

444

ASTÉRISQUE

2023

PARAMETRIX FOR WAVE EQUATIONS
ON A ROUGH BACKGROUND

IV

CONTROL OF THE ERROR TERM

Jérémie SZEFTTEL

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 444, 2023

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Tarifs

Vente au numéro: 60 € (\$ 90)
Abonnement Europe: 665 €, hors Europe : 718 € (\$ 1077)
Des conditions spéciales sont accordées aux membres de la SMF.

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ISSN: 0303-1179 (print) 2492-5926 (electronic)
ISBN 978-2-85629-978-4
10.24033/ast.1203

Directeur de la publication : Fabien Durand

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Texte soumis en avril 2012 et accepté en avril 2023.

Mathematical Subject Classification (2010). — 83C05, 35S30, 58J40.

Keywords. — Einstein equations, wave equation, parametrix, Fourier integral operator, rough solutions.

Mots-clefs. — Équations d'Einstein, équation des ondes, paramétrix, opérateur intégral de Fourier, solutions peu régulières.

PARAMETRIX FOR WAVE EQUATIONS ON A ROUGH BACKGROUND

IV CONTROL OF THE ERROR TERM

by Jérémie SZEFTTEL

Abstract. — This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, where \mathbf{g} is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor \mathbf{R} of \mathbf{g} is a major step of the proof of the bounded L^2 curvature conjecture proposed in [6], and solved jointly with S. Klainerman and I. Rodnianski in [11]. On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Résumé. (Parametrix pour l'équation des ondes sur un espace-temps peu régulier : IV. Contrôle du terme d'erreur) — Cet ouvrage est dédié à la construction et au contrôle d'une paramétrix pour l'équation des ondes homogène $\square_{\mathbf{g}}\phi = 0$, où \mathbf{g} est une métrique peu régulière satisfaisant les équations d'Einstein dans le vide. Le contrôle d'une telle paramétrix ainsi que du terme d'erreur associé lorsque l'on suppose seulement des bornes L^2 sur le tenseur de courbure \mathbf{R} de \mathbf{g} est une étape cruciale de la preuve de la conjecture de courbure L^2 proposée dans [6], et résolue conjointement avec S. Klainerman et I. Rodnianski dans [11]. Plus généralement, cet ouvrage concerne le contrôle de l'équation eikonale sur un espace-temps peu régulier et la dérivation de bornes L^2 pour des opérateurs intégraux de Fourier sur des variétés avec une phase et un symbole peu réguliers, et possède de ce point de vue un intérêt propre.

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CHAPTER 1

INTRODUCTION

We consider the Einstein vacuum equations,

$$(1.1) \quad \mathbf{R}_{\alpha\beta} = 0,$$

where $\mathbf{R}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathcal{M}, \mathbf{g})$. The Cauchy problem consists in finding a metric \mathbf{g} satisfying (1.1) such that the metric induced by \mathbf{g} on a given space-like hypersurface Σ_0 and the second fundamental form of Σ_0 are prescribed. The initial data then consists of a Riemannian three dimensional metric g_{ij} and a symmetric tensor k_{ij} on the space-like hypersurface $\Sigma_0 = \{t = 0\}$. Now, (1.1) is an overdetermined system and the initial data set (Σ_0, g, k) must satisfy the constraint equations

$$(1.2) \quad \begin{cases} \nabla^j k_{ij} - \nabla_i \text{Tr}k = 0, \\ R - |k|^2 + (\text{Tr}k)^2 = 0, \end{cases}$$

where the covariant derivative ∇ is defined with respect to the metric g , R is the scalar curvature of g , and $\text{Tr}k$ is the trace of k with respect to the metric g .

The fundamental problem in general relativity is to study the long term regularity and asymptotic properties of the Cauchy developments of general, asymptotically flat, initial data sets (Σ_0, g, k) . As far as local regularity is concerned it is natural to ask what are the minimal regularity properties of the initial data which guarantee the existence and uniqueness of local developments. In [11], we obtain the following result which solves bounded L^2 curvature conjecture proposed in [6]:

Theorem 1.1 (Theorem 1.10 in [11]). — *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum Equations (1.1) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t . Let $r_{\text{vol}}(\Sigma_t, 1)$ the volume radius on scales ≤ 1 of Σ_t ⁽¹⁾. Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \varepsilon, \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \text{ and } r_{\text{vol}}(\Sigma_0, 1) \geq \frac{1}{2}.$$

1. See Remark 1.5 below for a definition.

Then, there exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the following control holds on $0 \leq t \leq 1$:

$$\|\mathbf{R}\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \|k\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} \lesssim \varepsilon \text{ and } \inf_{0 \leq t \leq 1} r_{\text{vol}}(\Sigma_t, 1) \geq \frac{1}{4}.$$

Remark 1.2. — While the first nontrivial improvements for well posedness for quasi-linear hyperbolic systems (in spacetime dimensions greater than $1 + 1$), based on Strichartz estimates, were obtained in [2], [1], [19], [20], [7], [9], [12], Theorem 1.1, is the first result in which the full nonlinear structure of the quasilinear system, not just its principal part, plays a crucial role. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to its causal geometry, i.e., L^2 bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of null hypersurfaces. We refer the reader to Section 1 in [11] for more motivations and historical perspectives concerning Theorem 1.1.

Remark 1.3. — The regularity assumptions on Σ_0 in Theorem 1.1—i.e., R and ∇k bounded in $L^2(\Sigma_0)$ —correspond to an initial data set $(g, k) \in H_{\text{loc}}^2(\Sigma_0) \times H_{\text{loc}}^1(\Sigma_0)$.

