes k \otimes 1)\Sigma_{13}V_{13}) = (id \otimes id \otimes \omega)((1 \otimes k \otimes 1)\Sigma_{13}Y_{12}^*X_{23}Y_{12}X_{23}^*) = (id \otimes id \otimes \omega)((1 \otimes k \otimes 1)Y_{32}^*X_{21}\Sigma_{13}Y_{12}X_{23}^*)$. On conclut grace à l'égalité $\overline{lin}\,\{(id \otimes \omega)(\Sigma) \mathbin{/} \omega \in \mathcal{L}(H)_*\} = \mathcal{K}$.

Supposons maintenant que V soit semi-régulier. Comme (Y, X) est une représentation covariante de W on a par le résultat précédent $\overline{lin}\,(\mathcal{C}(W) \odot \mathcal{K}) = \overline{lin}\,(1 \odot \mathcal{K})Y_{21}(\mathcal{K}(K) \odot 1)X(1 \odot \mathcal{K})$. Nous avons $\mathcal{K}(K \otimes H) \subset \overline{lin}\,(\mathcal{C}(W) \odot \mathcal{K})$. En effet, il suffit de montrer qu'un opérateur compact $z \in \mathcal{K}(K \otimes H)$ de la forme $z = (1 \otimes h_1)X^*(k \otimes h_2)X(1 \otimes h_3)$ où $k \in \mathcal{K}(K)$, $h_i \in \mathcal{K}$, appartient à $\overline{lin}\,(\mathcal{C}(W) \odot \mathcal{K})$. Or, par la semi-régularité de V, on peut supposer qu' on a $X^*(k \otimes h_2) = (1 \otimes c)Y_{21}(k' \otimes 1)$ avec $c \in \mathcal{C}(V)$ et $k' \in \mathcal{K}(K)$, d'où la semi-régularité de W.

∎

3.16. LEMME. — *Soit (X,Y) une représentation covariante dans l'espace de Hilbert K de l'unitaire multiplicatif semi-régulier V, notons W l'unitaire multiplicatif dans K associé. Nous avons :*
a) $S_W = S_Y$ *et* $\widehat{S_W} = \widehat{S_X}$.

b) $S_V = \overline{lin}\ \{(\omega' \otimes id)(X)\ /\ \omega' \in \mathcal{L}(K)_*\}$ et $\widehat{S_V} = \overline{lin}\ \{(id \otimes \omega')(Y)\ /\ \omega' \in \mathcal{L}(K)_*\}$.

Démonstration. a) Y étant une représentation de W dans H, l'adhérence normique dans $\mathcal{L}(K)$ de l'espace vectoriel engendré par $\{(\omega\otimes\omega'\otimes id)(Y_{13}W_{23})\ /\ \omega \in \mathcal{L}(H)_*, \omega' \in \mathcal{L}(K)_*\}$ est S_W. Or, la relation de covariance $X_{12}W_{13}Y_{23} = Y_{23}X_{12}$ entraine que l'adhérence normique dans $\mathcal{L}(K)$ de l'espace vectoriel engendré par $\{(\omega' \otimes \omega \otimes id)(W_{13}Y_{23})\ /\ \omega \in \mathcal{L}(H)_*, \omega' \in \mathcal{L}(K)_*\}$ est S_Y. Comme S_W et S_Y sont des sous-C*-algèbres de $\mathcal{L}(K)$, il y a égalité.

La deuxième égalité résulte de la précédente en remplaçant V par $\Sigma V^*\Sigma$ et (X,Y) par $(\Sigma Y^*\Sigma, \Sigma X^*\Sigma)$.

b) (Y,X) étant une représentation covariante de W dont l'unitaire multiplicatif associé est V, le b) résulte alors du a). ∎

3.17. LEMME. — *Soient (V,U) un unitaire multiplicatif équilibré, (X,Y) une représentation covariante de V. Nous avons :*

a) $Y_{12} = (\widehat{Y}X)^*_{23}V_{13}(\widehat{Y}X)_{23}$.

b) $X_{23} = (Y\widetilde{X})_{12}V_{13}(Y\widetilde{X})^*_{12}$.

Si, de plus, V est semi-régulier, nous avons :

c) $Y_{23}W_{13}\widetilde{X}_{21} = \widetilde{X}_{21}Y_{23}$.

d) $\widehat{Y}_{32}W_{13}X_{12} = X_{12}\widehat{Y}_{32}$.

e) $W_{13} = (Y\widetilde{X})_{21}Y_{23}(Y\widetilde{X})^*_{21}$.

Démonstration. a) résulte de (3.9 b)) et de la propriété de covariance ; on démontre de même le b).

L'assertion resp. (3.9 b)) peut s'ecrire également $V_{31}X_{21}\widetilde{X}_{32} = \widetilde{X}_{32}V_{31}$ dans $\mathcal{L}(H \otimes K \otimes H)$. Dans $\mathcal{L}(H \otimes K \otimes K \otimes H)$, nous avons :

$$
\begin{aligned}
Y_{43}W_{23}\widetilde{X}_{42}Y_{13} &= Y_{43}X^*_{21}Y_{13}X_{21}Y^*_{13}\widetilde{X}_{42}Y_{13} \\
&= X^*_{21}Y_{43}Y_{13}X_{21}\widetilde{X}_{42} \\
&= X^*_{21}V^*_{41}Y_{13}V_{41}X_{21}\widetilde{X}42 \\
&= \widetilde{X}_{42}V^*_{41}\widetilde{X}^*_{42}Y_{13}\widetilde{X}_{42}V_{41} \\
&= \widetilde{X}_{42}V^*_{41}Y_{13}V_{41} = \widetilde{X}_{42}Y_{43}Y_{13}
\end{aligned}
$$

d'où le c). En conjuguant c) par l'unitaire $(1 \otimes U \otimes 1)$, on obtient d). L'assertion e) résulte de c) et du fait que Y est une représentation de W. ∎

Comme conséquence des résultats précédents, nous avons :

3.18. THÉORÈME. — *Soient V un unitaire multiplicatif dans H équilibré et semi-birégulier et (X,Y) une représentation covariante de V dans l'espace de Hilbert K. Notons W l'unitaire multiplicatif dans K associé. Nous avons :*

a) *Munie du morphisme δ_W (resp. $\widehat{\delta_W}$) donné par $\delta_W(x) = W(x \otimes 1)W^*$ (resp. $\widehat{\delta_W}(x) = W^*(1 \otimes x)W$), la C^*-algèbre S_W (resp. $\widehat{S_W}$) est une C^*-algèbre de Hopf bisimplifiable.*

b) *Il existe un unique isomorphisme de C^*-algèbres de Hopf π (resp. $\widehat{\pi}$) de S_V (resp. $\widehat{S_V}$) sur S_W (resp. $\widehat{S_W}$) vérifiant $(id \otimes \pi)(V) = Y$ (resp. $(\widehat{\pi} \otimes id)(V) = X$).*

c) *$(\widehat{\pi} \otimes \pi)(V) = W$.*

Démonstration. La coreprésentation Y de V étant stablement équivalente (3.17 a)) à la coreprésentation régulière, il existe un unique isomorphisme π de C*-algèbres de S_V sur (3.16 a)) la C*-algèbre $S_Y = S_W$ vérifiant $\pi(L_V(\omega)) = L_Y(\omega)$, donc $(id \otimes \pi)(V) = Y$.

Remplaçant V par $\Sigma V^* \Sigma$ et (X,Y) par $(\Sigma Y^* \Sigma, \Sigma X^* \Sigma)$, on déduit de ce qui précède un unique isomorphisme $\widehat{\pi}$ de C*-algèbres de $\widehat{S_V}$ sur la C*-algèbre $\widehat{S_X} = \widehat{S_W}$ vérifiant $\widehat{\pi}(\rho_V(\omega)) = \rho_X(\omega)$, donc $(\widehat{\pi} \otimes id)(V) = X$. Appliquant $(\widehat{\pi} \otimes id \otimes \pi)$ à la relation pentagonale $V_{12}V_{13}V_{23} = V_{23}V_{12}$, on obtient l'égalité $W = (\widehat{\pi} \otimes \pi)(V)$.

Soit $x = (\omega' \otimes id)(X)$ où $\omega' \in \mathcal{L}(K)_*$, nous avons :

$$\begin{aligned} W(\pi(x) \otimes 1)W^* &= (\omega' \otimes id \otimes id)(W_{23}W_{12}W_{23}^*) \\ &= (\omega' \otimes id \otimes id)(W_{12}W_{13}) \\ &= (\omega' \otimes id \otimes id)(\widehat{\pi} \otimes \pi \otimes \pi)(V_{12}V_{13}) \\ &= (\pi \otimes \pi)(V(L_V(\omega' \circ \widehat{\pi}) \otimes 1)V^*) \\ &= (\pi \otimes \pi)(\delta_V(x)) \end{aligned}$$

Par (3.12) et le calcul précédent, il résulte que S_W, munie de $\delta_W(x) = W(x \otimes 1)W^*$, est une C*-algèbre de Hopf bisimplifiable, et que π est un isomorphisme de C*-algèbres de Hopf de (S_V, δ_V) sur (S_W, δ_W).

Le reste de l'assertion (resp.) se déduit comme précédemment. ∎

Grâce aux résultats du paragraphe 4, on voit qu'avec les hypothèses précédentes, l'unitaire multiplicatif W n'est pas toujours (stablement) équivalent à V. Cependant, nous avons :

3.19. PROPOSITION. — *Soient (V,U) un unitaire multiplicatif dans H semi-régulier et équilibré, (X,Y) une représentation covariante de V dans l'espace de Hilbert K ; notons W l'unitaire multiplicatif dans K associé. Alors l'unitaire multiplicatif $W_{13}W_{23}$ dans $K \otimes K$ est stablement équivalent à l'unitaire multiplicatif $\widehat{Y}_{12}V_{24}\widehat{Y}_{12}^*$ dans $K \otimes H$.*

Démonstration. Montrons d'abord que l'unitaire multiplicatif (cf.3.21) $W_{13}Y_{23}$ dans $K \otimes H$ est équivalent à l'unitaire multiplicatif $\widehat{Y}_{12}V_{24}\widehat{Y}_{12}^*$ dans $K \otimes H$. Nous avons dans $\mathcal{L}(K \otimes H \otimes K \otimes H)$:

$$\begin{aligned}(\widehat{Y}X)_{12}(\widehat{Y}X)_{34}W_{13}Y_{23}(\widehat{Y}X)^*_{12}(\widehat{Y}X)^*_{34} &= (\widehat{Y}X)_{34}\widehat{Y}_{12}Y_{23}\widehat{Y}^*_{12}(\widehat{Y}X)^*_{34} \\ &= \widehat{Y}_{12}\widehat{Y}_{34}Y_{23}V_{24}\widehat{Y}^*_{34}\widehat{Y}^*_{12} = \widehat{Y}_{12}V_{24}\widehat{Y}^*_{12}\end{aligned}$$

Il suffit donc de montrer que l'unitaire multiplicatif $W_{13}W_{23}$ dans $K \otimes K$ est stablement équivalent à l'unitaire multiplicatif $W_{13}Y_{23}$ dans $K \otimes H$. Pour cela, soit $T \in \mathcal{L}(K \otimes H, H \otimes K)$ l'unitaire défini par $T = s(Y\widetilde{X})^*_{21}$ où $s \in \mathcal{L}(K \otimes H, H \otimes K)$ est la volte. Nous avons, d'aprés (3.17) , dans $\mathcal{L}\mathcal{H} \otimes \mathcal{H})$ où $\mathcal{H} = K \otimes H \otimes K$:

$$T_{56}T_{23}W_{14}W_{24}T^*_{23}T^*_{56} = W_{14}T_{23}W_{24}T^*_{23} = W_{14}Y_{24}$$

■

3.20. Corollaire. — *Soit (V, U) un unitaire multiplicatif dans H semi-régulier et équilibré vérifiant $[\widehat{V}_{12}, V_{23}] = 0$. Alors V est stablement équivalent à $W_{13}W_{23}$, où W est l'unitaire multiplicatif associé à toute représentation covariante (X, Y) de V dans un espace de Hilbert K.*

Démonstration. On a avec les notations de (3.9), $[\widehat{Y}_{12}, V_{23}] = 0$, d'où le résultat. ■

3.21. Remarque. — Si V est un unitaire multiplicatif dans H et X une coreprésentation (resp. représentation) de V dans l'espace de Hilbert K, il est facile de voir que l'unitaire $X_{23}V_{24}$ (resp. $V_{13}X_{23}$) dans $K \otimes H$ (resp. $H \otimes K$) est multiplicatif.

Nous terminons ce paragraphe par un résultat (3.23) dans le cas irréductible (cf.[4] 6.2), qui nous sera utile au paragraphe 4.

Donnons nous un unitaire multiplicatif irréductible (V, U) et supposons que V soit semi-birégulier, i.e V et $\widehat{V}$ semi-réguliers. Nous avons :

3.22. Lemme. — *L'unitaire V est un multiplicateur de la C^*-algèbre $\overline{\mathcal{C}(\widehat{V})} \otimes \mathcal{K}$.*

Démonstration. Pout tout $\omega \in \mathcal{L}(H)_*$ et tout $k \in \mathcal{K}$, nous avons :

$$\begin{aligned}(id \otimes \omega \otimes id)(V_{13}\Sigma_{12}\widehat{V_{12}}(1 \otimes 1 \otimes k)) &= (id \otimes \omega \otimes id)(\Sigma_{12}V_{23}\widehat{V_{12}}(1 \otimes 1 \otimes k)) \\ &= (id \otimes \omega \otimes id)(\Sigma_{12}\widehat{V_{12}}V_{23}(1 \otimes 1 \otimes k))\end{aligned}$$

on conclut grace à l'égalité $V(\mathcal{K} \otimes \mathcal{K}) = \mathcal{K} \otimes \mathcal{K}$.

■

Il résulte clairement du lemme précédent qu'on a $\overline{lin}\ \widehat{A}(V)\mathcal{C}(\widehat{V}) = \overline{\mathcal{C}(\widehat{V})}$.

3.23. Proposition. — *Soit (V, U) un unitaire multiplicatif irréductible et semi-birégulier. Alors l'espace vectoriel fermé engendré par $\{L(\omega')\,\rho(\omega)\ /\ \omega, \omega' \in \mathcal{L}(H)_*\}$ est une C^*-algèbre qui coincide avec la fermeture normique de l'algèbre $\mathcal{C}(\widehat{\widehat{V}})$.*

Démonstration. Par l'irréductibilité ([4] 6.2 a)) de V il est facile de voir qu'on a :

$$\overline{lin}\ \{(id\otimes\omega)(\widehat{V}V)\ /\ \omega\in\mathcal{L}(H)_*\} = U\overline{\mathcal{C}(V)}U = \overline{\mathcal{C}(\widehat{\widehat{V}})}$$

Remplaçant V par $\widehat{V}$ dans (3.22) et utilisant (3.10), on obtient :

$$\begin{aligned}\overline{\mathcal{C}(\widehat{\widehat{V}})} &= \overline{lin}\ \{(id\otimes\omega)((x\otimes 1)\widehat{V}V)\ /\ \omega\in\mathcal{L}(H)_*, x\in\widehat{A}(\widehat{V})\}\\ &= \overline{lin}\ \{L(\omega')\,\rho(\omega)\ /\ \omega,\omega'\in\mathcal{L}(H)_*\}\end{aligned}$$

■

3.24. Remarque. —

a) Soit V un unitaire multiplicatif équilibré et semi-birégulier. Si la C*-algèbre de Hopf S agit dans une C*-algèbre A par une coaction δ_A, alors comme dans le cas régulier ([4] appendice), on peut définir le produit croisé "max" $A\rtimes_{max}\widehat{S}$: c'est la C*-algèbre dont les représentations sont données par les couples (π, u) où $\pi : A \to \mathcal{L}(K)$ est une représentation de A dans l'espace de Hilbert K, et $u\in\mathcal{L}(K\otimes S)$ est une coreprésentation unitaire ([4] 0.3) de S vérifiant la condition de covariance $(\pi\otimes id)(\delta_A(a)) = u(\pi(a)\otimes 1)u^*$ pour tout $a\in A$.

Comme dans le cas des groupes, on a un plongement canonique $\pi : A \to M(A\rtimes_{max}\widehat{S})$ (resp. $\widehat{\theta} : \widehat{S_p} \to M(A\rtimes_{max}\widehat{S})$). Procédant comme dans ([4] A.6), on peut montrer qu'il existe un unitaire $W\in M(A\rtimes_{max}\widehat{S}\otimes S)$ tel que $A\rtimes_{max}\widehat{S} = \overline{lin}\ \{\pi(a)(id\otimes\omega)(W)\ /\ a\in A, \omega\in\mathcal{L}(H)_*\}$.

Si de plus l'unitaire multiplicatif V est moyennable ([4] appendice), par les résultats de (3.17), il est facile de montrer qu'on a $A\rtimes_{max}\widehat{S} = A\rtimes\widehat{S}$.

b) Avec les hypothèses de (3.23), le produit croisé réduit $S\rtimes\widehat{S}$ est isomorphe à la C*-algèbre $\overline{lin}\ \{\rho(\omega)\,L(\omega')\ /\ \omega,\omega'\in\mathcal{L}(H)_*\}$ et donc contient la C*-algèbre des opérateurs compacts. En effet, pour tout $\omega\in\mathcal{L}(H)_*$, on a ([4] 6.1 (2)) $\widehat{V}\delta(L(\omega))\widehat{V}^* = 1\otimes L(\omega)$ et ([4] 6.5 c)) $[\widehat{V}, 1\otimes\rho(\omega)] = 0$. Notons que dans le cas où l'unitaire multiplicatif est de plus moyennable, on a $S\rtimes\widehat{S} = S\rtimes_{max}\widehat{S}$; dans ce cas les représentations de la C*-algèbre $S\rtimes\widehat{S}$ sont les représentations covariantes de l'unitaire multiplicatif V.

4. Groupe quantique $E_\mu(2)$ de Woronowicz

Dans ce paragraphe, nous montrons que la représentation régulière de $E_\mu(2)$ est un unitaire multiplicatif semi-birégulier (mais non régulier) et irréductible. Pour cela, nous commençons par construire la mesure de Haar du groupe quantique $E_\mu(2)$. Nous donnons ensuite les représentations covariantes de cet unitaire multiplicatif et nous montrons que la C*-algèbre $S\rtimes\widehat{S} = S\rtimes_{max}\widehat{S}$ est une extension des compacts par les compacts. Enfin, nous explicitons la théorie modulaire des mesures de Haar de $E_\mu(2)$ et de son dual de Pontrjagyn.

Une partie des résultats de cette section ont été annoncés dans [3] ; nous donnons également les preuves de ces résultats.

a) Mesure de Haar du groupe quantique $E_\mu(2)$

Commonçons par rappeler la définition de la C*-algèbre de Hopf [25] des "fonctions continues sur $E_\mu(2)$ tendant vers 0 à l'infini" que nous noterons dans la suite (A,δ).

Fixons un nombre réel $\mu > 1$ et posons $\mathbf{C}_\mu = \{z \in \mathbf{C} / \mid z \mid \in \mu^Z\} \cup \{0\}$. La C*-algèbre A est la C*-algèbre universelle engendrée par un unitaire $\mathbf{v} \in M(A)$ et un opérateur normal $\mathbf{n}$ de spectre $\mathbf{C}_\mu$, affilié ([2], [25]) à A, vérifiant $\mathbf{v}^*\mathbf{n}\mathbf{v} = \mu\mathbf{n}$. Autrement dit, A est le produit croisé $A = C_0(\mathbf{C}_\mu) \rtimes \mathbf{Z}$ par l'action α de $\mathbf{Z}$ dans $\mathbf{C}_\mu$ définie par $\alpha(f)(\zeta) = f(\mu^{-1}\zeta)$.

Dans tout ce qui suit, nous notons L la représentation (fidèle) de A dans $l^2(\mathbf{Z}^3)$ définie par $L(\mathbf{v}) = 1 \otimes v_0 \otimes v_0$ et $L(\mathbf{n}) = v_0 \otimes n_0 \otimes 1$ où v_0 (resp. n_0) est l'opérateur dans $l^2(\mathbf{Z})$ défini par $v_0 e_n = e_{n+1}$ (resp. $n_0 e_n = \mu^n e_n$) ; (e_n) étant la base canonique de $l^2(\mathbf{Z})$.

Le coproduit δ est l'unique homomorphisme non dégénéré de A dans $M(A \otimes A)$ vérifiant $\delta(\mathbf{v}) = \mathbf{v} \otimes \mathbf{v}$ et $\delta(\mathbf{n})$ est la fermeture de l'opérateur $\mathbf{v} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v}^*$, voir [25].

Fixons maintenant quelques notations qui seront constamment utilisées dans ce paragraphe. Posons $H = l^2(\mathbf{Z}^3)$. Pour tout $a,b,c \in \mathbf{Z}$, nous notons $\omega_{a,b,c}$ la forme $x \to (e_{c,a+b,a} \mid L(x)e_{0,b,0})$ sur A où $(e_{a,b,c})$ désigne la base canonique de $l^2(\mathbf{Z}^3)$. Pour tout i,j , notons $f_{i,j} \in C_0(\mathbf{C}_\mu)$ la fonction à support dans $\{z \in \mathbf{C}_\mu / \mid z \mid = \mu^i\}$ définie par $f_{i,j}(z) = (\frac{z}{\mid z \mid})^j$ si $\mid z \mid = \mu^i$. Il est clair que $\omega_{a,b,c}(\mathbf{v}^i f_{j,k}(\mathbf{n})) = \delta^{ijk}_{abc}$ et que la forme $x \to (e_{a,b,c} \mid L(x)e_{a',b',c'})$ sur A coincide avec la forme $\delta^{c-c'}_{b-b'}\omega_{c-c',b',a-a'}$. Enfin, pour tout $\xi \in H$, notons ω_ξ la forme $\omega_{\xi,\xi} \circ L$.

Posons $\Phi = \sum_{-\infty}^{\infty} \mu^{2j}\omega_{0j0}$. Il est clair que Φ est un poids s.c.i et normiquement semi-fini sur A. Soit $M = L(A)''$ l'algèbre de von Neumann engendrée par $L(A)$; par abus de notation, nous notons également Φ le poids normal et ultrafaiblement semi-fini sur M défini par $x \to \sum_n \mu^{2n} (e_{0n0} \mid x e_{0n0})$.

Soit ν la mesure positive sur $\mathbf{C}_\mu$ définie pour toute fonction $f \in \mathbf{C}_0(C_\mu)$ à valeurs positives par :

$$\nu(f) = \sum_n \mu^{2n} \int_{S^1} f(\mu^n \zeta)\, d\zeta$$

où $d\zeta$ désigne la mesure de Haar normalisée de S^1.

4.1. PROPOSITION. — *Avec ces notations nous avons :*

a) *Le poids Φ sur M est le poids dual ([15], [22]) de la mesure ν.*

b) *La formule $e_{ijk} = \mu^{k-j}\Lambda_\Phi \mathbf{v}^k f_{j-k,i}(\mathbf{n})$ identifie l'espace de Hilbert H_Φ de la représentation GNS de Φ à H et la représentation π_Φ correspondante à L.*

c) *L'opérateur modulaire* Δ_Φ *et l'opérateur* J_Φ *canoniquement associés à* Φ , *sont respectivement donnés par* $\Delta_\Phi e_{ijk} = \mu^{2k} e_{ijk}$ *et* $J_\Phi e_{ijk} = e_{-i,j-k,-k}$.

d) $\sigma_t^\Phi(L(A) = L(A)$ *et pour tout* $a \in A$, $t \to \sigma_t^\Phi(L(a))$ *est normiquement continue.*

Démonstration. a) Soit $E : M \to L^\infty(C_\mu, \nu)$ l'espérance conditionnelle associée à l'action α. On a $\omega_{e_{0n0};e_{0n0}} \circ E = \omega_{e_{0n0};e_{0n0}}$ pour tout entier n, donc $\Phi \circ E = \Phi$. Par ailleurs, pour tout $f \in \mathbf{C}_0(C_\mu)$, on a $(e_{0k0} \mid f(\mathbf{n}) e_{0k0}) = \int_{S^1} f(\mu^k \zeta)\, d\zeta$, donc si f est à valeurs positives, on a $\Phi(f(\mathbf{n})) = \nu(f)$ d'où le a).

b) Soit T l'opérateur unitaire de H sur H_Φ définie par $T(e_{ijk}) = \mu^{k-j} \Lambda_\Phi \mathbf{v}^k f_{j-k,i}(\mathbf{n})$, il est facile de voir qu'on a $\pi_\Phi(\mathbf{v}) = TL(\mathbf{v})T^*$ et que pour $f \in \mathbf{C}_0(C_\mu)$, on a $\pi_\Phi(f(\mathbf{n})) = TL(f(\mathbf{n}))T^*$.

c) résulte de a), b) et de [15] ; d) se déduit facilement de a),b) et c). ∎

4.2. THÉORÈME. — *Le poids* Φ *est une mesure de Haar à gauche et à droite pour la* C^**-algèbre de Hopf* (A, δ).

Démonstration. Vérifions d'abord que le poids Φ satisfait les conditions a) et b) de (2.2). Pour cela, par (4.1 a) et d)), il suffit de montrer que le prolongement canonique $\bar{\Phi}$ de Φ à l'algèbre de von Neumann M coincide avec le poids $\tilde{\nu}$ sur M dual de la mesure ν. On a $\Phi = \bar{\Phi} \circ L = \tilde{\nu} \circ L$ sur A^+, comme $\bar{\Phi}$ est [1] le régularisé s.c.i de Φ, il résulte que $\bar{\Phi} \circ \sigma_t^{\tilde{\nu}} = \bar{\Phi}$. Par [19], il est clair que $\tilde{\nu} = \bar{\Phi}$.

Pour montrer la condition c) de (2.2), nous allons d'abord calculer le produit de convolution des formes ω_{abc}.

Soit g_μ la fonction continue [25] sur $\mathbf{C}_\mu$ définie par :

$$g_\mu(\zeta) = \prod_{j=0}^{\infty} \frac{1 + \mu^{-1-2j}\zeta}{1 + \mu^{-1-2j}\bar{\zeta}}$$

et soit $A(m,n) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g_\mu(\mu^m e^{it}) e^{-int}\, dt$, le nième coefficient de Fourier de la restriction de g_μ au cercle $\{z \in C_\mu / \mid z \mid = \mu^m\}$.

4.3. PROPOSITION. — *Nous avons :*

$$\omega_{abc} * \omega_{a'b'c'} = \delta_{c+c'}^{a-a'} \sum_n A(b-b', n-b')\, A(b-b'+a-a', n-b'+c) \omega_{c+a',n,c+c'}$$

Démonstration. Posons $a = \mathbf{v} \otimes \mathbf{n}$, $b = \mathbf{n} \otimes \mathbf{v}^*$. Il est facile de voir que les opérateurs a et b vérifient les conditions de ([28] thm.2.3) ; donc $\delta(\mathbf{n}) = \overline{a+b} = g_\mu(c)\, a\, g_\mu(c)^*$ où c est l'opérateur normal fermeture de l'opérateur $\mu^{-1} a^{-1} b$. Pour tout i, j, k nous en déduisons :

$$\begin{aligned}\delta(f_{jk}(\mathbf{n})) &= g_\mu(c)f_{jk}(\mathbf{v}\otimes\mathbf{n})g_\mu(c)^* \\ &= g_\mu(c)(\mathbf{v}^k\otimes f_{jk}(\mathbf{n}))g_\mu(c)^* \\ &= \sum_{l,m,n} A(l-n,j-n)A(l-n+k,j-n+m) \\ &\qquad (\mathbf{v}^{k-m}f_{lm}(\mathbf{n})\otimes\mathbf{v}^{-m}f_{n,k-m}(\mathbf{n}))\end{aligned}$$

En évaluant la forme $\omega_{abc}*\omega_{a'b'c'} = (\omega_{abc}\otimes\omega_{a'b'c'})\circ\delta$ en $x=\mathbf{v}^i f_{jk}(\mathbf{n})$, on obtient l'égalité cherchée.

■

4.4. LEMME. — *Soit $\xi\in H$ de composantes (ξ_{abc}) relativement à la base canonique de H. Pour tout $n\in\mathbf{Z}$, on a :*

a) $\omega_\xi*\omega_{0n0} = \sum\limits_{m,p,q}\omega_{\xi^{p,q}_{m,n}}$ *avec* $\xi^{p,q}_{m,n} = \sum\limits_{a,b,c}\delta_p^{a-b}\delta_q^{a-c}\xi_{abc}A(b-n,a+m-n)\,e_{a,m+a,c}.$

b) $\omega_{0n0}*\omega_\xi = \sum\limits_{m,p,q}\omega_{\xi^{m,n}_{p,q}}$ *avec* $\xi^{m,n}_{p,q} = \sum\limits_{a,b,c}\delta_p^{a+c}\delta_q^{b-c}\xi_{abc}A(n-b,m-a-b)\,e_{a,m-a,c}.$

Démonstration. Calculons d'abord $f*\omega_{0n0}$ où f est la forme définie par $f(x)=(e_{a,b,c}\mid L(x)e_{a',b',c'})$. Nous avons :

$$\begin{aligned} f*\omega_{0n0} &= \delta_{b-b'}^{c-c'}\,\omega_{c-c',b',a-a'}*\omega_{0n0} \\ &= \delta_{b-b'}^{c-c'}\delta_{a-a'}^{c-c'}\sum_m A(b'-n,m-n)A(b'-n+a-a',m-n+a-a')\,\omega_{c-c',m,a-a'} \\ &= \delta_{b-b'}^{c-c'}\delta_{a-a'}^{c-c'}\sum_m A(b'-n,a'+m-n)A(b-n,a+m-n)\,\omega_{c-c',m+a',a-a'} \\ &= \delta_{b-b'}^{c-c'}\delta_{a-a'}^{c-c'}\sum_m A(b'-n,a'+m-n)A(b-n,a+m-n)\,\omega_{e_{a,m+a,c};e_{a',m+a',c'}}\circ L \quad (1)\end{aligned}$$

Pour tout entier $N\geq 0$ et tout $p,q\in\mathbf{Z}$, soient χ_N la fonction caractéristique de l'ensemble $\{\ (a,b,c)\in\mathbf{Z}^3/\ |\,a\,|+|\,b\,|+|\,c\,|\leq N\ \}$ et l^N_{pq} la fonction définie par $l^N_{pq}(a,b,c)=\chi_N(a,b,c)\delta_p^{a-b}\,\delta_q^{a-c}\,\xi_{abc}$, posons :

$$\xi^{p,q,N}_{m,n} = \sum_{a,b,c} l^N_{pq}(a,b,c)\,A(b-n,a+m-n)\,e_{a,m+a,c}.$$

Par (A.3) on a $\sum\limits_{m,p,q}\|\xi^{p,q}_{m,n}-\xi^{p,q,N}_{m,n}\|^2 = \sum\limits_{|a|+|b|+|c|>N}|\,\xi_{abc}\,|^2\to 0$ quand $N\to\infty$ d'où

$\sum\limits_{m,p,q}\omega_{\xi^{p,q}_{m,n}} = \lim\limits_{N\to\infty}\sum\limits_{m,p,q}\omega_{\xi^{p,q,N}_{m,n}}$. D'autrepart, par (1), nous avons :

$$\sum_{m,p,q}\omega_{\xi^{p,q,N}_{m,n}} = \sum_{a,b,c}\sum_{a',b',c'}\chi_N(a,b,c)\,\chi_N(a',b',c')\,\overline{\xi_{abc}}\,\xi_{a'b'c'}\,\omega_{e_{a,b,c};e_{a',b',c'}}\circ L*\omega_{0n0}$$

qui converge normiquement quand $N\to\infty$ vers $\omega_\xi*\omega_{0n0}$, d'où le a).

Démonstration analogue pour le b).

■

4.5. LEMME. — *Soit $\xi \in H$ un vecteur de composantes (ξ_{abc}) relativement à la base canonique de H. Supposons que pour tout p, q , on ait $\sum_a \mu^a \;|\xi_{a,a+p,a+q}| < \infty$ (resp. $\sum_c \mu^c \;|\xi_{c+p,c+q,c}| < \infty$) , alors pour tout $x \in A^+$, on a :*

$$\Phi(x*\omega_\xi) = \|\xi\|^2 \Phi(x) \quad (resp.\ \Phi(\omega_\xi * x) = \|\xi\|^2 \Phi(x))$$

Démonstration. Nous avons avec les notations du lemme précédent :

$$\begin{aligned}\Phi(x*\omega_\xi) &= \sum_n \mu^{2n} (\omega_\xi * \omega_{0n0})(x) = \sum_{m,n,p,q} \mu^{2n}\, \omega_{\xi^{p,q}_{m,n}}(x)\\ &= \sum_{m,n,p,q} \mu^{2n} \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, \xi_{abc}\, A(b-n, a+m-n)\, \omega_{\xi^{p,q}_{m,n};e_{a,m+a,c}}(L(x))\end{aligned}$$

Par (A.2) , nous en déduisons :

$$\Phi(x*\omega_\xi) = \sum_{m,n,p,q} \mu^{2n} \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, \xi_{abc} (-\mu)^{a+m-n} A(-p-m, n-m-a)$$

$$\omega_{\xi^{p,q}_{m,n};e_{a,m+a,c}}(L(x)) \qquad (1)$$

Par (A.2) , nous avons également :

$$\begin{aligned}\|\xi^{p,q}_{m,n}\|^2 &= \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, |\xi_{abc}|^2\, A(b-n, a+m-n)^2\\ &= \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, |\xi_{abc}|^2\, \mu^{2(a+m-n)} A(-p-m, n-m-a)^2\\ &\le \mu^{2(m-n)} \sum_a \mu^{2a}\, |\xi_{a,a-p,a-q}|^2 < \infty\end{aligned}$$

Comme $\sum_n |A(x,n)| < \infty$, il vient que :

$$\begin{aligned}&\sum_n \mu^{2n} \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, \xi_{abc} (-\mu)^{a+m-n} A(-p-m, n-m-a)\, \omega_{\xi^{p,q}_{m,n};e_{a,m+a,c}}(L(x))\\ &= \sum_{a,b,c} \delta^{a-b,a-c}_{p,q}\, \xi_{abc} (-\mu)^{a+m} \sum_n (-\mu)^n A(-p-m, n-m-a)\, \omega_{\xi^{p,q}_{m,n};e_{a,m+a,c}}(L(x)) \qquad (2)\end{aligned}$$

En remplaçant $\xi^{p,q}_{m,n}$ par son expression dans la base canonique de H, nous obtenons :

$$\begin{aligned}&\sum_n (-\mu)^n A(-p-m, n-m-a)\, \omega_{\xi^{p,q}_{m,n};e_{a,m+a,c}}(L(x))\\ &= \sum_{a',b',c'} \delta^{a'}_a\, \delta^{a'-b',a'-c'}_{p,q}\, \overline{\xi_{a'b'c'}}\, (-\mu)^{a'+m}\, \omega_{e_{a',m+a',c'};e_{a,m+a,c}}(L(x))\end{aligned}$$

Tenant compte de (1),(2) et de l'égalité précédente , nous avons :

$$\Phi(x*\omega_\xi) = \sum_{m,p,q}\sum_{a,b,c} \delta_{p,q}^{a-b,a-c} \, |\xi_{abc}|^2 \, \mu^{2(a+m)} \, \omega_{e_{a,m+a,c}}(x)$$
$$= \sum_m \sum_{a,b,c} |\xi_{abc}|^2 \, \mu^{2(a+m)} \, \omega_{e_{0,m+a,0}}(x) = \|\xi\|^2 \, \Phi(x).$$

Démonstration analogue pour l'assertion (resp.). ∎

4.6. LEMME. — *Pour toute forme positive normale $\omega \in \mathcal{L}(H)_*$ et tout $x \in A^+$, nous avons :*

$$\Phi(x*\omega \circ L) = \Phi(\omega \circ L*x) = \omega(1)\,\Phi(x)$$

Démonstration. Il suffit de montrer que pour tout $\xi \in H$, nous avons $\Phi(x*\omega_\xi) = \Phi(\omega_\xi * x) = \|\xi\|^2 \, \Phi(x)$. Nous allons montrer que $\Phi(x*\omega_\xi) = \|\xi\|^2 \, \Phi(x)$; la preuve de l'égalité $\Phi(\omega_\xi * x) = \|\xi\|^2 \, \Phi(x)$ étant tout à fait analogue.

Supposons que $\Phi(x) < \infty$. Pour tout $N \geq 0$, posons $y_N = \sum_{|n| \leq N} \mu^{2n} (\omega_{0n0} * x)$. Alors la suite croissante d'opérateurs positifs $(L(y_N))$ converge faiblement vers $\Phi(x).1$. En effet, pour tout $\eta \in H$ à support fini, nous avons grace à (4.5) :

$$\omega_\eta(y_N) = \sum_{|n| \leq N} \mu^{2n} \, \omega_{0n0} \, (x*\omega_\eta) \leq \Phi(x*\omega_\eta) = \|\eta\|^2 \, \Phi(x)$$

Nous en déduisons que $\sup_N \|L(y_N)\| < \infty$, donc la suite $(L(y_N))$ converge faiblement vers un opérateur positif y. Mais, pour tout $\eta \in H$ à support fini, on a par (4.5) :

$$\omega_{\eta;\eta}(y) = \sup_N \omega_\eta(L(y_N)) = \Phi(x * \omega_\eta) = \|\eta\|^2 \, \Phi(x).$$

donc $y = \Phi(x).1$. Pour tout $\xi \in H$, nous avons alors :

$$\Phi(x*\omega_\xi) = \sum_n \mu^{2n} \, \omega_{0n0} \, (x*\omega_\xi)$$
$$= \sup_N \sum_{|n| \leq N} \mu^{2n} \, \omega_{0n0} \, (x*\omega_\xi)$$
$$= \sup_N \omega_\xi(y_N) = \|\xi\|^2 \, \Phi(x)$$

Pour terminer la preuve, il suffit de montrer que si $\eta \in H$ est non nul et vérifie $\Phi(x*\omega_\eta) < \infty$, alors $\Phi(x) < \infty$.

Par (4.5), nous avons $\Phi(x*\omega_{000}) = \Phi(x)$. Si η est un vecteur de H non nul de composantes (η_{abc}), nous avons :

$$\Phi(x* \omega_\eta) = \Phi(x*\omega_\eta*\omega_{000}) = \sum_{m,p,q} \Phi(x*\omega_{\eta_{m,0}^{p,q}})$$

Fixons m, p, q et posons $\eta^{p,q}_{m,0} = \xi$; d'aprés (4.4) et (A.2), les composantes (ξ_{abc}) de ξ sont données par :

$$\begin{aligned}\xi_{abc} &= \delta^{a+m,a-q}_{b,c}\eta_{a,a-p,a-q}A(a-p,a+m)\\ &= \delta^{a+m,a-q}_{b,c}\ \eta_{a,a-p,a-q}\,(-\mu)^{a+m}\ A(-p-m,-a-m)\end{aligned}$$

Il est clair qu'on peut trouver m, p, q de façon que le vecteur $\eta^{p,q}_{m,0} = \xi$ soit non nul et vérifie les hypotheses de (4.5). On a alors :

$$\|\eta^{p,q}_{m,0}\|^2\,\Phi(x) = \Phi(x*\omega_{\eta^{p,q}_{m,0}}) \leq \Phi(x*\omega_\eta) < \infty$$

■

Pour terminer la preuve du théorème, nous montrons dans l'appendice B que pour toute forme positive f sur A , nous avons $\Phi(x*f) = \Phi(f*x) = \|f\|\,\Phi(x)$ pour tout $x \in A_+$.

Notons V l'isométrie multiplicative (2.4) de l'espace de Hilbert $H_\Phi \otimes H_\Phi$ définie pour tout $x, y \in \mathfrak{N}_\Phi$ par $V(\Lambda_\Phi x \otimes \Lambda_\Phi y) = \Lambda_{\Phi\otimes\Phi}\delta(x)(1\otimes y)$. Il est évident que pour tout $x \in A$ nous avons $(L\otimes L)(\delta(x))V = V(L(x)\otimes 1)$ Pour montrer que V est surjective, notons d'abord qu'à travers l'identification $H = H_\Phi$ donnée par (4.1b)), nous avons :

$$V(e_{abc}\otimes e_{xyz}) = \sum_{l,m} \mu^{l-b+c+y} A(l, b-c-y)A(a+l, b-y+m)$$
$$(e_{c+m,a+y+l-m,a-m} \otimes e_{x+a-c-m,y-m,z-m})$$

Introduisons l'unitaire $V'' \in \mathcal{L}(H\otimes H)$ défini par $V''(e_{abc}\otimes e_{xyz}) = e_{abc}\otimes e_{x,y+c-a,z+c-a}$. Alors la formule précédente nous dit exactement que $VV''^* = g_\mu(X)g_\mu(Y)$ où $X = \mathbf{v}^*(\mathbf{n}^*)^{-1}\otimes \mathbf{v}^*\mathbf{n}^*$ (resp. $Y = -(v_0 n_0 \otimes n_0^{-1}\otimes v_0^*)\otimes \mathbf{v}^*\mathbf{n}^*$), donc V est un unitaire multiplicatif dans H et nous avons pour tout $x \in A$, $(L\otimes L)(\delta(x)) = V(L(x)\otimes 1)V^*$.

Posons $W = g_\mu(X)V''$ et $Q = -(n_0\otimes v_0 \otimes 1)$ dans H ; remarquons que les opérateurs normaux X et Y vérifient les conditions de ([28] thm.2.3) et que la fermeture de l'opérateur normal $\mu^{-1}X^{-1}Y$ est égale à l'opérateur $Q\otimes 1_H$.

4.7. PROPOSITION. — *Avec ces notations, nous avons :*

a) $V = (g_\mu(Q)\otimes 1_H)W(g_\mu(Q)^*\otimes 1_H)$.

b) V *est semi-régulier mais non régulier.*

c) $S_V = L(A)$

Démonstration. a) Par ([28] thm.2.3), nous avons

$$g_\mu(X)g_\mu(Y) = (g_\mu(Q)\otimes 1)g_\mu(X)(g_\mu(Q)^*\otimes 1)$$

Comme V'' et $g_\mu(Q)\otimes 1_H$ commutent, on a donc $V = (g_\mu(Q)\otimes 1_H)W(g_\mu(Q)^*\otimes 1_H)$.

b) En s'appuyant sur la formule :

$$W(e_{abc}\otimes e_{xyz})=\sum_n A(c-a-b+y,n)e_{a+n,b-n,c-n}\otimes e_{x-n,y+c-a-n,z+c-a-n}$$

il est facile de voir que l'adhérence normique E de l'espace vectoriel $\{\ (id\otimes\omega)(\Sigma W)\ /\ \omega\in\mathcal{L}(H)_*\}$ coincide avec celle de $\{\ T_{xyz;x'y'z'}\ /\ (x,y,z)\in\mathbf{Z}^3,(x',y',z')\in\mathbf{Z}^3\}$ où chaque opérateur $T_{xyz;x'y'z'}$ est défini par $T_{xyz;x'y'z'}\,e_{abc}=\delta_{x,y}^{a+c,b-c}A(z-c,-y')e_{a+x',b+y',c+z'}$. Par (A.4 b)), on a $\mathcal{K}(H)\subset E$ donc $\mathcal{K}(H)\subset\overline{\mathcal{C}(V)}$. Comme $\lim\limits_{a\to-\infty}A(a,0)=1$, l'inclusion est stricte.

c) Par le a), il est facile de voir que $S_V=\overline{lin}\ \{\ L_{abc}\ /\ (a,b,c)\in\mathbf{Z}^3\}$ où chaque opérateur L_{abc} est défini par $L_{abc}\,e_{xyz}=A(y+a,b)e_{x-b,y+c,z+c}$. Par (A.4 b)), on a donc $S_V\subset L(A)$. D'autrepart, par (A.4 b)) et le théorème de Stone-Weierstrass, l'espace vectoriel engendré par les produits finis de fonctions sur $\mathbf{C}_\mu$ de la forme $g(\mu^x z)=f(x)z^j$ et $g(0)=0$ où $f\in C_0(\mathbf{Z})$ et $j\in\mathbf{Z}$, et $g(\mu^x z)=A(x,0)$ et $g(0)=1$, est dense dans $C_0(\mathbf{C}_\mu)$. Il en résulte que $L(A)\subset S_V$. ∎

Soit U l'unitaire dans H défini par $Ue_{abc}=(-1)^a e_{a,b-c,-c}$. Nous avons :

4.8. PROPOSITION. — *(V,U) est un unitaire multiplicatif irréductible et semi-birégulier.*

Démonstration. Il est clair que $U=U^*$ et que $U^2=1$. Montrons que l'unitaire $\widehat{V}=\Sigma(U\otimes 1)V(U\otimes 1)\Sigma$ est multiplicatif.

Soit V' la co-isométrie dans $H_\Phi\otimes H_\Phi$ définie pour tout $x,y\in\mathfrak{N}_\Phi$ par $V'^*(\Lambda_\Phi x\otimes\Lambda_\Phi y)=\Lambda_{\Phi\otimes\Phi}(\delta(y)(x\otimes 1))$. Par la coassociativité de δ, V' est multiplicative. A travers l'identification $H=H_\Phi$, on trouve :

$$\widehat{V}(e_{abc}\otimes e_{xyz})=\sum_{l,n}\mu^{y+l-n}A(b-x-y-l,n-l-y)A(b-y,n-y)$$
$$(e_{a-l,b-x-z-l,c-x-z-l,}\otimes e_{x+l,n,z+l})$$

Utilisant la formule précédente et (4.1b)), on vérifie directement que $V'=\widehat{V}$. Procédant comme dans (4.7b)), on montre que $\widehat{V}$ est semi-régulier, donc V est semi-birégulier.

Il reste à montrer l'égalité $\widehat{V}V\widetilde{V}=(U\otimes 1)\Sigma$ où $\widetilde{V}=\Sigma(1\otimes U)V(1\otimes U)\Sigma$.

Utilisant (A.2 et A.3), on trouve :

$$V\widetilde{V}(e_{abc}\otimes e_{xyz})=\sum_{k,n}(-1)^{n+\alpha}A(\alpha+x,-n)A(a+n,\alpha+n+k)$$
$$(e_{k,a+b-\alpha-k,a+b-\alpha-y-k}\otimes e_{a+x-k,b-\alpha-n-k,c+x-k})$$

où on a posé $\alpha = b - c + z - y - x$. En composant avec $\widehat{V}$ et en utilisant (A.2), on obtient :

$$\widehat{V}V\widetilde{V}(e_{abc} \otimes e_{xyz}) = \sum_{l,n} (-1)^l (-\mu)^{-\alpha-n+l} A(\alpha + x, -n) A(-x + n - l, \alpha + n - l)$$
$$(e_{-l,-x+y-z-l,-x-z-l} \otimes e_{a+x+l,b,c+x+l})$$

Par (A.2 et A.3), on a $\sum_n (-\mu)^{-\alpha-n+l} A(\alpha+x,-n)A(-x+n-l,\alpha+n-l) = \delta_0^{l+x}$, d'où finalement $\widehat{V}V\widetilde{V}(e_{abc} \otimes e_{xyz}) = (-1)^x \, e_{x,y-z,-z} \otimes e_{abc}$. ∎

Comme conséquence de (3.10), V est un multiplicateur de la C*-algèbre $\widehat{S} \otimes S$. Nous allons préciser cette propriété.

Notons α' l'action de S^1 dans $\mathbf{C}_{\sqrt{\mu}} = \{z \in \mathbf{C} \;/\; \mid z \mid^2 \in \mu^{\mathbf{Z}}\} \cup \{0\}$ définie pour $f \in C_0(\mathbf{C}_{\sqrt{\mu}})$ par $\alpha'_z(f)(\zeta) = f(z^{-2}\zeta)$ et posons $B' = C_0(\mathbf{C}_{\sqrt{\mu}}) \rtimes_{\alpha'} S^1$. Soit $z \to u_z$ le plongement de S^1 dans $M(B')$; pour tout $\lambda \in \mathbf{C}\backslash\{0\}$, on obtient par prolongement analytique un multiplicateur non borné u_λ de B'. Notons $\mathbf{b}'$ la fonction $\zeta \to \zeta$ sur $\mathbf{C}_{\sqrt{\mu}}$; avec ces notations nous avons :

4.9. PROPOSITION. — *Il existe un unique homomorphisme ρ de B' sur $\widehat{S}$ vérifiant $\rho(u_\mu) = n_0 \otimes 1 \otimes n_0^{-1}$ et $\rho(\mathbf{b}')$ est l'opérateur normal fermeture de l'opérateur $(v_0^{-1} n_0^{-\frac{1}{2}} \otimes v_0 n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}}) - (v_0^{-1} n_0^{\frac{1}{2}} \otimes n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}})$ dans H.*

Démonstration. Montrons d'abord que $\rho(\mathbf{b}')$ est bien défini. Soit $X' = v_0^{-1} n_0^{-\frac{1}{2}} \otimes v_0 n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}}$ et $Y' = -v_0^{-1} n_0^{\frac{1}{2}} \otimes n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}}$. Il est facile de voir que les opérateurs normaux X' et Y' dans H, vérifient les conditions de ([28] thm.2.3), donc la fermeture Z' de l'opérateur $(v_0^{-1} n_0^{-\frac{1}{2}} \otimes v_0 n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}}) - (v_0^{-1} n_0^{\frac{1}{2}} \otimes n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}})$ est un opérateur normal. De plus, comme la fermeture de l'opérateur $\mu^{-1} X'^{-1} Y'$ est l'opérateur $Q = -(n_0 \otimes v_0 \otimes 1)$, par ([28] thm.2.3) nous avons $g_\mu(Q)^* Z' g_\mu(Q) = X'$. Nous en déduisons que la représentation ρ de B' dans H est bien définie et unique.

Pour voir que la C*-algèbre $\widehat{S}$ est l'image de ρ, introduisons l'opérateur unitaire T dans H définie par $T e_{abc} = e_{a+b+c,b,b+c}$. Il est facile de voir que $T^* X' T = v_0^{-2} n_0^{-\frac{1}{2}} \otimes v_0 n_0^{-1} \otimes 1$ et $T^* g_\mu(Q)^* (n_0 \otimes 1 \otimes n_0^{-1}) g_\mu(Q) T = T^*(n_0 \otimes 1 \otimes n_0^{-1}) T = n_0 \otimes 1 \otimes 1$. Nous avons $\widehat{S} = \overline{lin}\,\{\rho_V(\omega_{ijk}) \;/\; (i,j,k) \in \mathbf{Z}^3\}$. Utilisant (4.7), on vérifie directement que les représentations $B' \to \mathcal{L}(H) : x \to T^* g_\mu(Q)^* \rho(x) g_\mu(Q) T$ et $\widehat{S} \to \mathcal{L}(H) : x \to T^* g_\mu(Q)^* x g_\mu(Q) T$ ont même image, d'où le résultat. ∎

Dans toute la suite, posons $\mathbf{b} = \rho(\mathbf{b}')$ et $\mathbf{x} = \rho(u_\mu^{-1})$. Il résulte de (4.9) que les opérateurs $\mathbf{b}$ et $\mathbf{x}$ sont affiliés à $\widehat{S}$. Notons également h la fonction sur $\mathbf{C}_\mu \backslash \{0\}$ définie par $h(\mu^m z) = z^{-m}$.

4.10. THÉORÈME. — *Nous avons $V = g_\mu(\mathbf{b}^* \mathbf{x}^{-\frac{1}{2}} \otimes \mathbf{v}^* \mathbf{n}^*) h(\mathbf{x}^{-1} \otimes \mathbf{v})$.*

Démonstration. Il suffit de remarquer qu'avec les notations de (4.7), la fermeture de l'opérateur $\mathbf{b}^*\mathbf{x}^{-\frac{1}{2}} \otimes \mathbf{v}^*\mathbf{n}^*$ coincide avec celle de $X+Y$. Par ([28] thm.2.3), nous avons :

$$g_\mu(\mathbf{b}^*\mathbf{x}^{-\frac{1}{2}} \otimes \mathbf{v}^*\mathbf{n}^*) = g_\mu(X)g_\mu(Y) = VV''^*$$

D'autre part, il est clair que $V'' = h(\mathbf{x}^{-1} \otimes \mathbf{v})$. ∎

Notons β la représentation de $\widehat{S}$ dans $\mathcal{L}(l^2(\mathbf{Z}^2))$, apparue dans la preuve de (4.9), donnée par $x \rightarrow T^*g_\mu(Q)^*xg_\mu(Q)T$ où T désigne toujours l'unitaire de $\mathcal{L}(H)$ défini par $Te_{abc} = e_{a+b+c,b,b+c}$.

4.11. COROLLAIRE. — *La C^*-algèbre de Hopf $\widehat{S}_V$ munie du coproduit opposé $\widehat{\delta}^0 = \sigma \circ \widehat{\delta}$ est isomorphe à la C^*-algèbre de Hopf des "fonctions continues tendant vers 0 à l'infini" sur le dual de Pontrjagyn de $E_\mu(2)$ (au sens de Woronowicz).*

Démonstration. Soit $(B, \widehat{\Phi})$ la C*-algèbre de Hopf des "fonctions continues tendant vers 0 à l'infini" sur le dual de Pontryagin de $E_\mu(2)$, voir [26]. Il résulte de (4.9) que la représentation β définit un isomorphisme de C*-algèbres de $\widehat{S}$ sur B. Pour montrer que β est un morphisme de C*-algèbres de Hopf, introduisons la représentation γ de S dans $\mathcal{L}(l^2(\mathbf{Z}^2))$ définie par $\gamma(\mathbf{v}) = v_0^{-1} \otimes 1$ et $\gamma(\mathbf{n}) = n_0^{-1} \otimes v_0$. Alors par (4.10), $(\beta \otimes \gamma)(V)$ est la représentation fondamentale $\mathcal{W}$ donnée par Woronowicz. Il en résulte que β est un isomorphisme de C*-algèbres de Hopf. ∎

4.12. Remarque. — S'appuyant sur (4.7a)) et (A.2 et A.3), on peut voir que $\widehat{\delta}(\mathbf{b}) = V^*(1 \otimes \mathbf{b})V$ est l'opérateur normal ([28] thm.2.3) fermeture de l'opérateur $\mathbf{b} \otimes \mathbf{x}^{\frac{1}{2}} + \mathbf{x}^{-\frac{1}{2}} \otimes \mathbf{b}$. Ce résultat est également une conséquence du corollaire précédent et de [26]. D'autrepart, il est facile de voir qu'on a $\widehat{\delta}(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$.

b) Représentations de $S \rtimes \widehat{S}$

Nous allons montrer que la C*-algèbre $S \rtimes \widehat{S}$ est une extension de la C*-algèbre $\mathcal{K}$ des opérateurs compacts par $\mathcal{K}$, donc n'admet que deux représentations. Il en résulte que (3.24) l'unitaire multiplicatif V n'admet que deux représentations covariantes : la représentation régulière (V, V) et une deuxième (X, Y), dont l'unitaire multiplicatif correspondant est non régulier et non irréductible, que nous expliciterons.

Commençons par décrire la C*-algèbre $S \rtimes \widehat{S}$.

Par (3.24), nous savons que la C*-algèbre $S \rtimes \widehat{S}$ s'identifie à l'espace vectoriel fermé engendré par les opérateurs $\{ L(\omega)\rho(\omega') \ / \ \omega, \omega' \in \mathcal{L}(H)_* \}$, nous en déduisons un plongement canonique de la C*-algèbre S (resp. $\widehat{S}$) dans $M(S \rtimes \widehat{S})$. En particulier les opérateurs $L(\mathbf{v}), L(\mathbf{n}), \mathbf{b}, \mathbf{x}$ sont ([2], [25]) affiliés à $S \rtimes \widehat{S}$. Notons simplement dans cette section $\mathbf{v}$ (resp. $\mathbf{n}$), l'opérateur $L(\mathbf{v})$ (resp. $L(\mathbf{n})$), posons $\mathbf{u} = \text{Phase}\,\mathbf{n}$, $\mathbf{w} = \mathbf{u}\,\mathbf{v}$ et soit $\mathbf{y}$ l'opérateur normal fermeture ([28] thm.2.3) de l'opérateur $\mathbf{b} - \mathbf{v}\mathbf{n}^{-1}\mathbf{x}^{\frac{1}{2}}$; nous verrons plus loin que les opérateurs $\mathbf{u}$, $\mathbf{y}$ et $\mathbf{w}$ sont également affiliés à la C*-algèbre $S \rtimes \widehat{S}$.

Nous avons :

4.13. THÉORÈME. — *La C^*-algèbre $S \rtimes \widehat{S}$ est une extension de la C^*-algèbre $\mathcal{K}$ des opérateurs compacts par $\mathcal{K}$.*

Par (3.23 et 3.24 b)), il résulte de l'irréductibilité de V et de (4.7b)) que la C*-algèbre $S \rtimes \widehat{S}$ contient $\mathcal{K}$; pour montrer que le quotient de $S \rtimes \widehat{S}$ par $\mathcal{K}$ s'identifie également à $\mathcal{K}$, nous allons introduire une C*-algèbre D (qui est clairement une extension de $\mathcal{K}$ par $\mathcal{K}$), et montrer que l'extension D de $\mathcal{K}$ par $\mathcal{K}$ est canoniquement isomorphe à celle définie par $S \rtimes \widehat{S}$ et l'idéal $\mathcal{K} \subset S \rtimes \widehat{S}$.

Pour toute fonction $f \in C_0(\mathbf{Z} \times \mathbf{Z} \times (\mathbf{Z} \cup \{-\infty\}))$, notons T_f l'opérateur dans H défini par $T_f\, e_{abc} = f(a-c, b, c)\, e_{abc}$. Nous avons :

4.14. PROPOSITION. — $S \rtimes \widehat{S} = \overline{lin}\, (\mathcal{K} + \{\mathbf{u}^i\, \mathbf{v}^j\, T_f \;/\; f \in C_0(\mathbf{Z} \times \mathbf{Z} \times (\mathbf{Z} \cup \{-\infty\})), (i,j) \in \mathbf{Z}^2\})$.

Démonstration. Posons $E = \overline{lin}\, (\mathcal{K} + \{\mathbf{u}^i\, \mathbf{v}^j\, T_f\ \})$ et montrons que $E \subset S \rtimes \widehat{S}$. Ecrivant que les opérateurs $\mathbf{v}^x f_{y,z}(\mathbf{n})\, \rho_{ijk}\, \mathbf{v}^{x'} f_{y',z'}(\mathbf{n})$ appartiennent à $S \rtimes \widehat{S}$ et utilisant (A.2), on obtient que les opérateurs T de la forme :

$$T\, e_{abc} = \delta^x_{a-c}\, \delta^y_b\, A(c+z, y'-y)\, e_{a+x'y'c}$$

où $x, x'y, y'$ et z sont des entiers arbitraires, appartiennent à $S \rtimes \widehat{S}$. Par (A.4 b)), il est clair que $E \subset S \rtimes \widehat{S}$.

Pour montrer l'autre inclusion, nous avons besoin du lemme suivant :

4.15. LEMME. — *Pour tout $(x,y,z) \in \mathbf{Z}^3$ et tout $n \in \mathbf{Z}$, posons $\Phi_n(b,c) = A(n+x, b+x)A(b-c+y, n-c+z)$. Nous avons :*

$$\sum_b \Phi_n(b,c)^2 \leq \mu^{2(x+z-y)}\, \mu^{2n}$$

Démonstration. Par (A.2 et A.3), nous avons :

$$\begin{aligned}
\sum_b \Phi_n(b,c)^2 &= \sum_b \mu^{2(b+x)}\, A(n-b, -b-x)^2\, A(b-c+y, n-c+z)^2 \\
&\leq \sum_b \mu^{2(b+x)}\, A(b-c+y, n-c+z)^2 \\
&= \mu^{2(c+x-y)} \sum_b \mu^{2b}\, A(b, n-c+z)^2 \\
&= \mu^{2(x+z-y)}\, \mu^{2n}
\end{aligned}$$

Fin de la démonstration de la proposition . Par (4.7a)), la C*-algèbre $S \rtimes \widehat{S}$ est l'espace vectoriel fermé engendré par les opérateurs $L_{xyz}\, \rho_V(\omega_{ijk})$, où L_{xyz} (resp. $\rho_V(\omega_{ijk})$) est l'opérateur défini par $L_{xyz}\, e_{abc} = A(x+b,y)\, e_{a-y,b+z,c+z}$ (resp. $\rho_V(\omega_{ijk})\, e_{abc} = (-1)^k\, \delta_{a-c}^{k-i} \sum_n (-1)^{n-b} A(n-i-j, b-i-j) A(b-c-j, n-c-j-k)\, e_{a-k,n,c+k}$).

Pour montrer qu'un opérateur T de la forme $T = L_{xyz}\, \rho_{ijk}$ appartient à E, il suffit de montrer, grâce à (A.2) et au lemme précédent, que pour chaque $n \in \mathbf{Z}$, on a $T_n \in E$, où T_n est un opérateur de la forme $T_n\, e_{abc} = \delta_{a-c}^{x}\, \delta_b^y\, A(c+j, n-k-y)\, e_{a-k,n,c+k}$. Or, si $y = n-k$, T_n est clairement un opérateur de E ; si par contre $y \neq n-k$, par (A.4 b)) T_n est compact.

∎

Soit α l'action du groupe $S^1 \times \mathbf{Z}$ sur l'espace $\mathbf{C}_{\sqrt{\mu}} \times \mathbf{Z}$ définie par $\alpha_{z,n}(g)(\zeta,p) = g(z^{-2}\zeta, p-n)$ où $g \in C_0(\mathbf{C}_{\sqrt{\mu}} \times \mathbf{Z})$. Posons $C = C_0(\mathbf{C}_{\sqrt{\mu}} \times \mathbf{Z}) \rtimes (S^1 \times \mathbf{Z})$ et soit $(z,n) \to u_{z,n}$ le plongement de $S^1 \times \mathbf{Z}$ dans $M(C)$; notons $\mathbf{y}'$ (resp. $\mathbf{p}$) la fonction $(\zeta,p) \to \zeta$ (resp. $(\zeta,p) \to \mu^p$) sur $\mathbf{C}_{\sqrt{\mu}} \times \mathbf{Z}$. Munissons la C*-algèbre C de l'action du groupe $\mathbf{Z}$ donnée par l'automorphisme γ de C défini par :

$$\gamma(g)(\zeta,p) = g(\mu^{-\frac{1}{2}}\, \zeta, p+1)\ ,\ g \in C_0(\mathbf{C}_{\sqrt{\mu}} \times \mathbf{Z})\ ;\ \gamma(u_{z,n}) = z^{-1}\, u_{z,n}$$

Posons finalement $D = C \rtimes \mathbf{Z}$. Nous avons :

4.16. LEMME. — *D est une extension de $\mathcal{K}$ par $\mathcal{K}$.*

Démonstration. Soit J l'idéal de D défini par $\mathbf{C}_{\sqrt{\mu}} \setminus \{0\}$. Comme $\mathbf{C}_{\sqrt{\mu}} \setminus \{0\}$ s'identifie à $\mathbf{Z} \times S^1$, l'idéal J est isomorphe à $\mathcal{K}$ ainsi que le quotient de D par cet idéal.

∎

démonstration du théorème

Il suffit de construire un homomorphisme surjectif $\pi : D \to S \rtimes \widehat{S}$ de C*-algèbres tel que $\pi(J) = \mathcal{K}$ où J est l'idéal de D défini par $\mathbf{C}_{\sqrt{\mu}} \setminus \{0\}$.

Soit π la représentation de C dans H définie par $\pi(\mathbf{y}') = -v_0^{-1} n_0^{\frac{1}{2}} \otimes n_0^{-1} \otimes v_0 n_0^{\frac{1}{2}}$, $\pi(\mathbf{p}) = \mid \mathbf{n} \mid = 1 \otimes n_0 \otimes 1$ et $\pi(u_\mu, 1) = \mathbf{x}^{-1}\mathbf{w} = \mathbf{w}\mathbf{x}^{-1}$. Pour tout $x \in C$, on a $\pi(\gamma(x)) = \mathbf{v}^*\, \pi(x)\, \mathbf{v}$; nous en déduisons une représentation de D dans H notée toujours π. La description de la C*-algèbre $S \rtimes \widehat{S}$ donnée par (4.14) permet de voir sans peine que $\pi(D) = S \rtimes \widehat{S}$ et que $\pi(J) = \mathcal{K}$.

∎

4.17. COROLLAIRE. — *L'opérateur normal $\mathbf{y}$ est affilié à la C*-algèbre $S \rtimes \widehat{S}$.*

Démonstration. L'opérateur $\mathbf{y}'$ est affilié à la C*-algèbre D et on a $\pi(\mathbf{y}') = \mathbf{y}$.

∎

Explicitons pour terminer la représentation covariante de V correspondant à l'homo-
morphisme quotient $p : S \rtimes \widehat{S} \to S \rtimes \widehat{S}/\mathcal{K} \simeq \mathcal{K}$.

Soit θ (resp. $\widehat{\theta}$), la représentation de S (resp. $\widehat{S}$) dans $l^2(\mathbf{Z}^2)$ définie par $\theta(\mathbf{v}) = 1 \otimes v_0$ et $\theta(\mathbf{n}) = v_0 \otimes n_0$ (resp. $\widehat{\theta}(\mathbf{b}) = v_0^{-1} n_0^{-\frac{1}{2}} \otimes v_0 n_0^{-\frac{1}{2}}$ et $\widehat{\theta}(\mathbf{x}) = n_0^{-1} \otimes n_0$). Posons $X = (\widehat{\theta} \otimes id)(V)$, $Y = (id \otimes \theta)(V)$ et $W' = (\widehat{\theta} \otimes \theta)(V)$.

4.18. PROPOSITION. — *L'homorphisme quotient p est la représentation de $S \rtimes \widehat{S}$ correspondant à la représentation covariante (X, Y) de V. De plus l'unitaire multiplicatif W' associé est non régulier et non irréductible.*

Démonstration. Il est clair que X (resp. Y) est une représentation (resp. coreprésenta--tion) de V. Par un calcul directe utilisant (4.7) et (A.2 et A.3), on obtient la relation de covariance $Y_{12} V_{13} X_{23} = X_{23} Y_{12}$. Nous avons :

$$W'(e_{ab} \otimes e_{xy}) = \sum_n A(y-a, n) e_{a+n, b-n} \otimes e_{x-n, y+b-a-n}$$

S'appuyant sur la formule précédente, il est facile de voir que W' est non régulier et que le commutant de $S_{W'}$ est commutatif, donc W' est non irréductible. Soit p' la représentation (3.24) de $S \rtimes \widehat{S}$ dans $l^2(\mathbf{Z}^2)$ correspondant à cette représentation covariante ; on a $p'(\mathbf{y}) = 0$, donc $p'(S \rtimes \widehat{S}) = \mathcal{K}(l^2(\mathbf{Z}^2)$ et p' est unitairement équivalente à p.

■

c) Mesures de Haar duales et théorie modulaire

Pour expliciter les mesures de Haar duales, introduisons la famille (ω^{xyz}) de formes continues sur $\widehat{S_V}$ définie par

$$\omega^{xyz}(\rho_V(\omega_{ijk})) = (-\mu)^{-k}\ \delta^{x,z}_{k-i,k}\ A(i+j-y, k) = (f_{-z,y+z,z-x} \mid \rho_V(\omega_{ijk}) f_{0,y,-x})$$

où $f_{xyz} = g_\mu(Q) e_{xyz}$.

4.19. PROPOSITION. — *Nous avons :*
a) $\omega_{f_{x'y'z'}, f_{xyz}} = \delta^{z-x'}_{y'-y}\ \delta^{z'-z}_{y'-y}\ \omega^{x-z, y, x-x'}$ sur $\widehat{S_V}$.

b) $\omega^{xyz} * \omega^{x'y'z'} = \sum_n A(y'-x-y, n) A(y'-x-y+z'+z, n+z)\, \omega^{x+x', y'-x-n, z+z'}$

Démonstration. Nous avons $\widehat{S_V} = \overline{lin}\{\ \rho_V(\omega_{ijk} / (i,j,k) \in \mathbf{Z}^3\ \}$. Utilisant (4.7a)), il est facile de vérifier l'égalité a) pour tout $x = \rho_V(\omega_{ijk})$.

b) Pour tout i, j, k, posons $T = \rho_V(\omega_{ijk})$; nous avons :

$$\begin{aligned}
\omega^{xyz} * \omega^{x'y'z'}(T) &= (\omega^{xyz} \otimes \omega^{x'y'z'} \otimes \omega_{ijk})(V_{12}^* V_{23} V_{12}) \\
&= (\omega^{xyz} \otimes \omega^{x'y'z'} \otimes \omega_{ijk})(V_{13} V_{23}) \\
&= (\omega_{e_{-z,y+z,z-x}; e_{0,y,-x}} \otimes \omega_{e_{-z',y'+z',z'-x'}; e_{0,y',-x'}} \otimes \omega_{ijk})(W_{13} W_{23}) \\
&= \sum_{n,m} A(j-x'-y', n) A(j-x'-x-y-n, m) \delta_n^{-z'} \delta_m^{-z} \delta_{n+m}^{-k} \delta_{x+x'}^{k-i} \\
&= \delta_{x+x'}^{k-i}\ \delta_{z+z'}^{k}\ A(j-x'-y', -z')\, A(j-x'-x-y+z', -z)
\end{aligned}$$

Utilisant (A.2), on obtient $A(j-x'-y',-z') = A(x'+y'-j,x'+y'-z'-j)$ et $A(j-x-x'-y+z',-z) = A(x+x'+y-z'-j,-z+x+x'+y-z'-j)$. Nous en déduisons :

$$\omega^{xyz}*\omega^{x'y'z'}(T) = \delta^{k-i}_{x+x'}\,\delta^{k}_{z+z'}\,A(z'+y'+z-x-i-j,y'+z-x-i-j)A(z+y-i-j,y-i-j)$$

Posons $\xi = y'-x-y+z+z'$, $\alpha = z+z'$, $\eta = y'-x-y$, $\beta = z'$, $\gamma = x+z'-y'+i+j$, on obtient grace à (A.4 a)) :

$$\begin{aligned}\omega^{xyz}*\omega^{x'y'z'}(T) &= \delta^{k-i}_{x+x'}\,\delta^{k}_{z+z'}\,A(\xi+\beta-\gamma-\eta,\alpha-\gamma)A(\alpha-\gamma-\eta,\beta-\gamma-\eta)\\ &= \delta^{k-i}_{x+x'}\,\delta^{k}_{z+z'}\,(-\mu)^{-\gamma}\sum_n(-\mu)^{-n}A(\xi,n+\alpha)A(\eta,n+\beta)A(\xi-\eta,n+\gamma)\\ &= \delta^{k-i}_{x+x'}\,\delta^{k}_{z+z'}\,(-\mu)^{-k}\sum_n A(\xi,n+\alpha)A(\eta,n+\beta)A(n+\gamma,\xi-\eta)\ (\text{ A.2 a}))\\ &= \sum_n A(y'-x-y,n)A(y'+z'-x-y+z,n+z)\omega^{x+x',y'-x-n,z+z'}(T)\end{aligned}$$

■

Posons $\tau = \sum_{a,b}\mu^{-a-2b}\,\omega^{a,b,0}$ et avec les notations de (4.9), $\widehat{F} = \rho(u_\mu)$.

4.20. PROPOSITION. — *Avec les notations précédentes, nous avons :*

a) *τ est une trace s.c.i, semi-finie normiquement sur $\widehat{S}_V$ et fidèle sur le bicommutant de $\widehat{S}_V$.*

b) *$\widehat{\Phi} = \tau(\widehat{F}^{-1}.)$ et $\Psi = \tau(\widehat{F}.)$ sont des poids s.c.i, semi-finis normiquement sur $\widehat{S}_V$ et fidèles sur le bicommutant de $\widehat{S}_V$.*

c) *La formule $\Lambda_{\widehat{\Phi}}\,\rho_V(\omega_{abc}) = (-1)^c\,\mu^{-b-c}\,e_{-c,a+b,a}$ identifie l'espace de Hilbert $H_{\widehat{\Phi}}$ de la représentation GNS de $\widehat{\Phi}$ à $H = l^2(\mathbf{Z}^3)$ et la représentation $\pi_{\widehat{\Phi}}$ correspondante à ρ_V.*

d) *L'opérateur modulaire $\Delta_{\widehat{\Phi}}$ et l'opérateur $J_{\widehat{\Phi}}$ canoniquement associés à $\widehat{\Phi}$, sont respectivement donnés par $\Delta_{\widehat{\Phi}}\,e_{abc} = \mu^{-2a}\,e_{abc}$ et $J_{\widehat{\Phi}}\,e_{abc} = (-1)^a\,e_{-a,b,c}$.*

e) *$\rho_V(\widehat{F})$ est affilié au centralisateur du poids $\widehat{\Phi}$ et on a $\widehat{\Phi}(\widehat{F}^2.) = \widehat{\Psi}$.*

Démonstration. a) Par un calcul directe, on voit que pour $x = \rho_V(\omega_{ijk})$, on a $x \in \mathcal{N}_\tau \cap \mathcal{N}_\tau^*$ et que les formes $x\tau$ et τx se prolongent de façon unique au bicommutant de $\widehat{S}$. Il en résulte que τ est semi-finie. Utilisant l'inclusion $U\widehat{S}U \subset \widehat{S}'$ donnée par ([4] 6.5 c)), on voit facilement que τ est fidèle sur l'algèbre de von Neumann S_V''. Comme pour tout $x = \rho_V(\omega_{ijk})$ et tout $y = \rho_V(\omega_{i'j'k'})$, on a $\tau(xy) = \tau(yx)$, τ est une trace sur S_V''.

b) Par (4.9), on sait que $\widehat{F} = \rho(u_\mu)$ est affilié à la C*-algèbre $\widehat{S}$; le b) résulte alors du a) et de [19].

c) Résulte de (4.3), $\rho_V(\omega_{abc})^* = (-\mu)^{-c}\rho_V(\omega_{a,b,-c})$ et d'un calcul directe.

d) et e) résultent du a) et du b) ; voir [22].

■

Il résulte facilement de (4.19 b)) que pour tout x, y on a $\widehat{\Psi} * \omega^{x,y,0} = \widehat{\Psi}$ (resp. $\omega^{x,y,0} * \widehat{\Phi} = \widehat{\Phi}$) ; en particulier on a $\widehat{\Psi} * \omega^{0,0,0} = \widehat{\Psi}$ (resp. $\omega^{0,0,0} * \widehat{\Phi} = \widehat{\Phi}$). Procédant comme dans (4.2), on montre :

4.21. THÉORÈME. — *Le poids $\widehat{\Phi}$ (resp. $\widehat{\Psi}$ est une mesure de Haar à gauche (resp. à droite) sur la C*-algèbre de Hopf $\widehat{S}_V$.*

Nous avons muni chacune des C*-algèbres de Hopf S_V et $\widehat{S_V}$ canoniquement associées à l'unitaire multiplicatif irréductible (V, U), de mesures de Haar : S_V admet une mesure de Haar Φ à gauche et à droite ; $\widehat{S_V}$ admet une mesure de Haar $\widehat{\Phi}$ à gauche et une mesure de Haar $\widehat{\Psi}$ à droite, cette dernière étant un poids de densité $\widehat{F}^2$ relativement à $\widehat{\Phi}$. D'autrepart, le carré de l'antipode κ (resp. $\widehat{\kappa}$) engendre un groupe à un paramètre d'automorphismes $(\kappa^2)^{it}$ (resp. $(\widehat{\kappa}^2)^{it}$) sur S_V (resp. $\widehat{S_V}$) implémenté par l'opérateur autoadjoint positif K (resp. $\widehat{K}$) donné par $K = \lambda(\widehat{F})\rho(\widehat{F})$ (resp. $\widehat{K} = K^{-1}$).

Il résulte alors de (4.1) et (4.20) qu'on a (cf. [21]) :

4.22. PROPOSITION. — *a)* $U = J_\Phi J_{\widehat{\Phi}} = J_{\widehat{\Phi}} J_\Phi$

b) les opérateurs modulaires des poids $\widehat{\Phi}$, Φ et $\widehat{\Psi}$ sont donnés par :

$$\Delta_\Phi = \lambda(\widehat{F})\rho(\widehat{F}^{-1}) \, , \; \Delta_{\widehat{\Phi}} = \lambda(\widehat{F}^{-1})\rho(\widehat{F}^{-1}) \, , \; \Delta_\Psi = \lambda(\widehat{F})\rho(\widehat{F}).$$

c) $\delta \circ \sigma_t^\Phi = ((\kappa^2)^{it} \otimes \sigma_t^\Phi) \circ \delta = (\sigma_t^\Phi \otimes (\kappa^2)^{-it}) \circ \delta$ *sur* S_V.

d) $\widehat{\delta} \circ \sigma_t^{\widehat{\Phi}} = ((\widehat{\kappa}^2)^{it} \otimes \sigma_t^{\widehat{\Phi}}) \circ \widehat{\delta} \, , \; \widehat{\delta} \circ \sigma_t^{\widehat{\Psi}} = (\sigma_t^{\widehat{\Psi}} \otimes (\widehat{\kappa}^2)^{-it}) \circ \widehat{\delta}$ *sur* $\widehat{S_V}$.

e) $V \sigma_t^{\widehat{\Phi} \otimes \Phi}(V^*) = \rho(\widehat{F}^{2it}) \otimes 1$.

Procédant comme dans [4], on peut construire le double quantique de tout unitaire multiplicatif irréductible (V, U) semi-birégulier : on obtient un unitaire multiplicatif irréductible semi-birégulier.

Dans le cas du groupe quantique $E_\mu(2)$, l'algèbre réduite $S \otimes \widehat{S}$ (et l'algèbre réduite duale) admettent des mesures de Haar.

4.23. THÉORÈME. — *le poids $\Phi \otimes \widehat{\Phi}$ est une mesure de Haar à gauche et à droite pour le double quantique de $E_\mu(2)$.*

Démonstration. Soit $\tau : S \otimes \widehat{S} \to \widehat{S} \otimes S$ l'inversion donnée par $\tau(x \otimes y) = V(y \otimes x)V^*$, il suffit de montrer que $(\widehat{\Phi} \otimes \Phi) \circ \tau = \Phi \otimes \widehat{\Psi}$. Or par (4.22 e)), pour tout t on a $(D((\widehat{\Phi} \otimes \Phi) \circ \tau) : D(\Phi \otimes \widehat{\Psi}))_t = 1$, où $(D((\widehat{\Phi} \otimes \Phi) \circ \tau) : D(\Phi \otimes \widehat{\Psi}))_t$ désigne la dérivée de Radon-Nikodym [10] du poids $(\widehat{\Phi} \otimes \Phi) \circ \tau$ relativement au poids $\Phi \otimes \widehat{\Psi}$ sur l'algèbre de von Neumann $S'' \otimes \widehat{S}''$.

∎

Appendice A

Dans cette première appendice, nous donnons les preuves des propriétés des coefficients $A(m,j)$ utilisées principalement dans le paragraphe 4.

Considérons l'espace vectoriel réel (de dimension 2) E des suites $(b(m,j))_{(m,j)\in\mathbf{Z}}$ à valeurs dans $\mathbf{R}$ vérifiant pour tout (m,j) :

$$(E.1) \qquad b(m-1,j)\,\mu^j = b(m,j) + b(m,j+1)\,\mu^{m-1}$$

$$(E.2) \qquad b(m,j) = b(m-1,j)\,\mu^{-j} + b(m-1,j-1)\,\mu^{m-j}$$

Nous avons :

A.1 Lemme - *Soit $(b(m,j))$ une suite de E.*
a) $b(m,j) = (-\mu)^j\,b(m-j,-j) = b(-m,j-m) = (-\mu)^{j-m}\,b(j,m)$

b) Les nombres $b_{i,j,k} = (-\mu)^{-j}\,b(i-k,j-k)$ sont invariants par permutation des entiers i,j,k.

Démonstration. Posons $u(m,j) = (-\mu)^j\,b(m-j,-j)$ (resp. $v(m,j) = b(-m,j-m)$). Il est facile de voir que les suites $u(m,j)$ et $v(m,j)$ appartiennent à E. Comme $u(m,0) = b(m,0)$ et $v(0,j) = b(0,j)$ pour tout $m,j\in\mathbf{Z}$, on a donc $u = b = v$; les autres assertions se déduisent immédiatement de ce résultat.

■

Dans le paragraphe 4, nous avons noté g_μ, la fonction introduite par Woronowicz dans [25], définie par :

$$g_\mu(\zeta) = \prod_{j=0}^{\infty} \frac{1+\mu^{-1-2j}\zeta}{1+\mu^{-1-2j}\bar{\zeta}}$$

La fonction g_μ est continue sur $\mathbf{C}_\mu$ et, pour tout $m\in\mathbf{Z}$, la fonction $z\to g_\mu(\mu^m z)$ définie sur S^1, se prolonge en une fonction méromorphe [25] dans $\mathbf{C}\backslash\{0\}$ dont les pôles sont les nombres $-\mu^{|m|-q}$ où $q = 1,3,5,\ldots.$ Nous noterons dans ce qui suit $g_\mu(m,.)$ ce prolongement ; dans [25], il est noté $U(m,.)$. Rappelons également qu' on a posé $A(m,j)$ pour le jième coefficient de Fourier de la restriction de la fonction g_μ au cercle $\{z\in C_\mu/\mid z\mid=\mu^m\}$.

A.2 Proposition - *Nous avons :*
a) La suite $(A(m,j))$ appartient à E.

b) $\lim_{m\to-\infty} A(m,j) = \delta_0^j$ *uniformément en j.*

c) Si $(b(m,j))$ est une suite bornée de E et vérifie $\lim_{m\to-\infty} b(m,0) = 1$, alors nous avons $b(m,j) = A(m,j)$ pour tout (m,j).

Démonstration. a) Pour z de module 1, on a $g_\mu(m,\bar{z}) = \overline{g_\mu(m,z)}$, on en déduit que $A(m,j)$ est réel. La relation (E.1) (resp. (E.2)) résulte de la relation $g_\mu(m-1,\mu z) = (1+\mu^{m-1}z^{-1})\,g_\mu(m,z)$ (resp. $g_\mu(m,z) = (1+\mu^{m-1}z)\,g_\mu(m-1,\mu^{-1}z)$, voir [25].

b) Résulte de $\lim_{m\to-\infty} g_\mu(\mu^m z) = 1$ uniformément en z de module 1.

c) Il résulte de a) et b), qu'il suffit de montrer qu' il existe une suite de E non bornée. Remarquons que toute suite $(b(m,j))$ de E est déterminée par la suite $(b(m,0))$. Posons alors $c_m = b(m,0)$. Il résulte de $(E.1)$ et de $(E.2)$ que la suite (c_m) vérifie la relation de récurrence :

$$c_{m+1} = -c_{m-1} + c_m(2 - \mu^{-2m})$$

Posons $\lambda = \mu^{-2}$ et soit $N \geq 1$ un entier tel que $\dfrac{\lambda^N}{(1-\lambda)^2} \leq 1$. Notons (x_m) la suite définie par $x_N = 0$, $x_{N+1} = 1$ et la relation de récurrence précédente. S'appuyant sur l'égalité :

$$x_m - x_{m-1} = x_1 - x_0 - \sum_{j=1}^{m-1} \lambda^j x_j$$

il est facile de montrer que la suite (x_m) est non bornée.

■

Nous rassemblons dans la proposition suivante quelques propriétés de sommabilité de la suite$(A(m,n))$.

A.3 Proposition - *Nous avons :*

a) Pour tout entiers $m, x \in \mathbf{Z}$, on a $\sum_n A(m,n+x)\,A(m,n) = \delta_0^x$.

b) Pour tout entiers $n, y \in \mathbf{Z}$, on a $\sum_m \mu^{2m}\,A(m+y,n)\,A(m,n) = \mu^{2n}\,\delta_0^y$

c) Pour tout entiers a_i, h_i $(i = 1,2)$, on a :

$$\sum_n A(a_1, n+h_1)\,A(a_2, n+h_2)\,A(a_1 - n, a_2 - n) = A(h_1,h_2)\,A(a_1+h_2, a_2+h_1).$$

Démonstration. a) Pour tout z de module 1, on a $|\,g_\mu(\mu^m z)\,| = 1$; le a) résulte alors de l'égalité de Parseval.

b) Par (A.2), on a $A(m,n) = (-\mu)^{n-m}\,A(n,m)$; le b) découle alors immédiatement du a).

c) Fixons h_1, h_2 et posons $\alpha(m,j) = \sum_n A(m,n+h_1)A(j,n+h_2)A(m-n,j-n)$ (resp. $\beta(m,j) = A(m+h_2, j+h_1)$). Il résulte de $(E.1)$ et de $(E.2)$ que les deux suites $(\alpha(m,j))$ et $(\beta(m,j))$ vérifient les deux relations de récurrence :

$$(E'.1) \qquad b(m-1,j)\,\mu^{j+h_1-h_2} = b(m,j)\,\mu^{-h_2} + b(m,j+1)\,\mu^{m-1}$$
$$(E'.2) \qquad b(m,j)\,\mu_1^h = b(m-1,j)\,\mu^{-j} + b(m-1,j-1)\,\mu^{m-j+h_2}$$

Par le a), il est clair que la suite $(\alpha(m,j))$ est bornée. Il résulte alors de (A.2 c)) que les deux suites $(\alpha(m,j))$ et $(\beta(m,j))$ sont proportionnelles. Posons alors $\alpha(m,j) = \lambda(h_1,h_2)\,\beta(m,j)$ où $\lambda(h_1,h_2) \in \mathbf{R}$. Or pour tout $m \in \mathbf{Z}$, on a $\sum_n |\,A(m,n)\,| < \infty$; en faisant $j = -h_1$, il découle alors de (A.2 a) et b)) qu' on a :

$$\lim_{m\to-\infty} \sum_n A(m,n+h_1)A(-h_1,n+h_2)A(m-n,-h_1-n) = A(-h_1,-h_1+h_2)$$
$$= A(h_1,h_2).$$

comme $\lim_{m\to-\infty} A(m+h_2,0)=1$, on a bien $\lambda(h_1,h_2)=A(h_1,h_2)$. ∎

A.4 Remarques -

a) Il résulte d'un calcul facile s'appuyant sur (A.2 a)) que la formule :

$$\sum_n A(a_1,n+h_1)\,A(a_2,n+h_2)\,A(a_1-n,a_2-n) = A(h_1,h_2)\,A(a_1+h_2,a_2+h_1).$$

est équivalente à la relation :

$$\sum_n (-\mu)^{-n} A(x,n+a)\;A(y,n+b)\;A(x-y,n+c)$$
$$= (-\mu)^c A(a-c-y,b-c-y)\;A(b-c-y+x,a-c)$$

b) Soit $n\in\mathbf{Z}$, posons $q_n(a)=A(a,n)$. Pour tout $j\in\mathbf{Z}$, notons τ_j l'opérateur de translation par j et soit $\mathcal{A}_n$ l'algèbre des polynômes sans termes constants en $\{\tau_j q_n \;/\; j\in\mathbf{Z}\}$. Si $n\neq 0$, il résulte facilement du théorème de Stone-Weierstrass que l'algèbre $\mathcal{A}_n$ est dense dans $C_0(\mathbf{Z})$. Dans le cas où $n=0$, on voit de même que $\mathcal{A}_n$ est dense dans $C_0(\mathbf{Z}\cup\{-\infty\})$.

Appendice B

Dans cette seconde appendice, nous complétons la preuve du théorème (4.2) et nous montrons l'unicité de la mesure de Haar de $E_\mu(2)$. Nous conservons les notations du paragraphe 4.

B.1 Lemme - *Soit f une forme linéaire positive sur S_V. Alors pour tout $\xi\in H$, il existe une famille (ξ_i) (resp. (ξ_i') dans H telle que $f*\omega_\xi=\sum_i\omega_{\xi_i}$ (resp. $\omega_\xi * f=\sum_i\omega_{\xi_i'}$) et $\|f\|\|\xi\|^2=\sum_i\|\xi_i\|^2$ (resp. $\|f\|\|\xi\|^2=\sum_i\|\xi_i'\|^2$.*

Démonstration. On a par (3.10), $\widehat{V}\in M(S_V\otimes\mathcal{K})$ et pour tout $x\in S_V$, on a aussi $\delta(x)=\widehat{V}^*(1\otimes x)\widehat{V}$. Notons alors (π_f,H_f,ξ_f) la représentation GNS de la forme positive f et posons $(\pi_f\otimes id)(\widehat{V})(\xi_f\otimes\xi)=\sum_i\eta_i\otimes\xi_i$, où (η_i) est une base orthonormée de l'espace de Hilbert H_f. On a alors pour tout $x\in S_V$:

$$\begin{aligned}(f*\omega_\xi)(x) &= (f\otimes\omega_\xi)(\delta(x))\\ &= <\xi_f\otimes\xi,(\pi_f\otimes id)(\widehat{V})^*(1\otimes x)(\pi_f\otimes id)(\widehat{V})>\\ &= \sum_i <\xi_i,x\xi_i>\end{aligned}$$

L'assertion resp. se démontre de la même façon en remarquant (3.10) que $V \in M(\mathcal{K} \otimes S_V)$ et (3.12) que $\delta(x) = V(x \otimes 1)V^*$ pour tout $x \in S_V$.

■

fin de la démonstration de (4.2). Soit f une forme positive sur $A = S_V$, posons $f * \omega_{000} = \sum\limits_i \omega_{\xi_i}$ avec $\|f\| = \sum_i \|\xi_i\|^2$. Par (4.6), on a alors :

$$\begin{aligned}\Phi(x*f) &= \Phi(x*f*\omega_{000}) \\ &= \sum_i \Phi(x*\omega_{\xi_i}) \\ &= \sum_i \|\xi_i\|^2 \Phi(x) = \|f\| \, \Phi(x)\end{aligned}$$

On montre de même que $\Phi(f*x) = \|f\| \, \Phi(x)$.

■

B.2 Remarque - : Soit ϕ un poids s.c.i normiquement sur A. Alors pour tout $\xi \in H$, il existe une famille de vecteurs (ξ_i) (resp. (η_i) de H telle que $\omega_\xi * \phi = \sum\limits_i \omega_{\xi_i}$ (resp. $\phi * \omega_\xi = \sum\limits_i \omega_{\eta_i}$. En effet, par [19], il existe une famille de formes positives sur A telle que $\phi = \sum\limits_i f_i$; il suffit alors d'appliquer B.1 aux formes $\omega_\xi * f_i$ et $f_i * \omega_\xi$.

B.3 Proposition - *Soit ϕ un poids s.c.i et semi-fini normiquement sur A invariant à gauche, i.e pour tout vecteur ξ de H on a $\omega_\xi * \phi = \|\xi\|^2 \, \phi$. Si ϕ est invariant par le groupe à un paramètre d'automorphismes $(\kappa^2)^{it}$, alors il exite $\lambda \geq 0$ tel que $\phi = \lambda \, \Phi$.*
Démonstration. Il résulte de la formule (4.22 c)) $\delta \circ \sigma_t^\Phi = (\sigma_t^\Phi \otimes (\kappa^2)^{-it}) \circ \delta$ et de l'invariance de ϕ par $(\kappa^2)^{it}$ qu' on a $\phi = \phi \circ \sigma_t^\Phi$ pour tout t. Il en résulte que $\phi = \phi \circ E$ où $E : A \to C_0(\mathbf{C}_\mu)$ est l'espérance conditionnelle associées à l'action duale donc, ϕ étant semi-fini, sa restriction à $C_0(\mathbf{C}_\mu)^+$ est une mesure positive qui le détermine complétement. Comme $(\kappa^2)^{it}(\mathbf{n}) = \mu^{2it}\,\mathbf{n}$ et que $\phi = \phi \circ (\kappa^2)^{it}$, la restriction de ϕ à chaque cercle $\{z \in \mathbf{C}_\mu \; / \; | z |= \mu^j\}$ est proportionnelle à sa mesure sa mesure de Haar. Soit $\alpha_j \in [0, \infty]$ le facteur de proportionnalité, posons $\beta_j = \alpha_j \, \mu^{-2j}$; nous allons montrer que la suite (β_j) est constante.