Remark 1.4. — In [11], our main result is stated for corresponding large data. We then reduce the proof to the small data statement of Theorem 1.1 relying on a truncation and rescaling procedure, the control of the harmonic radius of Σ_0 based on Cheeger-Gromov convergence of Riemannian manifolds together with the assumption on the lower bound of the volume radius of Σ_0 , and the gluing procedure in [5], [4]. We refer the reader to Section 2.3 in [11] for the details.

Remark 1.5. — We recall for the convenience of the reader the definition of the volume radius of the Riemannian manifold Σ_t . Let $B_r(p)$ denote the geodesic ball of center p and radius r . The volume radius $r_{\text{vol}}(p, r)$ at a point $p \in \Sigma_t$ and scales $\leq r$ is defined by

$$r_{\text{vol}}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r'^3},$$

with $|B_r|$ the volume of B_r relative to the metric g_t on Σ_t . The volume radius $r_{\text{vol}}(\Sigma_t, r)$ of Σ_t on scales $\leq r$ is the infimum of $r_{\text{vol}}(p, r)$ over all points $p \in \Sigma_t$.

The proof of Theorem 1.1, obtained in the sequence of papers [11], [15], [16], [17], [18], [14], relies on the following ingredients⁽²⁾:

- A** Provide a system of coordinates relative to which (1.1) exhibits a null structure.
- B** Prove appropriate bilinear estimates for solutions to $\square_{\mathbf{g}} \phi = 0$, on a fixed Einstein vacuum background⁽³⁾.

2. We also need trilinear estimates and an $L^4(\mathcal{M})$ Strichartz estimate (see the introduction in [11]).

3. Note that the first bilinear estimate of this type was obtained in [8].

C Construct a parametrix for solutions to the homogeneous wave equations $\square_{\mathbf{g}}\phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature tensor is bounded in L^2 .

Steps **A** and **B** are carried out in [11]. In particular, the proof of the bilinear estimates rests on a representation formula for the solutions of the wave equation using the following plane wave parametrix ⁽⁴⁾:

$$(1.3) \quad Sf(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \quad (t, x) \in \mathcal{M},$$

where $u(\cdot, \cdot, \omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ on \mathcal{M} such that $u(0, x, \omega) \sim x \cdot \omega$ when $|x| \rightarrow +\infty$ on Σ_0 ⁽⁵⁾. Therefore, in order to complete the proof of the bounded L^2 curvature conjecture, we need to carry out step **C** with the parametrix defined in (1.3).

Remark 1.6. — Note that the parametrix (1.3) is invariantly defined ⁽⁶⁾, i.e., without reference to any coordinate system. This is crucial since coordinate systems consistent with L^2 bounds on the curvature would not be regular enough to control a parametrix.

Remark 1.7. — In addition to their relevance to the resolution of the bounded L^2 curvature conjecture, the methods and results of step **C** are also of independent interest. Indeed, they deal on the one hand with the control of the eikonal equation $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ at a critical level ⁽⁷⁾, and on the other hand with the derivation of L^2 bounds for Fourier integral operators with significantly lower differentiability assumptions both for the corresponding phase and symbol compared to classical methods (see in particular the discussion below (1.6)).

In view of the energy estimates for the wave equation, it suffices to control the parametrix at $t = 0$ (i.e., restricted to Σ_0)

$$(1.4) \quad Sf(0, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(0, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \quad x \in \Sigma_0$$

and the error term

$$(1.5) \quad Ef(t, x) = \square_{\mathbf{g}} Sf(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \square_{\mathbf{g}} u(t, x, \omega) f(\lambda \omega) \lambda^3 d\lambda d\omega, \quad (t, x) \in \mathcal{M}.$$

4. (1.3) actually corresponds to a half-wave parametrix. The full parametrix corresponds to the sum of two half-parametrix. See [16] for the construction of the full parametrix.

5. The asymptotic behavior for $u(0, x, \omega)$ when $|x| \rightarrow +\infty$ is used in [16] to generate with the parametrix any initial data set for the wave equation.

6. Our choice is reminiscent of the one used in [12] in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations. Note however that the construction in that paper is coordinate dependent.

7. We need at least L^2 bounds on the curvature to obtain a lower bound on the radius of injectivity of the null level hypersurfaces of the solution u of the eikonal equation, which in turn is necessary to control the local regularity of u (see [17]).

This requires the following ingredients, the two first being related to the control of the parametrix restricted to Σ_0 (1.4), and the two others being related to the control of the error term (1.5):

- C1** *Make an appropriate choice for the equation satisfied by $u(0, x, \omega)$ on Σ_0 , and control the geometry of the foliation generated by the level surfaces of $u(0, x, \omega)$ on Σ_0 .*
- C2** *Prove that the parametrix at $t = 0$ given by (1.4) is bounded in $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$ using the estimates for $u(0, x, \omega)$ obtained in **C1**.*
- C3** *Control the geometry of the foliation generated by the level hypersurfaces of u on \mathcal{M} .*
- C4** *Prove that the error term (1.5) satisfies the estimate $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ using the estimates for u and $\square_{\mathbf{g}}u$ proved in **C3**.*

Step **C1** has been carried out in [15], step **C2** has been carried out in [16], and step **C3** has been carried out in [17]. In the present paper, we focus on step **C4**. Note that the error term (1.5) is a Fourier integral operator (FIO) with phase $u(t, x, \omega)$ and symbol $\square_{\mathbf{g}}u(t, x, \omega)$. Now, we only assume L^2 bounds on the curvature tensor \mathbf{R} in order to be consistent with the statement of Theorem 1.1. This severely limits the regularity (t, x) we are able to obtain in step **C3** for the solution $u(t, x, \omega)$ of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on \mathcal{M} (see [17] and Section 2.4). Although \mathbf{R} does not depend on the parameter ω , the regularity in ω we are able to obtain in step **C3** for $u(t, x, \omega)$ is very limited as well⁽⁸⁾. In particular, we obtain for the symbol of E in (1.5):

$$(1.6) \quad \sup_{\omega, u} \left(\|\square_{\mathbf{g}}u\|_{L^\infty(\mathcal{H}_u)} + \|\mathbf{D}\square_{\mathbf{g}}u\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega \square_{\mathbf{g}}u\|_{L^2(\mathcal{H}_u)} \right) \lesssim \varepsilon,$$

where \mathcal{H}_u denotes the level hypersurfaces of the function $u(t, x, \omega)$. Let us note that the classical arguments for proving L^2 bounds for FIO are based either on a TT^* argument, or a T^*T argument, which requires in our setting taking at least 3 derivatives of the symbol in $L^\infty(\mathcal{M} \times \mathbb{S}^2)$ either with respect to (t, x) for T^*T , or with respect to (λ, ω) for TT^* (see for example [13]). Both methods would fail by a large margin, in particular in view of the regularity (1.6) obtained for the symbol of the error term E . In order to obtain the control required in step **C4** with the regularity of the symbol of the FIO E given by (1.6), we rely in particular on the following ingredients:

- geometric integrations by parts taking full advantage of the better regularity properties in certain directions tied to the level hypersurfaces \mathcal{H}_u of u ,
- the standard first and second dyadic decomposition in frequency and angle (see [13]),
- an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [10].

8. This is due to the fact that our estimates are sensitive to certain directions tied to the u -foliation of \mathcal{M} . Now, after differentiation with respect to ω , derivatives in “good” directions pick up a nonzero component along “bad” directions (see [17] for details).

Acknowledgments. — The author wishes to express his deepest gratitude to Sergiu Klainerman and Igor Rodnianski for stimulating discussions and constant encouragements during the long years where this work has matured. He also would like to stress that the basic strategy of the construction of the parametrix and how it fits into the whole proof of the bounded L^2 curvature conjecture has been done in collaboration with them. Finally, he would like to mention the influential work [12] providing construction and control of parametrices for $H^{2+\epsilon}$ solutions of quasilinear wave equations.

CHAPTER 2

MAIN RESULTS

The error term E in (1.5) is a Fourier integral operator on \mathcal{M} with phase $u(t, x, \omega)$ and symbol $\square_g u(t, x, \omega)$. The regularity assumptions on $u(t, x, \omega)$ will be crucial to complete step **C4**, that is prove the following estimate for the error term:

$$\|Ef\|_{L^2(\mathcal{M})} \lesssim \|\lambda f\|_{L^2(\mathbb{R}^3)}.$$

In this chapter, we state our assumptions on $u(t, x, \omega)$ before stating our main result.

2.1. Maximal foliation on \mathcal{M}

We foliate the space-time \mathcal{M} by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t . Denoting by T the unit, future oriented, normal to Σ_t and k the second fundamental form

$$(2.1) \quad k_{ij} = -\langle \mathbf{D}_i T, \partial_j \rangle$$

we find,

$$k_{ij} = -\frac{1}{2} \mathcal{L}_T g_{ij}$$

with \mathcal{L}_X denoting the Lie derivative with respect to the vectorfield X . Let $\text{Tr}(k) = g^{ij} k_{ij}$ where g is the induced metric on Σ_t and Tr is the trace. In order to be consistent with the statement of Theorem 1.1, we impose a maximal foliation

$$(2.2) \quad \text{Tr}(k) = 0.$$

We also define the lapse n as

$$(2.3) \quad n^{-1} = T(t).$$

We have (see for example [17]):

$$(2.4) \quad \mathbf{D}_T T = n^{-1} \nabla n,$$

where ∇ denotes the gradient with respect to the induced metric on Σ_t .

Finally, the lapse n satisfies the following elliptic equation on Σ_t (see [3] p. 13):

$$(2.5) \quad \Delta n = |k|^2 n,$$

where one uses (2.1), (2.4), Einstein vacuum Equations (1.1) and the fact that the foliation generated by t on \mathcal{M} is maximal (2.2).

2.2. Geometry of the foliation generated by u on \mathcal{M}

Remember that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on \mathcal{M} depending on an extra parameter $\omega \in \mathbb{S}^2$. The level hypersurfaces $u(t, x, \omega) = u$ of the optical function u are denoted by \mathcal{H}_u . Let L' denote the space-time gradient of u , i.e.:

$$(2.6) \quad L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha.$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$(2.7) \quad \mathbf{D}_{L'}L' = 0,$$

which implies that L' is the geodesic null generator of \mathcal{H}_u .

We have:

$$T(u) = \pm|\nabla u|,$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t . Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$(2.8) \quad T(u) = |\nabla u|.$$

We denote by $P_{t,u}$ the surfaces of intersection between Σ_t and \mathcal{H}_u . They play a fundamental role in our discussion.

Definition 2.1 (Canonical null pair). — *Let*

$$(2.9) \quad L = bL' = T + N, \quad \underline{L} = 2T - L = T - N,$$

where L' is the space-time gradient of u (2.6), b is the lapse of the null foliation (or shortly null lapse)

$$(2.10) \quad b^{-1} = -\langle L', T \rangle = T(u),$$

and N is a unit normal, along Σ_t , to the surfaces $P_{t,u}$. Since u satisfies the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on \mathcal{M} , this yields $L'(u) = 0$ and thus $L(u) = 0$. In view of the definition of L and (2.8), we obtain:

$$(2.11) \quad N = -\frac{\nabla u}{|\nabla u|}.$$

Definition 2.2. — *A null frame e_1, e_2, e_3, e_4 at a point $p \in P_{t,u}$ consists, in addition to the null pair $e_3 = \underline{L}, e_4 = L$, of arbitrary orthonormal vectors e_1, e_2 tangent to $P_{t,u}$.*

Definition 2.3 (Ricci coefficients). — Let e_1, e_2, e_3, e_4 be a null frame on $P_{t,u}$ as above. The following tensors on $P_{t,u}$

$$(2.12) \quad \begin{aligned} \chi_{AB} &= \langle \mathbf{D}_A e_4, e_B \rangle, & \underline{\chi}_{AB} &= \langle \mathbf{D}_A e_3, e_B \rangle, \\ \zeta_A &= \frac{1}{2} \langle \mathbf{D}_3 e_4, e_A \rangle, & \underline{\zeta}_A &= \frac{1}{2} \langle \mathbf{D}_4 e_3, e_A \rangle, \\ \xi_A &= \frac{1}{2} \langle \mathbf{D}_3 e_3, e_A \rangle \end{aligned}$$

are called the Ricci coefficients associated to our canonical null pair.