Par ([3] 2.), on a pour tout i, j entiers $f_{j0}(\mathbf{n}) * \omega_{0i0} = \sum\limits_n A(i-n, j-n)^2 \, f_{n0}(\mathbf{n})$.

Par invariance à gauche de ϕ, on en déduit :

$$\begin{aligned}\alpha_j &= \sum_n A(i-n, j-n)^2 \, \alpha_n \\ &= \sum_n A(i-j, n-j)^2 \, \mu^{2(j-n)} \, \alpha_n \qquad \text{par} \quad (A.2)\end{aligned}$$

donc $\beta_j = \sum\limits_n A(i-j, n-j)^2 \, \beta_n$. Comme les entiers i, j sont arbitraires, on a donc :

$$\begin{aligned}\beta_j &= \sum_n A(i, n-j)^2 \beta_n \\ &= \sum_n A(-i, n-j-i)^2 \beta_n \qquad \text{par} \quad (A.2) \\ &= \beta_{i+j}\end{aligned}$$

■

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Quantum group– and Poisson–deformation of $SU(2)$

Anne Bauval

Introduction

Woronowicz ([W1], [W2]) defined a family (in the set-theoretical sense) of quantum groups $(SU_\mu(2))_{\mu\in\mathbf{R}^*}$. (For $\mu = 1$, the C^*-algebra A_μ underlying $SU_\mu(2)$ is merely the algebra $C(SU(2))$ of continuous functions from the classical group $SU(2)$ into $\mathbf{C}$).

"Forgetting" the group structure of $SU(2)$, Sheu ([S1]) used the Weyl calculus to construct a continuous deformation of the Poisson structure of $C^\infty(SU(2))$, where the fibres are precisely the C^*-algebras A_μ.

Unifying these two points of view, we shall do the following :

(§**1**) put Woronowicz's $SU_\mu(2)$'s together into a continuous field of quantum groups,

(§**2**) construct a deformation of Poisson-$SU(2)$ in the underlying continuous field of C^*-algebras A_μ,

(§**3**) prove that such a deformation is unique among deformations fulfilling suitable requirements,

(§**4**) prove that Sheu's deformation fulfills these requirements, and compare it in detail with our deformation.

Paragraphs 2, 3 and 4 will be achieved by working, as Sheu did, at the more elementary level of Poisson-deformations of the disc, which is a "slice" of $SU(2)$.

1 Continuous structure on the family of quantum groups $SU_\mu(2)$

Definition 1.1 *([W1], [W2]) For any $\mu \in \mathbf{R}$, A_μ is the enveloping C^*-algebra of the involutive $\mathbf{C}$-algebra $\mathcal{A}_\mu$ defined by two generators α_μ, γ_μ and relations :*

$$\begin{aligned} \alpha_\mu^*\alpha_\mu + \gamma_\mu^*\gamma_\mu &= 1 \quad (1_\mu) \\ \alpha_\mu\alpha_\mu^* + \mu^2\gamma_\mu^*\gamma_\mu &= 1 \quad (2_\mu) \end{aligned} \qquad \begin{aligned} \gamma_\mu^*\gamma_\mu &= \gamma_\mu\gamma_\mu^* \quad (3_\mu) \\ \alpha_\mu\gamma_\mu &= \mu\gamma_\mu\alpha_\mu \quad (4_\mu) \\ \alpha_\mu\gamma_\mu^* &= \mu\gamma_\mu^*\alpha_\mu \quad (5_\mu) \end{aligned}$$

and if $\mu \neq 0$, the quantum group $SU_\mu(2)$ is defined by the unitary matrix

$$\begin{pmatrix} \alpha_\mu & -\mu\gamma_\mu^* \\ \gamma_\mu & \alpha_\mu^* \end{pmatrix}$$

In order to endow the family of these $SU_\mu(2)$'s with a structure of continuous field of quantum groups ([B1], [B2]), we just have to endow the family $(A_\mu)_{\mu\in\mathbf{R}^*}$ with a structure of continuous field of C^*-algebras in such a way that the sections $\mu \mapsto \alpha_\mu$, $\mu \mapsto \gamma_\mu$ are continuous. (We shall even do a little more : this field will also be defined at $\mu = 0$).

Definition 1.2 *A is the universal C^*-algebra defined by three generators α, γ, μ and relations :*

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= 1 \quad (1) \\ \alpha\alpha^* + \mu^2\gamma^*\gamma &= 1 \quad (2) \end{aligned} \qquad \begin{aligned} \gamma^*\gamma &= \gamma\gamma^* \quad (3) \\ \alpha\gamma &= \mu\gamma\alpha \quad (4) \\ \alpha\gamma^* &= \mu\gamma^*\alpha \quad (5) \end{aligned}$$

$$\begin{array}{ccll} \mu & \textit{commutes with} & \alpha, \gamma & (6) \\ -1 & \leq \mu \leq & 1 & (7). \end{array}$$

$\mathcal{A}$ is the *-subalgebra generated by α, γ, μ.

The restriction of the parameter μ to $[-1;1]$ is harmless since for $\mu \neq 0$, there is an isomorphism of *quantum groups* (not only of C^*-algebras) between $SU_\mu(2)$ and $SU_{1/\mu}(2)$ (sending α_μ to $\alpha_{1/\mu}{}^*$ and γ_μ to $-\frac{1}{\mu}\gamma_{1/\mu}{}^*$) : using this isomorphism it is then easy to extend to $\mathbf{R}$ the field on $[-1;1]$ which we are going to construct.

Moreover, such a restriction of the parameter is necessary, otherwise the generator μ would not be bounded, hence the involutive algebra defined by these generators and relations would not have a C^*-envelope.

We shall construct a field of C^*-algebras over $[-1;1]$, using the natural morphism from $C([-1;1])$ into the center of A. (This morphism is given by relations (6) and (7)). By a slight generalization of the Dauns-Hofmann theorem, proved by Dupré and Gilette ([DG], proposition 1.3 and corollary 2.2) and quoted in [Ri], there is a unique upper semi-continuous field related to the $C([-1;1])$-C^*- algebra A in the following way.

Definition 1.3 *ξ is the upper semi-continuous field of C^*-algebras on $[-1;1]$ such that :*

- *the fiber of ξ at x is A/xA (x denotes here both a point in $[-1;1]$ and the ideal of functions in $C([-1;1])$ vanishing at this point)*
- *the total space $\sqcup_{x\in[-1;1]} A/xA$ of ξ is endowed with a topology such that the continuous sections of ξ are the sections of the form $x \mapsto a \bmod xA$, for any $a \in A$.*

Using the universal properties of A and of the A_μ's, one easily proves the following relationship between our field ξ and Woronowicz's family $(A_\mu)_{\mu\in[-1;1]}$.

Proposition 1.4 *For any $\mu \in [-1;1]$, the fiber at μ of the field ξ is naturally isomorphic to the C^*-algebra A_μ. This family of isomorphisms identifies the two continuous sections of the field ξ associated to $\alpha, \gamma \in A$ with the two sections $\mu \mapsto \alpha_\mu, \mu \mapsto \gamma_\mu$ of the family $(A_\mu)_{\mu\in[-1;1]}$.*

Before introducing another field ζ with more elementary fibers, and proving the (lower) continuity of both fields ξ and ζ, let us first get rid of the case $\mu \leq 0$: we shall prove that the study of $\xi_{|[-1,0]}$ may be reduced to the study of $\xi_{|[0;1]}$ (and conversely), by a property with "fractal" flavour.

Proposition 1.5 *Let $\varepsilon_{i,j}$ $(1 \le i,j \le 2)$ be the canonical generators of $M_2(\mathbf{C})$ and $i_\mu : A_{-\mu} \to M_2(\mathbf{C}) \otimes A_\mu$ the morphism defined by :*

$$i_\mu(\alpha_{-\mu}) = (\varepsilon_{1,2} + \varepsilon_{2,1}) \otimes \alpha_\mu, \qquad i_\mu(\gamma_{-\mu}) = (\varepsilon_{1,1} - \varepsilon_{2,2}) \otimes \gamma_\mu.$$

For any $\mu \in \mathbf{R}$, i_μ is an embedding.

Proof. Let D be the subalgebra of $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ generated by the two elements $P = \varepsilon_{1,1} \otimes \varepsilon_{1,1} + \varepsilon_{2,2} \otimes \varepsilon_{2,2}$ and $Q = \varepsilon_{1,2} \otimes \varepsilon_{1,2} + \varepsilon_{2,1} \otimes \varepsilon_{2,1}$ and similarly, D' the subalgebra generated by $P' = \varepsilon_{1,1} \otimes \varepsilon_{2,2} + \varepsilon_{2,2} \otimes \varepsilon_{1,1}$ and $Q' = \varepsilon_{1,2} \otimes \varepsilon_{2,1} + \varepsilon_{2,1} \otimes \varepsilon_{1,2}$. Let $\varphi : D \to \mathbf{C}$ be the morphism such that $\varphi(P) = \varphi(Q) = 1$. One easily checks that the image of $j = (\mathrm{id}_{M_2(\mathbf{C})} \otimes i_{-\mu}) \circ i_\mu$ is included in $(D \oplus D') \otimes A_{-\mu}$ and that $((\varphi \oplus 0) \otimes \mathrm{id}_{A_{-\mu}}) \circ j = \mathrm{id}_{A_{-\mu}}$.

We shall now reduce the study of the A_μ's (quantum $SU(2)$) to the study of more elementary C^*-algebras B_μ (quantum disc). Let us recall the two results which naturally led us to this reduction.

Theorem 1.6 *([W2] appendix 2) A_μ is isomorphic to A_0, for any $\mu \in]-1,1[$.*

The isomorphism $T_\mu : A_\mu \xrightarrow{\sim} A_0$ was defined by Woronowicz as follows :

$$\begin{aligned} T_\mu(\alpha_\mu) &= \textstyle\sum_{n=0}^{\infty} \frac{(1-\mu^2)\mu^{2n}}{\sqrt{1-\mu^{2n+2}}+\sqrt{1-\mu^{2n}}} \alpha_0^{*n} \alpha_0^{n+1} \\ \text{and} \quad T_\mu(\gamma_\mu) &= \textstyle\sum_{n=0}^{\infty} \mu^n \alpha_0^{*n} \gamma_0 \alpha_0^n \end{aligned}$$

Theorem 1.7 *([S1] proposition 1.1) Let*

$$0 \longrightarrow C(D) \longrightarrow C(\overline{D}) \xrightarrow{\sigma_1} C(\mathbf{T}) \longrightarrow 0$$

be the exact sequence of the unit disc and

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{S}) \xrightarrow{\sigma_0} C(\mathbf{T}) \longrightarrow 0$$

be the Tœplitz exact sequence. Set $B_1 = C(\overline{D})$ and $B_0 = C^*(\mathcal{S})$. For $\mu = 1$ or 0, A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

For $\mu = 1$, the isomorphism consists in identifying $SU(2)$ with a family of discs $(D_u)_{u \in \mathbf{T}}$, glued together along their boundary circle :

$$(u, Z) \in \mathbf{T} \times \overline{D} \text{ is identified to } \begin{pmatrix} Z & -\overline{u}c \\ uc & \overline{Z} \end{pmatrix}, \text{ with } c = \sqrt{1 - |Z|^2}.$$

(This "slicing" of $SU(2)$ is compatible with the Poisson structure, cf §3 and 4).

For $\mu = 0$, let us recall the Tœplitz exact sequence. $C^*(\mathcal{S})$ is the C^*-algebra generated by the unilateral shift operator $\mathcal{S}$. $\mathcal{S}\mathcal{S}^*$ is equal to $1 - p$, p being a rank one projection. The closed ideal of $C^*(\mathcal{S})$ generated by p is the algebra $\mathcal{K}$ of compact operators, and $C^*(\mathcal{S})/\mathcal{K}$ is isomorphic to $C(\mathbf{T})$, the isomorphism sending the unitary generator $(\mathcal{S} \bmod \mathcal{K}) \in C^*(\mathcal{S})/\mathcal{K}$ to $\mathrm{id}_{\mathbf{T}}$.

In both cases $\mu = 1$ or 0, the embedding $A_\mu \to C(\mathbf{T}, B_\mu)$ sends

α_μ to $(u \mapsto \overline{\alpha}_\mu)$ and γ_μ to $(u \mapsto u\overline{\gamma}_\mu)$, with

$$\begin{aligned} \overline{\alpha}_1 &= (Z \mapsto Z), & \overline{\gamma}_1 &= (Z \mapsto \sqrt{1 - |Z|^2}) \\ \overline{\alpha}_0 &= \mathcal{S}^*, & \overline{\gamma}_0 &= p. \end{aligned}$$

Using the proof of theorem 1.6, one gets the following "generalization" of theorem 1.7 for free (we pass from the case $\mu = 0$ to the "more general" case $|\mu| < 1$ by a rather silly renaming ; the only nontrivial assertion of the following corollary is the first one, which justifies this renaming).

Corollary 1.8 *For $|\mu| < 1$, let us denote by*

- $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ *the elements of $C^*(\mathcal{S})$ defined as series in $\overline{\alpha}_0, \overline{\gamma}_0$ by the same formulas as in theorem 1.6, where $T_\mu(\alpha_\mu), T_\mu(\gamma_\mu)$ were defined as series in α_0, γ_0*
- $\mathcal{B}_\mu$ *the involutive subalgebra generated by $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and*
- B_μ *its closure in $C^*(\mathcal{S})$.*

For $|\mu| < 1$, B_μ is equal to $C^*(\mathcal{S})$ and the closed ideal of B_μ generated by $\overline{\gamma}_\mu$ is $\mathcal{K}$. Moreover, for $-1 < \mu \leq 1$, there is a morphism $\sigma_\mu : B_\mu \to C(\mathbf{T})$ such that $\sigma_\mu(\overline{\alpha}_\mu^*) = \mathrm{id}_\mathbf{T}$ and such that the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B_\mu \xrightarrow{\sigma_\mu} C(\mathbf{T}) \longrightarrow 0$$

is exact, and A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

Remark. For $\mu = 1$, the morphism $\sigma_1 : C(\overline{D}) \to C(\mathbf{T})$ which we are choosing is *not the mere restriction* but (in order to make the notations fit together) $\sigma_1(f)(v) = f(\overline{v})$. This remark will be important in lemma 4.5.

Since $\overline{\gamma}_0 = p \geq 0$, the definition of $\overline{\gamma}_\mu$ as a series makes it self adjoint. If $\mu \geq 0$ we even get : $\overline{\gamma}_\mu \geq 0$, hence (under the identification of A_μ given above) $|\gamma_\mu| = \left|(u \mapsto u\overline{\gamma}_\mu)\right| = (u \mapsto \overline{\gamma}_\mu)$. Using this fact and the universal property of A_μ, one easily proves the following proposition.

Proposition 1.9 *For $0 \leq \mu \leq 1$, B_μ is isomorphic to the universal C^*-algebra defined by generators $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and relations :*

the relations (1_μ)–(5_μ) (cf definition 1.1)

the additional relation : $\overline{\gamma}_\mu \geq 0$.

Remark. Instead of adding a relation and looking at B_μ as a quotient of A_μ, one may also prove (but this will not be used) that B_μ is isomorphic to the C^*-subalgebra of A_μ generated by α_μ, and characterize B_μ as the universal C^*-algebra defined by one generator α_μ and one relation (deduced from relations (1_μ) and (2_μ) by eliminating γ_μ) ([NN]).

Paraphrasing definitions 1.2 and 1.3 and proposition 1.4, we can now define the field of B_μ's.

Definition 1.10 • *B is the C^*-algebra defined by generators $\overline{\alpha}, \overline{\gamma}$ and $\overline{\mu}$ and relations :*

$$\begin{array}{rcll} \overline{\alpha}^*\overline{\alpha} + \overline{\gamma}^2 & = & 1 & (\overline{1}) \\ \overline{\alpha}\,\overline{\alpha}^* + \overline{\mu}^2\overline{\gamma}^2 & = & 1 & (\overline{2}) \end{array} \qquad \begin{array}{rcll} \overline{\gamma} & \geq & 0 & (\overline{3}) \\ \overline{\alpha}\overline{\gamma} & = & \overline{\mu}\,\overline{\gamma}\,\overline{\alpha} & (\overline{4}) \end{array}$$

$$\begin{array}{cccl} \overline{\mu} & \textit{commutes with} & \overline{\alpha}, \overline{\gamma} & (\overline{6}) \\ 0 & \leq \overline{\mu} \leq & 1 & (\overline{7}) \end{array}$$

- *$\mathcal{B}$ is the involutive subalgebra of B generated by $\overline{\alpha}, \overline{\gamma}, \overline{\mu}$ and*
- *ζ is the (upper semi-continuous) field on* $[0;1]$ *associated to B, with fibers B_μ.*

Proposition 1.11 *ζ is trivial on* $[0;1)$ *and ξ is trivial on* $(-1;1)$.

Proof. By uniqueness of the field defined by a total family of sections ([DG], proposition 1.3), in order to prove the first assertion it suffices to show that for any $b \in \mathcal{B}$ (viewed as a continuous section b of ζ) the restriction of b to $[0;1)$ is a continuous section of the trivial field $C^*(\mathcal{S}) \times [0;1)$ i.e. b is a continuous map from $[0;1)$ to $C^*(\mathcal{S})$. It is sufficient to prove this for $b = \overline{\alpha}$ or $b = \overline{\gamma}$. This is then a consequence of the definition of $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ in corollary 1.8. This proves the first assertion. The trivialness of ξ on $[0;1)$ is then a consequence of corollary 1.8, and its trivialness on $(-1;0]$ may be deduced from this and from proposition 1.5. (There is also a direct proof of the second assertion, analogous to that of the first one).

Before studying the behaviour of these two fields at the extreme points, let us make explicit the representations of the B_μ's and A_μ's deduced from the canonical representation of $C^*(\mathcal{S})$ by corollary 1.8. $\mathcal{S}$ acts on the space $\mathcal{H}$ with Hilbert basis $(e_n)_{n\in\mathbf{N}}$ by $\mathcal{S}(e_n) = e_{n+1}$, $\overline{\alpha}_0 = \mathcal{S}^*$ and $\overline{\gamma}_0 = p$ hence by definition of $\overline{\alpha}_\mu$, $\overline{\gamma}_\mu$ (corollary 1.8) we get, for $|\mu| < 1$:

$$\overline{\alpha}_\mu(e_n) = \sqrt{1-\mu^{2n}}\,e_{n-1} \qquad \overline{\gamma}_\mu(e_n) = \mu^n e_n.$$

¿From this faithful representation of B_μ on $\mathcal{H}$ we deduce a faithful representation π_μ of $C(\mathbf{T}, B_\mu)$ (hence of A_μ) on $L^2(\mathbf{T}, \mathcal{H})$: since $\alpha_\mu(u) = \overline{\alpha}_\mu$ and $\gamma_\mu(u) = u\overline{\gamma}_\mu$ ($\forall u \in \mathbf{T}$), we get, on the basis $(\psi_{n,k})_{k\in\mathbf{Z}, n\in\mathbf{N}}$ of $L^2(\mathbf{T}, \mathcal{H})$ defined by $\psi_{n,k}(u) = u^k e_n$:

$$\alpha_\mu(\psi_{n,k}) = \sqrt{1-\mu^{2n}}\,\psi_{n-1,k} \qquad \gamma_\mu(\psi_{n,k}) = \mu^n \psi_{n,k+1}.$$

($\pi_{-\mu}$ and π_μ are related by the morphism i_μ defined in proposition 1.5 : $\pi_\mu \circ i_\mu$ is equivalent to $\pi_{-\mu} \oplus \pi_{-\mu}$). Let us treat similarly the case $\mu = \pm 1$. (π_{-1} and π_1 will be related by the same formula hence it is sufficient to check that π_1 is faithful, which will be obvious by construction). The representations of B_μ and A_μ for $\mu = \pm 1$ will be constructed as direct sums : $\bigoplus_{t\in[0;1]} \sigma_{\mu,t}$ for B_μ and $\pi_\mu = \bigoplus_{t\in[0;1]} \sigma'_{\mu,t}$ for A_μ. Let us first define $\sigma_{1,t}$ and $\sigma'_{1,t}$. For any $t \in [0;1]$, let $\sigma_{1,t}$ be the representation of $B_1 = C(\overline{D})$ on $\mathcal{H}' = L^2(\mathbf{T})$ given by restricting an element of $C(\overline{D})$ to the circle of radius $\sqrt{1-t^2}$ and making this element of $C(\mathbf{T})$ act on $\mathcal{H}'$ by multiplication. Since $\overline{\alpha}_1(Z) = Z$ and $\overline{\gamma}_1(Z) = \sqrt{1-|Z|^2}$, we get, on the basis $(e'_n)_{n\in\mathbf{Z}}$ of $\mathcal{H}'$ defined by $e'_n(v) = v^{-n}$:

$$\sigma_{1,t}(\overline{\alpha}_1)(e'_n) = \sqrt{1-t^2}\,e'_{n-1} \qquad \sigma_{1,t}(\overline{\gamma}_1)(e'_n) = t e'_n.$$

¿From this representation $\sigma_{1,t}$ of B_1 on $\mathcal{H}'$ we deduce a representation $\sigma'_{1,t}$ of $C(\mathbf{T},B_1)$ (hence of A_1) on $L^2(\mathbf{T},\mathcal{H}')$. On the basis $(\psi'_{n,k})_{k,n\in\mathbf{Z}}$ of $L^2(\mathbf{T},\mathcal{H}')$ defined by $\psi'_{n,k}(u) = u^k e'_n$,

$$\sigma'_{1,t}(\alpha_1)(\psi'_{n,k}) = \sqrt{1-t^2}\psi'_{n-1,k} \qquad \sigma'_{1,t}(\gamma_1)(\psi'_{n,k}) = t\psi'_{n,k+1}.$$

The definitions of $\sigma_{-1,t}$ and $\sigma'_{-1,t}$ are analogous, but with t multiplied by $(-1)^n$.

These representations of the A_μ's (for $|\mu|\le 1$) are the ones used in [W2] (theorem 1.2) to show that $\{\alpha_\mu^k\gamma_\mu^m\gamma_\mu^{*n}|k,m,n\in\mathbf{N}\}\cup\{\alpha_\mu^{*k}\gamma_\mu^m\gamma_\mu^{*n}|k,m,n\in\mathbf{N},k\neq 0\}$ is a linear basis of $\mathcal{A}_\mu$, for $0<|\mu|\le 1$. This theorem is deducible from the following lemma, which may be proved by the same arguments, using the representations of the B_μ's which we just presented.

Lemma 1.12 *Let* $\beta_\mu^+ = \{\overline{\alpha_\mu}^p\overline{\gamma_\mu}^q|p,q\in\mathbf{N}\}$ *and* $\beta_\mu^- = \{\overline{\alpha_\mu}^{*p}\overline{\gamma_\mu}^q|p,q\in\mathbf{N},p\neq 0\}$. *For* $0<|\mu|\le 1$, $\beta_\mu := \beta_\mu^+\cup\beta_\mu^-$ *is a linear basis of* $\mathcal{B}_\mu$.

There remains to prove the lower semi-continuity of ζ and ξ at the extreme points. It is not really necessary to prove it here, since it will be a corollary of our theorem 4.1. But a direct proof, applying the ideas of [Ri] to the continuous field of Haar measures, may be found in the two preprints [S2] and [NN], which I did not yet know at Algop in July. I thank E.Blanchard for explaining to me [B1] and [B2], and A.Sheu, G.Skandalis and G.Nagy for sending me [S2], [NN], and [N1], [N2], and I include the sketch of this proof, for the reader's convenience.

Proposition 1.13 *([S2], [NN]) The field ζ is lower semi-continuous at* 1 *and the field ξ is lower semi-continuous at* ± 1.

(Sketch of) proof. As in the proof of proposition 1.11, the properties of ξ may be either deduced from those of ζ or proved directly by the same method. So we just have to prove the lower semi-continuity of ζ at 1. For $|\mu|<1$, let $\overline{h_\mu}$ be the state on B_μ defined by the same formulas as the Haar measure h_μ on A_μ ([W1], appendix 1) :

$$\overline{h_\mu}(b) = (1-\mu^2)\sum_{n=0}^{\infty}\mu^{2n}\langle e_n|b(e_n)\rangle .$$

Let $\overline{h_1}$ be the normalised Lebesgue measure on the disc. (The state h_μ on $A_\mu\subset C(\mathbf{T},B_\mu)$ is obtained from the state $\overline{h_\mu}$ on B_μ by integrating along the circle $\mathbf{T}$. Conversely, viewing B_μ as a subalgebra of A_μ as in the remark after proposition 1.9, $\overline{h_\mu}$ is obtained from h_μ by restriction). One proves that $\overline{h_\mu}$ is a continuous field of states by checking continuity on polynomials $b\in\mathcal{B}$. By faithfulness of the associated G.N.S. representations, this yields lower semi-continuity.

2 Poisson-deformation of $SU(2)$

Let us reformulate the two definitions of a "strict deformation" and of an "operator deformation" ([S1] p. 223). The first one corresponds to Rieffel's definition, the second one is a more flexible version, allowing to consider Sheu's construction as a deformation.

Definition 2.1 *Let $\mathcal{C}$ be an involutive Poisson* $\mathbf{C}$*-algebra, endowed with a C^*-norm. A Poisson-deformation of $\mathcal{C}$ is :*

(a) *a continuous field of C^*-algebras $(C_h)_{0\leq h<\varepsilon}$, such that C_0 is the completion of $\mathcal{C}$,*

(b) *for any $f \in \mathcal{C}$, a section $(h \mapsto \rho_h(f))$ such that $\rho_0(f) = f$*

satisfying :

(c) *the $\rho(f)$'s form a total family of continuous sections of this field*

(d) $\lim\limits_{h\to 0} \frac{[\rho_h(f),\rho_h(g)]}{ih} = \{f,g\} \qquad (\forall f, g \in \mathcal{C})$

(e) *the map ρ is linear*

(f) *for any h, the map ρ_h is injective*

and satisfying moreover one of the two conditions :

(SD) *for any h, $\rho_h(\mathcal{C})$ is closed under multiplication ("strict deformation")*

(OD) *ρ preserves the involution ("operator deformation").*

Remarks.

- $\mathcal{C}$ is then necessarily commutative
- In Rieffel's original definition ([S1]), one starts from a Poisson manifold M and chooses to deform some Poisson subalgebra $\mathcal{C}$ of $C_b^\infty(M)$ (smooth bounded functions), such that $\mathcal{C}$ contains the subalgebra $C_c^\infty(M)$ (smooth functions with compact support). In our case $\mathcal{C}$ will only be dense in $C(M)$ (and M will be compact), but $\mathcal{C}$ will not contain $C^\infty(M)$.
- In this reformulation, the data (a) and (b) are redundant : the topology of the field is fully determined by the total family of continuous sections.
- The limit in condition (d) has a meaning in the total space of the field.
- Condition (a) is much stronger than Rieffel's original definition ([S1]). One part of this strength is irrelevant : we could have required the field to be continuous only as a field of Banach spaces outside $h = 0$ (this would have caused no change in the rest of this paper). But the other part is crucial : at $h = 0$ we really want the multiplication and the involution to be continuous.
- This way of reformulating the definition forced us to make condition (f) explicit, whereas the injectivity of the ρ_h's was originally implicit. This condition does not seem relevant (the rest of the paper is true if we drop it), but we shall keep it since it will be fulfilled in our case. (This property is sometimes technically useful, to check condition (SD) : see [S1], [N1]).

We shall construct such a deformation for $M = SU(2)$ and $\mathcal{C} = \mathcal{A}_1$. It will be both a strict- and operator- deformation, in contrast with Sheu's one which was only an operator deformation. But Sheu deformed the whole algebra $C^\infty(SU(2))$, whereas our $\mathcal{A}_1$ is only dense in $C(SU(2))$.

"The" Poisson structure on $SU(2)$ will always be the one chosen by Sheu [S1] and described by Lu and Weinstein [LW]. In order to deform it, we first deform the unit disc D, which is a "slice" of it (theorem 1.7). The restriction to $\mathcal{A}_1$ of the Poisson bracket on $C^\infty(SU(2))$ is completely determined by :

$$\begin{array}{rclrcl} \{\alpha_1,\alpha_1^*\} & = & -\mathrm{i}\gamma_1\gamma_1^*, & \{\alpha_1,\gamma_1\} & = & \frac{\mathrm{i}}{2}\gamma_1\alpha_1, \\ \{\alpha_1,\gamma_1^*\} & = & \frac{\mathrm{i}}{2}\gamma_1^*\alpha_1, & \{\gamma_1,\gamma_1^*\} & = & 0. \end{array}$$

Hence it is induced by the Poisson bracket on $\mathcal{B}_1$ defined by :

$$\{\overline{\alpha}_1,\overline{\alpha}_1^*\} \;=\; -\mathrm{i}\overline{\gamma}_1^2,\;\; \{\overline{\alpha}_1,\overline{\gamma}_1\} \;=\; \tfrac{\mathrm{i}}{2}\overline{\gamma}_1\,\overline{\alpha}_1.$$

(These two brackets are related by : $\{f,g\}(u) = \{f(u),g(u)\} \quad \forall f,g \in \mathcal{A}_1, \forall u \in \mathbf{T}$.)

The main trick to deform these brackets is to notice that in the field ζ, if we choose a good change of parameter like $h(\mu) = 1-\mu^2$, condition (d) is fulfilled for $(f,g)=(\overline{\alpha}_1,\overline{\alpha}_1^*)$ or $(\overline{\alpha}_1,\overline{\gamma}_1)$ as soon as $\rho_\mu(\overline{\alpha}_1) = \overline{\alpha}_\mu$, $\rho_\mu(\overline{\alpha}_1^*) = \overline{\alpha}_\mu^*$ and $\rho_\mu(\overline{\gamma}_1) = \overline{\gamma}_\mu$. (This comes from the relations between $\overline{\alpha}_\mu$ and $\overline{\gamma}_\mu$).

There remains to choose $\rho(f)$ for any polynomial f in the (commuting) variables $\overline{\alpha}_1,\overline{\alpha}_1^*,\overline{\gamma}_1$. By lemma 1.12, it suffices to define $\rho(f)$ for any element f of the linear basis β_1. $\rho(f)$ may be chosen as a polynomial in the (non commuting) variables $\overline{\alpha},\overline{\alpha}^*,\overline{\gamma}$, and must be such that $\rho(f)_1 = f$. We shall present the simplest example of such a ρ, and prove that it yields a deformation. (The proof will use a combinatoric property of this choice, but many other choices satisfy this property).

Proposition 2.2 *Let $\rho : \mathcal{B}_1 \to \mathcal{B}$ be the linear involutive map such that*

$$\rho(\overline{\alpha}_1^p\overline{\gamma}_1^q) = \overline{\alpha}^p\overline{\gamma}^q \qquad (\forall p,q \in \mathbf{N}).$$

(ζ,ρ) is a strict- and operator- deformation of $\mathcal{B}_1$, for $\mu \in (0;1]$ and for any change of parameter $h(\mu)$ equivalent to $1-\mu^2$ when $\mu \to 1^-$.

Proof. All conditions except (d) are fulfilled by construction. ((c) and (f) are true by lemma 1.12). Let us prove (d). It suffices to prove it for $f,g \in \beta_1$, and we shall do this by induction on the length $\ell(fg) = \ell(f)+\ell(g)$ of the "word" fg on the "alphabet" $\{\overline{\alpha}_1,\overline{\alpha}_1^*,\overline{\gamma}_1\}$. If $\ell(fg) \le 2$, (d) holds by construction. If $\ell(fg) > 2$ we have $\ell(f) \ge 2$ or $\ell(g) \ge 2$, let us say for instance $\ell(f) \ge 2$. Take $x,y \in \beta_1$ of lengths $< \ell(f)$ such that $f = xy$ and $\rho(f) = \rho(x)\rho(y)$. From these two equalities we deduce :

$$\{f,g\} = x\{y,g\} + \{x,g\}y,$$

$$\text{and} \qquad [\rho(f),\rho(g)] = \rho(x)[\rho(y),\rho(g)] + [\rho(x),\rho(g)]\rho(y).$$

Since $\ell(yg) < \ell(fg)$, we may assume (by induction) that $\lim\limits_{h\to 0} \frac{[\rho_h(y),\rho_h(g)]}{\mathrm{i}h} = \{y,g\}$. Since $\rho(x)$ is a continuous section of ζ such that $\rho_0(x) = x$, the upper semi-continuity of ζ at $h = 0$ entails :

$$\lim_{h\to 0} \frac{\rho_h(x)[\rho_h(y),\rho_h(g)]}{\mathrm{i}h} = x\{y,g\}.$$

Similarly (using $\ell(xg) < \ell(fg)$), we get :

$$\lim_{h\to 0} \frac{[\rho_h(x), \rho_h(g)]\rho_h(y)}{ih} = \{x, g\}y,$$

and add up these two equalities.

Corollary 2.3 *Let $\rho' : \mathcal{A}_1 \to \mathcal{A}$ be the linear involutive map such that*

$$\rho'(\alpha_1^p \gamma_1^q \gamma_1^{*r}) = \alpha^p \gamma^q \gamma^{*r} \qquad (\forall p, q, r \in \mathbf{N}).$$

(ξ, ρ') is a strict- and operator- deformation of $\mathcal{A}_1$, for $\mu \in (0; 1]$ and for any change of parameter $h(\mu)$ equivalent to $1 - \mu^2$ when $\mu \to 1^-$.

Proof. By proposition 2.2, $\forall f, g \in \mathcal{A}_1$, $\lim_{h\to 0} \frac{[\rho_\mu(f(u)), \rho_\mu(g(u))]}{ih} = \{f(u), g(u)\}$, hence $\lim_{h\to 0} \frac{[\rho'_\mu(f), \rho'_\mu(g)]}{ih}(u) = \{f, g\}(u)$. A careful inspection of the proof of proposition 2.2 reveals that this holds *uniformly* in $u \in \mathbf{T}$.

Remark. Extending the field as explained just after definition 1.2 and letting μ tend to 1 from above, one gets a deformation of the opposite Poisson structure of $SU(2)$.

3 Relative uniqueness of the Poisson-deformation of R^2

We shall prove that the Poisson-deformation of $\mathcal{B}_1$ we have just constructed is, in some sense, "the only one". (We referred to $\mathbf{R}^2$ in the title because, as will be explained in the next paragraph, $\mathcal{B}_1$ is isomorphic to a Poisson-subalgebra of $C^\infty(\mathbf{R}^2)$).

Theorem 3.1 *Let (Θ, τ) be an operator deformation of $\mathcal{B}_1$ such that :*

- *Θ is trivial on $\mathbf{R}^*$, with fiber $C^*(\mathcal{S})$*
- *$\sigma_0(\tau_h(\overline{\alpha}_1)) = \sigma_1(\overline{\alpha}_1)$ for any $h \neq 0$*
- *$\tau_h(\overline{\alpha}_1) = R_h \mathcal{S}^*$, for any $h \neq 0$, with R_h selfadjoint, diagonal in the basis $(e_n)_{n\in\mathbf{N}}$ for which $\mathcal{S}(e_n) = e_{n+1}$.*

Then Θ is isomorphic to ζ.

Remark. In fact we shall use the continuity of Θ (in lemma 3.2) but only the upper semi-continuity of ζ, hence we do not need proposition 1.13.

Proof. Let us set $h = 1 - \mu^2$ and abbreviate $\lim_{h\to 0} x_h - y_h = 0$ by $x_h \asymp y_h$. We must prove that $\forall b \in \mathcal{B}_1, \tau_h(b) \asymp \rho_\mu(b)$. Let $w_h = \tau_h(\overline{\alpha}_1)$, and $v_h = \tau_h(\overline{\gamma}_1)$. It suffices to prove that (1) $w_h \asymp \overline{\alpha}_\mu$ and (2) $v_h \asymp \overline{\gamma}_\mu$. Let us first assume (1) and deduce (2) from it. *Using only the upper semi-continuity of ζ and Θ*, since $\overline{\gamma}_1 = \sqrt{1 - \overline{\alpha}_1^* \overline{\alpha}_1}$ we get : $\overline{\gamma}_\mu^2 \asymp 1 - \overline{\alpha}_\mu^* \overline{\alpha}_\mu \asymp 1 - w_h^* w_h \asymp v_h^2$ hence $\overline{\gamma}_\mu \asymp |v_h|$, and since $v_0 = \overline{\gamma}_1 \geq 0$ we get : $v_h \asymp |v_h|$. This yields (2). Let us now prove (1). Let $y_n(h)$ be $\sqrt{1 - \mu^{2(n+1)}}$ and Q_μ be the operator such that $Q_\mu(e_n) = y_n(h)e_n$. Since τ_h is a Poisson-deformation, we have : $\frac{[w_h,w_h^*]}{h} \asymp 1 - w_h^* w_h$, hence $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h)$, which (by elementary calculus) leads to : $R_h^2 \asymp Q_\mu^2$. From this, using the hypothesis that R_h is equal to 1 modulo $\mathcal{K}$, we deduce $R_h \asymp Q_\mu$ by the following elementary lemma. Since $\overline{\alpha}_\mu = Q_\mu \mathcal{S}^*$ by §1 (before lemma 1.12), this yields (1).

Lemma 3.2 *Let $x_n(h), y_n(h) \in \mathbf{R}$, for $n \in \mathbf{N}$ and $h \in [0, \varepsilon]$, be such that :*

- $x_n^2(h) - y_n^2(h) \to 0$ *when* $h \to 0$, *uniformly in* n,
- *for any fixed* n, $y_n(h)$ *is an increasing positive function of* h, *with limit* 0 *at* $h = 0$, *and* $x_n(h)$ *is a continuous function of* h,
- *for any fixed* h, *the sequence* $x_n(h)$ *is eventually positive.*

Then $x_n(h) - y_n(h) \to 0$ when $h \to 0$, uniformly in n.

4 Comparision with Sheu's deformation

Let us recall the deformation defined in [S1]. Let the unit disc D be equipped with the Poisson structure deduced from the usual one on $\mathbf{R}^2$ by the change of variables :

$$(z \in \mathbf{R}^2) \mapsto (Z = \frac{z}{|z|}\sqrt{1 - e^{-|z|^2/2}} \in D).$$

The Poisson structure on $SU(2)$ is related to this Poisson-D by the "slicing" described after theorem 1.7, and Sheu's deformation of Poisson-$SU(2)$ is naturally deduced from the following deformation of Poisson-$\mathbf{R}^2$.

Let S^0 be the algebra of Weyl symbols of order 0 on $\mathbf{R}^2$. For any $a \in S^0$ and any $h > 0$, let $W_h(a)$ be the operator on $L^2(\mathbf{R})$ defined by

$$W_h(a)u(y) = \frac{1}{2\pi h}\int \mathcal{A}(x, (y+y')/2)\, \exp(\frac{ix(y-y')}{h})\; u(y')\; dx\; dy',$$

$$\text{with} \quad \mathcal{A}(x,y) = a(y,x).$$

(In [S1], $\mathcal{A} = a$. We rectify the formula according to [Vo], in order to really get, as claimed by Sheu, a deformation of the Poisson-bracket on $\mathbf{R}^2$ defined by $\{f, g\} = \partial_1 f\ \partial_2 g - \partial_2 f\ \partial_1 g$ and not of the opposite one. The only point of [S1] spoiled by this mistake is proposition 2.1, which we shall put right in our lemma 4.5).

$W_h(a)$ is unitarily equivalent to $W_h'(a) := W(a_h)$, with $W = W_1$ and $a_h(z) = a(\sqrt{h}z)$ ([S1] p.224), and Sheu proved that W_h is a deformation of S^0. Hence W_h' is also a deformation of S^0. We call ζ' the continuous field associated to W_h' and ξ' the continuous field associated to the induced deformation of $C^\infty(SU(2))$.

Theorem 4.1 *The fields ζ and ζ' are isomorphic ; so are the fields ξ and ξ'.*

Proof. The second point is a consequence of the first one, which comes from theorem 3.1 and from lemma 4.2, 4.5 and 4.6 below.

Lemma 4.2 *ζ' is trivial on $\mathbf{R}^{+*}$, with fibers $C^*(\mathcal{S})$.*

Proof. The fibers are $C^*(\mathcal{S})$ by [S1] (proposition 2.1). By Howe's proof of theorem 3.1.3 in [Ho], there exists a constant C such that for any $a \in S^0$, if we define $\|a\|_{sup4} = \max_{|\gamma|\leq 4} \|\partial^\gamma a\|_\infty$, we get : $\|W(a)\| \leq C\|a\|_{sup4}$, hence for any $h > 0$, $\|W'_h(a) - W'_t(a)\| = \|W(a_h - a_t)\| \leq C\|a_h - a_t\|_{sup4} \xrightarrow[t\to h]{} 0$, hence the map $(t \mapsto W'_t(a))$ is continuous at h.

Remark. The above proof is inspired by [S1], page 225, first paragraph, which contains a hint of proof for the fact that the map $(t \mapsto W_t(a))$ is also continuous (Sheu uses a weaker seminorm on S^0 but this does not matter). So the field associated to W_h is, like ζ', trivial on $\mathbf{R}^{+*}$. But these two fields on $\mathbf{R}$ are different (see remark 2 after lemma 4.5).

Lemma 4.3 *ζ' is upper semi-continuous at* 0.

Proof. (This fact was already proved in [S1] p.225, hence this lemma is not necessary, but it is worth noticing that the inequality we just mentionned yields a much simpler proof). Call $\sharp_h$ the product on S_0 such that $W_h(f)W_h(g) = W_h(f\sharp_h g)$. Let $a \in S^0$ and $\varepsilon > \|a\|_\infty$. Choose M such that $\|a\|_\infty < M < \varepsilon$ and let $c = \sqrt{M^2 - |a|^2}$ and $r(h) = c^*\sharp_h c + a^*\sharp_h a - M^2$. When $h \to 0$, $\frac{r(h)}{h} = \frac{r(h)-r(0)}{h}$ is bounded in S^{-2} ([Vo] p. 124), hence $r(h) \to 0$ for the semi-norms of S^{-2} and a fortiori for the semi-norm $\|\ \|_{sup4}$ defined in the proof of the previous lemma. Therefore, we also have $r(h)_h \to 0$ for this semi-norm. This yields :

$$\begin{aligned}
\|W_h(a)\|^2 - M^2 &\leq \|W_h(a)^*W_h(a)\| - \|M^2 - W_h(c)^*W_h(c)\| \\
&\leq \|W_h(a)^*W_h(a) + W_h(c)^*W_h(c) - M^2\| \\
&= \|W_h(r(h))\| = \|W'_h(r(h))\| = \|W(r(h)_h)\| \\
&\leq C\|r(h)_h\|_{sup4} \xrightarrow[h\to 0]{} 0,
\end{aligned}$$

so for h small enough, $\|W'_h(a)\| = \|W_h(a)\| < \varepsilon$.

Let us recall ([Gu], §7 and 8) some facts and notations needed for the next three lemmas. Let m be the gaussian measure on $\mathbf{C}$, defined by :

$$\mathrm{d}m(x,y) = \frac{1}{\pi}\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\ \mathrm{d}y.$$

(By the change of variable described at the beginning of this paragraph, m corresponds to the normalized Lebesgue measure on D). The Fock space $\mathcal{F}$ is the Hilbert subspace of $L^2(\mathbf{C}, m)$ of holomorphic functions. For any $\lambda \in \mathbf{C}$, let $\varepsilon_\lambda(z) = \mathrm{e}^{\overline{\lambda}z}$: this defines $\varepsilon_\lambda \in \mathcal{F}$. The Bargmann transform is an isomorphism between $\mathcal{F}$ and $L^2(\mathbf{R})$, which sends the "Berezin basis" $(\varepsilon_\lambda)_{\lambda\in\mathbf{C}}$ of $\mathcal{F}$ to the "Berezin basis" $(c_\lambda f_\lambda)_{\lambda\in\mathbf{C}}$ of $L^2(\mathbf{R})$, with $f_\lambda(x) = \exp(\sqrt{2}\ \overline{\lambda}x - \frac{x^2}{2})$ and c_λ is a constant (such that the Bargmann transform is unitary). The Bargmann transform also sends the Hilbert basis $(e'_n)_{n\in\mathbf{N}}$ of $\mathcal{F}$ defined by $e'_n(z) = z^n/\sqrt{n!}$ to the Hilbert basis $(e_n)_{n\in\mathbf{N}}$ of Hermite functions in $L^2(\mathbf{R})$.

Any bounded operator $T^\sharp$ on $\mathcal{F}$ is completely determined by its "Berezin symbol" $\sigma_{T^\sharp}$ defined by :

$$\sigma_{T^\sharp}(\lambda,\mu) \quad = \quad \frac{\langle T^\sharp\varepsilon_\lambda, \varepsilon_\mu\rangle}{\langle \varepsilon_\lambda, \varepsilon_\mu\rangle} \quad = \quad \mathrm{e}^{-\overline{\lambda}\mu}\ \langle T^\sharp\varepsilon_\lambda, \varepsilon_\mu\rangle.$$

(We take Guillemin's notations, where $\langle f, g\rangle$ is linear in f and antilinear in g). (Since the map $\sigma_{T^\sharp}(\lambda,\mu)$ is holomorphic in μ and antiholomorphic in λ, it is determined by its restriction to the diagonal, denoted by $\sigma_{T^\sharp}$ again).

If T is the operator on $L^2(\mathbf{R})$ related to $T^\sharp$ by the Bargmann transform, the Berezin symbol of $T^\sharp$ may be calculated by :

$$\sigma_{T^\sharp}(\lambda,\mu) \quad = \quad \frac{\langle T(f_\lambda), f_\mu\rangle}{\langle f_\lambda, f_\mu\rangle} \quad = \quad \frac{\mathrm{e}^{-(\overline{\lambda}+\mu)^2/2}}{\sqrt{\pi}}\ \langle T(f_\lambda), f_\mu\rangle.$$

This holds in particular when $T = \mathcal{A}_\tau$ is the operator associated to a Weyl symbol $\mathcal{A} \in S^0$ (i.e. $\mathcal{A}_\tau = W(a)$, with $\mathcal{A}(x,y) = a(y,x)$).

Lemma 4.4 *With the preceeding notations,*

i)

$$\sigma_{W(a)^\sharp}(\lambda,\mu) = \mathrm{e}^{-2\overline{\lambda}\mu}\int a(\overline{z})\mathrm{e}^{\sqrt{2}(\overline{\lambda}z+\mu\overline{z})}\ \mathrm{d}m(z)$$

ii)

$$\langle W(a)(e_m), e_n\rangle = \sqrt{m!n!}\sum_{k=0}^{\inf(m,n)}\frac{(-1)^k I_{m-k,n-k}}{k!(m-k)!(n-k)!},$$

with

$$I_{r,s} = \int a(\overline{z})(\sqrt{2}z)^r(\sqrt{2}\overline{z})^s\ \mathrm{d}m(z).$$

Proof. i) Let us generalize and rectify Guillemin's calculus ([Gu] pp. 186–187). By definition of $\mathcal{A}_\tau = W(a)$ and of f_λ,

$$\langle \mathcal{A}_\tau(f_\lambda), f_\mu\rangle = \frac{1}{2\pi}\int \mathcal{A}(p, \frac{x+y}{2})\exp(\sqrt{2}(\overline{\lambda}y+\mu x) - \frac{x^2+y^2}{2} + \mathrm{i}p(x-y))\ \mathrm{d}x\mathrm{d}y\mathrm{d}p.$$

If we make the change of variable $q = \frac{x+y}{2}$ with x and p fixed, the integral above becomes

$$\frac{1}{\pi}\int \mathcal{A}(p,q)[\ldots]\exp(2\sqrt{2}\ \overline{\lambda}q - 2q^2 - 2\mathrm{i}pq)\ \mathrm{d}q\ \mathrm{d}p,$$

$$\text{with} \qquad [\ldots] = \int \exp((\sqrt{2}(\mu-\overline{\lambda}) + 2(q+\mathrm{i}p))x - x^2)\ \mathrm{d}x = \sqrt{\pi}\ \exp((\frac{\mu-\overline{\lambda}}{\sqrt{2}} + q + \mathrm{i}p)^2),$$

hence (after some simplifications) $\langle \mathcal{A}_\tau(f_\lambda), f_\mu\rangle =$

$$\frac{\exp((\overline{\lambda}+\mu)^2/2)}{\sqrt{\pi}}\int \mathcal{A}(p,q)\exp(-\overline{(\sqrt{2}\lambda - (q-\mathrm{i}p))}(\sqrt{2}\mu - (q-\mathrm{i}p))\ \mathrm{d}q\ \mathrm{d}p,$$

and the result follows.
ii) From i) we deduce :

$$\langle \mathcal{A}^\sharp_\tau \varepsilon_\lambda, \varepsilon_\mu \rangle = \mathrm{e}^{-\overline{\lambda}\mu} \int a(\overline{z}) \sum_{r,s\geq 0} \frac{(\sqrt{2}\,\overline{\lambda} z)^r}{r!} \frac{(\sqrt{2}\mu\overline{z})^s}{s!}\, \mathrm{d}m(z).$$

Since a is bounded and $\int \sum_{r,s\geq 0} \frac{|\sqrt{2}\,\overline{\lambda} z|^r}{r!} \frac{|\sqrt{2}\mu\overline{z}|^s}{s!}\, \mathrm{d}m(z) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{(|\lambda|+|\mu|)^2/2} < \infty$, this expression becomes :

$$\mathrm{e}^{-\overline{\lambda}\mu} \sum_{r,s\geq 0} \frac{\overline{\lambda}^r \mu^s I_{r,s}}{r!s!} = \sum_{r,s,k\geq 0} \frac{(-1)^k \overline{\lambda}^{r+k} \mu^{s+k} I_{r,s}}{r!s!k!},$$

hence $\langle \mathcal{A}^\sharp_\tau \varepsilon_\lambda, \varepsilon_\mu \rangle$ is equal to the series (with infinite bi-radius of convergence)

$$\sum_{m,n\geq 0} \overline{\lambda}^m \mu^n \sum_{k=0}^{\inf(m,n)} \frac{(-1)^k I_{m-k,n-k}}{k!(m-k)!(n-k)!}.$$

But it is also equal to

$$\sum_{m,n\geq 0} \langle \varepsilon_\lambda, e'_m \rangle \langle \mathcal{A}^\sharp_\tau(e'_m), e'_n \rangle \langle e'_n, \varepsilon_\mu \rangle = \sum_{m,n\geq 0} \frac{\overline{\lambda}^m}{\sqrt{m!}} \langle \mathcal{A}^\sharp_\tau(e'_m), e'_n \rangle \frac{\mu^n}{\sqrt{n!}},$$

with bi-radius at least $(1,1)$ (since $|\langle \mathcal{A}^\sharp_\tau(e'_m), e'_n \rangle| \leq \|\mathcal{A}^\sharp_\tau\|$). The result follows by identification.

Lemma 4.5 *For any $a \in S^0$ and $h > 0$, $\sigma_0(W'_h(a)) = \sigma_1(a)$.*

Proof. This lemma rectifies the proposition 2.1 of [S1], which states an analogous formula but with σ instead of σ_1 : $\sigma(a)(v) = \lim_{r\to\infty} a(rv)$, $\sigma_1(a)(v) = \sigma(a)(\overline{v})$ (see our remark after corollary 1.8). The mistake in [S1] came from two reasons : the wrong orientation (which we compensated by replacing a by $\mathcal{A}$ in the definition of W), and the use of an erroneous formula in [Gu], which we just rectified in lemma 4.4.i : in [Gu] p. 187, the result must be replaced by

$$\sigma_{\mathcal{A}^\sharp_\tau}\left(\frac{a+ib}{\sqrt{2}}\right) = \int \mathcal{A}(p,q) \mathrm{e}^{-(q-a)^2-(p+b)^2}\, \mathrm{d}m(q-\mathrm{i}p) = (\mathrm{e}^{-\Delta}\mathcal{A})(-b,a),$$

$$\text{hence} \quad \sigma(\mathcal{A}^\sharp_\tau) = \mathrm{e}^{-\Delta/2}\mathcal{B}, \quad \text{with} \quad \mathcal{B}(x,y) = \mathcal{A}(-\sqrt{2}y, \sqrt{2}x).$$

In formula 8.20 p.187 of [Gu] and in the subsequent pages $\mathcal{A}$ must be replaced by $\mathcal{B}$. Apart from this, the proof is the same as in [S1] so we do not repeat it.
Remarks.

1. If we do not replace a by $\mathcal{A}$, i.e. if we deform the opposite Poisson bracket, (as Sheu did) we find $\sigma_0(W'_h(a))(z) = \sigma(a)(\mathrm{i}\, z)$, not $\sigma(a)(z)$.

2. From this lemma we deduce : $\sigma_0(W_h(a)) = \sigma_1(a(.,h.))$. This shows that one cannot hope $\lim_{h\to 0} W_h(a) - W'_h(a) = 0$ to hold in general (i.e. for all $a \in S^0$),

because this would imply $\lim_{h\to 0} b(\frac{\cos\theta+i\ h\sin\theta}{\sqrt{\cos^2\theta+h^2\sin^2\theta}}) = b(\cos\theta + i\ \sin\theta)$ (uniformly in θ) for all $b = \sigma_1(a) \in C(\mathbf{T})$. This may be roughly expressed by saying that the two (isomorphic) fields on $\mathbf{R}^+$ associated to W_h and W'_h, which both consist of the same trivial field on $\mathbf{R}^{+*}$, "glued" to the same C^*-algebra at $h = 0$, are "not glued in the same way".

3. The isomorphism between $\overline{W'_h(S^0)}$ and $C^*(\mathcal{S})$, proved in [S1] proposition 2.1 and used in ourlemma 4.2, can now be made explicit in its correct version : it is simply an equality, when $C^*(\mathcal{S})$ is realized as a concrete algebra of operators on $L^2(\mathbf{R})$, letting $\mathcal{S}$ be the unilateral shift operator on the Hilbert basis of Hermite functions e_n. Moreover, $\sigma_0(W'_h(\overline{\alpha}_1)) = \sigma_1(\overline{\alpha}_1) = (z \mapsto \overline{z}) = \sigma_0(\mathcal{S}^*)$, thus modulo compact operators, $W'_h(\overline{\alpha}_1)$ is congruent to $\mathcal{S}^* = \overline{\alpha}_0$. By definition of B_μ (corollary 1.8), $\overline{W'_h(S^0)}$ is also equal to B_μ and $W'_h(\overline{\alpha}_1)$ is congruent to $\overline{\alpha}_\mu$ (modulo compact operators), for any values $h > 0$ and $\mu \in (-1;1)$.

Lemma 4.6 *Let $(e_n)_{n\in\mathbf{N}}$ be the Hilbert basis of $L^2(\mathbf{R})$ of Hermite functions and $\mathcal{S}$ be the unilateral shift on this basis. $W'_h(\overline{\alpha}_1)$ is of the form $R_h\mathcal{S}^*$ with R_h selfadjoint and diagonal in this basis.*

Proof. $\overline{\alpha}_1(z) = \frac{z}{|z|}\sqrt{1-e^{-|z|^2/2}}$, hence (taking the polar decomposition $z = \rho u$) $(\overline{\alpha}_1)_h(\rho u) = u\ b(h\rho^2)$ with $b(t) = \sqrt{1-e^{-t/2}}$. More generally, let us apply the result of lemma 4.4.ii to a symbol a_h such that $a_h(\rho u) = u^d b(h\rho^2)$ for some integer d and some function b. Under this hypothesis, $I_{r,s}$ will be equal to

$$\int u^{r-s-d} b(h\rho^2)(\sqrt{2}\ \rho)^{r+s}\ dm(\rho u),$$

hence it will be 0 if $r \neq s+d$, and

$$I_{s+d,d} = \int b(ht)(2t)^{s+\frac{d}{2}} e^{-t}\ dt.$$

Hence $\langle W(a_h)(e_m), e_n\rangle = 0$ if $m \neq n+d$ and

$$\langle W(a_h)(e_{n+d}), e_n\rangle = \sqrt{(n+d)!n!} \sum_{k=0}^{\inf(n+d,n)} \frac{(-1)^k I_{n+d-k,n-k}}{k!(n+d-k)!(n-k)!}$$

will be real if b takes real values.

Now we may consider that ρ and W' are two deformations of Poisson-$\mathcal{B}_1$ within the same field ζ', and that for any $a \in \mathcal{B}_1$, $\rho_h(a) - W'_h(a) \to 0$ when $h \to 0^+$ (using the identification of ζ and ζ' given by theorem 4.1 and the identification of the Poisson-disk and Poisson-$\mathbf{R}^2$). Moreover, the product induced by W'_h (on $\mathcal{B}_1$, and even on S^0) admits an asymptotic expansion. The following last proposition gives an analogous asymptotic expansion for the product induced by ρ_h, and a nice description of $\rho_h(a) - W'_h(a)$ for "good choices" of $\mu \mapsto h(\mu)$ (among functions equivalent to $1-\mu^2$ when $\mu \to 1^-$, cf proposition 2.2)

Proposition 4.7 *Let $\varphi : \mathcal{B}_1 \to \mathcal{B}_1$ be the linear map such that*

$$\begin{array}{rrcll} & \varphi(\overline{\alpha}_1^p \overline{\gamma}_1^q) & = & \frac{i}{2}\,(p-1)\,q\,\overline{\alpha}_1^p \overline{\gamma}_1^q & \forall p,q \in \mathbf{N} \\ \text{and} & \varphi(f^*) & = & -\varphi(f)^* & \forall f \in \mathcal{B}_1, \end{array}$$

and $C = \{\ ,\ \} + \partial\varphi$, with $\partial\varphi(f,g) = f\varphi(g) + \varphi(f)g - \varphi(fg)$.

i) $\rho_h(f)\rho_h(g) = \rho_h(fg + \frac{i\,h}{2}C(f,g)) + o(h)$

ii) *If $h = 2\frac{1-\mu^2}{1+\mu^2} + o(h^2)$ (or $\mu = \frac{1-h/4}{1+h/4} + o(h^2)$, which is equivalent), then*

$$\rho_h(f) = W_h'(f + \frac{i\,h}{2}\varphi(f)) + o(h) \qquad \forall f \in \mathcal{B}_1.$$

Proof. For $f, g \in \beta_1$ (cf lemma 1.12), one easily finds $C(f,g)$ such that (i) holds, using the relations between $\overline{\alpha}_\mu$ and $\overline{\gamma}_\mu$ (definition 1.10). (I do not reproduce this calculus here because it is not nice-looking, except that $C(g^*, f^*) = -C(f,g)^*$ since ρ is $*$-preserving). Let $D = C - \{\ ,\ \}$ (hence $D(g^*, f^*) = -D(f,g)^*$). One may check by induction on $p, q, p', q' \in \mathbf{N}$ that

$$\begin{array}{rcl} \{\overline{\alpha}_1^p \overline{\gamma}_1^q, \overline{\alpha}_1^{p'} \overline{\gamma}_1^{q'}\} & = & \frac{i}{2}(pq' - qp')\overline{\alpha}_1^{p+p'} \overline{\gamma}_1^{q+q'} \qquad \text{and} \\ \{\overline{\alpha}_1^p \overline{\gamma}_1^q, \overline{\alpha}_1^{*p'} \overline{\gamma}_1^{q'}\} & = & \frac{i}{2}\overline{\gamma}_1^{q+q'}[(pq' + qp' + 2pp')\overline{\alpha}_1^p \overline{\alpha}_1^{*p'} - 2pp'\overline{\alpha}_1^{p-1}\overline{\alpha}_1^{*p'-1}]. \end{array}$$

Then one may calulate $D(f,g)$ for $f \in \beta_1^+$ and $g \in \beta_1$, and check that it is equal to $\partial\varphi(f,g)$. This proves (i). We shall prove (ii) in three steps : prove that it holds for $f = \overline{\gamma}_1$ (step 1) and for $f = \overline{\alpha}_1$ or $\overline{\alpha}_1^*$ (step 2), and then use (i) to extend (ii) to any $f \in \mathcal{B}_1$ (step 3).

Step1. Applying the formulas of the proof of lemma 4.6 to $a(z) = \overline{\gamma}_1(z) = e^{-\frac{1}{4}|z|^2}$ we get :

$$W_h'(\overline{\gamma}_1)(e_n) = \frac{(1-\frac{h}{4})^n}{(1+\frac{h}{4})^{n+1}}\, e_n.$$

Comparing this with $\overline{\gamma}_\mu(e_n) = \mu^n e_n$, one easily gets that if $\mu = \frac{1-h/4}{1+h/4} + o(h^2)$ then

$$\overline{\gamma}_\mu = (1 + h/4)W_h'(\overline{\gamma}_1) + o(h) = W_h'(\overline{\gamma}_1 + \frac{i\,h}{2}\varphi(\overline{\gamma}_1)) + o(h).$$

Step 2. Let $w_h = W_h'(\overline{\alpha}_1)$ and let $x_n(h)$ be the real numbers such that $w_h(e_{n+1}) = x_n(h)e_n$. The application of the formulas of the proof of lemma 4.6 leads to rather complicated results for $a = \overline{\alpha}_1$ (in order to get the exact values of $x_n(h)$), but the results are much simpler for $a = \overline{\alpha}_1^2$:

$$W_h'(\overline{\alpha}_1^2)(e_{n+2}) = \frac{2n+3}{2\sqrt{(n+1)(n+2)}}[1 \ - \ \frac{(1-h/2)^{n+1}}{(1+h/2)^{n+2}}(1 + \frac{h}{2(2n+3)})]e_n.$$

Using the fact that $w_h^2 = W_h'(\overline{\alpha}_1^2) + o(h)$, the above formula proves that for h sufficiently small, $\forall n \in \mathbf{N}, x_n(h)x_{n+1}(h) > 0$ hence (since $\lim_{n\to\infty} x_n(h) = 1$) the numbers $x_n(h)$ are all positive. This simplifies the proof of theorem 3.1 in the particular case $\tau = W'$ (making lemma 3.2 superfluous). Moreover, this allows a refinement of this proof : if $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h^2)$ (instead of only $o(h)$) then (by the same calculus) $w_h - \overline{\alpha}_\mu = o(h)$ (instead of only $\varepsilon(h)$). Using asymptotic expansions [Vo], an elementary calculus shows that this condition $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h^2)$ is obtained when $1 - \mu^2 - \frac{h}{2}(1 + \mu^2) = o(h^2)$. Thus we get

$$\overline{\alpha}_\mu = W_h'(\overline{\alpha}_1) + o(h) = W_h'(\overline{\alpha}_1 + \frac{\mathrm{i}\,h}{2}\varphi(\overline{\alpha}_1)) + o(h).$$

Since W' and ρ are $*$-preserving and $\varphi(\overline{\alpha}_1^*) = -\varphi(\overline{\alpha}_1) = 0$, we deduce the same property for $\overline{\alpha}_1^*$.

Step 3. More generally, let ρ and W' be two deformations, in the same field, of some algebra $\mathcal{B}_1$ generated by some subset X, such that $\forall f, g \in \mathcal{B}_1$,

$$\begin{aligned} \rho_h(f) &= W_h(f) + \varepsilon(h) \\ \rho_h(f)\rho_h(g) &= \rho_h(fg + \tfrac{\mathrm{i}\,h}{2}C(f,g)) + o(h), \\ W_h'(f)W_h'(g) &= W_h'(fg + \tfrac{\mathrm{i}\,h}{2}C'(f,g)) + o(h), \end{aligned}$$

$C = C' + \partial\varphi$ for some linear map φ, and (ii) holds for any generator $f \in X$. In order to extend (ii) to any $f \in \mathcal{B}_1$, it suffices to show that if (ii) holds for f and g then it holds for fg. Let us do it :

$$\begin{aligned} \rho_h(fg) &= \rho_h(f)\rho_h(g) - \rho_h(\tfrac{\mathrm{i}\,h}{2}C(f,g))) + o(h) \\ &= W_h'(f + \tfrac{\mathrm{i}\,h}{2}\varphi(f))W_h'(g + \tfrac{\mathrm{i}\,h}{2}\varphi(g)) - W_h'(\tfrac{\mathrm{i}\,h}{2}C(f,g))) + o(h) \\ &= W_h'(fg + \tfrac{\mathrm{i}\,h}{2}[f\varphi(g) + \varphi(f)g + C'(f,g) - C(f,g)]) + o(h) \\ &= W_h'(fg + \tfrac{\mathrm{i}\,h}{2}\varphi(fg)) + o(h). \end{aligned}$$

Remark. The first assertion (i) in this proposition gives another way to prove proposition 2.2.

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Lattices in semi-simple Lie groups, and multipliers of group C^*-algebras

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Lattices in semi-simple Lie groups, and multipliers of group C^*- algebras

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1 Introduction, and some history.

Let G be a locally compact group, and H be a closed subgroup. Viewing $L^1(G)$ as a two-sided ideal in the measure algebra $M(G)$, and viewing elements of $L^1(H)$ as measures on G supported inside H, we obtain an action of $L^1(H)$ on $L^1(G)$ as double centralizers. It is easy to check (see e.g. proposition 4.1 in [Rie]) that this action extends to an action of the full group C^*-algebra $C^*(H)$ as double centralizers on C^*(G); this corresponds to a $*$-homomorphism $j_H : C^*(H) \to M(C^*(G))$, where $M(C^*(G))$ denotes the multiplier C^*-algebra of $C^*(G)$. We now quote from p. 209 of Rieffel's Advances paper [Rie]:

It does not seem to be known whether this homomorphism j_H is injective. It will be injective if and only if every unitary representation of H is weakly contained in the restriction to H of some unitary representation of G [Fe2]. J.M.G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [Fe2]) depended on the completeness of the classification of the irreducible representations of $SL_3(\mathbb{C})$ given in [GeN], and there is now some doubt that this classification is complete [Ste].

Probably this quotation requires some word of explanation. In [Fe2], Fell studies extensions to the topological framework of Frobenius reciprocity for finite groups. Thus he introduces a list of weak Frobenius properties, the last and weakest one being (WF3):

The locally compact group G satisfies property (WF3) if, for any closed subgroup H of G, every representation σ in the dual $\hat{H}$ is weakly contained in the restriction $\pi|_H$ of some unitary representation π of G.

Property (WF3) is indeed equivalent to the injectivity of j_H for any closed subgroup H; for completeness, we shall give a proof in Proposition 2.1 below. In §6 of [Fe2], Fell wishes to show that even (WF3) may fail, by taking $G = SL_3(\mathbb{C})$ and $H = SL_2(\mathbb{C})$; to this end he appeals to the incomplete description of $\hat{G}$ given in [GeN]; Fell's proof was recently corrected in Remark 1.13(i) of [BLS].

In this paper, we take for G a semi-simple Lie group with finite centre and without compact factor, and as closed subgroup a lattice Γ. In section 3, we prove:

THEOREM 1.1 *Let G be a semi-simple Lie group without compact factors, with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be a non-trivial irreducible unitary representation of Γ of finite dimension n. Then σ*

determines a direct summand of $C^*(\Gamma)$ *which is contained in the kernel of* $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$*; this direct summand is isomorphic to the algebra* $M_n(\mathbb{C})$ *of* n*-by-*n *matrices.*

If G is a non-compact *simple* Lie group with finite centre, then G has property (T) unless G is locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$ (see [HaV]). For these two families, we prove in section 4:

THEOREM 1.2 *Let* G *be locally isomorphic either to* $SO_o(n,1)$ *or* $SU(n,1)$*, for some* $n \geq 2$*. Let* Γ *be a lattice in* G*. Denote by* $\hat{\Gamma}_f$ *the set of (classes of) irreducible, finite-dimensional unitary representations of* Γ*. If the trivial representation* 1_Γ *is not isolated in* $\hat{\Gamma}_f$ *(for the induced Fell-Jacobson topology), then infinitely many elements of* $\hat{\Gamma}_f$ *are not weakly contained in the restriction to* Γ *of any unitary representation of* G*. In particular* $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ *is not injective.*

In view of Theorems 1.1 and 1.2, it seems natural to formulate the following

Conjecture. If Γ is a lattice in a non-compact semi-simple Lie group G, then $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.

This conjecture means that, if ρ is a representation of G which is faithful on $M(C^*(G))$ (e.g. take for ρ either the universal representation of G, or the direct sum of all its irreducible representations), then $\rho|_\Gamma$ is never faithful on $C^*(\Gamma)$; this has bearing on a question of de la Harpe in his paper in these Proceedings (see immediately after Problem 13 in [Har]). In §5, we give examples of lattices Γ in $SO_o(n,1)$ or $SU(n,1)$ such that 1_Γ is not isolated in $\hat{\Gamma}_f$; this is the case for any lattice in $SL_2(\mathbb{R})$, any non-uniform lattice in $SL_2(\mathbb{C})$, and any arithmetic lattice in $SO_o(n,1)$ for $n \neq 3$, 7.

In the final §6, we come back to property (WF3) and show that it always fails for almost connected, non-amenable groups:

THEOREM 1.3 *Let* G *be an almost connected, locally compact group. The following properties are equivalent:*

(i) G *has Fell's property (WF3);*
(ii) G *is amenable.*

Observe that Theorem 1.3 cannot hold for any locally compact group. Indeed, any discrete group G satisfies property (WF3) since, given a subgroup H of G, one checks easily that $C^*(H)$ is a C^*-subalgebra of $C^*(G) = M(C^*(G))$.

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A word about terminology: as usual, semi-simple Lie groups are assumed to be connected and non-trivial; group representations are assumed to be unitary, strongly continuous, and on non-zero Hilbert spaces.

2 On multipliers of C^*-algebras.

For a C^*-algebra B, we denote by $M(B)$ its multiplier algebra.

PROPOSITION 2.1 *Let A, B be C^*-algebras, and let $j : A \to M(B)$ be a $*$-homomorphism. The following properties are equivalent:*

(i) j is one-to-one;
(ii) for any $\sigma \in \hat{A}$, there exists a non-degenerate $*$-representation π of B such that σ is weakly contained in $\tilde{\pi} \circ j$, where $\tilde{\pi}$ denotes the extension of π to $M(B)$;
(iii) any $\sigma \in \hat{A}$ is weakly contained in $\{\tilde{\pi} \circ j \mid \pi \in \hat{B}\}$.

Fell's property (WF3), mentioned in §1, is deduced from property (ii) above by taking $B = C^*(G)$ and $A = C^*(H)$, for any closed subgroup H of the locally compact group G.

Proof of Proposition 2.1. (i) $\Rightarrow$ (ii) Let us assume that j is injective, so that we may identify A with a C^*-subalgebra of $M(B)$. Let π be a faithful representation of B. It is known that the extension $\tilde{\pi}$ of π to $M(B)$ is also faithful ([Ped], 3.12.5). Thus any representation of A is weakly contained in the restriction of $\tilde{\pi}$ to A.

(ii) $\Rightarrow$ (iii) This follows from decomposition theory.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Fix a non-zero element x of A; choose $\sigma \in \hat{A}$ such that $\sigma(x) \neq 0$. Our assumption says that $Ker\,\sigma$ contains $\bigcap_{\pi\in\hat{B}} Ker\,\tilde{\pi} \circ j = Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$; in particular $x \notin Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$. It follows that $j(x) \neq 0$, i.e. that j is one-to-one.

3 Proof of theorem 1.1

We slightly generalize Theorem 1.1 in the following form:

THEOREM 3.1 *Let G be a non-compact semi-simple Lie group with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be an irreducible representation of Γ of finite dimension n, which is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G. Then σ determines a direct summand of $C^*(\Gamma)$ isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices, which moreover is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$.*

Observe that Theorem 1.1 is an immediate consequence of Theorem 3.1: indeed, if G has no compact factor, then any unitary, finite-dimensional representation of G is trivial.

Proof of Theorem 3.1. Since G has property (T), so has Γ (see [HaV], Théorème 4 in Chapter 3). Let σ be an irreducible representation of Γ, of finite dimension n. By Theorem 2.1 in [Wan], σ is isolated in the dual $\hat{\Gamma}$, hence determines a direct sum decomposition of $C^*(\Gamma)$:

$$C^*(\Gamma) = J \oplus M_n(\mathbb{C})$$

where J is the C^*-kernel of σ.

We assume from now on that σ is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G, and wish to prove that the direct summand $M_n(\mathbb{C})$ lies in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$. Suppose by contradiction that j_Γ is non-zero on $M_n(\mathbb{C})$. Choose $\pi \in \hat{G}$ such that $\tilde{\pi} \circ j_\Gamma$ is non-zero, hence faithful on $M_n(\mathbb{C})$ (here $\tilde{\pi}$ denotes the extension of π to $M(C^*(G))$, as in Proposition 2.1). Then the C^*-kernel of $\tilde{\pi} \circ j_\Gamma$ is contained in J, which means that σ is weakly contained in the restriction $\pi|_\Gamma$. As σ is isolated in $\hat{\Gamma}$, this implies that σ is actually a subrepresentation of $\pi|_\Gamma$ (see Corollary 1.9 in [Wan]). Our assumption shows that π is infinite-dimensional. Two cases may occur:

(a) π is a discrete series representation of G (if any); this would imply that σ is an irreducible subrepresentation of the left regular representation of Γ, which in turn implies that Γ is finite - and this is absurd.

(b) π is not in the discrete series of G; then, by a result of Cowling and Steger (Proposition 2.4 in [CoS]), the restriction $\pi|_\Gamma$ is irreducible, which contradicts the fact that σ is a finite-dimensional subrepresentation.

With a contradiction reached in both cases, the proof of Theorem 3.1 is complete. We thank G. Skandalis for a helpful conversation that led to a more explicit version of Theorem 3.1.

Remark. Let us show that there are countably many finite-dimensional elements $\sigma \in \hat{\Gamma}$ satisfying the assumptions of Theorem 3.1.

Thus, let $G/Z(G)$ be the adjoint group of G; this is a linear group. Denote by Γ_1 the image of Γ in $G/Z(G)$; as a finitely generated linear group, Γ_1 is residually finite (see [Mal]); a non-trivial irreducible representation σ of Γ that factors through a finite quotient of Γ_1 cannot be contained in the restriction to Γ of a finite-dimensional unitary representation of G.

This argument shows that $Ker[j_\Gamma : C^*(\Gamma) \to M(C^*(G))]$ contains the C^*-direct sum of countably many matrix algebras.

4 The cases $SO_o(n,1)$ and $SU(n,1)$.

We begin with the following result, which is certainly known to many experts (see [Moo], Proposition 3.6; compare also with [Mar], Chap. III, (1.12), Remark 1).

PROPOSITION 4.1 *Let G be a simple Lie group with finite centre, and let Γ be a lattice in G. Denote by γ the quasi-regular representation of G on $L^2(G/\Gamma)$, and by γ_0 the restriction of γ to $L^2_o(G/\Gamma) = \{f \in L^2(G/\Gamma) | < f|1 >= 0\}$.*

(a) There exists $N \in \mathbb{N}$ such that the N-fold tensor product $\gamma_0^{\otimes N}$ is weakly contained in the left regular representation λ_G of G.

(b) The trivial representation 1_G is not weakly contained in γ_0.

Proof. (a) Suppose first that G has Kazhdan's property (T). Then, by Theorems 2.4.2 and 2.5.3 in [Cow], there exists $N \in \mathbb{N}$ such that $\pi^{\otimes N}$ is weakly contained in

λ_G for any unitary representation π of G which does not contain 1_G. This implies the result.

Suppose now that G is locally isomorphic either to $SO_o(n,1)$ or to $SU(n,1)$. Let K be a maximal compact subgroup of G. Let $\hat{G}_1 = \{\pi \in \hat{G} \mid \pi|_K \text{ contains } 1_K\}$ be the set of all spherical representations of G. Observe that $\hat{G}_1$ is open in $\hat{G}$ (because $\pi \in \hat{G}_1$ if and only if there exists $\xi \in H_\pi$ such that $\int_K < \pi(k)\xi|\xi > dk \neq 0$). For a unitary representation σ of G, set $Supp\,\sigma = \{\pi \in \hat{G} \mid \pi \text{ is weakly contained in } \sigma\}$. By Proposition 3.6 in [Moo], the existence of $N \in \mathbb{N}$ such that $\sigma^{\otimes N}$ is weakly contained in λ_G is equivalent to $1_G \notin Supp\,\sigma \cap \hat{G}_1$ (the proof of this uses the explicit description of the unitary duals of $SO_o(n,1)$ and $SU(n,1)$). So we must prove that 1_G is not in $Supp\,\gamma_0 \cap \hat{G}_1$ or, equivalently, that 1_G is isolated in $Supp\gamma \cap \hat{G}_1$.

Recall the standard parametrization of $\hat{G}_1$. Let ρ be half the sum of the positive roots associated with a maximal split torus of G. Then $\hat{G}_1$ identifies (topologically) with $i\mathbb{R}^+ \cup [0,\rho]$, the representations π_s with $s \in i\mathbb{R}^+$ being the spherical principal series representations, those π_s with $s \in]0,\rho[$ being the spherical complementary series representations, and π_ρ being the trivial representation 1_G.

Let X be the Riemannian symmetric space associated with G. The Laplace-Beltrami operator Δ on X is invariant for the left action of G, so it descends to a positive, unbounded operator on $L^2(\Gamma\backslash X)$. It is well-known that π_s is weakly contained in γ if and only if $\rho^2 - s^2$ belongs to the spectrum of Δ on $L^2(\Gamma\backslash X)$ (see §4 of Chap. I in [GGP] for $G = SL_2(\mathbb{R})$ and Γ uniform, or Theorem 1.7.10 in [GaV] for the general case; note that this Theorem is stated there for the quasi-regular representation of G on $L^2(X)$, but the proof extends word for word to our representation γ).

Denoting by $\lambda_1(\Gamma\backslash X)$ the bottom of the spectrum of the restriction of Δ to the orthogonal of constants in $L^2(\Gamma\backslash X)$, we see that our result follows from $\lambda_1(\Gamma\backslash X) > 0$. In turn, this is a consequence of the facts that the continuous spectrum of Δ on $L^2(\Gamma\backslash X)$ is the half-line $[\rho^2, \infty[$ (see [OsW]), and that its discrete spectrum is a sequence increasing to ∞ (see Theorem 3 in [BoG]). In our case, $\lambda_1(G\backslash X) > 0$ can also be deduced from the fact that $\lambda_1(M) > 0$ for any complete Riemannian manifold M with finite volume and pinched negative sectional curvature (see [Dod]).

(b) This follows from (a) and non-amenability of G.

Proof of Theorem 1.2

We shall use several times Fell's inner hull-kernel topology, which is defined on sets of unitary (not necessarily irreducible) representations of a locally compact group (cf. [Fel], section 2): a net $(\pi_i)_{i\in I}$ of representations converges to a representation π if and only if π is weakly contained in $\{\pi_j \mid j \in J\}$ for each subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$.

Assume that G and Γ satisfy the assumptions of Theorem 1.2. We are going to show that Fell's property (WF3) fails for the pair (G,Γ); i.e., we shall produce some $\sigma \in \hat{\Gamma}$ such that σ is not weakly contained in the set $\{\pi|_\Gamma \mid \pi \in \hat{G}\}$.

Since 1_Γ is not isolated in $\hat{\Gamma}_f$, there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\hat{\Gamma}_f - \{1_\Gamma\}$ that converges to 1_Γ .

1st step: There exists a sequence of integers $n_1 < n_2 < \ldots$, and spherical complementary series representations π_{n_k} of G such that π_{n_k} is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any k, and $\lim_{k\to\infty} \pi_{n_k} = 1_G$.

Indeed, by continuity of induction ([Fel], Theorem 4.1),

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = Ind_\Gamma^G 1_\Gamma = \gamma.$$

Since 1_G is a subrepresentation of γ, we also have

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = 1_G.$$

This implies that there exists integers $n_1 < n_2 < \ldots$, and irreducible representations π_{n_k} of G such that π_{n_k} is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any k, and such that $\lim_{k\to\infty} \pi_{n_k} = 1_G$ (cf. proof of Lemme 2, §1, in [Bur]). Since the spherical dual $\hat{G}_1$ is open in $\hat{G}$, and since G is not amenable, we can clearly assume that π_{n_k} is either 1_G, or a spherical complementary series representation. To exclude the case $\pi_{n_k} = 1_G$, we are going to show that 1_G is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$; this can be viewed as a form of Frobenius reciprocity.

Indeed, since σ_{n_k} is finite-dimensional, σ_{n_k} does not contain 1_G weakly. Moreover, we know by Proposition 4.1(b) that 1_G is isolated in $Supp\,\gamma$. Hence, by a result of Margulis ([Mar], Chap. III, (1.11)(b)), 1_G is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$. This proves the 1st step.

Let $\pi_{n_k} \in \hat{G}$ be a sequence as above. By Proposition 4.1(a), there exists $N \in \mathbb{N}$ such that $\gamma_0^{\otimes N}$ is weakly contained in λ_G. Since $\lim_{k\to\infty} \pi_{n_k} = 1_G$, we see that $\pi_{n_l}^{\otimes N}$ is not weakly contained in λ_G for $l \in \mathbb{N}$ big enough. Fix such an l, and set $\sigma = \sigma_{n_l}$ and $\pi = \pi_{n_l}$.

2nd step: σ is not weakly contained in $\{\rho|_\Gamma : \rho \in \hat{G}\}$. Indeed, assume by contradiction that there exists a sequence $\rho_n \in \hat{G}$ with $\lim_{n\to\infty} \rho_n|_\Gamma = \sigma$.

Then $\lim_{n\to\infty} Ind_\Gamma^G \rho_n|_\Gamma = Ind_\Gamma^G \sigma$. Hence, since π is weakly contained in $Ind_\Gamma^G \sigma$:

$$\lim_{n\to\infty} Ind_\Gamma^G (\rho_n|_\Gamma) = \pi\,.$$

But

$$Ind_\Gamma^G (\rho_n|_\Gamma) = \rho_n \otimes Ind_\Gamma^G 1_\Gamma = \rho_n \oplus (\rho_n \otimes \gamma_0).$$

Since π is irreducible, this implies (upon passing to a subsequence) that either $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ or $\lim_{n\to\infty} \rho_n = \pi$.

We first exclude the case $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$. Indeed, $(\rho_n \otimes \gamma_0)^{\otimes N} = \rho_n^{\otimes N} \otimes \gamma_0^{\otimes N}$ is weakly contained in λ_G. Hence, $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ would imply that $\pi^{\otimes N} = \lim_{n\to\infty} (\rho_n \otimes \gamma_0)^{\otimes N}$ is weakly contained in λ_G; this would contradict our choice of π.

It remains to exclude the case $\lim_{n\to\infty} \rho_n = \pi$. Since the set $\hat{G_1}^c = \{\pi_s \mid s \in]0, r[\}$ of all spherical complementary series representations is open in $\hat{G}$ and since $\pi \in \hat{G_1}^c$, we can clearly assume that $\rho_n \in \hat{G_1}^c$ for all n. Then, there exists $s_o \in]0, r[$ such that, for all n:

$$\rho_n \in \{\pi_s : 0 < s < s_o\}.$$

Therefore, there exists $M \in \mathbb{N}$ such that $\rho_n^{\otimes M}$ is weakly contained in λ_G, for all $n \in \mathbb{N}$. Hence

$$\sigma^{\otimes M} = \lim_{n\to\infty} (\rho_n^{\otimes M})|_\Gamma$$

is weakly contained in λ_Γ. Since $\sigma^{\otimes M}$ is finite-dimensional, this contradicts non-amenability of G. This concludes the proof of Theorem 1.2.

Remark: In our previous paper [BeV], Theorem 1.2 was already proved for $G = PSL_2(\mathbb{R})$ and Γ the fundamental group of a closed Riemann surface of genus 2.

5 Some examples of lattices in $SO_o(n, 1)$ and $SU(n, 1)$.

Let Γ be a lattice in a simple Lie group locally isomorphic either to $SO_o(n, 1)$ or $SU(n, 1)$. Let us denote by $\hat{\Gamma}_{fq}$ the set of elements of $\hat{\Gamma}$ that factor through some finite quotient of Γ.

DEFINITION 5.1 *We say that Γ satisfies property (*) if the trivial representation 1_Γ is not isolated in $\hat{\Gamma}_{fq}$, for the induced Fell-Jacobson topology.*

Our property (*) is precisely the negation of property $(T; R(\Im))$ in the notation of Lubotzky-Zimmer [LuZ], where a lucid discussion of this property appears on pp. 291-292. Since $\hat{\Gamma}_{fq}$ is a subset of $\hat{\Gamma}_f$, it is clear that, if Γ satisfies (*), then 1_Γ is not isolated in $\hat{\Gamma}_f$; note the question at the bottom of p. 291 of [LuZ] whether or not the converse implication holds.

The purpose of this section is to give examples of lattices with property (*), i.e. for which Theorem 1.2 is true. We begin with a sufficient condition for property (*).

PROPOSITION 5.2 *If Γ has a finite index subgroup Γ_o that maps homomorphically onto $\mathbb{Z}$, then Γ has property (*).*

Proof. Let $(\chi_m)_{m\in\mathbb{N}}$ be a sequence of non-trivial characters of finite order of $\mathbb{Z}$, viewed as characters of Γ_o, that converges to the trivial character. Set:

$$\pi_m = Ind_{\Gamma_o}^{\Gamma}\chi_m \, .$$

Claim: π_m factors through some finite quotient of Γ. Indeed, since χ_m has finite order, the subgroup $Ker\,\chi_m$ of Γ_o has finite index in Γ, so there exists a normal subgroup N_m of Γ, of finite index and contained in $Ker\,\chi_m$. Then π_m factors through the finite group Γ/N_m , which establishes the claim.

The rest of the proof is similar in spirit to the first step of the proof of Theorem 1.2, but considerably easier: by continuity of induction, the sequence $(\pi_m)_{m\in\mathbb{N}}$ converges to the quasi-regular representation λ_o of Γ on $l^2(\Gamma/\Gamma_o)$. Since λ_o contains the trivial representation 1_Γ, we may select for any $m \in \mathbb{N}$ an irreducible component σ_m of π_m in such a way that the sequence $(\sigma_m)_{m\in\mathbb{N}}$ converges to 1_Γ in $\hat{\Gamma}$. By the claim, each σ_m lies in $\hat{\Gamma}_{fq}$; finally, no σ_m may be trivial, by Frobenius reciprocity. This shows that 1_Γ is not isolated in $\hat{\Gamma}_{fq}$.

Because Γ is finitely generated, the condition that Γ_o maps homomorphically onto $\mathbb{Z}$ is equivalent to the non-vanishing of the first cohomology $H^1(\Gamma_o, \mathbb{C})$. This is known to have deep representation-theoretic consequences, as it gives information on the decomposition of $L^2(G/\Gamma_o)$ into irreducibles (see the whole of Chapter VII in [BoW], and especially Propositions 4.9 and 4.11). There is a conjecture, sometimes attributed to Thurston (see e.g. [Bor], 2.8), according to which any uniform lattice Γ in $SO_o(n,1)$ ($n \geq 2$) admits a finite index subgroup Γ_o such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$. Next proposition summarizes what we know about this problem, both in the uniform and non-uniform cases.

PROPOSITION 5.3 *The following lattices Γ in $SO_o(n,1)$ admit a finite index subgroup Γ_o such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$, and hence satisfy property (*):*

(i) any lattice in $PSL_2(\mathbb{R}) \simeq SO_o(2,1)$;
(ii) any non-uniform lattice in $PSL_2(\mathbb{C}) \simeq SO_o(3,1)$;
(iii) any uniform lattice Γ in $PSL_2(\mathbb{C})$ such that, for some x in the 3-dimensional real hyperbolic space $H_3(\mathbb{R})$, the orbit $\Gamma.x$ is invariant under some orientation-reversing involutive isometry of $H_3(\mathbb{R})$;
(iv) any arithmetic lattice, provided $n \neq 3, 7$; any non-uniform arithmetic lattice, without restriction on n.

Proof. The proof is compilation; however, it makes constant use of Selberg's lemma asserting that any lattice has a torsion-free subgroup of finite index.

(i) A torsion-free lattice in $PSL_2(\mathbb{R})$ is either a surface group (in the uniform case) or a non-abelian free group (in the non-uniform case); in any case, it surjects onto $\mathbb{Z}$.
(ii) Any torsion-free non-uniform lattice in $PSL_2(\mathbb{C})$ surjects onto $\mathbb{Z}$, by Propositions 5.1 and 3.1 in [Lub].
(iii) See Theorem 3.3 and Corollary 3.4 in [Hem]. Explicit examples of such lattices are given in §4 of [Hem].
(iv) The first statement is the main result of [LiM]. For the second one, combine the main result in [Mil] with the remarks on p. 365 of [LiM].

Concerning uniform lattices in $PSL_2(\mathbb{C})$, it seems appropriate to mention here the connection with a somewhat (in)famous question which is for sure due to Thurston (question 18 in [Thu]): does any complete, finite-volume, hyperbolic 3-manifold have a finite-sheeted cover that fibers over the circle S^1? An affirmative answer would imply that any lattice Γ in $PSL_2(\mathbb{C})$ satisfies property (*) (indeed, let Γ_1 be a torsion-free subgroup of finite index in Γ; then Γ_1 is the fundamental group, $\pi_1(M)$, of a complete finite-volume hyperbolic 3-manifold M; if N is a finite-sheeted cover of M which fibers over S^1, then $\Gamma_o = \pi_1(N)$ is a finite-index subgroup of Γ_1 that maps onto $\pi_1(S^1) = \mathbb{Z}$). For an example of a compact hyperbolic 3-manifold that does not fiber over S^1 but with a finite-sheeted cover that does, see example 2.1 in [Gab][1].

[1]Clearly, for a 3-dimensional closed hyperbolic manifold M, fibering over S^1 is a much stronger condition than having non-zero first Betti number. Algebraically, this can be seen by Stallings'fibration theorem [Sta]: if N is a normal subgroup of $\pi_1(M)$ such that $\pi_1(M)/N = \mathbb{Z}$, then N comes from a fibration of M over S^1 if and only if N is a finitely generated subgroup. Also, surface groups in $\pi_1(M)$ that come from some finite-sheeted cover of M fibering over S^1 (so-called virtual fibre groups) have been characterized algebraically in Corollary 1 of [Som].

In contrast with Proposition 5.3, we are not aware of any "large" class of lattices in $SU(n,1)$ that satisfies property (*). Essentially the only result we know is that, for any $n \geq 2$, there exists a uniform arithmetic lattice Γ in $SU(n,1)$ such that $H^1(\Gamma,\mathbb{C}) \neq 0$ (see Theorem 1 in [Kaz], or Theorem 1.4(b) in [Li]).