We decompose χ and $\underline{\chi}$ into their trace and traceless components.

$$(2.13) \quad \text{tr} \chi = \mathbf{g}^{AB} \chi_{AB}, \quad \text{tr} \underline{\chi} = \mathbf{g}^{AB} \underline{\chi}_{AB},$$

$$(2.14) \quad \widehat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} \text{tr} \chi \mathbf{g}_{AB}, \quad \widehat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2} \text{tr} \underline{\chi} \mathbf{g}_{AB}.$$

Observe that all tensors defined above are $P_{t,u}$ -tangent.

Definition 2.4. — We decompose the symmetric traceless 2 tensor k into the scalar δ , the $P_{t,u}$ -tangent 1-form ϵ , and the $P_{t,u}$ -tangent symmetric 2-tensor η as follows:

$$(2.15) \quad \begin{cases} k_{NN} = \delta, \\ k_{AN} = \epsilon_A, \\ k_{AB} = \eta_{AB}. \end{cases}$$

Note that $\text{Tr}(k) = \text{tr}(\eta) + \delta$ which together with the maximal foliation assumption (2.2) yields:

$$(2.16) \quad \text{tr}(\eta) = -\delta.$$

The following Ricci equations can be easily derived from the properties of T (2.1) (2.4), the fact that L' is geodesic (2.7), and the Definition (2.12) of the Ricci coefficients (see [3] p. 171):

$$(2.17) \quad \begin{aligned} \mathbf{D}_A e_4 &= \chi_{AB} e_B - \epsilon_A e_4, & \mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + \epsilon_A e_3, \\ \mathbf{D}_4 e_4 &= -\bar{\delta} e_4, & \mathbf{D}_4 e_3 &= 2\underline{\zeta}_A e_A + \bar{\delta} e_3, \\ \mathbf{D}_3 e_4 &= 2\zeta_A e_A + (\delta + n^{-1} \nabla_N n) e_4, & \mathbf{D}_3 e_3 &= 2\underline{\xi}_A e_A - (\delta + n^{-1} \nabla_N n) e_3, \\ \mathbf{D}_4 e_A &= \nabla_4 e_A + \zeta_A e_4, & \mathbf{D}_3 e_A &= \nabla_3 e_A + \zeta_A e_3 + \xi_A e_4, \\ \mathbf{D}_B e_A &= \nabla_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4, \end{aligned}$$

where ∇_3, ∇_4 denote the projection on $P_{t,u}$ of \mathbf{D}_3 and \mathbf{D}_4 , ∇ denotes the induced covariant derivative on $P_{t,u}$ and $\bar{\delta}, \bar{\epsilon}_A$ are defined by:

$$(2.18) \quad \bar{\delta} = \delta - n^{-1} N(n), \quad \bar{\epsilon}_A = \epsilon_A - n^{-1} \nabla_A n.$$

Also,

$$(2.19) \quad \begin{aligned} \underline{\chi}_{AB} &= -\chi_{AB} - 2k_{AB}, \\ \underline{\zeta}_A &= -\bar{\epsilon}_A, \\ \underline{\xi}_A &= \epsilon_A + n^{-1}\nabla_A n - \zeta_A. \end{aligned}$$

Let θ is the second fundamental form of $P_{t,u}$ in Σ_t . Since $L = T + N$, θ is connected to the second fundamental form k of Σ_t and the null second fundamental form χ of $P_{t,u}$ through the formula:

$$(2.20) \quad \theta_{AB} = \chi_{AB} + \eta_{AB}.$$

In view of the Ricci Equations (2.17), we have:

$$(2.21) \quad \begin{cases} \nabla_A N = \theta_{AB} e_B, \\ \nabla_N N = -b^{-1}\nabla b. \end{cases}$$

Recall that $\text{tr}\chi$ satisfies a transport equation called the Raychaudhuri equation:

$$(2.22) \quad L(\text{tr}\chi) + \frac{1}{2}(\text{tr}\chi)^2 = -|\widehat{\chi}|^2 - \bar{\delta}\text{tr}\chi.$$

We also recall the transport equation satisfied by the null lapse b :

$$(2.23) \quad L(b) = -\bar{\delta}b.$$

The following lemma will allow us to identify the symbol $\square_{\mathbf{g}}u$ of the error term (1.5):

Lemma 2.5. — *For any scalar function ϕ on \mathcal{M} , we have:*

$$(2.24) \quad \square_{\mathbf{g}}\phi = -\underline{L}(L(\phi)) + \underline{\Delta}\phi + 2\zeta \cdot \nabla\phi + (\delta + n^{-1}\nabla_N n)L(\phi) + \frac{1}{2}\text{tr}\chi\underline{L}(\phi) + \frac{1}{2}\text{tr}\underline{\chi}L(\phi).$$

Proof. — We have:

$$(2.25) \quad \square_{\mathbf{g}}\phi = \mathbf{D}^\alpha \mathbf{D}_\alpha \phi = -\mathbf{D}^2 \phi(L, \underline{L}) + \mathbf{D}^A \mathbf{D}_A \phi.$$

Now, using the Ricci Equations (2.17), we obtain:

$$\begin{aligned} \mathbf{D}_A \mathbf{D}_B \phi &= e_A(e_B(\phi)) - \mathbf{D}_{\mathbf{D}_A e_B} \phi \\ &= e_A(e_B(\phi)) - \nabla_{\nabla_A e_B} \phi - \frac{1}{2}\chi_{AB} \underline{L}(\phi) - \frac{1}{2}\underline{\chi}_{AB} L(\phi) \\ &= \nabla_A \nabla_B \phi - \frac{1}{2}\chi_{AB} \underline{L}(\phi) - \frac{1}{2}\underline{\chi}_{AB} L(\phi). \end{aligned}$$

Taking the trace, this yields:

$$(2.26) \quad \mathbf{D}^A \mathbf{D}_A \phi = \underline{\Delta}\phi - \frac{1}{2}\text{tr}\chi\underline{L}(\phi) - \frac{1}{2}\text{tr}\underline{\chi}L(\phi).$$

Using again the Ricci Equations (2.17), we also have:

$$(2.27) \quad \mathbf{D}^2 u(L, \underline{L})\phi = \underline{L}(L(\phi)) - \mathbf{D}_{\mathbf{D}_L} \phi = \underline{L}(L(\phi)) - 2\zeta \cdot \nabla\phi - (\delta + n^{-1}\nabla_N n)L(\phi).$$

Finally, (2.25), (2.26) and (2.27) yield the conclusion of the lemma. \square

We conclude this section with the identification of the symbol $\square_{\mathbf{g}}u$ of the error term (1.5). In view of (2.9), (2.10), the fact that $e_A, A = 1, 2$ are tangent to $P_{t,u}$, and the fact that $\underline{\Delta}$ is the Laplace-Beltrami on $P_{t,u}$, we have:

$$L(u) = 0, e_A(u) = 0, A = 1, 2, \underline{\Delta}(u) = 0, \text{ and } \underline{L}(u) = 2b^{-1}.$$

Together with (2.24), this yields:

$$(2.28) \quad \square_{\mathbf{g}}u = b^{-1}\text{tr}\chi.$$

Thus, we may rewrite the error term E as:

$$(2.29) \quad Ef(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} b^{-1}(t, x, \omega) \text{tr}\chi(t, x, \omega) f(\lambda\omega) \lambda^3 d\lambda d\omega.$$

2.3. Commutation formulas

From the Ricci Equations (2.17), we immediately deduce the following four useful commutation formulas:

Lemma 2.6. — *Let f a scalar function on \mathcal{M} . Then,*

$$(2.30) \quad \nabla_B \nabla_4 f - \nabla_4 \nabla_B f = \chi_{BC} \nabla_C f - n^{-1} \nabla_B n \nabla_4 f,$$

$$(2.31) \quad \nabla_B \nabla_3 f - \nabla_3 \nabla_B f = \underline{\chi}_{BC} \nabla_C f - \underline{\xi}_B \nabla_4 f - b^{-1} \nabla_B b \nabla_3 f,$$

$$(2.32) \quad [\underline{L}, L]f = -\bar{\delta} \nabla_3 f + (\delta + n^{-1} \nabla_N n) \nabla_4 f + 2(\zeta_B - \underline{\zeta}_B) \nabla_B f.$$

Finally, (2.30), (2.31) together with the fact that $N = \frac{1}{2}(L - \underline{L})$ yield:

$$(2.33) \quad \nabla_B \nabla_N f - \nabla_N \nabla_B f = (\chi_{BC} + k_{BC}) \nabla_C f - b^{-1} \nabla_B b \nabla_N f.$$

For some applications we have in mind, we would like to get rid of the term containing a ∇_4 derivative in the Right-hand side of (2.30). This is achieved by considering the commutator $[\nabla, \nabla_{nL}]$ instead of $[\nabla, \nabla_4]$:

$$(2.34) \quad \nabla_B \nabla_{nL} f - \nabla_{nL} \nabla_B f = n \chi_{BC} \nabla_C \Pi_A.$$

Also, we would like to get rid of the term containing a ∇_N derivative in the right-hand side of (2.33). This is achieved by considering the commutator $[\nabla, \nabla_{bN}]$ instead of $[\nabla, \nabla_N]$:

$$(2.35) \quad \nabla_B \nabla_{bN} f - \nabla_{bN} \nabla_B f = b(\chi_{BC} + k_{BC}) \nabla_C f.$$

2.4. Regularity assumptions on the phase $u(t, x, \omega)$

We define some norms on \mathcal{H} . For any $1 \leq p \leq +\infty$ and for any tensor F on \mathcal{H}_u , we have:

$$\|F\|_{L^p(\mathcal{H}_u)} = \left(\int_0^1 dt \int_{P_{t,u}} |F|^p d\mu_{t,u} \right)^{\frac{1}{p}},$$

where $d\mu_{t,u}$ denotes the area element of $P_{t,u}$. We also introduce the following norms:

$$\begin{aligned}\mathcal{N}_1(F) &= \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla F\|_{L^2(\mathcal{H}_u)} + \|\nabla_L F\|_{L^2(\mathcal{H}_u)}, \\ \mathcal{N}_2(F) &= \mathcal{N}_1(F) + \|\nabla^2 F\|_{L^2(\mathcal{H}_u)} + \|\nabla \nabla_L F\|_{L^2(\mathcal{H}_u)}.\end{aligned}$$

Let x' a coordinate system on $P_{0,u}$. By transporting this coordinate system along the null geodesics generated by L , we obtain a coordinate system (t, x') of \mathcal{H} . We define the following norms:

$$\begin{aligned}\|F\|_{L_x^\infty L_t^2} &= \sup_{x' \in P_{0,u}} \left(\int_0^1 |F(t, x')|^2 dt \right)^{\frac{1}{2}}, \\ \|F\|_{L_x^2 L_t^\infty} &= \left\| \sup_{0 \leq t \leq 1} |F(t, x')| \right\|_{L^2(P_{0,u})}.\end{aligned}$$

We now state our assumptions for the phase $u(t, x, \omega)$ and the symbol $b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega)$ of the error term E which is given by (2.29). These assumptions are compatible with the regularity obtained for the functions $u(t, x, \omega)$ constructed in [17] (this construction corresponds to step **C3**). The constant $\varepsilon > 0$ below is the one appearing in the statement of Theorem 1.1. In particular, it satisfies $0 < \varepsilon < 1$ and is small.