To conclude, let us indicate why, for a given lattice Γ in $SO_o(n,1)$, it is usually difficult to check that Γ satisfies property (*). Assume that Γ is arithmetic. It is then easy to construct elements of $\hat{\Gamma}_{fq}$: take a congruence subgroup $\Gamma(\wp)$ and consider irreducible representations of Γ that factor through the finite group $\Gamma/\Gamma(\wp)$ (for $\Gamma = SL_2(\mathbb{Z})$, these are representations that factor through some $SL_2(\mathbb{Z}/n\mathbb{Z})$). Denote by $\hat{\Gamma}_{arith}$ the subset of elements in $\hat{\Gamma}_{fq}$ that factor through some $\Gamma/\Gamma(\wp)$; then it follows from Selberg's inequality (see [Sel] for $n=2$, and corollary 1.3 in [BuS] for $n>2$) that the trivial representation 1_Γ is isolated in $\hat{\Gamma}_{arith}$. Thus, if Γ verifies property (*), any non-stationary net in $\hat{\Gamma}_{fq}$ that converges to 1_Γ will have to leave $\hat{\Gamma}_{arith}$ eventually.

We have been informed by M. Burger that, in unpublished work with P. Sarnak, similar phenomena have been obtained for a large class of arithmetic lattices in $SU(n,1)$.

6 Proof of Theorem 1.3

We begin with hereditary properties of the class of groups satisfying Fell's property (WF3).

LEMMA 6.1 *Let G be a locally compact group with property (WF3).*

(a) Any closed subgroup of G has property (WF3).

(b) Let K be a compact normal subgroup of G; then G/K has property (WF3).

Proof. (a) is obvious. To see (b), denote by $p: G \to G/K$ the quotient map. Let L be a closed subgroup of G/K; fix $\tau \in \hat{L}$. Set $H = p^{-1}(L)$ and $\sigma = \tau \circ (p|_H)$. Let π be a representation of G on a Hilbert space $\mathcal{H}$ such that $\pi|_H$ weakly contains σ. Let $\mathcal{H}^K$ be the space of K-fixed vectors in $\mathcal{H}$. Since K is a normal subgroup, $\mathcal{H}^K$ is an invariant subspace of π, and we denote by π_o the restriction of π to $\mathcal{H}^K$. Since K is compact and σ is irreducible, it is easy to see that σ is weakly contained in $\pi_o|_H$. But $\pi_o|_H$ can be viewed as a representation of $L = H/K$, that weakly contains τ.

Next lemma is probably well-known.

LEMMA 6.2 *Let G be a Lie group, and let S be a semisimple analytic subgroup. The closure $\bar{S}$ of S is reductive.*

Proof. We begin with a

Claim: Let h be a finite-dimensional Lie algebra, and let s be a semisimple ideal; then there exists an ideal j of h such that $h = s \oplus j$. Indeed, let $Der(s)$ be the Lie algebra of derivations of s. Since s is an ideal in h, we have a Lie algebra homomorphism:

$$\alpha : h \to Der(s) : X \to ad(X)_{|s}$$

the kernel of which is precisely the centralizer of s in h; set $j = Ker\,\alpha$. Since s is semisimple, $Der(s)$ is canonically isomorphic to s, so that α is onto and $h = s \oplus j$; this establishes the claim.

To prove Lemma 6.2, denote by s and h the Lie algebras of S and $\bar{S}$ respectively. Clearly $Ad(x)(s) = s$ for any x in S, so by density the same is true for any x in $\bar{S}$. This shows that s is an ideal in h. By the claim, there exists an ideal j of h such that $h = s \oplus j$. To see that h is reductive, it is enough to prove that j is central in h. But, for $X \in j$, we have $Ad(x)(X) = X$ for any x in S; again by density, this remains true for any x in $\bar{S}$; so X is central in h.

Proof of Theorem 1.3 It is easy to see that any amenable group G satisfies property (WF3); indeed, for a closed subgroup H of G, by amenability of H any representation of H is weakly contained in the left regular representation of H, which is itself contained in the restriction to H of the left regular representation of G; see also Corollary 1.5 in [BLS] for another proof.

Let us now prove the converse, namely that any almost connected locally compact group G with property (WF3) is amenable. In this proof, the stability of amenability under short exact sequences will be used constantly.

1st step: reduction to the connected case. Let G_0 be the connected component of the identity of G. By Lemma 6.1(a), G_0 has property (WF3). If G_0 is amenable, then so is G, since G/G_0 is compact.

2nd step: reduction to the Lie group case. Let G be a connected group with property (WF3). By the structure theory for connected groups, G admits a compact normal subgroup K such that G/K is a Lie group. By Lemma 6.1(b), G/K has property (WF3). If G/K is amenable, then so is G, since K is compact.

3rd step: reduction to the reductive case. Let G be a connected Lie group with property (WF3). Let $G = RS$ be a Levi decomposition, with R the solvable radical and S a semisimple analytic subgroup. Then the closure $\bar{S}$ is reductive with property (WF3), by Lemmas 6.1(a) and 6.2. If $\bar{S}$ is amenable, then so is $\bar{S}/(\bar{S} \cap R) = G/R$, hence so is G.

Coda. Let G be a connected, reductive Lie group with property (WF3). The adjoint group $G/Z(G)$ is a semisimple Lie group without centre, so it decomposes as a direct product

$$G/Z(G) = G_1 \times \cdots \times G_n$$

of simple Lie groups without centre. To prove that G is amenable, we have to show that G_j is compact for $j = 1, \cdots, n$. So suppose by contradiction that some G_j, say G_1, is not compact. By root theory, G_1 then contains a 3-dimensional analytic subgroup L which is locally isomorphic to $SL_2(\mathbb{R})$. Because G_1 is centreless, hence linear, L is closed in G_1 (any semisimple analytic subgroup in a linear group is closed, see Theorem 2 in [Got]). By the proof of Theorem 3 in [BeV], there exists a lattice Γ in L and a representation $\tau \in \hat{\Gamma}_f$ such that $\tau \otimes \bar{\tau}$ is not weakly contained in the restriction to Γ of any unitary representation of L. Denote by $p : G \to G_1$ the homomorphism obtained by composing the quotient map $G \to G/Z(G)$ with the projection of $G/Z(G)$ onto G_1. Set $H = p^{-1}(\Gamma)$ and $\sigma = \tau \circ (p|_H)$. Because H is closed in G, we find by property

(WF3) a net $(\rho_i)_{i\in I}$ in $\hat{G}$ such that

$$lim_i\, \rho_i|_H = \sigma\,.$$

Then:

$$lim_i(\rho_i \otimes \bar{\rho}_i)|_H = \sigma \otimes \bar{\sigma}\,.$$

But, since ρ_i is irreducible, the representation $\rho_i \otimes \bar{\rho}_i$ of G is trivial on $Z(G)$, so it factors through a representation π_i of $G/Z(G)$. Last formula then reads:

$$lim_i\, \pi_i|_\Gamma = \tau \otimes \bar{\tau}$$

and this contradicts our choice of G and τ.

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Tensor products of $C(X)$-algebras over $C(X)$

Etienne Blanchard

0 Introduction

Tensor products of C*-algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\min} B$ of two C*-algebras A and B always defines the minimal C*-norm on the algebraic tensor product $A \otimes_{alg} B$ of A and B over the complex field $\mathbf{C}$.

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$-algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$-algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$-algebras framework.

Let us introduce the following definition:

DEFINITION 0.1 *Given two $C(X)$-algebras A and B, we denote by $\mathcal{I}(A,B)$ the involutive ideal of the algebraic tensor product $A \otimes_{alg} B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.*

Our aim in the present article is to study the C*-norms on the algebraic tensor product $(A \otimes_{alg} B)/\mathcal{I}(A,B)$ of two $C(X)$-algebras A and B over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A,B) \subset A \otimes_{alg} B$ which contains $\mathcal{I}(A,B)$ such that every C*-semi-norm on $A \otimes_{alg} B$ which is zero on $\mathcal{I}(A,B)$ is also zero on $\mathcal{J}(A,B)$ and we prove that there always exist a minimal C*-norm $\| \|_m$ and a maximal C*-norm $\| \|_M$ on the quotient $(A \otimes_{alg} B)/\mathcal{J}(A,B)$.

We then study the following question of G.A. Elliott ([5]): when do the two ideals $\mathcal{I}(A,B)$ and $\mathcal{J}(A,B)$ coincide?

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1 Preliminaries

We briefly recall here the basic properties of $C(X)$-algebras.

Let X be a Hausdorff compact space and $C(X)$ be the C*-algebra of continuous functions on X. For $x \in X$, define the morphism $e_x : C(X) \to \mathbf{C}$ of evaluation at x and denote by $C_x(X)$ the kernel of this map.

DEFINITION 1.1 *([10]) A $C(X)$-algebra is a C*-algebra A endowed with a unital morphism from $C(X)$ in the center of the multiplier algebra $M(A)$ of A.*

We associate to such an algebra the unital $C(X)$-algebra $\mathcal{A}$ generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, denote by A_x the quotient of A by the closed ideal $C_x(X)A$ and by a_x the image of $a \in A$ in the fibre A_x. Then, as

$$\|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\},$$

the map $x \mapsto \|a_x\|$ is upper semi-continuous for all $a \in A$ ([14]).

Note that the map $A \to \oplus A_x$ is a monomorphism since if $a \in A$, there is a pure state ϕ on A such that $\phi(a^*a) = \|a\|^2$. As the restriction of ϕ to $C(X) \subset M(A)$ is a character, there exists $x \in X$ such that ϕ factors through A_x and so $\phi(a^*a) = \|a_x\|^2$.

Let $S(A)$ be the set of states on A endowed with the weak topology and let $\mathcal{S}_X(A)$ be the subset of states φ whose restriction to $C(X) \subset M(A)$ is a character, i-e such that there exists an $x \in X$ (denoted $x = p(\varphi)$) verifying $\varphi(f) = f(x)$ for all $f \in C(X)$. Then the previous paragraph implies that the set of pure states $P(A)$ on A is included in $\mathcal{S}_X(A)$.

Let us introduce the following notation: if $\mathcal{E}$ is a Hilbert A-module where A is a C*-algebra, we will denote by $\mathcal{L}_A(\mathcal{E})$ or simply $\mathcal{L}(\mathcal{E})$ the set of bounded A-linear operators on $\mathcal{E}$ which admit an adjoint ([9]).

DEFINITION 1.2 *([3]) Let A be a $C(X)$-algebra.*

A $C(X)$-representation of A in the Hilbert $C(X)$-module $\mathcal{E}$ is a morphism $\pi : A \to \mathcal{L}(\mathcal{E})$ which is $C(X)$-linear, i.e. such that for every $x \in X$, the representation $\pi_x = \pi \otimes e_x$ in the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{e_x} \mathbf{C}$ factors through a representation of A_x. Furthermore, if π_x is a faithful representation of A_x for every $x \in X$, π is said to be a field of faithful representations of A.

A continuous field of states on A is a $C(X)$-linear map $\varphi : A \to C(X)$ such that for any $x \in X$, the map $\varphi_x = e_x \circ \varphi$ defines a state on A_x.

If π is a $C(X)$-representation of the $C(X)$-algebra A, the map $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous since $\langle \xi, \pi(a)\eta \rangle \in C(X)$ for every $\xi, \eta \in \mathcal{E}$. Therefore, if A admits a field of faithful representations π, the map $x \mapsto \|a_x\| = \|\pi_x(a)\|$ is continuous for every $a \in A$, which means that A is a continuous field of C*-algebras over X ([4]).

The converse is also true ([3] théorème 3.3): given a separable $C(X)$-algebra A, the following assertions are equivalent:

1. A is a continuous field of C*-algebras over X,

2. the map $p : \mathcal{S}_X(A) \to X$ is open,
3. A admits a field of faithful representations.

2 C*-norms on $(A \otimes_{alg} B)/\mathcal{J}(A, B)$

DEFINITION 2.1 *Given two $C(X)$-algebras A and B, we define the involutive ideal $\mathcal{J}(A, B)$ of the algebraic tensor product $A \otimes_{alg} B$ of elements $\alpha \in A \otimes_{alg} B$ such that $\alpha_x = 0$ in $A_x \otimes_{alg} B_x$ for every $x \in X$.*

By construction, the ideal $\mathcal{I}(A, B)$ is included in $\mathcal{J}(A, B)$.

PROPOSITION 2.2 *Assume that $\| \ \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A \otimes_{alg} B$ of two $C(X)$-algebras A and B.*

If $\| \ \|_\beta$ is zero on the ideal $\mathcal{I}(A, B)$, then

$$\|\alpha\|_\beta = 0 \text{ for all } \alpha \in \mathcal{J}(A, B).$$

Proof: Let D_β be the Hausdorff completion of $A \otimes_{alg} B$ for $\| \ \|_\beta$. By construction, D_β is a quotient of $A \otimes_{\max} B$. Furthermore, if C_Δ is the ideal of $C(X \times X)$ of functions which are zero on the diagonal, the image of C_Δ in $M(D_\beta)$ is zero.

As a consequence, the map from $A \otimes_{\max} B$ onto D_β factors through the quotient $A \overset{M}{\otimes}_{C(X)} B$ of $A \otimes_{\max} B$ by $C_\Delta \times (A \otimes_{\max} B)$.

But an easy diagram-chasing argument shows that $(A \overset{M}{\otimes}_{C(X)} B)_x = A_x \otimes_{\max} B_x$ for every $x \in X$ ([3] corollaire 3.17) and therefore the image of $\mathcal{J}(A, B) \subset A \otimes_{\max} B$ in $A \overset{M}{\otimes}_{C(X)} B$ is zero. □

2.1 The maximal C*-norm

DEFINITION 2.3 *Given two $C(X)$-algebras A_1 and A_2, we denote by $\| \ \|_M$ the C*-semi-norm on $A_1 \otimes_{alg} A_2$ defined for $\alpha \in A_1 \otimes_{alg} A_2$ by*

$$\|\alpha\|_M = \sup\{\|(\sigma_1^x \otimes_{\max} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$.

As $\| \ \|_M$ is zero on the ideal $\mathcal{J}(A_1, A_2)$, if we identify $\| \ \|_M$ with the C*-semi-norm induced on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$, we get:

PROPOSITION 2.4 *The semi-norm $\| \ \|_M$ is the maximal C*-norm on the quotient $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: By construction, $\| \ \|_M$ defines a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Moreover, as the quotient $A_1 \overset{M}{\otimes}_{C(X)} A_2$ of $A_1 \otimes_{\max} A_2$ by $C_\Delta \times (A_1 \otimes_{\max} A_2)$ maps injectively in

$$\bigoplus_{x \in X} (A_1 \overset{M}{\otimes}_{C(X)} A_2)_x = \bigoplus_{x \in X} ((A_1)_x \otimes_{\max} (A_2)_x),$$

the norm of $A_1 \overset{M}{\otimes}_{C(X)} A_2$ coincides on the dense subalgebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ with $\| \|_M$. But we saw in proposition 2.2 that if $\| \|_\beta$ is a C*-norm on the algebra $(A \otimes_{alg} B)/\mathcal{J}(A, B)$, the completion of $(A \otimes_{alg} B)/\mathcal{J}(A, B)$ for $\| \|_\beta$ is a quotient of $A \overset{M}{\otimes}_{C(X)} B$. □

2.2 The minimal C*-norm

DEFINITION 2.5 *Given two $C(X)$-algebras A_1 and A_2, we define the semi-norm $\| \|_m$ on $A_1 \otimes_{alg} A_2$ by the formula*

$$\|\alpha\|_m = \sup\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$ and we denote by $A_1 \overset{m}{\otimes}_{C(X)} A_2$ the Hausdorff completion of $A_1 \otimes_{alg} A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1 \overset{m}{\otimes}_{C(X)} A_2)_x \to (A_1)_x \otimes_{\min} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\| \|_m$ induces a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. We are going to prove that this C*-norm defines the minimal C*-norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Let us introduce some notation.

Given two unital $C(X)$-algebras A_1 and A_2, let $P(A_i) \subset \mathcal{S}_X(A_i)$ denote the set of pure states on A_i and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples (ω_1, ω_2) such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_i) \to X$ is the restriction to $P(A_i)$ of the map $p : \mathcal{S}_X(A_i) \to X$ defined in section 1.

LEMMA 2.6 *Assume that $\| \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A_1 \otimes_{alg} A_2$ of two unital $C(X)$-algebras A_1 and A_2 which is zero on the ideal $\mathcal{J}(A_1, A_2)$ and define the closed subset $S_\beta \subset P(A_1) \times_X P(A_2)$ of couples (ω_1, ω_2) such that*

$$\Big|(\omega_1 \otimes \omega_2)(\alpha)\Big| \leq \|\alpha\|_\beta \text{ for all } \alpha \in A_1 \otimes_{alg} A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Proof: Define for $i = 1, 2$ the adjoint action ad of the unitary group $\mathcal{U}(A_i)$ of A_i on the pure states space $P(A_i)$ by the formula

$$[(ad_u)\omega](a) = \omega(u^* a u).$$

Then S_β is invariant under the product action $ad \times ad$ of $\mathcal{U}(A_1) \times \mathcal{U}(A_2)$ and we can therefore find non empty open subsets $U_i \subset P(A_i)$ which are invariant under the action of $\mathcal{U}(A_i)$ such that $(U_1 \times U_2) \cap S_\beta = \emptyset$.

Now, if K_i is the complement of U_i in $P(A_i)$, the set

$$K_i^\perp = \{a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i\}$$

is a non empty ideal of A_i and furthermore, if $\omega \in P(A_i)$ is zero on $K_i^\perp$, then ω belongs to K_i ([8] lemma 8,[15]).

As a consequence, if (φ_1, φ_2) is a point of $U_1 \times_X U_2$, there exist non zero self-adjoint elements $a_i \in K_i^\perp$ such that $\varphi_i(a_i) = 1$. If $x = p(\varphi_i)$, this implies in particular that $(a_1)_x \otimes (a_2)_x \neq 0$, and hence $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$. □

LEMMA 2.7 *([15] theorem 1) Let A_1 and A_2 be two unital $C(X)$-algebras.*

If the algebra A_1 is an abelian algebra, there exists only one C*-*norm on the quotient* $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Proof: Let $\| \ \|_\beta$ be a C*-semi-norm on $A_1 \otimes_{alg} A_2$ such that for all $\alpha \in A_1 \otimes_{alg} A_2$, $\|\alpha\|_\beta = 0$ if and only if $\alpha \in \mathcal{J}(A_1, A_2)$.

If $\rho \in P(A_\beta)$ is a pure state on the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for the semi-norm $\| \ \|_\beta$, then for every $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$,

$$\rho(a_1 \otimes a_2) = \rho(a_1 \otimes 1)\rho(1 \otimes a_2)$$

since $A_1 \otimes 1$ is included in the center of $M(A_\beta)$. Moreover, if we define the states ω_1 and ω_2 by the formulas $\omega_1(a_1) = \rho(a_1 \otimes 1)$ and $\omega_2(a_2) = \rho(1 \otimes a_2)$, then ω_2 is pure since ρ is pure, and $(\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)$. It follows that $P(A_\beta)$ is isomorphic to S_β.

In particular, if $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$ verifies

$$(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \text{ for all couples } (\omega_1, \omega_2) \in S_\beta,$$

the element $a_1 \otimes a_2$ is zero in A_β and therefore belongs to the ideal $\mathcal{J}(A_1, A_2)$. Accordingly, the previous lemma implies that $P(A_1) \times_X P(A_2) = S_\beta = P(A_\beta)$.

As a consequence, we get for every $\alpha \in A_1 \otimes_{alg} A_2$

$$\begin{aligned}\|\alpha\|_\beta^2 &= \sup\{\rho(\alpha^*\alpha), \rho \in P(A_\beta)\} \\ &= \sup\{(\omega_1 \otimes \omega_2)(\alpha^*\alpha), (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)\}\end{aligned}$$

But that last expression does not depend on $\| \ \|_\beta$, and hence the unicity. □

PROPOSITION 2.8 *([15] theorem 2) Let A_1 and A_2 be two unital $C(X)$-algebras.*

If $\| \ \|_\beta$ is a C*-*semi-norm on $A_1 \otimes_{alg} A_2$ whose kernel is $\mathcal{J}(A_1, A_2)$, then*

$$\forall \alpha \in A_1 \otimes_{alg} A_2, \quad \|\alpha\|_\beta \geq \|\alpha\|_m.$$

Proof: If we show that $S_\beta = P(A_1) \times_X P(A_2)$, then for every $\rho \in S_\beta$ and every α in $A_1 \otimes_{alg} A_2$, we have $\rho(s^*\alpha^*\alpha s) \leq \rho(s^*s)\|\alpha\|_\beta^2$ for all $s \in A_1 \otimes_{alg} A_2$. Therefore

$$\begin{aligned}\|\alpha\|_m{}^2 &= \sup\Big\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|^2, x \in X\Big\} \\ &= \sup\Big\{\frac{(\omega_1 \otimes \omega_2)(s^*\alpha^*\alpha s)}{(\omega_1 \otimes \omega_2)(s^*s)}, (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2) \text{ and} \\ &\qquad\qquad s \in A_1 \otimes_{alg} A_2 \text{ such that } (\omega_1 \otimes \omega_2)(s^*s) \neq 0\Big\} \\ &\leq \|\alpha\|_\beta{}^2.\end{aligned}$$

Suppose that $S_\beta \neq P(A_1) \times_X P(A_2)$. Then there exist thanks to lemma 2.6 self-adjoint elements $a_i \in A_i$ and a point $x \in X$ such that $(a_1)_x \otimes (a_2)_x \neq 0$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Let B be the unital abelian $C(X)$-algebra generated by $C(X)$ and a_1 in A_1. The preceding lemma implies that $B\overset{m}{\otimes}_{C(X)}A_2$ maps injectively into the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for $\| \ \|_\beta$.

Consider pure states $\rho \in P(B_x)$ and $\omega_2 \in P((A_2)_x)$ such that $\rho(a_1) \neq 0$ and $\omega_2(a_2) \neq 0$ and extend the pure state $\rho \otimes \omega_2$ on $B\overset{m}{\otimes}_{C(X)}A_2$ to a pure state ω on A_β. If we set $\omega_1(a) = \omega(a \otimes 1)$ for $a \in A_1$, then ω_1 is pure and $\omega(\alpha) = (\omega_1 \otimes \omega_2)(\alpha)$ for all $\alpha \in A_1 \otimes_{alg} A_2$ since ω and ω_2 are pure ([15] lemma 4). As a consequence, $(\omega_1, \omega_2) \in S_\beta$, which is absurd since $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = \rho(a_1)\omega_2(a_2) \neq 0$. □

PROPOSITION 2.9 *Given two $C(X)$-algebras A_1 and A_2, the semi-norm $\| \ \|_m$ defines the minimal* C**-norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: Let $\| \ \|_\beta$ be a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Thanks to the previous proposition, all we need to prove is that one can extend $\| \ \|_\beta$ to a C*-norm on $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the unital $C(X)$-algebras associated to the $C(X)$-algebras A_1 and A_2 (definition 1.1).

Consider the Hausdorff completion D_β of $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ and denote by π_i the canonical representation of A_i in $M(D_\beta)$ for $i = 1, 2$. Let us define the representation $\tilde{\pi}_i$ of $\mathcal{A}_i$ in $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ by the following formulas:

$$\begin{aligned} \tilde{\pi}_1(b_1 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_1(b_1) + g)\alpha \oplus (b_1 + f)a_1 \oplus fa_2 \oplus fg \\ \tilde{\pi}_2(b_2 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_2(b_2) + g)\alpha \oplus fa_1 \oplus (b_2 + f)a_2 \oplus fg \end{aligned}$$

For $i = 1, 2$, let $\varepsilon_i : \mathcal{A}_i \to C(X)$ be the map defined by

$$\varepsilon_i[a + u(f)] = f \text{ for } a \in A_i \text{ and } f \in C(X).$$

Then using the maps $(\varepsilon_1 \otimes \varepsilon_2)$, $(\varepsilon_1 \otimes id)$ and $(id \otimes \varepsilon_2)$, one proves easily that if $\alpha \in \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2$, $(\tilde{\pi}_1 \otimes \tilde{\pi}_2)(\alpha) = 0$ if and only if α belongs to $\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$.

Therefore, the norm of $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ restricted to the subalgebra $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$ extends $\| \ \|_\beta$. □

Remark: As the $C(X)$-algebra A is nuclear if and only if every fibre A_x is nuclear ([12]), $A\overset{M}{\otimes}_{C(X)}B \simeq A\overset{m}{\otimes}_{C(X)}B$ for every $C(X)$-algebra B if and only if A is nuclear.

3 When does the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ hold?

Given two $C(X)$-algebras A and B, Giordano and Mingo have studied in [6] the case where the algebra $C(X)$ is a von Neumann algebra: their theorem 3.1 and lemma 1.5 of [10] imply that in that case, we always have the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Our purpose in this section is to find sufficient conditions on the $C(X)$-algebras A and B in order to ensure this equality and to present a counter-example in the general case.

PROPOSITION 3.1 *Let X be a second countable Hausdorff compact space and let A and B be two $C(X)$-algebras.*
If A is a continuous field of C*-*algebras over X, then $\mathcal{I}(A,B) = \mathcal{J}(A,B)$.*

Proof: Let us prove by induction on the non negative integer n that if

$$s = \sum_{1\le i\le n} a_i \otimes b_i \in \mathcal{J}(A,B),$$

then s belongs to the ideal $\mathcal{I}(A,B)$.

If $n = 0$, there is nothing to prove. Consider therefore an integer $n > 0$ and suppose the result has been proved for any $p < n$.

Fix an element $s = \sum\limits_{1\le i\le n} a_i \otimes b_i \in \mathcal{J}(A,B)$ and define the continuous positive function $h \in C(X)$ by the formula $h(x)^{10} = \sum \|(a_k)_x\|^2$.

The element $a'_k = h^{-4}a_k$ is then well defined in A for every k since $a_k^* a_k \le h^{10}$. Consequently, the function $f_k(x) = \|(a'_k)_x\|$ is continuous.

For $1 \le k \le n$, let D_k denote the separable $C(X)$-algebra generated by 1 and the ${a'_k}^* a'_j$, $1 \le j \le n$, in the unital $C(X)$-agebra $\mathcal{A}$ associated to A (definition 1.1). Then D_k is a unital continuous field of C*-algebras over X (see for instance [3] proposition 3.2).

Consider the open subset $\mathcal{S}^k = \{\psi \in \mathcal{S}_X(D_k)/\, \psi[{a'_k}^* a'_k] > \psi(f_k^2/2)\}$. If we apply lemma 3.6 *b)* of [3] to the restriction of $p : \mathcal{S}_X(D_k) \to X$ to $\mathcal{S}^k$, we may construct a continuous field of states ω_k on D_k such that $\omega_k[{a'_k}^* a'_k] \ge f_k^2/2$.

Now, if we set $s' = \sum_i a'_i \otimes b_i$, as $({a'_k}^* \otimes 1)s'$ belongs to $\mathcal{J}(D_k, B)$,

$$(\omega_k \otimes id)[({a'_k}^* \otimes 1)s'] = \omega_k[{a'_k}^* a'_k]b_k + \sum_{j\ne k} \omega_k[{a'_k}^* a'_j]b_j = 0.$$

Noticing that f_k^3 is in the ideal of $C(X)$ generated by $\omega_k[{a'_k}^* a'_k]$, we get that $f_k^3 b_k$ belongs to the $C(X)$-module generated by the b_j, $j \ne k$, and thanks to the induction hypothesis, it follows that $(f_k^3 \otimes 1)s' \in \mathcal{I}(A,B)$ for each k.

But $h^2 = \sum_k f_k^2$ and so $h^4 \le n \sum_k f_k^4$ is in the ideal of $C(X)$ generated by the f_k^3, which implies
$s = (h^4 \otimes 1)s' \in \mathcal{I}(A,B)$. □

Remarks: **1.** As a matter of fact, it is not necessary to assume that the space X satisfies the second axiom of countability thanks to the following lemma of [11]: if $P(a) \in C(X)$ denotes the map $x \mapsto \|a_x\|$ for $a \in \mathcal{A}$, there exists a separable C*-subalgebra $C(Y)$ of $C(X)$ with same unit such that if D_k is the separable unital C*-algebra generated by $C(Y).1 \subset \mathcal{A}$ and the ${a'_k}^* a'_j$, $1 \le j \le n$, then $P(D_k) = C(Y)$. Furthermore, if $\Phi : X \to Y$ is the transpose of the inclusion map $C(Y) \hookrightarrow C(X)$ restricted to pure states, the map $D/(C_{\Phi(x)}(Y)D) \to A_x$ is a monomorphism for every $x \in X$ since $\mathcal{A}$ is continuous.

Consider now a continuous field of states $\omega_k : D_k \to C(Y)$ on the continuous field D_k over Y; if $\sum c^j \otimes d^j \in D_k \otimes_{alg} B$ is zero in $A_x \otimes_{alg} B_x$ for $x \in X$, then $\sum \omega_k(c^j)(x) d^j_x = 0$ in B_x, which enables us to conclude as in the separable case.

2. J. Mingo has drawn the author's attention to the following result of Glimm ([7] lemma 10): if $C(X)$ is a von Neumann algebra and A is a $C(X)$-algebras, then $P(a)^2 = \min\{z \in C(X)_+,\ z \geq a^*a\}$ is continuous for every $a \in \mathcal{A}$. Therefore we always have in that case the equality $\mathcal{I}(A,B) = \mathcal{J}(A,B)$ thanks to the previous remark.

COROLLARY 3.2 *Let A and B be two $C(X)$-algebras and assume that there exists a finite subset $F = \{x_1, \cdots, x_p\} \subset X$ such that for all $a \in A$, the function $x \mapsto \|a_x\|$ is continuous on $X \backslash F$.*

Then the ideals $\mathcal{I}(A,B)$ and $\mathcal{J}(A,B)$ coincide.

Proof: Fix an element $\alpha = \sum_{1 \leq i \leq n} a_i \otimes b_i \in A \otimes_{alg} B$ which belongs to $\mathcal{J}(A,B)$. In particular, we have $\sum_i (a_i)_x \otimes (b_i)_x = 0$ in $A_x \otimes_{alg} B_x$ for each $x \in F$.

As a consequence, thanks to theorem III of [13], we may find complex matrices $(\lambda^m_{i,j})_{i,j} \in M_n(\mathbf{C})$ for all $1 \leq m \leq p$ such that, if we define the elements $c^m_k \in A$ and $d^m_k \in B$ by the formulas

$$c^m_k = \textstyle\sum_i \lambda^m_{i,k} a_i \text{ and } d^m_k = b_k - \sum_j \lambda^m_{k,j} b_j,$$

we have $(c^m_k)_{x_m} = 0$ and $(d^m_k)_{x_m} = 0$ for all k and all m.

Consider now a partition $\{f_l\}_{1 \leq l \leq p}$ of $1 \in C(X)$ such that for all $1 \leq l, m \leq p$, $f_l(x_m) = \delta_{l,m}$ where δ is the Kronecker symbol and define for all $1 \leq k \leq n$ the elements $c_k = \sum_m f_m c^m_k$ and $d_k = \sum_m f_m d^m_k$. Thus,

$$\begin{aligned} \alpha &= (\textstyle\sum_i a_i \otimes d_i) + \left(\sum_{i,j,m} \lambda^m_{i,j} a_i \otimes f_m b_j\right) \\ &= (\textstyle\sum_i a_i \otimes d_i) + \left(\sum_j c_j \otimes b_j\right) + \left(\sum_{i,j,m} \lambda^m_{i,j}(a_i \otimes f_m b_j - f_m a_i \otimes b_j)\right) \end{aligned}$$

and there exists therefore an element $\beta \in \mathcal{I}(A,B)$ such that $\alpha - \beta$ admits a finite decomposition $\sum_i a'_i \otimes b'_i$ with $a'_i \in C_0(X \backslash F)A$ and $b'_i \in C_0(X \backslash F)B$.

But $C_0(X \backslash F)A$ is a continuous field. Accordingly, proposition 3.1 implies that $\alpha - \beta \in \mathcal{I}(C_0(X \backslash F)A, C_0(X \backslash F)B) \subset \mathcal{I}(A,B)$. □

Remark: If $\widetilde{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbf{N}$ and if A and B are two $C(\widetilde{\mathbf{N}})$-algebras, the corollary 3.2 implies the equality $\mathcal{I}(A,B) = \mathcal{J}(A,B)$.

Let us now introduce a counter-example in the general case.

Consider a dense countable subset $X = \{a_n\}_{n \in \mathbf{N}}$ of the interval $[0,1]$. The C*-algebra $C_0(\mathbf{N})$ of sequences with values in $\mathbf{C}$ vanishing at infinity is then endowed with the $C([0,1])$-algebra structure defined by:

$$\forall f \in C([0,1]), \forall \alpha = (\alpha_n) \in C_0(\mathbf{N}), \quad (f.\alpha)_n = f(a_n)\alpha_n \text{ for } n \in \mathbf{N}.$$

If we call A this $C([0,1])$-algebra, then $A_x = 0$ for all $x \notin X$.

Indeed, assume that $x \notin X$ and take $\alpha \in A$. If $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $|\alpha_n| < \varepsilon$ for all $n \geq N$. Consider a continuous function $f \in C([0,1])$ such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(a_i) = 1$ for every $1 \leq i \leq N$; we then have:

$$\|\alpha_x\| \leq \|(1-f)\alpha\| < \varepsilon.$$

Let $Y = \{b_n\}_{n\in\mathbf{N}}$ be another dense countable subset of $[0,1]$ and denote by B the associated $C([0,1])$-algebra whose underlying algebra is $C_0(\mathbf{N})$.

Then, if $X\cap Y = \emptyset$, the previous remark implies that for every $x \in [0,1]$, $A_x\otimes_{alg} B_x = 0$ and hence, $\mathcal{J}(A,B) = A\otimes_{alg} B$. What we therefore need to prove is that the ideal $\mathcal{I}(A,B)$ is strictly included in $A\otimes_{alg} B$.

Let us fix two sequences $\alpha \in A$ and $\beta \in B$ whose terms are all non zero and suppose that $\alpha\otimes\beta$ admits a decomposition $\sum_{1\leq i\leq k}[(f_i\alpha^i)\otimes\beta^i - \alpha^i\otimes(f_i\beta^i)]$ in $A\otimes_{alg} B$. Then for every $n, m \in \mathbf{N}$,

$$\alpha_n\beta_m = \sum_{1\leq i\leq k} \alpha_n^i\beta_m^i[f_i(a_n) - f_i(b_m)].$$

Now, if we set $\phi_i(a_n) = \alpha_n^i/\alpha_n$ and $\psi_i(b_n) = \beta_n^i/\beta_n$ for $1\leq i\leq k$ and $n\in\mathbf{N}$, this equality means that for all $(x,y)\in X\times Y$,

$$1 = \textstyle\sum_{1\leq i\leq k}\psi_i(x)\varphi_i(y)[f_i(x) - f_i(y)].$$

But this is impossible because of the following proposition:

PROPOSITION 3.3 *Let X and Y be two dense subsets of the interval $[0,1]$ and let n be a non negative integer.*

Given continuous functions f_i on $[0,1]$ and numerical functions $\psi_i : X\to\mathbf{C}$ and $\varphi_i : Y\to\mathbf{C}$ for $1\leq i\leq n$, if there exists a constant $c\in\mathbf{C}$ such that

$$\forall (x,y)\in X\times Y, \qquad \sum_{1\leq i\leq n}\psi_i(x)\varphi_i(y)[f_i(x) - f_i(y)] = c$$

then $c = 0$.

Proof: We shall prove the proposition by induction on n.

If $n = 0$, the result is trivial. Take therefore $n > 0$ and assume that the proposition is true for any $k < n$.

Suppose then that the subsets X and Y of $[0,1]$, the functions f_i, ψ_i and φ_i, $1\leq i\leq n$, satisfy the hypothesis of the proposition for the constant c.

For $x\in X$, let $p(x)\leq n$ be the dimension of the vector space generated in $\mathbf{C}^n$ by the $\left(\varphi_i(y)[f_i(x) - f_i(y)]\right)_{1\leq i\leq n}$, $y\in Y$.

If $p(x) < n$, there exists a subset $F(x)\subset\{1,\ldots,n\}$ of cardinal $p(x)$ such that for every $j\notin F(x)$:

$$\varphi_j(y)[f_j(x) - f_j(y)] = \textstyle\sum_{i\in F(x)}\lambda_i^j(x)\varphi_i(y)[f_i(x) - f_i(y)] \text{ for all } y\in Y,$$

where the $\lambda_i^j(x)\in\mathbf{C}$ are given by the Cramer formulas. As a consequence,

$$\textstyle\sum_{i\in F(x)}\Big(\psi_i(x) + \sum_{j\notin F(x)}[\lambda_i^j(x)\psi_j(x)]\Big)\varphi_i(y)[f_i(x) - f_j(y)] = c.$$

Now, if $p(x) < n$ for every $x \in X$, there exists a subset $F \subset \{1, \ldots, n\}$ of cardinal $p < n$ such that the interior of the closure of the set of those x for which $F(x) = F$ is not empty and contains therefore a closed interval homeomorphic to $[0,1]$. The induction hypothesis for $k = p$ implies that $c = 0$.

Assume on the other hand that $x_0 \in X$ verifies $p(x_0) = n$. We may then find $y_1, \cdots y_n$ in Y such that if we set

$$a_{i,j}(x) = \varphi_i(y_j)[f_i(x) - f_i(y_j)],$$

the matrix $(a_{i,j}(x_0))$ is invertible. There exists therefore a closed connected neighborhood I of x_0 on which the matrix $(a_{i,j}(x))$ remains invertible.

But for each $1 \le j \le n$, $\sum_i a_{i,j}(x)\psi_i(x) = c$ and therefore the $\psi_i(x)$ extend by the Cramer formulas to continuous functions on the closed interval I.

For $y \in Y \cap I$, let $q(y)$ denote the dimension of the vector space generated in $\mathbf{C}^n$ by the $\big(\psi_i(x)[f_i(x) - f_i(y)]\big)_{1 \le i \le n}$, $x \in X \cap I$.

If $q(y) < n$ for every y, then the induction hypothesis implies $c = 0$. But if there exists y_0 such that $q(y_0) = n$, we may find an interval $J \subset I$ homeomorphic to $[0,1]$ on which the φ_i extend to continuous functions; evaluating the starting formula at a point $(x,x) \in J \times J$, we get $c = 0$. □

4 The associativity

Given three $C(X)$-algebras A_1, A_2 and A_3, we deduce from [3] corollaire 3.17:

$$[(A_1 \overset{M}{\otimes}_{C(X)} A_2) \overset{M}{\otimes}_{C(X)} A_3]_x = (A_1 \overset{M}{\otimes}_{C(X)} A_2)_x \otimes_{\max} (A_3)_x = (A_1)_x \otimes_{\max} (A_2)_x \otimes_{\max} (A_3)_x,$$

which implies the associativity of the tensor product $\cdot \overset{M}{\otimes}_{C(X)} \cdot$ over $C(X)$.

On the contrary, the minimal tensor product $\cdot \overset{m}{\otimes}_{C(X)} \cdot$ over $C(X)$ is not in general associative. Indeed, Kirchberg and Wassermann have shown in [11] that if $\widetilde{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbf{N}$, there exist separable continuous fields A and B such that

$$(A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B)_\infty \neq A_\infty \otimes_{\min} B_\infty.$$

If we now endow the C*-algebra $D = \mathbf{C}$ with the $C(\widetilde{\mathbf{N}})$-algebra structure defined by $f.a = f(\infty)a$, then for all $C(\widetilde{\mathbf{N}})$-algebra D', we have

$$[D \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D']_{\hat{n}} = \begin{cases} 0 & \text{if } n \text{ is finite,} \\ (D')_\infty & \text{if } n = \infty. \end{cases}$$

Therefore, $[(A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B) \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D]_\infty \simeq (A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B)_\infty$ whereas $[A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} (B \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D)]_\infty$ is isomorphic to $A_\infty \otimes_{\min} B_\infty$.

However, in the case of (separable) continuous fields, we can deduce the associativity of $\cdot \overset{m}{\otimes}_{C(X)} \cdot$ from the following proposition:

PROPOSITION 4.1 *Let A and B be two $C(X)$-algebras.*

Assume π is a field of faithful representations of A in the Hilbert $C(X)$-module $\mathcal{E}$, then the morphism $a \otimes b \to \pi(a) \otimes b$ induces a faithful $C(X)$-linear representation of $A \overset{m}{\otimes}_{C(X)} B$ in the Hilbert B-module $\mathcal{E} \otimes_{C(X)} B$.

Proof: Notice that for all $x \in X$, we have $(\mathcal{E} \otimes_{C(X)} B) \otimes_B B_x = \mathcal{E}_x \otimes B_x$.

Now, as B maps injectively in $B_d = \oplus_{x \in X} B_x$, $\mathcal{L}_B(\mathcal{E} \otimes_{C(X)} B)$ maps injectively in $\oplus_{x \in X} \mathcal{L}_{B_x}(\mathcal{E}_x \otimes B_x) \subset \mathcal{L}_{B_d}(\mathcal{E} \otimes_{C(X)} B \otimes_B B_d)$ and therefore if $\alpha \in A \otimes_{alg} B$, we have $\|(\pi \otimes id)(\alpha)\| = \sup_{x \in X} \|(\pi_x \otimes id)(\alpha_x)\| = \|\alpha\|_m$. □

Accordingly, if for $1 \leq i \leq 3$, A_i is a separable continuous field of C*-algebras over X which admits a field of faithful representations in the $C(X)$-module $\mathcal{E}_i$, the $C(X)$-representations of $(A_1 \overset{m}{\otimes}_{C(X)} A_2) \overset{m}{\otimes}_{C(X)} A_3$ and $A_1 \overset{m}{\otimes}_{C(X)} (A_2 \overset{m}{\otimes}_{C(X)} A_3)$ in the.Hilbert $C(X)$-module $(\mathcal{E}_1 \otimes_{C(X)} \mathcal{E}_2) \otimes_{C(X)} \mathcal{E}_3 = \mathcal{E}_1 \otimes_{C(X)} (\mathcal{E}_2 \otimes_{C(X)} \mathcal{E}_3)$ are faithful, and hence the maps $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to (A_1 \overset{m}{\otimes}_{C(X)} A_2) \overset{m}{\otimes}_{C(X)} A_3$ and $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to A_1 \overset{m}{\otimes}_{C(X)} (A_2 \overset{m}{\otimes}_{C(X)} A_3)$ have the same kernel.

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Ergodic actions of compact matrix pseudogroups on C^*-algebras

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Ergodic Actions of Compact Matrix Pseudogroups on C*-algebras

Florin P. Boca

Dedicated to Professor Masamichi Takesaki on the ocasion of his 60th birthday

Let G be a compact group acting on a unital C^*-algebra $\mathcal{M}$. The action is said to be ergodic if the fixed point algebra $\mathcal{M}^G$ reduces to scalars. The first breakthrough in the study of such actions was the finiteness theorem of Høegh-Krohn, Landstad and Størmer [HLS]. They proved that the multiplicity of each $\pi \in \widehat{G}$ in $\mathcal{M}$ is at most $\dim(\pi)$ and the unique G-invariant state on $\mathcal{M}$ is necessarily a trace. When combined with Landstad's result [L] that finite-dimensionality for the spectral subspaces of actions of compact groups implies that the crossed product is a type I C^*-algebra (and in fact, as pointed out in [Wa1] is a direct sum of algebras of compact operators), the finiteness theorem shows that the crossed product of a unital C^*-algebra by an ergodic action of a compact group is necessarily equal to $\oplus_i \mathcal{K}(\mathcal{H}_i)$.

The study of such actions was essentialy pushed forward by Wassermann. He developed an outstanding machinery based on the notion of multiplicity maps, establishing a remarkable connection with the equivariant K-theory. This approach allowed him to prove, among other important things, the strong negative result that $SU(2)$ cannot act ergodically on the hyperfinite II_1 factor [Wa3].

The aim of this note is to study ergodic actions of Woronowicz's compact matrix pseudogroups on unital C^*-algebras, extending some of the previous results. In this insight we prove in §1 the analogue of the finiteness theorem. More precisely, if $G = (A, u)$ is a compact matrix pseudogroup acting ergodically on a unital C^*-algebra by a coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ such that $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$, then there is a decomposition of the *-algebra of σ-finite elements into isotypic subspaces $\mathcal{M}_0 = \oplus_{\alpha \in \widehat{G}} \mathcal{M}_\alpha$, orthogonal with respect to the scalar product induced on $\mathcal{M}$ by the unique σ-invariant state ω. Moreover, the spectral subspaces $\mathcal{M}_\alpha$ are finite dimensional and $\dim(\mathcal{M}_\alpha) \leq M_\alpha^2$, M_α being the *quantum dimension* of $\alpha \in \widehat{G}$. If the Haar measure is faithful on A, then $\mathcal{M}_0$ is dense in the GNS Hilbert space $\mathcal{H}_\omega$.

Although ω is not in general a trace, we prove the existence of a multiplicative linear map $\Theta : \mathcal{M}_0 \to \mathcal{M}_0$ such that $\omega(xy) = \omega(\Theta(y)x)$ for all $x \in \mathcal{M}, y \in \mathcal{M}_0$ and Θ is a scalar multiple of the modular operator F_α when restricted to each irreducible σ-invariant subspace of the spectral subspace $\mathcal{M}_\alpha$.

The crossed products by such coactions are studied in §2 where we prove, using the Takesaki-Takai type duality theorem of Baaj and Skandalis [BS], that they are isomorphic all the time to a direct sum of C^*-algebras of compact operators. As a corollary, if a compact matrix pseudogroup with underlying nuclear C^*-algebra acts ergodically on a unital C^*-algebra $\mathcal{M}$, then $\mathcal{M}$ follows nuclear.

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§1. The isotypic decomposition and finiteness of multiplicities for ergodic actions

We start with a couple of definitions.

Definition 1 ([BS]) *A coaction of a unital Hopf C^*-algebra (A, Δ_A) on a unital C^*-algebra $\mathcal{M}$ is a unital one-to-one *-homomorphism $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ (the tensor product will be all the time the minimal C^*-one) that makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\sigma} & \mathcal{M} \otimes A \\ \sigma \uparrow & & \uparrow \quad id_{\mathcal{M}} \otimes \Delta_A \\ \mathcal{M} \otimes A & \xrightarrow{\sigma \otimes id_A} & \mathcal{M} \otimes A \otimes A \end{array}$$

A C^*-algebra $\mathcal{M}$ with a coaction σ of (A, Δ_A) is called an A-algebra if σ is one-to-one and $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$.

Definition 2 *The fixed points of the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ are the elements of $\mathcal{M}^\sigma = \{x \in \mathcal{M} \mid \sigma(x) = x \otimes 1_A\}$. The coaction σ is called ergodic if $\mathcal{M}^\sigma = \mathbf{C}1_{\mathcal{M}}$.*

We denote by $\mathcal{M}^*$ the set of continuous linear functionals on $\mathcal{M}$.

Definition 3 *$\phi \in \mathcal{M}^*$ is called σ-invariant if*
$(\phi \otimes \psi)(\sigma(x)) = (\phi \otimes \psi)(x \otimes 1_A) = \psi(1_A)\phi(x)$ *for all* $\psi \in A^*$.

Let $G = (A, u)$ be a compact matrix pseudogroup group with comultiplication $\Delta_A : A \to A \otimes A$, smooth structure $\mathcal{A}$ and coinverse $\kappa : \mathcal{A} \to \mathcal{A}$ (cf [Wor]). Then A^* is an algebra with respect to the convolution $\phi * \psi = (\phi \otimes \psi)\Delta_A$, $\phi, \psi \in A^*$ and there exists a unique state h on A, called the Haar measure of G, so that $\phi * h = h * \phi = \phi(1_A)h$ for all $\phi \in A^*$. Let $\mathcal{M}$ be a unital C^*-algebra which is an A-algebra via the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ and consider $\theta = (id_{\mathcal{M}} \otimes h)\sigma$.

Lemma 4 *i)* $\theta(x) \in \mathcal{M}^\sigma$ *for all* $x \in \mathcal{M}$. *Moreover* θ *is a conditional expectation from* $\mathcal{M}$ *onto* $\mathcal{M}^\sigma$.

ii) If σ *is ergodic, then* $\theta(x) = \omega(x)1_{\mathcal{M}}$ *and* ω *is the only* σ*-invariant state on* $\mathcal{M}$.

Proof. i) Note that

$$\begin{aligned}\sigma(\theta(x)) &= \sigma((id_{\mathcal{M}} \otimes h)(\sigma(x)) = (\sigma \otimes h)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(\sigma(x)), \qquad x \in \mathcal{M}.\end{aligned}$$

Since $(id_A \otimes h)(\Delta_A(a)) = h * a = h(a)1_A$, $a \in A$ ([Wor, 4.2]), we obtain further :

$$\begin{aligned}&(\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(y \otimes a)) \\ &\qquad = (\psi_1 \otimes \psi_2)(y \otimes (id_A \otimes h)(\Delta_A(a))) \\ &\qquad = (\psi_1 \otimes \psi_2)(y \otimes h(a)1_A) = \psi_1(y)\psi_2(1_A)h(a) \\ &\qquad = (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)(y \otimes a) \otimes 1_A),\end{aligned}$$

for all $y \in \mathcal{M}, a \in A, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$. Therefore for $x \in \mathcal{M}, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$ we have :

$$(\psi_1 \otimes \psi_2)(\sigma(\theta(x)) = (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)\sigma(x) \otimes 1_A) = (\psi_1 \otimes \psi_2)(\theta(x) \otimes 1_A)$$

and consequently $\sigma(\theta(x)) = \theta(x) \otimes 1_A$ for all $x \in \mathcal{M}$. θ is a norm one projection since

$$\theta(x) = (id_{\mathcal{M}} \otimes h)(\sigma(x)) = (id_{\mathcal{M}} \otimes h)(x \otimes 1) = x, \qquad x \in \mathcal{M}^\sigma.$$

ii) Let $\psi \in A^*$. Then we get for all $x \in \mathcal{M}$:

$$\begin{aligned}(\omega \otimes \psi)(\sigma(x)) &= ((id_{\mathcal{M}} \otimes h)\sigma \otimes \psi)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \psi)((id_{\mathcal{M}} \otimes h)\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((\sigma \otimes id_A)\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((id_{\mathcal{M}} \otimes \Delta_A)\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes (h \otimes id_A)\Delta_A)(\sigma(x)) \\ &= \psi(1_A)(id_{\mathcal{M}} \otimes h)(\sigma(x)) = \psi(1_A)\omega(x),\end{aligned}$$

therefore ω is a σ-invariant state on $\mathcal{M}$. Finally, assume that ϕ is a σ-invariant state on $\mathcal{M}$. Then for all $x \in \mathcal{M}$:

$$\phi(x) = (\phi \otimes h)(\sigma(x)) = \phi((id_{\mathcal{M}} \otimes h)(\sigma(x))) = \phi(\theta(x)) = \phi(\omega(x)1_{\mathcal{M}}) = \omega(x). \qquad \square$$

Remarks. 1). The proof of the previous statement doesn't use the faithfulness of σ but only the equality $(\sigma \otimes id_A)\sigma = (id_{\mathcal{M}} \otimes \Delta_A)\sigma$.

2). Since the tensor product of two faithful completely positive maps is still faithful [T], it follows that if σ is one-to-one and h is faithful on A, then ω is a faithful state on $\mathcal{M}$.

3). Although one can easily pass from a compact matrix pseudogroup to the reduced one, which has faithful Haar measure, as indicated at page 656 in [Wor], it turns out that the Haar measure is faithful in several important examples (e.g. on commutative CMP, on reduced cocommutative CMP or, cf. [N], on $SU_\mu(N)$).

Denote by $\mathcal{H}_\omega$ the completion of $\mathcal{M}$ with respect to the inner product $\langle x, y\rangle_2 = \omega(y^*x)$ and let $\mathcal{M}$ acting on $\mathcal{H}_\omega$ in the GNS representation. Consider the C^*-Hilbert module $\mathcal{H}_\omega \otimes A$ with the A-valued inner product $\langle x_\omega \otimes a, y_\omega \otimes b\rangle_A = \omega(y^*x)b^*a$, for $x, y \in \mathcal{M}$, $a, b \in A$, which can be viewed as $\mathcal{M} \otimes A$ in the Stinespring representation of the completely positive map $\omega \otimes id_A$. Define also $V : \mathcal{H}_\omega \otimes A \to \mathcal{H}_\omega \otimes A$ by :

$$V(\sum_i (x_i)_\omega \otimes a_i) = \sum_i \sigma(x_i)(1_\omega \otimes a_i), \quad x_i \in \mathcal{M}, a_i \in A.$$

Lemma 5 *V is a unitary in $\mathcal{L}(\mathcal{H}_\omega \otimes A) = M(\mathcal{K}(\mathcal{H}_\omega) \otimes A)$ (the multipliers of the C^*-algebra $\mathcal{K}(\mathcal{H}_\omega) \otimes A$) and $\sigma(x) = V(x \otimes 1_A)V^*$, $x \in \mathcal{M}$.*

Proof. For any $\phi \in A^*, a, b \in A$, denote by $\phi(a \cdot b) \in A^*$ the linear functional $\phi(a \cdot b)(x) = \phi(axb)$, $x \in A$. The σ-invariance of ω yields :

$$\begin{aligned}\phi((\omega \otimes id_A)((1_\mathcal{M} \otimes b^*)\sigma(x)(1_\mathcal{M} \otimes a)) &= (\omega \otimes \phi(b^* \cdot a))(\sigma(x)) \\ &= \phi(b^* \cdot a)(1_A)\omega(x) = \phi(b^*a)\omega(x), \quad x \in \mathcal{M}, a, b \in A, \phi \in A^*,\end{aligned}$$

therefore we have for all $a, b \in A, x, y \in \mathcal{M}$:

$$\begin{aligned}\langle V(x_\omega \otimes a), V(y_\omega \otimes b)\rangle_A &= \langle \sigma(x)(1_\omega \otimes a), \sigma(y)(1_\omega \otimes b)\rangle_A \\ &= (\omega \otimes id_A)((1_\mathcal{M} \otimes b^*)\sigma(y^*x)(1_\mathcal{M} \otimes a)) = \omega(y^*x)b^*a = \langle x_\omega \otimes a, y_\omega \otimes b\rangle_A\end{aligned}$$

and V follows isometry on $\mathcal{H}_\omega \otimes A$. Furthermore V is unitary since $\sigma(\mathcal{M})(1_\mathcal{M} \otimes A)$ is dense in $\mathcal{M} \otimes A$ and the relation $V(x \otimes 1_A) = \sigma(x)V$, $x \in \mathcal{M}$, is obvious. □

Definition 6 ([BS]) *A corepresentation of the Hopf C^*-algebra (A, Δ_A) is a unitary $V \in \mathcal{L}(\mathcal{H}_V \otimes A) = M(\mathcal{K}(\mathcal{H}_V) \otimes A)$ such that*

$$V_{12}V_{13} = (id_{\mathcal{L}(\mathcal{H}_V)} \otimes \Delta_A)(V).$$

All the corepresentations throughout this paper will be unitary unless specified otherwise. Note that in the case when $\dim\mathcal{H}_V < \infty$ V is called in [Wor] a (finite dimensional) representation of the quantum matrix pseudogroup $G = (A, u)$. Thus the representations of the quantum matrix pseudogroup $G = (A, u)$ are the corepresentations of (A, Δ_A) and we will call them simply the corepresentations of A.

Lemma 7 *The unitary V from Lemma 5 is a corepresentation of A.*

Proof. Let $T = (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \Delta_A)(V) \in \mathcal{L}(\mathcal{H}_\omega \otimes A \otimes A) = M(\mathcal{K}(\mathcal{H}_\omega) \otimes A \otimes A)$. Since $1_\omega \otimes 1_A$ is fixed by V, then $\langle V(1_\omega \otimes 1_A), x_\omega^* \otimes 1_A\rangle_A = (\omega(x\cdot) \otimes id_A)(V) = \omega(x)1_A$ and consequently :

$$\begin{aligned}&(\omega(x\cdot) \otimes id_{A\otimes A})(id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \Delta_A)(V) = (\omega(x\cdot) \otimes \Delta_A)(V)\\ &\quad = \Delta_A((\omega(x\cdot) \otimes id_A)(V)) = \Delta_A(\omega(x)1_A) = \omega(x)1_{A\otimes A}, \quad x \in \mathcal{M}.\end{aligned}$$

Then, for any $a, a', b, b' \in A$:

$$\begin{aligned}\langle T(1_\omega \otimes a \otimes b), x_\omega \otimes a' \otimes b'\rangle_A &= (\omega \otimes id_{A\otimes A})((x^* \otimes a'^* \otimes b'^*)T(1_M \otimes a \otimes b))\\ &= (a'^* \otimes b'^*)((\omega(x^*\cdot) \otimes id_{A\otimes A})(T))(a \otimes b)\\ &= \omega(x^*)a'^*a \otimes b'^*b = \langle 1_\omega \otimes a \otimes b, x_\omega \otimes a' \otimes b'\rangle_A,\end{aligned}$$

therefore $T(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$. Furthermore, since T is unitary, we also have $T^*(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$ and for any $x \in \mathcal{M}, a, b \in A$:

$$\begin{aligned}V_{12}V_{13}(x_\omega \otimes a \otimes b) &= V_{12}(\sigma(x)_{13}(1_\omega \otimes a \otimes b)) = (\sigma \otimes id_A)(\sigma(x))(1_\omega \otimes a \otimes b)\\ &= (id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))(1_\omega \otimes a \otimes b)\\ &= T(x \otimes 1_A \otimes 1_A)T^*(1_\omega \otimes a \otimes b) = T(x_\omega \otimes a \otimes b).\end{aligned}$$

Remark 8 *Let $W \in \mathcal{L}(\mathcal{H}_W \otimes A)$ be a corepresentation, $\mathcal{H}_V \subset \mathcal{H}_W$ be a closed subspace of $\mathcal{H}_W$ and denote by P the orthogonal projection from $\mathcal{H}_W$ onto $\mathcal{H}_V$. If $(P \otimes id_A)W = W(P \otimes id_A)$, then $V = W(P \otimes id_A) \in \mathcal{L}(\mathcal{H}_V \otimes A)$ is by definition a subcorepresentation of W. Clearly $V^\perp = W(P^\perp \otimes id_A)$, where $P^\perp = I_{\mathcal{L}(\mathcal{H}_W)} - P$, is also a subcorepresentation of W.*

For $V \in M(\mathcal{K}(\mathcal{H}_V) \otimes A)$ and $\rho \in A^*$ define as in [Wor] $V_\rho = (id_{\mathcal{L}(\mathcal{H}_V)} \otimes \rho)V \in \mathcal{L}(\mathcal{H}_V)$. Then Lemma 6 yields for any $\rho, \rho' \in A^*$:

$$\begin{aligned}V_\rho V_{\rho'} &= (id \otimes \rho)(V)(id \otimes \rho')(V) = (id \otimes \rho \otimes \rho')(V_{12}V_{13})\\ &= (id \otimes \rho \otimes \rho')(id \otimes \Delta_A)(V) = (id \otimes (\rho * \rho'))(V) = V_{\rho*\rho'}.\end{aligned}$$

One checks immediately as in [Wor, 4.3] that $E = V_h$ is the projection of $\mathcal{H}_V$ onto the subspace $\{\xi \in \mathcal{H}_V \mid V_\rho\xi = \rho(1_A)\xi\ , \forall \rho \in A^*\}$ of all V-invariant vectors of $\mathcal{H}_V$.

The tensor product of the corepresentations $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ and $W \in \mathcal{L}(\mathcal{H}_W \otimes A)$ is defined as in the finite dimensional case by $V \odot W = V_{13}W_{23} \in \mathcal{L}(\mathcal{H}_V \otimes \mathcal{H}_W \otimes A)$ and is still a corepresentation since

$$\begin{aligned}(id \otimes \Delta_A)(V \odot W) &= (id \otimes \Delta_A)(V_{13})(id \otimes \Delta_A)(W_{23})\\ &= (V_{13}W_{23})_{12}(V_{13}W_{23})_{13} = (V \odot W)_{12}(V \odot W)_{13}.\end{aligned}$$

When $\dim\mathcal{H}_W < \infty$, W^c denotes as in [Wor] its contragradient corepresentation, acting on the conjugate Hilbert space $\mathcal{H}'_W$.

Lemma 9 *If the Haar measure is faithful on A and $V \in M(\mathcal{K}(\mathcal{H}_V) \otimes A)$ is a corepresentation of A such that $(V \odot \alpha^c)_h = 0$ for all $\alpha \in \widehat{G}$, then $V = 0$.*

Proof. Denote d_α =dim(α). Then $\alpha = \sum_{i,j=1}^{d_\alpha} e_{ij} \otimes u_{ij}^\alpha$, $u_{ij} \in \mathcal{A}$, where $\{e_{ij}\}_{1\le i,j\le d_\alpha}$ is the matrix unit of $\mathcal{L}(\mathcal{H}_\alpha)$ and $\alpha^c = \sum_{i,j=1}^{d_\alpha} e_{ij}^T \otimes u_{ji}^{\alpha *} \in \mathcal{L}(\mathcal{H}'_\alpha) \otimes \mathcal{A}$. If ϕ_{ij} denotes the linear functional on $\mathcal{L}(\mathcal{H}'_\alpha)$ defined by $\phi_{ij}(e_{kl}^T) = \delta_{ik}\delta_{jl}$ we obtain for all $\phi \in \mathcal{L}(\mathcal{H}_V)_*$:

$$(\phi \otimes \phi_{ij} \otimes id_A)(id_{\mathcal{L}(\mathcal{H}_V \otimes \mathcal{H}'_\alpha)} \otimes h)(V \odot \alpha^c) = h((\phi \otimes id_A)(V) u_{ji}^{\alpha *}).$$

Since $\{u_{ij}^\alpha\}_{1\le i,j\le d_\alpha, \alpha\in\widehat{G}}$ is a linear basis in $\mathcal{A}$, then $h((\phi\otimes id_A)(V)a) = 0$ for all $a \in \mathcal{A}$ and therefore for all $a \in A$. But h is faithful on A hence $(\phi \otimes id_A)(V) = 0$ for all $\phi \in \mathcal{L}(\mathcal{H}_V)_*$ and therefore $V = 0$. □

Corollary 10 *i) If the Haar measure is faithful on A, then there are no irreducible infinite dimensional corepresentations of A.*

ii) All the finite dimensional corepresentations (not necessarily unitary) of A are smooth (compare with a related question at page 636 in [Wor]).

Proof. i) Let V be an infinite dimensional corepresentation of A on $\mathcal{H}_V$. By the previous Lemma, there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$. Pick a nonzero element $S \in \text{Ran}(V \otimes \alpha^c)_h$. Since α is finite dimensional, $S \in \text{Mor}(\alpha, V)$. But Ker$S$ is α-invariant and α irreducible, thus S is one-to-one and we get the V-invariant finite-dimensional subspace Ran$S \subset \mathcal{H}_V$, contradicting the irreducibility of V.

ii) Consider now a finite dimensional corepresentation V of A, not necessarily unitary. The statement in the previous Lemma holds true, thus either V is completely degenerate or there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$ and $0 \neq S \in \text{Mor}(\alpha, V)$. It follows that $\mathcal{H}_V$ contains a V-invariant subspace $\mathcal{K} = \text{Ran}S$ and $V\mid_{\mathcal{K}}$ is equivalent to α, therefore it is smooth, irreducible and nondegenerate. By [Wor, Prop.4.6] $\mathcal{K}$ has a V-invariant complement and the process continues until we write $\mathcal{H}_V$ as a direct sum of linear subspaces $\mathcal{H}_V = \oplus_i \mathcal{H}_i \oplus \mathcal{H}_0$, each subspace being V-invariant, $V\mid_{\mathcal{H}_i}$ being equivalent to a corepresentation from $\widehat{G}$ and $V\mid_{\mathcal{H}_0}$ completely nondegenerate. □

The statements in the previous Corollary are implicit in S. L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups, *Invent. Math. 93(1988), 35-76*, as the referee informed as.

For any $\alpha \in \widehat{G}$ consider as in [Wor, §5] $\rho_\alpha \in A^*$, $\rho_\alpha(a) = M_\alpha h((f_1 * \chi_\alpha)^* a)$. Then $\rho_\alpha * \rho_\beta = \delta_{\alpha\beta}\rho_\alpha$ for all $\alpha, \beta \in \widehat{G}$. Therefore, if $U \in M(\mathcal{K}(\mathcal{H}_U) \otimes A)$ is a representation of A, $P_\alpha = U_{\rho_\alpha} = (id_{\mathcal{L}(\mathcal{H}_U)} \otimes \rho_\alpha)(U)$ are mutually orthogonal projections, but they are not self-adjoint in general. Consider also the bounded linear maps $P_\alpha : \mathcal{M} \to \mathcal{M}$, $P_\alpha(x) = (id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))$. Then we have for all $x \in \mathcal{M}$:

$$\begin{aligned}
P_\alpha P_\beta(x) &= (id_\mathcal{M} \otimes \rho_\alpha)(\sigma((id_\mathcal{M} \otimes \rho_\beta)(\sigma(x)))\\
&= (id_\mathcal{M} \otimes \rho_\alpha)((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\beta)(\sigma \otimes id_A)(\sigma(x)))\\
&= (id_\mathcal{M} \otimes \rho_\alpha)((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\beta)(id_\mathcal{M} \otimes \Delta_A)(\sigma(x)))\\
&= (id_\mathcal{M} \otimes \rho_\alpha(id_A \otimes \rho_\beta)\Delta_A)(\sigma(x))\\
&= (id_\mathcal{M} \otimes (\rho_\alpha * \rho_\beta))(\sigma(x)) = \delta_{\alpha\beta} P_\alpha(x).
\end{aligned}$$

We denote by $\mathcal{M}_\alpha$ the closed subspace $P_\alpha(\mathcal{M})$ of $\mathcal{M}$, $\alpha \in \widehat{G}$ and call it the *spectral subspace* associated with α.

The following lemma contains a couple of properties of the functionals $\rho_\alpha \in A^*$, $\alpha \in \widehat{G}$.

Lemma 11 *i)* $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$ *for all* $\alpha \in \widehat{G}$.
ii) $(h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0$ *for all* $a, b \in A, \alpha \neq \beta$ *in* $\widehat{G}$.

Proof. i) Let $x \in \mathcal{M}_\alpha$. Then $x = (id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))$ and we have to check that $x^* = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x^*))$. Obvious computations which are implicit in [Wor, 5.6] show that $(f_1 * \chi_{\alpha^c})^* = \chi_\alpha * f_{-1}$ and $M_\alpha = M_{\alpha^c}$, where $M_\alpha = f_{-1}(\chi_\alpha) = f_1(\chi_\alpha)$, therefore we have for any $a \in A$:

$$\begin{aligned}\overline{\rho_\alpha(a)} &= M_\alpha \overline{h((f_1 * \chi_\alpha)^* a)} = M_\alpha h(a^*(f_1 * \chi_\alpha)) \\ &= M_\alpha h(a^*(f_1 * \chi_\alpha * f_{-1} * f_1)) = M_\alpha h((\chi_\alpha * f_{-1})a^*) \\ &= M_\alpha h((f_1 * \chi_{\alpha^c})^* a^*) = M_{\alpha^c} h((f_1 * \chi_{\alpha^c})^* a^*) = \rho_{\alpha^c}(a^*).\end{aligned}$$

This shows that $x^* = (id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))^* = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x)^*) = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x^*))$.

ii) Since $span\{u_{ij}^\gamma\}_{1 \leq i,j \leq d_\gamma, \gamma \in \widehat{G}}$ is norm dense in A it is enough to take $a = u_{ij}^\gamma, b = u_{kl}^\theta$ for some $\gamma, \theta \in \widehat{G}$ and to remark that :

$$\begin{aligned}&(h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(u_{ij}^\gamma)_{12}\Delta_A(u_{kl}^\theta)_{13}) \\ &\quad = (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} u_{ir}^\gamma u_{ks}^\theta \otimes u_{rj}^\gamma \otimes u_{sl}^\theta) \\ &\quad = \sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} h(u_{ir}^\gamma u_{ks}^\theta)\delta_{\beta^c\gamma}\delta_{rj}\delta_{\alpha\theta}\delta_{sl} \\ &\quad = h(u_{ij}^{\beta^c} u_{kl}^\alpha) = h(u_{ij}^{\beta *} u_{kl}^\alpha) = 0. \qquad \square\end{aligned}$$

Corollary 12 $\omega(P_\beta(y)^* P_\alpha(x)) = \omega(P_\alpha(x)P_\beta(y)^*) = 0$ *for all* $x, y \in \mathcal{M}, \alpha \neq \beta$ *in* $\widehat{G}$.

Proof. Denote $x_0 = P_\alpha(x), y_0 = P_\beta(y)$. By the previous lemma $x_0^* \in \mathrm{Ran}P_{\alpha^c}$ and $y_0^* \in \mathrm{Ran}P_{\beta^c}$. Then we obtain :

$$\begin{aligned}\sigma(x_0) &= \sigma((id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathcal{M}\otimes A} \otimes \rho_\alpha)(id_\mathcal{M} \otimes \Delta_A)(\sigma(x)) \\ &= (id_\mathcal{M} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))\end{aligned}$$

and the similar relations for the pairs $(x_0^*, \alpha^c), (y_0, \beta), (y_0^*, \beta^c)$. The σ-invariance of ω and the previous relations yield :

$$\begin{aligned}\omega(y_0^* x_0) &= (\omega \otimes h)(\sigma(y_0^*)\sigma(x_0)) \\ &= (\omega \otimes h)((id_\mathcal{M} \otimes (id_A \otimes \rho_{\beta^c})\Delta_A)(\sigma(y^*))(id_\mathcal{M} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))),\end{aligned}$$

which is equal to 0 by Lemma 11 ii) since we have for any $a, b \in A$:

$$h((id_A \otimes \rho_{\beta^c})(\Delta_A(a))(id_A \otimes \rho_\alpha)(\Delta_A(b)) = (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0.$$

The equality $\omega(x_0 y_0^*) = 0$ follows similarly. □

The previous corollary actually shows that the spectral subspaces $\mathcal{M}_\alpha$, $\alpha \in \widehat{G}$ are mutually orthogonal with respect to both scalar products $\langle x, y\rangle_{2,\omega} = \omega(y^* x)$ and $\langle x, y\rangle_{1,\omega} = \omega(xy^*)$ on $\mathcal{M}$.

The next statement is the analogue of the decomposition of a representation of a compact Lie group into isotypic subrepresentations.

Proposition 13 *i) The spectral subspaces $\mathcal{M}_\alpha$, $\alpha \in \widehat{G}$ are σ-invariant. Moreover*

$$\sigma(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha \otimes \mathcal{A}_\alpha = span\{x_{ij} \otimes u_{ij}^\alpha \mid x_{ij} \in \mathcal{M}_\alpha, \quad 1 \le i, j \le d_\alpha\}$$

and if V is a finite dimensional σ-invariant subspace of $\mathcal{M}_\alpha$, then α is the only irreducible subcorepresentation of $\sigma|_V$.

ii) Any finite dimensional subspace of $\mathcal{M}_\alpha$ is contained in a finite dimensional σ-invariant subspace of $\mathcal{M}_\alpha$.

iii) If the Haar measure is faithful on A, then $\mathcal{M}_0 = span\{\mathcal{M}_\alpha \mid \alpha \in \widehat{G}\}$ is dense in the Hilbert space $\mathcal{H}_\omega$.

Proof. i) Remark first that

$$\begin{aligned}\sigma(P_\alpha(x)) &= \sigma((id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\alpha)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x)), \qquad x \in \mathcal{M}.\end{aligned}$$

Since $(id_A \otimes \rho_\alpha)(\Delta_A(A)) \subset \mathcal{A}_\alpha$ it follows that $\sigma(\mathcal{M}_\alpha) \subset \mathcal{M} \otimes \mathcal{A}_\alpha$. Using again the density of $span\{\mathcal{A}_\beta \mid \beta \in \widehat{G}\}$ in A and the equality

$$(\rho_\alpha \otimes id_A)(\Delta_A(u_{ij}^\beta)) = \delta_{\alpha\beta} u_{ij}^\beta = (id_A \otimes \rho_\alpha)(\Delta_A(u_{ij}^\beta)),$$

we obtain $(\rho_\alpha \otimes id_A)\Delta_A = (id_A \otimes \rho_\alpha)\Delta_A$ and furthermore

$$\begin{aligned}(P_\alpha \otimes id_A)(\sigma(x)) &= ((id_{\mathcal{M}} \otimes \rho_\alpha)\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \rho_\alpha \otimes id_A)(\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \rho_\alpha \otimes id_A)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes (\rho_\alpha \otimes id_A)\Delta_A)(\sigma(x)) = \sigma(P_\alpha(x)), \qquad x \in \mathcal{M}.\end{aligned}$$

This shows in particular that $\sigma(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha \otimes \mathcal{A}_\alpha$. If $V \subset \mathcal{M}_\alpha$ is finite dimensional and σ-invariant, then $\sigma|_V$ contains only copies of α since $(id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x)) = x$, $x \in V$.

ii) It is enough to prove that for any $x \in \mathcal{M}_\alpha$, there exists $V_x \subset \mathcal{M}_\alpha$ finite dimensional σ-invariant space that contains x. Set $V_x = \{(id_{\mathcal{M}} \otimes \rho)(\sigma(x)) \mid \rho \in A^*\}$, subspace of $\mathcal{M}_\alpha$ which contains x since $x = (id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x))$ and is finite dimensional since

$\sigma(x) = \sum_{i,j=1}^{d_\alpha} x_{ij} \otimes u^\alpha_{ij}$ for some $x_{ij} \in \mathcal{M}_\alpha$. Therefore $V_x \subset span\{x_{ij} \mid 1 \leq i,j \leq d_\alpha\}$ and finally V_x is σ-invariant since for any $\rho, \rho' \in A^*$, $x \in \mathcal{M}$ we have :

$$\begin{aligned}(id_{\mathcal{M}} \otimes \rho')(\sigma((id_{\mathcal{M}} \otimes \rho)(\sigma(x))) &= (id_{\mathcal{M}} \otimes \rho')((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))) \\ &= (id_{\mathcal{M}} \otimes \rho'((id_A \otimes \rho) \circ \Delta_A))(\sigma(x)) = (id_{\mathcal{M}} \otimes (\rho' * \rho))(\sigma(x)).\end{aligned}$$

iii) One can easily check as in the case when both corepresentations are finite dimensional that for any $\alpha \in \widehat{G}$ and any corepresentation $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ of A we still have:

$$\begin{aligned}Mor(\alpha, V) &= \{\xi \in \mathcal{H}_V \otimes \mathcal{H}'_\alpha = \mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_V) \mid (V \odot \alpha^c)_\rho \xi = \rho(1_A)\xi, \quad \forall \rho \in A^*\} \\ &= (V \odot \alpha^c)_h(\mathcal{H}_V \odot \mathcal{H}'_\alpha).\end{aligned}$$

Assume now that $\overline{\mathcal{M}_0} \neq \mathcal{H}_\omega$. Then, since $U \mid_{\overline{\mathcal{M}_0}}$ is a subcorepresentation of U it follows that there exists $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ corepresentation of A with $0 \neq \mathcal{H}_V \subset \mathcal{H}_\omega \ominus \overline{\mathcal{M}_0}$. By Lemma 9 there exists $\beta \in \widehat{G}$ such that $(V \odot \beta^c)_h \neq 0$ and we find a nonzero $S \in \mathcal{L}(\mathcal{H}_\beta, \mathcal{H}_V)$ with $(S \otimes id_A)\beta = V(S \otimes id_A)$. Since $KerS$ is β-invariant and β irreducible, S follows one-to-one. But $RanS$ is a finite dimensional V-invariant subspace of $\mathcal{H}_V$, thus $\beta \preceq V$. Since $\mathcal{H}_V$ is orthogonal to $\mathcal{M}_0$ we have $(id_{\mathcal{L}(\mathcal{H}_\beta)} \otimes \rho_\alpha)(\beta) = 0$ for all $\alpha \in \widehat{G}$, therefore $\beta = 0$ by [Wor, 5.8]. But σ is one-to-one, thus V is nondegenerate and we get a contradiction. Consequently $\overline{\mathcal{M}_0} = \mathcal{H}_\omega$. $\square$

Proposition 14 *$\mathcal{M}_0$ is a *-algebra and $\mathcal{M}_\alpha \mathcal{M}_\beta \subset span\{\mathcal{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$ for all $\alpha, \beta \in \widehat{G}$.*

Proof. Fix $\alpha, \beta \in \widehat{G}, u^\alpha \in \alpha, u^\beta \in \beta$ and $\mathcal{H}_\alpha \subset \mathcal{M}_\alpha$, $\mathcal{H}_\beta \subset \mathcal{M}_\beta$ irreducible finite dimensional σ-invariant subspaces. Let $\{e_i\}_{1 \leq i \leq d_\alpha}$ and $\{f_r\}_{1 \leq r \leq d_\beta}$ be orthonormal basis in $\mathcal{H}_\alpha$ and respectively $\mathcal{H}_\beta$ such that $\sigma(e_i) = \sum_{j=1}^{d_\alpha} e_j \otimes u^\alpha_{ji}$ and $\sigma(f_r) = \sum_{s=1}^{d_\beta} f_s \otimes u^\beta_{sr}$. The restriction of σ to $\mathcal{H}_\alpha$ (respectively to $\mathcal{H}_\beta$), $\widehat{V}_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha \otimes A$ (respectively $\widehat{V}_\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes A$) implements u^α (respectively u^β). Then $\widehat{V} : \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \otimes A$, $\widehat{V}(x \otimes y) = \widehat{V}_\alpha(x)_{13} \widehat{V}_\beta(y)_{23} = \sigma(x)_{13}\sigma(y)_{23}$ implements $u^\alpha \odot u^\beta$. Consider the onto linear operator $S : \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to span\mathcal{H}_\alpha\mathcal{H}_\beta$, $S(x \otimes y) = xy$. Since :

$$\begin{aligned}\widehat{V}(e_i \otimes f_r) &= \sum_{j=1}^{d_\alpha} \sum_{s=1}^{d_\beta} e_j f_s \otimes u^\alpha_{ji} u^\beta_{sr} = (\sum_{j=1}^{d_\alpha} e_j \otimes u^\alpha_{ji})(\sum_{s=1}^{d_\beta} e_s \otimes u^\beta_{sr}) \\ &= \sigma(e_i)\sigma(f_r) = \sigma(e_i f_r) = \sigma S(e_i \otimes f_r),\end{aligned}$$

the following diagram is commutative

$$\begin{array}{ccc}\mathcal{H}_\alpha \otimes \mathcal{H}_\beta & \xrightarrow{\widehat{V}} & \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \otimes \mathcal{A} \\ S \uparrow & & \uparrow S \otimes id_A \\ span\mathcal{H}_\alpha\mathcal{H}_\beta & \xrightarrow{\sigma} & span\mathcal{H}_\alpha\mathcal{H}_\beta \otimes A.\end{array}$$

Therefore $S \in Mor(u^\alpha \odot u^\beta, \sigma \mid_{span\mathcal{H}_\alpha\mathcal{H}_\beta})$ and any irreducible subcorepresentation of $\sigma \mid_{span\mathcal{H}_\alpha\mathcal{H}_\beta}$ should appear in $\alpha \odot \beta$, hence decomposing the last corepresentation into irreducible components we get $\mathcal{M}_\alpha\mathcal{M}_\beta \subset span\{\mathcal{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$. $\mathcal{M}_0$ is *-closed since $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$ by Lemma 11. □

Denote by $\mathcal{H}_h$ the completion of A with respect to the scalar product $\langle a, b\rangle_{2,h} = h(b^*a)$, $a, b \in A$ and consider $W(x_\omega \otimes a_h) = (\sigma(x)(1_\mathcal{M} \otimes a))(1_\omega \otimes 1_h)$, $x \in \mathcal{M}, a \in A$. Clearly $W \in \mathcal{L}(\mathcal{H}_\omega \otimes \mathcal{H}_h)$ is a unitary. Since $\rho_\alpha = M_\alpha h((f_1 * \chi_\alpha)^* \cdot)$ and $f_1 * \chi_\alpha \in A$, it follows that $\rho_\alpha \in \mathcal{L}(\mathcal{H}_h)_*$ and ρ_α coincides on $\mathcal{L}(\mathcal{H}_h)$ with the vector form $M_\alpha\langle \quad \cdot 1_h, (f_1 * \chi_\alpha)_h\rangle_{2,h}$. Therefore it makes sense to consider $p(\alpha) = (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \rho_\alpha)(W) \in \mathcal{L}(\mathcal{H}_\omega)$. A straightforward computation yields for any $x, y \in \mathcal{M}$:

$$\begin{aligned}
&\langle P_\alpha(x)_\omega, y_\omega\rangle_2 = \omega(y^* P_\alpha(x)) = \omega(y^*(id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x)))\\
&= \omega((id_\mathcal{M} \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x)))\\
&= (\omega \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x)) = (\omega(y^* \cdot) \otimes h(M_\alpha(f_1 * \chi_\alpha)^* \cdot))\sigma(x)\\
&= \langle \sigma(x)(1_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h}\\
&= \langle W(x_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h}\\
&= \langle (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \rho_\alpha)(W)x_\omega, y_\omega\rangle_2 = \langle p(\alpha)(x_\omega), y_\omega\rangle_2.
\end{aligned}$$

We obtain :

Remark 15 *P_α extends to the bounded operator $p(\alpha) \in \mathcal{L}(\mathcal{H}_\omega)$ and $p(\alpha)$ is a projection from $\mathcal{H}_\omega$ onto $\mathcal{H}_\alpha = \{x_\omega \mid x \in \mathcal{M}_\alpha\}$. Moreover, it is easy to see that $p(\alpha)$ is self-adjoint.*

Lemma 16 $\sum_{i=1}^{d_\alpha} u_{pi}^{\alpha^*} \kappa^2(u_{qi}^\alpha) = \delta_{pq} 1_A = \sum_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha) u_{iq}^{\alpha^*}$ *for all $\alpha \in \widehat{G}, 1 \le p, q \le d_\alpha$.*

Proof. Using the antimultiplicativity of κ and $\kappa(u_{ij}^\alpha) = u_{ji}^{\alpha^*}$ we obtain :

$$\sum_{i=1}^{d_\alpha} u_{pi}^{\alpha^*} \kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(u_{ip}^\alpha)\kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(\kappa(u_{qi}^\alpha)u_{ip}^\alpha) = \kappa(\sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha)u_{ip}^\alpha) = \delta_{pq}1_A$$

$$\sum_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha)u_{iq}^{\alpha^*} = \sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha \kappa(u_{ip}^\alpha)) = \delta_{pq}1_A. \qquad \square$$

Theorem 17 *If $G = (A, u)$ is a compact matrix pseudogroup and $\mathcal{M}$ is a unital C^*-algebra with an A-algebra structure given by the ergodic coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$, then the spectral subspaces $\mathcal{M}_\alpha$ are finite dimensional and $dim\mathcal{M}_\alpha \le M_\alpha^2$.*

Proof. Let $\alpha \in \widehat{G}$ and fix $u^\alpha \in \alpha$ acting on $\mathcal{H}_\alpha$ and an orthonormal basis $\xi_1, ..., \xi_{d_\alpha}$ in $\mathcal{H}_\alpha$ such that $u^\alpha(\xi_i) = \sum_{r=1}^{d_\alpha} \xi_r \otimes u^\alpha_{ri}$. Let $V_1, ..., V_N \subset \mathcal{M}_\alpha$ be mutually $\langle\ ,\ \rangle_{2,\omega}$ orthogonal irreducible σ-invariant subspaces and let $U_k \in \mathcal{L}(\mathcal{H}_\alpha, V_k)$, $1 \leq k \leq N$ be unitaries such that $\sigma U_k = (U_k \otimes id_A)\alpha$. Denoting $e_i^{(k)} = U_k\xi_i$ it follows that $e_1^{(k)}, ..., e_{d_\alpha}^{(k)}$ is an orthonormal basis in V_k and

$$\sigma(e_i^{(k)}) = \sum_{r=1}^{d_\alpha} e_r^{(k)} \otimes u^\alpha_{ri}, \tag{1}$$

thus

$$\sigma(\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)*}) = \sum_{i,r,s=1}^{d_\alpha} e_r^{(k)} e_s^{(l)*} \otimes u^\alpha_{ri} u^{\alpha*}_{si} = \sum_{r=1}^{d_\alpha} e_r^{(k)} e_r^{(l)*} \otimes 1_A$$

and $\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)*} \in \mathbf{C}1_{\mathcal{M}}$.

Moreover, the equality $(F_\alpha \otimes id_A)\alpha = \alpha^{cc} F_\alpha$ and the previous Lemma yield :

$$\begin{aligned}
\sigma(\sum_{i=1}^{d_\alpha}(e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)})) &= \sigma(\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1}(\xi_i)) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)*})(U_l F_\alpha^{-1} U_l^* \otimes id_A)\alpha^{cc}(\xi_i) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)*}) \sum_{s=1}^{d_\alpha} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \kappa^2(u^\alpha_{si}) \\
&= \sum_{r,s=1}^{d_\alpha} e_r^{(k)*} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \sum_{i=1}^{d_\alpha} u^{\alpha*}_{ri} \kappa^2(u^\alpha_{si}) \\
&= \sum_{r=1}^{d_\alpha} e_r^{(k)*} U_l F_\alpha^{-1} U_l^*(e_r^{(l)}) \otimes 1_A,
\end{aligned}$$

thus

$$\begin{aligned}
\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)}) &= \omega(\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)}))1_{\mathcal{M}} \\
&= \sum_{i=1}^{d_\alpha} \langle U_l F_\alpha^{-1} U_l^*(e_i^{(l)}), e_i^{(k)}\rangle_{2,\omega} 1_{\mathcal{M}} = \delta_{kl} Tr(F_\alpha^{-1}) 1_{\mathcal{M}} = \delta_{kl} M_\alpha 1_{\mathcal{M}}.
\end{aligned} \tag{2}$$

Since the modular operator F_α associated with u^α is positive ([Wor, 5.4]), the matrix $C = (\langle F_\alpha^{-1}\xi_i, \xi_j\rangle)_{ij} \in Mat_{d_\alpha}(\mathbf{C})$ is positive definite.

Denote $C^{1/2} = (\lambda_{ij})_{ij}$, $x_{rl} = \sum_{j=1}^{d_\alpha} \lambda_{rj} e_j^{(l)}$ and $X = (x_{rl})_{rl} \in Mat_{d_\alpha, N}(\mathcal{M})$. Then

$$\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} \xi_i = \sum_{i,j=1}^{d_\alpha} \langle F_\alpha^{-1}\xi_i, \xi_j\rangle e_i^{(k)*} e_j^{(l)} = \sum_{r,i,j=1}^{d_\alpha} \overline{\lambda_{ri}} \lambda_{rj} e_i^{(k)*} e_j^{(l)} = \sum_{r=1}^{d_\alpha} x_{rk}^* x_{rl},$$

or in other words $X^*X = M_\alpha I$ in $\mathcal{M} \otimes Mat_N(\mathbf{C})$. This yields $XX^* \leq M_\alpha I$ in $\mathcal{M} \otimes Mat_{d_\alpha}(\mathbf{C})$.

Consider the positive linear functional $\phi \in Mat_{d_\alpha}(\mathbf{C})_*$, $\phi(Y) = Tr(C^{-1}Y)$, Tr being the normalized trace on $Mat_{d_\alpha}(\mathbf{C})$. Then

$$\|\phi\| = \phi(I) = Tr(C^{-1}) = \sum_{i=1}^{d_\alpha}(C^{-1})_{ii} = \sum_{i=1}^{d_\alpha}\langle F_\alpha\xi_i, \xi_i\rangle = Tr(F_\alpha) = M_\alpha$$

and the last inequality yields :

$$\begin{aligned} M_\alpha^2 &\geq \|\omega\otimes\phi\|\cdot\|XX^*\| \geq (\omega\otimes\phi)(XX^*) = \sum_{i,j=1}^{d_\alpha}\sum_{k=1}^{N}\phi(e_{ij})\omega(x_{ik}x_{jk}^*) \\ &= \sum_{k=1}^{N}\sum_{i,j,r,s=1}^{d_\alpha}\lambda_{sj}(C^{-1})_{ji}\lambda_{ir}\omega(e_r^{(k)}e_s^{(k)*}) = \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\omega(e_r^{(k)}e_r^{(k)*}) \\ &= \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\|e_r^{(k)}\|_{1,\omega}^2. \end{aligned}$$

This inequality plays a crucial role since it shows that if $W_\alpha \subset \mathcal{M}_\alpha$ is a σ-invariant subspace and A_{W_α} the positive invertible operator on W_α (with respect to $\langle\ ,\ \rangle_{2,\omega}$) such that $\langle A_{W_\alpha}(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W_\alpha$, then

$$(3) \qquad Tr(A_{W_\alpha}) \leq M_\alpha^2.$$

Consider now a finite-dimensional σ-invariant subspace $W \subset \mathcal{M}_0$ and let A_W be the unique positive invertible operator on W with $\langle A_W(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W$. Since $\langle \mathcal{M}_\alpha, \mathcal{M}_\beta\rangle_{1,\omega} = 0$ for $\alpha \neq \beta$, it follows that A_W invariates each spectral subspace W_α of W and

$$(4) \qquad Tr(A_W) = \sum_{\alpha\prec W} Tr(A_{W_\alpha}) \leq \sum_{\alpha\prec W} M_\alpha^2.$$

If in addition A_W is *-invariant and $Jx = x^*$, $x \in \mathcal{M}$, then

$$\langle JA_W^{-1}Jx, y\rangle_{2,\omega} = \langle y^*, A_W^{-1}(x^*)\rangle_{1,\omega} = \langle y^*, x^*\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega} = \langle A_W(x), y\rangle_{2,\omega}, \quad x, y \in W.$$

Therefore $JA_W^{-1}J = A_W$ and in particular if λ is an eigenvalue of A_W with multiplicity m_λ, so is λ^{-1} with the same multiplicity. It follows that

$$(5) \qquad Tr(A_W) = \sum_\lambda m_\lambda\lambda = m_1 + \sum_{\lambda>1} m_\lambda(\lambda+\lambda^{-1}) \geq m_1 + \sum_{\lambda>1} 2m_\lambda = dim(W).$$

Finally let $V \subset \mathcal{M}_\alpha$ be a finite-dimensional σ-invariant subspace. Since $V^* \subset \mathcal{M}_{\alpha^c}$ it follows that $W = V + V^*$ is σ-invariant. If α is self-conjugate, then $W \subset \mathcal{M}_\alpha$, $W = W^*$ and therefore $dim(V) \leq dim(W) \leq M_\alpha^2$. In the case when $\alpha \neq \alpha^c$ in $\widehat{G}$ the sum $V+V^*$ is orthogonal and by (4) and (5) we get $2dim(V) = dim(W) \leq M_\alpha^2 + M_{\alpha^c}^2 = 2M_\alpha^2$. □

The next statement describes the modularity of ω.