Assumption 1 (regularity with respect to (t, x)):

$$(2.36) \quad \|n - 1\|_{L^\infty(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty(\mathcal{H}_u)} + \|\nabla^2 n\|_{L_t^\infty L_x^2} + \|\nabla T(n)\|_{L_t^\infty L_x^2} \lesssim \varepsilon,$$

$$(2.37) \quad \mathcal{N}_1(k) + \|\nabla_L \epsilon\|_{L^2(\mathcal{H}_u)} + \|\underline{L}(\delta)\|_{L^2(\mathcal{H}_u)} + \|\epsilon\|_{L_x^\infty L_t^2} + \|\delta\|_{L_x^\infty L_t^2} \lesssim \varepsilon,$$

$$(2.38) \quad \|b - 1\|_{L^\infty(\mathcal{H}_u)} + \mathcal{N}_2(b) + \|L(b)\|_{L_x^2 L_t^\infty} + \|\nabla(b)\|_{L_x^2 L_t^\infty} + \|\underline{L}(b)\|_{L_x^4 L_t^\infty} \lesssim \varepsilon,$$

$$(2.39) \quad \|\text{tr} \chi\|_{L^\infty(\mathcal{H}_u)} + \|\nabla \text{tr} \chi\|_{L_x^2 L_t^\infty} + \|\underline{L} \text{tr} \chi\|_{L_x^2 L_t^\infty} \lesssim \varepsilon,$$

$$(2.40) \quad \|\widehat{\chi}\|_{L_x^2 L_t^\infty} + \mathcal{N}_1(\widehat{\chi}) + \|\nabla \underline{L} \widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon,$$

$$(2.41) \quad \|\zeta\|_{L_x^2 L_t^\infty} + \mathcal{N}_1(\zeta) \lesssim \varepsilon.$$

Assumption 2 (regularity with respect to ω):

$$(2.42) \quad \|\partial_\omega N\|_{L^\infty(\mathcal{H}_u)} \lesssim 1,$$

$$(2.43) \quad \| |N(x, \omega) - N(x, \omega')| - |\omega - \omega'| \| \lesssim \varepsilon |\omega - \omega'|, \forall x \in \sigma, \omega, \omega' \in \mathbb{S}^2,$$

$$(2.44) \quad \|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)} + \|\partial_\omega \chi\|_{L_x^2 L_t^\infty} \lesssim \varepsilon.$$

Furthermore, we have the following decomposition for $\widehat{\chi}$:

$$(2.45) \quad \widehat{\chi} = \chi_1 + \chi_2,$$

where χ_1 and χ_2 are two symmetric traceless $P_{t,u}$ -tangent 2-tensors satisfying:

$$(2.46) \quad \mathcal{N}_1(\chi_1) + \|\partial_\omega \chi_1\|_{L_t^\infty L_x^2} + \mathcal{N}_1(\chi_2) + \|\chi_2\|_{L_x^\infty L_t^2} + \|\partial_\omega \chi_2\|_{L_t^\infty L_x^2} \lesssim \varepsilon$$

and for any $2 \leq p < +\infty$, we have:

$$(2.47) \quad \|\chi_1\|_{L_t^p L_x^\infty} + \|\partial_\omega \chi_2\|_{L^{6-}(\mathcal{H}_u)} \lesssim \varepsilon.$$

Assumption 3 (additional regularity with respect to x):

We introduce the family of intrinsic Littlewood-Paley projections P_j which have been constructed in [10] using the heat flow on the 2-surfaces $P_{t,u}$ (see Section 2.6). There exists a function μ in $L^2(\mathbb{R})$ satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \leq 1,$$

such that for all $j \geq 0$, we have:

$$(2.48) \quad \|\nabla_N P_j \nabla_N \text{tr} \chi\|_{L^2(\mathcal{H}_u)} \lesssim 2^j \varepsilon + 2^{\frac{j}{2}} \varepsilon \mu(u).$$

Remark 2.7. — In Assumptions 1-3, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω . Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.

Remark 2.8. — The fact that we may take a small constant $\varepsilon > 0$ in Assumptions 1-3 is directly related to the conclusions of Theorem 1.1.

Remark 2.9. — In the flat case, we have $\mathcal{M} = (\mathbb{R}^{1+3}, \mathbf{m})$, where \mathbf{m} is the Minkowski metric, $u(t, x, \omega) = t + x \cdot \omega$, $b = 1$, $N = -\omega$, $L = \partial_t - \omega \cdot \partial_x$, $\underline{L} = \partial_t + \omega \cdot \partial_x$, and $\chi = \underline{\chi} = \zeta = \underline{\zeta} = \xi = \underline{\xi} = k = 0$. Thus, Assumptions 1-3 are clearly satisfied with $\varepsilon = 0$.

2.5. Estimates on $P_{t,u}$ and \mathcal{M}

In this section, we state the embeddings on $P_{t,u}$ and \mathcal{M} that will be needed for the proof of the main theorem. We refer to Section 3 in [17], as well as [10] for (2.49), for their proof within the regularity assumptions of Section 2.4.

We have the following Gagliardo-Nirenberg inequality on $P_{t,u}$ (see [10]). For an arbitrary tensorfield F on $P_{t,u}$ and any $2 \leq p < \infty$, we have:

$$(2.49) \quad \|F\|_{L^p(P_{t,u})} \lesssim \|\nabla F\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} \|F\|_{L^2(P_{t,u})}^{\frac{2}{p}} + \|F\|_{L^2(P_{t,u})}.$$

We have the classical Sobolev inequality on \mathcal{H} (see [17]):

Lemma 2.10. — *For any tensor F on \mathcal{H}_u , we have:*

$$(2.50) \quad \|F\|_{L^6(\mathcal{H}_u)} \lesssim \mathcal{N}_1(F),$$

and

$$(2.51) \quad \|F\|_{L_t^\infty L_x^4} \lesssim \mathcal{N}_1(F).$$

On \mathcal{H}_u , we also have the following estimate of the $L_t^\infty L_x^2$ norm for any tensor F on \mathcal{H}_u :

$$(2.52) \quad \|F\|_{L_x^2, L_t^\infty}^2 \lesssim \int_{\mathcal{H}_u} |F| |\mathbf{D}_L F| dt d\mu_{t,u} + \|F\|_{L^2(\mathcal{H}_u)}^2.$$

The following lemma will be useful to estimate transport equations (see [17] for a proof).

Lemma 2.11. — *Let W and F two $P_{t,u}$ -tangent tensors such that $\nabla_L W = F$. Then, for any $p \geq 1$, we have:*

$$(2.53) \quad \|W\|_{L_x^p, L_t^\infty} \lesssim \|W(0)\|_{L^p(P_{0,u})} + \|F\|_{L_x^p, L_t^1}.$$

Finally, we have the Sobolev embedding on \mathcal{M} . Given an arbitrary tensorfield F on \mathcal{M} , we have (see [17])

$$(2.54) \quad \|F\|_{L^4(\mathcal{M})} \lesssim \|\mathbf{D}F\|_{L^2(\mathcal{M})}.$$

2.6. Geometric Littlewood-Paley projections on $P_{t,u}$

In [10], Littlewood-Paley projections have been constructed relying on the heat flow on 2-surfaces, within our low regularity assumptions. They recover the basic properties of the standard Littlewood-Paley projections. We denote by P_j such a Littlewood-Paley projection on the 2-surface $P_{t,u}$. In particular, we have from [10]

$$(2.55) \quad \sum_j P_j = I.$$

Also, the following properties of the LP-projections P_j have been proved in [10]:

Theorem 2.12. — *The LP-projections P_j verify the following properties:*

i) L^p -boundedness For any $1 \leq p \leq \infty$, and any interval $I \subset \mathbb{Z}$,

$$(2.56) \quad \|P_I F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})}.$$

ii) Bessel inequality

$$\sum_j \|P_j F\|_{L^2(P_{t,u})}^2 \lesssim \|F\|_{L^2(P_{t,u})}^2.$$

iii) Finite band property For any $1 \leq p \leq \infty$.

$$(2.57) \quad \begin{aligned} \|\Delta P_j F\|_{L^p(P_{t,u})} &\lesssim 2^{2j} \|F\|_{L^p(P_{t,u})}, \\ \|P_j F\|_{L^p(P_{t,u})} &\lesssim 2^{-2j} \|\Delta F\|_{L^p(P_{t,u})}. \end{aligned}$$

In addition, the L^2 estimates

$$(2.58) \quad \begin{aligned} \|\nabla P_j F\|_{L^2(P_{t,u})} &\lesssim 2^j \|F\|_{L^2(P_{t,u})}, \\ \|P_j F\|_{L^2(P_{t,u})} &\lesssim 2^{-j} \|\nabla F\|_{L^2(P_{t,u})} \end{aligned}$$

hold together with the dual estimate

$$\|P_j \nabla F\|_{L^2(P_{t,u})} \lesssim 2^j \|F\|_{L^2(P_{t,u})}.$$

iv) Weak Bernstein inequality For any $2 \leq p < \infty$

$$\|P_j F\|_{L^p(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^2(P_{t,u})},$$

$$\|P_{<0} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})},$$

together with the dual estimates

$$\|P_j F\|_{L^2(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^{p'}(P_{t,u})},$$

$$\|P_{<0} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^{p'}(P_{t,u})}.$$

We also have the following sharp Bernstein inequality for a scalar function f on $P_{t,u}$:

$$(2.59) \quad \|P_j f\|_{L^\infty(P_{t,u})} \lesssim 2^j \|f\|_{L^2(P_{t,u})},$$

$$(2.60) \quad \|P_{<0} f\|_{L^\infty(P_{t,u})} \lesssim \|f\|_{L^2(P_{t,u})}$$

and the following Bochner inequality:

$$(2.61) \quad \int_{P_{t,u}} |\nabla^2 f|^2 \lesssim \int_{P_{t,u}} |\Delta f|^2 + \varepsilon \int_{P_{t,u}} |\nabla f|^2.$$

There is an equivalent of (2.61) for tensors, which yields the following consequence. There exists a function μ in $L^2(\mathbb{R})$ satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \lesssim 1,$$

such that for any scalar function f on $P_{t,u}$, we have:

$$(2.62) \quad \|\nabla^3 f\|_{L^2(P_{t,u})} \lesssim \|\nabla \Delta f\|_{L^2(P_{t,u})} + \mu(t)\varepsilon \|\Delta f\|_{L^2(P_{t,u})} + \mu^2(t)\varepsilon \|\nabla f\|_{L^2(P_{t,u})}.$$

Finally, we have the following lemma (see Lemma 5.10 in [17]):

Lemma 2.13. — For any 1-form F on $P_{t,u}$, for any $1 < p \leq 2$ and for all $j \geq 0$, we have:

$$(2.63) \quad \|P_j \operatorname{div}(F)\|_{L^2(P_{t,u})} \lesssim 2^{\frac{2}{p}j} \|F\|_{L^p(P_{t,u})}.$$

2.7. Commutator estimates

In this section, we state the commutator estimates, as well as two additional estimates for $\operatorname{tr}\chi$, that will be needed for the proof of the main theorem. We refer to Section 9 in [17] for their proof within the regularity assumptions of Section 2.4.