Proposition 18 *There exists $\Theta : \mathcal{M}_0 \to \mathcal{M}_0$ linear multiplicative map such that $\omega(x\Theta(y)) = \omega(yx)$, $x \in \mathcal{M}, y \in \mathcal{M}_0$. Moreover Θ invariates all the irreducible σ-invariant subspaces of $\mathcal{M}_0$ and is a scalar multiple of the modular operator F_α on such a subspace of $\mathcal{M}_\alpha$ for all $\alpha \in \widehat{G}$.*

Proof. Let $V_1, ..., V_N$ be the mutually orthogonal irreducible σ-invariant subspaces of $\mathcal{M}_\alpha$ and let $e_1^{(k)}, ..., e_d^{(k)}$ be an orthonormal basis in V_k ($d = dim(\alpha)$) such that $\sigma(e_i^{(k)}) = \sum_{r=1}^{d} e_r^{(k)} \otimes u_{ri}^\alpha$, $1 \leq i \leq d, 1 \leq k \leq N$. Then $\sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} \in \mathcal{M}^\sigma$, therefore there exist $\lambda_{kl} \in \mathbf{C}$, $1 \leq k, l \leq N$ such that $\sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} = \lambda_{kl} 1_\mathcal{M}$. The matrix $\Lambda = (\lambda_{kl})_{1 \leq k, l \leq 1, N}$ is positively definite, hence there exists a unitary $S = (s_{kl})_{1 \leq k,l \leq N} \in Mat_N(\mathbf{C})$ and $\lambda_1, ..., \lambda_N > 0$ such that $S \Lambda S^* = diag(\lambda_1, ..., \lambda_N)$. Replacing $e_i^{(k)}$ by $\sum_{p=1}^{N} s_{kp} e_i^{(p)}$ we get :

$$\text{(6)} \qquad \sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} = \delta_{kl} \lambda_k 1_\mathcal{M}, \qquad 1 \leq k, l \leq N.$$

Fix now $1 \leq k, l \leq N$ and consider $T_{kl} : V_k \to V_l$, $T_{kl}(x) = \sum_{j=1}^{d} \omega(x e_j^{(l)*}) e_j^{(l)}$, $x \in V_k$. Combining the σ-invariance of ω, the orthogonality relations (5.25) in [Wor] and (6) we get for $1 \leq i \leq d$:

$$\begin{aligned} T_{kl} e_i^{(k)} &= \sum_{j=1}^{d} \omega(e_i^{(k)} e_j^{(l)*}) e_j^{(l)} = \sum_{j,p,q=1}^{d} (\omega \otimes h)(e_p^{(k)} e_q^{(l)*} \otimes u_{pi}^\alpha u_{qj}^{\alpha *}) e_j^{(l)} \\ &= \sum_{j,p,q=1}^{d} \omega(e_p^{(k)} e_q^{(l)*}) \tfrac{\delta_{pq}}{M_\alpha} f_1(u_{ji}^\alpha) e_j^{(l)} = \sum_{j,p=1}^{d} \omega(e_p^{(k)} e_p^{(l)*}) \tfrac{f_1(u_{ji}^\alpha)}{M_\alpha} e_j^{(l)} \\ &= \tfrac{\delta_{kl}\lambda_k}{M_\alpha} \sum_{j=1}^{d} f_1(u_{ji}^\alpha) e_j^{(k)} = \tfrac{\delta_{kl}\lambda_k}{M_\alpha} (id_\mathcal{M} \otimes f_1)(\sigma(e_i^{(k)})), \end{aligned}$$

thus $T_{kl} = \frac{\delta_{kl}\lambda_k}{M_\alpha} (id_\mathcal{M} \otimes f_1) \circ \sigma$. We check now that $T_k = T_{kk} \in Mor(\sigma |_{V_k}, \sigma^{cc} |_{V_k})$. The previous formula for T_k yields :

$$\sigma T_k = \frac{\lambda_k}{M_\alpha} \sigma((id_\mathcal{M} \otimes f_1)\sigma) = \frac{\lambda_k}{M_\alpha} (id_\mathcal{M} \otimes ((f_1 \otimes id_A)\Delta_A))\sigma$$

and

$$(T_k \otimes id_A)\sigma = \frac{\lambda_k}{M_\alpha}((id_\mathcal{M} \otimes f_1)\sigma \otimes id_A)\sigma = \frac{\lambda_k}{M_\alpha}(id_\mathcal{M} \otimes ((f_1 \otimes id_A)\Delta_A))\sigma.$$

Since $\kappa^2(a) = f_{-1} * a * f_1$, $a \in \mathcal{A}$ we also have :

$$\begin{aligned} (id_\mathcal{M} \otimes \kappa^2)(id_\mathcal{M} \otimes (id_A \otimes f_1) \circ \Delta_A)(y \otimes a) &= y \otimes \kappa^2(f_1 * a) = y \otimes (a * f_1) \\ = (id_\mathcal{M} \otimes (f_1 \otimes id_A) \circ \Delta_A)(y \otimes a), &\qquad y \in \mathcal{M}, a \in \mathcal{A}, \end{aligned}$$

hence $(T_k \otimes id_A)\sigma = (id_\mathcal{M} \otimes \kappa^2)\sigma T_k = \sigma^{cc} T_k$ on V_k.

But $Mor(\alpha, \alpha^{cc}) = \{\lambda F_\alpha \mid \lambda \in \mathbf{C}\}$, therefore $\omega(e_i^{(k)} e_j^{(l)*}) = \delta_{kl} \lambda'_k (F_\alpha)_{ji}$. Comparing the previous relation with (7) we get :

$$\langle e_i^{(k)}, e_j^{(l)} \rangle_{1,\omega} = \omega(e_i^{(k)} e_j^{(l)*}) = \frac{\delta_{kl}\lambda_k}{M_\alpha}(F_\alpha)_{ji} = \frac{\delta_{kl}\lambda_k}{M_\alpha} \langle F_\alpha(e_i^{(k)}), e_j^{(l)} \rangle_{2,\omega},$$

thus $\omega(xy^*) = \delta_{kl}\omega(y^*\Theta(x)) = \omega(y^*\Theta(x))$ for all $x \in V_k, y \in V_l$, where we let $\Theta\mid_{V_k} = \frac{\lambda_k}{M_\alpha}F_\alpha \in \mathcal{L}(V_k)$. The orthogonality of the spectral subspaces $\mathcal{M}_\alpha$ implies now that $\omega(xy) = \omega(y\Theta(x))$ for all $x \in \mathcal{M}_0, y \in \mathcal{M}$. Clearly Θ is linear by definition and follows multiplicative by the previous equality. □

§2. The structure of the crossed product

Using the finite dimensionality of the spectral subspaces $\mathcal{M}_\alpha$ we prove that the reduced C^*-crossed product $\mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$ (as defined in [BS]) of a unital C^*-algebra $\mathcal{M}$ by an ergodic coaction of a compact matrix pseudogroup $G = (A, u)$ with faithful Haar measure h on A, turning $\mathcal{M}$ into an A-algebra, is isomorphic to a direct sum of algebras of compact operators, generalizing Proposition 2 in [L] and Corollary 2 in [Wa1, §1.4]. The main ingredient is the Takesaki-Takai duality type theorem of Baaj and Skandalis and the line of the proof follows Wassermann's one for the case when $A = C(G)$, the commutative C^*-algebra of continuous functions on a compact group G and $\mathcal{N} = \mathcal{M} \times_\sigma G$.

Remark first that A is an A-algebra via the comultiplication $\Delta_A : A \to A \otimes A$, that Δ_A is ergodic and that h is the Δ_A-invariant state on A. Denote $\mathcal{H} = \mathcal{H}_h$ and consider as in Remark 15 the operator $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, $V(a_h \otimes b_h) = (\Delta_A(a)(1_A \otimes b))(1_h \otimes 1_h)$, $a, b \in A$. Then V is a biregular irreducible multiplicative unitary. Moreover, the partial isometry U (no relationship with the unitary associated to the coaction σ in §1) in the polar decomposition $T = U \mid T \mid$ of the closure of the preclosed operator T_0 defined by $T_0(a_h) = \kappa(a)_h$, $a \in \mathcal{A}$ is a unitary on $\mathcal{H}$ with $U^2 = I_{\mathcal{H}}$. In fact $Ux_h = (f_1 * \kappa(x))_h$, $x \in \mathcal{A}$ and $(\mathcal{H}, V, U)$ is a Kac system (see [BS, §6]). Consider the reduced C^*-crossed product $\mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$, defined in [BS, §7] as the C^*-algebra generated by products of type $\sigma_L(x)(1_A \otimes \rho(\omega))$, $x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*$. Then, there exists a dual coaction $\widehat{\sigma} = \sigma_{\mathcal{M}\times_\sigma\widehat{A}}$ on $\mathcal{N}$, which transforms this way into an $\widehat{A}$-algebra. Let A acting in the GNS representation of h on $\mathcal{H}$ and denote $\sigma_R(x) = (id_{\mathcal{M}} \otimes AdU)(\sigma(x))$, $x \in \mathcal{M}$, $\lambda(\omega) = (id_{\mathcal{L}(\mathcal{H})} \otimes \omega)(V)$, $\omega \in \mathcal{L}(\mathcal{H})_*$. Then, by Theorem 7.5 in [BS], $\mathcal{M} \times_\sigma \widehat{A}$ identifies with the norm closure of $span\{\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*\}$, $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$ with the closure of $span\{\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)a) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A\}$ and denoting $\sigma'(x \otimes k) = V_{23}\sigma(x)_{13}(1_{\mathcal{M}} \otimes k \otimes 1_A)V_{23}^*$, $x \in \mathcal{M}$, $k \in \mathcal{K}(\mathcal{H})$, the double crossed-product $((\mathcal{M} \times_\sigma \widehat{A}) \times_{\widehat{\sigma}} A, \widehat{\sigma})$ is isomorphic to $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}), \sigma')$ as A-algebras. Moreover, the last two equalities in the proof of that theorem show that via the previous identification $\mathcal{M} \times_\sigma \widehat{A} \subset (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$.

The proof of the structure of the crossed-product will make use of the next Lemma:

Lemma ([Wa1, §1.4]). *Let A be a (not necessarily unital) C^*-algebra with an approximate identity $p_1 \le p_2 \le \ldots$ consisting of finite rank projections (i.e. $dim(p_iAp_i) < \infty$). Then A is isomorphic to a direct sum of algebras of compact operators.*

Theorem 19 *Let $G = (A, u)$ be a compact matrix pseudogroup and let $\mathcal{M}$ be a unital C^*-algebra which is an A-algebra via the ergodic coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$. Then $\mathcal{M} \times_\sigma \widehat{A} \simeq \oplus_i \mathcal{K}(\mathcal{H}_i)$.*

Proof. Replacing A by its reduced C^*-algebra, the crossed product $\mathcal{M} \times_\sigma \widehat{A}$ is still unchanged (we owe this remark to the referee), thus one may assume that that the Haar measure is faithful on A. For each $\alpha \in \widehat{G}$ consider the projections $p(\alpha)$ from $\mathcal{H}$ onto $\mathcal{H}_\alpha$ defined in Remark 15. Since $p(\alpha) = (id_{\mathcal{L}(\mathcal{H})} \otimes \rho_\alpha)(V)$ and $\rho_\alpha \in \mathcal{L}(\mathcal{H})_*$ it follows that $p(\alpha) \in \widehat{A}$. But $\mathcal{M}$ is unital, therefore $1_{\mathcal{M}} \otimes p(\alpha) \in \mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$. Since $\sum_{\alpha\in\widehat{G}} p(\alpha) = 1_{\widehat{A}}$ in the strict topology, it follows that writing $\widehat{G} = \cup_n F_n$ with $card(F_n) < \infty$ we get an approximate unit $p_n = p(F_n) = \sum_{\alpha\in F_n} p(\alpha)$, $n \geq 1$ of $\mathcal{N}$. Finally we check that each $p(F_n)$ has finite rank, or equivalently that $dim(p(\alpha)\mathcal{N}p(\pi)) < \infty$ for all $\alpha, \pi \in \widehat{G}$. Since $1_{\mathcal{M}} \otimes p(\alpha) \in \mathcal{N}$ and $\mathcal{N} \subset (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$, we obtain :

$$p(\alpha)\mathcal{N}p(\pi) \subset p(\alpha)(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}p(\pi) \subset (\mathcal{M} \otimes p(\alpha)\mathcal{K}(\mathcal{H})p(\pi))^{\sigma'} = (\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))^{\sigma'}.$$

If $\alpha_0 \in \widehat{G}$ is fixed, then $\sigma' |_{\mathcal{M}^c_{\alpha_0} \otimes \mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)}$ is a corepresentation of A that coincides with the tensor product corepresentation $\sigma |_{\mathcal{M}^c_0} \odot \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)}$ and therefore

$$\begin{aligned}(id_{\mathcal{M}\otimes\mathcal{L}(\mathcal{H})} \otimes h)(\sigma'(\mathcal{M}^c_0 \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))) &= Ran(\sigma |_{\mathcal{M}^c_0} \odot \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)})_h \\ &= Mor(\sigma |_{\mathcal{M}^c_0}, \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)})^T.\end{aligned}$$

Since there are only finitely many $\alpha_0 \in \widehat{G}$ that appear in $\sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)}$, the space $(\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))^{\sigma'} = (id_{\mathcal{M}} \otimes h)(\sigma'(\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha)))$ follows finite dimensional. □

Remark 20 *Although we didn't use it in proving the last theorem, it is easy to check that in fact* $\mathcal{M} \times_\sigma \widehat{A} = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$ *via the realization of the crossed-product in* $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$.

This is the case since the canonical conditional expectation $E = (id_{\mathcal{M}\otimes\mathcal{K}(\mathcal{H})} \otimes h)\sigma'$ carries total sets in $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$ onto total sets in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$. In particular the set

$$\mathcal{S} = \{E(\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)a)) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A\}$$

is total in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$. But the formula for σ' and $(id_A \otimes h)\Delta_A(a) = h(a)1_A$, $a \in A$, yield for any $x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A$:

$$\begin{aligned}&E(\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)a)) \\ &\quad= (id_{\mathcal{M}\otimes\mathcal{K}(\mathcal{H})} \otimes h)((\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)) \otimes 1_A)(1_{\mathcal{M}} \otimes \Delta_A(a)) \\ &\quad= \sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega))(1_{\mathcal{M}} \otimes (id_{\mathcal{L}(\mathcal{H})} \otimes h)\Delta_A(a)) \\ &\quad= h(a)\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)),\end{aligned}$$

therefore $\mathcal{S} \subset \mathcal{M} \times_\sigma \widehat{A}$. □

Let B and C be unital C^*-algebras, $G = (A, u)$ be a compact matrix pseudogroup with A nuclear and $\sigma : C \to C \otimes A$ be a unital *-morphism. By the associativity of the maximal tensor product and the nuclearity of A, the diagram :

$$\begin{array}{ccc} B\otimes_{\max} C & \overset{\tilde{\sigma} = id_B \otimes_{\max} \sigma}{\longrightarrow} & B\otimes_{\max}(C\otimes A) = (B\otimes_{\max} C)\otimes A \\ \tilde{\sigma} \uparrow & & \uparrow id_{B\otimes_{\max} C}\otimes \Delta_A = id_B \otimes_{\max} (id_C \otimes \Delta_A) \\ B\otimes_{\max}(C\otimes A) = (B\otimes_{\max} C)\otimes A & \overset{id_B\otimes_{\max}(\sigma\otimes id_A)}{\longrightarrow} & (B\otimes_{\max} C)\otimes A\otimes A = B\otimes_{\max}(C\otimes A\otimes A)\ , \end{array}$$

is commutative although $\tilde{\sigma}$ may not be one-to-one in general (note that $\tilde{\sigma}\otimes_{\max} id_A = id_B\otimes_{\max}(\sigma\otimes id_A)$). But Lemma 4.1 i) holds true in such a case, thus $\tilde{E} = (id_{B\otimes_{\max}C}\otimes h)\tilde{\sigma}$ is a conditional expectation from $B\otimes_{\max} C$ onto $(B\otimes_{\max} C)^{\tilde{\sigma}}$ and it turns out that $\tilde{E} = id_B\otimes_{\max} E$, where $E = (id_C\otimes h)\sigma$ is conditional expectation from C onto C^{σ}. In particular, this shows

Remark 21 $(B\otimes_{\max} C)^{\tilde{\sigma}} = B\otimes_{\max} C^{\sigma}$.

The next statement, whose proof follows the line of [Wa2, Lemma22], shows that if A is nuclear, then $\mathcal{M}$ itself follows nuclear.

Proposition 22 *Let $(\mathcal{M},\sigma)$ be an A-algebra such that $\mathcal{M}\times_\sigma \widehat{A}$ and A are nuclear C^*-algebras. Then $\mathcal{M}$ is nuclear.*

Proof. We prove that given any unital C^*-algebra B, the natural *-morphism θ that maps $B\otimes_{\max}\mathcal{M}$ onto $B\otimes\mathcal{M}$ is one-to-one.

Consider the coaction $\sigma' : \mathcal{M}\otimes\mathcal{K}(\mathcal{H}) \to \mathcal{M}\otimes\mathcal{K}(\mathcal{H})\otimes A$ and define as in the previous remark $\widetilde{\sigma'} : B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H})) \to B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))\otimes A$.

The map $\Theta = \theta\otimes id_{\mathcal{K}(\mathcal{H})} : B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H})) \to B\otimes\mathcal{M}\otimes\mathcal{K}(\mathcal{H})$ is A-equivariant, i.e. $(id_B\otimes\sigma')\Theta = \Theta\widetilde{\sigma'}$ and using the canonical conditional expectations onto fixed point algebras

$$E^{\widetilde{\sigma'}} : B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H})) \to (B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H})))^{\widetilde{\sigma'}} = B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{\widetilde{\sigma'}}$$

and

$$E^{id_B\otimes\sigma'} : B\otimes\mathcal{M}\otimes\mathcal{K}(\mathcal{H}) \to (B\otimes\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{id_B\otimes\sigma'} = B\otimes(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{\sigma'},$$

we get $\Theta E^{\widetilde{\sigma'}} = E^{id_B\otimes\sigma'}\Theta$. Consequently

$$(Ker\theta\otimes\mathcal{K}(\mathcal{H}))^{\widetilde{\sigma'}} = Ker(B\otimes_{\max}(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{\sigma'} \xrightarrow{\Theta} B\otimes(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{\sigma'}).$$

Since $(\mathcal{M}\otimes\mathcal{K}(\mathcal{H}))^{\sigma'} \simeq \mathcal{M}\times_\sigma\widehat{A}$ is nuclear, it follows that $(Ker\theta\otimes\mathcal{K}(\mathcal{H}))^{\widehat{\sigma'}} = 0$ and therefore $Ker\theta = 0$. □

Corollary 23 *If $G = (A,u)$ is a compact matrix pseudogroup, the C^*-algebra A is nuclear and $\mathcal{M}$ is a unital C^*-algebra which is an A-algebra via an ergodic coaction, then $\mathcal{M}$ is nuclear.*

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Simplicity of crossed products from ergodic actions of compact matrix pseudogroups.

Magnus B. Landstad

Appendix to: "Ergodic Actions of Compact Matrix Pseudogroups on C*-algebras" by Florin Boca.

Introduction. For an ergodic covariant system $(\mathcal{M}, \rho, G)$ of a compact group G it was shown in [L2,Theorem 8] and [Wal,Theorem 15] that the crossed product $\mathcal{M} \times_\rho G$ is a simple C*–algebra (or a factor in the von Neumann algebra case) $\Longleftrightarrow$ the multiplicity of each $\pi \in \widehat{G}$ in ρ equals $\dim(\pi)$, and in this case $\mathcal{M} \times_\rho G$ is isomorphic to the algebra of compact operators on $L^2(G)$. We shall here study the corresponding result for an ergodic coaction $(\mathcal{M}, \sigma, \mathcal{A})$ of a compact matrix pseudogroup $G = (\mathcal{A}, u)$ with faithful Haar measure as defined in F. Boca's article.

The main tool used in the group case is the construction of a fundamental eigenoperator $U \in M(\mathcal{M} \otimes C^*(G))$ satisfying $\rho_x \otimes i(U) = U1 \otimes L_x$ for $x \in G$. We shall construct a similar operator Y in Lemma A1. In the group case the multiplicity of each $\pi \in \widehat{G}$ in ρ is always $\leq \dim(\pi)$, hence U can be considered a partial isometry over $L^2(\mathcal{M}, \omega) \otimes L^2(G)$, with ω the invariant trace on $\mathcal{M}$. Since the bound M_α of the multiplicity obtained in Theorem 17 can be larger than d_α, we have to be more careful with the domain and the range of the eigenoperator Y, it turns out that Y is a partial isometry from a subspace of $L^2(\mathcal{M}, \omega) \otimes L^2(\mathcal{A}, h)$ onto $L^2(\mathcal{M}, \omega) \otimes L^2(\mathcal{M}, \omega)$, here h and ω are the canonical invariant states on $\mathcal{A}$, respectively $\mathcal{M}$. It also has to be taken into account that the invariant state ω is not a trace. It was shown above in Proposition 18 by F. Boca that the modular operator Θ leaves the finite dimensional spaces $\mathcal{M}_\alpha$ invariant and that $\Theta|\mathcal{M}_\alpha \cong \Lambda_\alpha \otimes F_\alpha$, where F_α is the fundamental matrix corresponding to α and Λ_α is a $N \times N$–matrix, N being the multiplicity of α in σ.

The main result, Theorem A, can then be stated as follows: $\mathcal{M} \times_\sigma \widehat{\mathcal{A}}$ is a simple C*-algebra $\Longleftrightarrow$ $\mathrm{Tr}(\Lambda_\alpha) = \mathrm{Tr}(F_\alpha)$ for all $\alpha \in \widehat{G}$, and in this case $\mathcal{M} \times_\sigma \widehat{\mathcal{A}}$ is isomorphic to the algebra of compact operators on $L^2(\mathcal{M}, \omega)$. Therefore, if we define the quantum dimension of α to be $\mathrm{Tr}(F_\alpha)$, it is natural to define the quantum multiplicity of α in σ as $\mathrm{Tr}(\Lambda_\alpha)$. We then get a generalisation of the result for ordinary compact groups.

All unexplained notation and references are as in Boca's article.

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Notation. Let $\mathcal{H}_\alpha = \alpha$-part of $\mathcal{H}_h = L^2(\mathcal{A}, h)$, this is generated by $\{(u^\alpha_{ij})_h | i, j \leq d\}$ and has dimension d^2. Similarily we have that $\mathcal{M}_\alpha = \alpha$-part of $\mathcal{M}$ has dimension dN.

Define the following partial isometries:

$$\begin{array}{lll} A^\alpha_{ij}:\mathcal{H}_\alpha\to\mathcal{H}_\alpha & A^\alpha_{ij}((u^\alpha_{kl})_h)=\delta_{jl}(u^\alpha_{ki})_h & i,j\le d\\ B^\alpha_{ij}:\mathcal{M}_\alpha\to\mathcal{M}_\alpha & B^\alpha_{ij}((e^{(k)}_l)_\omega)=\delta_{jk}(e^{(i)}_l)_\omega & i,j\le N\\ C^\alpha_{ij}:\mathcal{H}_\alpha\to\mathcal{M}_\alpha & C^\alpha_{ij}((u^\alpha_{kl})_h)=\delta_{jl}(e^{(i)}_k)_\omega & i\le N,\ j\le d. \end{array}$$

The following formulas should then be easy to verify:

$$\begin{array}{lll} A^\alpha_{ij}A^\alpha_{kl}=\delta_{jk}A^\alpha_{il} & B^\alpha_{ij}B^\alpha_{kl}=\delta_{jk}B^\alpha_{il}, & \\ B^\alpha_{ij}C^\alpha_{kl}A^\alpha_{mn}=\delta_{jk}\delta_{lm}C^\alpha_{in} & C^\alpha_{ij}C^{\alpha *}_{kl}=\delta_{jl}B^\alpha_{ik} & C^{\alpha *}_{ij}C^\alpha_{kl}=\delta_{ik}A^\alpha_{jl}. \end{array}$$

With P_α the ortogonal projction $\mathcal{H}_h\to\mathcal{H}_\alpha$ and V as in Remark 15, let $V(\alpha)=P_\alpha\otimes 1V=VP_\alpha\otimes 1$. Then over $\mathcal{H}_\alpha\otimes\mathcal{H}_h$ we have that $V(\alpha)((u^\alpha_{ij})_h\otimes a_h)=\Delta(u^\alpha_{ij})1_h\otimes a_h=\sum_k(u^\alpha_{ik})_h\otimes u^\alpha_{kj}a_h$. So we have that

$$V(\alpha)=\sum_{jk}A^\alpha_{jk}\otimes u^\alpha_{jk}.$$

Definition. *$\sigma\otimes i$ is the action on $\mathcal{M}\otimes\mathcal{K}$ given by $\sigma\otimes i(m\otimes k)=\sigma(m)_{13}1\otimes k\otimes 1$, so $\sigma'=Ad(V_{23})\sigma\otimes i$. Next, let λ_i be as in Proposition 18, i.e. $\sum_i e^{(k)}_i e^{(l)^*}_i=\delta_{kl}\lambda_k 1_\mathcal{M}$ and take*

$$Y(\alpha)=\sum_{ik}\lambda_k^{-\frac{1}{2}}e^{(k)}_i\otimes C^\alpha_{ki}\in\mathcal{M}\otimes\mathcal{L}(\mathcal{H}_\alpha,\mathcal{M}_\alpha).$$

Lemma A1.

(1) $Y(\alpha)Y(\alpha)^*=1_\mathcal{M}\otimes 1_{\mathcal{L}(\mathcal{M}_\alpha)}$

(2) $\sigma\otimes i(Y(\alpha))=(Y(\alpha)\otimes 1)(1\otimes V(\alpha))$

(3) *Z satisfies (2)* $\iff$ *$Z=(1\otimes D)Y(\alpha)$ for some $D\in\mathcal{L}(\mathcal{M}_\alpha)$*

Proof: (1): $Y(\alpha)Y(\alpha)^*=\sum\lambda_k^{-\frac{1}{2}}\lambda_l^{-\frac{1}{2}}e^{(k)}_i e^{(l)^*}_i\otimes B^\alpha_{kl}=1_\mathcal{M}\otimes 1_{\mathcal{L}(\mathcal{M}_\alpha)}$.
Then for (2):

$$\begin{aligned}\sigma\otimes i(Y(\alpha))=&\ \sum\lambda_k^{-\frac{1}{2}}e^{(k)}_j\otimes C^\alpha_{ki}\otimes u^\alpha_{ji}\\ =&\ \sum\lambda_k^{-\frac{1}{2}}(e^{(k)}_j\otimes C^\alpha_{kj}\otimes 1)(1\otimes A^\alpha_{si}\otimes u^\alpha_{si})=(Y(\alpha)\otimes 1)(1\otimes V(\alpha)).\end{aligned}$$

And for (3): If $Z\in\mathcal{M}\otimes\mathcal{L}(\mathcal{H}_\alpha,\mathcal{M}_\alpha)$ satisfies (2), then the "$\mathcal{M}$-part" of Z must be in $\mathcal{M}_\alpha$, i.e. $Z\in\mathcal{M}_\alpha\otimes\mathcal{L}(\mathcal{H}_\alpha,\mathcal{M}_\alpha)$, so we can write $Z=\sum e^{(l)}_r\otimes E_{lr}$ for some maps $E_{lr}\in\mathcal{L}(\mathcal{H}_\alpha,\mathcal{M}_\alpha)$. If (2) holds we get

$$\sum e^{(l)}_j\otimes E_{lr}\otimes u^\alpha_{jr}=\sum e^{(l)}_j\otimes E_{lj}A^\alpha_{sr}\otimes u^\alpha_{sr},$$

thus $E_{lj}A^\alpha_{sr}=\delta_{js}E_{lr}$. Taking $D=\sum\lambda_j^{\frac{1}{2}}E_{j1}C^{\alpha *}_{j1}\in\mathcal{L}(\mathcal{M}_\alpha)$ we get

$$\begin{aligned}(1\otimes D)Y(\alpha)=&\ \sum\lambda_k^{-\frac{1}{2}}e^{(k)}_i\otimes\lambda_j^{\frac{1}{2}}E_{j1}C^{\alpha *}_{j1}C_{ki}\\ =&\ \sum e^{(k)}_i\otimes E_{k1}A_{1i}=\sum e^{(k)}_i\otimes E_{ki}=Z.\end{aligned}$$

□

An element Z satisfying (2) is called an *α-eigenoperator* for the action. So Lemma A1 tells us that $Y(\alpha)$ generates all α-eigenoperators by the formula (3). We shall also need the universal eigenoperator $Y = \sum^{\oplus} Y(\alpha)$, this is a map from $\mathcal{H}_\omega \otimes \mathcal{H}_h$ to $\mathcal{H}_\omega \otimes \mathcal{H}_\omega$ satisfying

$$\sigma \otimes i(Y) = Y_{12}V_{23} \qquad YY^* = 1_{\mathcal{M}} \otimes 1_{\mathcal{M}}.$$

It then follows that $\sigma'(Y^*aY) = Y_{12}^*\sigma \otimes i(a)Y_{12}$ for all a.

Lemma A2. *Let Θ be as in Proposition 18 and let Θ_α be its restriction to $\mathcal{M}_\alpha$. With Λ_α the matrix given by $(\Lambda_\alpha)_{kl}1_{\mathcal{M}} = \frac{1}{M_\alpha}\sum_j e_j^{(k)}e_j^{(l)*}$ we have*

$$\begin{aligned}
&(1) \quad Tr(\Theta_{\alpha^c}) = Tr(\Theta_\alpha^{-1}) \\
&(2) \quad Tr(\Theta_\alpha) = Tr(\Lambda_\alpha)M_\alpha = \textstyle\sum \lambda_k \\
&(3) \quad Tr(\Theta_\alpha^{-1}) = Tr(\Lambda_\alpha^{-1})M_\alpha = M_\alpha^2 \textstyle\sum \lambda_k^{-1} \\
&(4) \quad Tr(\Lambda_{\alpha^c}) = M_\alpha \textstyle\sum \lambda_k^{-1}
\end{aligned}$$

Proof: (1) follows from the fact that $\Theta(x) = \lambda x \Longrightarrow \Theta(x^*) = \lambda^{-1}x^*$. In Proposition 18 it is proved that $\Theta_\alpha \cong \Lambda_\alpha \otimes F_\alpha$, hence (2) and (3). Combining these three properties with the fact that $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$, then $M_{\alpha^c} = M_\alpha$ and (4) follows.

□

We are now ready to prove the main result:

Theorem A. *With the same assumptions as in Theorem 19 and if $\mathcal{M}_\alpha \neq 0$ for all α, then the following conditions are equivalent:*

$$\begin{aligned}
&(1) \quad \mathcal{N} = \mathcal{M} \times_\sigma \widehat{\mathcal{A}} \quad \text{is a simple } C^*\text{-algebra} \\
&(2) \quad \mathcal{N} \cong \mathcal{K}(\mathcal{H}_\omega) \\
&(3) \quad Y(\alpha)^*Y(\alpha) = 1_{\mathcal{M}} \otimes 1_{\mathcal{L}(\mathcal{H}_\alpha)} \text{ for all } \alpha \in \widehat{G} \\
&(4) \quad Tr(\Lambda_\alpha) = Tr(F_\alpha) \text{ for all } \alpha \in \widehat{G}.
\end{aligned}$$

Proof: (3) $\Longrightarrow$ (2): In this case Y is a unitary eigenoperator between $\mathcal{H}_\omega \otimes \mathcal{H}_h$ and $\mathcal{H}_\omega \otimes \mathcal{H}_\omega$, so

$$\begin{aligned}
\mathcal{N} &\cong (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_h))^{\sigma'} = [Y^*(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega)Y]^{\sigma'} = Y_{12}^*(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega))^{\sigma \otimes i}Y_{12} \\
&= Y_{12}^*\mathcal{M}^\sigma \otimes \mathcal{K}(\mathcal{H}_\omega)Y_{12} = Y_{12}^*1 \otimes \mathcal{K}(\mathcal{H}_\omega)Y_{12} \cong \mathcal{K}(\mathcal{H}_\omega).
\end{aligned}$$

Note that from Lemma 4 there is a conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^\sigma$, so we have that $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega))^{\sigma \otimes i} = \mathcal{M}^\sigma \otimes \mathcal{K}(\mathcal{H}_\omega)$.
(2) $\Longrightarrow$ (1) is obvious.
(1) $\Longrightarrow$ (3): If $\mathcal{N}$ is simple, so is $1 \otimes p(\alpha)_*\mathcal{N}1 \otimes p(\alpha)_* = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$. Now $\mathcal{J} = Y(\alpha)^*1 \otimes \mathcal{K}(\mathcal{M}_\alpha)Y(\alpha)$ is a 2-sided ideal in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$:

If $A \in \mathcal{K}(\mathcal{H}_\alpha)$, $B \in (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$ then $Y(\alpha)B$ satisfies (2) in Lemma A1, so $Y(\alpha)B = 1 \otimes CY(\alpha)$ for some $C \in \mathcal{K}(\mathcal{M}_\alpha)$. Therefore $Y(\alpha)^*1 \otimes AY(\alpha)B = Y(\alpha)^*1 \otimes ACY(\alpha) \in \mathcal{J}$, and since $\mathcal{J}^* = \mathcal{J}$, $\mathcal{J}$ is a 2-sided ideal.

If $\mathcal{J} = \{0\}$ then $\mathcal{M}_\alpha = \{0\}$, so simplicity gives us that $\mathcal{J} = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$. Thus $1 \in \mathcal{J}$, hence $Y(\alpha)^*Y(\alpha) = 1$.
(3) $\Longleftrightarrow$ (4): Since $Y(\alpha)^*Y(\alpha)$ always is a projection and ω is faithful, we have for all α:

$$\begin{aligned}(3) \quad &\iff \omega \otimes i(Y(\alpha)^* Y(\alpha)) = 1_{\mathcal{L}(\mathcal{H}_\alpha)} \iff \sum \lambda_k^{-1} \omega\Big(e_i^{(k)^*} e_j^{(k)}\Big) A_{ij}^\alpha = 1_{\mathcal{L}(\mathcal{H}_\alpha)} \\ &\iff \sum \lambda_k^{-1} = 1 \iff \mathrm{Tr}(\Lambda_{\alpha^c}) = M_\alpha \iff \mathrm{Tr}(\Lambda_{\alpha^c}) = \mathrm{Tr}(F_{\alpha^c}).\end{aligned}$$

□

Remark. $\mathrm{Tr}(F_\alpha) = M_\alpha$ is called the quantum dimension of α. ¿From Theorem A it then seems reasonable to call $\mathrm{Tr}(\Lambda_\alpha)$ the quantum multiplicity of α in σ. Since $Y(\alpha)^* Y(\alpha)$ always is a projection we have from the proof of (3) $\iff$ (4) the following:

Corollary. *With $(\mathcal{M}, \sigma, G)$ as in Theorem 19 one has always $\mathrm{Tr}(\Lambda_\alpha) \leq \mathrm{Tr}(F_\alpha)$ with equality $\iff$ (1)–(4) in Theorem A hold.*

Addtional reference:

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HOMOTOPY OF PROJECTIONS IN C^*-ALGEBRAS OF STABLE RANK ONE

LAWRENCE G. BROWN[1]

1. DISCUSSION AND INTRODUCTION

My talk at the Orleans conference concerned extremally rich C^*-algebras, based on joint work with G. K. Pedersen. A unital C^*-algebra A is called *extremally rich* if its closed unit ball, A_1, is the convex hull of its extreme points, and a non-unital C^*-algebra A is *extremally rich* if and only if $\widetilde{A}$, the result of adjoining an identity, is extremally rich. For other equivalent definitions and most of the properties the reader is referred to our forthcoming papers [4] and [5]. The theory of extremal richness is not involved in the first theorem of this paper, but some of it will be discussed in this section, mainly for heuristic purposes and partly as a research announcement.

A theorem of Rørdam [15] implies that every C^*-algebra of stable rank one is extremally rich. In fact $tsr(A) = 1$ if and only if A is extremally rich and every extreme point of $\widetilde{A}_1$ is unitary. (The second condition is a kind of finiteness condition – see further discussion in §3. Of course every unitary is extremal.) There is an intermediate concept, isometric richness, and A is isometrically rich if and only if it is extremally rich and every extreme point of $\widetilde{A}_1$ is an isometry or co-isometry. Thus for prime C^*-algebras, in particular for primitive or simple C^*-algebras, isometric richness is equivalent to extremal richness. Rørdam [16, 4.5] and Pedersen [12, 10.1] state (without the terminology) that every purely infinite simple C^*-algebra is extremally rich. In fact a simple C^*-algebra is extremally rich if and only if it is either purely infinite or of stable rank one.

A guiding philosophy is that if a property of C^*-algebras A is known both when A is purely infinite simple and when $tsr(A) = 1$, then one can hope to prove it for A extremally rich. This hope is substantially realized with respect to the invariance properties of extremal richness (strong Morita equivalence, passage to hereditary subalgebras, behavior under extensions) and with respect to certain non-stable K-theoretic properties, but this hope is very possibly wrong with respect to other non-stable K-theoretic properties. Moreover, the facts stated in the abstract show that it is wrong with respect to the question of Zhang stated there, though there is a way to re-formulate the question (see §3). Actually the issue of how close is the relationship between extremal richness and non-stable K-theory is very much up in the air. Conceivably all of the results already proved could be special cases of more

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general results that do not mention extremal richness, but this seems unlikely to me at present.

A (possibly non-unital) C^*-algebra A is said to have *weak cancellation* if whenever p and q are projections in $A \otimes \mathcal{K}$ such that each of p, q generates the same (closed, two-sided) ideal J and p and q have the same image in $K_0(J)$, then $p \sim q$. Here $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space and $\sim$ denotes Murray–von Neumann equivalence. Of course, C^*-algebras of stable rank one satisfy the strong cancellation property that if p and q have the same image in $K_0(A)$, then $p \sim q$ (Blackadar [1, 6.5.1]). (I am here not stating the cancellation property in quite the usual way, but the discrepancy is only technical and not important when $tsr(A) = 1$.) A related fact is that $tsr(A) = 1$ implies that the natural map from $K_0(J)$ to $K_0(A)$ is injective. This is a known result (folklore) for which I do not know a reference, and the corresponding statement for extremally rich C^*-algebras is false. Cuntz [7] showed that purely infinite simple C^*-algebras have weak cancellation.

A unital C^*-algebra A is said to have *K_1-surjectivity* if the natural map from $U(A)/U_0(A)$ to $K_1(A)$ is surjective, *K_1-injectivity* if this map is injective, and *K_1-bijectivity* if it is bijective. Here $U(A)$ is the unitary group of A and $U_0(A)$ the connected component of the identity. Cuntz [7] showed that purely infinite simple C^*-algebras have K_1-bijectivity (the same is true of $\widetilde{A}$ if A is non-unital), and Rieffel [14] showed that $tsr(A) = 1$ implies K_1-bijectivity.

It will be shown in [5] that every isometrically rich C^*-algebra has weak cancellation and also every extremally rich C^*-algebra of real rank zero. There are other positive theorems, but it is not known whether every extremally rich C^*-algebra has weak cancellation. Every extremally rich C^*-algebra with weak cancellation also has K_1-surjectivity and certain other properties. It was shown by Lin [10] that every C^*-algebra of real rank zero has K_1-injectivity. There is presently no real evidence of a relationship between K_1-injectivity and extremal richness.

Zhang [21] formulated the question of the abstract as follows. For a projection p in A let $\mathcal{G}_p = \{upu^{-1} : u \in U(\widetilde{A})\}$. The question is whether $\mathcal{G}_p$ is connected. Note that a projection q is in $\mathcal{G}_p$ if and only if $q \sim p$ and $1 - q \sim 1 - p$. Thus if $tsr(A) = 1$, q is in $\mathcal{G}_p$ if and only if $q \sim p$. Also the results of Cuntz [7] show that if A is purely infinite simple, then q is in $\mathcal{G}_p$ if and only if $q \sim p$ and either $p = q = 1$ or $p \neq 1$ and $q \neq 1$. It is well known that if p and q are homotopic projections in A (i.e., p and q are in the same connected component of the set of projections of A), then $q = upu^{-1}$ for some u in $U_0(A)$ or $U_0(\widetilde{A})$. All of this must be kept in mind since the results in the literature prior to [21] are not formulated in the same way.

2. First Theorem and example

The proof of the theorem has much in common with the proof of the quoted result of Effros and Kaminker [8, 2.4]. A different proof, perhaps more suggestive, is sketched in §3.

Theorem 1. *If p is a projection in a C^*-algebra A and if $tsr(A) = 1$, then $\mathcal{G}_p$ is connected. In other words, unitarily equivalent projections in A are homotopic.*

Proof. Without loss of generality we may assume A is unital. Let I be the ideal of A generated by p, and let $B = I + \mathbb{C} \cdot 1$. Then $tsr(B) = 1$ by results of Rieffel [13]. Let q be an element of $\mathcal{G}_p$. Since $q \sim p$ in A, q is in I and $q \sim p$ in I. Thus q is unitarily equivalent to p in B (as indicated above, because of the cancellation property for B). Let u be in $U(B)$ such that $q = upu^{-1}$, and let α be the image of u in $K_1(B)$. Because pIp is a full hereditary C^*-subalgebra of I the natural map from $K_1(pIp)$ to $K_1(I)$ is a bijection (see the following remark). Also $K_1(B) = K_1(\widetilde{I}) = K_1(I)$. Thus α is the image of an element β of $K_1(pIp)$. Now $tsr(pIp) = 1$ (see the following remark) and hence β is the image of a unitary v_1 in $U(pIp)$ by a result of Rieffel [14] (K_1-surjectivity). Let $v = v_1 + 1 - p$ and $w = uv^{-1}$. Then w is in $U(B)$, $q = wpw^{-1}$, and the image of w in $K_1(B)$ is 0. Thus the K_1-injectivity result of [14] implies w is in $U_0(B)$. This clearly implies that q is homotopic to p.

Remark. I have not been able to find a reference for the fact that inclusions of full hereditary C^*-algebras induce isomorphisms of K-groups in full generality. Paschke [11, 1.2] proves this when both algebras are σ-unital. The general case can be deduced from this by using direct limits of separable subalgebras.

I also do not know a reference for the fact that $tsr(A) = 1$ implies $tsr(B) = 1$ when B is a hereditary C^*-subalgebra of A, though surely this fact is known. If I is the ideal of A generated by B, then $tsr(I) = 1$ by Rieffel [13] and I is strongly Morita equivalent to B. Rieffel also proves in [13] that the property $tsr = 1$ is preserved by stable isomorphism, which is the same as strong Morita equivalence when both algebras are σ-unital. A proof that the property $tsr = 1$ is preserved by strong Morita equivalence in general will be found in [4], and also a direct (and short) proof that $tsr = 1$ passes to hereditary subalgebras can be obtained from [4].

EXAMPLE. Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis for a Hilbert space H. Let U in $B(H)$ be the bilateral shift defined by $Ue_n = e_{n+1}$, let P in $B(H)$ be the projection on $\overline{\{e_n : n \geq 0\}}$, and let P_0 be the projection on $\overline{\{e_n : n > 0\}}$. Let $A = \{T \in B(H) : PT(1-P), (1-P)TP \in \mathcal{K}\}$. Then A is a C^*-algebra, $U, P, P_0 \in A$, $UPU^{-1} = P_0$, and P is not homotopic to P_0 in A.

Only the last assertion needs proof. Note that $A \supset \mathcal{K}$ and $A/\mathcal{K} \cong \mathcal{Q} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is the Calkin algebra. Thus every element V of $U_0(A)$ has an image $v_1 \oplus v_2$ in $\mathcal{Q} \oplus \mathcal{Q}$ and $v_1, v_2 \in U_0(\mathcal{Q})$. It follows that PVP and $(1-P)V(1-P)$ have index 0 as operators on PH and $(1-P)H$. In particular, $U \notin U_0(A)$. But if P and P_0 were homotopic, there would be V in $U_0(A)$ such that $VPV^{-1} = P_0$, and thus $V = UW$ with $WP = PW$. Since the last equation implies $W \in U_0(A)$, we have a contradiction.

Now A has real rank zero but is not extremally rich. That $RR(A) = 0$ follows from the facts that $RR(\mathcal{K}) = RR(A/\mathcal{K}) = 0$ and projections lift. That A is not extremally rich follows from [4] and the fact that there is an extremal element u in $A/\mathcal{K}$ which is not the image of an extremal element of A. Namely, $u = v \oplus v^*$ where v is a proper isometry in $\mathcal{Q}$. To get the promised example, we replace A with a C^*-subalgebra B such that U and P, and hence P_0, are in B. Then P and P_0 are still unitarily equivalent projections in B which are not homotopic.

Let D in $B(H)$ be defined by $De_n = \lambda^n e_n$, where λ is a complex number of absolute value 1 which is not a root of unity. Note that D is unitary and $DP = PD$. Let

$$B_1 = C^*(PD, PUP) \text{ and } B_2 = C^*((1-P)D, (1-P)U(1-P)) .$$

Then $B_1 \supset \mathcal{K}(PH)$ and $B_2 \supset \mathcal{K}((1-P)H)$, since PUP and $(1-P)U(1-P)$ are respectively a unilateral shift and a backward shift. Since $DU = \lambda UD$, it is easy to see that $B_1/\mathcal{K}(PH) \cong C^*(u_1, d_1)$ and $B_2/\mathcal{K}(1-P)H) \cong C^*(u_2, d_2)$, where u_i and d_i are unitary and $d_i u_i = \lambda u_i d_i$. Then $C^*(u_1, d_1)$ and $C^*(u_2, d_2)$ are isomorphic to the same irrational rotation algebra C. That $tsr(C) = 1$ and $RR(C) = 0$ follows *a fortiori* either from Blackadar, Kumjian, and Rørdam [2] or from Elliott and Evans [9]. (This subject has a distinguished history, and various combinations of earlier papers could have been used for constructing an example.) Now let $B = \{T \in A : PTP \in B_1 \text{ and } (1-P)T(1-P) \in B_2\}$. Thus $\mathcal{K} \subset B$ and $B/\mathcal{K} \cong C \oplus C$. Since $RR(\mathcal{K}) = RR(B/\mathcal{K}) = 0$, to show that $RR(B) = 0$ we need only show that every projection in $B/\mathcal{K}$ is the image of a projection in B (see [3, 3.14], [18, 2.4], or [20, 2.3]). The fact that projections always lift when the ideal is $\mathcal{K}$ follows from a well known result of Calkin [6]. It will be shown in [4] that whenever a C^*-algebra B has an ideal which is a dual C^*-algebra such that the quotient algebra is isometrically rich, then B is extremally rich. This applies to the algebra called B here. Note also that B is primitive, so that B is even isometrically rich.

3. More discussion and second theorem

Let p be a fixed projection in a unital C^*-algebra A. Proposition 3.2 of Zhang [21] implies $\mathcal{G}_p$ is connected if and only if the natural map from

$$U(pAp)/U_0(pAp) \oplus U((1-p)A(1-p))/U_0((1-p)A(1-p))$$

to $U(A)/U_0(A)$ is surjective.

Now let I and J be the ideals of A generated by p and q, respectively, and let $s(I, J)$ be the natural map from $K_1(I) \oplus K_1(J)$ to $K_1(A)$. Then it is clear from the previous paragraph that a necessary condition for connectedness of $\mathcal{G}_p$ is that certain elements of $K_1(A)$, namely those in the image of $U(A)/U_0(A)$, be in the range of $s(I, J)$. If A has K_1-surjectivity, it is necessary that $s(I, J)$ be surjective. Note that $s(I, J)$ is surjective if and only if the natural map from $K_0(I \cap J)$ to $K_0(I) \oplus K_0(J)$ is injective and that this is so if $tsr(A) = 1$ (or merely if either $tsr(I) = 1$ or $tsr(J) = 1$). If $s(I, J)$ is surjective and if both pAp and $(1-p)A(1-p)$ have K_1-surjectivity, then for any u in $U(A)$ there is a v such that $vp = pv$ and uv^{-1} is in the kernel of the map from $U(A)/U_0(A)$ to $K_1(A)$. (Here we use the facts that $K_1(pAp) \cong K_1(I)$, $K_1((1-p)A(1-p)) \cong K_1(J)$, cf. proof of Theorem 1.) Then if also A has K_1-injectivity, we see that $\mathcal{G}_p$ is connected.

Thus if we make enough hypotheses about K_1-surjectivity and –injectivity, then $\mathcal{G}_p$ is connected if and only if $s(I, J)$ is surjective. Of course, the example in the previous section was deliberately constructed so that $s(I, J)$ would not be surjective. Now the surjectivity of $s(I, J)$ strikes me as an interesting condition, but it may be desirable to eliminate it from consideration in order to concentrate on the non-stable K-theory. This can be done by making some additional hypothesis. I note that the cited results of Effros-Kaminker and Zhang explicitly assumed that $p \neq 0$ and $p \neq 1$ and that if A is simple this is equivalent to $I = J = A$. Thus the following theorem seems a reasonable offering.

Theorem 2. *If A is a (not necessarily unital) extremally rich C^*-algebra of real rank zero and if p is a projection in A such that either p generates A as an ideal or $1-p$ generates $\widetilde{A}$ as an ideal, then $\mathcal{G}_p$ is connected.*

Proof. The results about extremally rich C^*-algebras announced in §1 and the K_1-injectivity result of Lin [10] imply all the K_1-surjectivity and –injectivity needed. Since $\mathcal{G}_p$ is homeomorphic to $\mathcal{G}_{1-p}$, both the proof given for Theorem 1 and the argument sketched above can be adapted to the present situation.

Theorem 2 is certainly not a best possibly result, as is clear from the above, but it seems unpleasant to state a theorem with a lot of *ad hoc* hypotheses even if that theorem would be stronger. I think attempts to formulate the optimal theorem should wait until more is known. Some speculative discussion follows.

Although none of the results in this paper suggests it, it seems possible that there is a theorem on connectedness of $\mathcal{G}_p$ in which the main hypothesis is $RR(A) = 0$. It could be that $RR(A) = 0$ and surjectivity of $s(I, J)$ suffices, or perhaps $RR(A) = 0$ and some finiteness hypothesis. I am not making any conjectures, just suggesting questions.

With regard to finiteness, the theory of extremal richness in [4] and [5] emphasizes the fact that there is more than one useful finiteness condition for non-simple C^*-algebras. There are at least four concepts, which I define only in the unital case for simplicity:

(F1) A has a faithful family of finite traces.
(F2) A is *finite* if it contains no proper isometries.
(F3) A is *residually finite* if every quotient algebra is finite.
(F4) Every extreme point of A_1 is unitary.

Then each of (F1) and (F3) implies (F4), which in turn implies (F2); and neither of (F1) and (F3) implies the other.

Question 4.16 of Rieffel [13] asks whether $tsr(M(A)) = tsr(A)$ if $M(A)$ is finite in a suitable sense (open-ended). For a straightforward example let A be a non-unital finite matroid C^*-algebra. Then $tsr(A) = 1$; and $tsr(M(A)) = \infty$, since $M(A)/A$ is purely infinite simple. Also $M(A)$ is not extremally rich, though it is of real rank zero. $M(A)$ has a faithful finite trace but obviously is not residually finite.

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Operator algebras, free groups and other groups

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OPERATOR ALGEBRAS, FREE GROUPS AND OTHER GROUPS

PIERRE DE LA HARPE

1. *INTRODUCTION.*

Let Γ be a group. We denote by $\mathbb{C}[\Gamma]$ the group algebra of complex linear combinations of elements of Γ, given together with the involution

$$X = \sum_{\gamma \in \Gamma} z_\gamma \gamma \mapsto X^* = \sum_{\gamma \in \Gamma} \overline{z_\gamma} \gamma^{-1} .$$

The **operator algebras** of interest here are various completions of $\mathbb{C}[\Gamma]$. Non abelian **free groups** are among the most studied examples of groups in this context. We denote by F_n the non abelian free group on n generators, where n is either an integer, $n \geq 2$, or $n = \infty$, meaning an infinite countable number of generators.

Our guiding principle is that the special case of free groups indicates typical behaviours which hold in many other cases of geometrical interest. This has suggested the three main aspects of the report below :

a survey of some properties of operator algebras associated to the F_n 's,

an exploration of "geometric" groups giving rise to algebras with similar properties,

a list of open problems (some of them are numbered, from 1 to 19, and others appear in the text).

We shall concentrate on groups Γ which are lattices in semi-simple Lie groups ([Rag], [Mas]) or hyperbolic [Gr1], and on algebras which are either von Neumann algebras or C^*-algebras. But we shall mention on occasions other groups and other algebras. Unless explicitely stated otherwise, Γ denotes a **countable** group and operator algebras are **separable** in the appropriate sense.

Many important developments are left untouched. In particular, we say very little on K-theory and KK-theory related to group C^*-algebras, and nothing at all on the Novikov conjecture.

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2. THE VON NEUMANN ALGEBRA $W^*_\lambda(\Gamma)$.

2.1. *Generalities.*

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the involutive algebra of bounded operators on $\mathcal{H}$ and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. Any **unitary representation** $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ of a group Γ gives rise to a morphism of involutive algebras $\mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ which is again denoted by π, and defined by

$$\pi\left(\sum_{\gamma\in\Gamma} z_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} z_\gamma \pi(\gamma) \ .$$

We denote by $W^*_\pi(\Gamma)$ the weak closure of $\pi(\mathbb{C}[\Gamma])$ in $\mathcal{L}(\mathcal{H})$.

Consider in particular the space $l^2(\Gamma)$ of square summable complex valued functions on Γ and the left regular representation

$$\lambda : \Gamma \to \mathcal{U}\left(l^2(\Gamma)\right)$$

where $(\lambda(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $\gamma, x \in \Gamma$ and for all $\xi \in l^2(\Gamma)$. The weak closure $W^*_\lambda(\Gamma)$ of $\lambda(\mathbb{C}[\Gamma])$ is the **von Neumann algebra** of Γ.

There is a **finite normal trace** $\tau : W^*_\lambda(\Gamma) \to \mathbb{C}$ which extends the map $\mathbb{C}[\Gamma] \to \mathbb{C}$ given by $\sum_{\gamma\in\Gamma} z_\gamma\gamma \mapsto z_1$, and this trace is faithful. Thus the von Neumann algebra $W^*_\lambda(\Gamma)$ is finite, of the form $W_I \oplus W_{II} = (\oplus_{i=1}^\infty W_i) \oplus W_{II}$ with each W_i of type I_i, say with unit e_i, and with W_{II} of type II_1, say with unit e.

One has $e = 0$ if and only if Γ contains an abelian group of finite index. Let Γ_f denote the subgroup of Γ of elements with finite conjugacy classes and let $D\Gamma_f$ denote its commutator subgroup; then one has $e = 1$ if and only if either $[\Gamma : \Gamma_f] = \infty$ or $[\Gamma : \Gamma_f] < \infty$ and $|D\Gamma_f| = \infty$. See [Kan], [Sm1] and [Tho]. In case Γ is finitely generated, one has either $e = 0$ or $e = 1$. (This appears in [Ka2], but it is also a straightforward consequence of [Kan]. Indeed $e \neq 1$ implies $[\Gamma : \Gamma_f] < \infty$ by [Kan, Satz 1]; as Γ_f is also finitely generated in this case, the centre of Γ_f is of finite index in Γ_f [Tom, Corollary 1.5], and thus also in Γ; consequently $e = 0$.) But there are already in [Kap] examples, due to B.H. Neumann, which show that one may have $0 \neq e \neq 1$. Here is one of these examples : for each $i \in \mathbb{N}$, denote by D_i a copy of the dihedral group of order 8 and by C_i its center, which is of order 2 and which is also its derived group; let B be the direct sum of the B_i 's and let C be the subgroup of elements $(c_i)_{i\in\mathbb{N}} \in B$ such that $c_i \in C_i$ for each $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} c_i = 1$; this B.H. Neumann example is the quotient A/B; its von Neumann algebra is the direct product of $\mathbb{C}$ (with $\tau(e_1) = 1/2$) and of a factor of type II_1 (with $\tau(e) = 1/2$).

For a group Γ, Kaplansky has observed that $\tau(e_1)$ is the inverse of the order of the derived group of Γ [Kap, Theorem 1]. There are formulas giving $1 - e$ [Fo2]. The sum $1 - e = \sum e_i$ is finite, and indeed $e_i = 0$ whenever $i^2 > |\Gamma/\Gamma_f|$ [Sm2]. When $e \neq 1$, one has $W_I \approx W^*_\lambda(\Gamma/\Gamma_0)$, where the von Neumann kernel Γ_0 of Γ is defined as $\bigcap_\pi Ker(\pi : \Gamma \to U(n))$, the intersection being over all finite dimensional representations of Γ [Sch, Satz 1].

Let Γ be a group such that $0 \neq e \neq 1$; I do not know whether there exists a group Γ_{II} naturally associated to Γ and such that $W_{II} \approx W^*_\lambda(\Gamma_{II})$. Here is a similar question : let Γ be a group such that $W^*_\lambda(\Gamma)$ is *not* a II_1-factor but contains a central projection c such that $cW^*_\lambda(\Gamma)$ is a II_1-factor; does there exist a group Γ_c naturally associated to Γ and such that $cW^*_\lambda(\Gamma) \approx W^*_\lambda(\Gamma_c)$?

Observe that, in case Γ is a hyperbolic group, Γ_f is precisely the so-called *virtual center* of Γ, denoted by $Z_{virt}(\Gamma)$ in [Cha].

2.2. *Free groups.*

Historically, the first examples of factors of type II_1 are given by Murray and von Neumann in [MNI], as crossed products which involve abelian groups (indeed subgroups of $\mathbb{R}$) acting ergodically on appropriate spaces. Several years later, they give a new construction which is "considerably simpler than our previous procedures, but it is clearly related to them" [MNIV, Introduction, §5]. Among other things, they show the following results. Recall that a group Γ has **infinite conjugacy classes,** or in short is icc, if *all* its conjugacy classes distinct from $\{1\}$ are infinite; for example, F_n is icc for all $n \geq 2$.

Theorem 1 (Murray and von Neumann).
(i) Let Γ be a group. Then $W^*_\lambda(\Gamma)$ is a factor if and only if Γ is icc.
(ii) For each $n \geq 2$ the factor $W^*_\lambda(F_n)$ does not possess Property Gamma.

This is shown in [MNIV] : see Lemma 5.3.4 for (i), Definition 6.1.1 for Property Gamma and §6.2 for (ii) when $n = 2$; moreover Lemma 6.3.1 shows that $W^*_\lambda(\Gamma_1 \star \Gamma_2)$ is a factor which does not possess Property Gamma whenever Γ_1 [respectively Γ_2] is a group containing at least two [resp. three] elements (the star denotes a free product). About the meaning of (ii), let us recall that a von Neumann algebra M does *not* have Property Gamma if and only if it is **full**, namely if and only if the group $Int(M)$ of its inner automorphisms is closed in the group $Aut(M)$ of all its automorphisms (see [Co74, Corollary 3.8] and [Co76, Theorem 2.1]).

Though we do *not* consider twisted crossed products in this report, let us at least mention that many of the results discussed here have "twisted formulations". For example, for claim (i) of Theorem 1 above, see [Pac, Proposition 1.3].

Claim (ii) suggests immediately the following, which is Problem 4.4.44 in [Sak].

Problem 1. *Does it happen that $W^*_\lambda(F_n) \approx W^*_\lambda(F_{n'})$ for $n \not\approx n'$?*

Though Problem 1 is still open, progress has been obtained recently, using Voiculescu's theory of freeness in noncommutative probability spaces (see among others [Vo2], [VDN] and [Sk2]). For example, one must have

either $W^*_\lambda(F_n) \approx W^*_\lambda(F_{n'})$ for all n, n' such that $2 \leq n, n' \leq \infty$
or $W^*_\lambda(F_n) \not\approx W^*_\lambda(F_{n'})$ for all n, n' such that $2 \leq n < n' \leq \infty$.

This has been first proved for $n, n' < \infty$, independently by K. Dykema and F. Radulescu; moreover, this holds for $n, n' \leq \infty$ by [Ra5, Corollary 4.7]. Let us also mention that

$$W_\lambda^*(\star_{n=1}^\infty \Gamma_n) \approx W_\lambda^*(F_\infty)$$

whenever Γ_n is a nontrivial amenable group for all $n \geq 1$ (see [Vo2, Corollary 3.5] and [Dy2, Corollary 5.4]), and that

$$W_\lambda^*(\Gamma \star \Gamma') \approx W_\lambda^*(F_2)$$

when Γ , Γ' are infinite amenable groups [Dy2, particular case of Corollary 5.3]. See also [HaVo].

One of the novelties connected with the results above is the discovery, due independently to K. Dykema [Dy1] and F. Radulescu [Ra4], of a continuous family of II_1-factors $L(F_r)$ interpolating the free group factors. In the next theorem, we denote by $M_1 \star M_2$ the II_1-factor which is the reduced free product of two finite factors M_1, M_2; this is a crucial notion in Voiculescu's approach [Vo2].

Theorem 2 (Dykema, Radulescu, Voiculescu). *For each extended real number r such that $1 < r \leq \infty$, there exists a II_1-factor $L(F_r)$ such that*

$L(F_r) \star L(F_{r'}) \approx L(F_{r+r'})$ *for all* $r, r' \in]1, \infty]$,
$p(L(F_r) \otimes M_n(\mathbb{C})) \approx L(F_{1+\gamma^{-2}(r-1)})$ *for any* $r \in]1,\infty]$ *and any projection* $p \in L(F_r) \otimes M_n(\mathbb{C})$ *of trace* $\gamma \in]0,\infty[$ *(where n is large enough),*
$L(F_n) \approx W_\lambda^*(F_n)$ *for all* $n \in \{2,3,...,\infty\}$,
$L(F_r) \otimes M \approx L(F_{r'}) \otimes M$ *for all* $r, r' \in]1,\infty[$ *whenever M is either* $\mathcal{L}(\mathcal{H})$, *or* R, *or* $W_\lambda^*(F_\infty)$,
the isomorphism class of $L(F_r) \otimes L(F_{r'})$ *depends only on* $(r-1)(r'-1)$, *for all* $r, r' \in]1,\infty]$.

In the theorem, $\mathcal{L}(\mathcal{H})$ denotes the factor of type I_∞ and R denotes the hyperfinite factor of type II_1; moreover p is of trace γ for the trace of value 1 on the unity of $L(F_r)$. The first result quoted after Problem 1 is in fact

either $L(F_r) \approx L(F_{r'})$ for all r, r' such that $1 < r, r' \leq \infty$

or $L(F_r) \not\approx L(F_{r'})$ for all r, r' such that $1 < r < r' \leq \infty$.

Let us mention that some attention has been paid to free groups on uncountably many generators : if $F_{!!}$ denotes such a free group, then $W_\lambda^*(F_{!!})$ hasn't any "regular MASA"; also (we anticipate here on Section 3) the reduced C^*-algebra $C_\lambda^*(F_{!!})$, which clearly is not separable, has only separable abelian *-subalgebras [Po1, Section 6].

2.3. *Other groups.*

Considerable effort has been devoted to understand whether various factors of the form $W_\lambda^*(\Gamma)$ are or are not isomorphic to each other. The oldest result of this kind follows from Claim (ii) of Theorem 1 above on one hand and from the consideration of locally finite groups which are icc on the other hand; this result, which is the existence of two non isomorphic factors of type II_1, is recorded as the achievement of Chapters V and VI in [MNIV, Theorem XVI]. Later, the same construction $\Gamma \mapsto W_\lambda^*(\Gamma)$ has been used by D. McDuff to show that there are uncountably many pairwise non isomorphic type II_1 factors (see [McD] or [Sak, 4.3.10]).

We shall review now how some properties of an icc group are reflected in properties of the corresponding factor.

2.4. *Injectivity.*

For an icc group Γ, A. Connes has shown that $W^*_\lambda(\Gamma)$ is the *unique* injective factor of type II_1 if and only if Γ is amenable [Co76, in particular Corollary 7.2].

There is a very large number of pairwise non isomorphic icc amenable countable groups. Let us mention icc locally finite groups, as in [MNIV, Lemma 5.6.1], and icc solvable groups, such as the group

$$\Gamma_{\mathbb{K}} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{K}) \ : \ a \in \mathbb{K}^* \ , \ b \in \mathbb{K} \right\}$$

where $\mathbb{K}$ is a countable infinite field (the so-called $ax+b$ group associated to $\mathbb{K}$). It is known that there exist uncountably many pairwise nonisomorphic groups which are locally finite and icc (indeed simple) [KeW, Corollary 6.12]. It is also easy to check that two groups $\Gamma_{\mathbb{K}}$ and $\Gamma_{\mathbb{K}'}$ as above are isomorphic if and only if the fields $\mathbb{K}$ and $\mathbb{K}'$ are isomorphic, and there are uncountably many pairwise nonisomorphic countable fields (examples : the fields $\mathbb{K}_S = \mathbb{Q}\left((\sqrt{p})_{p\in S}\right)$ where S is a set of prime numbers; $\mathbb{K}_S \approx \mathbb{K}_{S'}$ if and only if $S = S'$, as it follows from Kummer's theory [Bou, V, p. 85, Théorème 4]; I am grateful to M. Ojanguren for explanations on this). There are many other ways to construct uncountable families of icc amenable groups; the way suggested in [Wat] provides groups with pairwise nonisomorphic C^*-algebras.

Yet all these groups provide the same factor.

2.5. *Fullness.*

The proof in [MNIV] of Theorem 1.ii above uses arguments which go much beyond free products. Indeed, one has the following, for which we refer to Effros [Ef1] and to [BdH]. As the terminology is unfortunately *not* uniform (compare [Pat, page 84]), let us recall that, *here*, a group is **inner amenable** if there exists a finitely additive measure $\mu : \mathcal{P}(\Gamma - \{1\}) \to [0,1]$ defined on all subsets of $\Gamma - \{1\}$, which is normalized by $\mu(\Gamma - \{1\}) = 1$ and which satisfies $\mu(\gamma D \gamma^{-1}) = \mu(D)$ for all $\gamma \in \Gamma$ and $D \subset \Gamma - \{1\}$. A group which has a finite conjugacy class distinct from $\{1\}$ is inner amenable.

Proposition 1 (Effros). *If Γ is a group which is not inner amenable, the algebra $W^*_\lambda(\Gamma)$ is a full factor.*

Problem 2. *Does there exist an icc group Γ which is inner amenable and such that the von Neumann algebra $W^*_\lambda(\Gamma)$ is a full factor ?*

There are many examples of families of groups which are known to be not inner amenable, and thus to give rise to full factors. Here are some of them.

Let G be a connected semi-simple real Lie group without centre and without compact factor. It is a simple corollary of Borel density theorem that a lattice Γ in G is an icc group. (See e.g. [BkH, Proposition 2], which proves a slight strengthening of this; for the density theorem, see [Bo1] or [Zim, Theorem 3.2.5].) Also Γ is not amenable (indeed, C.C. Moore has shown this for any Zariski-dense subgroup of G [Zim, 4.1.11 and 4.1.15]). It can be shown that such a lattice Γ is never inner amenable, so that $W^*_\lambda(\Gamma)$ is a full factor; this carries over to lattices in adjoint semi-simple Lie groups over local fields [HS3].

In case G is moreover a simple real Lie group of rank one, it is also known that an icc subgroup of G which is discrete (not necessarily a lattice) and which is not amenable cannot be inner amenable (Georges Skandalis, private communication of December, 1992, and [Sk1]).

Let now Γ be a group which is hyperbolic and non elementary (a hyperbolic group is said to be elementary if it contains a cyclic subgroup of finite index). Such a group is not necessarily icc, for example because there may exist a subgroup Γ_0 of Γ such that Γ is the direct product of Γ_0 and of a non trivial finite subgroup. Consider however

$$\Gamma_f = \{\gamma \in \Gamma : \text{ the centralizer of } \gamma \text{ in } \Gamma \text{ is of finite index in } \Gamma\}.$$

Then Γ_f is a finite normal subgroup in Γ and the quotient $\Gamma_{icc} = \Gamma/\Gamma_f$ is icc [Cha, cor. 2.2.2]. If Γ is moreover torsion free (this implies $\Gamma_f = \{1\}$), then Γ is icc and is not inner amenable [Har4], so that $W^*_\lambda(\Gamma)$ is a full factor. I do not know if this holds under the more general condition $\Gamma_f = \{1\}$.

Let B_n denote the Artin braid group on n strings, let C_n denote its centre (which is isomorphic to $\mathbb{Z}$) and let DB_n denote is commutator group. It has been shown in [GiH] that $W^*_\lambda(B_n/C_n)$ and $W^*_\lambda(DB_n)$ are full factors for all $n \geq 3$. From the same paper, we repeat here the following.

Problem 3. *Let $K \subset \mathbb{S}^3$ be a piecewise linear knot which is not a torus knot and let Γ_K denote the fundamental group of the knot complement $\mathbb{S}^3 - K$. Show that Γ_K is icc and not inner amenable.*

One may of course repeat for the groups above the question of Problem 1 : in particular, if K and K' are two such knots, when are $W^*_\lambda(\Gamma_K)$ and $W^*_\lambda(\Gamma_{K'})$ isomorphic ? (Compare with Problems 4 and 6 below.) One may also formulate similar problems for other classes of groups appearing in geometry, such as mapping class groups, or infinite irreducible Coxeter groups which are neither finite nor affine. (The latter have free subgroups [Har3]; for many examples, see the references quoted in [Har5, nos 78-81].) Ditto for various notions of generic or random groups [Cha], [Gr2, § 9].

2.6. *Fundamental groups.*

The **fundamental group** of a factor M of type II_1 with trace τ is the group of positive real numbers

$$\begin{aligned}\mathcal{F}(M) = \Big\{ t \in \mathbb{R}_+^* \ &: \ \text{there exist a projection } e \in M \otimes \mathcal{L}(\mathcal{H}) \\ &\text{such that } \tau(e) = t \ \text{ and } \ e\big(M \otimes \mathcal{L}(\mathcal{H})\big)e \approx M \Big\} \\ = \Big\{ t \in \mathbb{R}_+^* \ &: \ \text{there exists } \alpha \in Aut\big(M \otimes \mathcal{L}(\mathcal{H})\big) \\ &\text{such that } \tau(\alpha(x)) = t\tau(x) \ \text{ for all } \ x \in M \otimes \mathcal{L}(\mathcal{H}) \Big\}\end{aligned}$$

defined by Murray and von Neumann. Chapter V of [MNIV] ends with the disappointing observation that "as to $\mathcal{F}(M)$, we know nothing beyond Theorem XV" (which shows that the fundamental group of the injective II_1 factor is $\mathbb{R}_+^*$). Today, we know at least two more things

if Γ has Kazhdan Property (T), then $\mathcal{F}(W_\lambda^*(\Gamma))$ is a countable subgroup of $\mathbb{R}_+^*$ (see [Co80], and [Po5] for a generalization),

the fundamental group of $W_\lambda^*(F_\infty)$ is $\mathbb{R}_+^*$ [Ra1]; more precisely there exists a one parameter group $(\alpha_t)_{0<t<\infty}$ of automorphisms of $M = W_\lambda^*(F_\infty) \otimes \mathcal{L}(\mathcal{H})$ such that $\tau(\alpha_t(x)) = t\tau(x)$ for all $t \in \mathbb{R}_+^*$ and $x \in M$, where τ denotes the canonical trace on M [Ra2].

In particular $W_\lambda^*(\Gamma) \not\approx W_\lambda^*(\Gamma')$ if Γ has Property (T) and if Γ' is amenable or is F_∞.

Problem 4. *Does one have $\mathcal{F}(W_\lambda^*(\Gamma)) = \mathcal{F}(W_\lambda^*(F_2))$ whenever Γ is one of the following groups ?*

a non elementary Fuchsian group,
a lattice in $SO^0(1,n)$ or in $PSU(1,n)$ for some $n \geq 2$,
a quotient B_n/C_n of a braid group by its centre ($n \geq 3$).

Dykema and Radulescu have shown that

either $\mathcal{F}(W_\lambda^*(F_n)) = \{1\}$ for all n such that $2 \leq n < \infty$,
and then $W_\lambda^*(F_n) \not\approx W_\lambda^*(F_{n'})$ for all $n, n' \in \{2,3,...\}$ such that $n \neq n'$,
or $\mathcal{F}(W_\lambda^*(F_n)) = \mathbb{R}_+^*$ for all n such that $2 \leq n < \infty$,
and then $W_\lambda^*(F_n) \approx W_\lambda^*(F_{n'})$ for all $n, n' \in \{2,3,...\}$.

Then Radulescu has shown that the second possibility implies $W_\lambda^*(F_n) \approx W_\lambda^*(F_\infty)$ for all $n \geq 2$. For Fuchsian groups, see [HaVo].

2.7. *Jones' invariants.*

Let M be a separable factor of type II_1. V. Jones has defined the invariant

$$\mathcal{I}(M) = \{ r \in [1, \infty] \ : \ \text{there exists a } II_1\text{-subfactor } N \text{ of } M \text{ with index } r \ \}$$

and has shown that it satisfies the following properties :

$r_1, r_2 \in \mathcal{I}(M) \Longrightarrow r_1 r_2 \in \mathcal{I}(M)$,

$\{n^2\}_{n=1,2,\dots} \cup \{\infty\} \subset \mathcal{I}(M)$,

for each $x \in \mathcal{F}(M)$ one has $x + 2 + x^{-1} \in \mathcal{I}(M)$,

$\mathcal{I}(M) \subset \{4\cos^2 \frac{\pi}{n}\}_{n=3,4,\dots} \cup [4,\infty] = \mathcal{I}(R)$,

where R denotes the injective II_1-factor. Moreover, if Γ is an icc group

$\mathcal{I}(\Gamma) \subset \mathcal{I}(W^*_\lambda(\Gamma))$

where $\mathcal{I}(\Gamma) = \{n \in \mathbb{N}$: there exists a subgroup of Γ of index $n\}$. For all this, see [Jo83]. It is also known that

$\mathcal{I}(M)$ is countable if M has Property (T),

for example if $M = W^*_\lambda(\Gamma)$ for an icc group Γ with Property (T) (see [PiP], and a generalization in [Po5]).

A recent computation of Radulescu shows that

$\mathcal{I}(W^*_\lambda(F_\infty)) = \mathcal{I}(R)$.

Moreover, let $s \in \mathcal{I}(R)$ be the index of a subfactor of R with trivial relative commutant obtained by iteration of Jones' "basic construction" from a commuting square; then $s \in \mathcal{I}(W^*_\lambda(F_n))$ for each $n \geq 2$; in particular $\mathcal{I}(W^*_\lambda(F_n)) \cap [1,4] = \mathcal{I}(R) \cap [1,4]$ for each $n \geq 2$. For all this, see [Ra3], [Ra5].

Problem 5. *Let Γ be an icc group which has Kazhdan Property (T). Compute $\mathcal{F}(W^*_\lambda(\Gamma))$ and $\mathcal{I}(W^*_\lambda(\Gamma))$.*

The question about $\mathcal{F}(W^*_\lambda(\Gamma))$ appears in [Co90, section 3.10, problème 3] and [Co93, Section V.11]. One could of course add a (probably even more difficult) problem about the invariant $\mathcal{C}(M) = \{r \in [1,\infty]$: there exists a II_1-subfactor N of M with index r and with trivial relative commutant$\}$.

2.8. *The constants of Cowling and Haagerup.*

In their work on completely bounded multipliers, M. Cowling and U. Haagerup have defined constants $\Lambda(\Gamma)$, $\Lambda(G)$, $\Lambda(M) \in [1,\infty]$, associated respectively to a discrete group Γ, a second countable locally compact group G and a finite von Neumann algebra M. Moreover :

$\Lambda(G) = 1$ if G is amenable,

$\Lambda(\Gamma) = \Lambda(G)$ if Γ is a lattice in G,

$\Lambda(\Gamma) = \Lambda(W^*_\lambda(\Gamma))$,

$\Lambda(M_1) \leq \Lambda(M_2)$ if M_1 is a subalgebra of the finite algebra M_2,

$\Lambda(G) = 1$ if G is locally isomorphic to one of $SO(1,n)$ or $SU(1,n)$ for some $n \geq 2$,

$\Lambda(G) = 2n - 1$ if G is locally isomorphic to $Sp(1,n)$ for some $n \geq 2$,

$\Lambda(G) = \infty$ if G is a connected simple real Lie group with finite centre which is non compact and which is of real rank at least 2.

This provides many examples of pairs of lattices $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ such that $W^*_\lambda(\Gamma_1)$ is not isomorphic to any subalgebra of $W^*_\lambda(\Gamma_2)$. For all this, see [Haa3] and [CoH], as well as [LHa] for the universal covering of $SU(1,n)$; there is also a nice review by Cowling [Cw1]. Let us finally mention that $\Lambda(G) = 1$ for a locally compact group acting properly on a locally finite simplicial tree (a result of Szwarc, see [Va3, Proposition 6]) or for various free products of amenable groups amalgamated onver a common open compact subgroup [BoP], and that $\Lambda(\Gamma) = 1$ for a Coxeter group Γ in the so-called "right-angled" class [Va4]; it is believed that $\Lambda(\Gamma) = 1$ for any Coxeter group Γ.

Superrigidity à la Margulis suggests the following problem, again due to A. Connes. See [Co90, section 3.10, problème 2], [Co93, Section V.11], and also the last question in [CoH] about lattices in $Sp(1,11)$ and in $F_{4(-20)}$.

Problem 6. *Let Γ_1, Γ_2 be two icc groups which have Kazhdan Property (T). Show that $W^*_\lambda(\Gamma_1)$ and $W^*_\lambda(\Gamma_2)$ are isomorphic if and only if Γ_1 and Γ_2 are isomorphic.*

Here is a related problem concerning rigid groups. I believe it is also due to A. Connes.

Problem 7. *Find an icc group Γ such that any automorphism of the factor $W^*_\lambda(\Gamma)$ is inner.*

If such a group Γ exists, it has to be perfect and any automorphism of Γ itself has to be inner [Beh1, Theorems 5.1 and 5.2], [Ka1, Remark 2.3].

There is a notion of Property (T) for von Neumann algebras [CoJ] which is well adapted to the groups we discuss here : if a group Γ is icc, or more generally if the subgroup Γ_f of these elements of Γ which have a finite conjugacy class is finite, then $W^*_\lambda(\Gamma)$ has Property (T) if and only if Γ has Property (T). But Jolissaint has observed that, if Γ is a group which has Property (T) and which is such that Γ_f is infinite, then $W^*_\lambda(\Gamma)$ does *not* have Property (T) of [CoJ]. (See [Jo3], which gives also a characterization in terms of $W^*_\lambda(\Gamma)$ of Property (T) for an arbitrary group Γ; for examples, due to Serre, of groups Γ which have Property (T) and which have infinite centres, see [HaVa, § 3.d].)

It is known that a II_1-factor with Property (T) cannot be isomorphic to a subfactor of $W^*_\lambda(F_2)$ [CoJ, Corollary 4]. Moreover, if Γ is a group which has Property (T), any homomorphism from Γ to the unitary group of $W^*_\lambda(F_2)$ has an image whose strong closure is a compact subgroup of $\mathcal{U}(W^*_\lambda(F_2))$; this is a particular case of a result in [Rob].

One may ask whether there exists a sequence $(T_1) = (T)$, (T_2) , ... of strengthenings of Property (T) such that, if Γ or $W^*_\lambda(\Gamma)$ has and if Γ' or $W^*_\lambda(\Gamma')$ has not Property (T_n), then $W^*_\lambda(\Gamma)$ cannot be a subfactor of $W^*_\lambda(\Gamma')$. (This is a suggestion of M. Gromov.)

Let us finally repeat here an old problem which is still open (see e.g. [Po4, § 4.3]).

Problem 8. *Let M be a factor of type II_1 which is not injective. Does there exist a subfactor of M isomorphic to $W^*_\lambda(F_2)$?*

2.9. *The χ invariant and other invariants.*

For any factor M with separable predual, A. Connes has defined an abelian Borel group $\chi(M)$, which is the centre of the image in the outer automorphism group $Out(M) = Aut(M)/Int(M)$ of the group $\overline{Int}(M)$ of approximately inner automorphisms of M. He has shown that $\chi(M) = \{1\}$ if M is the injective factor of type II_1 or if $M = W^*_\lambda(F_n)$ for some integer $n \geq 2$. The invariant is meant (among other things) as an obstruction to a factorization as a tensor product of a full factor and a hyperfinite factor. A. Connes has constructed examples M such that $\chi(M) \neq \{1\}$ (some of these examples can be realized as group factors, but this is not used for the computation of their χ). He has also used the invariant χ to show that there exist factors of type II_1 which are *not* anti-isomorphic to themselves, and in particular *not* of the form $W^*_\lambda(\Gamma)$. See [Co75], as well as [C76b, Section 3.10], [Jo79], [Jo80] and [Kaw]. It seems appropriate to formulate explicitely the following question.

Problem 9. *What can be said about $\chi\,(W^*_\lambda(\Gamma))$ for other icc groups Γ ?*

In [Po2], S. Popa has made a detailed study of the maximal injective von Neumann subalgebras of $W^*_\lambda(F_n)$. He has shown in particular that each free generator of F_n generates such a maximal injective subalgebra, abelian and isomorphic to $W^*_\lambda(\mathbb{Z}) \approx L^\infty(\mathbb{S}^1)$. He asks moreover the following problem.

Problem 10. *Classify up to isomorphism the maximal injective von Neumann subalgebras of the II_1-factors.*

Let us define a factor M of type II_1 to be **tensorially indecomposable** if it cannot be written as any tensor product $A \otimes B$ of two factors A, B of type II_1. In [Po1, Corollary 6.6], Popa shows that the nonseparable factor $W^*_\lambda(F_{!!})$ of a free group $F_{!!}$ on uncountably many generators is tensorially indecomposable. In [Po6], he asks for examples of separable II_1-factors which have this property, and asks in particular the following.

Problem 11. *Are the $W^*_\lambda(F_n)$'s tensorially indecomposable ?*

Among other invariants of factors of the form $M = W^*_\lambda(\Gamma)$ which should be investigated, obvious candidates are cohomology spaces, in particular $H^k_c(M, M)$ and $H^k_c(M, M_*)$, where the subscript c indicates cohomology with norm-continuous cochains and where M_* denotes the predual of M. For example, one has $H^k_c(M, M) = \{0\}$ for all $k \geq 1$ if M is hyperfinite, and more generally if M is isomorphic to its tensor product with the hyperfinite II_1-factor; does this hold in general ? (it does for $k = 1$). See [Rin], [Co78]. Recently, A. Sinclair and co-workers have observed that the Gromov bounded cohomology group $H^k_b(\Gamma)$ injects in $H^k_c\left(l^1(\Gamma), l^1(\Gamma)\right)$ for all $k \geq 2$, so that one has for example $H^2_c\left(l^1(F_n), l^1(F_n)\right) \neq \{0\}$. There is now some effort to try and produce an example with

$$H^{2 \text{ or } 4}_c\,(W^*_\lambda(\Gamma), W^*_\lambda(\Gamma)) \overset{?}{\neq} \{0\}$$

but all this is quite conjectural at the time of writing. (Moreover specialists don't all agree about which way to conjecture; see for example [Po6].)

Here is another open problem, stated in [FaH], slightly related to cohomology considerations : if γ is a free generator of F_n, does there exist $X, Y \in W^*_\lambda(F_n)$ such that $\lambda(\gamma) = XY - YX$?

A more exotic project would be to study homotopy groups of the unitary group (with the strong topology) of II_1-factors of the form $M = W^*_\lambda(\Gamma)$; such a unitary group is contractible if M is hyperfinite or if $\Gamma = F_\infty$ [PoT]. The homotopy groups of the unitary group of a II_1-factor M with respect to the *norm* topology are known; firstly $\Pi_1(\mathcal{U}(M)_{\text{norm}}) = \mathbb{R}$ by [ASS]; secondly $\Pi_{2k+1}(\mathcal{U}(M)_{\text{norm}}) = \Pi_1(\mathcal{U}(M)_{\text{norm}}) = \mathbb{R}$ and $\Pi_{2k}(\mathcal{U}(M)_{\text{norm}}) = \{0\}$ for all $k \geq 1$ by [Sc1]; see also [Sc2].

Finally, we would like at least to mention the impressive entropy computations of E. Størmer for automorphisms of the factor $W^*_\lambda(F_\infty)$ [Sto].

2.10. *Other representations.*

Let Γ be an irreducible lattice in a connected semi-simple real Lie group G without centre and without compact factor, let ρ be an irreducible unitary representation of G and let $\rho|\Gamma$ denote the restriction of ρ to Γ. If ρ is *not* in the discrete series of G, then $\rho|\Gamma$ is irreducible [CoS, Proposition 2.5], so that $W^*_{\rho|\Gamma}(\Gamma) = \mathcal{L}(\mathcal{H}_\rho)$ by Shur's Lemma. If ρ is in the discrete series, then $W^*_{\rho|\Gamma}(\Gamma) \approx W^*_\lambda(\Gamma)$ by [GHJ, Section 3.3.c].

Let Γ be an infinite group such that $C^*_\lambda(\Gamma)$ is simple (see below) and let M be an infinite hyperfinite factor (e.g. a Powers factor R_λ for some $\lambda \in]0,1[$). It follows from a result of O. Maréchal that there exists a representation π of Γ which is weakly equivalent to the regular representation and such that $W^*_\pi(\Gamma) \approx M$ [Mar]. It is also known that *any* properly infinite von Neumann algebra is of the form $W^*_\pi(PSL_2(\mathbb{Z}))$ [Beh2], and that the same holds for large classes of finite von Neumann algebras [Beh3].

In case $\Gamma = F_n$, finite factors of the form $W^*_\pi(F_n)$ are precisely the II_1-factors which can be generated by n unitaries. In particular, any icc group Γ given together with a set of n generators provides a factor $M = W^*_\lambda(\Gamma)$ and a representation $\pi : F_n \to \Gamma \to \mathcal{U}(l^2(\Gamma))$ such that $W^*_\pi(F_n) = M$. For example, for each $k \geq 2$, the group $PGL_k(\mathbb{Z})$ can be generated by 2 elements [CxM, Chapter 7]; thus there exists a representation π of F_2 such that $W^*_\pi(F_2) \approx W^*_\lambda(PGL_k(\mathbb{Z}))$. I don't know of any II_1-factor which could not be generated by two unitaries (see also the end of 4.1 below); nor do I know of any II_1-factor without Cartan subalgebras. (If a II_1-factor with separable predual has a Cartan subalgebra, then it is generated by two unitaries [Po3, Theorem 3.4]; however, given such a factor M, the existence of a Cartan subalgebra of M is "in general" an open problem.)

Given a group Γ and a class of von Neumann algebras, an ambitious project is to classify the representations π of Γ such that $W^*_\pi(\Gamma)$ is in the given class. The appropriate kind of classification is up to quasi-equivalence : π and π' are quasi-equivalent if there exists an isomorphism Φ from $W^*_\pi(\Gamma)$ onto $W^*_{\pi'}(\Gamma)$ such that $\Phi(\pi(\gamma)) = \pi'(\gamma)$ for all $\gamma \in \Gamma$. For the class of finite factors, such a classification can be rephrased in terms of normalized characters of finite type, namely of functions of positive type

$\xi : \Gamma \to \mathbb{C}$ which are normalized ($\xi(1) = 1$), central ($\xi(\gamma\gamma') = \xi(\gamma'\gamma)$) and indecomposable ($\xi = a\xi' + (1-a)\xi''$ with $a \in]0,1[$ and ξ', ξ'' normalized central of positive type implies $\xi = \xi' = \xi''$); see [DC*, corollaire 6.7.4 and proposition 17.3.5].

In case Γ is locally finite, factors of the form $W^*_\pi(\Gamma)$ are either the injective II_1-factor or finite dimensional factors. Characters have been classified for a few groups such as the group $S(\infty)$ of permutations with finite supports of an infinite countable set, and related groups. These results are due to Thoma, Vershik-Kerov and Nazarov [Naz].

3. *THE REDUCED C^*–ALGEBRA $C^*_\lambda(\Gamma)$, AND SIMPLICITY.*

3.1. *Generalities.*

Notations being as in the beginning of Chapter 2, the norm closure of $\pi : \mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ is denoted by $C^*_\pi(\Gamma)$. In particular, if π is the left regular representation λ, one obtains the **reduced C^*-algebra** $C^*_\lambda(\Gamma)$ of Γ. One may also choose the universal representation π_{un} of Γ (say here that π_{un} is the direct sum of all cyclic representations of Γ, up to equivalence), and one obtains the **full C^*-algebra** $C^*(\Gamma)$ of Γ. For any representation π, one has a natural morphism from $C^*(\Gamma)$ onto $C^*_\pi(\Gamma)$ which is again denoted by π. One has in particular a morphism $C^*(\Gamma) \to C^*_\lambda(\Gamma)$, and this is an isomorphism if and only if Γ is amenable, by a theorem of Hulanicki and Reiter [Ped, Th. 7.3.9].

In the classical case $\Gamma = \mathbb{Z}$, the algebra $C^*_\lambda(\Gamma) \approx C^*(\Gamma)$ is isomorphic via Fourier transform to the algebra $\mathcal{C}(\mathbb{T})$ of continuous functions on the one-dimensional torus $\mathbb{T}$. This carries over to any discrete abelian group Γ and its compact Pontryagin dual : $C^*_\lambda(\Gamma) \approx \mathcal{C}(\hat{\Gamma})$.

The oldest published reference I know involving reduced C^*-algebras of locally compact groups is [Seg].

3.2. *Free groups.*

The following result was published in 1975 [Pow], seven years after it was found [Va2, page 489]. Recall first that a normalized trace on a C^*-algebra A with unit is a linear map $\tau : A \to \mathbb{C}$ such that $\tau(1) = 1$, $\tau(a^*a) \geq 0$ and $\tau(ab - ba) = 0$ for all $a, b \in A$; it follows that $|\tau(a)| \leq ||a||$ for all $a \in A$ [DC*, Proposition 2.1.4]. The **canonical trace** on the reduced C^*-algebra of a group Γ is the extension to $C^*_\lambda(\Gamma)$ of the map $\mathbb{C}[\Gamma] \to \mathbb{C}$ which applies $\sum_{\gamma\in\Gamma} z_\gamma \gamma$ to z_1, as in 2.1 above.

Theorem 3 (Powers). *The reduced C^*-algebra $C^*_\lambda(F_n)$ of a free group on $n \geq 2$ generators is simple and has a unique normalized trace.*

There is a proof in Appendix 2 below. Another formulation of this theorem is that any unitary representation π of F_n which is weakly contained in the regular representation λ of F_n is in fact weakly equivalent to λ, and in particular is such that $C^*_\pi(F_n) \approx C^*_\lambda(F_n)$; for more of this point of view, see [BkH]; it is moreover true that

π and λ as above are approximately equivalent in the sense of [Vo1, see in particular Corollary 1.4].

There is no analogue to Problem 1 here because computations of K_1-groups show that the algebras $C^*_\lambda(F_n)$ are pairwise nonisomorphic [PiV, Corollary 3.7]. Concerning pairs Γ_1, Γ_2 of non-isomorphic groups such that $C^*_\lambda(\Gamma_1) \approx C^*_\lambda(\Gamma_2)$, there are several known examples but apparently no systematic study. Let us describe two classes of such examples, the algebras being commutative in one case and simple in the other.

Let first Γ be any infinite countable abelian torsion group. Its Pontryagin dual $\hat{\Gamma}$ is compact, metrisable, totally disconnected and without isolated point [because the locally compact abelian group Γ is respectively discrete, countable, torsion, and infinite]. Consequently $\hat{\Gamma}$, viewed as a topological space, is homeomorphic to the triadic Cantor set K. Hence the C^*-algebra of Γ is isomorphic to the C^*-algebra $\mathcal{C}(K)$ of continuous functions on K, and thus does not depend on the detailed structure of Γ.

The second example is due to G. Skandalis. Let F_1, F_2 be two non isomorphic finite abelian groups of the same order, say n, for example $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. We identify both $C^*(F_1)$ and $C^*(F_2)$ to the same algebra A, isomorphic to $\mathbb{C}^n$. For $i \in \{1,2\}$, set $\Gamma_i = F_i \star \mathbb{Z}$. The full C^*-algebra of Γ_i is canonically isomorphic to a free product with amalgamation over $\mathbb{C}$, and one has more precisely $C^*(\Gamma_i) \approx A \underset{\mathbb{C}}{\star} C^*(\mathbb{Z})$ in the sense of [Bro]. Moreover, the canonical trace $\tau_i : C^*(\Gamma_i) \to \mathbb{C}$ is independent on i when viewed as a map $A \underset{\mathbb{C}}{\star} C^*(\mathbb{Z}) \to \mathbb{C}$. Consequently the ideal $\mathcal{K}_i = \{x \in C^*(\Gamma_i) \mid \tau_i(x^*x) = 0\}$ and the quotient $C^*_\lambda(\Gamma_i) = C^*(\Gamma_i)/\mathcal{K}_i$ are both independent on i. On the other hand, Proposition 2 below shows that $C^*_\lambda(\Gamma_i)$ is a simple C^*-algebra with unique trace.

There exists however an uncountably infinite family $(\Gamma_\iota)_{\iota \in I}$ of countable groups such that the C^*-algebras $C^*_\lambda(\Gamma_\iota)$ are pairwise non-isomorphic, each being simple with a unique normalized trace; this follows easily from McDuff's result quoted in 2.3 above [AkL, Corollary 9].

It is known that some group Banach algebras determine the group Γ. (Examples : $L^1(\Gamma)$, $A(\Gamma)$ and $B(\Gamma)$; this holds indeed for a locally compact group; see [Lep], [Wa1], [Wa2] and [Wen].) But we do not discuss these algebras further here.

3.3. *Other groups with simple reduced C^*-algebras.*

Theorem 3 has begotten many generalizations : see among others [Ake], [AkL], [Be1], [Be2], [BCH], [BN1], [BN2], [Har2], [HSk], [HoR], [PaS], [Ros]. Let us indicate some of these results.

Proposition 2. *Let Γ be a group which admits at least one of the following descriptions :*

(a) a free product $\Gamma_1 \star \Gamma_2$ where $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$,

(b) a Zariski-dense subgroup in a semi-simple connected real Lie group without centre and without compact factor,

(c) a group $PSL(n, K)$ for some integer $n \geq 2$ and for some field K which is either of characteristic zero, or of characteristic p and not algebraic over $\mathbb{F}_p$.

(d) a group of K-rational points $\mathbb{G}(K)$ for some field K of characteristic zero and for some connected semi-simple algebraic group $\mathbb{G}$ defined over K,

(e) a torsionfree hyperbolic group which is not elementary.

Then $C^*_\lambda(\Gamma)$ is simple with unique normalized trace.

See [PaS] for (a), [BCH] for (b) and (d), [HoR] and [Ros] for (c), and [Har4] for (e). In (b), the subgroup is not supposed to be discrete in the ambient Lie group, but it is viewed as a discrete (possibly uncountable) group in the statement about $C^*_\lambda(\Gamma)$. There is also in [BCH] a proof of (c) for the particular case of a field K of characteristic zero.

The following repeats [Har2, Section 2, Question (2)]. (Question (3) of the same reference, same section, has been answered in [Bel].)

Problem 12. *Does there exist a group Γ such that $C^*_\lambda(\Gamma)$ is simple but has several traces? or such that $C^*_\lambda(\Gamma)$ has a unique trace but is not simple ?*

We collect below two observations resulting from a conversation with A. Valette. Before these, consider an amenable normal subgroup N of a group Γ. As the identity representation of N is weakly contained in the regular representation of N, one sees by induction from N to Γ that the regular representation of Γ/N (viewed as a representation of Γ) is weakly contained in the regular representation of Γ. In other words, one has a morphism of C^*-algebras π from $C^*_\lambda(\Gamma)$ onto $C^*_\lambda(\Gamma/N)$. This applies for example to the normal subgroup Γ_f of elements with finite conjugacy classes in Γ, which is an amenable group [Tom, Corollary 1.5].

Observation 1 : if $C^*_\lambda(\Gamma)$ is simple, then Γ is an icc group, and more generally any amenable normal subgroup N of Γ is reduced to $\{1\}$. Indeed, the morphism π defined above has to be injective, so that $N = \{1\}$. Observe also that, if Γ was not icc, the centre of $C^*_\lambda(\Gamma)$ would be strictly larger than the scalar multiples of the identity (the characteristic function of any finite conjugacy class of Γ is in this centre).

Observation 2 : if $C^*_\lambda(\Gamma)$ has a unique trace, then (again) any amenable normal subgroup N of Γ is reduced to $\{1\}$. Indeed, the composition of the morphism π and of the canonical trace of $C^*_\lambda(\Gamma/N)$ has to coïncide with the canonical trace of $C^*_\lambda(\Gamma)$.

All this being said, problems of simplicity of reduced C^*-algebras of groups should not conceal other problems. In particular, it would be pleasant to know "many" examples of groups Γ which are *not* C^*_λ-simple but for which two-sided ideals of $C^*_\lambda(\Gamma)$ are *classified* in some way (a few examples appear in Theorem 4 of [BCH]). The corresponding program for the Fourier algebra $A(\Gamma)$ of a discrete group Γ is the subject of [For, see in particular Theorem 3.20].

3.4. *Other representations*

The diversity of C^*-algebras of the form $C^*_\pi(F_n)$ has no limit. Indeed, let A be a separable C^*-algebra with unit acting in some Hilbert space $\mathcal{H}$, and assume that A

is generated as a C^*-algebra by a finite or infinite sequence $(a_1, a_2, ...)$ of length n. As any element in a C^*-algebra with unit is a linear combination of unitaries [Ped, 1.1.11], there is no loss of generality if we assume that the a_i 's are unitary. Consider now a free set of generators s_i of F_n, and define a representation $\pi : F_n \to \mathcal{U}(\mathcal{H})$ by $\pi(s_i) = a_i$. Then $C^*_\pi(F_n) = A$.

In particular, there are C^*-algebras with unit of the form $C^*_\pi(\Gamma)$ which cannot be reduced C^*-algebras of discrete groups. A specific example is the algebra $\mathbb{C} \oplus \mathcal{K}$ acting on a separable infinite dimensional Hilbert space, which is generated by the identity and by the compact operators : it is $C^*_\pi(F_2)$ for an appropriate π [HRV1, Section A], and it cannot be of the form $C^*_\lambda(\Gamma)$. (Indeed, assume firstly that Γ has an element γ of infinite order. Then γ generates a sub-C*-algebra isomorphic to $C^*\left(\gamma^{\mathbb{Z}}\right) \approx \mathcal{C}(\mathbb{T})$, so that the spectrum of γ is the circle $\mathbb{T}$. But the spectrum of any element in $\mathbb{C} \oplus \mathcal{K}$ is countable, and thus $\mathcal{C}(\mathbb{T})$ cannot be isomorphic to a subalgebra of $\mathbb{C} \oplus \mathcal{K}$. Assume secondly that Γ has an element γ of some finite order $k > 1$. As Γ is infinite, it follows that any kth root of 1 appears in the spectrum of γ with infinite multiplicity. This cannot happen in $\mathbb{C} \oplus \mathcal{K}$.)

It would be interesting to understand better, for a given group Γ,

(i) which are the algebras of the form $C^*_\pi(\Gamma)$,
(ii) what are the automorphism groups of these $C^*_\pi(\Gamma)$,
(iii) for which π these algebras $C^*_\pi(\Gamma)$ are simple.

About (ii), it is known that $\overline{Inn}(A)/Inn(A)$ is uncountable for any separable C^*-algebra A which does not have continuous trace, e.g. for $A = C^*_\lambda(\Gamma)$ whenever Γ is not of type I. (See [Phi, Theorem 3.1], and compare with Problem 7 above.) It is also known that $\overline{Inn}\left(C^*_\lambda(F_2)\right)/\overline{Inn_0}\left(C^*_\lambda(F_2)\right)$ is non trivial, where $\overline{Inn_0}(A)$ denotes the closure of the group of inner automorphisms of A determined by unitaries connected to 1 in the group of automorphisms of the C^*-algebra A, closure for the topology of pointwise convergence [ElR, 4.13].

About (iii), see Proposition 4 of Appendix 2.

Representations of free groups are discussed in [FTP]; see also [FTN], [Sz1] and [Sz3].

In a remarkable paper of the early 50's, Yoshizawa has constructed an irreducible representation π of F_2 which weakly contains *any* irreducible representation of F_2, namely which is such that the natural morphism $C^*(F_2) \to C^*_\pi(F_2)$ is an isomorphism [Yos, § 3]. In other words, the C^*-algebra $C^*(F_2)$ is primitive, namely has a representation which is both irreducible and faithful [Ped, 3.13.7].

Problem 13. *What are the groups with primitive full C^*-algebras ?*

Let Γ be a group given as a discrete subgroup of some Lie group G. A natural way to obtain representations of Γ is to consider a representation ρ of G and its restriction $\rho|\Gamma$ to Γ. If Γ and G are as in 2.10 above, it is a natural question to ask about properties of $C^*_{\rho|\Gamma}(\Gamma)$. Here is a partial and easy answer, from [BkH] : let G be a simple connected real Lie group which is non compact and with centre reduced to

$\{1\}$, let ρ be a unitary representation of G in the principal series and let Γ be a lattice in G; then $C^*_{\rho|\Gamma}(\Gamma) \approx C^*_\lambda(\Gamma)$.

Another natural representation of Γ to consider is $\rho_{un}|\Gamma$, where ρ_{un} denotes the universal representation of the Lie group G; if this is again a non compact simple connected real Lie group without centre, it is conjectured that $C^*(\Gamma) \to C^*_{\rho_{un}|\Gamma}(\Gamma)$ is *never* an isomorphism, and this has been proved in many cases (e.g. if G has Property (T)) in [BeV].

4. *THE C^*-ALGEBRAS $C^*_\lambda(\Gamma)$ AND $C^*(\Gamma)$: SOME OTHER PROPERTIES.*

4.1. *Nuclearity and other finiteness conditions.*

Given two C^*-algebras A and B, there are in general several ways to complete the algebraic tensor product of A and B to obtain a C^*-tensor product. The algebra A is said to be **nuclear** if these ways coincide, for any B. For more on this, see [La2] and [Ta2]. Nuclearity for $C^*_\lambda(\Gamma)$ is settled by the following result (of which (ii) has the remarkable property that the "only if" part does *not* extend to the locally compact case).

Theorem 4 (Takesaki, 1964, and Lance, 1973).
(i) The algebra $C^*_\lambda(F_2)$ is not nuclear.
(ii) Let Γ be a group. Then $C^*_\lambda(\Gamma)$ is nuclear if and only if Γ is amenable.

It follows easily from Claim (ii) that $C^*(\Gamma)$ is nuclear if and only if Γ is amenable. (Indeed, if Γ is amenable, then $C^*(\Gamma)$ is isomorphic to $C^*_\lambda(\Gamma)$, which is nuclear. If $C^*(\Gamma)$ is nuclear, then $C^*_\lambda(\Gamma)$ is nuclear, because any quotient of a nuclear algebra is nuclear by a result of Choi and Effros [La2], so that Γ is amenable by (ii).)

Besides amenability $\Longleftrightarrow$ nuclearity, there are only few known exact translations between properties of Γ and properties of $C^*_\lambda(\Gamma)$ or of $C^*(\Gamma)$. For example, does

$$\Gamma \text{ finitely generated } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \text{ finitely generated}$$

hold ? One may ask what is the smallest number of generators for $C^*_\lambda(\Gamma)$ or $C^*(\Gamma)$. S. Wasserman [Was2, Section 6] has observed that this number is at least 2 for $C^*(F_2)$, because $C^*(F_2)$ has a quotient isomorphic to $\mathcal{C}(\mathbb{T}^2)$ and because the 2-torus $\mathbb{T}^2$ is not planar. On the other hand, it is known that $W^*_\lambda(F_2)$ and the hyperfinite II_1-factor are both singly generated [Sai, Theorem 2.3 and following example].

And does

$$\Gamma \text{ locally finite } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \text{ approximately finite (AF)}$$

hold ? Known examples of groupoid C^*-algebras which are AF for non obvious reasons [Kum] may suggest that $\Leftarrow$ does *not* hold.

What about

$$\Gamma \text{ f.g. and of polynomial growth } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \text{ ess. of polynomial growth}$$

("f.g." holds for finitely generated and "ess." for essentially) ? See [KiV] for algebras of essential polynomial growth.

The answer to the question

$$\Gamma \text{ residually finite } \overset{?}{\Longleftrightarrow} C^*(\Gamma) \text{ residually finite dimensional [ExL]}$$

is negative, but can one slightly change the question to have an affirmative answer ? The negative answer follows from properties of the group $\Gamma = SL(2, \mathbb{Z}[1/p])$ where p is a prime, as shown to me by M. Bekka. Indeed, one one hand Γ is residually finite because it is both finitely generated and linear [Mal]. On the other hand Γ does not have Kazhdan's Property (T) because Γ is dense in $SL(2, \mathbb{R})$ [HaV, Propositions 1.6 and 3.6], but the unit representation of Γ is isolated in the set of all its finite dimensional unitary representations [LuZ]; these facts imply that finite dimensional representations of $C^*(\Gamma)$ do not separate elements of this C^*-algebra. However, I do not know whether $\Leftarrow$ holds or not.

What are the properties of $C^*_\lambda(\Gamma)$, or of $C^*(\Gamma)$, which are equivalent to the group being finitely presented ? with solvable word problem ? hyperbolic ? small cancellation ? of finite cohomological dimension (say over $\mathbb{Q}$) ? a torsion group ? solvable ? Dually, what are the properties of Γ which are equivalent to $C^*_\lambda(\Gamma)$ being simple ? generated by one element ? to $C^*(\Gamma)$ having Hausdorff spectrum ? (These lists can be extended at will.)

4.2. *Exactness.*

The C^*-algebra A is said to be **exact** if, given any short exact sequence

$$0 \to J \to B \to B/J \to 0$$

of C^*-algebras, the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is also exact, where $\otimes$ denotes the minimal (or spatial) tensor product. For a proof of the following result, we refer to [Was1], [Ki1], [Ki2], [Ki3] and [HRV2].

Theorem 5 (S. Wassermann, Kirchberg and others).

(i) The algebra $C^*(F_2)$ is not exact.

(ii) Let Γ be a group; assume that Γ is isomorphic to a subgroup of some locally compact group G such that $C^*(G)$ is a nuclear C^*-algebra. Then $C^*(\Gamma)$ is exact if and only if Γ is amenable.

Recall that the C^*-algebra $C^*(G)$ of a locally compact group G is nuclear as soon as G is almost connected [Co76].

In sharp contrast, there is no known example of a group Γ such that $C^*_\lambda(\Gamma)$ is not exact. For example, let Γ be a group and assume that Γ embeds as a discrete subgroup in some second countable locally compact group G having a closed amenable subgroup P with G/P compact (a connected real Lie group G would do); then $C^*_\lambda(\Gamma)$ embeds in the nuclear C^*-algebra $\mathcal{C}(G/P) \rtimes_r \Gamma$, and in particular $C^*_\lambda(\Gamma)$ is exact. (This is an unpublished result of A. Connes which was circulating in the 1980 Kingston's Conference.) Also $C^*_\lambda(\Gamma)$ is exact for any hyperbolic group Γ (unpublished result of Hilsum-Renault-Skandalis).

Problem 14. *(i) If Γ is any group (not necessarily isomorphic to a subgroup of a C^*-nuclear locally compact group) such that $C^*(\Gamma)$ is exact, does it follow that Γ is amenable ?*

(ii) Does there exist a group Γ such that $C^*_\lambda(\Gamma)$ is not exact ?

Question (ii) is the open problem (P1) of [Ki3].

4.3. *Non-existence of idempotents.*

Given a torsionfree group Γ, it is an old question to know whether $\mathbb{C}[\Gamma]$ may have zero divisors, and in particular idempotents distinct from 0 and 1. This is often attributed to Kaplansky : see [Kou, Problem I.3], and also [Far]. The oldest result I know on this is that of Higman [Hig, particular case of Theorem 12].

Theorem 6 (Higman). *The algebra $\mathbb{C}[F_n]$ has no zero divisor.*

Here is a more recent result, which is a particular case of [Fo1, Theorem 9] and [Bas, § 9]: if Γ is a torsionfree finitely generated linear group, then $\mathbb{C}[\Gamma]$ has no idempotent distinct from 0 and 1.

A C^*-algebra distinct from $\mathbb{C}$ has always zero divisors (this is easy to check via functional calculus) but it is a conjecture going back to Kadison and Kaplansky that

$$C^*_\lambda(\Gamma) \text{ has no idempotent, except } 0 \text{ and } 1$$

for any torsionfree group Γ. This would follow from a more general conjecture of P. Baum and A. Connes [BaC] which involves the K-theory groups $K_i(C^*_\lambda(\Gamma))$. For all this, see the discussion in [Va2].

Theorem 7 (Pimsner-Voiculescu). *For each $n \geq 2$, the C^*-algebra $C^*_\lambda(F_n)$ has no idempotent distinct from 0 and 1.*

Theorem 7 has first appeared in [PiV]. There is a very nice proof of it in [Co86, Section I.1], in terms of a Fredholm module over $C^*_\lambda(F_2)$ associated to the standard action of F_2 on the homogeneous tree of degree 4. (Theorem 7 for F_n follows from the result for F_2 because F_n is a subgroup of F_2.) Conne's proof is so nice that minor variations of it have appeared in semi-popularization journals [Ef2]. There is another proof by Cuntz [Cu2], using the easy result that $C^*(F_2)$ has no idempotent [Cho], [Cu1].

It has been shown that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1 for Γ a torsionfree discrete subgroup in a connected Lie group whose semi-simple part is locally isomorphic to a product of compact groups, of Lorentz groups $SO(n,1)$ [Kas], and of groups $SU(n,1)$ [JuK]. The published proofs use KK-theory.

The following problem suggests another approach which could work for more groups. It appears in [Co90, section 2.5, problème 11]. It is open even if G is one of the groups $SO(n,1)$ or $SU(n,1)$ dealt with by Kasparov and Julg. This has been explained to me by P. Julg and A. Valette.

Problem 15. *Let Γ be a torsionfree subgroup of a connected semi-simple real Lie group G. Show that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1 by analyzing the appropriate Fredholm module and its Chern character.*

The same problem holds for an arbitrary torsionfree hyperbolic group.

We refer to [Co90] for explanations about the "appropriate" Fredholm module; see also page 77 of the same reference, and [Co93, Section IV.3].

Let us mention that the following algebras have also been shown to be without non trivial idempotent

$C^*_\lambda(\Gamma)$ for Γ abelian torsionfree (this is Pontryagin Theory [Va2, Theorem 2]),

$C^*_\lambda(\Gamma)$ for Γ locally nilpotent torsion free [KaT] (see also [Ji1, Theorem 5.1]),

$C^*_\lambda(\Gamma)$ for Γ a discrete subgroup of a connected simply connected solvable group [BaC],

$C^*_\lambda(\Gamma)$ for various groups Γ acting on trees as in [Pim],

$l^1(\Gamma)$ for Γ torsionfree hyperbolic [Ji2, Theorem 4.2],

$C^*(\Gamma)$ for a group Γ which is free [Coh1], [Cho], or a free product of torsionfree abelian groups [Cu1], or a free product of torsionfree amenable groups (and a few other cases) [JiP].

Note that $C^*(\Gamma)$ *does* have non trivial projections if Γ has Property (T) by [Va1]; see also [Va5].

These no idempotent results have applications on the structure of various spectra : one appears in [Sun]; another one is the observation (suggested to me by L. Guillopé) following Theorem 8 below. Let Γ be a group given together with a symmetric probability measure, namely with a function $p : \Gamma \to [0,1]$ in $l^1(\Gamma)$ such that $p(\gamma^{-1}) = p(\gamma)$ for all $\gamma \in \Gamma$ and such that $\sum_{\gamma\in\Gamma} p(\gamma) = 1$. To avoid trivialities, assume moreover that the support of p generates Γ. The **Markov operator** of the associated random walk on Γ is the operator $M(p) : l^2(\Gamma) \to l^2(\Gamma)$ of convolution to the right $\xi \mapsto \xi \star p$. It is obvious that $M(p)$ is self-adjoint and that $||M(p)|| \leq 1$, it is easy to check that $||M(p)|| = max\{\lambda \in \mathbb{R} \, : \, \lambda \text{ is in the spectrum of } M(p)\}$ (see [HRV1, Lemma 8]), and it is a result of H. Kesten that $||M(p)|| = 1$ if and only if Γ is amenable [Ke2]. For p equidistributed on a symmetric set S of generators of Γ, this has been reformulated in terms of "cogrowth" by Grigorchuck and Cohen (see [Coh2], [Sz2], [Woe]).

Theorem 8 (Kesten). *Let Γ be a group generated by a finite set $S = \{s_1, ..., s_n\}$ with $n \geq 2$. Let $p : \Gamma \to [0,1]$ be defined by $p(\gamma) = \frac{1}{|S\cup S^{-1}|}$ if $\gamma \in S \cup S^{-1}$ and $p(\gamma) = 0$ otherwise. Then :*

(i) one has $||M(p)|| \geq \frac{1}{n}\sqrt{2n-1}$,

(ii) Γ *is free on* S *if and only if* $||M(p)|| = \frac{1}{n}\sqrt{2n-1}$,
(iii) if Γ *is free on* S, *then the spectrum of* $M(p)$ *is the interval* $[-\frac{1}{n}\sqrt{2n-1}\ ,\ \frac{1}{n}\sqrt{2n-1}]$.

Suppose now moreover that Γ is such that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1, and let $p : \Gamma \to [0,1]$ be a symmetric probability measure as above. It is an immediate corollary of functional calculus that

the spectrum of $M(p)$ is an interval

as it is the case in Theorem 8.iii. About the following problem, see [KaV] and [HRV1].

Problem 16. *Compute the spectrum and the multiplicity function of* $M(p)$ *for other pairs* (Γ, p), *for example when* Γ *is a Fuchsian group; compute also the spectral measure of* $M(p)$.

It is easy to check that the spectrum of $M(p)$ is finite if and only if the group Γ is finite [HRV3, Section 2.1]. One may also consider non-symmetric probability measures; it is then an open problem to know whether there exists a pair (Γ, p) with Γ infinite and the spectrum of $M(p)$ finite [HRV3].

Of course, group C^*-algebras in general do have projections. For an analysis of the case of $C^*_\lambda(\mathbb{Z}/n\mathbb{Z} \star \mathbb{Z}/m\mathbb{Z})$, see [ABH].

5. *RAPIDLY DECREASING FUNCTIONS ON A FINITELY GENERATED GROUP.*

Let Γ be a group given together with a length function $L : \Gamma \to \mathbb{R}_+$. For simplicity, we will moreover assume here that Γ is generated by a finite set S and that, for each $\gamma \in \Gamma$, the length $L(\gamma)$ is the smallest integer n such that $\gamma = s_1 ... s_n$ with $s_1, ..., s_n \in S \cup S^{-1}$. For each $s \in \mathbb{R}$, define the Sobolev space

$$H^s(\Gamma) = \left\{ \xi : \Gamma \to \mathbb{C} \;\middle|\; \sum_{\gamma \in \Gamma} |\xi(\gamma)|^2 (1 + L(\gamma))^{2s} \ < \ \infty \right\}$$

which is a Hilbert space for the obvious scalar product. The space of **rapidly decreasing functions** on Γ is the Fréchet space

$$H^\infty(\Gamma) = \bigcap_{s \in \mathbb{R}} H^s(\Gamma) \ .$$

It is easy to show that the isomorphism classes of these spaces do not depend on the choice of the finite generating set S.

By definition, $H^\infty(\Gamma)$ is a subspace of $l^2(\Gamma)$. But $H^\infty(\Gamma)$ need not be a convolution algebra. (There may exist $\xi, \eta \in H^\infty(\Gamma)$ such that the convolution $\xi \star \eta$, which is always well defined and in $c_0(\Gamma)$, is not in $H^\infty(\Gamma)$: this happens for example if Γ is amenable and not of polynomial growth [Jo1, Proposition B].)

Proposition 3. *Let Γ be a finitely generated group.*

(i) The space $H^\infty(\Gamma)$ is a subspace of $C^*_\lambda(\Gamma)$ if and only if it is a subalgebra of $C^*_\lambda(\Gamma)$.

(ii) If the conditions of (i) hold, then the inclusion $H^\infty(\Gamma) \subset C^*_\lambda(\Gamma)$ induces an isomophism in K-theory.

We refer to [Jo1] and [Jo2] for the proof. Claim (ii) is due to A. Connes, and is an important step in one application of these ideas to differential topology [CoM]. A finitely generated group is said to have **Property (RD)** if it satisfies the conditions of Claim (i). The following appears in [Haa1, Lemma 1.5]; see also [CaH], and the exposition in [Haa2].

Theorem 9 (Haagerup). *Let n be an integer such that $2 \leq n < \infty$; then $H^2(F_n) \subset C^*_\lambda(F_n)$, and in particular $H^\infty(F_n) \subset C^*_\lambda(F_n)$.*

Various groups have been shown to have Property (RD), and it is in particular the case for groups of polynomial growth, for which $H^s(\Gamma) \subset C^*_\lambda(\Gamma)$ for some s depending on the growth, and for hyperbolic groups, for which $H^2(\Gamma) \subset C^*_\lambda(\Gamma)$ (see mainly [Jo1], and also [JoV] and [Har4]). The known proofs are quite different for these two classes, and it is an interesting open problem to find an argument covering both $\mathbb{Z}^n$ (say) and hyperbolic groups. More generally, we formulate the following question (even though the hope for a positive answer is very small).

Problem 17. *Let Γ be a lattice in $G = PSL_n(\mathbb{R})$ such that G/Γ is compact (and $n \geq 3$); does Γ have Property (RD) ?*

The cocompactness hypothesis is crucial, because $PSL_n(\mathbb{Z})$ does not have Property (RD) as soon as $n \geq 3$ [Jo1, Corollary 3.1.9]. The question of Problem 17 is also open for most Coxeter groups.

Other rapid decay algebras have been introduced by Ogle [Ogl].

APPENDIX 1. *ON FREE GROUP.*

It is almost a tautology to say that free groups play a central role in combinatorial group theory, but this should not conceal the important role of free groups in other parts of mathematics. As an example of an old appearance of free groups in traditional subjects, we may quote Schottky groups, which are free subgroups of $PSL_2(\mathbb{C})$ [Kle, page 200].

Here are three criteria for recognizing free groups. For the first one, a convenient reference is the recent book of Serre [Ser, § I.3], though the result itself is quite old : it appears for example in [Rei, Section 4.20], but "it is, of course, very difficult to claim that something is *not* due to Poincaré" [ChM, page 96].

Criterium 1. *A group which operates freely on a tree is a free group.*

Let us also recall the result of Stallings and Swan according to which a torsionfree group which has a free subgroup of finite index is itself a free group [Swa].

We state now the "Table Tennis Lemma", essentially due to F. Klein: see [Mac], [Tit] and [Har1].

Criterium 2. *Let G be a group acting on a set X, let Γ_1, Γ_2 be two subgroups of G and let X_1, X_2 be two subsets of X; assume that $|\Gamma_2| \geq 3$. Assume that*

$$\begin{aligned} \gamma(X_2) \subset X_1 \quad \text{for all} \quad \gamma \in \Gamma_1 \ , \ \gamma \neq 1 \ , \\ \gamma(X_1) \subset X_2 \quad \text{for all} \quad \gamma \in \Gamma_2 \ , \ \gamma \neq 1 \ . \end{aligned}$$

Then the subgroup Γ of G generated by Γ_1 and Γ_2 is isomorphic to the free product $\Gamma_1 \star \Gamma_2$ of Γ_1 and Γ_2. (In particular, if Γ_1 and Γ_2 are free of rank n_1 and n_2 respectively, then Γ is free of rank $n = n_1 + n_2$.)

The next "quasi-geodesic criterium" is due to Gromov [Gr1, 7.2.C]; see also [GhH, § 5.3], or the much shorter proof in [Del].

Criterium 3. *Let Γ be a δ-hyperbolic group and let $\gamma_1, \gamma_2 \in \Gamma$ be such that the word-length relations*

$$\begin{aligned} |\gamma_j^2| \geq |\gamma_j| + 2\delta + 1 \quad & \text{for} \quad j \in \{1, 2\} \\ |\gamma_1^\epsilon \gamma_2^\eta| \geq \max(|\gamma_1|, |\gamma_2|) + 2\delta + 1 \quad & \text{for} \quad \epsilon, \eta \in \{1, -1\} \end{aligned}$$

hold. Then the subgroup of Γ generated by γ_1, γ_2 is free of rank 2.

This criterium is related to the following fact (see [Gr1, 5.3.B] and [Del]). Let Γ be a hyperbolic group which is torsionfree and non elementary. Then there exists a finite sequence $\Gamma_1, ..., \Gamma_k$ of subgroups of Γ such that any pair of elements of Γ generates a subgroup which is either free of rank two or conjugated to one of the Γ_j's.

For other examples of free subgroups of geometrically significant groups, see among many others [BeL, appendice] [Bo2], [DeS], [Gla], [Har1], [Hau], [MyW] and [Wag].

APPENDIX 2. *PROOFS OF THEOREM 3 AND PROPOSITION 2(e).*

The litterature contains a large number of proofs of Theorem 3 (see e.g. the references quoted before Proposition 2). On one hand, most of these proofs are minor variations of Powers' original proof. On the other hand however, each proof extends to some other groups than free groups. The following proof is convenient for the discussion below of some research activity on this subject between 1975 (Powers' paper) and now (see also [BCH]). Our first lemma is straightforward.

Lemma 1. *Let $X_1, ..., X_k$ be a finite sequence of operators on a Hilbert space such that the image subspaces $Im(X_1), ..., Im(X_k)$ are pairwise orthogonal. Then*

$$||X_1 + ... + X_k|| \leq \sqrt{k} \max_{1 \leq j \leq k} ||X_j|| \ .$$

Proof of Theorem 3.

Write Γ for F_n.

Step one. Let τ_c denote the canonical trace on $C^*_\lambda(\Gamma)$. Let $\mathcal{J}$ be a non-zero two-sided ideal in $C^*_\lambda(\Gamma)$ and choose $V \in \mathcal{J}$, $V \neq 0$. Upon multiplying V by some $z \in \mathbb{C}$, $z \neq 0$ and by some $\gamma \in \Gamma$, we may assume that $V = 1 + W$ where $\tau_c(W) = 0$. We shall show that $\mathcal{J}$ contains a sum of conjugates of V which is an invertible element, and that any trace on $C^*_\lambda(\Gamma)$ vanishes on W.

Let ϵ be a real number such that $0 < \epsilon \leq \frac{1}{3}$. As $\lambda_\Gamma(\mathbb{C}[\Gamma])$ is dense in $C^*_\lambda(\Gamma)$, we may choose $X \in \lambda_\Gamma(\mathbb{C}[\Gamma])$ such that $||X - W|| \leq \epsilon$ and $\tau_c(X) = 0$. We may write $X = \sum_{x\in F} z_x x$, where F is a finite subset of $\Gamma - \{1\}$ and where the z_x 's are complex numbers.

Step two. Choose a system $\{s_1, ..., s_n\}$ of free generators of Γ. For a large enough number m, the reduced words $s_1^m x s_1^{-m}$ begin and end with a non-zero power of s_1 for all $x \in F$ (this is [Pow, Lemma 4]). Let C be the subset of Γ of reduced words which begin by s_1^{-m} (followed by a non-zero power of some s_j , $j \neq 1$, or by nothing at all) and set $D = \Gamma - C$. Then one has $xC \cap C = \emptyset$ for all $x \in F$. For each integer $j \geq 1$, set $\gamma_j = s_2^j s_1^m$; one has $\gamma_i D \cap \gamma_j D = \emptyset$ whenever $i \neq j$.

Step three. Choose an integer $k \geq 1$. For each $j \in \{1, ..., k\}$, let P_j denote the orthogonal projection of $l^2(\Gamma)$ onto the subspace $l^2(\gamma_j D)$ of functions $\Gamma \to \mathbb{C}$ with supports inside $\gamma_j D$. As $xC \cap C = \emptyset$ for all $x \in F$, one has $(1-P_j)\gamma_j X \gamma_j^{-1}(1-P_j) = 0$. Thus

$$\gamma_j X \gamma_j^{-1} = P_j X_j' + (P_j X_j'')^* \quad \text{with} \quad \begin{cases} X_j' &= \gamma_j X \gamma_j^{-1} \\ X_j'' &= \gamma_j X^* \gamma_j^{-1}(1 - P_j) \end{cases}$$

for each $j \in \{1, ..., k\}$. Set $Y = \frac{1}{k}\sum_{1\leq j\leq k} \gamma_j X \gamma_j^{-1}$. As $\gamma_i D \cap \gamma_j D = \emptyset$, Lemma 1 implies

$$||Y|| \leq \frac{1}{k} \| \sum_{1\leq j\leq k} P_j X_j' \| + \frac{1}{k} \| \sum_{1\leq j\leq k} P_j X_j'' \| \leq \frac{2}{\sqrt{k}} \| X \| .$$

For k large enough, one has consequently $||Y|| \leq \frac{1}{3}$ and

$$\| \frac{1}{k} \sum_{1\leq j\leq k} \gamma_j W \gamma_j^{-1} \| \leq \| W - X \| + \| Y \| \leq \frac{2}{3} < 1 .$$

It follows that $\frac{1}{k}\sum_{1\leq j\leq k} \gamma_j V \gamma_j^{-1} = 1 + \frac{1}{k}\sum_{1\leq j\leq k} \gamma_j W \gamma_j^{-1}$ is invertible. As this element is obviously in $\mathcal{J}$, one has $\mathcal{J} = C^*_\lambda(\Gamma)$. Thus $C^*_\lambda(\Gamma)$ is a simple C^*-algebra.

Step four. Let τ be any normalized trace on $C^*_\lambda(\Gamma)$. One has

$$|\tau(W) - \tau(X)| \leq \| W - X \| \leq \epsilon$$

and $|\tau(X)| = |\tau(Y)| \leq \frac{2}{\sqrt{k}} \| X \|$. As this holds for all $\epsilon > 0$ and for all $k \geq 1$, one has $\tau(W) = 0$. Thus $\tau = \tau_c$, and $C^*_\lambda(\Gamma)$ has a unique trace.

One strategy of proof of Proposition 2 is to extend the validity of the previous proof.

A first attempt has been do define a **Powers group** as a group Γ which possesses the following property :

for each finite subset $F \subset \Gamma - \{1\}$ and for each integer $k \geq 1$
there exist a partition $\Gamma = C \coprod D$ and elements $\gamma_1, ..., \gamma_k$ in Γ
such that $xC \cap C = \emptyset$ for all $x \in F$ and $\gamma_i D \cap \gamma_j D = \emptyset$ for all $i \neq j$ in $\{1, ..., k\}$.

Step two in the previous proof shows precisely that F_n is a Powers group.

Lemma 2. *If Γ is a Powers group, the C^*-algebra $C^*_\lambda(\Gamma)$ is simple with unique trace.*

Proof. see steps one, three and four in the previous proof.

The conclusion of Lemma 2 can be proved with weaker hypothesis. For example, in the definition of "Powers group", one could replace "for all finite subset $F \subset \Gamma - \{1\}$" by "for all finite subset F inside a conjugacy class distinct from $\{1\}$", so that the lemma applies to the so-called "weak Powers groups". (And direct products such as $F_2 \times F_2$ are weak Powers groups which are not Powers groups; see [BN1, Proposition 1.4] and [Pro, Proposition 3.2].) Or one may consider reduced crossed products $A \rtimes_r \Gamma$ where A is a C^*-algebra with unit which does not have any non trivial Γ-invariant ideal nor any Γ-invariant trace [HS2]. Or one may also cope with twisted reduced crossed products $A \rtimes_{c,r} \Gamma$, where $c : \Gamma \times \Gamma \to \mathcal{U}$ is a 2-cocycle with values in the unitary group of the centre of A [BN2], or even in the unitary group of A itself [Be1], [Be2].

Now comes (at least) some geometry. Let Γ be a group acting by homeomorphisms on a compact topological space Ω. Say the action is **strongly faithful** if, for every finite subset $F \subset \Gamma - \{1\}$, there exists $\omega_0 \in \Omega$ such that $x\omega_0 \neq \omega_0$ for all $x \in F$. Recall that the action is **minimal** if every orbit $\Gamma\omega$ is dense in Ω. Say that $\gamma \in \Gamma$ is **hyperbolic** if there exist two fixed points s_γ, r_γ with the following properties : given neighbourhoods S_γ of s_γ and R_γ of r_γ in Ω, there exists an integer $k \geq 1$ such that

$$\gamma^l(\Omega - S_\gamma) \subset R_\gamma \quad \text{and} \quad \gamma^{-l}(\Omega - R_\gamma) \subset S_\gamma$$

for all integers $l \geq k$. Two hyperbolic elements $\gamma, \gamma' \in \Gamma$ are **transverse** if the four points s_γ , r_γ , $s_{\gamma'}$, $r_{\gamma'}$ are distinct. Say finally that the action of Γ on Ω is **strongly hyperbolic** if, for each integer $k \geq 1$, there exist pairwise transverse hyperbolic elements $\gamma_1, ..., \gamma_k$ in Γ.

Lemma 3. *Let Γ be a group acting by homeomorphisms on a compact space Ω. Assume that the action is strongly faithful, minimal and strongly hyperbolic. Then Γ is a Powers group.*

Proof. see [Har2, Lemma 4].

PROOF OF PROPOSITION 2(e). It follows easily from the three previous lemmas. See [Har2] and [Har4] for more details.

Similar arguments may be used to prove Proposition 2(a), or 2(b) in case the ambient Lie group has real rank 1. It is probably possible to extend this proof to any hyperbolic group Γ with virtual centre Γ_f reduced to $\{1\}$. For this it should be checked that the action of such a group Γ on its Gromov boundary $\partial\Gamma$ is strongly faithful.

However it does not seem possible to extend the proof above to cover, for example, groups such as $PSL_n(\mathbb{Z})$ and $PSL_n(\mathbb{R})$ when $n \geq 3$. (There is an argument in [Ha2] for $n = 3$, but it does not work when $n \geq 4$.) This suggest the following problem.

Problem 18. *Given a group Γ and an element $\gamma \in \Gamma$ of infinite order, describe obstructions to the existence of a compact space Ω on which Γ acts by homeomorphisms in such a way that γ is hyperbolic. Particular problem : $\Gamma = PSL_n(\mathbb{Z})$ with $n \geq 3$. (These obstructions vanish if Γ is hyperbolic; see e.g.* [GhH, § 8.2].*)*

A second attempt to extend the validity of the proof of Theorem 3 has been in terms of the following notions. Say that a group Γ is **naively permissive** if, for any finite subset $F \subset \Gamma - \{1\}$, there exists an element $y \in \Gamma$ of infinite order such that the canonical morphism $< x, y > \to < x > \star < y >$ is an isomorphism for each $x \in F$ (where $< x, y >$ [respectively $< x >, < y >$] denotes the subgroup of Γ generated by $\{x, y\}$ [resp. x, y]). One may observe that Lemma 6.3.2 or [MNIV] says that a naively permissive group is an icc group. We leave it to the reader to check that a torsionfree non elementary hyperbolic group is naively permissive.

One may show on one hand that $C^*_\lambda(\Gamma)$ has a unique normalized trace and that it is a simple C^*-algebra if Γ is naively permissive. One may show on the other hand that some of the groups of Proposition 2 are naively permissive [BCH]. However the following is still open.

Problem 19. *If Γ is as in Proposition 3, is Γ naively permissive ? In particular, is $PSL_n(\mathbb{Z})$ naively permissive for all $n \geq 2$?*

A third attempt to extend the validity of the proof of Theorem 3 is in term of other *permissive* properties of groups. For this and for the proof of Proposition 2, we refer to [BCH].

Finally, in connection with Section 3.4, we state and prove the following, due to M. Bekka.

Proposition 4. *Let G be a non compact simple connected real Lie group without centre, let ρ be an irreducible representation of G distinct from the trivial representation of G in $\mathbb{C}$, let Γ be a lattice in G and let $\rho|\Gamma$ denote the restriction of ρ to Γ. Then $C^*_{\rho|\Gamma}(\Gamma)$ does not have any non trivial two sided ideal of finite codimension.*

Proof. Let π be a representation of Γ such that $C^*_\pi(\Gamma)$ has a non trivial two-sided ideal $\mathcal{J}$ of finite codimension. The closure of $\mathcal{J}$ is non trivial (because $C^*_\pi(\Gamma)$ has a unit) and self-adjoint [DC*, proposition 1.8.2]; denote by A the C^*-algebra quotient $C^*_\pi(\Gamma)/\overline{\mathcal{J}}$. There exists an integer $n \geq 1$ and a quotient of A isomorphic to $M_n(\mathbb{C})$. The resulting C^*-morphism $C^*(\Gamma) \to M_n(\mathbb{C})$ defines a finite dimensional representation $\sigma : \Gamma \to U(n)$ which is weakly contained in π, and we write this $\sigma \prec \pi$.

Let now ρ be as in the statement to be proved. There exists a real number p such that ρ is of class L^p. (See [Cw2, théorème 2.5.2, lemmes 2.2.5 et 3.1.2] if the Lie group is of real rank 1; see [Cw2, théorème 2.4.2] if the Lie group is of real rank at least 2.) This implies that there exists an integer $k \geq 1$ such that the tensor product $\rho^{\otimes k}$ is weakly contained in the regular representation λ_G of G. (See [Cw1], [CHH] and [How, pages 288 and 285].)

Suppose now ab absurdo that $\pi = \rho|\Gamma$. Then

$$\sigma^{\otimes k} \prec \pi^{\otimes k} = \left(\rho^{\otimes k}\right)|\Gamma \prec \lambda_G|\Gamma \prec \lambda_\Gamma$$

where the last weak containement $\lambda_G|\Gamma \prec \lambda_\Gamma$ follows from [DC*, proposition 18.3.5]. As $\sigma^{\otimes k}$ is finite dimensional, this implies that Γ is amenable, which is absurd.

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A problem on the II_1-factors of Fuchsian groups

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A PROBLEM ON THE II_1-FACTORS OF FUCHSIAN GROUPS

P. DE LA HARPE AND D. VOICULESCU

Unexplained notations are as in the previous paper [Har].

Let $G = PSL_2(\mathbb{R})$ be viewed as the group of orientation preserving isometries of $\mathcal{P}$, the Poincaré half-plane (= the connected simply connected complete Riemannian manifold of dimension 2 and constant curvature -1). Consider in G a discrete subgroup Γ (= a Fuchsian group) which is finitely generated and not elementary (namely neither a finite group nor a finite extension of $\mathbb{Z}$). If Γ_1, Γ_2 are two such groups which are isomorphic as abstract groups, then it is known that their covolumes (in the sense defined below) are equal. Very briefly, the question we ask is : does one have $W^*_\lambda(\Gamma) \approx L(F_r)$ for the appropriate r ? Let us recall some classical facts, precise the meaning of "the appropriate", and list cases where the answer to the previous question is known to be positive.

The group Γ is described by the following data : integers $g, q, s, t \geq 0$ and integers $\nu_1, ..., \nu_q \geq 2$, submitted to the unique condition

$$\frac{1}{2\pi} Cov(\Gamma) \doteq 2g - 2 + s + t + \sum_{j=1}^{q} \left(1 - \frac{1}{\nu_j}\right) \ > \ 0 \ .$$

The data are summarized in the *signature* $(g : \nu_1, ..., \nu_q; s, t)$ of the group, written also $(g; s, t)$ if $q = 0$. In case Γ is a lattice in G (a lattice is automatically finitely generated and non elementary), then $t = 0$ and Γ is *completely* described by its signature, up to conjugation by an orientation preserving quasiconformal homeomorphism of $\mathcal{P}$.

Algebraically, the group Γ has a presentation with $2g + q + s + t$ generators

$$a_1, b_1, ..., a_g, b_g, e_1, ..., e_q, p_1, ..., p_s, h_1, ..., h_t$$

and $1 + q$ relations

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{q} e_j \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1 \ ,$$
$$e_j^{\nu_j} = 1 \quad (1 \leq j \leq q).$$

There are precisely q classes of maximal finite cyclic subgroups of Γ, each of these classes containing a group generated by one of the e_j 's. (In particular, Γ is torsionfree if and only if $q = 0$.) Each parabolic element of Γ is conjugated to exactly one power of exactly one of the p_j 's.

It is easy to check that the conjugacy classes of Γ distinct from $\{1\}$ are all infinite, so that the von Neumann algebra $W^*_\lambda(\Gamma)$ is a II_1-factor. Moreover, it is a full factor (see [Har]).

Geometrically, the group Γ has a limit set $\mathcal{L}_\Gamma$ in the boundary $\partial\mathcal{P}$ of $\mathcal{P}$. (The space $\mathcal{P} \cup \partial\mathcal{P}$ is naturally homeomorphic to a closed 2-disc, and $\partial\mathcal{P} = \mathbb{R} \cup \{\infty\} \approx \mathbb{S}^1$ if $\mathcal{P} = \{z \in \mathbb{C} | Im(z) > 0\}$.) The *Nielsen region* $\overline{N}$ is the hyperbolic convex hull of $\mathcal{L}_\Gamma$ in $\mathcal{P} \cup \partial\mathcal{P}$. The inside part $N = \overline{N} \cap \mathcal{P}$ is a Γ-invariant closed subset of $\mathcal{P}$. The quotient N/Γ is a Riemann surface obtained from a compact surface of genus g with t connected components in its boundary by removing s inner points. Moreover there are distinct points $x_1, ..., x_q$ in the interior of N/Γ such that the covering $N \to N/\Gamma$ has ramification of order ν_j over x_j (and no other ramification point). The number $Cov(\Gamma)$ defined above is then the hyperbolic area of N/Γ, and one has $Cov(\Gamma) = -2\pi\chi(N/\Gamma)$, where χ denotes the appropriate Euler-Poincaré characteristics, in the sense of orbifolds. (This is a consequence of Gauss-Bonet formula, and this explains also why $Cov(\Gamma_1) = Cov(\Gamma_2)$ if Γ_1, Γ_2 are isomorphic as abstract groups.) There is also a purely algebraic way to define the Euler-Poincaré characteristics of the virtually torsion-free group Γ, described in [Ser]. (This is straightforward if G/Γ is not compact, because then Γ contains a free subgroup of finite index; if G/Γ is compact, see n^o 3.2 in [Ser].)

Observe that N/Γ is compact if and only if $s = 0$. Observe also that, for Fuchsian groups which are finitely generated (as discussed here), the four following are equivalent : $N = \mathcal{P}$, the group Γ is a lattice in G (namely the space $\mathcal{P}/\Gamma$ is of finite aerea), the Fuchsian group Γ is "of the first kind" (namely $\mathcal{L}_\Gamma = \partial\mathcal{P}$), $t = 0$ in the signature of Γ.

Here are a few examples of signatures and related groups:

(1) $(g; 0, 0) \Longrightarrow \Gamma \approx \Pi_1(\Sigma_g)$, where Σ_g denotes a closed surface of genus g,

(2) $(g; s, t)$ with $s + t > 0 \Longrightarrow \Gamma$ is isomorphic to the free group $F_{2g+s+t-1}$,

(3) $(0 : 2, \nu; 1, 0)$ with $\nu \geq 3 \Longrightarrow \Gamma \approx (\mathbb{Z}/\nu\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z})$ is one of the *Hecke groups*,

(4) $(g : \nu_1, ..., \nu_q; s, t)$ with $s + t > 0 \Longrightarrow \Gamma$ is a free product of cyclic groups $F_n \star (\mathrm{F}_{1 \leq j \leq q}(\mathbb{Z}/\nu_j\mathbb{Z}))$, with $n = 2g + s + t - 1$.

Of course, (4) generalizes both (2) and (3). For all this, see e.g. [Bea] and [Gre].

The following is a wild formulation of the problem to understand how far the factors W^*_λ(Fuchsian groups) are from the W^*_λ(free groups). The factors $L(F_r)$ are defined by K. Dykema [Dy1] and F. Radulescu [Rad] for all real numbers r such that $r > 1$ and for $r = \infty$.

Problem. *Let $\Gamma \subset PSL_2(\mathbb{R})$ be a finitely generated Fuchsian group as above, and set $r = 1 + \frac{1}{2\pi}Cov(\Gamma)$. Does one have $W^*_\lambda(\Gamma) \approx L(F_r)$?*

In the torsion-free case (the group is then either free or the fundamental group of a closed surface), the problem can be rephrased as

$$W^*_\lambda(\Pi_1(\Sigma_g)) \overset{?}{\approx} W^*_\lambda(F_{2g-1}) \, .$$

For the groups of example (4) above with $q = 1$, the conjecture holds by [Voi, Theorem 3.3]. For the Hecke groups of (3), it holds by [Dy2, Corollary 5.3]; more precisely, if $\Gamma \approx (\mathbb{Z}/\nu\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z})$, then $W^*_\lambda(\Gamma) \approx W^*_\lambda\left(F_{3/2-1/\nu}\right)$. The general case of (4) holds by [Dy2, Proposition 2.4].

REMARKS. (i) A factor M of type II_1 with trace τ is said to have the *Haagerup Approximation Property* if the identity on the Hilbert space $L^2(M,\tau)$ can be approximated by compact unital trace-preserving completely positive maps. Let Γ be an infinite conjugacy class group, and assume that there exists a function $\psi : \Gamma \to \mathbb{R}$ which is conditionally of negative type and which tends to infinity at infinity. For each integer $n \geq 1$, the function $\phi_n = exp\left(-\frac{1}{n}\psi\right)$ is of positive type by Schoenberg's Theorem, and thus defines a multiplier on $W^*_\lambda(\Gamma)$ which is completely positive (see e.g. [CaH, Proposition 4.2]). It follows that the factor $W^*_\lambda(\Gamma)$ has the Haagerup Approximation Property. Examples of such groups Γ include non abelian free groups [Haa], various groups acting on trees or real trees, infinite Coxeter groups, and discrete subgroups of Lie groups in the families $SO(1,n)$ and $SU(1,n)$; see e.g. [HaV, chapitres 5, 6].

The problem phrased above is a way of testing how much the class of full II_1-factors which have the Haagerup Approximation Propery is larger than the class of free group factors. A similar problem appears in [Pop].

(ii) As covolumes of lattices in G are related to Murray-von Neumann coupling constants [GHJ, Section 3.3.d], there may be an approach to the problem above using the index of appropriate subfactors.

(iii) It is known that a Fuchsian group Γ which is not finitely generated is a free product of an infinite sequence of cyclic groups. For such a group, one has $W^*_\lambda(\Gamma) \approx W^*_\lambda(F_\infty)$ by [Dy2, Corollary 5.4].

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Exact operator spaces

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EXACT OPERATOR SPACES

Gilles Pisier

Plan

Introduction

In this paper, we study *operator spaces* in the sense of the theory developed recently by Blecher-Paulsen [BP] and Effros-Ruan [ER1]. By an operator space, we mean a closed subspace $E \subset B(H)$, with H Hilbert. In the category of operator spaces, the morphisms are the *completely bounded maps* for which we refer the reader to [Pa1]. Let $E \subset B(H)$, $F \subset B(K)$ be operator spaces (H, K Hilbert). A map u: $E \to F$ is called completely bounded (*c.b.* in short) if

$$\sup_{n\geq 1} \|I_{M_n} \otimes u\|_{M_n(E)\to M_n(F)} < \infty$$

where $M_n(E)$ and $M_n(F)$ are equipped with the norms induced by $B(\ell_2^n(H))$ and $B(\ell_2^n(K))$ respectively. We denote

$$\|u\|_{cb} = \sup_{n\geq 1} \|I_{M_n} \otimes u\|_{M_n(E)\to M_n(F)}.$$

The map u is called a complete isomorphism if it is an isomorphism and if u and u^{-1} are *c.b.*. We say that u: $E \to F$ is a complete isometry if for each $n \geq 1$ the map $I_{M_n} \otimes u$: $M_n(E) \to M_n(F)$ is an isometry. We refer to [Ru, ER2-7, B1, B2] for more information on the rapidly developing Theory of Operator Spaces.

We will be mainly concerned here with the "geometry" of *finite dimensional* operator spaces. In the Banach space category, it is well known that every separable space embeds isometrically into ℓ_∞. Moreover, if E is a finite dimensional normed space then for each $\varepsilon > 0$, there is an integer n and a subspace $F \subset \ell_\infty^n$ which is $(1+\varepsilon)$-isomorphic to E, i.e. there is an isomorphism u: $E \to F$ such that $\|u\| \, \|u^{-1}\| \leq 1+\varepsilon$.

Here of course, n depends on ε, say $n = n(\varepsilon)$ and usually (for instance if $E = \ell_2^k$) we have $n(\varepsilon) \to \infty$ when $\varepsilon \to 0$.

Quite interestingly, it turns out that this fact is not valid in the category of operator spaces: although every operator space embeds completely isometrically into $B(H)$ (the non-commutative analogue of ℓ_∞) it is not true that a finite dimensional operator space must be close to a subspace of M_n (the non-commutative analogue of ℓ_∞^n) for some n. The main object of this paper is to study this phenomenon.

We will see that this phenomenon is very closely related to the remarkable work of E. Kirchberg on exact C^*-algebras. We will show that some of Kirchberg's ideas can be developed in a purely "operator space" setting. Our main result in the first section is Theorem 1, which can be stated as follows.

Let $B = B(\ell_2)$ and let $K \subset B$ be the ideal of all the compact operators on ℓ_2.

If X, Y are operator spaces, we denote by $X \otimes_{\min} Y$ their minimal (or spatial) tensor product. If $X \subset B(H)$ and $Y \subset B(K)$, this is just the completion of the linear tensor product $X \otimes Y$ for the norm induced by $B(H \otimes_2 K)$.

Let $\lambda \geq 1$ be a fixed constant.

The following properties of an operator space X are equivalent:

(i) The sequence

$$\{0\} \to K \otimes_{\min} X \to B \otimes_{\min} X \to (B/K) \otimes_{\min} X \to \{0\}$$

is exact and the map

$$T_X\colon\ (B \otimes_{\min} X)/(K \otimes_{\min} X) \to (B/K) \otimes_{\min} X$$

has an inverse T_X^{-1} with norm $\|T_X^{-1}\| \leq \lambda$.

(ii) for each $\epsilon > 0$ and each finite dimensional subspace $E \subset X$, there is an integer n and a subspace $F \subset M_n$ such that $d_{cb}(E, F) < \lambda + \epsilon$.

Here $d_{cb}(E, F)$ denotes the *c.b.* analogue of the Banach-Mazur distance (see (0) below for a precise definition.) We will denote by $d_{SK}(E)$ the infimum of $d_{cb}(E, F)$ when F runs over all operator spaces F which are subspaces of M_k for some integer k.

One of the main results in section 2 can be stated as follows (see Theorem 7 below).

Consider $F \subset M_k$ with $\dim F = n$ and $k \geq n$ arbitrary, then for any linear isomorphism $u\colon\ \ell_\infty^n \to F^*$ we have

$$\|u\|_{cb}\|u^{-1}\|_{cb} \geq n[2(n-1)^{1/2}]^{-1}.$$

In particular this is > 1 for any $n \geq 3$. Here the space F^* is the dual of F with its "dual operator space structure" as explained in [BP, ER1, B1, B2].

Equivalently, if we denote by E_1^n the operator space dual of ℓ_∞^n, (this is denoted by $\max(\ell_1^n)$ in [BP]) then we have

$$d_{SK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

We also show a similar estimate for the space which is denoted by $R_n + C_n$ in [P1]. Moreover, we show that the n-dimensional operator Hilbert space OH_n (see [P1]) satisfies

$$d_{SK}(OH_n) \geq (\frac{n}{2\sqrt{n-1}})^{1/2}.$$

These estimates are asymptotically sharp in the sense that $d_{SK}(E_1^n)$ and $d_{SK}(R_n+C_n)$ are $O(n^{1/2})$ and $d_{SK}(OH_n)$ is $O(n^{1/4})$ when n goes to infinity.

Later on in the paper, we show that the operator space analogue of the "Banach Mazur compactum" is not compact and we prove various estimates related to that phenomenon. (The noncompactness itself was known, at least to Kirchberg.) We will include several simple facts on ultraproducts of finite dimensional operator spaces which are closely connected to the discussion of "exact" operator spaces presented in section 1. Let us denote by OS_n the set of all n-dimensional operator spaces. We consider that two spaces E, F in OS_n are the same if they are completely isometric. Then the space OS_n is a metric space when equipped with the distance

$$\delta(E, F) = \log d_{cb}(E, F).$$

We include a proof that OS_n is complete but not compact (at least if $n \geq 3$) and we give various related estimates. As pointed out to me by Kirchberg, it seems to be an open problem whether OS_n is a *separable* metric space.[1]

In passing, we recall that in [P1] we proved that $d_{cb}(E, OH_n) \leq n^{1/2}$ for any E in OS_n and therefore that

$$\sup\{d_{cb}(E, F) \mid E, F \in OS_n\} = n.$$

Actually, that supremum is attained on the subset $HOS_n \subset OS_n$ formed of all the *Hilbertian* operator spaces (i.e. those which, as normed spaces, are isometric to the Euclidean space ℓ_2^n). We also show that (at least for $n \geq 3$) HOS_n is a closed but non compact subset of OS_n. Perhaps the subset HOS_n is not even separable.[1] In section 5, we show the following result. Let E be any operator space and let $C \geq 1$ be a constant. Fix an integer $k \geq 1$. Then there is a compact set T and a subspace $F \subset C(T) \otimes_{\min} M_k$ such that $d_{cb}(E, F) \leq C$ iff for any operator space X and any $u \colon X \to E$ we have

$$\|u\|_{cb} \leq C\|u\|_k,$$

where $\|u\|_k = \|u\|_{M_k(X) \to M_k(E)}$.

Notation: Let (E_m) be a sequence of operator spaces. We denote by $\ell_\infty\{E_m\}$ the direct sum in the sense of ℓ_∞ of the family (E_m). As a Banach space, this means that $\ell_\infty\{E_m\}$ is the set of all sequences $x = (x_m)$ with $x_m \in E_m$ for all m with $\sup_m \|x_m\|_{E_m} < \infty$ equipped with the norm $\|x\| = \sup_m \|x_m\|_{E_m}$. The operator space structure on $\ell_\infty\{E_m\}$ is defined by the identity

$$\forall n \quad M_n(\ell_\infty\{E_m\}) = \ell_\infty\{M_n(E_m)\}.$$

[1] See the Note added at the end of this paper.

Equivalently, if $E_m \subset B(H_m)$ (completely isometrically) then $\ell_\infty\{E_m\}$ embeds (completely isometrically) into $B(\oplus_m H_m)$ as block diagonal operators.

We will use several times the observation that if F is an other operator space then $\ell_\infty\{E_m\} \otimes_{\min} F$ embeds completely isometrically in the natural way into $\ell_\infty\{E_m \otimes_{\min} F\}$. In particular, if F is finite dimensional these spaces can be completely isometrically identified.

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§1. Exact operator spaces.

Let E, F be operator spaces. We denote

$$(0) \qquad d_{cb}(E,F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\}$$

where the infimum runs over all isomorphisms $u\colon E \to F$. If E, F are not completely isomorphic we set $d_{cb}(E,F) = \infty$. This is the operator space version of the Banach Mazur distance. We will study the smallest distance of an operator space E to a subspace of the space $K = K(\ell_2)$ of all compact operators on ℓ_2. More precisely, this is defined as follows

$$(1) \qquad d_{SK}(E) = \inf\{d_{cb}(E,F) \mid F \subset K\}.$$

Let F be a finite dimensional subspace of K. By an entirely classical perturbation argument one can check that for each $\varepsilon > 0$ there is an integer n and a subspace $\widetilde{F} \subset M_n$ such that $d_{cb}(F, \widetilde{F}) < 1 + \varepsilon$. It follows that for any finite dimensional operator space E we have

$$(1)' \qquad d_{SK}(E) = \inf\{d_{cb}(E,F) \mid F \subset M_n, \quad n \geq 1\}.$$

In his remarkable work on exact C^*-algebras (cf. [Ki]) Kirchberg introduces a quantity which he denotes by $locfin(E)$ for any operator space E. His definition uses completely positive unit preserving maps. The number $d_{SK}(E)$ appears as the natural "*c.b.*" analogue of Kirchberg's $locfin(E)$. Note that $d_{SK}(E)$ is clearly an invariant of the operator space E and we have obviously

$$(1)'' \qquad d_{SK}(E) \leq d_{SK}(F) d_{cb}(E,F)$$

for all operator spaces E, F.

Let X be an operator space. We will say that X is exact if the sequence

$$(2) \qquad \{0\} \to K \otimes_{\min} X \to B \otimes_{\min} X \to (B/K) \otimes_{\min} X \to \{0\}$$

is exact. In other words, X is an exact operator space if the natural completely contractive map

$$B \otimes_{\min} X \to (B/K) \otimes_{\min} X$$

in onto and its kernel coincides with $K \otimes_{\min} X$.

Note in particular that every finite dimensional space is trivially exact. Following Kirchberg, we will measure the "degree of exactness" of X via the number $ex(X)$ defined as follows: we consider the map

$$T_X\colon (B \otimes_{\min} X)/(K \otimes_{\min} X) \to (B/K) \otimes_{\min} X \tag{3}$$

associated to the exact sequence (2) and we define

$$ex(X) = \|T_X^{-1}\|. \tag{4}$$

Clearly $ex(X)$ is a (completely) isomorphic invariant of X in the following sense: if X and Y are completely isomorphic operator spaces we have

$$ex(X) \leq ex(Y) d_{cb}(X, Y). \tag*{(4)'}$$

The main result of this section is the following which is proved by adapting in a rather natural manner the ideas of Kirchberg [Ki]. One simply needs to substitute everywhere in his argument "completely positive unital" by "completely bounded" and to keep track of the $c.b.$ norms. The resulting proof is very simple.

Theorem 1. *For every finite dimensional operator space E, one has*

$$ex(E) = d_{SK}(E). \tag{5}$$

More generally, for any operator space X

$$ex(X) = \sup\{d_{SK}(E) \mid E \subset X, \dim E < \infty\}, \tag{6}$$

and X is exact iff the right side of (6) is finite.

Remarks. (i) If X is a C^*-algebra then the maps appearing in (2) are (C^*-algebraic) representations. Recall that a representation necessarily has closed range and becomes isometric when we pass to the quotient modulo its kernel (cf. e.g. [Ta, p.22]). Hence if X is a C^*-algebra (2) is exact iff the kernel of the map

$$B \otimes_{\min} X \to (B/K) \otimes_{\min} X$$

coincides with $K \otimes_{\min} X$.

By a known argument, a sufficient condition for this to hold is a certain "slice map" property (cf. [W2]) which is a consequence of the $CBAP$ (see [Kr]). (Recall that X has the $CBAP$ if the identity on X is a pointwise limit of a net of finite rank maps $u_i\colon X \to X$ with $\sup_i \|u_i\|_{cb} < \infty$.)

Thus it is known that the reduced C^*-algebra of the free group $\mathbf{F}_N$ with N generators ($N \geq 2$) is exact, because by [DCH] it has the $CBAP$. On the other hand, it is known ([W1]) that the full C^*-algebras $C^*(\mathbf{F}_N)$ are not exact. This can also be derived from Theorem 7 below after noticing that the space $E_1^n = (\ell_\infty^n)^*$ appearing in Theorem 7 is completely isometric to a subspace of $C^*(\mathbf{F}_n)$. By the same argument (using Corollary 10 below) if an operator space E is completely isometric to the space OH_n introduced in [P1], then the C^*-algebra generated by E is not exact if $n \geq 3$.

(ii) As explained to me by Kirchberg, if X is a C^*-algebra then we have $ex(X) < \infty$ iff $ex(X) = 1$. Indeed since (3) is a representation, it is isometric if it is injective. This shows if a C^*-algebra is completely isomorphic to an exact operator space then it is exact as a C^*-algebra.

We will use the following simple fact.

Lemma 2. *Let E be a separable operator space. There are operators P_n: $E \to M_n$ such that*

(i) $\|P_n\|_{cb} \leq 1$ *for all* n.

(ii) The embedding J: $E \to \ell_\infty\{M_n\}$ *defined by* $J(x) = (P_n(x))_{n\in\mathbb{N}}$ *is a complete isometry.*

(iii) For all $k \leq n$, *there is a map* a_{kn}: $M_n \to M_k$ *with* $\|a_{kn}\|_{cb} \leq 1$ *such that* $P_k = a_{kn}P_n$.

(iv) Assume E finite dimensional. Then for some $n_0 \geq 1$, *the maps* P_n *are injective for all* $n \geq n_0$.

Proof. We can assume $E \subset B(\ell_2)$. Then let Q_n: $B(\ell_2) \to M_n$ be the usual projection (defined by $Q_n(e_{ij}) = e_{ij}$ if $i, j \leq n$ and $Q_n(e_{ij}) = 0$ otherwise). Let $P_n = Q_{n|E}$. Then (i), (ii), (iii) and (iv) are immediate. ∎

The point of the preceding lemma is that we can write for all $N \geq 1$ and all (a_{ij}) in $M_N(E)$

$$\|(a_{ij})\|_{M_N(E)} = \lim_{n\to\infty} \uparrow \|(P_n(a_{ij}))\|_{M_N(M_n)}. \tag{7}$$

Indeed, by (ii) we have

$$\|(a_{ij})\|_{M_N(E)} = \sup_n \|(P_n(a_{ij}))\|_{M_N(M_n)}$$

and by (iii) this supremum is monotone nondecreasing, whence (7).

The following two lemmas are well known to specialists.

Lemma 3. *If X, Y are exact operator spaces and if $X \subset Y$, then $ex(X) \leq ex(Y)$.*

Proof. We will identify $B \otimes_{\min} X$ (resp. $(B/K) \otimes_{\min} X$) with a subspace of $B \otimes_{\min} Y$ (resp. $(B/K) \otimes_{\min} Y$). Consider u in the open unit ball of $(B/K) \otimes_{\min} X$. By definition of $ex(Y)$, there is an element v in $B \otimes_{\min} Y$ such that $\|v\| < ex(Y)$ and if q: $B \otimes_{\min} Y \to (B/K) \otimes_{\min} Y$ is the canonical mapping, we have $q(v) = u$. On the other hand, since X is exact we know there is a $\tilde{u}$ in $B \otimes_{\min} X$ such that $q(\tilde{u}) = u$. Note that by exactness Ker $q = K \otimes_{\min} Y$, hence $v - \tilde{u} \in K \otimes_{\min} Y$. Let p_n be an increasing sequence of finite rank projections in B tending to the identity (in the strong operator topology). Consider the mapping $\sigma_n \colon B \to B$ defined by $\sigma_n(x) = (1 - p_n)x(1 - p_n)$. Clearly $\|\sigma_n\|_{cb} \leq 1$ and for all x in B we have $\sigma_n(x) - x \in K$. Moreover, for all x in K we have $\|\sigma_n(x)\| \to 0$. More generally, by equicontinuity, for any w in $K \otimes_{\min} Y$ we have $\|(\sigma_n \otimes I_Y)(w)\|_{K \otimes_{\min} Y} \to 0$ when $n \to \infty$.

Hence for any $\varepsilon > 0$, for some n large enough we have $\|(\sigma_n \otimes I_Y)(v - \tilde{u})\| < \varepsilon$. Therefore,

$$\|(\sigma_n \otimes I_Y)\tilde{u}\| \leq \|\sigma_n\|_{cb}\|v\| + \varepsilon < ex(Y) + \varepsilon.$$

But on the other hand, $\tilde{u} - (\sigma_n \otimes I_Y)\tilde{u} = ((1 - \sigma_n) \otimes I_Y)\tilde{u} \in K \otimes_{\min} X$ since $\tilde{u} \in B \otimes_{\min} X$. Hence $\operatorname{dist}(\tilde{u}, K \otimes_{\min} X) < ex(Y) + \varepsilon$ and we conclude $ex(X) < ex(Y) + \varepsilon$. ■

Lemma 4. *Let X be an operator space. Then $\sup\{\|T_E^{-1}\| \mid E \subset X, \dim E < \infty\}$ is finite iff X is exact and we have*

$$ex(X) = \sup\{\|T_E^{-1}\| \mid E \subset X, \dim E < \infty\}. \tag{8}$$

Proof. Let λ be the right side of (8). By Lemma 3 we clearly have $\lambda \leq ex(X)$ hence it suffices to show that if λ is finite X is exact and (8) holds. Assume λ finite. Then clearly T_X is onto. Let $q\colon B \otimes_{\min} X \to (B/K) \otimes_{\min} X$ be the natural map. Consider u in $\operatorname{Ker}(q)$. By density there is a sequence u_n in $B \otimes X$ such that $\|u - u_n\| < 2^{-n}$. Then $\|q(u_n)\| < 2^{-n}$. By definition of λ (since $u_n \in B \otimes E_n$ for some finite dimensional subspace $E_n \subset X$ and $\|T_{E_n}^{-1}\| \leq \lambda$) there is v_n in $B \otimes X$ which is a lifting of $q(u_n)$ so that $\|v_n\| < 2^{-n}\lambda$ and $u_n - v_n \in K \otimes X$. Therefore $\|u - (u_n - v_n)\| < 2^{-n} + 2^{-n}\lambda$ so that $u = \lim(u_n - v_n) \in K \otimes X$. This shows that $\operatorname{Ker}(q) = K \otimes_{\min} X$. Thus we have showed that $\lambda < \infty$ implies X exact. By definition of T_X^{-1} it is then easy to check that $\|T_X^{-1}\| \leq \lambda$. ■

Lemma 5. *For any operator space X*

$$ex(X) \leq \sup\{d_{SK}(E) \mid E \subset X, \dim E < \infty\}. \tag{9}$$

Proof. By the preceding lemma it suffices to show that a finite dimensional operator space E satisfies $\|T_E^{-1}\| \leq d_{SK}(E)$.

Now consider $F \subset M_n$. By Lemma 3 and by (4)′ we have

$$\begin{aligned}\|T_E^{-1}\| = ex(E) &\leq ex(F)d_{cb}(E,F)\\ &\leq ex(M_n)d_{cb}(E,F)\end{aligned}$$

but trivially $ex(M_n) = 1$ hence we obtain $\|T_E^{-1}\| \leq d_{cb}(E,F)$ and taking the infimum over F, $\|T_E^{-1}\| \leq d_{SK}(E)$. ■

Proof of Theorem 1. Let $E \subset X$ be finite dimensional. We will prove

$$d_{SK}(E) \leq ex(E).$$

This is the main point. To prove this claim we consider the maps P_n: $E \to M_n$ appearing in Lemma 2. Let $E_n = P_n(E) \subset M_n$. For $n \geq n_0$ we consider the isomorphism u_n: $E \to E_n$ obtained by considering P_n with range E_n instead of M_n. Since E is finite dimensional u_n^{-1} is *c.b.* for each $n \geq n_0$. We claim that we have

$$\limsup_{n\to\infty} \|u_n^{-1}\|_{cb} \leq ex(E). \tag{10}$$

¿From (10) it is easy to complete the proof of Theorem 1. Indeed, if (10) holds, we have

$$d_{SK}(E) \leq \limsup_{n\to\infty} \|u_n\|_{cb}\|u_n^{-1}\|_{cb} \leq ex(E).$$

By (9) we have conversely $ex(E) \leq d_{SK}(E)$, whence (5). Then by (8) X is exact iff the right side of (6) is finite and (6) follows from (8). Thus to conclude it suffices to prove our claim (10).

Consider $\varepsilon_n > 0$ with $\varepsilon_n \to 0$. For each $n \geq n_0$, we choose h_n in $M_{k(n)}(E)$ such that

$$\|(I_{M_{k(n)}} \otimes u_n)h_n\|_{M_{k(n)}(E_n)} = 1 \text{ and } \|h_n\|_{M_{k(n)}(E)} > \|u_n^{-1}\|_{cb} - \varepsilon_n. \tag{11}$$

Then we form the direct sum $B_1 = \ell_\infty\{M_{k(n)}\}$ and consider the corresponding element

$$h = (h_n)_{n\geq n_0} \quad \text{in} \quad B_1 \otimes_{\min} E = \ell_\infty\{M_{k(n)}(E)\}.$$

Let $K_1 \subset B_1$ be the subspace formed of all the sequences (x_n) with $x_n \in M_{k(n)}$ which tend to zero when $n \to \infty$. By suitably embedding B_1 into $B(\ell_2)$ and K_1 into $K(\ell_2)$ we find that the natural map

$$T_1\colon\ (B_1 \otimes_{\min} E)/(K_1 \otimes_{\min} E) \to (B_1/K_1) \otimes_{\min} E$$

satisfies (note that it is an isomorphism since $\dim E < \infty$)

$$\|T_1^{-1}\| \leq \|T_E^{-1}\| = ex(E). \tag{12}$$

Let q: $B_1 \otimes_{\min} E \to (B_1 \otimes_{\min} E)/(K_1 \otimes_{\min} E)$ be the quotient mapping. Observe that we have clearly

$$\limsup_{n\to\infty} \|h_n\| \leq \|q(h)\|. \tag{13}$$

On the other hand we have $q(h) = T_1^{-1}T_1 q(h)$ hence

$$\|q(h)\| \leq \|T_1^{-1}\| \; \|T_1 q(h)\| \tag{14}$$

and since $J\colon E \to \ell_\infty\{E_m\}$ is a complete isometry (cf. Lemma 2) we have

$$\|T_1 q(h)\| = \|(I_{B_1/K_1} \otimes J)T_1 q(h)\|_{(B_1/K_1)\otimes_{\min}\ell_\infty\{E_m\}}. \tag{15}$$

Let $q_1\colon B_1 \otimes_{\min} E \to (B_1/K_1) \otimes_{\min} E$ be the natural map. Clearly $q_1 = T_1 q$, and the right side of (15) is the same as the norm of the corresponding element in the space $\ell_\infty\{(B_1/K_1) \otimes_{\min} E_m\}$, hence the right side of (15) is equal to

$$\sup_m \|(I_{B_1/K_1} \otimes u_m)q_1(h)\|_{(B_1/K_1)\otimes_{\min} E_m}$$

which is clearly

$$\leq \sup_m \limsup_{n\to\infty} \|(I_{M_{k(n)}} \otimes u_m)(h_n)\|_{M_{k(n)}(E_m)}.$$

For $m \leq n$, we have by Lemma 2 $u_m = a_{mn} u_n$ with $\|a_{mn}\|_{cb} \leq 1$. By (11) this implies

$$\|(I_{M_{k(n)}} \otimes u_m)(h_n)\|_{M_{k(n)}(E_m)} \leq \|(I_{M_{k(n)}} \otimes u_n)(h_n)\|_{M_{k(n)}(E_n)} = 1.$$

Hence we conclude that (15) is ≤ 1. By (12), (13) and (14) we obtain that $\limsup_{n\to\infty} \|u_n^{-1}\|_{cb} = \limsup_{n\to\infty} \|h_n\| \leq ex(E)$. This proves (10) and concludes the proof of Theorem 1. ∎

Remark. The reader may have noticed that our definition of exact operator spaces is not the most natural extension of "exactness" in the category of operator spaces. However the more natural notion is easy to describe. Let us say that an operator space X is OS-exact if for any exact sequence of operator spaces (note: here the morphisms are c.b. maps)

$$\{0\} \to Y_1 \to Y_2 \to Y_3 \to \{0\}$$

the sequence

$$\{0\} \to Y_1 \otimes_{\min} X \to Y_2 \otimes_{\min} X \to Y_3 \otimes_{\min} X \to \{0\}$$

is exact.

Then we claim that X is OS-exact iff there is a constant C such that for any finite dimensional subspace $E \subset X$, the inclusion $i_E\colon E \to X$ admits for some n a factorization of the form

$$i_E\colon E \xrightarrow{a} M_n \xrightarrow{b} X$$

with $\|a\|_{cb}\|b\|_{cb} \leq C$.

Equivalently, this means that there is a net (u_i) of finite rank maps on X of the form $u_i = b_i a_i$ with $a_i\colon X \to M_{n_i}$ and $b_i\colon M_{n_i} \to X$ such that

$$\sup \|a_i\|_{cb}\|b_i\|_{cb} < \infty \quad \text{and} \quad u_i(x) \to x$$

for all x in X.

This result was known to E. Kirchberg and G. Vaillant. It can be proved as follows. First if X is OS-exact, it is a fortiori exact in the above sense so that by Theorem 1 there is a constant C_1 such that $d_{SK}(E) \le C_1$ for all finite dimensional subspaces $E \subset X$.

Secondly, if X is OS-exact, there is clearly a constant C_2 such that for any pair of finite dimensional operator spaces $E_1 \subset E_2$ (so that $E_1^{\perp} \subset E_2^*$) we have an isomorphism

$$T\colon\ (E_2^* \otimes_{\min} X)/(E_1^{\perp} \otimes_{\min} X) \longrightarrow (E_2^*/E_1^{\perp}) \otimes_{\min} X$$

such that $\|T^{-1}\| \le C_2$.

In other words, since $E_2^* \otimes_{\min} X = cb(E_2, X)$ we have an extension property associated to the following diagram:

$$\begin{tikzcd} E_2 \arrow[dr, "\widetilde{v}"] & \\ E_1 \arrow[u, phantom, "\cup"] \arrow[r, "v"] & X \end{tikzcd}$$

More precisely, for any $v\colon E_1 \to X$ there is an extension $\widetilde{v}\colon E_2 \to X$ such that $\|\widetilde{v}\|_{cb} \le C_2\|v\|_{cb}$. Consider now an arbitrary finite dimensional subspace $E \subset X$. Let $\varepsilon > 0$. Consider $E_1 \subset M_n$ such that there is an isomorphism $u\colon E_1 \to E$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \le d_{SK}(E) + \varepsilon \le C_1 + \varepsilon$.

Using the preceding extension property (with $E_2 = M_n$) we find an operator $b\colon M_n \to X$ extending u and such that $\|b\|_{cb} \le C_2\|u\|_{cb}$. Let $a\colon E \to M_n$ be the operator u^{-1} considered as acting into M_n. Then $i_E = ba$ and

$$\|a\|_{cb}\|b\|_{cb} \le C_2\|u\|_{cb}\|u^{-1}\|_{cb} \le C_2(C_1 + \varepsilon).$$

This proves our claim. Note in particular that if X is a C^*-algebra, it is OS-exact iff it is nuclear, by [P1, Remark before Theorem 2.10].

In the category of Banach spaces one can define a similar notion of exactness using the *injective* tensor product instead of the minimal one. Then a Banach space is "exact" iff it is a $\mathcal{L}_\infty$-space in the sense of [LR]. We refer the reader to [LR, Theorem 4.1 and subsequent Remark].

§2. Ultraproducts.

The notion of exactness for operator spaces is closely connected to a commutation property involving ultraproducts. To explain this let us recall a few facts about ultraproducts. Let $(F_i)_{i\in I}$ be a family of operator spaces and let $\mathcal{U}$ be a nontrivial ultrafilter on I. We denote by $\widehat{F} = \Pi F_i/\mathcal{U}$ the associated ultraproduct in the category of Banach spaces (cf. e.g. [Hei]). Recall that if $dim(F_i) = n$ for all i in I, then the ultraproduct $\widehat{F}$ clearly also is n-dimensional.

Clearly $\widehat{F}$ can be equipped with an operator space structure by defining

$$M_n(\widehat{F}) = \Pi M_n(F_i)/\mathcal{U}. \tag{16}$$

It is easy to check that Ruan's axioms [Ru] are satisfied so that $\widehat{F}$ with the matricial structure (16) is an operator space. Alternatively, one may view F_i as embedded into $B(H_i)$ (H_i Hilbert) and observe that $\widehat{F} \subset \Pi B(H_i)/\mathcal{U}$. Since C^*-algebras are stable by ultraproduct we obtain $\widehat{F}$ embedded in a C^*-algebra. It is easy to see that the resulting operator space structure is the same as the one defined by (16). Note that (16) can be written as a commutation property between ultraproducts and the minimal tensor product, as follows

$$M_n \otimes_{\min} [\Pi F_i/\mathcal{U}] = \Pi[M_n \otimes_{\min} F_i]/\mathcal{U}. \tag{17}$$

It is natural to wonder which operator spaces E can be substituted to M_n in this identity (17). It turns out that this property is closely related to the invariant $d_{SK}(E)$, as we will now show.

We first observe that there is for any finite dimensional operator space E a canonical map

$$v_E\colon\ \Pi(E \otimes_{\min} E_i)/\mathcal{U} \to E \otimes_{\min} \widehat{E} \tag{18}$$

with $\|v_E\| \le 1$. Indeed, we clearly have a norm one mapping

$$E \otimes_{\min} \ell_\infty\{E_i\} \to E \otimes_{\min} \widehat{E} \tag{18$'$}$$

but if E is finite dimensional $E \otimes_{\min} \ell_\infty\{E_i\} = \ell_\infty\{E \otimes_{\min} E_i\}$ and the map (18)' vanishes on the subspace of elements with $\mathcal{U}$ limit zero. Hence, after passing to the quotient by the kernel of (18)', we find the map (18) with norm ≤ 1. More generally (recall the isometric identity $F^* \otimes_{\min} E = cb(F, E)$, cf. [BP, ER1]) if $(E_i)_{i\in I}$ (resp. $(F_i)_{i\in I}$) is a family of n-dimensional (resp. m-dimensional operator spaces), we clearly have a norm one canonical map

$$\Pi cb(E_i, F_i)/\mathcal{U} \to cb(\widehat{E}, \widehat{F}), \tag{18$''$}$$

where $\widehat{E} = \Pi E_i/\mathcal{U}$ and $\widehat{F} = \Pi F_i/\mathcal{U}$.

Proposition 6. *Let E be a finite dimensional operator space and let $C \ge 1$ be a constant. The following are equivalent.*

(i) $d_{SK}(E) \le C$.

(ii) For all ultraproducts $\widehat{F} = \Pi F_i/\mathcal{U}$ the canonical isomorphism (which has norm ≤ 1)

$$v_E\colon\ \Pi(E \otimes_{\min} F_i)/\mathcal{U} \to E \otimes_{\min} (\Pi F_i/\mathcal{U})$$

satisfies $\|v_E^{-1}\| \le C$.

(iii) Same as (ii) but with all ultraproducts $(F_i)_{i\in I}$ on a countable set and such that $\sup\limits_{i\in I} \dim F_i \le \dim E$.

Proof. First observe that if $G \subset F$ are operator spaces then we have isometric embeddings
$$G \otimes_{\min} \widehat{F} \to F \otimes_{\min} \widehat{F}$$
and
$$\Pi(G \otimes_{\min} F_i)/\mathcal{U} \to \Pi(F \otimes_{\min} F_i)/\mathcal{U}.$$
Therefore in the finite dimensional case we have clearly $\|v_G^{-1}\| \leq \|v_F^{-1}\|$.

(i) $\Rightarrow$ (ii): Assume (i). Then consider $G \subset M_n$ isomorphic to E. We have clearly $\|v_{M_n}^{-1}\| = 1$ by (16), hence we can write

$$\begin{aligned}\|v_E^{-1}\| \leq d_{cb}(E,G)\|v_G^{-1}\| &\leq d_{cb}(E,G)\|v_{M_n}^{-1}\| \\ &\leq d_{cb}(E,G)\end{aligned}$$

hence $\|v_E^{-1}\| \leq d_{SK}(E)$, whence (ii).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): This is proved by an argument similar to the proof of (10) in Theorem 1. We merely outline the argument. Let $u_n\colon E \to E_n = P_n(E)$ be given by Lemma 2, as in the above proof of (10). For $n \geq n_0$ we consider $u_n^{-1}\colon E_n \to E$ and we identify u_n^{-1} with an element of $E_n^* \otimes E$. Recall $\|u_n^{-1}\|_{E_n^* \otimes_{\min} E} = \|u_n^{-1}\|_{cb(E_n,E)}$. Then

$$\|(u_n^{-1})_n\|_{\Pi(E_n^* \otimes_{\min} E)/\mathcal{U}} = \lim_{\mathcal{U}} \|u_n^{-1}\|_{cb}$$

and on the other hand since J is a complete isometry and since we have the monotonicity property (7) we have

$$\begin{aligned}\|(u_n^{-1})_n\|_{(\Pi E_n^*/\mathcal{U}) \otimes_{\min} E} &= \sup_m \|(P_m u_n^{-1})_n\|_{(\Pi E_n^*/\mathcal{U}) \otimes_{\min} M_m} \\ &= \sup_m \lim_{n,\mathcal{U}} \|P_m u_n^{-1}\|_{E_n^* \otimes_{\min} M_m} \\ &\leq \sup_m \lim_{n,\mathcal{U}} \|a_{mn}\|_{cb} \leq 1.\end{aligned}$$

Hence (iii) implies $\lim_{\mathcal{U}} \|u_n^{-1}\|_{cb} \leq C$ if $\mathcal{U}$ is any nontrivial ultrafilter on $\mathbf{N}$, and we conclude $d_{SK}(E) \leq \lim_{\mathcal{U}} \|u_n\|_{cb}\|u_n^{-1}\|_{cb} \leq C$. ■

§3. How large can $d_{SK}(E)$ be?

We now wish to produce finite dimensional operator spaces E with $d_{SK}(E)$ as large as possible. It follows from Theorem 9.6 in [P1] that for any n-dimensional operator space E we have
$$d_{SK}(E) \leq \sqrt{n}.$$
We will show that this upper bound in general cannot be improved, at least asymptotically, when n goes to infinity. We will consider the space ℓ_∞^n with its natural operator

space structure. We will denote by E_1^n the dual in the category of Banach spaces, so that as a Banach space E_1^n is the usual space ℓ_1^n, however it is embedded into $B(H)$ in such a way that the canonical basis $e_1, \dots, e_n$ of E_1^n satisfies for all $a_1, \dots, a_n$ in $B = B(\ell_2)$.

$$\left\|\sum_1^n e_i \otimes a_i\right\|_{E_1^n \otimes_{\min} B} = \sup_{\substack{u_i \\ \text{unitary}}} \left\|\sum_1^n u_i \otimes a_i\right\|_{B \otimes_{\min} B} \tag{19}$$

where the supremum runs over all unitary operators u_i in B.

Another remarkable representation of E_1^n appears if we consider the full C^*-algebra $C^*(\mathbf{F}_n)$ of the free group with n generators. If we denote by $\delta_1, \dots, \delta_n$ the generators of $\mathbf{F}_n$ viewed as unitary operators in $C^*(\mathbf{F}_n)$ in the usual way then $\|\sum \delta_i \otimes a_i\|_{C^*(\mathbf{F}_n)\otimes_{\min} B}$ is equal to (19), which shows that the map u: $E_1^n \to \mathrm{span}(\delta_i)$ which takes e_i to δ_i is a complete isometry. Our main result is the following.

Theorem 7. *For all $n \geq 2$*

$$d_{SK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

Hence in particular $d_{SK}(E_1^n) > 1$ for all $n \geq 3$.

Remark. It is easy to check (this was pointed out to me by Paulsen) that $d_{SK}(E_1^n) = 1$ for $n = 2$. Indeed, in that case (19) becomes (after multiplication by u_1^{-1})

$$\|e_1 \otimes a_1 + e_2 \otimes a_2\| = \sup_{u \text{ unitary}} \|I \otimes a_1 + u \otimes a_2\|$$

and since (I, u) generate a commutative C^*-algebra, this is the same as

$$\sup_{\substack{|z|=1 \\ z \in \mathbf{C}}} \|a_1 + z a_2\|$$

which shows that E_1^2 is completely isometric to the span of $\{1, e^{it}\}$ in $C(\mathbf{T})$. Therefore (since $C(\mathbf{T})$ is nuclear) $d_{SK}(E_1^2) = 1$. We will use an idea similar to Wassermann's argument in [W1]: we consider the direct sum $M = M_1 \oplus M_2 \oplus \cdots$, or equivalently $M = \ell_\infty\{M_n\}$ in our previous notation, and we denote by $I_{\mathcal{U}}$ the set

$$I_{\mathcal{U}} = \{(x_\alpha) \in M \mid \lim_{\mathcal{U}} \tau_\alpha(x_\alpha^* x_\alpha) = 0\}$$

where τ_α is the normalized trace on M_α and where $\mathcal{U}$ is a nontrivial ultrafilter on $\mathbf{N}$. Then the group von Neumann algebra $VN(\mathbf{F}_n)$ is isomorphic to a von Neumann subalgebra of the quotient $N = M/I_{\mathcal{U}}$. It is well known that N is a finite von Neumann algebra with normalized trace τ given by $\tau(x) = \lim_{\mathcal{U}} \tau_\alpha(x_\alpha)$ where x denotes the equivalence class in N of (x_α). Let us denote by

$$\Phi\colon\ M \to M/I_{\mathcal{U}}$$

the quotient mapping.

In the sequel, we will make use of the operator space version of the projective tensor product introduced in [ER5]. However, to facilitate the task of the reader, we include in the next few lines the simple facts that we use with indication of proof. Consider a finite dimensional algebra M_k equipped with the normalized trace which we denote by τ. We denote by $L_1(\tau)$ the space M_k equipped with the norm $\|x\|_1 = \tau(|x|)$. Let E be an operator space. Since $M_k = L_1(\tau)^*$ we have $(L_1(\tau) \otimes E)^* = M_k(E^*)$. We then denote by $L_1(\tau) \otimes_\wedge E$ the space $L_1(\tau) \otimes E$ equipped with the norm induced on $L_1(\tau) \otimes E$ by $M_k(E^*)^*$. We will use the following two facts which are easy to check:

(a) If $F \subset E$ (completely isometric embedding) then $L_1(\tau) \otimes_\wedge F \subset L_1(\tau) \otimes_\wedge E$ (completely isometric embedding).

(b) If $E = E_1^n$ and $e_1, e_2, \ldots, e_n$ is the canonical basis of $E_1^n = {\ell_\infty^n}^*$, then for any $x_1, \ldots, x_n$ in $L_1(\tau)$ we have

$$\left\| \sum_1^n x_i \otimes e_i \right\|_{L_1(\tau) \otimes_\wedge E_1^n} = \sum_1^n \|x_i\|_{L_1(\tau)}.$$

(c) We have a norm one inclusion

$$M_k(E) \to L_1(\tau) \otimes_\wedge E.$$

These facts can be checked as follows:

(a) follows by duality from the isometric identity

$$M_k(F^*) = M_k(E^*/F^\perp) = M_k(E^*)/M_k(F^\perp).$$

(b) follows again by duality from the identity

$$M_k(\ell_\infty^n) = \ell_\infty^n(M_k).$$

(c) follows from the inequality $\forall\ \xi_{ij} \in E^*\ \forall\ x_{ij} \in E$

$$(d) \qquad k^{-1} \left| \sum_{ij \le k} \xi_{ij}(x_{ij}) \right| \le \|(x_{ij})\|_{M_k(E)} \|(\xi_{ij})\|_{M_k(E^*)}.$$

The latter inequality can be checked using the factorization of *c.b.* maps (cf. [Pa1, p. 100]) since $\|(\xi_{ij})\|_{M_n(E^*)} = \|(\xi_{ij})\|_{cb(E,M_n)}$: if $\|(\xi_{ij})\|_{M_n(E^*)} \le 1$ then we can write $\xi_{ij}(x) = \langle \pi(x) x_j, y_i \rangle$ with π: $E \to B(H)$ restriction of a representation and with $x_j, y_i \in H$ such that $\| \sum \alpha_j x_j \| \le 1$ and $\| \sum \alpha_i y_i \| \le 1$ whenever $\sum |\alpha_j|^2 \le 1$. This implies $(\sum \|x_j\|^2 \sum \|y_i\|^2)^{1/2} \le k$ whence

$$\left| \sum \xi_{ij}(x_{ij}) \right| = \left| \sum \langle \pi(x_{ij}) x_j, y_i \rangle \right| \le k \|(x_{ij})\|_{M_k(E)}$$

which proves the inequality (d).

We denote below by $C_\lambda^*(\mathbf{F}_n)$ the reduced C^*-algebra associated to the left regular representation for the free group $\mathbf{F}_n$ with n generators. Then the key result for our subsequent estimates can be stated as follows.

Theorem 8. *Fix $n \geq 2$. There is a family of unitary matrices (u_i^α) with $u_i^\alpha \in M_\alpha$, $(i = 1, \ldots, n,\ \alpha \in \mathbf{N})$ and a nontrivial ultrafilter $\mathcal{U}$ on $\mathbf{N}$ such that for all $m \geq 1$ and all $x_1, \ldots, x_n$ in M_m we have*

$$\lim_{\mathcal{U}} \left\| \sum_{i=1}^{n} u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge M_m} \leq \left\| \sum_{1}^{n} \lambda(g_i) \otimes x_i \right\|_{C_\lambda^*(\mathbf{F}_n) \otimes_{\min} M_m} \tag{20}$$

Remark. By results included in [HP], the right side of (20) is

$$\leq 2 \max \left\{ \left\| \sum_{1}^{n} x_i^* x_i \right\|^{1/2}, \left\| \sum_{1}^{n} x_i x_i^* \right\|^{1/2} \right\}.$$

Proof of Theorem 8. As explained in [W1], for each i there is a unitary $u_i = (u_i^\alpha)_{\alpha \in \mathbf{N}}$ in M such that $\Phi(u_i) = \lambda(g_i)$. Let us denote

$$b = \sum \lambda(g_i) \otimes x_i \in C_\lambda^*(\mathbf{F}_n) \otimes_{\min} M_m.$$

Then $\sum u_i \otimes x_i$ is a lifting of b in $M \otimes M_m$ and $M_m(VN(\mathbf{F}_n))$ embeds isometrically (see [W1]) into $M_m(M/I_{\mathcal{U}})$ or equivalently into $M_m(M)/M_m(I_{\mathcal{U}})$. It follows that we have

$$\|b\| = \inf \left\{ \left\| \sum_{i=1}^{m} x_i \otimes u_i + \gamma \right\|_{M_m(M)} \mid \gamma \in M_m(I_{\mathcal{U}}) \right\}.$$

Hence there is a sequence $(\gamma^\alpha)_{\gamma \in \mathbf{N}}$ with $\gamma^\alpha \in M_m(M_\alpha)$ satisfying

$$\forall\, i, j \leq m \qquad \lim_{\substack{\alpha \to \infty \\ \mathcal{U}}} \tau_\alpha(\gamma_{ij}^{\alpha *} \gamma_{ij}^\alpha) = 0$$

and such that

$$\lim_{\mathcal{U}} \left\| \sum_{i=1}^{n} x_i \otimes u_i^\alpha + \gamma^\alpha \right\|_{M_m(M_\alpha)} \leq \|b\|. \tag{21}$$

Now observe that $M_m(M_\alpha) = M_\alpha(M_m)$. We will use the norm one inclusion (see fact (c) above) $M_\alpha(M_m) \longmapsto L_1(\tau_\alpha) \otimes_\wedge M_m$. Note that the inclusion $L_2(\tau_\alpha) \to L_1(\tau_\alpha)$ has norm ≤ 1 so that

$$\lim_{\substack{\alpha \to \infty \\ \mathcal{U}}} \|\gamma^\alpha\|_{L^1(\tau_\alpha) \otimes_\wedge M_m} = 0.$$

Therefore (21) yields

$$\lim_{\mathcal{U}} \left\| \sum\nolimits_1^n u_i^\alpha \otimes x_i \right\|_{L^1(\tau_\alpha) \otimes_\wedge M_m} \leq \|b\|, \tag{22}$$

which is the announced inequality. ∎

To prove Theorem 7, we will use the following lemma.

Lemma 9. *Consider the operator*

$$T_n\colon \ell_\infty^n \to C_\lambda^*(\mathbf{F}_n)$$

defined by $T_n(\alpha_1, \ldots, \alpha_n) = \sum_1^n \alpha_i \lambda(g_i)$. *Then for any* m, *any* n*-dimensional subspace* $F \subset M_m$ *and any factorization*

$$\begin{tikzcd} & F^* \arrow[dr, "b"] & \\ \ell_\infty^n \arrow[ur, "a"] \arrow[rr, "T_n"] & & C_\lambda^*(\mathbf{F}_n) \end{tikzcd}$$

with $T_n = ba$ *we have*

$$n \le \|a\|_{cb}\|b\|_{cb}. \tag{23}$$

Proof. Consider a, b as above. We identify $b\colon F^* \to C_\lambda^*(\mathbf{F}_n)$ with an element of $F \otimes_{\min} C_\lambda^*(\mathbf{F}_n)$. Then we can write $b = \sum_{i=1}^n x_i \otimes \lambda(g_i)$ with $x_i \in F$ such that $a^*(x_i) = e_i$. (Recall that e_i is the canonical basis of $E_1^n = (\ell_\infty^n)^*$.)
Now by Theorem 8 we have

$$\lim_{\mathcal{U}} \left\| \sum\nolimits_1^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha)\otimes_\wedge M_m} \le \|b\|_{cb}.$$

By fact (a) recalled above, this implies

$$\lim_{\mathcal{U}} \left\| \sum\nolimits_1^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha)\otimes_\wedge F} \le \|b\|_{cb},$$

hence since $\|a^*\|_{cb} = \|a\|_{cb}$

$$\lim_{\mathcal{U}} \left\| \sum\nolimits_1^n u_i^\alpha \otimes a^*(x_i) \right\|_{L_1(\tau_\alpha)\otimes_\wedge E_1^n} \le \|a\|_{cb}\|b\|_{cb}.$$

This gives the conclusion since $a^*(x_i) = e_i$ and by fact (b)

$$\left\| \sum\nolimits_1^n u_i^\alpha \otimes e_i \right\|_{L_1(\tau_\alpha)\otimes_\wedge E_1^n} = \sum\nolimits_1^n \|u_i^\alpha\|_{L_1(\tau_\alpha)} = n.$$

■

Remark. The same operator T_n as in Lemma 9 was already considered in [H]. By [H, Lemma 2.5] we have $\|T_n\|_{dec} = n$, but this does not seem related to (23).

Proof of Theorem 7. By [AO] (see [H] for more details) we have $\|T_n\|_{cb} \le 2\sqrt{n-1}$, hence for any $F \subset M_m$ we can write by (23)

$$n \le \|T_n\|_{cb} d_{cb}(F^*, \ell_\infty^n) \tag{24}$$

$$n \le \|T_n\|_{cb} d_{cb}(F^*, E_n^\lambda) \tag{25}$$

where $E_n^\lambda = \text{span}\{\lambda(g_i) \mid i = 1, \ldots, n\}$ in $C_\lambda^*(\mathbf{F}_n)$. Since $d_{cb}(F^*, \ell_\infty^n) = d_{cb}(F, \ell_\infty^{n\,*})$ and $\ell_\infty^{n\,*} = E_1^n$ we obtain

$$n(2\sqrt{n-1})^{-1} \leq d_{cb}(F, E_1^n)$$

so that Theorem 7 follows.

Remark. By the same argument we have

$$d_{SK}(E_n^{\lambda^*}) \geq n(2\sqrt{n-1})^{-1}.$$

Here again this is > 1 if $n \geq 3$ but $d_{SK}(E_n^{\lambda^*}) = 1$ if $n = 2$ for the same reason as above for E_1^n. We can also derive an estimate for the n dimensional operator Hilbert space which is denoted by OH_n. This space was introduced in [P1] to which we refer for more details. It is isometric to ℓ_2^n and has an orthonormal basis $(\theta_i)_{i \leq n}$ such that for all $a_1, \ldots, a_n$ in B we have

$$\left\| \sum_1^n \theta_i \otimes a_i \right\|_{OH_n \otimes_{\min} B} = \left\| \sum_1^n a_i \otimes \bar{a}_i \right\|_{B \otimes_{\min} \overline{B}}^{1/2}. \tag{26}$$

Corollary 10. *For each $n \geq 2$, we have*

$$d_{SK}(OH_n) \geq [n(2\sqrt{n-1})^{-1}]^{1/2}. \tag{27}$$

Proof. By (26) we have (we denote simply by $\| \ \|$ the minimal tensor norm everywhere)

$$\left\| \sum \theta_i \otimes \lambda(g_i) \right\| = \left\| \sum \lambda(g_i) \otimes \overline{\lambda(g_i)} \right\|^{1/2}$$

and by [AO] we have

$$\left\| \sum_1^n \lambda(g_i) \otimes \overline{\lambda(g_i)} \right\| = 2\sqrt{n-1}.$$

Hence we have a factorization of T_n of the form $\ell_\infty^n \xrightarrow{a} OH_n \xrightarrow{b} E_n$ with $\|b\|_{cb}^2 \leq 2\sqrt{n-1}$. On the other hand

$$\|a\|_{cb} = \|a^*\|_{cb} = \left\| \sum_1^n e_i \otimes \bar{e}_i \right\|_{E_1^n \otimes_{\min} \overline{E_1^n}}^{1/2} \leq n^{1/2}$$

hence we have by (23)

$$n \leq d_{SK}(OH_n) \|a\|_{cb} \|b\|_{cb}$$

which implies (27). ∎

Remark. It is easy to verify that these estimates are asymptotically best possible. More precisely, we have with the notation of [P1], $d_{SK}(E_1^n) \leq d_{cb}(E_1^n, R_n) \leq \sqrt{n}$.

Similarly, $d_{SK}(E_n^{\lambda^*}) \le d_{cb}(E_n^{\lambda^*}, R_n) \le \sqrt{n}$, and finally $d_{SK}(OH_n) \le d_{cb}(OH_n, R_n \cap C_n) \le n^{1/4}$.

Remark. It is natural to raise the following question: Is there a function $n \to f(n)$ and a constant C such that for any E in OS_n satisfying $d_{SK}(E) = 1$, there is a subspace $F \subset M_m$ with $m \le f(n)$ and $d_{cb}(F, E) \le C$? It is easy to derive from the preceding construction that the answer is negative (contrary to the commutative case with ℓ_∞^n in the place of M_n). A negative answer can also be derived easily from the fact (due to Szankowski [Sz]) that the space of compact operators on ℓ_2 fails the *uniform* approximation property.

Remark. It is rather natural to introduce the following quantity for E in OS_n.

$$d_{QSK}(E) = \inf\{d_{cb}(E, F)\}$$

where the infimum runs over all the spaces F which are quotient of a subspace of K, i.e. there are $S_2 \subset S_1 \subset K$ such that $F = S_1/S_2$. Clearly $d_{QSK}(E) \le d_{SK}(E)$. We do not know much about this new parameter. Note however that in view of the lifting property of E_1^n we have by Theorem 7

$$d_{QSK}(E_1^n) \ge \frac{n}{2\sqrt{n-1}}.$$

Moreover, it can be shown that OH_n embeds completely isometrically into the direct sum $L_\infty(R_n) \oplus L_\infty(C_n)$, and a fortiori into $L_\infty(M_{2n})$, so that $d_{QSK}(OH_n) = 1$, indeed this follows from the identity $OH_n = (R_n, C_n)_{1/2}$ proved in [P1, Corollary 2.6]. Note that for the quotient of a subspace case, the identity between (1) and (1)' might no longer hold, so that it might be necessary to distinguish between the quotients of a subspace of K and those of M_n for some $n \ge 1$.

§4. On the metric space of all n-dimensional operator spaces.

It will be convenient in the sequel to record in the next statement several elementary facts on ultraproducts.

Proposition 11. *Let E and F be two n-dimensional operator spaces. Let $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ be two families of n dimensional operator spaces. Let $\mathcal{U}$ be an ultrafilter on I and let $\widehat{E}$, $\widehat{F}$ be the corresponding ultraproducts.*

(i) We have

$$d_{cb}(\widehat{E}, \widehat{F}) \le \lim_{\mathcal{U}} d_{cb}(E_i, F_i). \tag{28}$$

Moreover, if $d_{cb}(E_i, F_i) \to 1$ then $\widehat{E}$ is completely isometric to $\widehat{F}$.

(ii) If $F_i = F$ for all $i \in I$ then $\widehat{F}$ is completely isometric to F.

(iii) If $d_{cb}(E_i, F) \to 1$ then $\widehat{E}$ is completely isometric to F.

(iv) If $d_{cb}(E,F)=1$, then E and F are completely isometric.

Proof. We have isomorphisms u_i: $E_i \to F_i$ such that $v_i = u_i^{-1}$ satisfies $\|u_i\|_{cb} \le 1$ and $\lim_{\mathcal{U}} \|v_i\|_{cb} = \lim_{\mathcal{U}} d_{cb}(E_i, F_i)$. Then the maps $\hat{u}$: $\widehat{E} \to \widehat{F}$ and $\hat{v}$: $\widehat{F} \to \widehat{E}$ (associated respectively to (u_i) and (v_i)) are inverse of each other and satisfy $\|\hat{u}\|_{cb}\|\hat{v}\|_{cb} \le \lim_{\mathcal{U}} d_{cb}(E_i, F_i)$. Hence we have (28). The preceding also shows that $\hat{u}$ is a complete isometry between $\widehat{E}$ and $\widehat{F}$ when $\lim_{\mathcal{U}} d_{cb}(E_i, F_i) = 1$, whence (i). But on the other hand it is easy to check that, if $F_i = F$ for all i, then $\widehat{F}$ and F are completely isometric via the map T: $\widehat{F} \to F$ defined by $T(x) = \lim_{\mathcal{U}} x_i$ if x is the equivalence class of (x_i) modulo $\mathcal{U}$. This justifies (ii). Then (iii) is clear. Finally, taking $E_i = E$ and $F_i = F$ for all i in what precedes, we obtain (iv). (Actually, (iv) is clear, by a direct argument based on the compactness of the unit ball of $cb(E,F)$.) ∎

In this section, we will include some remarks on the set OS_n formed of all n dimensional operator spaces. More precisely OS_n is the set of all equivalence classes when we identify two spaces if they are completely isometric. We equip OS_n with the metric
$$\delta(E,F) = \mathrm{Log}\ d_{cb}(E,F).$$
This is the analogue for operator spaces of the classical "Banach-Mazur compactum" formed of all n dimensional normed spaces equipped with the Banach-Mazur distance.

However, contrary to the Banach space situation the metric space OS_n is not compact. The next result was known, it was mentioned to me by Kirchberg.

Proposition 12. *The set OS_n equipped with the metric Log d_{cb} is a complete metric space, but it is not compact at least if $n \ge 3$.*

Proof. We sketch a proof using ultrafilters. Let (E_i) be a Cauchy sequence in OS_n. Let $\widehat{E}$ be an ultraproduct associated to a nontrivial ultrafilter $\mathcal{U}$ on N. Then by the Cauchy condition for each $\varepsilon > 0$ there is an i_0 such that for all $i, j > i_0$ we have
$$\mathrm{Log}\ d_{cb}(E_i, E_j) \le \varepsilon.$$
By Proposition 11 this implies $\forall\ i > i_0$
$$\mathrm{Log}\ d_{cb}(\widehat{E}, E_i) \le \varepsilon$$
hence $E_i \to \widehat{E}$ when $i \to \infty$ and OS_n is complete.

We now show that OS_n is not compact by exhibiting a sequence without converging subsequences if $n \ge 3$. Consider any space E_0 in OS_n such that

$$d_{SK}(E_0) > 1. \tag{29}$$

We know that such spaces exist if $n \geq 3$ by Theorem 7 and Corollary 10.

By Lemma 2, we can find a sequence of spaces $E_i \subset M_i$ with $\dim E_i = n$ such that for any nontrivial ultrafilter on $\mathbf{N}$ the ultraproduct $\widehat{E} = \Pi E_i/\mathcal{U}$ is completely isometric to E_0. (Indeed, this is clear by (7).) Assume that some subsequence of E_i converges in OS_n. Then, by Proposition 11 (ii), its limit must be $\widehat{E}$ which is the same as E_0. In other words the subsequence can only converge to E_0 but on the other hand by (29) we have (since $E_i \subset M_i$)

$$\forall\, i \in I \qquad 1 < d_{SK}(E_0) \leq d_{cb}(E_0, E_i),$$

which is the desired contradiction. ■

Remark. It is well known that every separable Banach space E embeds isometrically into the space of all continuous functions on the unit ball of X^* equipped with the weak*-topology. Let us denote simply by $\mathcal{C}$ the latter space, and let k: $E \to \mathcal{C}$ be the isometric embedding. Given P_n as in Lemma 2, we can introduce $\widetilde{P}_n$: $E \to \mathcal{C} \oplus_\infty M_n$ by setting $\widetilde{P}_n(x) = (k(x), P_n(x))$. Then each $\widetilde{P}_n$ is an *isometric* isomorphism of E into $\widetilde{E}_n \subset \mathcal{C} \oplus_\infty M_n$. Moreover the embedding $\widetilde{J}$: $E \to \ell_\infty\{\widetilde{E}_n\}$ is a completely isometric embedding with the same properties as in Lemma 2. Finally the C^*-algebras $\mathcal{C} \oplus_\infty M_n$ are nuclear. Using this it is easy to modify the preceding reasoning, replacing E_n by $\widetilde{E}_n$ (note $d_{SK}(\widetilde{E}_n) = 1$ since $\mathcal{C} \oplus_\infty M_n$ is nuclear) and to demonstrate the following

Corollary 13. *Let E_0 be any n dimensional operator space such that $d_{SK}(E_0) > 1$. Then the (closed) subset of OS_n formed of all the spaces isometric to E_0 is not compact.*

Remark. Actually we obtain a sequence E_i of spaces each isometric to E_0 and such that E_0 is completely isometric to $\widehat{E} = \Pi E_i/\mathcal{U}$ but $d_{cb}(E_i, E_0) \not\to 0$. More precisely (in answer to a question of S. Szarek), the preceding argument shows that the "metric entropy" of OS_n is quite large in the following sense: Let $\delta = d_{SK}(E_0)$ with E_0 as in Corollary 13. Then for any $\varepsilon > 0$ there is a sequence E_i in OS_n such that

$$d_{cb}(E_i, E_j) > \delta - \varepsilon \quad \text{for any} \quad i \neq j.$$

(Moreover, E_i is isometric to E_0 and the ultraproduct $\widehat{E} = \Pi E_i/\mathcal{U}$ is completely isometric to E_0). This suggests the following question: does there exist such a sequence if δ is equal to the diameter of the set OS_n (or of the subset of OS_n formed of all the spaces which are isometric to E_0)? If not, what is the critical value of δ?

Corollary 13 is of course particularly striking in the case $E_0 = OH_n$, it shows that the set of all possible operator space structures on the Euclidean space ℓ_2^n is very large. We refer to [Pa2] for more information of the latter set.

These results lead to the following question (due to Kirchberg).

Problem. Is the metric space OS_n separable[1]?

[1] See the Note added at the end of this paper.

Equivalently, is there a separable operator space X such that for any $\epsilon > 0$ and and any n-dimensional operator space E, there is a subspace $F \subset X$ such that $d_{cb}(E,F) < 1+\epsilon$?

More generally, let E be an n-dimensional normed space. Let $OS_n(E)$ be the subset of OS_n formed of all the spaces which are isometric to E. For which spaces E is the space $OS_n(E)$ compact or separable? Note that $OS_n(E)$ can be a singleton, this happens in the 2-dimensional case if $E = \ell_1^2$ or $E = \ell_\infty^2$, however it never happens if $dim(E) \geq 5$ (see [Pa2, Theorem 2.13]).

We will now characterize the spaces E_0 for which the conclusion of Corollary 13 holds. We will use the following simple observation.

Lemma 14. *Fix $n \geq 1$. Let $\widehat{E} = \Pi E_i/\mathcal{U}$ be an ultraproduct of n-dimensional spaces. Then $(\widehat{E})^* = \Pi E_i^*/\mathcal{U}$ completely isometrically.*

Proof. Let F_i be a family of m-dimensional spaces with $m \geq 1$ fixed and let $\widehat{F}$ be their ultraproduct. It is well known that

$$\Pi B(E_i, F_i)/\mathcal{U} = B(\widehat{E}, \widehat{F}) \tag{30}$$

isometrically. Now observe that for any integer k we have (by [Sm]) for any $u\colon \widehat{E} \to M_k$, associated to a family $(u_i)_{i\in I}$ with $u_i \in B(E_i, M_k)$

$$\|u\|_{cb} = \|I_{M_k} \otimes u\|_{M_k(\widehat{E}) \to M_k(M_k)}.$$

By (30) it follows that

$$\|u\|_{cb} = \lim_{\mathcal{U}} \|I_{M_k} \otimes u_i\|_{M_k(E_i) \to M_k(M_k)}.$$

Hence

$$\|u\|_{cb(\widehat{E}, M_k)} = \lim_{\mathcal{U}} \|u_i\|_{cb(E_i, M_k)}.$$

Equivalently

$$\begin{aligned} M_k((\widehat{E})^*) &= \Pi M_k(E_i^*)/\mathcal{U} \\ &= M_k(\Pi E_i^*/\mathcal{U}) \end{aligned}$$

and we conclude that $(\widehat{E})^*$ and $\Pi E_i^*/\mathcal{U}$ are completely isometric. ∎

Corollary 15. *Consider E in OS_n. The following are equivalent:*

(i) $d_{SK}(E) = d_{SK}(E^*) = 1$.

(ii) For any sequence E_i in OS_n such that $\widehat{E} = \Pi E_i/\mathcal{U}$ is completely isometric to E we have

$$\lim_{\mathcal{U}} d_{cb}(E, E_i) = 1.$$

(iii) Same as (ii) with each E_i isometric to E.

Proof. Assume (ii) (resp. (iii)). Then the proof of Proposition 12 (resp. Corollary 13) shows that necessarily $d_{SK}(E) = 1$. By Lemma 14 it is clear that (ii) and (iii) are self dual properties, hence we must also have $d_{SK}(E^*) = 1$. Conversely assume (i). We will use Proposition 6 together with the identity $E \otimes_{\min} F = cb(F^*, E)$ valid when F is finite dimensional. By Proposition 6 and Lemma 14, if $d_{SK}(E) = 1$ we have

$$\Pi cb(E_i, E)/\mathcal{U} = cb(\widehat{E}, E) \tag{31}$$

whenever $\widehat{E}$ is an ultraproduct of n dimensional spaces. Now if $d_{SK}(E^*) = 1$ this implies

$$\Pi cb(E_i^*, E^*)/\mathcal{U} = cb(\widehat{E}^*, E^*)$$

hence after transposition

$$\Pi cb(E, E_i)/\mathcal{U} = cb(E, \widehat{E}). \tag{32}$$

It is then easy to conclude: let u: $\widehat{E} \to E$ be a complete isometry. Let u_i: $E_i \to E$ be associated to u via (31) and let v_i: $E \to E_i$ be associated to u^{-1} via (32) in such a way that $\|u\|_{cb} = \lim_{\mathcal{U}} \|u_i\|_{cb} = 1$ and $\|u^{-1}\|_{cb} = \lim_{\mathcal{U}} \|v_i\|_{cb} = 1$.

Then $I_E = \lim_{\mathcal{U}} u_i v_i$ hence we have $\lim_{\mathcal{U}} \|I_E - u_i v_i\| = 0$, hence $(u_i v_i)^{-1}$ exists for i large and its norm tends to 1, so that u_i^{-1} exists and (since $\lim_{\mathcal{U}} \|v_i\| < \infty$) $\lim_{\mathcal{U}} \|u_i^{-1}\| < \infty$, whence $\lim_{\mathcal{U}} \|u_i^{-1} - v_i\| = 0$. Since all norms are equivalent on a finite dimensional space, we also have $\lim_{\mathcal{U}} \|I_E - u_i v_i\|_{cb} = 0$, finally (repeating the argument with the cb-norms) we obtain $\lim_{\mathcal{U}} \|u_i^{-1} - v_i\|_{cb} = 0$ and we conclude that

$$\lim_{\mathcal{U}} d_{cb}(E, E_i) \le \lim_{\mathcal{U}} \|u_i\|_{cb} \|u_i^{-1}\|_{cb} \le 1.$$

This shows that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is trivial. ■

Remark. The row and column Hilbert spaces R_n and C_n obviously satisfy the properties in Corollary 15. Also, the two dimensional spaces ℓ_1^2 and ℓ_∞^2, which (see [Pa2]) admit only one operator space structure (so that any space isometric to either one is automatically completely isometric to it), must clearly satisfy these properties. At the time of this writing, these are the only examples I know of spaces satisfying the properties in Corollary 15.

It is natural to describe the spaces appearing in Corollary 15 as points of continuity with respect to a weaker topology (on the metric space OS_n) which can be defined as follows. For any $k \ge 1$ and any linear map u: $E \to F$ between operator spaces, we denote as usual

$$\|u\|_k = \|I_{M_k} \otimes E\|_{M_k(E) \to M_k(F)}.$$

Then for any E, F in OS_n we define

$$d_k(E,F) = \inf\{\|u\|_k \|u^{-1}\|_k\}$$

where the infimum runs over all isomorphisms $u\colon E \to F$.

We will say that a sequence $\{E_i\}$ in OS_n tends weakly to E if, for each $k \geq 1$, $\operatorname{Log} d_k(E_i, E) \to 0$ when $i \to \infty$. This notion of limit clearly corresponds to a topology (namely to the topology associated to the metric $\tilde{\delta} = \sum_{k \geq 1} 2^{-k} \operatorname{Log} d_k$) which we will call the weak topology. Let us say that E, F are k-isometric if there is an isomorphism $u\colon E \to F$ such that $I_{M_k} \otimes u$ is an isometry. Clearly this holds (by a compactness argument) iff $d_k(E,F) = 1$. Moreover (again by a compactness argument) E and F are completely isometric iff they are k-isometric for all $k \geq 1$. This shows that the weak topology on OS_n is Hausdorff. We observe

Proposition 16. *Let E and E_i $(i = 1, 2 \ldots)$ be operator spaces in OS_n. Then E_i tends weakly to E iff for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the ultraproduct $\widehat{E} = \Pi E_i/\mathcal{U}$ is completely isometric to E.*

Proof. Clearly if E_i tends weakly to E then $\widehat{E}$ is k-isometric to E for each $k \geq 1$, hence $\widehat{E}$ is completely isometric to E. Conversely, if E_i does not tend weakly to E, then for some $k \geq 1$ and some $\varepsilon > 0$ there is a subsequence E_{n_i} such that

$$d_k(E_{n_i}, E) > 1 + \varepsilon \quad \text{for all} \quad i \neq j. \tag{33}$$

Let $\mathcal{U}$ be an ultrafilter refining this subsequence, let $\widehat{E}$ be the corresponding ultraproduct, and let $u\colon \widehat{E} \to E$ be any isomorphism. Clearly there are isomorphisms $u_i\colon E_i \to E$ which correspond to u and we have for each $k \geq 1$ (by compactness) $\|u\|_k = \lim_{\mathcal{U}} \|u_i\|_k$ and $\|u^{-1}\|_k = \lim_{\mathcal{U}} \|u_i^{-1}\|_k$. Therefore we obtain by (33) $d_k(\widehat{E}, E) \geq 1 + \varepsilon$ and we conclude that E and $\widehat{E}$ are not completely isometric. ■

We can now reformulate Corollary 15 as follows

Corollary 17. *Let $i\colon OS_n \to OS_n$ be the identity considered as a map from OS_n equipped with the weak topology to OS_n equipped with the metric d_{cb}. Then an element E in OS_n is a point of continuity of i iff $d_{SK}(E) = d_{SK}(E^*) = 1$.*

§5. On the dimension of the containing matrix space.

It is natural to try to connect our study of $d_{SK}(E)$ with a result of Roger Smith [Sm]. Smith's result implies that for any subspace $F \subset M_k$ we have for any operator space X and for any linear map $u\colon X \to F$

$$\|u\|_{cb} \leq \|u\|_k. \tag{34}$$

More generally, if T is any compact set and $F \subset C(T) \otimes_{\min} M_k$ then we also have (34).

Now let E be any finite dimensional operator space. For each $k \geq 1$ we introduce

$$\delta_k(E) = \inf\{d_{cb}(E, F) \mid F \subset C(T) \otimes_{\min} M_k\}$$

where the infimum runs over all possible compact sets T.

Note that

$$d_{SK}(E) = \inf_{k \geq 1} \delta_k(E).$$

Clearly if $C = \delta_k(E)$, then by (34) we have for all u: $X \to E$

$$\|u\|_{cb} \leq C \|u\|_k. \tag{35}$$

It turns out that the converse is true: if (35) holds for all X and all u: $X \to E$ then necessarily $\delta_k(E) \leq C$. This is contained in the next statement which can be proved following the framework of [P2], but using an idea of Marius Junge [J].

Theorem 18. *Let E be any operator space and let $C \geq 1$ be a constant. Fix an integer $k \geq 1$. Then the following are equivalent:*

(i) There is a compact set T and a subspace $F \subset C(T) \otimes_{\min} M_k$ such that $d_{cb}(E, F) \leq C$.

(ii) For any operator space X and any u: $X \to E$ we have

$$\|u\|_{cb} \leq C \|u\|_k.$$

(iii) For all finite dimensional operator spaces X, the same as (ii) holds.

Proof. (Sketch) (i) $\Rightarrow$ (ii) is Smith's result [Sm] as explained above. (ii) $\Rightarrow$ (iii) is trivial. Let us prove (iii) $\Rightarrow$ (i). Assume (iii). Note that (for k fixed) the class of spaces of the form $C(T) \otimes_{\min} M_k$ is stable by ultraproduct, since the class of commutative unital C^*-algebras is stable by ultraproducts. In particular we may and will assume (for simplicity) that E is finite dimensional. Let G be an other operator space and consider a linear map v: $E \to G$. We introduce the number $\alpha_k(v)$ as folows. We consider all factorizations of v of the form

$$E \xrightarrow{a} \ell_\infty^N \otimes_{\min} M_k \xrightarrow{b} G$$

where $N \geq 1$ is an arbitrary integer, and and we set

$$\alpha_k(v) = \inf\{\|a\|_{cb} \|b\|_{cb}\}$$

where the infimum runs over all possible N and all possible such factorizations.

Now, using Lemma 2 and an ultraproduct argument it suffices to prove that for any n, any $\varepsilon > 0$ and any map v: $E \to M_n$, there is an integer $N \geq 1$ and a factorization of v of the form

$$E \xrightarrow{a} \ell_\infty^N \otimes_{\min} M_k \xrightarrow{b} M_n$$

with $\|a\|_{cb}\|b\|_{cb} \leq C(1+\varepsilon)\|v\|_{cb}$.

In other words, to conclude the proof, it suffices to show that if (iii) holds we have for all n and all v: $E \to M_n$

$$\alpha_k(v) \leq C\|v\|_{cb}. \tag{36}$$

Now we observe that (for any G) α_k is a norm on $cb(E,G)$ (left to the reader, this is where the presence of ℓ_∞^N is used). Hence (36) is equivalent to a statement on the dual norms. More precisely, (36) is equivalent to the fact that for any T: $M_n \to E$ we have

$$|tr(vT)| \leq C\|v\|_{cb}\alpha_k^*(T), \tag{37}$$

where α_k^* is the dual norm to α_k, i.e.

$$\alpha_k^*(T) = \sup\{|tr(vT)| \mid v\colon\ E \to M_n,\ \alpha_k(v) \leq 1\}.$$

We will now use the operator space version of the absolutely summing norm which was first introduced in [ER6]. In the broader framework of [P2], these operators are called completely 1-summing and the corresponding norm is denoted by π_1^o. We will use a version of the "Pietsch factorization" for these maps which is presented in [P2]. As observed by M. Junge, in the present situation the proof of Theorem 2.1 and Remark 2.7 in [P2] yield a factorization of T of the following form

$$M_n \xrightarrow{w} X \xrightarrow{u} E$$

where X is an n^2-dimensional operator space and where the maps w and u satisfy

$$\pi_1^o(w) \leq 1 \quad \text{and} \quad \|u\|_k \leq \alpha_k^*(T).$$

Hence if (iii) holds we find

$$\pi_1^o(T) = \pi_1^o(uw) \leq \|u\|_{cb}\pi_1^o(w) \leq \|u\|_{cb} \leq C\|u\|_k \leq C\alpha_k^*(T).$$

But since T is defined on M_n, we clearly have by definition of π_1^o

$$|tr(vT)| \leq \pi_1^o(vT) \leq \pi_1^o(T)\|v\|_{cb},$$

hence we obtain (37). ■

Junge's idea can also be used to obtain many more variants. For instance let $k \geq 1$ be fixed. Consider the following property of an operator space E: There is a constant C such that for any n and any bounded operator v: $M_n \to M_n$ we have

$$\|v \otimes I_E\|_{M_n(E)\to M_n(E)} \leq C\|v\|_k = C\|v\|_{M_n(M_k)\to M_n(M_k)}.$$

Then this property holds iff there is an operator space F completely isomorphic to E with $d_{cb}(E,F) \leq C$ such that for some compact set T, F is a quotient of a subspace of $C(T) \otimes_{\min} M_k$. In particular if $k = 1$ this result answers a question that I had raised in the problem book of the Durham meeting in July 92. Here is a sketch of a proof: By ultraproduct arguments, an operator space E satisfies $d_{cb}(E,F) \leq C$ for some F subspace of a quotient of $C(T) \otimes_{\min} M_k$ for some T iff for any integer n and any maps v_1: $M_n^* \to E$ and v_2: $E \to M_n$ we have

$$\alpha_k(v_2 v_1) \leq C \|v_1\|_{cb} \|v_2\|_{cb}.$$

Equivalently, this holds iff

$$|tr(v_2 v_1 T)| \leq C \|v_1\|_{cb} \|v_2\|_{cb} \alpha_k^*(T)$$

for any T: $M_n \to M_n^*$. The proof can then be completed using the factorization property of α_k^* as above. (See [Her] for similar results in the category of Banach spaces.)

We refer the reader to M. Junge's forthcoming paper for more results of this kind.

Note. (added December 11 1993) In a very recent joint paper with Marius Junge, we prove that OS_n (and even HOS_n) is non-separable when $n > 2$. See [JP].

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Free-independent sequences in type II_1 factors and related problems

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Free-independent sequences in type II_1 factors and related problems

by Sorin Popa

Dedicated to Professor Ciprian Foias, on his 60'th birthday

Introduction

We will show in this paper that, unlike central sequences (i.e., commuting-independent sequences) which in general may or may not exist, free-independent sequences exist in any separable type II_1 factor.

More generally, we will in fact prove the following:

Theorem. *Let $N \subset M_\infty$ be an inclusion of separable type II_1 factors. Assume there exists an increasing sequence of von Neumann subalgebras $N \subset M_n \subset M_\infty$ such that $\overline{\bigcup_n M_n} = M_\infty$ and such that $N' \cap M_n$ is finite dimensional for all n. Then there exists a unitary element $v = (v_n)_n$ in the ultrapower algebra N^ω ([D1]) such that*

$$M_\infty \vee vM_\infty v^* = M_\infty \underset{N'\cap M_\infty}{*} vM_\infty v^*.$$

Here $P_1 \underset{B}{*} P_2$ denotes the finite von Neumann algebra free product with amalgamation, with its free trace $\tau_1 * \tau_2$, where (P_1, τ_1), (P_2, τ_2) are finite von Neumann algebras with their corresponding finite, normal, faithful traces, and with $B \subset P_1$, $B \subset P_2$ a common subalgebra such that $\tau_{1|_B} = \tau_{2|_B}$ ([Po6], [V2]).

In the particular case when $N \subset M = M_\infty$ are factors and $N' \cap M = \mathbb{C}$, for example when $N = M$, the amalgamated free product is a genuine free product ([V1]) and any element of the form vxv^*, $x \in M$ is free with respect to M. Thus we get:

Corollary. *If $N \subset M$ is an inclusion of type II_1 factors with trivial relative commutant then there exist unitary elements $(u_n)_n$ in N that are free independent with respect to M, i.e., such that $\tau(u_n^k) = 0$, $\forall n$, $\forall k \neq 0$, and $\lim_{n\to\infty} \tau(u_n^{k_1} b_1 u_n^{k_2} b_2 \cdots u_n^{k_\ell} b_\ell) = 0$ for any $\ell \geq 1$ and any $b_1, \cdots, b_\ell \in M$, $\tau(b_i) = 0$, $1 \leq i \leq \ell - 1$, $k_1, k_2, \cdots, k_\ell \in \mathbb{Z} \setminus \{0\}$.*

Since the notion of "independent events" in classical probability theory becomes "free independence" in the noncommutative probability of ([V3]), our result on the existence of free-independent sequences can be regarded as the "free" analogue of the results on the existence of nontrivial central sequences in a factor ([D1], [McD], [C2]) or a subfactor ([Bi]). There are thus some notable differences between central and free-independent sequences: Nontrivial central sequences may not exist in general, but they always form an algebra while free sequences always exist, though the set of all such sequences doesn't form an algebra. Also, the existence of noncommuting central sequences in a factor M implies that M splits off the hyperfinite type II_1 factor, i.e., $M \simeq M \otimes R$, but, although all factors have free-independent sequences, neither the hyperfinite nor the property T factors ([C3]) are free products of algebras (cf. [MvN], [Po5]). Along these lines note also that, while taking the free product $M * R$ of a property T factor M by R cancels the property T for $M * R$, the fundamental group of $M * R$ will remain countable (cf. [Po5]), yet $M \otimes R$ will have fundamental group $\mathbb{R}_+^*$. Thus, as also pointed out in ([V1,3]), the analogy between tensor and free products seems, in certain respects, rather limited.

The above theorem was first stated, without a proof, in Sec. 8 of [Po6]. But in fact it was obtained prior to the rest of the results in [Po6]. It was this theorem that led us to the construction of irreducible subfactors of arbitrary index s, $N^s(Q) \subset M^s(Q)$, by using free traces on amalgamated free product algebras. Indeed, when suitably interpreted the theorem shows that such inclusions $N^s(Q) \subset M^s(Q)$ can be asymptotically recovered in any other irreducible inclusion of same index s.

The paper is organized as follows. In Sec. 1 we prove the technical results needed for the proof of the theorem. The proofs are inspired from (2.1 in [Po4]), where a slightly weaker version of the results here were obtained. The proofs rely on the local quantization principle ([Po1, 7]) and on a maximality argument, like in [Po4]. Conversely, the results in [Po1, 7] are immediate consequences of the theorem and its corollary, giving some sharp estimates as a bonus. This fact will be explained in Sec. 2, where the main result of the paper, a generalization of the above stated theorem, is proved, (see 2.1) and some more immediate corollaries are deduced. We expect it in fact to be useful for approaching some other problems as well, an aspect on which we comment in Sec. 3. Thus, we speculate on the possibility of having a functional analytical characterization of the free group algebras, on the indecomposability of such algebras and their possible embedding into other algebras. We also include a construction of separable type II_1 factors M with the fundamental group $\mathcal{F}(M)$ countable but containing any prescribed countable subset of $(0, \infty)$, e.g., with $\mathcal{F}(M) \supset \mathbb{Q}$.

We are grateful to D. Voiculescu for stimulating us to write down the proof of the result announced in Sec. 8 of [Po4], through his constant interest and motivating comments.

1 Some technical results

In what follows all finite von Neumann algebras are assumed given with a normal, finite, faithful trace, typically denoted by τ. For standard notations and terminology, we refer the reader to [Po6, 7].

We will also often use the following:

1.1. Notation. Let B be a von Neumann algebra. If $v \in B$ is a partial isometry with $v^*v = vv^*$, $S \subset B$ is a subset and $k \leq n$ are nonnegative integers then denote $S_v^{0,n} \overset{\text{def}}{=} S$ and $S_v^{k,n} \overset{\text{def}}{=} \left\{ b_0 \underset{i=1}{\overset{k}{\pi}} v_i b_i \,|\, b_i \in S,\ 1 \leq i \leq k-1, \quad b_0,\ b_k \in S \cup \{1\} \text{ and } \quad v_i \in \{v^j | 1 \leq |j| \leq n\} \right\}$.

The next lemma is the crucial technical result needed to prove the theorem in this paper:

1.2. Lemma. *Let $N \subset M$ be an inclusion of type II_1 von Neumann algebras. Assume $N' \cap M$ is finite dimensional. Let $\varepsilon > 0$, n a positive integer, $F \subset M$ a finite set and $f \in N$ a projection of scalar central trace in N such that $E_{N' \cap M}(b) = 0$, for all $b \in fFf$. Then there exists a partial isometry v in fNf such that:*

a) $v^*v = vv^*$ and its central trace in N is a scalar $> \dfrac{\tau(f)}{4}$.

b) $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon, \quad x \in \bigcup_{k=1}^{n} F_v^{k,n}$.

Proof. Let $\delta > 0$. Denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Let $\mathcal{W} = \{v \in fNf | v$ partial isometry, $v^*v = vv^*$, the central trace of v^*v in N is a scalar, and $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon_k \tau(v^*v)$, for all $1 \leq k \leq n$, $x \in F_v^{k,n}\}$. Endow $\mathcal{W}$ with the order $\leq$ in which $v_1 \leq v_2$ iff $v_1 = v_2 v_1^* v_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) \leq \tau(f)/4$. If w is a partial isometry in pNp, where $p = f - v^*v$, and if $u = v + w$ then for $x = b_0 \underset{i=1}{\overset{k}{\pi}} u_i b_i \in F_u^{k,n}$ we have

$$(1) \qquad x = b_0 \prod_{i=1}^{k} v_i b_i + \sum_{\ell} \sum_{i} z_0^i \prod_{j=1}^{\ell} w_{i_j} z_j^i$$

where $k \geq \ell \geq 1$, $i = (i_1, \ldots, i_\ell)$ with $1 \leq i_1 < \cdots < i_\ell \leq k$, $w_{i_j} = w^s$ if $v_{i_j} = v^s$, $z_0^i = b_0 v_1 b_1 \cdots b_{i_1 - 1} p$, $z_j^i = p b_{i_j} v_{i_j + 1} \cdots v_{i_{j+1}} p$, for $1 \leq j < \ell$ and $z_\ell^i = p b_{i_\ell} v_{i_\ell + 1} \cdots v_k b_k$ and where the sum is taken over all $\ell = 1, 2, \cdots, k$ and all $i = (i_1, \ldots, i_\ell)$.

By (A.1.4 in [Po7]), given any $\alpha > 0$ there exists a projection q in pNp, of scalar central support in pNp (and thus in N), such that

$$\|qzq - E_{(N'\cap M)p}(z)q\|_{1,pMp} < \alpha\tau_{pMp}(q) \tag{2}$$

for all z of the form z^i_j, for some $\ell \geq 2$, some $i = (i_1, \cdots, i_\ell)$ and $1 \leq j \leq \ell - 1$.

In the case $\ell = 2$ and $i_1 = 1, \quad i_2 = k$, if we take the partial isometry $w \in pNp$ so that $w^*w = ww^* = q$, then we get for $z = pb_1v_2b_2 \cdots v_{k-1}b_{k-1}p$:

$$\begin{aligned}
(3)\quad & \|E_{N'\cap M}(b_0w_1b_1v_2b_2 \cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \\
&\leq \|w_1b_1v_2b_2 \cdots b_{k-1}w_k\|_1 \\
&= \|qb_1v_2b_2 \cdots b_{k-1}q\|_1 \\
&= \|qzq\|_1 = \|qzq\|_{1,pMp}\tau(p) \\
&\leq \left(\|E_{(N'\cap M)p}(z)q\|_{1,pMp} + \alpha\tau_{pMp}(q)\right) \tau(p) \\
&= \left(\|E_{(N'\cap M)p}(z)\|_{1,pMp}\tau(q)/\tau(p) + \alpha\tau(q)/\tau(p)\right) \tau(p) \\
&= \left(\|E_{(N'\cap M)}(z)\|_1\tau(p)^{-1} + \alpha\right) \tau(q).
\end{aligned}$$

But since for $x \in N' \cap M, \quad v$ and $p = vv^*$ commute with x we get by taking into account that $vb_1v_2b_2 \cdots v_{n-1}b_{n-1}v^* \in F_v^{k,n}$ and $b_1v_2b_2 \cdots v_{k-1}b_{k-1} \in F_v^{k-2,n}$ the following estimate:

$$\begin{aligned}
(4)\quad & \|E_{N'\cap M}(z)\|_1 \\
&= \sup\{|\tau(zx)| | x \in N' \cap M, \quad \|x\| \leq 1\} \\
&= \sup\{|\tau(pb_1v_2b_2 \cdots v_{k-1}b_{k-1}x)| | x \in N' \cap M, \quad \|x\| \leq 1\} \\
&\leq \sup\{|\tau(b_1v_2b_2 \cdots v_{k-1}b_{k-1}x)| | x \in N' \cap M, \quad \|x\| \leq 1\} \\
&\quad + \sup\{|\tau(vb_1v_2b_2 \cdots v_{k-1}b_{k-1}v^*x)| | x \in N' \cap M, \quad \|x\| \leq 1\} \\
&= \|E_{N'\cap M}(b_1v_2b_2 \cdots v_{k-1}b_{k-1})\|_1 \\
&\quad + \|E_{N'\cap M}(vb_1v_2b_2 \cdots b_{k-1}v^*)\|_1 \\
&\leq \varepsilon_{k-2}\tau(v^*v) + \varepsilon_k\tau(v^*v).
\end{aligned}$$

By combining (3) and (4) and noting that $\tau(v^*v) \leq \tau(f)/4$ implies $\tau(v^*v)/\tau(p) \leq \tau(f)/3$, it follows that if we take $\alpha \leq \delta/3 < (\varepsilon_k - \varepsilon_{k-2})/3$ then we get:

$$\|E_{N'\cap M}(b_0w_1b_1v_2b_2 \cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \leq 2/3\varepsilon_k\tau(q). \tag{5}$$

Note now that $z = pb_1v_2b_2 \cdots v_{k-1}b_{k-1}p$ is the only element of the form z^i_j for which $i_2 - i_1 = k - 1$ and that it appears in the sum (1) only once, in the writing of the element $b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k = b_0w_1zw_kb_k$. For all other elements z^i_j with $\ell \geq 2$ we have $i_2 - i_1 < k-1$.

Thus, if the partial isometry $w \in pNp$ is supported on q like before, i.e., $w^*w = ww^* = q$ then we get:

(6) $\|E_{N'\cap M}(z_0w_{i_1}z^i_1w_{i_2}z^i_2\cdots w_{i_\ell}z^i_\ell)\|_1$

$$\begin{aligned}
&\leq \|w_{i_1}z^i_1w_{i_2}\|_1 \\
&= \|qz^i_1q\|_1 = \|qz^i_1q\|_{1,pMp}\tau(p) \\
&\leq \left(\|E_{(N'\cap M)_p}(z^i_1)q\|_{1,pMp} + \alpha\tau_{pMp}(q)\right)\tau(p) \\
&= (\|E_{N'\cap M}\ (z^i_1)\|_1\tau(p)^{-1} + \alpha\big)\,\tau(q).
\end{aligned}$$

Since $b_{i_1}v_{i+1+1}\cdots v_{i_2-1}b_{i_2-1} \in F_v^{i_2-i_1-1,n}$, with $i_2 - i_1 - 1 \leq k-3$, and $vb_{i_1}v_{i_1+1}b_{i_1+1}\cdots v_{i_2-1}b_{i_2-1}v^* \in F_v^{i_2-i_1+1,n}$ with $i_2 - i_1 + 1 \leq k-1$, we obtain like in (3), (4) that

$$\|E_{N'\cap M}(z^i_1)\|_1 \leq (\varepsilon_{k-1} + \varepsilon_{k-3})\tau(v^*v). \tag{7}$$

Combining (6) and (7) we obtain for $\tau(v^*v) \leq \tau(f)/4$ and for $\alpha \leq \delta/3 \leq (\varepsilon_{k-1} - \varepsilon_{k-3})/3$, the estimate:

$$\|E_{N'\cap M}(z_0w_{i_1}z^i_1w_{i_2}z^i_2\cdots w_{i_\ell}z^i_\ell)\|_1 \leq 2/3\varepsilon_{k-1}\tau(q). \tag{8}$$

Since $2^{k+1}\varepsilon_{k-1} = \varepsilon_k$ and since there are at most $\sum\limits_{i=2}^{k}\binom{k}{i} = 2^k - k - 1$ elements in the sum in (1) for which $\ell \geq 2$ and $i_2 - i_1 \neq k-1$, we get

(9) $$\sum_{\ell\geq 2}\ \sum_{i_2-i_1\neq k-1}\left\|E_{N'\cap M}\left(z_0\prod_{j=1}^{\ell}w_{i_j}z^i_j\right)\right\|_1$$
$$\begin{aligned}
&\leq 2/3(2^k - k - 1)\varepsilon_{k-1}\tau(q) \\
&\leq 1/3\varepsilon_k\tau(q) - (2k+2)/3\varepsilon_{k-1}\tau(q).
\end{aligned}$$

Finally, from the sum on the right hand side of (1) we will now estimate the terms with $\ell = 1$. These are terms which are obtained from $b_0v_1b_1v_2b_2\cdots v_kb_k$ by replacing exactly one v_i by w_i, so there are k of them. Note at this point that for the above estimates we only used the fact that $w^*w = ww^* = q$ and not the form of w. We will make the appropriate choice for w now, to get the necessary estimates for these last terms. To do this, note that for any finite set of elements $Y \subset M$, any integer $n \geq 1$ and any $\beta > 0$ there exists $w \in qNq$, $w^*w = ww^* = q$, such that

$$(10) \qquad |\tau(w^i x)| < \beta\tau(q), \quad \forall x \in Y, \quad 0 < |i| \leq n.$$

To see this, take for instance $A \subset qNq$ to be a maximal abelian algebra which since qNq is of type II_1 will be diffuse, thus it will contain a separable subalgebra of the form $L^\infty(\pi, \ \mu)$ and let w_0 be its generator, so that $\tau(w_0^m) = 0$ for all $m \neq 0$. Thus w_0^m tends weakly to 0 so that $w = w_0^m$ for large enough m will do. If we take $Y = \{b_j v_{j+1} \cdots v_k b_k e^p_{rs} b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j | 1 \leq j \leq k, \quad r,s,p\}$, where $\{e^p_{rs}\}_{p,r,s}$ is a matrix unit for $N' \cap M$, then by (10) we get for $w_i \in \{w^j | 0 < |j| \leq n\}$

$$\begin{aligned}
(11) \quad & \|E_{N'\cap M}(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k)\|_1 \\
&= \sup\{|\tau(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k x)| \,|\, x \in N' \cap M, \quad \|x\| \leq 1\} \\
&\leq \sum_{p,r,s} |\tau(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k e^p_{rs})| \\
&< (\dim N' \cap M)\beta\tau(q).
\end{aligned}$$

Thus, if β is chosen such that

$$k(\dim N' \cap M)\beta < (2k+1)/3\varepsilon_{k-1}, \quad \forall k \leq n$$

then by adding up (5), (9) and (11) we get from (1) the following estimates for $x \in F^{k,n}_{v+w}$, $1 \leq k \leq n$:

$$\begin{aligned}
\|E_{N'\cap M}(x)\|_1 \leq \; & \left\| E_{N'\cap M}\left(b_0 \prod_{i-1}^{k} v_i b_i\right)\right\|_1 \\
& + \sum_{\ell \geq 2} \sum_i \left\| E_{N'\cap M}\left(z_0^i \prod_{j=1}^{\ell} w_{i_j} z_j^i\right)\right\|_1 \\
& + \sum_j \|E_{N'\cap M}(b_0 v_1 b_1 \cdots w_j \cdots v_k b_k)\|_1 \\
\leq \; & \varepsilon_k \tau(v^*v) \\
& + 2/3\varepsilon_k \tau(q) + 1/3\varepsilon_k\tau(q) - (2k+2)/3\varepsilon_{k-1}\tau(q) \\
& + (2k+2)/3\varepsilon_{k-1} \\
= \; & \varepsilon_k \tau(v^*v + q) = \varepsilon_k\tau((v+w)^*(v+w)).
\end{aligned}$$

But this contradicts the maximality of $v \in \mathcal{W}$.

We conclude that $\tau(v^*v) > \tau(f)/4$. If δ is taken so that $\varepsilon_n < \varepsilon$ then the statement follows. Q.E.D.

1.3. Lemma. *Let $\{N_m \subset M_m\}_m$ be a sequence of inclusions of type II_1 von Neumann algebras with $\dim(N'_m \cap M_m) < \infty$, for all m. Let ω be a free ultrafilter on $\mathbb{N}$. Given any nonzero projection f in ΠN_m of scalar central support and any countable set of elements $X \subset \prod_\omega M_m$, with $E_{\prod_\omega N'_m \cap \prod_\omega M_m}(b) = 0$, for all $b \in fXf$, there exists a nonzero partial isometry $v \in f(\prod_\omega N_m)f$ such that $v^*v = vv^*$ has scalar central support in ΠN_m and such that for all n, all $k \leq n$ and all $x \in X_v^{k,n}$ one has:*

$$E_{\prod_\omega N'_m \cap \prod_\omega M_m}(x) = 0.$$

Proof. Let $f = (f_m)_m$ be a representation of f in ΠN_m, with $f_m \in N_m$ projections of scalar central support. Let $X = \{x^k\}_k$ and let $x^k = (x_n^k)_n$ be a representation of x^k in $\prod_\omega M_n$, with $E_{N' \cap M_n}(x_n^k) = 0$, for all k. For each n apply Lemma 1.2 for the inclusion $N_n \subset M_n$, the positive element $\varepsilon = 2^{-n}$, the integer n, the projection f_n and the finite set $X_n = \{x_n^k | k \leq n\}$ to get a partial isometry v_n in $f_n N_n f_n$ such that $v_n^* v_n = v_n v_n^*$ has central trace in N_n a scalar $\geq \tau(f_n)/4$ and

$$\left\| E_{N'_n \cap M_n}(x) \right\|_1 \leq 2^{-n}, \qquad x \in \bigcup_{k \leq n} (X_n)_{v_n}^{k,n}.$$

Then $v = (v_n)$ clearly satisfies the conditions. Q.E.D.

1.4. Lemma. *Let $\{N_n \subset M_\infty^n\}_n$ be a sequence of inclusions of type II_1 von Neumann algebras and assume that for each n there exists an increasing sequence of von Neumann subalgebras $\{M_k^n\}_k$ of M_∞^n, containing N_n such that $M_\infty^n = \overline{\bigcup_k M_k^n}$ and $N'_n \cap M_k^n$ is atomic for each k. Let ω be a free ultrafilter on $\mathbb{N}$. Given any countable set $Y \subset \prod_\omega M_\infty^n$ with $E_{\prod_\omega N'_n \cap \prod_\omega M_\infty^n}(b) = 0$, for all $b \in Y$, there exists a unitary element $v \in \prod_\omega N_n$ such that $\tau(w) = 0$ for any word w of alternating letters a_i, b_i (i.e., $w = a_1 b_1 a_2 b_2 \cdots$, or $w = b_1 a_1 b_2 a_2 \cdots$, ending either with b_i or a_i) where $b_i \in Y$ and where each a_i is of the form v^{n_i}, for some $n_i \neq 0$, we have*

$$E_{\prod_\omega N'_n \cap \prod_\omega M_\infty^n}(w) = 0.$$

Proof. Since Y is countable there exist $k_1 < k_2 < \ldots$ and $p_n \in N'_n \cap M_{k_n}^n$ such that $Y \subset \prod_\omega p_n M_{k_n}^n p_n$, and $N'_n p_n \cap p_n M_{k_n}^n p_n$ is finite dimensional for each n. Let $\mathcal{W} = \{v \in \prod_\omega N_n | v$ partial isometry, $v^*v = vv^*$ has scalar central support in $\prod_\omega N_n$, $E_{\prod_\omega N'_n \cap \prod_\omega M_n}(x) = 0$, for all $k \leq n$ and all $x \in Y_v^{k,n}\}$, where $M_n = M_{k_n}^n$. Endow $\mathcal{W}$ with the order given by $v_1 \leq v_2$ iff $v_1 = v_2 v_1^* v_1$. Then $(\mathcal{W}, \leq)$ is clearly inductively ordered. Let v be a maximal element of $\mathcal{W}$. Assume v is not a unitary element and let $f = 1 - v^*v \neq 0$. Then $E_{(\Pi N_n)' \cap (\Pi M_n)}(fxf) = 0$, for all $x \in X \overset{\text{def}}{=} \bigcup_{k \leq n} Y_v^{k,n}$. Indeed, because for $y \in (\prod_\omega N_n)' \cap (\prod_\omega M_n)$ we have $\tau(fxfy) = \tau(fxy) = \tau((1-v^*v)xy) = \tau(xy) - \tau(vxv^*y) = 0$, since either $x \in Y_v^{k,n}$ begins or ends with a nonzero power of v (when $x = b_0 v_1 b_1 \cdots v_k b_k$ with either b_0 or b_k equal to 1) or $vxv^* \in Y_v^{k+2}$, so both $\tau(xy) = 0$, $\tau(vxv^*y) = 0$ by the fact that $v \in \mathcal{W}$ and by the definition of $\mathcal{W}$.

Thus Lemma 1.3 applies to get a nonzero partial isometry $u \in f \prod_\omega N_n f$ such that $u^*u = uu^*$ and $E_{(\Pi N_n)' \cap (\Pi M_n)}(x) = 0$ for all $x \in \bigcup_{k \leq n} X_u^{k,n}$. But then $v + u \in \mathcal{W}$, $v + u \geq v$ and $v + u \neq v$, thus contradicting the maximality of v.

We conclude that v must be a unitary element and it then clearly satisfies the conditions. Q.E.D.

2 The main results

We can now deduce the main result of this paper:

2.1. Theorem. *Let $\{N_n \subset M_\infty^n\}_n$ be a sequence of inclusions of type* II_1 *von Neumann algebras. Assume that for each n there exists an increasing sequence of subalgebras $\{M_k^n\}_k$ in M_∞^n, generating M_∞^n, containing N_n, and such that $N_n' \cap M_k^n$ is atomic for each k. Let $\{P_j\}_j$ be a countable family of separable von Neumann subalgebras of $\prod_\omega M_\infty^n$ with a common subalgebra $B \subset P_j$ such that $E_{(\prod_\omega N_n)' \cap (\Pi M_n^n)}(P_j) = B$ for all j, i.e., $(\Pi N_n)' \cap P_i = B$, $\forall i$, and such that for each i P_i, $(\Pi N_n)' \cap (\Pi M_\infty^n)$ and B satisfy the commuting square condition of ([Po3]). Then there exists a unitary element v in $\prod_\omega N_n$ such that $\bigvee_j v^j P_j v^{-j} = \ast_B v^j P_j v^{-j}$, i.e. the algebras $v^j P_j v^{-j}$ generate an amalgamated free product over B.*

Proof. Since $\{P_j\}_j$ are all separable, there exists a countable set $Y \subset \bigcup_j P_j$, with $E_{(\Pi N_n)' \cap (\Pi M_n)}(x) = E_B(x) = 0$ for $x \in Y$, dense in the norm $\| \ \|_2$ in the set $\bigcup_j (P_j \ominus B)$. Thus, if v satisfies the conditions in the conclusion of Lemma 1.4 for the set Y, then $v^j P_j v^{-j}$ will clearly generate an amalgamated free product over their common subalgebra B. Q.E.D.

Note that Theorem stated in the introduction is just the case $N_n = N$, $M_\infty^n = M_\infty$ for all n of Theorem 2.1 and that its Corollary is then trivial. Another feature of Theorem 2.1, the proof of which relied almost entirely on the local quantization principle of [Po1, 7], is that it can be used to get back that theorem, with some sharp estimate, coming from direct calculation (in the norm $\| \quad \|_2$ case) and from the spectral estimates in ([V1]) (in the norm $\| \quad \|$ case).

2.2. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors with trivial relative commutant and let ω be a free ultrafilter on $\mathbb{N}$. Let $B \subset M^\omega$ be a separable von Neumann subalgebra. Given any n there exists a partition of the unity $\{p_i\}_{1 \leq i \leq n}$ in N^ω, with projections of trace $1/n$, such that:*

(i) *For any $x \in B$, with $\tau(x) = 0$, and any i, one has:*

$$\|p_i x p_i\|_2^2 = \|x\|_2^2 1/n\tau(p_i).$$

(ii) *For any projection $f \in B$, with $\tau(f) \leq 1 - 1/n$, and any i one has:*

$$\|p_i(f - \tau(f)1)p_i\| = 1/n - 2\tau(f)/n + \sqrt{4\tau(f)(1-\tau(f))1/n(1-1/n)}.$$

(iii) *For any unitary element $v \in B$, with either $\tau(v^k) = 0, \quad \forall k \neq 0$, or $v^m = 1$ for some $m \geq 2$ and $\tau(v^k) = 0, \quad 0 < |k| < m$, and any i one has:*

$$\|p_i v p_i\| = 4/n(1 - 1/n).$$

Proof. By Theorem A there exists a separable diffuse abelian subalgebra A in N^ω which is free with respect to B. Then [V1] applies to get (ii) and (iii). Direct calculations then give (i), since $\|p_i x p_i\|_2^2 = \tau(p_i x p_i x^* p_i) = \tau(p_i x x^* p_i)\tau(p_i) = \|x\|_2^2 \tau(p_i)^2$. Q.E.D.

2.3. Remark. For each n, $N \subset M$ an inclusion of type II_1 factors with trivial relative commutant and $x \in M$ an element with $\tau(x) = 0$ let $c_n(N \subset M; x) = \inf\left\{ \sum_{i=1}^{n} \|p_i x p_i\|_2^2 | \{p_i\}_{1 \leq i \leq n} \subset N \text{ partition of the unity with projections of trace } 1/n\right\}$. Let $c_n = \sup\{c_n(N \subset M; x) | N \subset M, \quad N' \cap M = \mathbb{C} \text{ and } x \in M \text{ as above}\}$. The above Corollary shows that $c_n(N \subset M; x) \leq 1/n$, for all $N \subset M$ and $x \in M, \quad \|x\|_2 \leq 1$. If one takes $M = N * B$ for some B without atoms then for any $x \in B, \quad \|x\|_2 = 1, \quad \tau(x) = 0$, and any partition $\{p_i\}_{1 \leq i \leq n}$ in N, as above, $\sum_{i=1}^{n} \|p_i x p_i\|_2^2 = 1/n$. Thus $c_n = 1/n$. This shows in particular that $1/n + \varepsilon$, with ε arbitrarily small, are the best constants α for which given any inclusion $N \subset M$, with trivial relative commutant and any $x \in M$ as above, one can find projections q of trace $1/n$ so that $\|qxq\|_2^2 < \alpha\tau(q)$.

2.4. Corollary. *If $N \subset M$, with $N' \cap M = \mathbb{C}1$, and ω are as in the preceding corollary, then for any $x \in M^\omega$, $\quad \tau(x)1 \in \overline{\mathrm{co}}^n\{uxu^* | u \in \mathcal{U}(N^\omega)\}$.*

Proof. Since any $x \in M^\omega$ is in the norm closure of the linear combinations of projections in M^ω and since by (i),b) of Corollary 2.2 we have the statement for $x = f$ a projection, by taking $u = \Sigma\lambda^i p_i$, $\lambda = \exp 2\pi i/n$ and by using that $\sum_{i=1}^{n} p_i f p_i = \frac{1}{n}\sum_{k=1}^{n} u^k f u^{-k}$, it follows arguing like in Dixmier's theorem ([D2]) that we get the statement for any $x \in M^\omega$. Q.E.D.

Let us finally deduce here the result stated without proof in (8.1 of [Po6]), which we already mentioned in the introduction.

2.5. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors of finite index $[M : N] = s$. Assume $N \subset M$ is extremal ([PiPo],[Po7]), i.e., $E_{N' \cap M}(e_0) \in \mathbb{C}1$ for $e_0 \subset M$ a Jones projection (i.e., $E_N(e_0) = s^{-1}1$). Let $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$ be the associated Jones tower and $M_\infty = \overline{\bigcup_n M_n}$ be its enveloping algebra. Denote by $R = vN\{e_1, \cdots\}$, $R^s = vN\{e_2, \cdots\}$. If $Q_0 \subset M^\omega$ is a separable von Neumann algebra then there exists a unitary element $v \in N^\omega$ such that $Q = vQ_0v^*$ and R generate the algebra $M^s_\infty(Q) = (Q \otimes R^s) \underset{R^s}{*} R$ in a way that identifies $Q \subset Q \vee R$ with $Q \simeq (Q \otimes \mathbb{C}) * \mathbb{C} \subset M^s_\infty(Q)$ and $R \subset Q \vee R$ with $R \simeq \mathbb{C} * R \subset M^s_\infty(Q)$. Moreover $M^s_\infty(Q)$ makes commuting squares with N^ω, M^ω and M^ω_n, $n \geq 1$, and one has $N^\omega \cap M^s_\infty(Q) = N^s(Q)$, $M^\omega \cap M^s_\infty(Q) = M^s(Q)$, $M^\omega_n \cap M^s_\infty(Q) = M^s_n(Q)$, $n \geq 1$, where $N^s(Q) \subset M^s(Q) \subset M^s_1(Q) \subset \cdots$ are the subalgebras of $M^s_\infty(Q)$ defined in* [Po6].

Proof. Since $N \subset M$ is extremal, $E_{M' \cap M_1}(e_1) \in \mathbb{C}1$, so that $E_{M' \cap M_\infty}(R) = R^s \subset M' \cap M_\infty$ and the conditions of Theorem 2.1 are fulfilled for $P_1 = Q_0 \vee R^s$ and $P_2 = R$ and $B = R^s$. The rest follows by the definitions of $M^s_n(Q)$ in [Po6]. Q.E.D.

3 Some problems and comments

3.1. Embeddings of $L(\mathbb{F}_n)$. The following is a problem for which the above results may be useful. It was first posed in [Po5] and it can be regarded as the operator algebra analogue of von Neumann's problem on the embeddings of free groups into nonamenable groups.

3.1.1. Problem. Does any nonhyperfinite type II_1 factor contain copies of $L(\mathbb{F}_2)$? Do all the non Γ factors, or at least the property T factors, necessarily contain copies of $L(\mathbb{F}_2)$?

Upon inspection of the proof of 2.1 one can see that a main obstruction for getting back in M the asymptotically free sequences in M^ω seems to be to obtain a "rigidity" type result that would insure $\tau(w) = 0$ for any word w in u and v once one has some u', v' for which $\tau(w') = 0$ for words w' in u', v' of length less than some n. The necessary rigidity feature could be provided by the property non Γ (or T) assumption on the factor M. Related to this, note that if $u \in M$ is the generator of a Cartan subalgebra A of M then by 2.1 there are $v \in M^\omega$ which are free with respect to u but there are no such v in M. Indeed, if $N \subset M$ is any von Neumann subalgebra of M that contains A then A is Cartan in N (see e.g. [JPo]]), but if $N = \{u, v\}''$ with v free with respect to u then A is singular in N (see e.g. [Po2]).

3.2. Functional analytical characterization of $L(\mathbb{F}_n)$. We recall that there exists no satisfactory functional analytical characterization of the free group factors.

Related to this, we propose here, the following working conjecture (see also [dlHV] and [dlH] in these Proceedings for related problems and comments):

3.2.1. Problem. Are all separable non Γ (or full [C1]) type II_1 factors which have the Haagerup approximation property mutually isomorphic? Are all non Γ subfactors of $L(\mathbb{F}_\infty)$ isomorphic to $L(\mathbb{F}_\infty)$?

Recall that M has the Haagerup approximation property [H] if the identity on M can be approximated by compact unital trace-preserving completely positive maps, in the point-$\| \ \|_2$ topology. Note that this property is clearly inherited by subalgebras. One can strengthen this property in the above assumption by requiring the compact maps to have only unitaries from M as eigenvectors.

While at this stage this problem seems hopelessly difficult, due to Voiculescu's noncommutative probability approach, the problem of the mutual isomorphism of the $L(\mathbb{F}_n), 2 \leq n \leq \infty$, which is a particular case of 3.2.1, may soon be resolved. In fact, by the recent results in [Ra1, 2, 3], in order to show that all the $L(\mathbb{F}_\kappa)$ are isomorphic it is necessary and sufficient to prove that for some n the fundamental group of $L(\mathbb{F}_n)$ is nontrivial or that $L(\mathbb{F}_n) \otimes \mathcal{B}(H) \simeq L(\mathbb{F}_\infty) \otimes \mathcal{B}(H)$. A related problem that was not yet solved is whether $L(\mathbb{F}_n) \otimes R$ is isomorphic to $L(\mathbb{F}_\infty) \otimes R$ or not and whether this would be sufficient to insure the isomorphism of $L(\mathbb{F}_n)$ with $L(\mathbb{F}_\infty)$.

3.2.2. Problem. Are all non Γ type II_1 factors with the Haagerup approximation property stably isomorphic?

3.2.3. Problem. If M is non Γ and is approximable by subfactors isomorphic to $L(\mathbb{F}_\infty)$ is then M itself isomorphic to $L(\mathbb{F}_\infty)$? Is at least the stabilized version of this true?

Related to this note that by [C4] there are plenty of examples of nonisomorphic nonhyperfinite type II_1 factors M with $M \simeq M \otimes R$ (and even factorial $M' \cap M^\omega$, i.e., without hypercentral sequences) which have the Haagerup approximation property.

If one seeks to give a negative answer to 3.2.1 then a class of factors that could be tested are the factors $M^s(Q)$ of [Po6], since they are non Γ by [Po6] and have the Haagerup approximation property when Q has it by [Bo]. For each $Q = L(\mathbb{F}_x)$, as defined in [Ra2], [Dy] for $x \in (1, \infty]$, and for each $s \in \{4\cos^2 \pi/n | n \geq 4\}$, the factors $M^s(Q)$ were proved to be of the form $L(\mathbb{F}_y)$, for some $y \in (1, \infty]$ depending on x and s, in [Ra2]. But for $s > 4$ and/or $Q = L(\mathbb{F}_1) = L^\infty(\mathbb{T}, \mu)$, $\quad Q = R$, the problem of identifying $M^s(Q)$ (or its generalizations [Ba], [Ra3]) as some $L(\mathbb{F}_y)$ is still open. If answered positively, this would prove the existence of irreducible subfactors of any index $s > 4$ in $L(\mathbb{F}_x)$, $\quad 1 < x \leq \infty$.

3.3. Primeness (or indecomposability) for factors. (see also [dlH]). Besides implying the mutual isomorphism of all $L(\mathbb{F}_n)$, $\quad 2 \leq n \leq \infty$, a positive answer to 3.2.1 would imply that $L(\mathbb{F}_\ltimes) \otimes \mathbb{L}(\mathbb{F}_\gtrdot) \simeq \mathbb{L}(\mathbb{F}_\infty)$, $\quad \nvDash \leq \ltimes, \gtrdot \leq \infty$, in particular that the type II_1 factors that are associated to free groups are decomposable i.e., that can be written as a tensor product of type II_1 factors, in other words, that they are not "prime". However, the result in [Po2] showing that for uncountable sets of generators S the type II_1 factors $L(\mathbb{F}_S)$ are prime (or indecomposable) gives an indication that the same result may be true for finitely and countably many generators as well. Thus, the algebras $L(\mathbb{F}_n) \otimes L(\mathbb{F}_m)$ are good candidates for giving a negative answer to Problem 3.2.1 and the following seems important to settle:

3.3.1. Problem. Are $L(\mathbb{F}_n)$, $\quad 2 \leq n \leq \infty$, prime factors? Do there exist separable prime factors at all? Do property T factors have a decomposition into a (unique?) product of prime factors?

In case $L(\mathbb{F}_n)$ turns out to be prime, then one probably has to add primeness to the conditions in Problem 3.2.1.

On the other hand, as many situations show, results holding true in nonseparable situations can hardly suggest what the answer to the similar separable situation would be, and vice versa. We will illustrate this by two examples:

For instance, it is showed in [Po1] that if $M \subset P$ are separable type II_1 factors with $M' \cap P = \mathbb{C}$ then M contains maximal abelian subalgebras of P But that if M is non Γ and $P = M^\omega$ then no maximal abelian subalgebra of M is maximal abelian in M^ω (although $M' \cap M^\omega = \mathbb{C}$). So if M^ω is regarded as a union of an increasing net of separable type II_1 factors M_i that contain M then for each i there is a masa of M_i in M but at the limit there is none.

Let us consider a second example when a certain property that holds true in a separable situation "blows" when passing to nonseparability, showing one more time that one could hardly guess whether or not $L(\mathbb{F}_n)$ are prime, just from the fact that $L(\mathbb{F}_S)$ are prime for uncountable sets S.

We first need 2 simple observations:

3.3.2. Lemma. *If $N \subset M$ are type II_1 factors and N has the property T then $N' \cap M^\omega = (N' \cap M)^\omega$.*

Proof. If $x = (x_n) \in N' \cap M^\omega$, then $\lim_{n\to\infty} \|[y, x_n]\|_2 = 0$, for any element $y \in N$, so that by the property T of N this holds true uniformly for $y \in M$, $\|y\| \leq 1$. Thus ([Po5]) $\lim_{n\to\omega} \|x_n - E_{N'\cap M}(x_n)\| = 0$. Q.E.D.

3.3.3. Lemma. *If M is a type II_1 factor then $\overline{\mathcal{F}(M)} \subset \mathcal{F}(M^\omega)$.*

Proof. Let $t \in \overline{\mathcal{F}(M)}$, $0 < t < 1$, and $t_n \in (0,1) \cap \mathcal{F}(M)$ be such that $\lim_n t_n = t$. Let $p_n \in \mathcal{F}(M)$ be projections with $\frac{\tau(p_n)}{\tau(1-p_n)} = t_n$ and $\theta_n : p_n M_{p_n} \simeq (1-p_n)M(1-p_n)$ onto isomorphisms. Then $p = (p_n) \in M^\omega$, $\tau(p) = t$ and $\theta = \prod \theta_n$ is an isomorphism of $pM^\omega p$ onto $(1-p)M^\omega(1-p)$. Thus $t \in \mathcal{F}(M^\omega)$. Q.E.D.

3.3.4. Corollary. *If M is a separable type II_1 factor containing a subfactor Q with the property T such that $Q' \cap M = \mathbb{C}$ then any separable subfactor $P \subset M^\omega$ that contains Q has countable fundamental group, although M^ω itself may have fundamental group $(0, \infty)$.*

Proof. By [Po6], given any $s \in (0,\infty)$ there exists a separable type II$_1$ factor M^s containing a subfactor Q with the property T and trivial relative commutant, and having s in its fundamental group. If $s_1, s_2 \in (0,\infty)$ are so that the multiplicative group generated by s_1, s_2 in $(0,\infty)$ is dense in $(0,\infty)$ then let $M = M^{s_1} \otimes M^{s_2}$, which is separable, has the subfactor $Q \otimes Q$ with the property T and trivial relative commutant, and has fundamental group containing s_1, s_2, thus dense in $(0,\infty)$. Then the above two lemmas apply, together with the result in [Po5], which shows that separable factors that contain a subfactor with the property T and trivial relative commutant have countable fundamental group. Q.E.D.

At this point it is worth mentioning a trivial consequence of the above construction: while in [Po6] we proved that given any finite set of elements $S \subset (0,\infty)$ there exist separable type II$_1$ factors with the fundamental group countable but containing that finite set, we can now prescribe countable sets as well.

3.3.5. Corollary. *Given any countable set $S \subset (0, \infty)$ there exist a separable type II_1 factor M with countable fundamental group $\mathcal{F}(M)$ containing the set S. In particular, by taking S to contain all the positive rationals, one gets examples of separable type II_1 factors M with $\mathcal{F}(M)$ countable and such that $M_{n \times n}(M) \simeq M$ for all n.*

Proof. Let Q be a type II_1 factor with the property T, $s_1, s_2 \in (0, \infty)$ generating a dense multiplicative subgroup in $(0, \infty)$ and let $N = M^{s_1}(Q) \otimes M^{s_2}(Q)$ ([Po6]). Since $s_i \in \mathcal{F}(M^{s_i}(Q))$ we have that $s_1, s_2 \in \mathcal{F}(N)$, so that $\mathcal{F}(N^\omega) = (0, \infty)$, by 3.3.3. Let $S = \{t_n\}_n$. For each n there exists a t_n-scaling automorphism θ_n of $N^\omega \otimes \mathcal{B}(H)$. Let $\tilde{M}$ be the von Neumann algebra generated by

$$\cup\{\theta_{k_1}^{i_1} \theta_{k_2}^{i_2} \cdots \theta_{k_j}^{i_j}(N \otimes \mathcal{B}(H)) | j \geq 0, k_1, \ldots, k_j \geq 1, i_1, \ldots, i_j \in \mathbb{Z}\}.$$

Then $\tilde{M}$ is separable and $\theta_n(\tilde{M}) = \tilde{M}$, for all n.

Since $\tilde{M} \supset N \otimes \mathcal{B}(H)$, there exists $M \supset N$, a separable type II_1 factor, such that $\tilde{M} = M \otimes \mathcal{B}(H)$. Since $Q \otimes Q$ has the property T and $(Q \otimes Q)' \cap N = \mathbb{C}$, by 3.3.2 $(Q \otimes Q)' \cap M \subset (Q \otimes Q)' \cap N^\omega = \mathbb{C}$. Thus M has countable $\mathcal{F}(M)$. But since $\theta_n(M \otimes \mathcal{B}(H)) = M \otimes \mathcal{B}(H)$ and θ_n is t_n-scaling the trace, we obtain that $t_n \in \mathcal{F}(M)$, thus $S \subset \mathcal{F}(M)$. Q.E.D.

Added in the proof: It was pointed out to us by D. Bisch and V. Jones that a result similar to Corollary 3.3.5 was already obtained by V. Ya. Golodets and N. I. Nessonov in their paper "T-property and nonisomorphic full factors of type II and III", J. Funct. Analysis, 70 (1987), 80-89, by using a different method.

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A type III_λ factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$

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A TYPE III_λ FACTOR WITH CORE ISOMORPHIC TO THE VON NEUMANN ALGEBRA OF A FREE GROUP, TENSOR $B(H)$.

FLORIN RĂDULESCU

In this paper we obtain a type III_λ factor by using the free product construction from [Vo1,Vo2] and show that its core ([Co]) is $\mathcal{L}(F_\infty) \otimes B(H)$. We will prove that

$$M_2(\mathbb{C}) * L^\infty([0,1],\nu)$$

is a type III_λ factor if $M_2(\mathbb{C})$ is endowed with a nontracial state. Moreover we will show that the core ([Co]) of this type III_λ factor (when tensorized by $B(H)$) is $\mathcal{L}(F_\infty) \otimes B(H)$ and we will give an explicit model for the associated (trace scaling) action of $\mathbb{Z}$ on the core (cf. [Co], [Ta]). Here $B(H)$ is the space of all linear bounded operators on a separable, infinite dimensional Hilbert space H.

Recall from [Vo1], that a family $(A_i)_{i\in I}$ of subalgebras in a von Neumann algebra M with state ϕ, is free with respect to ϕ if $\phi(a_1a_2...a_k) = 0$ whenever

$$\phi(a_i) = 0, a_i \in A_{j_i}, i = 1,2,...k, j_1 \neq j_2,...j_{k-1} \neq j_k.$$

Reciprocally given a family $(A_i, \phi_i), i \in I$ of von Neumann algebras with faithful normal states ϕ_i , one may construct (see[Vo1]) the (reduced) free product von Neumann algebra $*A_i$, which contains $A_i, i \in I$ and has a faithful normal state ϕ so that $\phi|_{A_i} = \phi_i$ and so that the algebras $(A_i)_{i\in I}$ are free with respect to ϕ.

The aim of this paper is to show the following result.

Theorem. *Let $\mathcal{E} = M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ be endowed with the free product state ϕ where $M_2(\mathbb{C})$ is endowed with the state ϕ_0 which is subject to the condition*

$$\phi_0(e_{11})/\phi_0(e_{22}) = \lambda \in (0,1) \text{ and } \phi(e_{12}) = \phi(e_{21}) = 0,$$

while $L^\infty([0,1],\nu)$ has the state given by Lebesgue measure on [0,1]. With these hypothesis, $M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ is a type III_λ factor and its core is isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$.

In the proof of the theorem we will also obtain a model for the core of $\mathcal{E} \otimes B(H)$ and for the corresponding (dual) action on the core, of the modular group of the weight $\phi \otimes tr$ (tr is the canonical semifinite trace on $B(H)$). This model will be a submodel of the one parameter action of $\mathbb{R}_+/\{0\}$ on $\mathcal{L}(F_\infty) \otimes B(H)$,that we have constructed in [Ra].

The model. *Model for the core of $(M_2(\mathbb{C}) * L^\infty([0,1],\nu)) \otimes B(H)$ and of the corresponding dual action on the core for the modular group of automorphism for the weight $\phi \otimes tr$:*

Let $\mathcal{A}_0$ be the subalgebra in the algebraic free product

$$L^\infty(\mathbb{R}) * (\mathbb{C}[X] * \mathbb{C}[Y])$$

generated by $\{pXp, pYp, p|\ p \text{ finite projection in } L^\infty(\mathbb{R})\}$ where $L^\infty(\mathbb{R})$ is endowed with the Lebesgue measure.

Let τ be the unique trace on $\mathcal{A}_0$ defined by the requirement that the restriction τ_p to the algebra generated in $p\mathcal{A}p$ by $pXp, pYp, pL^\infty(\mathbb{R})$ is subject to the following conditions:

(i) The three algebras generated respectively by pXp, pYp, $pL^\infty(\mathbb{R})$ are free with respect to τ_p

(ii) $\tau(p)^{-1/2}pXp, \tau(p)^{-1/2}pYp$ are semicircular (with respect τ_p)(see [Vo1] for the definition of a semicircular element).

Such a construction is possible because of the Theorem 1 in [Ra].

Assume that pXp, pYp are selfadjoint and let $\mathcal{A}$ be the weak completion of $\mathcal{A}_0$ in the G.N.S. representation for τ. Then (cf. [Ra]), $\mathcal{A}$ is a type II_∞ factor isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$ and the trace τ extends to a semifinite normal trace on $\mathcal{A}$ (which we also denote by τ).

Recall (by [Ra]) that in this case, there exists a one parameter group of automorphism $(\alpha_t)_{t\in\mathbb{R}_+\setminus\{0\}}$ on $\mathcal{A}$, scaling trace by t, for each $t \in \mathbb{R}_+ \setminus \{0\}$, which is induced by $d_t * M_t$ on $L^\infty(\mathbb{R}) * (\mathbb{C}[X] * [Y])$ where d_t is dilation by t on $L^\infty(\mathbb{R})$, while $M_t(X) = t^{-1/2}X; M_t(Y) = t^{-1/2}Y$, $t > 0$.

Let $\mathcal{B}$ the von Neumann subalgebra of $\mathcal{A}$ generated by

$$q_n = \chi_{[\lambda^{n-1},\lambda^n]}, n \in \mathbb{Z},$$

the characteristic functions of the intervals $[\lambda^{n-1}, \lambda^n]$ and by the following subsets of $\mathcal{A}$:

$$\tilde{X} = \{q_nXq_m | n,m \in \mathbb{Z}, |n-m| \leq 1\},$$
$$\tilde{Y} = \{q_nYq_n | n \in \mathbb{Z}\}.$$

Clearly $\mathcal{B}$ is invariant under $\{\alpha_{\lambda^n}\}_{n\in\mathbb{Z}}$ and by Lemma 3 in [Ra], $\mathcal{B}$ is isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$. Let $\beta_n = \alpha_{\lambda^n}|\mathcal{B}$.

Let $\mathcal{D} = \mathcal{B} \rtimes_\beta \mathbb{Z}$ be the cross product of $\mathcal{B}$ by the action $\mathbb{Z}$ given by β. Then by [Co], $\mathcal{D}$ is a type III_λ factor. Let $u \in \mathcal{D}$ be the unitary implementing the cross product. Moreover let ψ be the normal semifinite faithful weight on $\mathcal{D}$ obtained as the composition expectation from $\mathcal{D}$ onto $\mathcal{B}$.

We will prove that $\mathcal{B}$, with the action of $\mathbb{Z}$ given by $(\beta_n)_{n\in\mathbb{Z}}$ is isomorphic to the core of $\mathcal{E} \otimes B(H)$, with the dual action (on the core) for the modular group of automorphisms of the weight $\phi \otimes tr$ on $\mathcal{E} \otimes B(H)$. Our main result will be a consequence of the following proposition:

Proposition.

Let $\mathcal{E}$ be the von Neumann algebra free product $M_2(\mathbb{C}) * L^\infty([0,1],\nu)$, with the free product state $\phi = \phi_0 * \nu$, where $M_2(\mathbb{C}) = (e_{ij})_{i,j=1}^2$ is endowed with the normalized state ϕ_0 with $\phi(e_{11})/\phi(e_{22}) = \lambda$ and $\phi(e_{12}) = \phi(e_{21}) = 0$. Then, with the above notation $\mathcal{E}$ is isomorphic to $(q_o + q_1)\mathcal{D}(q_0 + q_1)$.

Moreover the state ϕ on $\mathcal{E}$ is (via this identification) the (normalized) restriction of ψ to $(q_o + q_1)\mathcal{D}(q_0 + q_1)$.

(Here $\mathcal{D} = \mathcal{B} \rtimes_\beta \mathbb{Z}$, where $\mathcal{B}$ is the von Neumann subalgebra in $\mathcal{A}$ generated by $\tilde{X} = \{q_n X q_m | n, m \in \mathbb{Z}, |n-m| \le 1\}$, $\tilde{Y} = \{q_n Y q_n | n \in \mathbb{Z}\}$ and the characteristic functions $q_n = \chi_{[\lambda^{n-1},\lambda^n]}, n \in \mathbb{Z}$, $q_n \in L^\infty(\mathbb{R}) \subseteq \mathcal{A}$. Moreover $\beta_n = \alpha_{\lambda^n}, n \in \mathbb{Z}$.)

Recall from above that the von Neumann algebra $\mathcal{A}$ is a type II_∞ factor isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$ and $\mathcal{A}$ is generated by

$$\{pXp, pYp, p|\ p \text{ finite projection in } L^\infty(\mathbb{R})\}.$$

Here $\alpha_t, t > 0$ acts as dilation by t on $L^\infty(\mathbb{R})$ and multiplies X, Y by $t^{-1/2}$. The trace on $\mathcal{A}$ is subject to the above conditions (i), (ii) and it is scaled by the automorphisms $\alpha_t, t > 0$.

This proposition will be a consequence of the following two lemmas.

Lemma 1.

With $\mathcal{A}, \mathcal{B}, \mathcal{D}, \psi, \tau$ and u as before let

$$e_{11} = q_1 u = u q_0; e_{11} = q_0; e_{22} = q_1$$

Let $a = x + y$, where

$$x = (q_0 + q_1)X(q_0 + q_1) - q_0 X q_0$$

$$y = q_0 Y q_0.$$

Then $M_2(\mathbb{C}) = (e_{ij})_{i,j=1}^2$ is free with respect to

$$\psi_1 = (\psi(q_0 + q_1))^{-1}\psi|_{(q_o+q_1)\mathcal{D}(q_0+q_1)},$$

to the semicircular element a, in the algebra $(q_0+q_1)\mathcal{D}(q_0+q_1)$ *with unit* q_0+q_1.

Proof. We have to check freeness, which means that the value of ψ_1 on certain monomials in a, u, e_{11}, e_{22} is null. Since by definition , ψ_1 vanishes the monomials containing a different number of u 's and u^* 's, we have only to check this if the number of occurrences for u is equal to the one for u^*.

Let $p_n = q_n + q_{n+1} = \chi_{[\lambda^{n-1},\lambda^{n+1}]}$.

Using the fact that u implements β_1 on $\mathcal{D}$ it follows that we only have to check $\psi_1(m) = 0$ if

$$m = p_0 f_1 q_{i_1} f_2 q_{i_2} f_3 \dots q_{i_n} f_{n+1} p_0$$

where the following conditions are fulfilled:

(a) i_{j+1} is either i_j or $i_j \pm 1$.

(b) Card $\{s | i_j = s, j = 1,2,\dots,n\}$ is even for every s.

(c) f_k is a product

$$f_1^k a_1^k \dots f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k,\ n_k \geq 1$$

where f_s^k, $s = 1,2,3\dots n_k$, is an element of null value under the state ψ_1 in the algebra generated by $\alpha_j(a)$ while a_s^k is an element of null trace in the algebra generated by q_j, q_{j+1}. Here j is an integer which is completely determined, for each k. If $i_k \neq i_{k+1}$ then j is the minimum of the i_k and i_{k+1}. If $i_k = i_{k+1}$ then j is either i_k if $i_{k-1} \leq i_k$ or either $i_k - 1$ if $i_{k-1} > i_k$.

To see that those are all the monomials of null state that may appear in the algebra generated by $M_2(\mathbb{C})$ and a it is sufficient to note that any string

$$f_1 e_{21} f_2 e_{21} \dots f_p e_{21} f_{p+1} e_{12} f_{p+2} e_{12} \dots e_{12} f_{2p+1} =$$

$$= f_1(q_1 u) f_2 q_1 u \dots f_p q_1 u f_{p+1} (u^* q_1) \dots (u^* q_1) f_{2p+1},$$

after cancelation, is equal to

$$f_1(q_1 u) f_2 \dots q_1 u f_p q_1 \beta_1(f_{p+1}) q_1 f_{p+2} (u^* q_1) \dots (u^* q_1) f_{2p+1} =$$

$$= f_1(q_1 u) f_2 \dots q_1 \beta_1(f_p) q_2 \beta_2(f_{p+1}) q_2 \alpha_1(f_{p+2}) q_1 \dots (u^* q_1) f_{2p+1} =$$

$$= f_1 q_1 \alpha_1(f_2) q_2 \dots \beta_{p-1}(f_p) q_p \beta_p(f_{p+1}) q_p \beta_{p-1}(f_{p+2}) \dots q_1 f_{2p+1}$$

and similarly for a string in which each $q_1 u$ is replaced by $u^* q_1$ and conversely.

Here the f_i 's are products of the form $f_1^i a_1^i f_2^i a_2^i \dots f_n^i$ where f_j^i are elements of null trace in the algebra generated by $a = (q_0+q_1)a(q_0+q_1)$, while a_j^i are elements of null trace in the algebra generated by q_0, q_1.

The monomials in the algebra generated by $M_2(\mathbb{C})$ and a that are to be checked for having zero value under ψ_1 are obtained by replacing certain f_j by other strings of this form, or by putting together such strings.

To show that the value of $\psi_1(m)$ is zero we will use the following observation which is a consequence of Lemma 3.1 in [Vo2]. This observation will be used to replace the elements $f_1, ..., f_{n+1}$ in the monomial m by elements of null trace.

Observation. *Let B be a W^*-algebra with trace τ, let X be a semicircular element and p a nontrivial projection that is free with X. Then any element of null trace in the algebra (with unit p) generated by pXp is a sum of monomials which are products either of elements of null trace in the algebra generated by pXp or either of the form $p - \tau(p)$, but no such monomial is $p - \tau(p)$ itself.*

Proof. Indeed if x is such an element then $pxp = x$, and moreover any other such monomial, which is different from $p - \tau(p)$, when multiplied with p, preserves the property of having null trace.

On the other hand
$$\tau(p(p - \tau(p))) = 1 - \tau(p) \neq 0.$$
This ends the proof of the observation.

To conclude the proof of Lemma 1 we let p a projection which is greater than the supremum of all the projections $\{q_i | i \in I_m\}$ that are involved in m.

We may then assume by construction that we are given a finite family of semicircular elements z^j so that $z^j = pz^jp$ and so that (modulo a multiplicative constant) $\alpha_j(a) = (q_j + q_{j+1})z^j(q_j + q_{j+1})$ for j in I_m.

Using the above observation we may express $f_k = f_1^k a_1^k ... f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k$ as a sum of products of null trace in the algebras generated by $\{q_j\}$ and $\{z_j\}$ (adjacent elements are allways in different algebras).

(Note that $(q_j + q_{j+1})(q_j - \tau(q_j)(\tau(q_j + q_{j+1}))^{-1}(q_j + q_{j+1})$ has always null trace).

Again the above observation shows that each of these monomials must contain at least on term in z^j. Since consecutive f^i involve different elements in the set $\{z^j\}$ it follows that $\psi_1(m) = 0$.

This ends the proof of Lemma 1.

Lemma 2. *With $\mathcal{B}, \mathcal{D}$ as before we have that $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$ coincides with the von Neumann subalgebra $\mathcal{C} \subseteq (q_o + q_1)\mathcal{D}(q_0 + q_1)$ (with unit $q_0 + q_1$) that is generated by the monomials with an equal number of e_{12} 's and e_{21} 's.*

Proof. We have to show that the subalgebra $(q_0+q_1)\mathcal{B}(q_0+q_1)$ coincides with the subalgebra $\mathcal{C} \subseteq (q_0+q_1)(\mathcal{B} \rtimes_\beta \mathbb{Z})(q_0+q_1) = (q_0+q_1)\{\mathcal{B}, u\}''(q_0+q_1)$ that is generated by monomials in a and $(e_{ij})_{i,j=1}^2$ containing an equal number of e_{12} 's and e_{21} 's.

Clearly $\mathcal{C}$ is invariant under the action of $\mathbb{R}$ (or $\mathbb{T}$) on $\mathcal{D}$ given by the modular group of ψ which acts by $\sigma_t^\psi(u) = \lambda^{it}u, \sigma_t^\psi|_\mathcal{B} = Id_\mathcal{B}$ so that $\mathcal{C} \subseteq (q_0+q_1)\mathcal{D}^\mathbb{R}(q_0+q_1) = (q_0+q_1)\mathcal{B}(q_0+q_1)$.

Hence we have to only prove the reverse inclusion. But due to the specific form of the generators in $\mathcal{B}$, we obtain that $\mathcal{B}$ is generated by elements of the form

$$m = f^1 q_{i_1} f^2 q_{i_2} \dots f^n q_{i_n} f^{n+1}$$

where the conditions on $i_1, \dots, i_n$ are

$$a)\ i_{j+1} \in \{i_j, i_j-1, i_j+1\},\ j=1,2,\dots,n,\ i_0, i_n \in \{0,1\}$$

$$b)\ \text{card}\ \{j | i_j = s\} \text{ is even,}$$

while f is one of the elements

$$\alpha_s(q_0Xq_0);\ \alpha_s(q_0Xq_1); \alpha(q_1Xq_0) \text{ or } \alpha_s(q_1Yq_1),$$

where s is either i_j or i_{j+1} if $i_j \neq i_{j+1}$. If $i_j = i_{j+1}$, then either $s = i_j$ and $f^j = \alpha_s(q_0Xq_0)$ or either $s = i_{j-1}$ and $f^j = \alpha_s(q_1Yq_1)$.

The assumptions we made are sufficient to show that in such a monomial we have some symbols corresponding to $\alpha_s(a)$ which are then necessary followed by symbols corresponding to $\alpha_{s+1}(a)$ (or to $\alpha_{s-1}(a)$). Moreover in m this sets of symbols are always separated by one of the projections q_p ($p \in \{s, s \pm 1\}$).

If we replace in m any such q_p by q_1u (or respectively by u^*q_1) and we replace the symbols from $\alpha_s(a)$ by the corresponding symbols in a we get the same m, but this time expressed as an element in the subalgebra of $\mathcal{C}$, generated by monomials with equal occurrence number of e_{12} 's and e_{21} 's. This ends the proof of Lemma 2.

To conclude the proof, we note the following observation:

Remark.

Let $\mathcal{B}, \mathcal{D} = \mathcal{B} \rtimes_\beta \mathbb{Z}$ and $u, \{q_i\}_{i\in\mathbb{Z}}$ be as before . Then $(q_0+q_1)\mathcal{D}(q_0+q_1)$ coincides with the algebra generated by $(q_0+q_1)\mathcal{B}(q_0+q_1)$ and $e_{12} = q_1u = uq_0$.

Proof. With q_0, q_1 as before we have to show that $q_0(u^*)^n bq_0 = q_0(u^*)^n q_n bq_0$ is contained in the algebra generated by $(q_0+q_1)\mathcal{B}(q_0+q_1)$. Assume $n > 1$; we may express q_nbq_0 as

$$q_nb_nq_{n-1}b_{n-1}q_{n-2}\dots q_1b_1q_0.$$

Then

$$q_0(u^*)^n b q_0 = q_0(u^*)^n q_n b_n q_{n-1} \dots q_1 b_1 q_0 =$$
$$= q_0 u^* \alpha_{n-1}(b_n) q_0 u^* \alpha_{n-2}(b_{n-1}) q_0 \dots q_0 u^* b_1 q_0$$

which is an element in the algebra generated by $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$ and uq_0.

Proof of the theorem.

Clearly the subalgebra generated by e_{12} and all the elements in

$$M_2(\mathbb{C}) * L^\infty([0,1],\nu)$$

with equal occurrence number of e_{12} 's and e_{21} 's coincides with the algebra itself. Thus $M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ with the free product state ϕ is identified with $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$ with the restriction of ψ (which is generated by $uq_0 = q_1 u$, and a). In particular the modular group of ϕ is $\sigma_t^\phi(e_{ij}) = \lambda^{it} e_{ij}$ and σ_t^ϕ is the identity on $L^\infty([0,1],\nu)$.

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Entropy in operator algebras

by Erling Størmer

1 Introduction

While entropy has for a third of a century been a central concept in ergodic theory, its non-Abelian counterpart is still in its adolescent stage with only a few signs of mature strength. The signs, however, are promising and show a potential of a subject of importance in operator algebras, so much that I am glad to use this opportunity to take the reader on a guided tour of its ideas and their resulting definitions and theorems. I have also included some open problems with the hope that they may inspire further development of the subject into maturity. In addition to giving the necessary definitions I shall mainly be concerned with explicit formulas for entropy of automorphisms. I shall therefore not discuss entropy of endomorphisms and completely positive maps, nor will I say much about applications to physics.

There is another very promising approach to non-Abelian entropy which we shall not discuss but is presently persued by Voiculescu [32,33]. The definitions are quite different from the ones we shall give, but the values of the entropies are closely related to ours in nice cases, but are essentially different in general, see section 5.

2 Definitions and basic results

Before we embark on the non-Abelian definition of entropy let us recall the classical definition. We are then given a probability space $(X, \mathcal{B}, \mu)$ and a nonsingular measure preserving transformation T of X. If $\mathcal{P} = (P_1, \ldots, P_n)$ is a measurable partition of X we shall often identify it with the finite dimensional Abelian algebra generated by the characteristic functions $\mathcal{X}_{P_i}$. The entropy of $\mathcal{P}$ is

$$H(\mathcal{P}) = \sum_{i=1}^{n} \eta(\mu(P_i)),$$

where η is the real function on the unit interval, $\eta(t) = -t \log t$ for $t \in (0,1]$, and $\eta(0) = 0$. If $\mathcal{P} \vee \mathcal{Q}$ is the partition generated by two partitions $\mathcal{P}$ and $\mathcal{Q}$ then $H(\mathcal{P} \vee \mathcal{Q}) \leq H(\mathcal{P}) + H(\mathcal{Q})$, so we have convergence of the sequence

$$\frac{1}{k} H\left(\bigvee_{i=0}^{k-1} T^{-i} \mathcal{P} \right).$$

We denote by $H(T,\mathcal{P})$ its limit, and define the entropy of T by

$$H(T) = \sup_{\mathcal{P}} H(T,\mathcal{P}), \tag{2.1}$$

where the sup is taken over all finite measurable partitions. The crucial result for computing $H(T)$ is the Kolmogoroff-Sinai Theorem [34, 4.17].

Theorem 2.2. *If $\mathcal{P}$ is a generator, i.e. the σ-algebra generated by $(T^{-i}\mathcal{P})_{i\in\mathbb{Z}}$ equals $\mathcal{B}$, written $\bigvee_{-\infty}^{\infty} T^{-i}\mathcal{P} = \mathcal{B}$, then $H(T) = H(T,\mathcal{P})$.*

There is another version of this theorem which will be of interest in the sequel [34, 4.22].

Theorem 2.3. *If $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is an increasing sequence of partitions of X with $\bigvee_{n=1}^{\infty} \mathcal{P}_n = \mathcal{B}$ then*

$$H(T) = \lim_n H(T,\mathcal{P}_n)\,.$$

If one wants to extend the above definition of entropy to von Neumann algebras one is immediately confronted with a major obstacle. While there is a natural extension of the concept of finite partitions, namely finite dimensional von Neumann algebras, there is no natural candidate for the analogue of the partition $\mathcal{P} \vee \mathcal{Q}$ generated by $\mathcal{P}$ and $\mathcal{Q}$. Remember that the von Neumann algebra generated by two finite dimensional algebras can easily be infinite dimensional. However, if one considers the function $H\left(\bigvee_{i=1}^{k} \mathcal{P}_i\right)$ with $\mathcal{P}_i$ finite partitions as a function $H(\mathcal{P}_1,\ldots,\mathcal{P}_k)$ of k-variables, one can try to generalize this function. This will now be done following [8] for a von Neumann algebra M with a faithful normal finite trace τ such that $\tau(1) = 1$. In section 6 we shall see how this definition can be extended to general C^*-algebras and states.

For each $k \in \mathbb{N}$ denote by S_k the set of multiple indexed finite partitions of unity of M^+, $(x_{i_1\ldots i_k})_{i_j\in\mathbb{N}}$, i.e. each $x_{i_1\ldots i_k} \in M^+$, zero except for a finite number of indices and satisfying $\sum_{i_1,\ldots,i_k} x_{i_1\ldots i_k} = 1$.

For $x \in S_k$, $\ell \in \{1,\ldots,k\}$, $i_\ell \in \mathbb{N}$, we put

$$x^{\ell}_{i_\ell} = \sum_{i_1,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k} x_{i_1\ldots i_k}\,.$$

If $N \subset M$ is a von Neumann subalgebra we denote by E_N the unique τ-invariant conditional expectation $E_N : M \to N$ defined by the identity

$$\tau(E_N(x)y) = \tau(xy), \qquad x \in M,\ y \in N\,.$$

Definition 2.4. Let $N_1, \ldots, N_k$ be finite dimensional von Neumann subalgebras of M. Then

$$H(N_1, \ldots, N_k) = \sup_{x \in S_k} \left\{ \sum_{i_1, \ldots, i_k} \eta(\tau(x_{i_1 \ldots i_k})) - \sum_{\ell=1}^{k} \sum_{i_\ell} \tau(\eta(E_{N_\ell}(x^{\ell}_{i_\ell}))) \right\}.$$

Since the trivial partition $x = (1)$ gives the value zero, $H \geq 0$. Also it is clear that H is symmetric in the N's. Furthermore H satisfies the following nice requirements

(A) $H(N_1, \ldots, N_k) \leq H(P_1, \ldots, P_k)$ when $N_j \subset P_j$, $j = 1, \ldots, k$.

(B) $H(N_1, \ldots, N_k) \leq H(N_1, \ldots, N_j) + H(N_{j+1}, \ldots, N_k)$ for $1 \leq j < k$.

(C) $N_1, \ldots, N_j \subset N \Rightarrow H(N_1, \ldots, N_j, N_{j+1}, \ldots, N_k) \leq H(N, N_{j+1}, \ldots, N_k)$.

(D) For any family of minimal projections of N, $(e_\alpha)_{\alpha \in I}$ such that $\sum_{\alpha \in I} e_\alpha = 1$ one has $H(N) = \sum_{\alpha \in I} \eta\tau(e_\alpha)$.

(E) If $(N_1 \cup \cdots \cup N_k)''$ is generated by pairwise commuting von Neumann subalgebras P_j of N_j then

$$H(N_1, \ldots, N_k) = H\big((N_1 \cup \cdots \cup N_k)''\big).$$

The crucial technical ingredient in the proof of the above properties, and in particular of (C), is the relative entropy of two states, or rather positive operators in our case, defined by

$$S(x|y) = \tau(x(\log x - \log y)), \qquad x, y \in M^+, \ x \leq \lambda y$$

for some $\lambda > 0$. For general normal states of von Neumann algebras the relative entropy is defined by Araki [1] via the relative modular operator of the two states, and by Pusz and Woronowicz [25] for states of C^*-algebras. The main property of S is that it is a jointly convex function in x and y [16], see also [15] and [25].

Having H it is now an easy matter to extend the classical definition (2.1) of entropy. We look at the measure preserving transformation T on $(X, \mathcal{B}, \mu)$ as an automorphism α_T of the Abelian von Neumann algebra $L^\infty(X, \mathcal{B}, \mu)$ defined by $\alpha_T(f) = f \circ T^{-1}$, and partitions as finite dimensional algebras.

Definition 2.5 Let α be an automorphism of M such that $\tau \circ \alpha = \tau$. If $N \subset M$ is finite dimensional we let

$$H(\alpha, N) = \lim_{k \to \infty} \tfrac{1}{k} H(N, \alpha(N), \ldots, \alpha^{k-1}(N)),$$

where as in the classical case the sequence converges by the subadditivity of H, property (B). The *entropy* of α is

$$H(\alpha) = \sup_{N} H(\alpha, N),$$

where the sup is taken over all finite dimensional subalgebras $N \subset M$.

Remark 2.6. If $P \subset M$ is a von Neumann subalgebra such that $\alpha(P) = P$, it is immediate from the definition that the restriction $\alpha|P$ satisfies $H(\alpha|P) \leq H(\alpha)$.

Remark 2.7. If α is periodic then $H(\alpha, N) = 0$ for all N, hence $H(\alpha) = 0$. More generally, if α is contained in a compact subgroup of $\mathrm{Aut}(M)$ then Besson [2] has shown that we still have $H(\alpha) = 0$.

To compute $H(\alpha)$ it is as in the classical case necessary to reduce the choice of N's. The following concept is helpful for this purpose.

Definition 2.8. If N and P are finite dimensional von Neumann subalgebras of M their *relative entropy* is

$$H(N|P) = \sup_{x \in S_1} \sum_i (\tau\eta E_P(x_i) - \tau\eta E_N(x_i)).$$

$H(N|P)$ has the following nice properties:

(F) $H(N_1, \ldots, N_k) \leq H(P_1, \ldots, P_k) + \sum_{j=1}^{k} H(N_j|P_j)$.

(G) $H(N|Q) \leq H(N|P) + H(P|Q)$.

(H) $H(N|P)$ is increasing in N and decreasing in P.

(I) If N and P commute then

$$H(N|P) = H((N \cup P)''|P) = H(H \cup P)'') - H(P).$$

Properties (F), (G), (H) are easy to prove, while (I) is a consequence of the Lieb-Ruskai second strong subadditivity property [17]. The relative entropy is continuous in the following sense.

Theorem 2.9. *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all pairs of von Neumann subalgebras N and P of M with $\dim N = n$, we have*

$$N \overset{\delta}{\subset} P \Rightarrow H(N|P) < \varepsilon .$$

Here $N \overset{\delta}{\subset} P$ means that for all $x \in N$, $\|x\| \leq 1$, there exists $y \in P$ with $\|y\| \leq 1$ such that $\|x - y\|_2 < \delta$, where $\|z\|_2 = \tau(z^*z)^{1/2}$. This result together with property (F) is very useful in restricting the choice of N in the definition of $H(\alpha)$. An example is the proof of the generalization of the Kolmogoroff-Sinai Theorem (2.3).

Theorem 2.10. *Suppose M is hyperfinite with an increasing sequence $(P_n)_{n \in \mathbb{N}}$ of finite dimensional subalgebras with union weakly dense in M. Then if $\alpha \in \mathrm{Aut}(M)$ and $\tau \circ \alpha = \tau$ we have*

$$H(\alpha) = \lim_n H(\alpha, P_n) .$$

Proof. Given N and $\varepsilon > 0$ choose by Theorem 2.9 P_n such that $H(N|P_n) < \varepsilon$. Then for $k \in \mathbb{N}$, by property (F),

$$\begin{aligned} H(\alpha, N) &= \lim_k \tfrac{1}{k} H(N, \alpha(N), \ldots, \alpha^{k-1}(N)) \\ &\leq \lim_k \tfrac{1}{k} H(P_n, \alpha(P_n), \ldots, \alpha^{k-1}(P_n)) + \overline{\lim_k} \tfrac{1}{k} \sum_{j=0}^{k-1} H(\alpha^j(N)|\alpha^j(P_n)) \\ &\leq H(\alpha, P_n) + \varepsilon, \end{aligned}$$

since $H(\alpha^j(N)|\alpha^j(P_n)) = H(N|P_n)$. The theorem follows.

3 Entropy of shifts

The definition of entropy as given in section 2 had originally as its main model the n-shift. Therefore, to make the results of section 2 easier to understand let me describe the n-shift in some detail.

Let $n \in \mathbb{N}$ and $M^i = M_n(\mathbb{C})$ for $i \in \mathbb{Z}$. Let $\tau_i = \frac{1}{n} Tr$ be the normalized trace on M^i. Let

$$R = \bigotimes_{-\infty}^{\infty} (M^i, \tau_i)$$

be the von Neumann algebra tensor product of the M^i with respect to the traces τ_i. Then R is the hyperfinite II_1-factor with trace $\tau = \otimes \tau_i$. Even though it is not needed for the computation of the entropy of the shift let me as an illustration introduce a partition which will give the right value for $H(N_1, \ldots, N_k)$.

Let e_j be the minimal projection in the diagonal in $M_n(\mathbb{C})$ with matrix which is 0 except in the j^{th} row, where it is 1. Let $k \in \mathbb{N}$ and

$$x_{i_1 \ldots i_k} = \cdots \otimes 1 \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes 1 \otimes \cdots,$$

where $e_{i_j} \in M^j$. Then $(x_{i_1 \ldots i_k}) \in S_k$,and

$$x^\ell_{i_\ell} = \cdots \otimes 1 \otimes e_{i_\ell} \otimes 1 \otimes \cdots .$$

Put

$$N_i = \cdots \otimes 1 \otimes M^i \otimes 1 \otimes \cdots \subset R,$$

i.e. N_i is M^i imbedded in R. Let σ be the n-shift on R, i.e. σ is the shift automorphism on $\otimes M^i$ which maps each factor one factor to the right. Since $\eta(e) = 0$ for each projection e, we have

$$\begin{aligned} H(N_1, \sigma(N_1), \ldots, \sigma^{k-1}(N_1)) &= H(N_1, \ldots, N_k) \\ &\geq \sum \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{N_\ell}(x^\ell_{i_\ell}) \\ &= n^k \eta(n^{-k}) - \sum_\ell \sum_{i_\ell} \tau\eta(e_{i_\ell}) \\ &= k \log n - 0, \end{aligned}$$

so that $H(\sigma, N_1) \geq \log n$.

Now $(N_1 \cup N_2 \cup \cdots \cup N_k)''$ is a factor of type I_{n^k}, hence we have by properties (D) and (E)

$$\tfrac{1}{k} H(N_1, \ldots, N_k) = \tfrac{1}{k} \log n^k = \log n\,,$$

so the sup is indeed obtained for our choice $(x_{i_1} \ldots i_k)$ in S_k. To show that $H(\sigma) = \log n$ we put

$$P_q = \cdots \otimes 1 \otimes \bigotimes_{-q}^{q} M^i \otimes 1 \otimes \cdots = \Big(\bigcup_{-q}^{q} N_i\Big)''.$$

Then the sequence (P_q) satisfies the requirements of the Kolmogoroff-Sinai Theorem 2.10. Since

$$(P_q \cup \sigma(P_q) \cup \cdots \cup \sigma^{k-1}(P_q))'' = \Big(\bigcup_{-q}^{q+k-1} N_i\Big)'',$$

we have by property (E) that

$$\tfrac{1}{k} H(P_q, \sigma(P_q), \ldots, \sigma^{k-1}(P_q)) = \tfrac{1}{k} \log n^{2q+k} = \frac{2q+k}{k} \log n$$

Therefore

$$H(\sigma) = \lim H(\sigma, P_q) = \log n\,.$$

It should be noted from the above computation that if D denotes the diagonal in P_q then we have

$$\begin{aligned} H(P_q, \sigma(P_q), \ldots, \sigma^{k-1}(P_q)) &= H(D, \sigma(D), \ldots, \sigma^{k-1}(D)) \\ &= H(D \cup \sigma(D) \cup \cdots \cup \sigma^{k-1}(D))'')\,. \end{aligned}$$

Thus the shift σ is not much different from the classical n-shift. This situation prevails for a large class of non-Abelian extensions of classical shift operators. Using the above notation we define them as follows. Let

$$X = \prod_{-\infty}^{\infty} X_i\,, \qquad X_i = \{1, \ldots, n\} \quad \text{for all } i\,.$$

Let S be the shift on X given by $S(x_i) = (x_{i+1})$, and suppose μ is an invariant probability measure on X. If D_n denotes the diagonal in $M_n(\mathbb{C})$ we identify $C(X_0)$ with D_n, and thus $C(X)$ with the infinite tensor product $D = \bigotimes_{-\infty}^{\infty} D_n^i$, where $D_n^i = D_n$. S induces a shift σ_0 on D in the obvious way, and μ defines likewise a σ_0-invariant state ω on D. Let E_n denote the canonical trace invariant expectation of $M_n(\mathbb{C})$ on D_n, and let $E = \bigotimes_{-\infty}^{\infty} E_n^i$ with $E_n^i : M^i \to D_n^i$ equal to E_n. Let $A = \bigotimes_{-\infty}^{\infty} M^i$ be the C^*-algebraic infinite tensor product. Then E is a conditional expectation of A on D. Extend ω to a state ρ on A by $\rho = \omega \circ E$. If σ_1 denotes the shift on A then $\rho \circ \sigma_1 = \rho$ and $\sigma_0 \circ E = E \circ \sigma_1$. Let (π, ξ, H) be the GNS-representation of ρ, and put $M = \pi(A)''$. Then σ_1 extends to an ω_ξ-invariant automorphism of M, hence restricts to an ω_ξ-invariant automorphism σ of the centralizer M_ρ of ω_ξ in M. If the measure μ is sufficiently ergodic, M is a factor, and even M_ρ is often a factor with trace $\tau = \omega_\xi$. From its construction M_ρ is the hyperfinite II_1-factor R, and σ is an ergodic automorphism of R. In the cases which have been computed it turns out that the entropy $H(\sigma)$ is equal to the classical entropy $H(S)$ of S with respect to μ.

The above construction was first performed by Connes and myself for Bernoulli shifts in [8], then by Besson [3] for Markov shifts, however both results were known to Krieger (unpublished). Later on these results were generalized by Quasthoff [26], who used a different, however equivalent construction to define shifts on R. He also showed such a result for an extension to flows of Markov shifts [27].

There is an interesting shift automorphism on R which arises from the theory of subfactors. For $\lambda \in (0,1]$ let $\{e_0, e_1, e_2, \ldots\}$ be a sequence of projections in R satisfying the axioms

a) $e_i e_{i\pm1} e_i = \lambda e_i$,

b) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$,

c) $\tau(we_i) = \lambda\tau(w)$ if w is a word on $\{1, e_0, \ldots, e_{i-1}\}$.

It was shown by Jones [14] that such a sequence exists and generates R if and only if $\lambda \in \left(0, \frac{1}{4}\right] \cup \left\{\frac{1}{4}\sec^2\frac{\pi}{n} : n \geq 3\right\}$. Furthermore, if R_λ is the subfactor generated by $\{1, e_1, e_2, \ldots\}$ then the index $[R : R_\lambda] = \lambda^{-1}$.

We now reindex the projections e_i so the index set is $\mathbb{Z}$, and we denote by θ_λ the shift automorphism of R defined by $\theta_\lambda e_i = e_{i+1}$. It follows from the relations a)–c) that θ_λ is mixing with respect to the trace, hence is ergodic. This automorphism was studied by Pimsner and Popa in [22]. They showed by explicit construction of a sequence (e_i) that if $\lambda < 1/4$ then θ_λ is a Bernoulli shift as described above, and $H(\theta_\lambda) = \eta(t) + \eta(1-t)$, where $t(1-t) = \lambda$. If $\lambda > 1/4$ the situation is quite different and they showed $H(\theta_\lambda) = -\frac{1}{2}\log\lambda$. The same result was then showed by Choda [4] when $\lambda = 1/4$.

4 Inner automorphisms of the hyperfinite II_1-factor

In the last section we saw how to extend many shifts to outer automorphisms of the hyperfinite II_1-factor R, and it was remarked that the entropy of the extended shift was the same as that of the original (classical) shift. One can also via crossed products extend ergodic transformations to inner automorphisms of R. Let $(X, \mathcal{B}, \mu)$ be a probability space and T a nonsingular ergodic measure preserving transformation of X. Then T defines an automorphism α_T on $L^\infty(X,\mu)$ by $(\alpha_T f)(x) = f(T^{-1}x)$ for $f \in L^\infty(X,\mu)$, $x \in X$. The crossed product $L^\infty(X,\mu) \times_{\alpha_T} \mathbb{Z}$ equals R, and α_T extends in a natural way to an inner automorphism $\operatorname{Ad} U_T$ on R.

Since the restriction $\operatorname{Ad} U_T | L^\infty(X,\mu) = \alpha_T$, and $H(\alpha_T) = H(T)$, it follows from Remark 2.6 that

$$H(\operatorname{Ad} U_T) \geq H(T), \tag{4.1}$$

where $\operatorname{Ad} U_T$ is considered as an automorphism of R.

Problem 4.2. Do we have equality in (4.1)?

The first result on this problem is due to Besson [2], who showed that if the unitary operator V_T on $L^2(X,\mu)$ defined by $(V_T f)(x) = f(T^{-1}x)$, $f \in L^2(X,\mu)$, $x \in X$, has pure point spectrum then $H(\operatorname{Ad} U_T) = 0$.

Let T be as above and let A denote the Abelian von Neumann algebra generated by U_T. Using the Fourier expansion $\sum_{-\infty}^{\infty} a_n U_T^n$ with $a_n \in L^\infty(X,\mu)$ of an element $x \in A' \cap R$ it follows easily from the ergodicity of T that each a_n is a scalar, hence $x \in A$. Thus A is a maximal Abelian subalgebra of R. It turns out that the entropy $H(\operatorname{Ad} U_T)$ depends essentially on the size of the normalizer of A in R, i.e. the set of unitaries $u \in R$ such that $uAu^* = A$. If the normalizer of A generates R as a von Neumann algebra, A is called a *Cartan* (or regular) *subalgebra*. The next result [30] shows that if $H(T) > 0$ then A is not a Cartan subalgebra.

Theorem 4.3. *Let u be a unitary operator contained in a Cartan subalgebra of R. Then the entropy $H(\operatorname{Ad} u) = 0$.*

In the proof of this result one uses the Connes-Feldman-Weiss Theorem [6], [24] on the uniqueness of Cartan subalgebras, so we may assume $R = \bigotimes_{i=1}^{\infty} (M^i, \tau_i)$ and $u \in \bigotimes_{i=1}^{\infty} D^i$, and use the subalgebras $P_q = \left(\bigotimes_1^q M^i\right) \otimes 1 \otimes \cdots$ in the application of the Kolmogoroff-Sinai Theorem 2.10. One can show that the function $H(P_q, \operatorname{Ad} u(P_q), \ldots, \operatorname{Ad} u^{k-1}(P_q))$ grows as $\log k$, hence $H(\operatorname{Ad} u, P_q) = 0$, and therefore $H(\operatorname{Ad} u) = 0$.

In particular if $T = T_\theta$ is the irrational rotation by an angle θ on the circle $\mathbb{T}$, and V_φ denotes the multiplication operator on $L^2(\mathbb{T})$ by $e^{i\varphi}$, then in the crossed product $L^\infty(\mathbb{T}) \times_{\alpha_T} \mathbb{Z}$ $U_T V_\varphi = e^{i\theta} V_\varphi U_T$. In particular $V_\varphi A V_\varphi^{-1} = A$ with A as before the Abelian von Neumann subalgebra generated by U_T. Since V_φ and U_T generate R, A is a Cartan subalgebra of R, hence by Theorem 4.3 we have $H(\operatorname{Ad} U_T) = 0$. In particular we have a positive solution to Problem 4.2 in this case. It should be remarked that this argument can be generalized greatly. It would be interesting to see what kind of maximal Abelian subalgebra U_T generates when $H(T)$ is large, as it is for example for Bernoulli shifts.

5 The free shift

So far we have only looked at the hyperfinite case, in which the Kolmogoroff-Sinai Theorem is applicable. I'll now discuss a completely different case, that of the II_1-factor $L(\mathbb{F}_\infty)$ obtained from the left regular representation of the free group $\mathbb{F}_\infty$ in infinite number of generators. Let G denote the set of generators of $\mathbb{F}_\infty$. Then each bijection of G gives rise to an automorphism of $\mathbb{F}_\infty$ and hence of $L(\mathbb{F}_\infty)$. We say a bijection α is *free* if each orbit $\{\alpha^n(x) : x \in G\}$ is infinite, or equivalently the map $n \longmapsto \alpha^n(x)$ of $\mathbb{Z}$ into G is injective for all $x \in G$. An example is the *free shift*, which is defined by indexing G by $\mathbb{Z}$ and letting α correspond to the shift $n \longmapsto n+1$ of $\mathbb{Z}$. The free shift is an extremely ergodic automorphism; indeed Popa [22] showed that the only globally invariant injective subalgebra of $L(\mathbb{F}_\infty)$ is the scalars, and recently Gaure [12] showed that each nontrivial globally invariant von Neumann subalgebra is a full

II_1-factor. It is therefore rather surprising that we have, see [29].

Theorem 5.1. *Let α be an automorphism of $L(\mathbb{F}_\infty)$ defined by a free bijection of the set of generators G on itself. Then its entropy $H(\alpha) = 0$.*

This result makes my comment at the beginning of section 3, that the definition of entropy was modelled on the n-shift, more profound. The theorem shows that entropy is a function of independence with respect to commutativity of the translates by the automorphism, rather than the degree of ergodicity.

Since each bijection of G on itself is a combination of free and periodic maps, and each periodic automorphism has entropy zero (Remark 2.7), I would expect that all automorphisms of $L(\mathbb{F}_\infty)$ arising from bijections of G have entrópy zero. Even more may be true.

Problem 5.2. *Suppose α is an automorphism of $L(\mathbb{F}_\infty)$ arising from an automorphism of $\mathbb{F}_\infty$. Is $H(\alpha) = 0$?*

Note that $L(\mathbb{F}_\infty)$ has lots of inner automorphisms with positive entropy. This follows from Remark 2.6 since the hyperfinite II_1-factor R is a subfactor of $L(\mathbb{F}_\infty)$, and R has, as we saw in section 4, many inner automorphisms with positive entropy.

The proof of Theorem 5.1 is quite different from the other proofs we have seen. Since we don't have a Kolmogoroff-Sinai Theorem at our disposal we must use Definition 2.4 of the entropy function $H(N_1, \ldots, N_k)$ directly, hence we have to study the behaviour of partitions of unity in $L(\mathbb{F}_\infty)$. Given a finite dimensional subalgebra $N \subset L(\mathbb{F}_\infty)$ we may use freeness of the automorphism α together with approximation results to assume that for some $p \in \mathbb{N}, N, \alpha^p(N), \alpha^{2p}(N), \ldots, \alpha^{(k-1)p}(N)$ belong to subfactors $L(\mathbb{F}_{S_i})$ corresponding to the free subgroups of $\mathbb{F}_\infty$ obtained from disjoint subsets $S_0, \ldots, S_{k-1}$ of G. Then we show a uniform estimate on the set S_k of all partitions of unity, namely,

Lemma 5.3. *Given $\varepsilon > 0$ there exists $r = r(\varepsilon, N) \in \mathbb{N}$ such that for all partitions $(x_{i_1 \ldots i_k}) \in S_k$ and all k there is a set $J \subset \mathbb{N}$ with card $J \leq r$ such that*

$$\sum_{i_\ell} \|E_{N_\ell}(x^\ell_{i_\ell}) - \tau(x^\ell_{i_\ell})1\| < \varepsilon \qquad \text{for } \ell \notin J\,,$$

where $N_\ell = \alpha^{p(\ell-1)}(N)$.

Thus the projection of $x^\ell_{i_\ell}$ on N_ℓ is almost a scalar for all $\ell \notin J$, and this happens for ℓ outside a set of cardinality less than r independently of the partition. This is the crucial idea in the proof of Theorem 5.1, because if we assume $E_{N_\ell}(x^\ell_{i_\ell}) = \tau(x^\ell_{i_\ell}) = E_{\mathbb{C}}(x_{i_\ell} =^\ell)$ for all i_ℓ in Definition 2.4, we get

$$\begin{aligned}
&\sum \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{N_\ell}(x^\ell_{i_\ell}) \\
&= \sum \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{\mathbb{C}}(x^\ell_{i_\ell}) \\
&\leq H(\mathbb{C}, \ldots, \mathbb{C}) = H(\mathbb{C}) = 0\,.
\end{aligned}$$

As mentioned in the introduction Voiculescu [32, 33] has introduced another concept of entropy, called the *perturbation theoretic* entropy. In the nicer cases like the classical case and Bernoulli shifts this entropy gives values close to ours. However, for $L(\mathbb{F}_\infty)$ it is quite different; indeed he showed [33] that the perturbation theoretic entropy of the free shift is $+\infty$. Thus it is plausible that the perturbation theoretic entropy is a better concept of entropy for studying automorphisms of highly non-Abelian factors like $L(\mathbb{F}_\infty)$.

6 Entropy in C^*-algebras

A natural problem that arose after the introduction of entropy in finite von Neumann algebras, was its extension to general von Neumann algebras and even C^*-algebras. Using the quantum mechanical entropy of states $-S(\rho) = Tr(\eta(\Omega_\rho))$ if p is a normal state on $B(H)$ defined by a trace class operator Ω_ρ – and the delicacy of the relative entropy, an attempt was made in [9]. But all we could do was to show that the definition we came up with, gave the classical entropy in the Abelian case. Then Evans [10] modified Definition 2.4 to AF-algebras and defined the topological entropy by taking the sup over all tracial states. By taking the shift on an AF-algebra defined by an aperiodic $n \times n$ matrix A with entries 0 or 1, he could show that the topological entropy= of the shift was the logarithm of the spectral radius of A. For more recent work on topological entropy see the thesis of Hudetz [13].

Connes [5] was the first who succeeded in giving a general definition for normal states of von Neumann algebras. Essentially what he did was to replace the trace in Definition 2.4 by a given normal state φ, and instead of considering partitions $(x_{i_1 \dots i_k}) \in S_k$ to consider finite sets of positive linear functionals φ_i with $\sum \varphi_i = \varphi$. After this breakthrough Connes together with Narnhofer and Thirring [7] succeeded in extending the definition to general C^*-algebras. This was done as follows.

We are given a C^*-algebra A with a state φ and want to define the analogue of $H(N_1, \dots, N_k)$ as in Definition 2.4. Since A need not have any nontrivial finite dimensional subalgebras we replace the N's by finite dimensional C^*-algebras $C_1, \dots, C_k$ which are not necessarily subalgebras of A, but have unital completely positive maps $\gamma_j : C_j \to A$. As we have seen in the previous sections the entropy considered, has a built in Abelianness in it. This is made more explicit in the new definition. We let B be a finite dimensional Abelian C^*-algebra and $P : A \to B$ a unital positive linear map such that there is a state μ on B with $\mu \circ P = \varphi$. We now define a concept called entropy defect for P, which will later on be used for some maps from the C_j's and not for P.

Let $p_1, \dots, p_k$ be the minimal projections in B. Then there are states $\varphi_1, \dots, \varphi_r$ on A such that

$$P(x) = \sum_{i=1}^{r} \varphi_i(x) p_i ,$$

and

$$\varphi = \sum_{i=1}^{r} \mu(p_i)\varphi_i$$

is φ written as a convex sum of states, cf. Connes' von Neumann algebra definition, where he considered families $\{\varphi_i\}$ with $\varphi = \sum \varphi_i$. We now introduce the important relative entropy of states, and use a general definition of Pusz and Woronwicz [28],so [15]. Put

$$\varepsilon_\mu(P) = \sum_{i=1}^{r} \mu(p_i) S(\varphi|\varphi_i),$$

and let the entropy defect be

$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P),$$

where $S(\mu) = \sum \eta(\mu(p_i))$ is the entropy of μ.

In order to come back to the original finite dimensional algebras C_j and completely positive maps γ_j we let $B_1, \ldots, B_k$ be C^*-subalgebras of B, and $E_j : B \to B_j$ the μ-invariant conditional expectation. Then the quadruple (B, E_j, P, μ) is called an *Abelian model* for $(A, \varphi, \gamma_1, \ldots, \gamma_n)$ and its entropy is defined to be

$$S\Big(\mu\Big| \bigvee_{\gamma=1}^{k} B_j\Big) - \sum_{j=1}^{k} s_\mu(P_j), \tag{6.1}$$

where $P_j = E_j \circ P \circ \gamma_j : C_j \to B_j$, and the definitionof $s_\mu(P_j)$ is the same as for P above, where we replace μ by $\mu|B_j$, P by P_j, and φ by $\varphi \circ \gamma_j$.

We can now define the entropy function H in analogy with Definition 2.4 as

$$H_\varphi(\gamma_1, \ldots, \gamma_k) = \sup \text{ of (6.1) over all Abelian models.}$$

Then we can show similar properties as (A)–(E) in section 2. From here on we continue as before. If $\alpha \in \operatorname{Aut}(A)$ is φ-invariant and C is a finite dimensional C^*-algebra with a unital completely positive map $\gamma : C \to A$ we denote by

$$h_{\varphi,\alpha}(\gamma) = \lim_{k\to\infty} \tfrac{1}{k} H_\varphi(\gamma, \alpha \circ \gamma, \ldots, \alpha^{k-1} \circ \gamma),$$

and define the entropy of α to be

$$h_\varphi(\alpha) = \sup_\gamma h_{\varphi,\alpha}(\gamma),$$

where the sup is taken over all pairs (C, γ). The interested reader will see from [7, Remark III.5.3] how the definition of $H_\varphi(\gamma_1, \ldots, \gamma_k)$ extends the definition in [5], and thus the original definition (2.4).

While in the finite von Neumann algebra case the Kolmogoroff-Sinai Theorem was true for hyperfinite algebras, in the C^*-algebra case it holds true for nuclear C^*-algebras [7, V.2].

Theorem 6.2. *Let A, φ, α be as above. Suppose there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ of unital completely positive maps $\theta_n : A \to A$ such that for each n there are finite dimensional C^*-algebras A_n and unital completely positive maps $\sigma_n : A \to A_n$, $\tau_n : A_n \to A$ for which*

$\theta_n = \tau_n \circ \sigma_n$, *and for all* $x \in A$

$$\lim_n \|\theta_n(x) - x\| = 0\,.$$

Then

$$\lim_n h_{\varphi,\alpha}(\tau_n) = h_\varphi(\alpha)\,.$$

In particular for an AF-algebra $A = \overline{\bigcup_{n-1}^{\infty} A_n}$ with $1 \in A_1 \subset A_2 \subset \cdots$, finite dimensional, if we let $\gamma_n : A_n \to A$ be the inclusion map, so we write A_n instead of γ_n, we have

$$h_\varphi(\alpha) = \lim_n h_{\varphi,\alpha}(A_n)\,.$$

This result has been extended to the setting of quasi local algebras in quantum statistical mechanics by Park and Shin [20].

If M is a von Neumann algebra and φ a normal state of M we can do the same as above assuming all functionals and maps to be normal. For states on a C^*-algebra A we get the same entropy as before if we use this approach to the GNS-representation of the state. This is often useful, since it is in many cases easier to compute entropy in the von Neumann algebra setting. The following result is analogous to those we discussed for shifts in section 3, see [7, VIII.8].

Theorem 6.3. *Let M be a von Neumann algebra with a faithful normal state φ. Let M_φ denote the centralizer of φ in M. Suppose $N_1, \ldots, N_k$ are finite dimensional subalgebras of M which contain Abelian subalgebras $A_j \subset N_j \cap M_\varphi$ which are pairwise commuting and generate a maximal Abelian subalgebra $A = \bigvee_{j=1}^{k} A_j$ in the von Neumann algebra $N = (N_1 \cup \cdots \cup N_k)''$. Then*

$$H_\varphi(N_1, \ldots, N_k) = S(\varphi|N)\,.$$

This result tells us in particular that the modular automorphism σ^φ comes into play, because $M_\varphi = \{x \in M : \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R}\}$.

One might believe that we need a large centralizer in order to have positive entropy. This is not so. Connes [5] has exhibited an example of a factor M of type III_1 with a state φ with $M_\varphi = \mathbb{C}$, and $h_\varphi(\alpha) > 0$ for a φ-invariant automorphism α.

In some cases one does not have the nice situation of Theorem 6.3 but only an approximation to it. The following is a result on such a situation [7, IX.1]. It is applicable to shift invariant states which are not necessarily product states, but almost so.

Theorem 6.4. *Let $M^i = M_q(\mathbb{C})$ for $i \in \mathbb{Z}$. Let A be the C^*-algebra $A = \bigotimes_{i=-\infty}^{\infty} M^i$ and σ the shift $\sigma : M^i \to M^{i+1}$. Let φ be a state with $\varphi \circ \sigma = \varphi$. For $I \subset \mathbb{Z}$ let $A(I) = \bigotimes_{i \in I} M^i$ and $\varphi_I = \varphi|A(I)$.= Assume that for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that with $I = [-N, n+N]$ then*

(1) $\quad \|\sigma^{\varphi}_{i/2}(\sigma^{\varphi_I}_{-i/2}(x)) - x\| < \varepsilon \, \|x = \|$ *for* $x \in A([0,n])$,

(2) $\quad \lim_{n\to\infty} \frac{N}{n} = 0$.

Then $h_\varphi(\sigma)$ *equals the* mean entropy $S(\varphi)$, *where per definition*

$$S(\varphi) = \lim_{|I|\to\infty} \frac{1}{|I|} S(\varphi|A(I)) .$$

Here $|I| = 2N + n + 1$ *is the length of* I.

At this point it is appropriate to mention that not everything seems to go smoothly with entropy. In the classical case if T_i is a nonsingular measure preserving transformation on a probability space $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1,2$, then the entropy of $T_1 \times T_2$ on $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, \mu_1 \times \mu_2)$ satisfies

$$H(T_1 \times T_2) = H(T_1) + H(T_2) .$$

Transformed into the language of C^*-algebras this says that if α_i is an automorphism of an Abelian C^*-algebra A_i with an invariant state φ_i, $i = 1,2$, then

$$h_{\varphi_1\otimes\varphi_2}(\alpha_1 \otimes \alpha_2) = h_{\varphi_1}(\alpha_1) + h_{\varphi_2}(\alpha_2) . \tag{6.5}$$

Problem 6.6. Does (6.5) hold if A_1 and A_2 are non-Abelian?

The inequality

$$h_{\varphi_1\otimes\varphi_2}(\alpha_1 \otimes \alpha_2) \geq h_{\varphi_1}(\alpha_1) + h_{\varphi_2}(\alpha_2)$$

is easy [31, Lem. 3.4], because we have many more choices of Abelian models to estimate the left side than the right, where we have to use tensor products of Abelian models in A_1 and A_2. The technical reason why (6.5) holds for Abelian algebras is that each pure state of $A_1 \otimes A_2$ is a product state $\rho_1 \otimes \rho_2$ with ρ_i pure. This is also true if one of the A_i's is Abelian, but not in general. I thus incline to the view that the answer to Problem 6.6 is negative when both A_1 and A_2 are non-Abelian.

7 The anti-commutation relations

The main example for which the C^*-algebra entropy has been computed, is that of quasifree states of the CAR-algebra and invariant Bogoliubov (or quasifree) automorphisms. Let us recall the definitions.

Let H be a complex Hilbert space. The CAR-algebra $\mathcal{A}(H)$ over H is a C^*-algebra with the property that there is a linear map $f \to a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C^*-algebra and satisfies the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f,g)1, \qquad f,g \in H ,$$

$$a(f)a(g) + a(g)a(f) = 0 ,$$

where $(\cdot,\cdot)$ is the inner product on H and 1 the unit of $\mathcal{A}(H)$. If $0 \le A \le 1$ is an operator on H, then the quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)$ given by

$$\omega_A(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det((Ag_i, f_i)) .$$

If U is a unitary operator on H then U defines an automorphism α_U on $\mathcal{A}(H)$, called a *Bogoliubov automorphism*, determined by

$$\alpha_U(a(f)) = a(Uf) .$$

If U and A commute it is an easy consequence of the above definition of ω_A that α_U is ω_A-invariant.

Problem 7.1. Compute $h_{\omega_A}(\alpha_U)$ when $[U, A] = 0$.

Connes suggested to me that if ω_A is the trace τ the answer should be

$$h_\tau(\alpha_U) = \frac{\log 2}{2\pi} \int_0^{2\pi} m(U)(\theta) d\theta , \tag{7.1}$$

where $m(U)$ is the multiplicity function of the absolutely continuous part U_a of U. Then Voiculescu and I [31] showed this and more by solving the problem when A has pure point spectrum. Later on Narnhofer and Thirring [19] and Park and Shin [21] independently extended the result to more general A. Before I state the results let us look more closely at the concepts and ideas in question.

If A has pure point spectrum there is an orthonormal basis (f_n) of H such that $Af_n = \lambda_n f_n$, $n \in \mathbb{N}$, $0 \le \lambda_n \le 1$. Define recursively operators

$$V_0 = 1, \qquad V_n = \prod_{i=1}^{n} \left(1 - 2a(f_i)^* a(f_i)\right), \qquad e_{11}^{(n)} = a(f_n) a(f_n)^*$$

$$e_{12}^{(n)} = a(f_n) V_{n-1}, \quad e_{21}^{(n)} = V_{n-1} a(f_n)^*, \quad e_{22}^{(n)} = a(f_n)^* a(f_n) .$$

Then the $e_{ij}^{(n)}$, $i,j = 1,2$ form a complete set of 2×2 matrix units generating a I_2-factor $M_2(\mathbb{C})_n$, and for distinct n and m $e_{ij}^{(n)}$ and $e_{k\ell}^{(m)}$ commute. Thus $\mathcal{A}(H) \simeq \bigotimes_1^\infty M_2(\mathbb{C})_n$, and ω_A is a product state $\omega_A = \bigotimes_1^\infty \omega_{\lambda_n}^0$ with respect to this factorization, where ω_λ^0 is the state on $M_2(\mathbb{C})$ given by

$$\omega_\lambda^0\left(\begin{pmatrix} a & b \\ c & a \end{pmatrix}\right) = (1-\lambda)a + \lambda d .$$

In case $A = \lambda 1$ we write ω_λ for ω_A. Then α_U is ω_A-invariant for all U. We first consider the entropy $h_{\omega_\lambda}(\alpha_U)$.

Each unitary U is a direct sum $U = U_a \oplus U_s$, where U_a has spectral measure absolutely continuous with respect to Lebesgue measure $d\theta$ on the circle, while U_s has spectral measure singular with respect to $d\theta$. We shall as above denote by $m(U)$ the multiplicity function of U_a. The idea is now to approximate the case when

$$U = U_s \oplus U_1 \oplus \cdots \oplus U_n ,$$

where each U_i acts on a Hilbert space H_i, $i = 1, \dots, n$, and U_i is unitarily equivalent to V^{p_i}, where V is a bilateral shift. Let us for simplicity ignore the complications due to the grading of $\mathcal{A}(H)$ as a direct sum of its even and odd parts. Then

$$\alpha_U = \alpha_{U_s} \otimes \alpha_{U_1} \otimes \cdots \otimes \alpha_{U_n}$$

and

$$\omega_\lambda = \omega_\lambda|\mathcal{A}(H_s) \otimes \omega_\lambda|\mathcal{A}(H_1) \otimes \cdots \otimes \omega_\lambda|\mathcal{A}(H_n).$$

Thus we could hope that

$$h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda|\mathcal{A}(H_s)}(\alpha_{U_s}) + \sum_{i=1}^{n} h_{\omega_\lambda|\mathcal{A}(H_i)}(\alpha_{U_i}), \tag{7.2}$$

and thus restrict attention to the case when U is singular or a power of a bilateral shift. We do have problems because of Problem 6.6, but life turns out nicely because we can as with the shift in section 3 restrict attention to the diagonal, and the diagonal is contained in the even CAR-algebra, where the tensor product formulas above hold. First we take care of the singular part U_s.

Lemma 7.3. *If U has spectral measure singular with respect to the Lebesgue measure, and α_U is φ-invariant for a state φ, then $h_\varphi(\alpha_U) = 0$.*

Thus in (7.2) we can forget about U_s. If $U = V^p$ with V a bilateral shift and $p \in \mathbb{Z}$, then

$$h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda}((\alpha_V)^p) = |p| h_{\omega_\lambda}(\alpha_V).$$

If we write $\mathcal{A}(H) = \bigotimes\limits_{n=-\infty}^{\infty} M_\alpha(\mathbb{C})_n$, then on the diagonal α_V is the shift, so like in section 3 we get

$$h_{\omega_\lambda}(\alpha_V) = \eta(\lambda) + \eta(1-\lambda).$$

Now $|p|$ is the multiplicity $m(U)$ of U, and since $\frac{1}{2\pi} d\theta$ is the normalized Haar measure on the circle, it is not surprising that we have

Theorem 7.4. *Let U be a unitary operator on H and $\lambda \in [0,1]$. Then $h_{\omega_\lambda}(\alpha_U) = \frac{1}{2\pi}\big(\eta(\lambda) + \eta(1-\lambda)\big) \int_0^{2\pi} m(U)(\theta) d\theta$.*

Note that if $\lambda = 1/2$, $\omega_\lambda = \tau$, so we get formula (7.1). For more general A we use direct integral theory with respect to the von Neumann algebra generated by U_a. If A commutes with U, $A = A_a \oplus A_s$, where $A_a = \int_0^{2\pi}{}^{\oplus} A(\theta) d\theta$, where $H = \int^{\oplus} H_\theta d\theta$, and $H_\theta = 0$ if $m(U)(\theta) = 0$, and $A(\theta) \in B(H_\theta)$. If A has pure point spectrum, $A = \sum\limits_{j \in I} \lambda_j e_j$ with J finite or countably infinite, and (e_j) is an orthogonal family of projections with sum 1, $\lambda_j \in [0,1]$. Denote by $U_j = U|e_j(H)$, and let Tr be the usual trace on $B(H_\theta)$. Writing $e_j = \int^{\oplus} e_j(\theta) d\theta$ we get

$$\begin{aligned} Tr\Big(\eta(A(\theta)) + \eta(1 - A(\theta))\Big) &= \sum_j (\eta(\lambda_j) + \eta(1-\lambda_j)) Tr(e_j(\theta)) \\ &= \sum_j (\eta(\lambda_j) + \eta(1-\lambda_j)) m(U_j)(\theta). \end{aligned}$$

Integrating and using Theorem 7.4 we have

$$\sum_{j \in J} h_{\omega_{\lambda_j}}(\alpha_{U_j}) = \frac{1}{2\pi} \int_0^{2\pi} Tr(\eta(A(\theta)) + \eta(1 - A(\theta)))d\theta \, .$$

Some extra work shows that the left side equals $h_{\omega_A}(\alpha_U)$, thus we have

Theorem 7.5. *Let $0 \le A \le 1$ have pure point spectrum and U a unitary commuting with A. Then*

$$h_{\omega_A}(\alpha_U) = \frac{1}{2\pi} \int_0^{2\pi} Tr(\eta(A(\theta)) + \eta(1 - A(\theta)))d\theta \, .$$

A natural problem is to extend this result to general A. This is nontrivial and was studied already by Narnhofer and Thirring in [18], where they showed, with some gaps in the proof, that the formula is true for a bilateral shift. There are recently two independent papers on the problem, by Narnhofer and Thirring [19] and Park and Shin [21]. They both concentrate attention to the case when U is absolutely continuous. Imposing technical assumptions on A they prove formulas like the one in the theorem. In [21] this was also done for the canonical commutation relations. The mathematics in [21] is quite involved. They use the definition of entropy directly and go through hard analysis to estimate the entropy defects. A perhaps easier approach would be to apply Theorem 6.4 directly. I believe this is done in [19], but I must admit, I have not understood the proof. Narnhofer and Thirring claim [19, Remark 3.4] that the assumptions they impose on A are so weak that they consider Problem 7.1 as settled.

In the above papers it is shown that when α_U is space translation on $\mathcal{A}(H)$ then $h_{\omega_A}(\alpha_U)$ is the mean entropy in the sense of Theorem 6.4. It should, however, be noted that Fannes [11] showed a formula like Theorem 7.5 for this mean entropy.

8 An alternative definition of entropy

Sauvageot and Thouvenot [28] have given an alternative definition of entropy which is very close to that of Connes, Narnhofer, and Thirring [7}, but which is closer in spirit and notation to the classical definition.

Let A be a C^*-algebra together with a state ρ. A *coupling* of (A, ρ) with an Abelian C^*-algebra is a pair (λ, B), where B is an Abelian C^*-algebra, and λ is a state on the C^*-algebra $A \otimes B$ whose restriction to A (identified with $A \otimes 1$) is ρ. We denote by μ the probability measure on B obtained from the restriction of λ to B. If B is finite dimensional then $B = C(X)$ with X a finite set, and for each $x \in X$ the characteristic function $\mathcal{X}_x (= \mathcal{X}_{\{x\}})$ is a minimal projection in B. We then get a state ρ_x on A defined by

$$\rho_x(a) = \mu(\{x\})^{-1} \lambda(a \otimes \mathcal{X}_x) \, ,$$

which gives ρ as a convex sum of states,

$$\rho = \sum_{x \in X} \mu(\{x\}) \rho_x \, .$$

In analogy with $\varepsilon_\mu(P)$ from section 6 we put

$$\varepsilon_\lambda(A,B)=\sum_{x\in X}\mu(\{x\})S(\rho|\rho_x)=S(\rho\otimes\mu|\lambda).$$

It follows that $\varepsilon_\lambda(A,B)$ measures how far λ is from being a product state by using the distance function relative entropy. The entropy defect s_μ now takes the form of a conditional entropy when we define it as before as

$$H_\lambda(B|A)=H_\mu(B)-\varepsilon_\lambda(A,B)=-\sum_{x\in X}S(\rho|\mu(\{x\})\rho_x).$$

Here $H_\mu(B)$ is the entropy of μ as a probability measure on X. If we identify each finite dimensional subalgebra of B with the partition consisting of its minimal projections we find for $\mathcal{P}$ and $\mathcal{Q}$ partitions of X, that

$$H_\lambda(\mathcal{P}\vee\mathcal{Q}|A)=H_\lambda(\mathcal{P}|A)+H_\lambda(\mathcal{Q}|A\otimes\mathcal{P}).$$

In the special case when $A=\mathbb{C}$ this identity reduces to the classical identity for conditional entropy

$$H(\mathcal{P}\vee\mathcal{Q})=H(\mathcal{P})+H(\mathcal{Q}|\mathcal{P}).$$

Let now α be a ρ-invariant automorphism of A, and (λ,B) a coupling with an Abelian C^*-algebra which is not necessarily finite dimensional. Suppose further that $\sigma\in Aut(B)$ and that $\alpha\otimes\sigma$ is λ-invariant. For a finite partition $\mathcal{P}$ of B denote its past by

$$\mathcal{P}^-=\bigvee_{i=1}^{\infty}\sigma^{-i}\mathcal{P}.$$

In the classical case we have

$$h(\sigma,\mathcal{P})=\lim_n\tfrac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1}\sigma^{-1}(\mathcal{P})\Big)=H_\mu(\mathcal{P}|\mathcal{P}^-).$$

Analogously we define two expressions

$$\begin{aligned}h(\lambda,\mathcal{P})&=H_\mu(\mathcal{P}|\mathcal{P}^-)-H_\lambda(\mathcal{P}|A\otimes\mathcal{P}^-)\\ h'(\lambda,\mathcal{P})&=H_\mu(\mathcal{P}|\mathcal{P}^-)-H_\lambda(\mathcal{P}|A).\end{aligned}$$

Then $h(\lambda,\mathcal{P})\geq 0$ and $h(\lambda,\mathcal{P})\geq h'(\lambda,\mathcal{P})$.

Sauvageot and Thouvenot now define the entropy of the dynamical system (A,α,ρ) to be

$$H_\rho(\alpha)=\sup h(\lambda,\mathcal{P}),$$

where the sup is taken over all couplings (λ,B), partitions $\mathcal{P}$ and automorphisms σ as above. Then they show that we get the same by using h', i.e.

$$H_\rho(\alpha)=\sup h'(\lambda,\mathcal{P}).$$

Furthermore they show

Theorem 8.1. *If A is nuclear the entropy $H_\rho(\alpha)$ is the same as the entropy $h_\rho(\alpha)$ defined in section 6.*

In addition to its resemblance to the classical case the above definition has another nice feature. Let A and ρ be as before and (π, H, ξ) the GNS-representation of ρ. If $\alpha \in Aut(A)$ is ρ-invariant it is implemented in the GNS-representation by the unitary U_ρ defined by $U_\rho \pi(x)\xi = \pi(\alpha(x))\xi$. The following result is analogous to Lemma 7.3.

Theorem 8.2. *If the spectral measure of U_ρ is singular with respect to the Lebesgue measure on the circle then $H_\rho(\alpha) = 0$.*

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Hypergroup structures associated with Gel'fand pairs of compact quantum groups

Leonid Vainerman

1 Introduction

A notion of a Gel'fand pair for compact quantum groups introduced by T.H. Koornwinder in [19] is a generalization of the classical one for a locally compact group G and its compact subgroup K such that for any irreducible unitary representation of G, the dimension of the space of K-bi-invariant matrix elements is not greater then 1; this is equivalent to the commutativity of the subalgebra of group algebra of G, formed by K-bi-invariant functions (see [11]). This classical notion of a Gel'fand pair can be formulated as the cocommutativity of the coproduct

$$\Delta(f)(g,h) := \int_K f(gkh)d\mu_K(k) \quad (\mu_K - \ Haar\ measure\ for\ K) \tag{1}$$

on the space of all K-bi-invariant functions on G. Considering such functions as functions on the set of double cosets $Q = K\backslash G/K$, one can rewrite (1) in the following form:

$$\Delta(f)(p,r) = \int_Q K(p,r,s)f(s)d\mu_Q(s) \qquad (p,r \in Q), \tag{2}$$

where $K(.,.,.)$ is some positive kernel, μ_Q is some positive Borel measure on Q (which can depend on p and r in general case). A function $\chi_\alpha(.)$ on Q (α is classifying parameter) is called a character of the coalgebra given by (2) if it satisfies a product formula:

$$\int_Q K(p,r,s)\chi_\alpha(s)d\mu_Q(s) = \chi_\alpha(p)\chi_\alpha(r) \qquad (p,r \in Q). \tag{3}$$

We will say that the coproduct (2) defines a hypergroup structure on the algebra of K-bi-invariant functions on G with the pointwise multiplication. One can find a discussion of hypergroups in [5],[6],[13],[22],[26] and in references given there. In many cases the χ_α are well known special functions. Very often we have a similar formula with respect to α - dual product formula. It shows that χ_α is also a character of a dual hypergroup by the variable α.

In this paper we consider double cosets of compact quantum group with respect to its subgroup and distinguish cases of a Gel'fand pair and a strict Gel'fand pair. We show that every strict Gel'fand pair of compact quantum groups generates a normal commutative hypercomplex system with compact basis [5],[6] and a commutative discrete hypergroup [13], which are in duality to one another, consider corresponding examples and describe characters of hypergroups in terms of q-orthogonal polynomials.

After this paper had gone to press, the essential development of the subject took place. On the one hand, Gel'fand pairs for non compact quantum groups were considered (see, for example, [29]). On the other hand, one can consider a notion of a quantum subgroup of a quantum group from more general point of view then in this paper, using a notion of a coideal (see, for example, [9],[12],[14],[15],[20], [21],[23],[28]). This permits to apply the Gel'fand pair approach to exceptionally interesting classes of q-special functions such as Macdonalds and Askey-Wilson polynomials and Jacksons q-Bessel functions. This development is described in the survey [28].

I would like to express my gratitude to Yu.A. Chapovsky, T.H. Koornwinder and A.U. Klimyk for many useful discussions.

2 Double cosets of quantum groups

2.1. Let $\mathbf{H}:=(H,d,1,\Delta,\varepsilon,S),\widehat{\mathbf{H}}:=(\widehat{H},\hat{d},\hat{1},\hat{\Delta},\hat{\varepsilon},\widehat{S})$ be two Hopf algebras over C [1], with multiplications $d,\hat{d}$, units $1,\hat{1}$, comultiplications $\Delta,\hat{\Delta}$, counits $\varepsilon,\hat{\varepsilon}$, antipodes $S,\widehat{S}$

Definition 1 *We say that $\mathbf{H},\widehat{\mathbf{H}}$ are in duality, if there exists a doubly non-degenerate pairing $\langle\cdot,\cdot\rangle$: $H\times\widehat{H}\to C$ such that:*

$$\langle 1,\zeta\rangle=\hat{\varepsilon}(\zeta),\ \langle ab,\zeta\rangle=\langle a\otimes b,\hat{\Delta}(\zeta)\rangle,\ \langle\Delta(a),\zeta\otimes\eta\rangle=\langle a,\zeta\eta\rangle,$$

$$\langle a,\hat{1}\rangle=\varepsilon(a),\ \langle S(a),\zeta\rangle=\langle a,\widehat{S}(\zeta)\rangle\qquad(\forall a,b\in H,\ \zeta,\eta\in\widehat{H}).$$

We can define elements $\zeta*a:=(id\otimes\zeta)\circ\Delta(a),\quad a*\zeta:=(\zeta\otimes id)\circ\Delta(a)$, where the pairing is used in the first, respectively second part of the tensor product. It is possible to rewrite the last equalities as $\langle\zeta*a,\eta\rangle=\langle a,\eta\zeta\rangle$, $\langle a*\zeta,\eta\rangle=\langle a,\zeta\eta\rangle$. These operations yield left and right algebra actions of $\widehat{H}$ on H:

$$(\zeta\eta)*a=\zeta*(\eta*a),\ a*(\zeta\eta)=(a*\zeta)*\eta\qquad(\forall a\in H,\ \eta,\zeta\in\widehat{H}).$$

Now let $\mathbf{H},\widehat{\mathbf{H}}$ be two Hopf algebras in duality, $\zeta\in\widehat{H}$. In what follows we will suppose that another pair $(\mathbf{H}_1,\widehat{\mathbf{H}}_1)$ of Hopf algebras in duality exists together with an epimorphism $\pi:\mathbf{H}\to\mathbf{H}_1$ and embedding $i:\widehat{\mathbf{H}}_1\to\widehat{\mathbf{H}}$ such that $\langle\pi(a),\zeta\rangle=\langle a,i(\zeta)\rangle$ $(\forall a\in H,\zeta\in\widehat{H}_1)$. Left and right coactions $\Delta^l:=(\pi\otimes id)\circ\Delta$, $\Delta^r:=(id\otimes\pi)\circ\Delta$ of $\mathbf{H}_1$ on H define the subsets of left-, right- and bi-ivariant elements: $H_1\backslash H:=\{h\in H|\Delta^l(h)=1_1\otimes h\}$, $H/H_1:=\{h\in H|\Delta^r(h)=h\otimes 1_1\}$, $H_1\backslash H/H_1:=H_1\backslash H\cap H/H_1$. All these sets are evidently unital algebras. Let an invariant integral ν_1 (such that $\nu_1(1)=1$) on Hopf algebra $\mathbf{H}_1$ [1] exist (it always exists when $\mathbf{H}_1$ is a compact quantum group in the sense of [34]). Then we can introduce two projections $\pi^l:=(\nu_1\circ\pi\otimes id)\circ\Delta$, $\pi^r:=(id\otimes\nu_1\circ\pi)\circ\Delta$ from H to $H_1\backslash H$ and H/H_1 correspondingly. They commute and $\pi^r\circ\pi^l$ is a projection from H to $H_1\backslash H/H_1$ (see [7],[19]). A new coproduct may be introduced on $H_1\backslash H/H_1$:

$$\tilde{\Delta}:=(id\otimes\nu_1\circ\pi\otimes id)\circ(id\otimes\Delta)\circ\Delta\tag{4}$$

This definition is a generalization of (1) for Hopf algebra case.

Theorem 1 *Let a mapping $\tilde{\Delta}$ be defined by (4). Then:*

(a) *$\tilde{\Delta}$ maps $H_1\backslash H/H_1$ into $H_1\backslash H/H_1 \otimes H_1\backslash H/H_1$;*

(b) *$\tilde{\Delta}$ is coassociative, i.e. $(id \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes id) \circ \tilde{\Delta}$;*

(c) *ε is a counit with respect to $\tilde{\Delta}$: $(\varepsilon_1 \otimes id) \circ \tilde{\Delta} = (id \otimes \varepsilon_1) \circ \tilde{\Delta} = id$;*

(d) *if ν is an invariant integral on $\mathbf{H}$, then ν is invariant with respect to $\tilde{\Delta}$:*

$$(\nu \otimes id) \circ \tilde{\Delta}(h) = (id \otimes \nu) \circ \tilde{\Delta}(h) = \nu(h) \cdot 1;$$

(e) *the following relation holds: $\tilde{\Delta} \circ S = \Pi \circ (S \otimes S) \circ \tilde{\Delta}$.*

PROOF. a) Evidently, $\tilde{\Delta} = (id \otimes \pi^l)\Delta = (\pi^r \otimes id)\Delta$. On the other hand, $(id \otimes \pi^r)\Delta = \Delta \circ \pi^r, (\pi^l \otimes \pi^l)\Delta = \Delta \circ \pi^l$. So for every $h \in H_1\backslash H/H_1$ we have $\tilde{\Delta}(h) \in H_1\backslash H/H_1 \otimes H$. Similarly we see that $\tilde{\Delta}(h) \in H \otimes H_1\backslash H/H_1$. b) Both sides of needed equality coincide with $(\pi^r \otimes id \otimes \pi^l)(\Delta \otimes id)\Delta$. c) $(\varepsilon \otimes id)\tilde{\Delta} = (\varepsilon \otimes \pi^l)\Delta = \pi^l$, so that ε is right counit. Similarly one can see that it is also left counit. d) Replacing ε by ν, we can prove this statement exactly as previous. e) This is implied by the following chain of equalities: $\tilde{\Delta} \circ S = (id \otimes \pi^l)\Delta \circ S = (id \otimes \pi^l)\Pi(S \otimes S)\Delta = (id \otimes \nu_1 \circ \pi \otimes id)$ $(\Pi \otimes id)(id \otimes \Pi)(S \otimes S \otimes S)(\Delta \otimes id)\Delta = \Pi(S \otimes S)(id \otimes \nu_1 \circ S_1 \circ \pi \otimes id)(\Delta \otimes id)\Delta = \Pi(S \otimes S)\tilde{\Delta}$. □

Two Hopf *-algebras $\mathbf{H}, \widehat{\mathbf{H}}$ are said to be in duality, if they are in duality as Hopf algebras and

$$\zeta^*(f) = \overline{\zeta(S(f^*))} \quad \forall \zeta \in \widehat{\mathbf{H}},\ f \in \mathbf{H},$$

where the same symbol denotes the involution in $\mathbf{H}$ and in $\widehat{\mathbf{H}}$. In what follows, we will be considering $\mathbf{H}, \mathbf{H}_1$ as Hopf *-algebras (see, for example, [25],[32],[34]) with the Hopf algebra structure and the involution *, π as an epimorphism of Hopf *-algebras, ν_1 as a state on the *-algebra H_1. Then $H_1\backslash H$, H/H_1 and $H_1\backslash H/H_1$ will be unital *-algebras, $\pi^l, \pi^r, \tilde{\Delta}$ map the cone of positive elements into the cones of positive elements of the corresponding *-algebras.

Definition 2 *A pair of Hopf algebras (resp. *-Hopf algebras) $(\mathbf{H}, \mathbf{H}_1)$ is called a Gel'fand pair if the coproduct $\tilde{\Delta}$ is cocommutative. A Gel'fand pair is called strict if the algebra $H_1\backslash H/H_1$ is commutative.*

2.2. Now let $\mathbf{H}$ be *-Hopf algebra associated with a compact quantum group and $\widehat{\mathbf{H}}$ is its algebraic dual. We know [34] that H can be represented as

$$H = \sum_{\alpha} \sum_{i,j=1}^{d_\alpha} \mathbf{C} u_{i,j}^{\alpha}, \tag{5}$$

where $u_{i,j}^{\alpha}$ are matrix elements of d_α- dimensional unitary corepresentation of $\mathbf{H}$ ($d_\alpha < \infty$ for all α running in some discrete set $\widehat{Q}$) and there exists an invariant integral ν on H, which is a state and such that α-sum in (5) defines an orthogonal decomposition in the sense of the inner product given by $\langle f, g\rangle := \nu(f \cdot g^*)$ after a suitable choice of an orthonormal basis for each representation space. In this case, the comodules $H_1\backslash H$, H/H_1 and also $H_1\backslash H/H_1$ may be given by

$$H_1\backslash H = \sum_\alpha \sum_{i=1}^{d'_\alpha} \sum_{j=1}^{d_\alpha} \mathbf{C}u^\alpha_{i,j}, \ H/H_1 = \sum_\alpha \sum_{i=1}^{d_\alpha} \sum_{j=1}^{d'_\alpha} \mathbf{C}u^\alpha_{i,j},$$

$$H_1\backslash H/H_1 = \sum_\alpha \sum_{i,j=1}^{d'_\alpha} \mathbf{C}u^\alpha_{i,j}$$

where $d'_\alpha \le d_\alpha$ for all α. A notion of a Gel'fand pair for compact quantum groups was introduced in [19] as a pair $(\mathbf{H},\mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$, such that for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.

Lemma 1 *A pair of compact quantum groups $(\mathbf{H},\mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ is a Gel'fand pair in the sense of Definition 2, iff for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.*

PROOF. Suppose that $2 \le d_\beta$ for some fixed $\beta \in \hat{Q}$. Set $\eta_1(u^\beta_{1,2}) := 1$, $\eta_1(u^\alpha_{i,j}) := 0$ otherwise and $\eta_2(u^\beta_{2,1}) := 1$, $\eta_2(u^\alpha_{i,j}) := 0$ otherwise. One can check that $\eta_1, \eta_2 \in (H_1\backslash H/H_1)^* \cap \widehat{H}_1$. Direct calculations show that $\langle\tilde{\Delta}(u^\beta_{1,1}), \eta_1 \otimes \eta_2\rangle \ne \langle\tilde{\Delta}(u^\beta_{1,1}), \eta_2 \otimes \eta_1\rangle$, i.e., $\tilde{\Delta}$ is not cocommutative. Conversely, if $d_\alpha = 1$ $\forall\alpha \in \hat{Q}$, then $\tilde{\Delta}$ is obviously cocommutative. □

3 Connections with hypercomplex systems and hypergroups

3.1. We will use notions of a spatial tensor product for C^*-algebras, a unital Hopf C^*-algebra, a morphism, and a counit for unital Hopf C^*-algebras, as well as notions of a coaction of a unital Hopf C^*-algebra on a unital C^*- algebra and finite Haar measure on a unital Hopf C^*-algebra (see[4],[10],[34]). If $\mathbf{H}$ is a unital Hopf C^*-algebra, then the coproduct defines a structure of a Banach algebra in the conjugate space H^* for the C^*-algebra H:

$$\tilde{\omega} * \omega := (\tilde{\omega} \otimes \omega) \circ \Delta, \quad \forall\omega, \tilde{\omega} \in H^*.$$

$\mathbf{H}$ has a counit if and only if H^* is a unital algebra.

As in Section 2, we denote $\forall a \in H, \omega \in H^*$:

$$\omega * a := (id \otimes \omega) \circ \Delta(a), \ a * \omega := (\omega \otimes id) \circ \Delta(a).$$

Let ν be finite Haar measure on a unital Hopf C^*-algebra $\mathbf{H}$. One can introduce by means of GNS-construction a structure of the Hilbert space $L_2(H,\nu)$ and the corresponding representation of Λ_ν of H in this space. For every compact quantum group the completion of the initial Hopf *-algebra with respect to the C^{**}-norm $|\cdot| = \sup_\rho ||\rho(\cdot)||$, where ρ runs over the set of all irreducible representations of H, give a unital Hopf C^*-algebra $\mathbf{H}$ (see [34]).

Now we consider the initial situation of section 2, in which: 1) $\mathbf{H}$ is a unital Hopf C*-algebra, having a finite Haar measure ν and a counit ε; 2) $\mathbf{H_1}$ is a unital Hopf C*-algebra, having a finite Haar measure ν_1 and a counit ε_1 (such measures and counits always exist when $\mathbf{H}, \mathbf{H_1}$ are compact quantum groups (see [34]); 3) π is an epimorphism in the category of unital Hopf C^*-algebras. Then one can consider all the above mentioned algebras as unital C^*-algebras, all the above mentioned coactions and positive mappings as coactions of unital Hopf C^*-algebras and positive mappings of C^*-algebras. Particularly, $H_1\backslash H$, H/H_1, $H_1\backslash H/H_1$ are C^*-subalgebras of H, $\pi^l, \pi^r, \tilde{\Delta}$ are positive mappings of unital C^*-algebras. All the statements of section 2 are valid for unital Hopf C^*-algebras except for those which involve an antipode.

3.2. Let Q be a locally compact Hausdorff space, and let M be a Banach space of complex valued functions on Q. We denote by $\{L^s | s \in Q\}$ a family of left generalized shift operators (below Delsarte-Levitan hypergroup or simply hypergroup) [22] acting in M. One can find a discussion of hypergroups and their special classes in [5],[6],[13],[26] and references given there. The notion of a hypergroup can be formulated in terms of the coassociativity of the coproduct $(\tilde{\Delta} f)(t,s) := L^s f(t)$. In the case when $M = C_b(Q)$ is the C^*-algebra of all bounded continuous functions on Q and $\tilde{\Delta}$ is a continuous mapping from M to $M \otimes M$, there exists a structure of a Banach algebra on the dual space $M^* = M(Q)$ of all finite regular Borel measures on Q with the convolution

$$\langle f, \delta_s * \delta_t \rangle = \langle L^s f,\ \delta_t \rangle = (L^s f)(t) \qquad \forall t, s \in Q, f \in M, \tag{6}$$

(where $\langle \cdot, \cdot \rangle$ is the pairing for M and M^*, δ_t is the delta function concentrated at the point t), and with a unity δ_e. The hypergroup is called commutative if the algebra M^* is commutative. We will call a function $\chi(\cdot) \in M$ *a character* of hypergroup $\{L^p\}$ if it is a character of the algebra M^*, i.e., if $(L^p \chi)(r) = \chi(p)\chi(r)\ \forall p, r \in Q$.

In many applications, hypergroups satisfy some special conditions:

a) the action of a hypergroup preserves positivity of functions and the function which identically equals to unity;

b) there exists an involutive homeomorphism $x \mapsto x^\vee$ of Q (the analogue of taking the inverse in a group) such that

$$(L^s f)(t) = (L^t f^\vee)(s), \quad \forall f \in M, r, s, t \in Q, f^\vee(t) := f(t^\vee)$$

and $e^\vee = e$. The hypergroup having a property b) is called involutive. In this case M^* is a Banach *-algebra with the involution extending the mapping $\delta_x \to \delta_{x^\vee}$. If $x = x^\vee \quad \forall x \in Q$, the corresponding hypergroup is called Hermitian, it is automatically commutative. The definition of a character of an involutive hypergroup contains a condition $\chi(r^\vee) = \overline{\chi(r)} \quad \forall r \in Q$.

Usually the existence of some special positive regular Borel measure ν on Q with the property $\int_Q L^p f(q) d\nu(q) = \int_Q f(q) d\nu(q)$ - the analogue of a Haar measure on a group is assumed (or is proved under some additional conditions [5],[6],[13]). Such a measure is also called a Haar measure. Then one can consider the hypergroup as a family of bounded linear operators acting in the spaces $L_p(Q,\nu), (1 \le p \le \infty)$. The important special classes of hypergroups with the described properties were studied by Yu.M. Berezanskii, S.G. Krein, A.A. Kalyuzhnyi (hypercomplex systems with a locally compact basis) and also by Ch.Dunkl, R. Jewett, R. Spector and others (DJS-hypergroups). See a discussion in [5],[6],[13],[26].

Let us suppose that $\mathbf{H}$ is a unital Hopf C^*-algebra with a finite Haar measure ν and $H_1\backslash H/H_1$ is a unital commutative C^*-algebra. Then the restristion of ν to this algebra is generated by some finite measure on its spectrum Q, which is a compact topological space. We use $\tilde{\Delta}$ and ν to denote the restrictions of the corresponding mappings to $H_1\backslash H/H_1$ (as well as the measure on Q, generating ν). Since $H_1\backslash H/H_1$ is isomorphic to the unital C^*-algebra $C(Q)$ of all continuous functions on Q and any $p \in Q$ can be identified with a continuous homomorphism $p : H_1\backslash H/H_1 \to \mathrm{C}$, there exists a family of operators $L^p : H_1\backslash H/H_1 \to H_1\backslash H/H_1$ given by

$$L^p(f) := (id \otimes p) \circ \tilde{\Delta}(f) = p *' f \qquad \forall f \in H. \tag{7}$$

One can see that these operators generate hypergroup with $e = \varepsilon, M = H_1\backslash H/H_1$. We use $H_1\backslash H/H_1$ and the measure ν to construct the Hilbert space $L_2(Q,\nu)$ and consider L^p as operators acting in this space, defined first on $L_2(Q,\nu) \cap L_\infty(Q,\nu)$.

Lemma 2 *Let the hypergroup $L^p, p \in Q$, be given by (7). Then:*

1) L^p may be extended to a bounded operator for all $p \in Q$ and the mapping $p \mapsto L^p$ is strongly continuous;

2) $L^\varepsilon = id$;

3) for any positive $f \in L_2(Q,\nu)$, $L^p(f)$ is positive for all $p \in Q$;

4) $L^p(1)(q) = 1$ for all $p, q \in Q$.

PROOF. 1) It follows from the definition of L^p and the positivity property of $\tilde{\Delta}$ that L^p are bounded with $||L^p|| \leq 1$. Moreover, for any $f \in H$ the mapping $p \mapsto L^p(f) = (p \otimes id)\tilde{\Delta}(f)$ is strongly continuous. 2) This is a direct consequence of part c) of the Theorem 1. 3) Since $\tilde{\Delta}$ is positive, this follows from the fact that p is a homomorphism and from the property 1). As a cosequence of this fact we have that L^p maps real functions into real. 4) Since $\tilde{\Delta}(1) = 1 \otimes 1$ and p, q are homomorphisms, we have $L^p(1)(q) = 1$. □

Remark 3.1. Let now S, S_1 be antipodes on $\mathbf{H}, \mathbf{H}_1$ respectively such that $\pi \circ S = S_1 \circ \pi$ and the restriction of S to $H_1\backslash H/H_1$ is continuous. Then one can define an involutive homeomorphism $\vee$ of Q: $p^\vee = p \circ S \quad \forall p \in Q$ such that the hypergroup has a property

5) $\varepsilon^\vee = \varepsilon$ and $(L^q S(f))(p) = (L^p f)(q)$ for all $f \in L_2(Q,\nu)$.

If additionally the Haar measure ν satisfies the relation

$$\nu((p *' a)^* b) = \nu(a^*(p *' b^\vee)), \qquad \forall a, b \in H_1\backslash H/H_1, p \in Q, \tag{8}$$

then the hypergroup have an additional property

6)$(L^p)^* = L^{p^\vee}$, where $(L^p)^*$ is the operator adjoint to L^p in $L_2(Q,\nu)$.

These considerations and Theorem 2.1 of [6] show that the described hypergroup satisfies all the properties of a commutative normal hypercomplex system with a compact basis and a basis unity. A dual hypergroup may be constructed using the considerations of [19] (in this paper a discrete, generally noncommutative, DJS-hypergroup was constructed for every Gel'fand pair of compact quantum groups, not obligatory strict).

Thus, every strict Gel'fand pair of quantum groups generates two commutative hypergroup structures dual to one another: a discrete DJS-hypergroup and a normal hypercomplex system with a compact basis.

3.3. The described construction of double cosets for compact quantum groups may be generalized, if we replace a Hopf algebra $\widehat{\mathbf{H}}_1$ by a coideal (see[23]) in a Hopf algebra $\widehat{\mathbf{H}}$. Such a generalization permits to establish general point of view based on the notion of strict Gel'fand pair, to the interesting examples of Askey-Wilson [3],[21] and Macdonald [23] polynomials (see also [12],[9]). It will be described in a separate paper. In this more general situation we do not know if the mapping $\tilde{\Delta}$ is positive, so that we can not refer neither to hypercoplex systems in the sense of [5],[6] nor to DJS-hypergroups [13]. We can only use a duality principle for real hypercomplex systems with compact and discrete basis described in [27].

4 Examples

4.1. It is known [25] that quantum group $SL_q(2,C)$ is generated by elements $\alpha,\beta,\gamma,\delta$ such that:

$$\alpha\beta = q\beta\alpha, \alpha\gamma = q\gamma\alpha, \beta\gamma = \gamma\beta, \beta\delta = q\delta\beta,$$
$$\gamma\delta = q\delta\gamma, \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma = 1,$$
$$\Delta(\alpha) := \alpha\otimes\alpha + \beta\otimes\gamma, \Delta(\beta) := \alpha\otimes\beta + \beta\otimes\delta,$$
$$\Delta(\gamma) := \gamma\otimes\alpha + \delta\otimes\gamma, \Delta(\delta) := \gamma\otimes\beta + \delta\otimes\delta,$$
$$\varepsilon(\alpha) = \varepsilon(\delta) := 1, \varepsilon(\beta) = \varepsilon(\gamma) := 0, \tag{9}$$
$$S(\alpha) := \delta,\ S_1(\beta) := -q^{-1}\beta,\ S(\gamma) := -q\gamma,\ S(\delta) := \alpha, \quad q \in \{C\backslash 0\}.$$

The dual Hopf algebra (quantized enveloping algebra) $U_q(sl(2,C))$ is generated by elements A, B, C, D and relations:

$$AB = qBA,\ AC = q^{-1}CA,\ AD = DA = \hat{1}_1,\ BC - CB = (q - q^{-1})^{-1}$$
$$(A^2 - D^2), \hat{\Delta}(A) := A\otimes A,\ \hat{\Delta}(B) := A\otimes B + B\otimes D, hat\Delta(C) := A\otimes C + C\otimes D,$$
$$\hat{\Delta}(D) := D\otimes D, \hat{\varepsilon}(A) = \hat{\varepsilon}(D) := 1,\ \hat{\varepsilon}(B) = \hat{\varepsilon}(C) := 0, \tag{10}$$
$$\hat{S}(A) := D,\ \hat{S}(B) := -q^{-1}B,\ \hat{S}(C) =:= -qC,\ \hat{S}(D) := A\ (q^2 \neq 1).$$

. In [31] it was shown that these Hopf algebras are in duality with respect to the pairing

$$\langle\alpha, A\rangle = \langle\delta, D\rangle := q^{\frac{1}{2}}, \langle\delta, A\rangle = \langle\alpha, D\rangle = q^{-\frac{1}{2}}, \langle\beta, B\rangle = \langle\gamma, C\rangle = 1, \tag{11}$$

which equals to 0 for other pairs of generators. A real form $\mathbf{H} = \mathbf{SU_q(2)}$ of $SL_q(2,C)$ distinguished by an involution $\alpha^* = \delta$, $\delta^* = \alpha$, $\beta^* = -q\gamma$, $\gamma^* = -q^{-1}\beta$ $(0 < q < 1)$ may be equipped with a structure of a compact quantum group in the sense of [34] (see also [31],[35]). Let $\mathbf{H}_1 = U(1)$ be a Hopf *-algebra generated by commuting variables $t, t^{-1} = t^*$ and mappings: $\Delta_1(t) := t\otimes t$, $S_1(t) := t^{-1}$, $\varepsilon_1(t) := 1$. An epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ is defined as $\pi(\alpha) := t$, $\pi(\delta) := t^{-1}$, $\pi(\beta) = \pi(\gamma) := 0$. One can consider π as an epimorphism of unital Hopf C^*-algebras and Δ as a positive continuous mapping of commutative unital C^*-algebra $H_1\backslash H/H_1$ to its tensor square. The spectrum of the C^*-algebra $H_1\backslash H/H_1$ is a compact Hausdorff space $Q = \{q^{2k}|k \in Z_+\} \cup \{0\}$ and restriction of the invariant integral ν of $SU_q(2)$ to this C^*-algebra is given by the Jackson integral:

$$\nu(f) = (1-q^2) \sum_{k\in Z_+} f(q^{2k})q^{2k} \qquad \forall f \in C(Q). \tag{12}$$

Theorem 2 *[19],[30] The strict Gel'fand pair $(SU_q(2), U(1))$ generates a Hermitian normal hypercomplex system with a compact basis Q, $e = 0$ and Haar measure of the form (12). The corresponding operation has a form*

$$(L^p f)(r) = (\Delta f)(p,r) = (1-q^2) \sum_{k\in Z_+} K(q^{2p'}, q^{2r'}; q^{2k}|q^2) f(q^{2k})q^{2k},$$

where $p', r' \in Z_+, p = q^{2p'}, r = q^{2r'} \in Q, K(q^{2p'}, q^{2r'}; q^{2k}|q^2)$ may be expressed by means of ${}_3\varphi_2$-q-hypergeometric series. The series in the right-hand side of the latter equality converges absolutely. The corresponding complete orthogonal in $L_2(Q,\nu)$ system of characters is formed by the little q-Legendre polynomials $p_n(z;1,1|q^2)$, $n \in Z_+, z \in Q$.

Corollary. We have the following product formula for the little q-Legendre polynomials:

$$(1-q^2) \sum_{k\in Z_+} K(q^{2p'}, q^{2r'}; q^{2k}|q^2) p_l(q^{2k}; 1,1|q^2) = p_l(q^{2p'}; 1,1|q^2) p_l(q^{2r'}; 1,1|q^2),$$

where $p', r', l \in Z_+$. One can find this formula and the expression for $K(q^{2p'}, q^{2r'}; q^{2k}|q^2)$ in [18].

Now we can find a dual hypercomplex system applying the general construction from [5],[6]. It has a discrete basis Z_+ and, hence, it is a DJS-hypergroup. Here we have

$$p_l(q^{2n}; 1,1|q^2) p_m(q^{2n}; 1,1|q^2) = \sum_{s=|l-m|}^{l+m} \widehat{K}(l,m,s) p_s(q^{2n}; 1,1|q^2),$$

where the numbers $\widehat{K}(l,m,s)$ are Clebsch-Gordan coefficients for the irreducible corepresentations of $SU_q(2)$. One can look at this formula as at the dual product formula for the little q-Legendre polynomials. The kernel $\widehat{K}(l,m,s)$ was expressed in [16] by means of q-Hahn polynomials.

4.2. Now consider a strict Gel'fand pair $(SU_q(n), U_q(n-1)), n \geq 2$. This is a generalization of the example considered in 4.1. It is known that [25] $U_q(n) := \mathrm{C}\langle t_{ij}, t, 1\rangle / I_R$, where $\mathrm{C}\langle t_{ij}, t, 1\rangle$ is a free algebra generated by the elements of the matrix $T = (t_{ij})$, $i,j = 1,\ldots,n$, the elements t, 1 and I_R is a two-sided ideal generated by the relations

$$RT_1T_2 = T_2T_1R,\ t\cdot t_{ij} = t_{ij}\cdot t,\ t\cdot \det_q(T) = \det_q(T)\cdot t = 1.$$

Here $T_1 = T \otimes I$, $T_2 = I \otimes T$, I is the identity matrix in R^n, the matrix R is given by

$$R := \sum_{1\leq i,j\leq n} q^{\Delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{1\leq i<j\leq n} e_{ij} \otimes e_{ji},$$

$e_{ij} \in Mat(n \times n)$ are matrix units, $\det_q(T) = \sum_{\sigma\in S_n}(-q)^{l(\sigma)}t_{1\sigma_1} \cdot \ldots \cdot t_{n\sigma_n}$, S_n is the permutation group, $l(\sigma)$ - the length of the permutation σ, and $q \in \mathrm{C}$.

For $q \in \mathrm{R}$ and $|q| < 1$, the structure of a $*$-Hopf is given by

$$\Delta(t_{i,j}) := \sum_k t_{i,k} \otimes t_{k,j},\ \Delta(t) := t \otimes t,\ \varepsilon(t_{i,j}) := \delta_{i,j},\ \varepsilon(t) := 1,$$

$$S(t_{i,j}) := (-q)^{i-j} \sum_{\sigma\in S_{n-1}} (-q)^{l(\sigma)} t_{0\sigma_0} \cdot \ldots \cdot t_{j-1\sigma_{j-1}} t_{j+1\sigma_{j+1}} \cdot \ldots \cdot t_{n-1\sigma_{n-1}},$$

where $\sigma_k \in [0, i-1] \cup [i+1, n-1]$, and the antipode $S(t) := \det_q(T)$, and the involution $t^*_{i,j} := S(t_{j,i})$, $t^* := t$.

The dual Hopf *-algebra, $U_q(u(n))$, is defined as [25] $U_q(u(n)) := \mathrm{C}\langle l^+_{ij}, l^-_{ij}, 1\rangle / I_{R^+}$, where the free algebra $\mathrm{C}\langle l^+_{ij}, l^-_{ij}, 1\rangle$ is generated by the elements l^+_{ij}, l^-_{ij}, 1, $i, j = 1, \ldots n$, and the two-sided ideal I_{R^+} is generated by the relations

$$R^+ L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R^+,\ R^+ L^+_1 L^-_2 = L^-_2 L^+_1 R^+,$$

where $L^\pm_1 = L^\pm \otimes I$, $L^\pm_2 = I \otimes L^\pm$, $R^+ = PRP$ $(P(l_1 \otimes l_2) = l_2 \otimes l_1)$. The coalgebra structure is given by

$$\hat{\Delta}(l^\pm_{ij}) := \sum_{k=1}^{n} l^\pm_{ik} \otimes l^\pm_{kj},$$

and a nondegenerate pairing is defined to be $\langle L^\pm, T_1 \ldots T_k\rangle := R^\pm_1 \cdot \ldots \cdot R^\pm_k$, where

$$T_i = I \otimes \ldots \otimes \underbrace{T}_{i} \otimes \ldots \otimes I,$$

$R^- = R^{-1}$, and $R^\pm_i$ acts as $R^\pm$ on the 0^{th} and i^{th} component of the tensor product $(\mathrm{R}^n)^{\otimes(k+1)}$. $SU_q(n)$ is a Hopf *-algebra distinguished by the condition $\det_q(T) = 1$, with the same $\Delta_1, \varepsilon_1, S_1, *$.

Let the generators of $U_q(n-1)$ be s_{ij}, s and Hopf *-algebra epimorphism π : $SU_q(n) \to U_q(n-1)$ be$\pi(t_{ij}) := s_{ij}$, $\pi(t_{0j}) := \pi(t_{i0}) = 0$, $i, j = 1, \ldots, n-1$, $\pi(t_{00}) := s$. It was shown in [8] that $U_q(n-1)\backslash SU_q(n)$ is generated by the elements $x_{ij} = t_{0i}t^*_{0j}$, $i, j = 0, \ldots, n-1$, $SU_q(n)/U_q(n-1)$ - by the elements $y_{ij} = t_{i0}t^*_{j0}$, $i, j = 0, \ldots, n-1$,$U_q(n-1)\backslash SU_q(n)/U_q(n-1)$ is generated by an element $z = t_{00}t^*_{00}$. Since the latter unital algebra is commutative and S_1 is trivial on it, ($SU_q(n)$,$U_q(n-1)$) is strict Gel'fand pair.

Let H be the completion of the algebra $U_q(n-1)\backslash SU_q(n)/U_q(n-1)$ with respect to C^*-norm $|\cdot| = \sup_\rho ||\rho(\cdot)||$, where ρ runs over the set of all the irreducible representations of $SU_q(n)$. The spectrum of this C^*-algebra is $Q = \{q^{2k}\}_{k\in Z_+} \cup \{0\}$ and the restriction to it of the Haar measure ν of $SU_q(n)$ is given by the Jackson integral:

$$\nu(F(z_1)) = (1 - q^{2n-2}) \sum_{k=0}^{\infty} q^{2k(n-1)} F(q^{2k}) \tag{13}$$

(see [32]). It was shown in [8], that we have here a structure of a hypercomplex system with compact basis Q, its characters φ_m can be expressed by the little q-Jacobi polynomials P_m $(m = 1, 2, \ldots)$:

$$\varphi_m(z_1) = A_m P_m(z_1/q^2; q^{2n-4}, 1|q^2),\ A_m = \frac{(1 - q^{2n-2})(1 - q^2)}{(1 - q^{-2m})(1 - q^{2m+2n-2})} \tag{14}$$

After that we can again consider a dual hypercomplex system with the discrete basis Z_+ (DJS-hypergroup), whose structural constants are the Clebsch-Gordan coefficients for the irreducible corepresentations of $SU_q(n)$. Thus we have:

Theorem 3 *There are two dual to each other structures associated with the strict Gelfand pair $(SU_q(n), U_q(n-1))$: a commutative hypercomplex system with the compact basis $\{q^{2k}\}_{k \in Z_+} \cup \{0\}$ and the Haar measure given by (13), and the discrete commutative DJS-hypergroup with the basis Z_+. Their characters are expressed by (14).*

For $n > 2$ the expressions for the kernels in the product formula for the little q-Jacobi polynomials and in dual product formula are not known.

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Operations on certain non-commutative operator-valued random variables

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OPERATIONS ON CERTAIN NON-COMMUTATIVE OPERATOR-VALUED RANDOM VARIABLES

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An example motivating the study of the addition of free pairs of "non-commutative operator-valued random variables" is provided by the computation of spectra of convolution operators on free groups.

Let G be the (non-commutative) free group on two generators g_1, g_2 and let λ denote the left regular representation on $l^2(G)$. To compute spectra of convolution operators

$$Y = \sum_{g \in G} c_g \lambda(g)$$

with $c_g \neq 0$ only for finitely many $g \in G$ it suffices to be able to decide whether such Y is invertible. This in turn is equivalent to deciding whether a certain operator

$$X = \sum_{k \in \mathbb{Z}} (\alpha_k \otimes \lambda(g_1^k) + \beta_k \otimes \lambda(g_2^k))$$

where $\alpha_k = \alpha_{k'}^*$, $\beta_{-k} = \beta_k^*$ are $n \times n$ matrices, is invertible. If $n = 1$, i.e. if the matrices are scalars, then the spectrum of X can be computed using our results on the addition of free pairs of non-commutative random variables [8]. Thus the computation of the spectrum of Y is reduced to a generalization of the addition of free pairs of non-commutative random variables to the case of "matrix-valued non-commutative random variables". (For a different approach to the question of computing the spectrum of Y see [1].)

The present paper deals with the extension of our previous work ([8],[9]) on addition and multiplication of free pairs of non-commutative random variables to, what might be called, the operator-valued case. This means that the field of complex numbers is replaced by an operator algebra, the free products are with amalgamation over this algebra and the specified states are replaced by specified conditional expectations. Also the natural frame-work of operator algebras with dual algebraic structure ([10]) for the considered operators in the "scalar" case has a corresponding extension to the "operator-valued" case.

Though our results are meant for applications to operator algebras and spectral theory, most of our considerations will be in a purely algebraic context, since we shall be mainly concerned with finding the formulae for computing the operation on the distributions of the random-variables. Concerning distributions of operator-valued non-commutative random-variables, let us only say that since the scalars $\mathbb{C}$ are replaced by an operator algebra B, the moments of the variable X are the expectation valued of monomials of the form $Xb_1Xb_2\dots Xb_{n-1}X$. It is an important fact for the computation of spectra that the addition of free pairs of B-valued random variables gives an operation among the symmetric parts of the distributions i.e. among the expectation values of monomials of the form $bXbX\dots bXb$. For the symmetric distributions the addition formulae closely resemble those in the scalar case with the generating series viewed as germs of maps $\mathbb{C} \to \mathbb{C}$ replaced by germs of maps $B \to B$.

The paper has eight sections.

The first section discusses free families of non-commutative B-valued random variables and distributions of such random variables.

The second and third section deal with the algebras $A(M)$ and the canonical form of a random variable with a given distribution. This is the analogue for the B-valued case of the special Toeplitz operators which we used in the scalar case for studying the addition of free pairs of non-commutative random variables. We also give formulae for the canonical form of a random variable after multiplication by elements in B.

The fourth section gives the solution to the addition problem for the symmetric parts of distributions of B-valued random variables. It is obtained by studying the differential equation for semigroups with respect to addition. The final formulae closely resemble those in the scalar case.

The fifth section deals with the differential equation for semigroups with respect to the multiplicative operation. We also introduce a corresponding free exponential map. Studying the differential equations we show that multiplicative free convolution is well defined for the symmetric distributions.

The sixth section presents the application to the computation of spectra of convolution operators on free groups.

Section seven is a brief outline of the necessary adaptations to make the operators on B-valued random variables fit in a framework of dual algebraic structures as in the scalar case.

Section eight deals with the free central limit theorem for B-valued random variables generalizing our results from the scalar case [7].

The present paper is an expanded version of our paper with the same title (preliminary version) INCREST Preprint No. 42/1986, Bucarest. This revised version consists of the material of the preliminary version (without changes) to which we have added 3.3.–3.7., 5.4.–5.10. and section 8.

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1. B-valued non-commutative random-variables.

1.1. Throughout B will denote a fixed unital algebra over $\mathbb{C}$ (this choice of the base field is inessential). Let A be another unital algebra over $\mathbb{C}$ containing B as a subalgebra (with the same unit) and let $\varphi : A \to B$ be a conditional expectation i.e. a linear map such that $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$ if $b_1, b_2 \in B$, $a \in A$ and $\varphi(b) = b$ if $b \in B$. An element $a \in A$, will be viewed as a B-valued random variable.

1.2. Definition. *Let (A, φ) be as in* 1.1 *and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras. The family $(A_i)_{i \in I}$ will be called free if*

$$\varphi(a_1 a_2 \dots a_n) = 0$$

whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$. A family of subsets $X_i \subset A$ (elements $a_i \in A$) where $i \in I$ will be called free if the family of subalgebras A_i generated by $B \cup X_i$ (respectively $B \cup \{a_i\}$) is free.

Free families of subalgebras arise in the C^*-algebraic context (in which case the conditional expectations are of norm one) from reduced free products with amalgamation (see §5 in [7]).

1.3. Proposition. *Let (A, φ) be as in* 1.1 *and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras such that A is generated by $\cup_{i \in I} A_i$ and $(A_i)_{i \in I}$ is a free family. Then φ is completely determined by the $\varphi_i = \varphi \mid A_i$ $(i \in I)$.*

Proof. By linearity it is sufficient to prove that we may compute $\varphi(a_1 \dots a_n)$ whenever $a_j \in A_{i_j}$ $(1 \leq j \leq n)$. We shall proceed by induction on the least non-negative integer such that $\varphi_{i_j}(a_j) = 0$ if $k < j$ and $i_{k+1} \neq i_{k+2} \neq \cdots \neq i_n$. If $k = 0$ then $\varphi(a_1 \dots a_n) = 0$. Assume our assertion has been established up to a certain k. Then for $k+1$ if $i_k \neq i_{k+1}$ we have

$$\begin{aligned} &\varphi(a_1 \dots a_n) = \varphi(a_1 \dots a_k(\varphi_{i_{k+1}}(a_{k+1}) a_{k+2}) a_{k+3} \dots a_n) + \\ &+ \varphi(a_1 \dots a_k a'_{k+1} a_{k+2} \dots a_n) \text{ where } a'_{k+1} = a_{k+1} - \varphi_{i_{k+1}}(a_{k+1}), \end{aligned}$$

so that the induction hypothesis applies.

If $i_k = i_{k+1}$ then we write

$$\begin{aligned} &\varphi(a_1 \dots a_n) = \varphi(a_1 \dots a_{k-1}(a_k a_{k+1} - \varphi_{i_k}(a_k a_{k+1})) a_{k+2} \dots a_n) + \\ &+ (a_1 \dots a_{k-1} \varphi_{i_k}(a_k a_{k+1}) a_{k+2} \dots a_n) \end{aligned}$$

which is again a reduction to the induction hypothesis. □

1.4. The algebra freely generated by B and an indeterminate X will be denoted by $B\langle X\rangle$. Let (A,φ) be as in 1.1 and $a \in A$ a B-valued random variable. The distribution of a is the conditional expectation $\mu_a : B\langle X\rangle \to B$ defined by $\mu_a = \varphi\circ\tau_a$ where $\tau_a : B\langle X\rangle \to A$ is the unique homomorphism such that $\tau_a(b) = b$ for $b \in B$ and $\tau_a(X) = a$. Quantities such as $\mu_a(b_0Xb_1X\dots b_{n-1}Xb_n)$ will be called moments. The set of all conditional expectations $\mu : B\langle X\rangle \to B$ will be denoted by Σ_B.

1.5. Let $\mathcal{G}_n$ denote the symmetric group and let

$$S_n(b_1\dots b_n) = \sum_{\sigma\in\mathcal{G}_n} b_{\sigma(1)}Xb_{\sigma(2)}\dots Xb_{\sigma(n)}$$

$S_1(b) = b$ and $S_0 = 1$. Let further

$$\begin{aligned} SB\langle X\rangle =& \ \mathrm{l.s.}\{S_n(b,\dots,b) \mid b\in B, n\geq 0\} = \\ =& \ \mathrm{l.s.}\{S_n(b_1,\dots,b_n) \mid b_j\in B, n\geq 0, n\geq j\geq 1\} \\ \widetilde{SB}\langle X\rangle =& \ \mathbb{C}X + X(SB\langle X\rangle)X \end{aligned}$$

where "l.s." denotes the vector space spanned by the given set.

Lemma. *We have*

$$B(SB\langle X\rangle)B = B + B(\widetilde{SB}\langle X\rangle)B.$$

Proof. The inclusion $\subset$ is obvious. To prove the converse remark that if $n\geq k+1$, $n\geq 3$, we have

$$\begin{aligned} &S_n(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1) - (n-k)kbXS_{n-2}(\underbrace{b,\dots,b}_{(k-1)\text{-times}},1,\dots,1)X- \\ &-(n-k)kXS_{n-2}(\underbrace{b,\dots,b}_{(k-1)\text{-times}},1,\dots,1)Xb- \\ &-k(k-1)bXS_{n-2}(\underbrace{b,\dots,b}_{(k-2)\text{-times}},1,\dots)Xb= \\ &=(n-k)(n-k-1)XS_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X. \end{aligned}$$

Taking into account that

$$n(n-1)XS_{n-2}(1,\dots,1)X = S_n(1,\dots,1)$$

the preceding recurrence relation applied for $k = 1,\dots,n-2$ can be used to prove inductively that for $n\geq 3$ and $1\leq k\leq n-2$ we have

$$XS_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X \in B(SB\langle X\rangle)B.$$

Also

$$X = 2^{-1}S_2(1,1) \in B(SB\langle X\rangle)B.$$

□

The above lemma implies that if $\mu \in \Sigma_B$ then $\mu \mid SB\langle X\rangle$ is completely determined by $\mu \mid \widetilde{SB}\langle X\rangle$ and conversely $\mu \mid \widetilde{SB}\langle X\rangle$ is completely determined by $\mu \mid SB\langle X\rangle$. We shall denote by $S\Sigma_B$ the set

$$S\Sigma_B = \{(\mu \mid B(SB\langle X\rangle)B) \mid \mu \in \Sigma_B\}$$

and we shall write $S\mu = \mu \mid B(SB\langle X\rangle)B$ if $\mu \in \Sigma_B$. If $a \in A$ is a random variable, then $S\mu_a$ will be called the symmetric distribution of a and quantities of the type $\mu_a(S(b_1, \ldots, b_n))$ or $\mu_a(XS(b_1, \ldots, b_n)X)$ will be called symmetric moments of a.

1.6. If $\{a_1, a_2\} \subset A$ is a free pair of B-valued random variables then it follows from Proposition 1.3 that $\mu_{a_1+a_2}$ and $\mu_{a_1a_2}$ depend only on μ_{a_1} and μ_{a_2}. For any given $\mu_1, \ldots, \mu_n \in \Sigma_B$ one can find a free family $\{a_1, \ldots, a_n\}$ of random-variables in some (A, φ) such that $\mu_{a_j} = \mu_j$. We shall not give an ad-hoc proof for this here since it will follow from our results on the canonical form of a random variable. This implies that there are well-defined operations, $\boxplus$ and $\boxtimes$ on Σ_B such that if $\{a_1, a_2\}$ is a free pair then

$$\mu_{a_1+a_2} = \mu_{a_1} \boxplus \mu_{a_2}$$
$$\mu_{a_1+a_2} = \mu_{a_1} \boxtimes \mu_{a_2}.$$

This gives two semigroup structures on Σ_B.

2. The algebra $A(M)$.

2.1. Let M be a right B-module and let $\mathcal{X}_n(M) = \mathcal{L}(M^{\otimes n}, B)$ be the n-linear B-valued maps of $M \times \ldots \times M$ into B (the $\otimes$ and linearity are over $\mathbb{C}$) and $\mathcal{X}_0(M) = B$. Let further $\mathcal{X}(M) = \oplus_{n\geq 0}\mathcal{X}_n(M)$ with its natural right B-module structure. If $\xi \in \mathcal{X}_n(M)$ we define the endomorphism $\lambda(\xi)$ of the right B-module $\mathcal{X}(M)$ by:

$$\begin{aligned}
&\lambda(\xi)\eta \in \mathcal{X}_{n+k}(M)\\
&(\lambda(\xi)\eta)(m_1 \otimes \cdots \otimes m_{n_k}) =\\
&= \eta(m_{n+1}\xi(m_1 \otimes \cdots \otimes m_n) \otimes m_{n+2} \otimes \cdots \otimes m_{n+k})
\end{aligned}$$

if $\deg \eta = k > 0$ where deg refers to the obvious grading of $\mathcal{X}(M)$ and

$$\lambda(\xi)\eta = \xi\eta$$

if $\deg\eta = 0$ i.e. $\eta \in B$. We also define $\lambda^*(m)$, where $m \in M$, by:

$$\begin{aligned}&\lambda^*(m)\eta = 0 \text{ if } \deg\eta = 0\\ &\deg\lambda^*(m)\eta = \deg\eta - 1\\ &(\lambda^*(m)\eta)(m_1 \otimes \cdots \otimes m_{k-1}) = \eta(m \otimes m_1 \otimes \cdots \otimes m_{k-1})\end{aligned}$$

if $\deg\eta = k > 0$.

$A(M)$ is the algebra of endomorphisms of the right B-module $\mathcal{X}(M)$ generated by

$$\{\lambda(\xi) \mid \xi \in \mathcal{X}_n(M), n \geq 0\} \cup \{\lambda^*(m) \mid m \in M\}.$$

Endowing $A(M)$ with the natural grading corresponding to its action on $\mathcal{X}(M)$ we have $\deg\lambda(\xi) = \deg\xi$ and $\deg\lambda^*(m) = -1$.

2.2. It is easy to check that the following equalities hold

$$\begin{aligned}\lambda(\xi_1)\lambda(\xi_2) &= \lambda(\lambda(\xi_1)\xi_2)\\ \lambda^*(m)\lambda(\xi) &= \lambda(\lambda^*(m)\xi) \text{ if } \deg\xi > 0\\ \lambda^*(m)\lambda(\xi) &= \lambda^*(m\xi) \text{ if } \deg\xi = 0.\end{aligned}$$

2.3. We define a linear map

$$\gamma : (\oplus_{n\geq 0}\mathcal{X}_n(M)) \oplus \left(\oplus_{k\geq 0}M^{\otimes k}\right) \to A(M)$$

by

$$\gamma(\xi \otimes (m_1 \otimes \cdots \otimes m_k)) = \lambda(\xi)\lambda^*(m_1)\ldots\lambda^*(m_k).$$

Lemma. *γ is a bijection.*

Proof. Clearly the range of γ contains the $\lambda(\xi)$'s and the $\lambda^*(m)$'s and using the relations 2.2 we easily infer that the range of γ is an algebra, so that γ is onto.

For the injectivity let

$$\alpha = \sum_{k_0\leq k\leq k_1}\sum_{i\in I_k}\xi_{i,k} \otimes \nu_{i,k} \neq 0$$

where $\xi_{i,k} \in \oplus_{n\geq 0}\mathcal{X}_n(M)$ and $\nu_{i,k} \in M^{\otimes k}$ for $i \in I_k$. Since $\alpha \neq 0$ we may assume the $\nu_{i,k}$'s are linearly independent and the $\xi_{i,k}$'s are non-zero. Then, fixing $i_0 \in I_{k_0}$ there is $\eta \in \mathcal{X}_{k_0}(M)$ such that $\eta(\nu_{i_0,k_0}) = 1 \in B$ and $\eta(\nu_{i,k_0}) = 0$ for $i \in I_{k_0}\backslash\{i_0\}$.

Let $\eta' \in \mathcal{X}_{k_0}(M)$ be defined by

$$\eta'(m_1 \otimes \cdots \otimes m_{k_0}) = \eta(m_{k_0} \otimes \cdots \otimes m_1).$$

We have

$$\begin{aligned}\gamma(\alpha)\eta' &= \gamma\left(\sum_{i\in I_{k_0}} \xi_{i,k_0}\otimes\nu_{i,k_0}\right)\eta' = \\ &= \sum_{i\in I_{k_0}} \lambda(\xi_{i,k_0})\eta(\nu_{i,k_0}) = \lambda(\xi_{i_0,k_0})1 = \\ &= \xi_{i_0,k_0} \neq 0.\end{aligned}$$

□

2.4. B identifies via $\lambda : \mathcal{X}_0(M) \simeq B \to A(M)$ with a subalgebra of $A(M)$ and there is a linear map $\varepsilon_M : A(M) \to B$ defined by $\varepsilon_M(\gamma(\xi_n \otimes \nu_k)) = 0$ if $n+k>0$ where $\xi_n \in \mathcal{X}_n(M)$, $\nu_k \in M^{\otimes k}$ and $\varepsilon_M(\gamma(\xi_0\otimes\eta_0)) = \gamma(\xi_0\otimes\eta_0) = \xi_0\nu_0 \in B$ if $\xi_0 \in \mathcal{X}_0(M) = B$ and $\nu_0 \in M^{\otimes 0} \simeq \mathbb{C}$. It is easily seen that ε_M is a conditional expectation i.e. that $\varepsilon_M(\lambda(b_1)a\lambda(b_2)) = b_1\varepsilon_M(a)b_2$ and $\varepsilon_M(\lambda(b)) = b$.

2.5. *Remark.* If $B=\mathbb{C}$ and $M=\mathbb{C}^n$ then $A(\mathbb{C}^n)$ is isomorphic with a certain dense subalgebra of an extension of the C^*-algebra O_n of Cuntz [3] realized on the Fock space for Boltzmann statistics ([6], [5], [4]).

2.6. It will be useful to consider a larger algebra $\bar{A}(M) \supset A(M)$ acting on $\bar{\mathcal{X}}(M) = \prod_{n\geq 0}\mathcal{X}_n(M)$ such that there is a bijection

$$\bar{\gamma} : \left(\prod_{n\geq 0}\mathcal{X}_n(M)\right) \otimes \left(\oplus_{k\geq 0}M^{\otimes k}\right) \to \bar{A}(M)$$

extending γ and the multiplication of the formal sums which constitute $A(M)$ is also determined by the formulae 2.2. The obvious extension of ε_M to $\bar{A}(M)$ will be denoted also by ε_M. We have for $T \in \bar{A}(M)$

$$\varepsilon_M(T) = (T1)_0$$

where $1 \in B = \mathcal{X}_0(M) \subset \bar{\mathcal{X}}(M)$ and $(\cdot)_0$ denotes the component of degree zero. Note also that along the same lines as in the proof of Lemma 2.3 it is easy to show that the representation of $A(M)$ on $\bar{\mathcal{X}}(M)$ is faithful.

2.7. If $M = M_1 \oplus M_2$ there are injections

$$\chi_j : \left(\prod_{n\geq 0}\mathcal{X}_n(M_j)\right)\otimes\left(\oplus_{k\geq 0}M_j^{\otimes k}\right) \to \left(\prod_{n\geq 0}\mathcal{X}_n(M)\right)\otimes\left(\oplus_{k\geq 0}M^{\otimes k}\right)$$

given by

$$\chi_j((\xi_n)_{n\geq 0}\otimes\nu_k) = (\xi_n \circ pr_j^{\otimes n})\otimes(i_j^{\otimes k}\nu_k)$$

where $i_j : M_j \hookrightarrow M$ are the natural inclusions and $pr_j : M \to M_j$ the proejctions onto the two summands and $\xi_0 \circ pr_j^{\otimes 0}$ means just ξ_0. Since the relations 2.2 determine the multiplication in the algebras $\bar{A}(\cdot)$ it is easy to check that the maps $h_j : \bar{A}(M_j) \to \bar{A}(M)$ such that $h_j \circ \bar{\gamma} = \bar{\gamma} \circ \chi_j$ are homomorphisms. Moreover we have $h_j(\lambda(b)) = \lambda(b)$ for $b \in B = \mathcal{X}_0(M_j) = \mathcal{X}_0(M)$ and $\varepsilon_M \circ h_j = \varepsilon_{M_j}$.

Proposition. *If $M = M_1 \oplus M_2$ then with h_1, h_2 as above, the pair of subalgebras $(h_j(\bar{A}(M)))_{j=1,2}$ is B-free in $(\bar{A}(M), \varepsilon_M)$.*

Proof. Write

$$\bar{\mathcal{X}}(M) = \Gamma_1 \oplus \Gamma_2 \oplus B$$

where

$$\Gamma_j = \prod_{n \geq 1} \mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$$

with $\mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$ identified with a subspace of $\mathcal{L}(M^{\otimes n}, B)$ via $\eta_n \rightsquigarrow \eta_n \circ (pr_j \otimes id_M \otimes \cdots \otimes id_M)$. If $T \in h_2(\bar{A}(M_1))$ and $\varepsilon_M(T) = 0$ then $T(\Gamma_2 \oplus B) \subset \Gamma_1$. Also the analogue of this with 1 and 2 interchanged holds. This easily implies our assertion. For instance if $T_j \in h_1(\bar{A}(M_1))$ and $S_j \in h_2(\bar{A}(M_j))$ and $\varepsilon_M(T_j) = \varepsilon_M(S_j) = 0$ then $T_1 1 \in \Gamma_1$, $S_1 T_1 1 \in \Gamma_2$ and continuing in this way we get $S_n T_n \dots S_1 T_1 1 \in \Gamma_2$ so that

$$\varepsilon_M(S_n T_n \dots S_1 T_1) = 0.$$

□

2.8. If $M = B^m$ we shall denote

$$\bar{A}(M) \text{ by } \bar{A}(m) \text{ and } \varepsilon_M \text{ by } \varepsilon_m.$$

3. The canonical form.

3.1. Elements $a \in \bar{A}(1)$ of the form

$$a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$$

where $\xi_n \in \mathcal{X}_n(B)$, will be called canonical.

Proposition. *Given a distribution $\mu \in \Sigma_B$ there is a unique canonical element*

$$a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$$

such that $\mu_a = \mu$.

Proof. We have $\mu_a(X) = \varepsilon_1(a) = \xi_0$ so that we must put $\xi_0 = \mu(X)$. If $n > 0$ we have

$$\begin{aligned}&\varepsilon_1(a\lambda(b_1)a\lambda(b_2)\dots a\lambda(b_n)a) =\\ &= \varepsilon_1(\lambda^*(1)\lambda(b_1)\lambda^*(1)\lambda(b_2)\dots\lambda^*(1)\lambda(b_n)\lambda(\xi_n))+\\ &+ E_n(\xi_0,\dots,\xi_{n-1})(b_1\otimes\dots\otimes b_n)\end{aligned}$$

where $E_n(\xi_0,\dots,\xi_{n-1}) \in \mathcal{L}(B^{\otimes n}, B)$ depends only on $\xi_0,\dots,\xi_{n-1}$. Remark that

$$\varepsilon_1(\lambda^*(1)\lambda(b_1)\dots\lambda^*(1)\lambda(b_n)\lambda(\xi_n)) = \xi_n(b_n\otimes\dots\otimes b_1).$$

We infer that ξ_n satisfies $\xi_n(b_n\otimes\dots\otimes b_1) = \mu(Xb_1Xb_2\dots Xb_nX) - E_n(\xi_0,\dots,\xi_{n-1})(b_1\otimes \dots\otimes b_n)$ which determines ξ_n inductively. □

The canonical element a in the above proposition will be called the canonical form of a random variable with distribution μ and we shall write $\xi_n = R_{n+1}(\mu)$.

3.2. Proposition. *Let*

$$a_k = \lambda^*(1) + \sum_{n\geq 0}\lambda(\xi_{n,k})$$

$k = 1,2,3$ *be canonical elements. Then* $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$ *if and only if*

$$\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$$

for all $n \geq 0$.

Proof. In view of the uniqueness of the canonical form it will be sufficient to prove that if $\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$ for all $n \geq 0$ then $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$. Passing to $\bar{A}(2)$ we have in view of 2.7 that $h_1(a_1) + h_2(a_2)$ has distribution $\mu_{a_1} \boxplus \mu_{a_2}$.

Let

$$\begin{aligned}Y = h_1(a_1) + h_1(a_2) = \lambda^*(1\oplus 1) + \sum_{n\geq 0}\lambda(\xi_{n,1}\circ pr_1+\\ + \xi_{n,2}\circ pr_2).\end{aligned}$$

Expanding

$$\begin{gathered}\varepsilon_2(Y\lambda(b_1)Y\dots Y\lambda(b_n)Y) \text{ and}\\ \varepsilon_1(a_3\lambda(b_1)a_3\dots a_3\lambda(b_n)a_3)\end{gathered}$$

our assertion is obtained from the following remark. Let

$$\varepsilon_2(S_1\lambda(b_1)S_2\dots S_n\lambda(b_n)S_{n+1})$$

where each S_j is an element of one of the following forms

$$\lambda^*(1 \oplus 1),\ \lambda(\beta_n \circ pr_1^{\otimes n}) \text{ or } \lambda(\beta_n \circ pr_2^{\otimes n}).$$

Then replacing S_j by S_j' where S_j' is obtained from S_j by replacing $\lambda^*(1 \oplus 1)$ by $\lambda^*(1)$, $\lambda(\beta_n \circ pr_k^{\otimes n})$ $(k = 1, 2)$ by $\lambda(\beta_n)$ it is easy to see that

$$\begin{aligned}&\varepsilon_1(S_1'\lambda(b_1)S_2' \dots S_n'\lambda(b_n)S_{n+1}') = \\ &= \varepsilon_2(S_1\lambda(b_1)S_2 \dots S_n\lambda(b_n)S_{n+1}).\end{aligned}$$

□

Thus we have proved that

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_1) + R_n(\mu_2)$$

for all $n \geq 1$ and $\mu_j \in \Sigma_B$, $j = 1, 2$.

3.3. The rest of this section will deal with the effect on the canonical form of the multiplication of a random variable by an element in B. We begin with some definitions this will require.

Since B is a $B-B$-bimodule, in addition to the right multiplication $\xi_n b$ by an element $b \in B$ of $\xi_n \in \mathcal{X}_n(M)$ we also may define $b\xi_n$ by

$$(b\xi_n)(m_1 \otimes \cdots \otimes m_n) = b(\xi_n(m_1 \otimes \cdots \otimes m_n)).$$

Further, if $M = B$ and $b \in B$, we shall also consider

$$d_n(b), s_n(b), \sigma_n(b) : \mathcal{X}_n(B) \to \mathcal{X}_n(B)$$

defined by the formulae

$$\begin{aligned}&(d_n(b)\xi_n)(b_1 \otimes \cdots \otimes b_n) = \xi_n(bb_1 \otimes \cdots \otimes bb_n) \qquad (n \geq 1)\\ &d_0(b)\xi_0 = \xi_0 \qquad (n = 0)\\ &(s_n(b)\xi_n)(b_1 \otimes \cdots \otimes b_n) = b\xi_n(b_1 \otimes b_2 b \otimes b_3 \otimes \cdots \otimes b_n) \qquad (n \geq 2)\\ &(s_1(b)\xi_1)(b_1) = b\xi_1(b_1)\\ &s_0(b)\xi_0 = \xi_0\\ &(\sigma_n(b)\xi_n)(b_1 \otimes \cdots \otimes b_n) = b\xi_n(b_1 b \otimes \cdots \otimes b_n b).\end{aligned}$$

Let further $d(b), s(b) : \mathcal{X}(B) \to \mathcal{X}(B)$ be given by

$$\begin{aligned}d(b) &= \oplus_{n \geq 0} d_n(b)\\ s(b) &= \oplus_{n \geq 0} s_n(b).\end{aligned}$$

Note that the somewhat unusual formulae for $s_n(b)$ (when compared to $d_n(b)$) are due to a certain asymmetry in our definition of $A(M)$, $\mathcal{X}(M)$.

3.4. The reader may easily check the following formulae for $\xi_n \in \mathcal{X}_n(B)$ $(n \geq 0)$

$$\begin{aligned} d(b)\lambda(\xi_n) &= \lambda(d_n(b)\xi_n)d(b) \\ \lambda^*(1)d(b) &= d(b)\lambda^*(b) \end{aligned}$$

and

$$\begin{aligned} \lambda(\sigma_n(b)\xi_n)s(b) &= s(b)\lambda(b)\lambda(\xi_n) \\ \lambda^*(1)s(b) &= s(b)\lambda(b)\lambda^*(1). \end{aligned}$$

3.5. Proposition. *Let a be a B-valued random variable and let $b \in B$ and let*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n)$$

be the canonical form of a. Then we have:

(i) *The canonical form of ab is*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(d_n(b)\xi_n b).$$

(ii) *The canonical form of ba is*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\sigma_n(b)\xi_n).$$

Proof. (i) The random variable ab has the same distribution like

$$(\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n))\lambda(b) = \lambda^*(b) + \sum_{n\geq 0} \lambda(\xi_n b)$$

where we have used 2.2.

To prove (i) we must show that $T = \lambda^*(b) + \sum_{n\geq 0} \lambda(\xi_n b)$ and $T_1 = \lambda^*(1) + \sum_{n\geq 0} \lambda(d_n(b)\xi_n b)$ have the same distribution in $(\bar{A}(1), \varepsilon_1)$. In view of 3.4 we have

$$d(b)T = T_1 d(b)$$

and hence

$$\begin{aligned} \varepsilon_1(T\lambda(b_1)\dots T\lambda(b_n)) &= (T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (d(b)T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (T_1 d(b)\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (T_1\lambda(b_1)d(b)T\lambda(b_2)\dots T\lambda(b_n)1)_0 = \end{aligned}$$

$$\begin{aligned}&= \cdots = (T_1\lambda(b_1)\dots T_1 d(b)\lambda(b_n)1)_0 = \\ &= \varepsilon_1(T_1\lambda(b_1)\dots T_1\lambda(b_n)).\end{aligned}$$

(ii) To prove (ii) we proceed similarly using the second group of formulae in 3.4. Let $T = \lambda(b)(\lambda^*(1) + \sum_{n\geq 0}\lambda(\xi_n))$ and let $T_1 = \lambda^*(1) + \sum_{n\geq 0}\lambda(\sigma_n(b)\xi_n)$. We have

$$s(b)T = T_1 s(b).$$

It is also easy to check that

$$s(b)\lambda(b_1) = \lambda(b_1)s(b).$$

We have

$$\begin{aligned}&\varepsilon_1(T\lambda(b_1)\dots T\lambda(b_n)) = (T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (s(b)T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (T_1\lambda(b_1)s(b)T\lambda(b_2)\dots T\lambda(b_n)1)_0 = \\ &= \cdots = (T_1\lambda(b_1)\dots T_1\lambda(b_n)s(b)1)_0 = \\ &= \varepsilon_1(T_1\lambda(b_1)\dots T_1\lambda(b_n)).\end{aligned}$$

□

3.6. Proposition. *Let (A, φ) be a B-probability space, let $a \in A$ be a random variable and let $e \in B$ be an idempotent $e = e^2 \neq 0$. Let*

$$T = \lambda^*(1) + \sum_{n\geq 0}\lambda(\xi_n) \quad (\xi_n \in \mathcal{X}_n(B))$$

be the canonical form of $eae \in A$ and let

$$S = \lambda^*(e) + \sum_{n\geq 0}\lambda(\eta_n)$$

be the canonical form of the eBe-valued random variable eae (i.e. $eae \in eAe$, where we consider the eBe-probability space $(eAe, \varphi(e\cdot e))$, so that e is the unit of eBe and $\eta_n \in \mathcal{X}_n(eBe)$). Then we have

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = \eta_n(eb_1e \otimes eb_2e \otimes \cdots \otimes eb_ne).$$

Proof. In view of Proposition 3.5 we have

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = e\xi_n(eb_1e \otimes \cdots \otimes eb_ne)e.$$

We identify $\prod_{n\geq 0} \mathcal{X}_n(eBe)$ with a subspace $\mathcal{Y} \subset \prod_{n\geq 0} \mathcal{X}_n(B)$ by identifying $\chi_n \in \mathcal{X}_n(eBe)$ with $\zeta_n \in \mathcal{X}_n(B)$ defined by

$$\zeta_n(b_1 \otimes \cdots \otimes b_n) = \chi_n(eb_1e \otimes \cdots \otimes eb_ne).$$

Remark that $T\mathcal{Y} \subset \mathcal{Y}$, $\lambda^*(1)\mathcal{Y} \subset \mathcal{Y}$, $\lambda(\xi_n)\mathcal{Y} \subset \mathcal{Y}$ and $\lambda^*(1) \mid \mathcal{Y} = \lambda^*(e) \mid \mathcal{Y}$. Moreover

$$\begin{aligned}
&(T\lambda(b_1)T\lambda(b_2)T\ldots\lambda(b_k)T1)_0 = \\
&= \varphi(eaeb_1eaeb_2e\ldots eb_keae) = \\
&= (T\lambda(eb_1e)T\lambda(eb_2e)\ldots\lambda(eb_ke)Te)_0 = \\
&= (S\lambda(eb_1e)S\lambda(eb_2e)\ldots\lambda(eb_ke)Se)_0.
\end{aligned}$$

To conclude from here that

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = \eta_n(eb_1e \otimes \cdots \otimes eb_ne)$$

one proceeds by induction on n. Let

$$\begin{aligned}
T_n &= \lambda^*(1) + \sum_{k=0}^{n} \lambda(\xi_k) \\
\tilde{T}_n &= \lambda^*(e) + \sum_{k=0}^{n-1} \lambda(\xi_k) + \lambda(\xi_n) \\
S_n &= \lambda^*(e) + \sum_{k=0}^{n} \lambda(\eta_k).
\end{aligned}$$

Assume we proved $\xi_k(b_1 \otimes \cdots \otimes b_k) = \eta_k(eb_1e \otimes \cdots \otimes eb_ke)$ for $k < n$. This implies $\lambda(\xi_k) \mid \mathcal{Y} = \lambda(\eta_k) \mid \mathcal{Y}$ for $k < n$. It follows that

$$\begin{aligned}
&(S_n\lambda(eb_1e)\ldots S_n\lambda(eb_ne)S_ne)_0 = \\
&= (S\lambda(eb_1e)\ldots S\lambda(eb_ne)Se)_0 = \\
&= (T\lambda(b_1)\ldots T\lambda(b_n)Te)_0 = \\
&= (T_n\lambda(b_1)\ldots T_n\lambda(b_n)T_ne)_0 = \\
&= (\tilde{T}_n\lambda(b_1)\ldots \tilde{T}_n\lambda(b_n)\tilde{T}_ne)_0.
\end{aligned}$$

Like in 3.1 we get from this equality

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = \eta_n(eb_1e \otimes \cdots \otimes eb_ne)$$

(since the terms involving $\eta_k (k < n)$ are identical). □

3.7. The next corollary outlines a standard application of the preceding result.

Corollary. *Let X_{ij} $(1 \le i, j \le n)$ be free random variables in a $\mathbb{C}$-probability space (A, τ). Consider the $\mathcal{M}_n(\mathbb{C})$-probability space $(A \otimes \mathcal{M}_n(\mathbb{C}), \tau \otimes id_n)$ and the $\mathcal{M}_n(\mathbb{C})$-valued random variable*

$$X = \sum_{1 \le i,j \le n} X_{ij} \otimes e_{ij}$$

where e_{ij} $(1 \le i, j \le n)$ are the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$. Let

$$R(X_{ij})(z) = \sum_{n=0}^{\infty} R_{n+1}(X_{ij}) z^n$$

be the R-series of X_{ij}. Then the canonical form $\lambda^*(1) + \sum_{k \ge 0} \lambda(\xi_k)$ of T is given by

$$\xi_k(M^{(1)} \otimes \cdots \otimes M^{(k)}) = \sum_{1 \le i,j \le n} R_{k+1}(X_{ij}) m_{ji}^{(1)} \ldots m_{ji}^{(k)} e_{ij}$$

where $M^{(s)} = \sum_{1 \le i,j \le n} m_{ij}^{(s)} e_{ij} \in \mathcal{M}_n(\mathbb{C})$.

Proof. The random variables $X_{ij} \otimes I_n$ are $\mathcal{M}_n(\mathbb{C})$-free in $(A \otimes \mathcal{M}_n(\mathbb{C}), \tau \otimes id_n)$ and hence $X_{ij} \otimes e_{ij}$ $(1 \le i, j \le n)$ are also $\mathcal{M}_n(\mathbb{C})$-free. Using the fact that the R-series gives the canonical form of a $\mathbb{C}$-valued random variable and 3.6, we get that the canonical form

$$\lambda^*(1) + \sum_{n \ge 0} \lambda(\eta_n)$$

of $X_{ij} \otimes e_{ii}$ is given by

$$\eta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = R_{n+1}(X_{ij}) m_{ii}^{(1)} \ldots m_{ii}^{(n)} e_{ii}.$$

If $\lambda^*(1) + \sum_{n \ge 0} \lambda(\zeta_n)$ is the canonical form of $X_{ij} \otimes e_{ij}$, then $X_{ij} \otimes e_{ij} = (X_{ij} \otimes e_{ii})(I \otimes e_{ij})$ together with 3.5 gives

$$\zeta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = \eta_n(e_{ij} M^{(1)} \otimes \cdots \otimes e_{ij} M^{(n)}) e_{ij}$$

so that

$$\zeta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = R_{n+1}(X_{ij}) m_{ji}^{(1)} m_{ji}^{(2)} \ldots m_{ji}^{(n)} e_{ij}.$$

To conclude the proof it suffices to use Proposition 3.2. □

4. The differential equation for $\boxplus$.

4.1. Lemma. *Let $T \in \bar{A}(1)$ and let $\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n) \in \bar{A}(1)$ be a canonical element. Then if $Y(\alpha) = h_1(\lambda^*(1) + \alpha \sum_{n\geq 0} \lambda(\xi_n))$, we have*

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b))(Y(\alpha) + h_2(T))\lambda(b))^m)|_{\alpha=0} =$$
$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0}\lambda(\xi_n((b(\varepsilon_1((T\lambda(b))^{k_n}))\otimes$$
$$\cdots \otimes (b\varepsilon_1((T\lambda(b))^{k_1}))(b)(T\lambda(b))^{k_{n+1}})$$

where $\alpha \in \mathbb{C}$ and $b \in B$.

Proof. The expression the derivative of which must be computed is a polynomial in $\alpha \in \mathbb{C}$ with coefficients in B, which shows also the sense in which this derivative should be understood.

We have $Y(\alpha) = \lambda^*(1 \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\xi'_n)$ where $\xi'_n = \xi_n \circ pr_1^{\otimes n}$. Let $\eta_n = \lambda(\xi'_n)b$, $S = h_2(T\lambda(b))$ and $X(\alpha) = \lambda^*(b \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\eta_n)$.

We have

$$\varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m) =$$
$$= \varepsilon_2(\lambda(b)(S + X(\alpha))^m)$$

and hence

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m|_{\alpha=0} =$$
$$= \sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S + \lambda^*(b \oplus 0))^j \sum_{n\geq 0} \lambda(\eta_n)(S + \lambda^*(b \oplus 0))^{m-1-j}).$$

For the computation which follows one should keep in mind that $\varepsilon_M(R) = (R1)_0$ and the proof of Proposition 2.7. We have

$$\sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S + \lambda^*(b \oplus 0))^j \sum_{n\geq 0} \lambda(\eta_n)(S + \lambda^*(b \oplus 0))^{m-1-j}) =$$
$$= \sum_{j=0}^{m-1} \sum_{n=0}^{j} \sum_{\substack{n+k_0+\cdots+k_n=j \\ k_0\geq 0,\ldots,k_n\geq 0}} b\varepsilon_2(S^{k_0}\lambda^*(b \oplus 0)\ldots S^{k_{n-1}}\lambda^*(b \oplus 0)S^{k_n}\lambda(\eta_n)S^{m-1-j}) =$$
$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} b\varepsilon_2(S^{k_0}\lambda^*(b \oplus 0)\ldots S^{k_{n-1}}\lambda^*(b \oplus 0)S^{k_n}\lambda(\eta_n)S^{k_{n+1}}) =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} b\varepsilon_2(S^{k_0}\lambda(\eta_n(((b\oplus 0)\varepsilon_2(S^{k_n})\otimes \ldots$$

$$\otimes ((b\oplus 0)\varepsilon_2(S^1)))S^{k_{n+1}})) =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0}\lambda(\xi_n((b\varepsilon_1((T\lambda(b))^{k_n})\otimes \ldots$$

$$\otimes (b\varepsilon_1((T\lambda(b))^{k_1}))))\lambda(b)(T\lambda(b))^{k_{n+1}}).$$

□

4.2. It is easy to see that the same computation yields the more general formula

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b_0)(Y(\alpha)+h_2(T))\lambda(b_1)\ldots\lambda(b_{m-1})(Y(\alpha)+h_2(T))\lambda(b_m))|_{\alpha=0} =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0\ldots k_{n+1}\geq 0}} \varepsilon_1(\lambda(b_0)T\lambda(b_1)\ldots T\lambda(b_{k_0})$$

$$\lambda(\xi_n(\varepsilon_1(\lambda(b_{k_0+\cdots+k_{n-1}+n})T\ldots\lambda(b_{k_0+\cdots+k_n+n-1})T\lambda(b_{k_0+\cdots+k_n+n}))\otimes$$

$$\cdots\otimes\varepsilon_1(\lambda(b_{k_0+1})T\ldots\lambda(b_{k_0+k_1+1})T\lambda(b_{k_0+k_1+2})))$$

$$\lambda(b_{k_0+\cdots+k_n+n+1})T\ldots\lambda(b_{k_0+\cdots+k_{n+1}+n})T\lambda(b_{k_0+\cdots+k_{n+1}+n+1}))$$

where $b_0,\ldots,b_m \in B$.

4.3. Before passing to the differential equations we have to discuss certain formal series which are the analogue of formal power series when maps $\mathbb{C}\to\mathbb{C}$ are replaced by maps $B\to B$.

Let $S\mathcal{X}_n(B)\subset\mathcal{X}_n(B)$ be the subspace of symmetric n-linear maps i.e. $\xi_n(b_1\otimes\cdots\otimes b_n)=\xi_n(b_{\sigma(1)}\otimes\cdots\otimes b_{\sigma(n)})$ for all $\sigma\in\mathcal{G}_n$. If $\eta\in\mathcal{X}_n(B)$ we denote by $S\eta\in S\mathcal{X}_n(B)$ the element such that $S\eta(b^{\otimes n})=\eta(b^{\otimes n})$. Elements of $S\mathcal{X}(B)=\prod_{n\geq 0}S\mathcal{X}_n(B)$ will be written

$$\sum_{n\geq 0}\xi_n$$

$S\mathcal{X}(B)$ is a ring with multiplication such that $(\xi_m\xi_n)(b^{\otimes(m+n)})=\xi_m(b^{\otimes m})\xi_n(b^{\otimes n})$. $S\mathcal{X}(B)$ has a natural filtering given by the powers of the ideal formed by elements of the form $\sum_{n\geq 1}\xi_n$. If $\varphi=\sum_{n\geq 0}\xi_n$ and $\psi=\sum_{n\geq 1}\eta_n$ then the composition $\varphi\circ\psi$ is easily seen to be well defined as follows $\varphi\circ\psi=\Sigma_{n\geq 0}\zeta_n$ where

$$\zeta_0=\xi_0 \text{ and if } k\geq 1$$

$$\zeta_k=\sum_{m\geq 1}\sum_{\substack{k_1+\cdots+k_m=k \\ k_1\geq 1,\ldots,k_m\geq 1}}\xi_m(\eta_{k_1}(b^{\otimes k_1})\otimes\cdots\otimes\eta_{k_m}(b^{\otimes k_m})).$$

The differential of $\varphi \in S\mathcal{X}(B)$ is an element of $\prod_{n\geq 0} S\mathcal{L}(B^{\otimes n}, \mathcal{L}(B,B))$ where $S\mathcal{L}(B^{\otimes n}, E)$ denotes the symmetric n-linear E-valued maps. If $\varphi = \sum_{n\geq 0} \xi_n$ then the differential is

$$D\varphi = \sum_{n\geq 1} D\xi_n$$

where $D\xi_n \in \mathcal{L}(B^{\otimes(n-1)}, \mathcal{L}(B,B))$ is such that $(D\xi_n(b^{\otimes(n-1)}))(\beta) = \sum_{k=0}^{n-1} \xi_n(\underbrace{b\otimes\cdots\otimes b}_{k\text{-times}}\otimes \beta\otimes b\otimes\cdots\otimes b)$.

We shall write formally also $\varphi(b) = \sum_{n\geq 0} \xi_n(b^{\otimes n})$, $(D\varphi)(b)[\beta]$ or $(D_b\varphi)(b)[\beta]$ and $\varphi(\psi(b))$.

If $\varphi = \sum_{n\geq 0} \xi_n$ and the ξ_n's depend on a parameter then the derivative of φ with respect to this parameter is meant component wise.

4.4. If $\mu \in S\Sigma_b$ we consider the formal series

$$G_\mu(b) = \sum_{n\geq 0} \mu(b(Xb)^n).$$

It will be also useful to consider

$$\Gamma_\mu(b) = \sum_{n\geq 0} \mu((Xb)^n X)$$

so that

$$G_\mu(b) = b + b\Gamma_\mu(b)b.$$

If $\mu \in \Sigma_B$ we shall write also G_μ for $G_{S\mu}$ for Γ_μ for $\Gamma_{S\mu}$. Also if μ is a distribution μ_T we shall write G_T and Γ_T instead of G_{μ_T} and Γ_{μ_T}.

4.5. Proposition. *Let $T \in \bar{A}(1)$ and let $Y(\alpha) = h_1(\lambda^*(1) + \alpha\sum_{n\geq 0}\lambda(\xi_n)) \in h_1(\bar{A}(1))$. Let $T(\alpha) = Y(\alpha) + h_2(T)$. Then we have $G_{T(0)} = G_T$ and*

$$\frac{\partial}{\partial\alpha}G_{T(\alpha)}(b) = (D_b G_{T(\alpha)})(b)[b\Xi(G_{T(\alpha)}(b))b],$$

where $\Xi(b) = \sum_{n\geq 0}\xi_n(b^{\otimes n})$.

Proof. If $\alpha = 0$ the equality of the terms which are of degree $m+1$ in b in the differential equation is precisely the equality established in Lemma 4.1. The general case can be reduced to the case $\alpha = 0$ since in view of 2.7 and 3.2 we have

$$\mu_{T(\alpha)} = \mu_T \boxplus \mu_{Y(\alpha)} = \mu_{T(\alpha_0)} \boxplus \mu_{Y(\alpha-\alpha_0)}.$$

$\square$

4.6. Corollary. *Let $\mu \in \Sigma_B$ and $a = \lambda^{*(1)} + \sum_{n\geq 0} \lambda(\xi_n)$ be the canonical element with distribution μ. Then the $(S\xi_n)_{n\geq 0}$ depend only on $S\mu$ and conversely $S\mu$ depends only on the $(S\xi_n)_{n\geq 0}$. In particular if $\mu_1, \mu_2 \in \Sigma_B$ then $S(\mu_1 \boxplus \mu_2)$ is completely determined by $S\mu_1$ and $S\mu_2$.*

Proof. Let $a(\alpha) = \lambda^*(1) + \alpha \sum_{n\geq 0} \lambda(\xi_n)$.

We have

$$\begin{aligned}\frac{d}{d\alpha}\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m) &= bS\xi_{m-1}(b\otimes\cdots\otimes b)b+ \\ &+ F(S\xi_j, \varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^j)b, 0 \leq j \leq m-2)\end{aligned}$$

where F is a "polynomial" of the quantities on which it depends. These differential equations with initial condition $\bar{\varepsilon}_1(\lambda(b)(a(0)\lambda(b))^m) = 0$ if $m \geq 1$ can be solved recurrently and we obtain that

$$\begin{aligned}&\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m) = \\ &= \alpha bS\xi_{m-1}(b\otimes\cdots\otimes b)b+ \\ &+ P(\alpha, b, S\xi_j, 0 \leq j \leq m-2)\end{aligned}$$

where P is "polynomial". Taking $\alpha = 1$ we see that $S\mu$ completely determines the $(S\xi_n)_{n\geq 0}$ and also that conversely the $(S\xi_n)_{n\geq 0}$ completely determine $S\mu$. The assertion concerning $S(\mu_1 \boxplus \mu_2)$ follows now from 3.2. □

4.7. The differential equation in 4.5 immediately implies the following fact: if $\mu_1, \mu_2 \in S\Sigma_B$ and $\mu(\alpha) \in S\Sigma_B$ is such that $SR_n(\mu(\alpha)) = SR_n(\mu_1) + \alpha SR_n(\mu_2)$ then

$$\frac{\partial}{\partial\alpha} G_{\mu(\alpha)}(b) = (D_b G_{\mu(\alpha)})(b)[b\Xi(G_{\mu(\alpha)}(b))b]$$

where $\Xi = \sum_{n\geq 0} \xi_n$ where $\xi_n = SR_{n+1}(\mu_2)$. Interpreting this equation as a system of ordinary differential equations as in 4.6 we see that with the initial condition $G_{\mu(0)} = G_{\mu_1}$ we have $G_{\mu(1)} = G_{\mu_1\boxplus\mu_2}$ which is completely determined by the differential equation.

4.8. We shall now assume B is a Banach algebra and Γ_μ and hence G_μ is an analytic function in some neighborhood of $0 \in B$. This implies that the symmetric moments of μ viewed as n-linear maps $B^n \to B$ are continuous and the formal series defining G_μ is absolutely convergent in some neighborhood of 0. For instance if T is a B-valued random-variable $T \in A$ where A is also a Banach algebra, $A \supset B$ with a continuous conditional expectation $\varphi : A \to B$ then $G_T(b) = \sum_{n\geq 0} \varphi(b(Tb)^n) = \varphi(b(1 - Tb)^{-1})$ satisfies these assumptions.

For the lemma which follows we shall denote by $\mathcal{M}$ the set of germs at $0 \in B$ of analytic B-valued maps and we shall use the notation F^{-1} only for multiplicative inverses, not for inverses with respect to composition.

Lemma. *Let $\Gamma, G \in \mathcal{M}$ be such that $G(b) = b + b\Gamma(b)b$ near 0.*

(i) *If $K \in \mathcal{M}$ is such that $K(G(b)) = G(K(b)) = b$ near 0, then there is $Q \in \mathcal{M}$ such that $K(b) = b + bQ(b)b$.*

(ii) *There is $R \in \mathcal{M}$ such that for some neighborhood V of $0 \in B$ we have $(K(b))^{-1} = b^{-1} + R(b)$ if $b \in V \cap GL(B)$. R is unique.*

Proof. (i) If $\|b\|$ is small enough, we have

$$b = G(b)(1 + \Gamma(b)b)^{-1} = G(b)(1 + \Gamma(K(G(b)))K(G(b)))^{-1}$$

so that there is $H \in \mathcal{M}$ for which

$$K(b) = bH(b).$$

Similarly there is $J \in \mathcal{M}$ so that

$$K(b) = J(b)b.$$

We have

$$b = G(b) - b\Gamma(b)b = G(b) - G(b)H(G(b))\Gamma(K(G(b)))J(G(b))G(b)$$

so $Q(b) = -H(b)\Gamma(K(b))J(b)$ will do.

(ii) Choosing V small enough, if $b \in V \cap GL(B)$ we have

$$\begin{aligned} K(b)^{-1} &= b^{-1}(1 + bQ(b))^{-1} = \\ &= b^{-1}Q(b)(1 + bQ(b))^{-1}. \end{aligned}$$

The uniqueness of R is easily seen from the fact that $R \mid (V \cap GL(B))$ determines the germ of R at 0. □

4.9. Theorem. *Assume B is a Banach algebra and $\mu \in S\Sigma_B$ is such that $\Gamma_\mu(b)$ is analytic in some neighborhood of $0 \in B$. Let K and R be germs of B-valued analytic functions at $0 \in B$ such that*

$$K(G_\mu(b)) = G_\mu(K(b)) = b$$

and

$$K(b)^{-1} = b^{-1} + R(b)$$

for $b \in GL(B)$ in some neighborhood of 0. Then we have

$$R(b) = \sum_{n \geq 0} SR_{n+1}(\mu)(b^{\otimes n})$$

where the $SR_{n+1}(\mu)$ are given by the canonical element with distribution μ.

Proof. Let

$$K(\alpha, b) = (b^{-1} + \alpha R(b))^{-1} = (b^{-1}(1 + \alpha b R(b)))^{-1} = (1 + \alpha b R(b))^{-1} b$$

which for $0 \le \alpha \le 1$ makes sense in some fixed neighborhood of 0, the last equality making the invertibility of b superfluous. There is a neighborhood of 0 independent of $0 \le \alpha \le 1$ for which $K(\alpha, b)$ has an inverse (with respect to composition) $K(\alpha, G(\alpha, b)) = G(\alpha, K(\alpha, b)) = b$.

We have

$$\frac{d}{d\alpha} G(\alpha, K(\alpha, b)) = 0$$

which with $b_1 = K(\alpha, b)$ gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha} G(\alpha, b_1) + (D_b G)(\alpha, b_1)[-b_1 R(b) b_1] = \\ &= \frac{\partial}{\partial \alpha} G(\alpha, b_1) - (D_b G)(\alpha, b_1)[b_1 R(G(\alpha, b_1)) b_1]. \end{aligned}$$

Moreover $G(0, b) = b$. Thus defining $\mu(\alpha) \in S\Sigma_B$ by $G_{\mu(\alpha)}(b) = G(\alpha, b)$ we have that $\frac{\partial}{\partial \alpha} G_{\mu(\alpha)}(b) = (D_b G_{\mu(\alpha)})(b)[bR(G_{\mu(\alpha)}(b))b]$ and $\mu(0)$ is the distribution of the 0 random-variable. In view of 4.7 this implies that G is the generating series for a symmetric distribution for which the corresponding symmetric parts of the components of the corresponding canonical element yield the series $R(b)$. Thus we have

$$R(b) = \sum_{n \ge 0} SR_{n+1}(\mu(1))(b^{\otimes n}).$$

On the other hand $G_{\mu(1)} = G_\mu$ so that $\mu(1) = \mu$. □

4.10. We have chosen in 4.8 and 4.9 to work in the Banach algebra context where we work with genuine functions since this is the situation for the applications to computations of spectra. On the other hand the reader will not find it difficult to transpose Lemma 4.8 and Theorem 4.9 in the framework of formal series and general B where similar statements hold.

4.11. Thus in the B-valued case the computation of $\mu_1 \boxplus \mu_2$, $\mu_j \in S\Sigma_B$ is done as in the scalar case: one forms G_{μ_j}, then the inverse K_{μ_j} then the multiplicative inverses $b^{-1} + R_{\mu_j}$. Then $R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$ and from $R_{\mu_1 \boxplus \mu_2}$ one goes back to $K_{\mu_1 \boxplus \mu_2}$ and $G_{\mu_1 \boxplus \mu_2}$.

5. The differential equation for $\boxtimes$.

5.1. Lemma. *Let $T_1, \dots, T_m \in \bar{A}(1)$, let $a(\tau) = \lambda^*(1) + \lambda(1) + \tau \sum_{n \geq 1} \lambda(\xi_n) \in \bar{A}(1)$ be a canonical element and let $Y(\tau) = h_1(a(\tau))$. Then we have*

$$\frac{d}{d\tau}\varepsilon_2(Y(\tau)h_2(T_1)Y(\tau)h_2(T_2)\dots Y(\tau)h_2(T_m))|_{\tau=0} =$$
$$= \sum_{p\geq 1} \sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0 \geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_1(T_1 \dots T_{j_0}\lambda(\xi_p(\varepsilon_1(T_{j_0+\dots+j_{p-1}+1}\dots T_{j_0+\dots+j_p})\otimes$$
$$\dots \otimes \varepsilon_1(T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots j_p+1}\dots T_{j_0+\dots+j_{p+1}}).$$

Proof. For the computation it will be convenient to put $\xi_p' = \xi_p \circ pr_1^{\otimes p}$ so that $\lambda(\xi_p') = h_1(\lambda(\xi_p))$ and $S_k = h_2(T_k)$.

We have

$$\frac{d}{d\tau}\varepsilon_2(Y(\tau)S_1\dots Y(\tau)S_m)|_{\tau=0} =$$
$$= \sum_{j=1}^{m} \varepsilon_2(Y(0)S_1\dots Y(0)S_{m-j}\sum_{p\geq 1}\lambda(\xi_p')S_{m-j+1}Y(0)S_{m-j+2}\dots Y(0)S_m) =$$
$$= \sum_{j=1}^{m} \varepsilon_2((\lambda^*(1\oplus 0)+\lambda(1))S_1\dots(\lambda^*(1\oplus 0)+\lambda(1))S_{m-j}\sum_{p\geq 1}\lambda(\xi_p')S_{m-j+1}\dots S_m) =$$
$$= \sum_{j=1}^{m}\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_p=m-j \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_p\geq 1}} \varepsilon_2(S_1\dots S_{j_0}\lambda^*(1\oplus 0)S_{j_0+1}\dots S_{j_0+j_1}\lambda^*(1\oplus 0)$$
$$\lambda^*(1\oplus 0)S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p}\lambda(\xi_p')S_{j_0+\dots+j_p+1}\dots S_m) =$$
$$= \sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_2(S_1\dots S_{j_0}\lambda(\xi_p(\varepsilon_2(S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p})\otimes$$
$$\dots\otimes\varepsilon_2(S_{j_0+1}\dots S_{j_0+j_1})))S_{j_0+\dots+j_{p+1}}\dots S_m) =$$
$$= \sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} \varepsilon_1(T_1\dots T_{j_0}\lambda(\xi_p(\varepsilon_1(T_{j_0+\dots+j_{p-1}+1}\dots T_{j_0+\dots+j_p})\otimes$$
$$\dots\otimes\varepsilon_1(T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots+j_{p+1}}\dots T_{j_0+\dots+j_{p+1}}).$$

□

5.2. Corollary. *Let $T \in \bar{A}(1)$, let*

$$a(\tau) = \lambda^*(1) + \lambda(1) + \tau\sum_{n\geq 1}\lambda(\xi_n) \in \bar{A}(1)$$

be a canonical element and let $Y(\tau) = h_1(a(\tau))$. Then we have

$$\frac{d}{d\tau}\varepsilon_2((\lambda(b)Y(\tau)h_2(T))^m)|_{\tau=0} =$$
$$= \sum_{p\geq 1} \sum_{\substack{j_0+\cdots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\ldots,j_{p+1}\geq 1}} \varepsilon_1(\lambda(b)(T\lambda(b))^{j_0}\lambda(\xi_p(\varepsilon_1((T\lambda(b)^{j_p})\otimes\ldots$$
$$\otimes\,\varepsilon_1((T\lambda(b))^{j_1}))T(\lambda(b)T)^{j_{p+1}-1})$$

where $b \in B$ and $\tau \in \mathbb{C}$.

5.3. Proposition. *Let $T \in \bar{A}(1)$ and let*

$$Y(\tau) = h_1(\lambda^*(1) + \lambda(1) + \tau\sum_{n\geq 1}\lambda(\xi_n)) \in \bar{A}(1).$$

Let $T(\tau) = Y(\tau)h_2(T)$. Then we have

$$\frac{d}{d\tau}(b\Gamma_{T(\tau)}(b))|_{\tau=0} = (D_b(b\Gamma_{T(0)}))(b)[b(\Theta(\Gamma_{T(0)}(b)b))]$$

where $\Theta(b) = \sum_{n\geq 1}\xi_n(b^{\otimes n})$.

Proof. The proposition is obtained from Corollary 5.2 by looking at the terms which are degree m in b. □

5.4. The use of differential equations in the study of the operator ⊠ relies on viewing

$$\Sigma_{1,B} = \{\mu \in \Sigma_B \mid \mu(X) = 1\}$$

as a kind of infinite dimensional Lie group with respect to multiplicative free convolution and to identify the solutions of the differential equations:

$$(*)\qquad \begin{aligned}&\frac{d}{d\tau}\mu(\tau)(Xb_1\ldots Xb_m) =\\ &= \sum_{p\geq 1} \sum_{\substack{j_0+\cdots+j_{p+1}=m \\ j_0\geq 0 \\ j_1\geq 1,\ldots,j_{p+1}\geq 1}} \mu(\tau)(Xb_1\ldots Xb_{j_0}\\ &\xi_p((\mu(\tau)(Xb_{j_0+\cdots+j_{p-1}+1}\ldots Xb_{j_0+\cdots+j_p}))\otimes\ldots\\ &\otimes(\mu(\tau)(Xb_{j_0+1}\ldots Xb_{j_0+j_1})))\\ &Xb_{j_0+\cdots+j_p+1}\ldots Xb_{j_0+\cdots+j_{p+1}})\end{aligned}$$

with the integral curves of right-invariant vector fields on this group. Here $\mu(\tau) \in \Sigma_{1,B}$, $\xi_p \in \mathcal{X}_p(B)$ and the equations are obtained from those in Lemma 5.1 by taking $T_j = T\lambda(b_j)$ with T a function of τ and $\mu(\tau)$ the distribution of T.

5.5. Before looking at the differential equations let us note a few facts about $(\Sigma_{1,B}, \boxtimes)$. The set $\Sigma_{1,B}$ has an obvious affine space structure. Denoting by $\Sigma_{1,B}^{(n)}$ the set of n-th order distributions (i.e. the restrictions of distributions to noncommutative monomials $b_1Xb_2X\dots b_mXb_{m+1}$ of degree $m \le n$) it is easily seen that $(\Sigma_{1,B}, \boxtimes)$ is the inverse limit of groups $(\Sigma_{1,B}^{(n)}, \boxtimes_n)$. We shall denote by μ_1 the distribution of 1, which is the neutral element of $(\Sigma_{1,B}, \boxtimes)$ and by $\mu_1^{(n)}$ the corresponding n-th order distribution.

It is easily seen that if $\mu, \nu \in \Sigma_{1,B}$ then

$$\begin{aligned}&(\mu \boxtimes \nu)(b_1Xb_2X\dots b_mXb_{m+1}) = \\ &= \mu(b_1Xb_2\dots Xb_{m+1}) + \nu(b_1Xb_2\dots Xb_{m+1}) + \\ &+ F(\mu^{(n-1)}, \nu^{(n-1)})\end{aligned}$$

where $F(\mu^{(n-1)}, \nu^{(n-1)})$ depends only on $\mu^{(n-1)}, \nu^{(n-1)}$. Note also that $\boxtimes_n$ is a polynomial map in the sense, that if $V_1, V_2 \subset \Sigma_{1,B}^{(n)}$ are finite-dimensional affine subspaces then there is an affine subspace $V_3 \subset \Sigma_{1,B}^{(n)}$ such that

$$V_1 \boxtimes_n V_2 \subset V_3$$

and the map

$$V_1 \times V_2 \to V_3$$

defined by $\boxtimes_n$ is a polynomial map of degree $\le n$. A more careful inspection of the map defined by $\boxtimes_n$, also yields the following fact which we record as a lemma, the proof of which is left to the reader as an exercise.

5.6. Lemma. *Let $V_1, V_2 \subset \Sigma_{1,B}^{(n)}$ be finite-dimensional affine subspaces. Then there is a finite-dimensional affine subspace $W \subset \Sigma_{1,B}^{(n)}$ such that*

$$V_2 \subset W \text{ and } V_1 \boxtimes_n W \subset W.$$

With these preparations we are ready to study the free exponential.

5.7. Definition. The free exponential is the map

$$\operatorname{fexp} : \prod_{n\ge 1} \mathcal{X}_n(B) \to \Sigma_{1,B}$$

defined by $\operatorname{fexp}((\xi_n)_{n\ge 1}) = \mu(1)$ where $\mu(\tau)$ satisfies $(*)$ with initial condition $\mu(0) = \mu_1$.

5.8. Proposition.

a) *The map* fexp *is a bijection.*

b) *If $\mu(\tau)$ satisfies $(*)$ with initial condition $\mu(0) = \nu \in \Sigma_{1,B}$, then*

$$\nu(1) = \text{fexp}((\xi_n)_{n\geq 1}) \boxtimes \nu.$$

Proof. a) Denote by $\mu(\tau; n)$ the element $(b_1 \otimes \cdots \otimes b_n) \to \mu(\tau)(Xb_1 \ldots Xb_n)$ of $\mathcal{X}_n(B)$. The system of differential equations $(*)$ for the $\mu(\tau; n)$ can be solved recurrently. It is easily seen by recurrence that the right-hand side of $(*)$ is a polynomial in τ of degree $< m$ with zeroth order term

$$\xi_{m-1}(b_{m-1} \otimes \cdots \otimes b_1)b_m$$

while the higher order terms depend only on $\xi_1, \ldots, \xi_{m-2}$. This easily gives that fexp is a bijection.

b) It will suffice to prove that

$$\lim_{n\to\infty} \mu_{Z(n;\tau)} = \mu(\tau)$$

where

$$Z(n; \tau) = h_{n+1}(Y(\tau/n)) \ldots h_2(Y(\tau/n))h_1(T),$$

$T \in \overline{A(1)}$ has $\mu_T = \nu$ and $h_1, \ldots, h_{n+1}$ are the homomorphisms $\bar{A}(1) \to \bar{A}(n+1)$, corresponding to the canonical summands in B^{n+1}. Note indeed that this means

$$\mu_{Z(n;\tau)} = \underbrace{\mu_{Y(\tau/n)} \boxtimes \cdots \boxtimes \mu_{Y(\tau/n)}}_{n\text{-times}} \boxtimes \nu$$

and that for the special case $T = \lambda(1)$, $\mu_T = \mu_1$ we also get

$$\lim_{n\to\infty} (\mu_{Y(\tau/n)})^{\boxtimes n} = \text{fexp}((\xi_n)_{n\geq 1}).$$

In all these considerations, the limits make sense in view of Lemma 5.6. Indeed, by Lemma 5.6 we have that for fixed $b_1, \ldots, b_m$ and all n

$$\mu_{Z(n;\tau)}(Xb_1 \ldots Xb_m)$$

takes values in a fixed finite dimensional vector subspace of B. The limit is with respect to the usual topology of this finite-dimensional complex vector space.

To prove the assertion we have to use again Lemma 5.6. Let $V_1, V_2 \subset \Sigma_{1,B}^{(m)}$ be finite-dimensional affine subspaces, such that

$$\mu_{Y(\tau)}^{(m)} \in V_1 \text{ for all } \tau \in \mathbb{C}$$

and $\nu^{(m)}, \mu_1^{(m)} \in V_2$. Let W be as in Lemma 5.4. Then

$$D(\tau) : W \to W$$

is a polynomial map of bounded degree, depending polynomially on $\tau \in \mathbb{C}$ and such that $D(0) = \mathrm{id}_W$. The first m equations $(*)$ describe in view of Lemma 5.1 precisely the integral curves of the vector field on W defined by

$$\frac{d}{d\tau} D(\tau)w|_{\tau=0} \text{ at } w \in W.$$

In this context, where all the maps etc. are polynomial it is immediate that

$$\lim_{n\to\infty} (D(\tau/n))^n \nu^{(m)} = (\mu(\tau))^{(m)}$$

which is the desired result. □

5.9. Proposition. *The symmetric part $S\mu(\tau)$ of a solution of $(*)$ depends only on the symmetric parts $(S\xi_n)_{n\geq 1}$ and on the symmetric part $S\mu(0)$ of the initial data. In particular, there is a map* fexp : $\prod_{n\geq 1} S\mathcal{X}_n(B) \to S\Sigma_{B,1}$ *(the free symmetric exponential) such that*

$$\begin{array}{ccc}
\prod_{n\geq 1} \mathcal{X}_n(B) & \xrightarrow{\text{fexp}} & \Sigma_{B,1} \\
\big\downarrow S & & \big\downarrow S \\
\prod_{n\geq 1} S\mathcal{X}_n(B) & \xrightarrow{\text{fexp}} & S\Sigma_{B,1}
\end{array}$$

is a commutative diagram and fexp *is a bijection. Moreover $S(\mu_1 \boxtimes \mu_2)$ depends only on $S\mu_1$ and $S\mu_2$.*

Proof. For the first assertion it is clearly sufficient to show that the system of differential equations $(*)$ yields a system of differential equations for $S\mu(\tau)$, which completely determines $S\mu(\tau)$ for a given initial condition and which involves only the symmetric parts of the ξ_n. Such a system of differential equations is provided by the equations $(*)$ with $b_1 = \cdots = b_{m-1} = b$ and $b_m = 1$. To see that this system involves only the symmetric parts of ξ_p, note that the sum inthe right-hand side sum of $(*)$ when restricted to fixed p, j_0 and j_{p+1} is a sum over $j_1 \geq 1, \dots, j_p \geq 1$, $j_1 + \cdots + j_p = m - j_0 - j_{p+1}$. The permutation group $\mathcal{G}_p$ acts on this set of p-tuples $(j_1, \dots, j_p)$ and the partial sums over its orbits involve only symmetrizations

$$\sum_{\sigma\in\mathcal{G}_p} \xi_p(\beta_{\sigma(1)} \otimes \cdots \otimes \beta_{\sigma(p)})$$

where $\beta_1, \dots, \beta_p \in B$, which are completely determined by $S\xi_p$. In turn, the partial sum for fixed $p, j_1, \dots, j_p$ is a sum of the form

$$\mu(\tau)(\beta XbXb\dots X) + \mu(\tau)(Xb\beta XbX\dots bX) +$$

$$\begin{aligned}&\cdots + \mu(\tau)(XbXb\ldots Xb\beta X) = \\ &= \beta\mu(\tau)(XbX\ldots bXbX) + \\ &\frac{d}{d\varepsilon}\mu(\tau)(Xb(1+\varepsilon\beta)Xb(1+\varepsilon\beta)\ldots Xb(1+\varepsilon\beta)X)|_{\varepsilon=0}.\end{aligned}$$

This concludes the proof of the first assertion (an alternative proof could have been based on 5.3 and 4.6). The existence of the map fexp is an immediate consequence.

To see that fexp is a bijection, remark that

$$\begin{aligned}&\frac{d}{d\tau}\mu(\tau)(XbX\ldots XbX) - \\ &-\xi_{m-1}(b\otimes\cdots\otimes b) = \\ &= \text{ right-hand side sum of } (*) \text{ restricted to } 1 \le p \le m-2\end{aligned}$$

and hence this difference depends only on $S\xi_1, \ldots, S\xi_{m-2}$. The last assertion is now an immediate corollary in view of 5.8. □

5.10. *Remark.* The differential equations for $S\mu(\tau)$, in view of the preceding proposition and of Proposition 5.3 can also be written in terms of "generating series" in the form:

$$\begin{aligned}&\frac{d}{d\tau}(B\Gamma_{S\mu(\tau)}(b)) = \\ &= (D_b(b\Gamma_{S\mu(\tau)}))(b)[b(\Theta(\Gamma_{S\mu(\tau)}(b)b))]\end{aligned}$$

where

$$\Gamma_{S\mu(\tau)}(b) = \sum_{n\ge 0} S\mu(\tau)(b(Xb)^n)$$

and

$$\Theta(b) = \sum_{n\ge 1}\xi_n(b^{\otimes n}).$$

This differential equation can also be used to compute the symmetric multiplication free convolution

$$S\mu_1 \boxtimes S\mu_2$$

provided we compute $(\text{fexp})^{-1}S\mu_1$.

6. Computation of spectra.

As we mentioned in the introduction the results concerning the operation $\boxplus$ provide a method for computing spectra of left convolution operators in $l^2(G)$ where $G = \mathbb{Z} * \mathbb{Z}$ is the free group on two generators g_1, g_2. Actually the same ideas provide a method for dealing with more complicated groups obtained by taking free products with amalgamation. We shall however stick here to the case of $\mathbb{Z} * \mathbb{Z}$ since we think this particular example will suffice to explain our approach.

6.1. Let $Y = \sum_{g \in G} c_g \lambda(g)$ where $c_g \in \mathbb{C}$, $c_g \neq 0$ only for finitely many $g \in G$ and where λ is the left regular representation of G on $l^2(G)$. To compute the spectrum of Y we have to provide a method for deciding whether $Y - zI$ is invertible for a given $z \in \mathbb{C}$. Since $Y - zI$ is of the same form as Y we may state our problem as deciding whether Y is invertible.

6.2. We recall one of the standard algebraic tricks with matrices.

Let A be a ring and let $c_0, \ldots, c_n$ and $u_1, \ldots, u_n$ be elements in A. Let further $y = c_0 + \sum_{k=1}^n c_k u_k \ldots u_1$ and $y_p = c_p + \sum_{k=p+1}^n c_k u_k \ldots u_{p+1}$. Then in the ring $\mathcal{M}_{n+1}(A)$ of $(n+1) \times (n+1)$ matrices over A we have:

$$\begin{pmatrix} 1 & y_1 & \cdots & y_n \\ & 1 & & \\ 0 & & \ddots & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ -u_1 & \ddots & & \\ 0 & & -u_n & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}.$$

This identity shows that y is invertible if and only if the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}$$

is invertible.

6.3. Let $C_r^*(G)$ be the reduced C^*-algebra of G. An application of 6.2 to the element Y in 6.1 with $A = C_r^*(G)$, $y = Y$, $c_j \in \mathbb{C}$, $u_j \in \{\lambda(g_1), \lambda(g_2), \lambda(g_1^{-1}), \lambda(g_2^{-1})\}$ shows that given Y there is $q \in \mathbb{N}$ and there are $\alpha_j, \beta_j \in \mathcal{M}_q(\mathbb{C})$, $(j = \pm 1)$ and $\gamma \in \mathcal{M}_q(\mathbb{C})$ with α_j, β_j and q depending on $\{g \in G \mid c_g \neq 0\}$ and γ a first order polynomial function of the c_g, such that: (Y invertible) $\Leftrightarrow$ ($\alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2) + \gamma \otimes 1$ invertible).

6.4. It will be convenient to make one further matrix transformation so as to be in the self-adjoint case. With the notations of 6.3 put

$$\begin{aligned} a &= \alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \gamma \otimes 1 \\ b &= \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2). \end{aligned}$$

Then we have

$$\left(a + b \text{ invertible}\right) \Leftrightarrow \left(\left(\begin{pmatrix} 0 & a+b \\ a^* + b^* & 0 \end{pmatrix}\right) \text{ invertible}\right).$$

So defining $X_1 = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ we have

$$(Y \text{ invertible}) \Leftrightarrow (X_1 + X_2 \text{ invertible}).$$

Moreover

$$X_j = X_j^* \in \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset \mathcal{M}_{2q}(\mathbb{C}) \otimes C_r^*(G) \quad (j = 1, 2)$$

and if $\{g \in G \mid c_g \neq 0\}$ is a fixed finite set then X_2 is constant and X_1 is a first order polynomial in c_g and $\bar{c}_g (g \in G)$. Also only $\lambda(g_j^{\pm 1})$ appear in the expression of X_j.

6.5. Let $B = \mathcal{M}_{2q}(\mathbb{C})$, $A = \mathcal{M}_{2q}(\mathbb{C}) \otimes C_r^*(G)$, $A_j = \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset A$ and $\varphi : A \to \mathcal{M}_{2q}(\mathbb{C})$ the conditional expectation $\varphi = \mathrm{id} \otimes \tau$ where τ is the canonical trace on $C_r^*(G)$. Then $\{A_1, A_2\}$ is a free pair of subalgebras in (A, φ) and hence $\{X_1, X_2\}$ is a free pair of $\mathcal{M}_{2q}(\mathbb{C})$-valued random variables. It is especially easy to compute G_{X_j} since $A_j \simeq \mathcal{M}_{2q}(\mathbb{C}) \otimes C(\mathbb{T})$. Using the results of section 4 we have a method for computing $G_{X_1+X_2}(b) = \varphi(b(I - (X_1 + X_2)b)^{-1})$ for $b \in \mathcal{M}_{2q}(\mathbb{C}) \otimes I \subset A$. Note that $\mathrm{Tr}_{2q} \circ \varphi$ is faithful on A so that taking $b = zI_{2q} \otimes I$, $z \in \mathbb{C}$ and $\mathrm{Tr}_{2q}(G_{X_1+X_2}(zI_{2q} \otimes I))$ gives us the generating series for the moments of $X_1 + X_2$ with respect to a faithful trace on A. Solving this moment problem one gets the spectrum of $X_1 + X_2$ and hence the possibility of deciding whether $X_1 + X_2$ is invertible.

7. Dual algebraic structures.

This section deals with the necessary adaptations that have to be performed in our considerations in [10] in order to fit the B-valued case.

7.1. The basic idea is to replace the category of unital pro-C^*-algebras in [10] by some other category. Corresponding to the two cases: the purely algebraic one when B is just a unital algebra over $\mathbb{C}$ and the C^*-algebraic one when B is a unital C^*-algebra, we will consider the categories $\mathcal{C}_B$ and respectively $\mathcal{C}_B^*$.

$\mathcal{C}_B$ is the category of unital algebras A over $\mathbb{C}$ containing B as a subalgebra $B \hookrightarrow A$, the inclusion being unital and the morphisms are homomorphisms for which the diagrams

$$\begin{array}{ccc} A_1 & \rightarrow & A_2 \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

are commutative.

$\mathcal{C}_B^*$ is the category of B-pro-C^*-algebras, i.e. unital C^*-algebras $(A, \| \quad \|)$ with $1 \in B \subset A$ and endowed with a family of C^*-seminorms $(\| \quad \|_\alpha)$ $\alpha \in I$ indexed by some directed set I so that $\|b\|_\alpha = \|b\|$ if $b \in B$ and $\alpha \leq \beta \Rightarrow \|x\|_\alpha \leq \|x\|_\beta$, $\|x\| = \sup_{\alpha \in I} \|x\|_\alpha$ if $x \in A$ and moreover

$$A_1 = \varprojlim_{\alpha \in I} A_{\alpha_1}$$

where the subscript 1 is for the unit ball and A_α is the quotient of A by the ideal annihilated by $\| \quad \|_\alpha$. Morphisms in $\mathcal{C}_B^*$ are morphisms of unital pro-C^*-algebras (see 1.4 in [10]) $A \rightarrow A'$ making the diagram

$$\begin{array}{ccc} A & \rightarrow & A' \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

commutative.

7.2. A dual algebraic structure is an algebraic structure in a category as defined in Chapter IV, §1 of [2]. We examined in [10] what a dual group structure means in the category of unital pro-C^*-algebras. In $\mathcal{C}_B$ and $\mathcal{C}_B^*$ we have a similar situation.

Let μ, j, χ be the binary, unary and nullary operations defining the dual group structure on A. Here

$$\begin{array}{ccc} A & \xrightarrow{\chi} & B \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

is commutative.

Also the free products with amalgamation over $\mathbb{C}$ have to be replaced by free products with amalgamation over B. Thus $\mu : A \rightarrow A \underset{B}{*} A$. If $A \in \mathcal{C}_B^*$, this free product is defined as follows: it is the inverse limit of the C^*-algebraic free product with amalgamation $A_\alpha \underset{B}{*} A_\alpha$.

7.3. If $A \in \mathcal{C}_B$ the state space of A denoted by $S(A)$ is the set of conditional expectations $\varphi : A \rightarrow B$. If $A \in \mathcal{C}_B^*$ then $S(A)$ is the set of conditional expectations $\varphi : A \rightarrow B$ such that $\|\varphi(a)\| \leq \|a\|_\alpha$ for one of the seminorms of A.

If $\varphi_j \in S(A_j)$ $(j = 1, 2)$ then there is a unique $\varphi \in S(A_1 \underset{B}{*} A_2)$ such that $\varphi(a_1 \dots a_n) = 0$ whenever $a_k \in A_j(k)$, $j(k) \in \{1, 2\}$, $\varphi_{j(k)}(a_k) = 0$, $j(k) \neq j(k+1)$ $(1 \leq k \leq n-1)$ and $\varphi \mid A_j = \varphi_j$.

Uniqueness of φ follows from 1.3 both in the $\mathcal{C}_B$ and $\mathcal{C}_B^*$ cases. Existence of φ in the $\mathcal{C}_B^*$ case is obtained from §5 of [7].

The existence of φ in the $\mathcal{C}_B$-case is seen as follows. Let $\overset{\circ}{A}_j = \ker \varphi_j$ and $D_n = \{(i_1, \dots, i_n) \mid i_j \in \{1, 2\}, i_k \neq i_{k+1}, 1 \leq j \leq n, 1 \leq k \leq n-1\}$. Then we have

$$A_1 \underset{B}{*} A_2 \simeq B \oplus \bigoplus_{n \geq 0} \bigoplus_{(i_1, \dots, i_n) \in D_n} \overset{\circ}{A}_{i_1} \otimes_B \overset{\circ}{A}_{i_2} \otimes_B \cdots \otimes_B \overset{\circ}{A}_{1_n}$$

and we define φ as the projection onto the B-summand.

We shall denote φ by $\varphi_1 * \varphi_2$.

7.4. If (A, μ, j, χ) is a dual group in $\mathcal{C}_B$ or $\mathcal{C}_B^*$ (actually dual semigroup would suffice) and if $\varphi_1, \varphi_2 \in S(A)$ then $(\varphi_1, \varphi_2) \rightsquigarrow \varphi_1 \Theta \varphi_2 = (\varphi_1 * \varphi_2) \circ \mu$ defines a semigroup structure on $S(A)$ with unit χ.

7.5. In $\mathcal{C}_B$ there is a dual group structure on $B\langle X \rangle$ defined by

$$\begin{aligned} &B\langle X \rangle \xrightarrow{\mu} B\langle X \rangle *_B B\langle X \rangle \simeq B\langle X_1, X_2 \rangle \\ &\mu(X) = X_1 + X_2, \ j(X) = -X \end{aligned}$$

and $\chi(b_0 X b_1 \dots b_{n-1} X b_n) = 0$ if $n \geq 1$ and $\chi(b) = b$. Then $S(B\langle X \rangle) = \Sigma_B$ and Θ is $\boxplus$.

Similarly in $\mathcal{C}_B^*$ a corresopnding dual group is $A = \mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B$ (with the notations of 5.1 [10]) and since $(\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B) *_B (\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B) = (\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} \mathbb{R}_{\mathbb{C},nc}) *_{\mathbb{C}} B$ we define μ from the dual operation of $\mathbb{R}_{\mathbb{C},nc}$.

It is easy to construct similar examples for other dual groups considered in [10] by taking free products with B.

7.6. There are also examples of a somewhat different nature involving tensor products. For instance let $B[X]$ be the polynomials in X with coefficients in B (X and $b \in B$ commute) and let

$$\mu(X) = X_1 + X_2$$

where X_j are the two images of X in $B[X] *_B B[X]$, $j(X) = -X$ and $\chi(bX^n) = 0$ if $n > 0$, $\chi(b) = b$.

8. The B-valued Central Limit Theorem.

The B-valued central limit theorem for free random variables is an immediate consequence of the properties of the canonical form.

8.1. Definition. A random variable a is called B-semicircular if its canonical form is

$$\lambda^*(1) + \lambda(\xi_0) + \lambda(\xi_1).$$

The distribution of such a random variable is also called B-semicircular. The B-semicircular random variable is centered if $\varphi(a) = 0$ (equivalently if $\xi_0 = 0$).

8.2. Since this paper concentrates on algebraic aspects, we will use the weakest kind of convergence for distributions. Clearly, there is a lot of room for improving the convergence side in our central limit result.

Definition. Let B be a Banach algebra and $\mu, \mu_n : B\langle X\rangle \to B$ ($n \in \mathbb{N}$) B-valued distributions. We shall say μ_n convergence pointwise to μ if for every $P \in B\langle X\rangle$ we have $\lim_{n\to\infty} \|\mu_n(P) - \mu(P)\| = 0$.

8.2. *Remark.* It is easy to see along the lines of 3.1 if $(\mu_\iota)_{\iota\in I}$ is a family of B-valued distributions (B a Banach algebra) with canonical forms

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_{n,\iota})$$

then the following two conditions are equivalent

(i) there are constants $C_0, \ldots, C_n$ such that

$$\sup_{\iota\in I} \|\mu_\iota(Xb_1X\ldots b_kX)\| \leq C_k\|b_1\|\ldots\|b_k\|$$

for all $b_1, \ldots, b_k \in B$, $0 \leq k \leq n$.

(ii) there are constants $D_0, \ldots, D_n$ such that

$$\sup_{\iota\in I} \|\xi_{k,\iota}(b_1 \otimes \cdots \otimes b_k)\| \leq D_k\|b_1\|\ldots\|b_k\|$$

for all $b_1, \ldots, b_k \in B$, $0 \leq k \leq n$.

8.3. *Remark.* Assume B is a Banach algebra, μ_j, μ are B-valued distributions ($j \in \mathbb{N}$) and assume the equivalent conditions of 8.2 are satisfied by $(\mu_j)_{j\in\mathbb{N}}$. Then the following are equivalent:

(i)
$$\lim_{j\to\infty} \mu_j(Xb_1X\ldots b_kX) = \mu(Xb_1X\ldots b_kX)$$

for all $0 \leq k \leq n$ and $b_1, \ldots, b_k \in B$.

(ii)
$$\lim_{j\to\infty} \xi_{k,j}(b_1 \otimes \cdots \otimes b_k) = \xi_k(b_1 \otimes \cdots \otimes b_k)$$

for all $0 \leq k \leq n$ and $b_1, \ldots, b_k \in B$.

8.4. Theorem. *Assume B is a Banach algebra and a_j ($j \in \mathbb{N}$) is a B-free sequence of random variables, such that*

$1°$. $\varphi(a_j) = 0$, $j \in \mathbb{N}$

$2°$. *there is a bounded linear map $\eta : B \to B$ such that*

$$\lim_{n\to\infty} n^{-1} \sum_{1\le j\le n} \varphi(a_j b a_j) = \eta(b)$$

$3°$. *there are constants C_k ($k \geq 1$) such that*

$$\sup_{j\in\mathbb{N}} \|\varphi(a_j b_1 a_j \dots b_k a_j)\| \leq C_k \|b_1\| \dots \|b_k\|.$$

Let $S_n = n^{-1/2}(a_1 + \cdots + a_n)$. Then the distribution of S_n converges pointwise to the semicircular distribution with canonical form

$$\lambda^*(1) + \lambda(\eta).$$

Proof. Let

$$\lambda^*(1) + \sum_{k\ge 0} \lambda(\xi_{k,j})$$

be the canonical form of a_j. We have

$$\xi_{0,j} = 0 \quad (j \in \mathbb{N})$$

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \xi_{1,j}(b) = \eta(b)$$

$$\sup_{j\in\mathbb{N}} \|\xi_{n,j}(b_1 \otimes \cdots \otimes b_n)\| \leq C_n \|b_1\| \dots \|b_n\| \quad (n \geq 1)$$

for some constants C_n ($n \in \mathbb{N}$).

Let

$$\lambda^*(1) + \sum_{k\ge 0} \lambda(\eta_{k,n})$$

be the canonical form of S_n. It follows from 3.2 and 3.5 that

$$\eta_{k,n} = n^{-(k+1)/2}(\xi_{k,1} + \cdots + \xi_{k,n}).$$

Clearly $\eta_{0,n} = 0$,

$$\lim_{n\to\infty} \eta_{1,n}(b) = \eta(b),$$

$$\|\eta_{k,n}(b_1 \otimes \cdots \otimes b_k)\| \leq C_k \|b_1\| \dots \|b_k\| n^{-(k-1)/2}.$$

and

$$\|\eta_{1,n}(b)\| \leq C_1 \|b\|.$$

In view of 8.3 and 8.2 this implies that the distribution of S_n converges pointwise to the distribution of

$$\lambda^*(1) + \lambda(\eta)$$

i.e. to a B-semicircular distribution. □

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