Let f a scalar function on \mathcal{M} . Then, we have the following commutator estimates:

$$(2.64) \quad \|[bN, P_j]f\|_{L^2(\mathcal{H}_u)} + 2^{-j} \|\nabla [bN, P_j]f\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon \mathcal{N}_1(f)$$

and

$$(2.65) \quad \|[nL, P_j]f\|_{L^2(\mathcal{H}_u)} + 2^{-j} \|\nabla [nL, P_j]f\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon \mathcal{N}_1(f).$$

We also have the following commutator estimates acting on $\text{tr}\chi$.

$$(2.66) \quad 2^j \|[nL, P_j]\text{tr}\chi\|_{L_t^1 L_x^2} + \|\nabla[nL, P_j]\text{tr}\chi\|_{L_t^1 L_x^2} \lesssim \varepsilon,$$

$$(2.67) \quad 2^{\frac{j}{2}} \|[nL, P_j]\text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + 2^{-\frac{j}{2}} \|\nabla[nL, P_j]\text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim \varepsilon,$$

and

$$(2.68) \quad 2^j \|[bN, P_j]\text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + \|\nabla[bN, P_j]\text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim 2^{\frac{j}{2}} \varepsilon.$$

Finally, we have the following estimate for $P_m \text{tr}\chi$:

$$(2.69) \quad \|P_m \text{tr}\chi\|_{L_{x'}^2 L_t^\infty} + \|P_m(nL \text{tr}\chi)\|_{L_{x'}^2 L_t^1} \lesssim 2^{-m} \varepsilon,$$

and the following estimate for $P_{\leq m} \text{tr}\chi$:

$$(2.70) \quad \|\nabla P_{\leq m} \text{tr}\chi\|_{L_{x'}^2 L_t^\infty} + \|\nabla(P_{\leq m}(nL \text{tr}\chi))\|_{L_{x'}^2 L_t^1} \lesssim \varepsilon.$$

2.8. Dependance of the norm $L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})$ on $\omega \in \mathbb{S}^2$

Let ω and ν in \mathbb{S}^2 such that

$$|\omega - \nu| \lesssim 2^{\frac{j}{2}}.$$

Let $u = u(\cdot, \omega)$ and $u_\nu = u(\cdot, \nu)$. In this section, we morally evaluate the norm in $L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})$ of the difference between various scalars and tensors evaluated at ω and their corresponding evaluation at ν . Consider a FIO where the integration in ω is localized in a patch of size $2^{-\frac{j}{2}}$ and of center ν . This will be used to morally replace the symbol of this FIO depending on ω by its value at the middle of the patch ν , so that one may take the symbol outside of the integral in ω . The following decompositions are proved in Section 8 of [17] within the regularity assumptions of Section 2.4.

— We have the following decomposition for $N(\cdot, \omega) - N(\cdot, \nu)$:

$$(2.71) \quad 2^{\frac{j}{2}}(N(\cdot, \omega) - N(\cdot, \nu)) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$\|F_1^j\|_{L^\infty} \lesssim 1$$

and where the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim 2^{-\frac{j}{2}}.$$

— We have following decomposition for $\text{tr}\chi$:

$$(2.72) \quad \text{tr}\chi(\cdot, \omega) = f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon$$

and where the scalar f_2^j satisfies:

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

— Let $p \in \mathbb{Z}$. We have following estimate for b^p :

$$(2.73) \quad \|b^p(\cdot, \omega) - b^p(\cdot, \nu)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon |\omega - \nu|.$$

— We have following decomposition for $\widehat{\chi}$:

$$(2.74) \quad \widehat{\chi}(\cdot, \omega) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$\|F_1^j\|_{L_{u\nu}^\infty, x', L_t^2} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

— We have following decomposition for χ_2 :

$$(2.75) \quad \|\chi_2(\cdot, \omega) - \chi_2(\cdot, \nu)\|_{L_u^\infty L^{4-}(\mathcal{H}_u)} \lesssim \varepsilon |\omega - \nu|.$$

— We have following decomposition for χ :

$$(2.76) \quad \chi(\cdot, \omega) = \chi_2(\cdot, \nu) + F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L_{u\nu}^\infty L_t^p L_{x'}^\infty} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

— We have following decomposition for $|\widehat{\chi}|^2$:

$$(2.77) \quad |\widehat{\chi}|^2(\cdot, \omega) = |\chi_2(\cdot, \nu)|^2 + \chi_2(\cdot, \nu) \cdot F_1^j + \chi_2(\cdot, \nu) \cdot F_2^j + f_3^j + f_4^j + f_5^j,$$

where the tensor F_1^j and the scalar f_3^j only depends on ν and satisfy:

$$\|F_1^j\|_{L_{u\nu}^\infty L_t^2 L^\infty(P_{t, u\nu})} + \|f_3^j\|_{L_{u\nu}^\infty L_t^2 L^\infty(P_{t, u\nu})} \lesssim \varepsilon,$$

where the tensor F_2^j , and the scalar f_4^j satisfy:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|f_4^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}$$

and where the scalar f_5^j satisfies:

$$\|f_5^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}.$$

— We have the following decomposition for $\widehat{\chi}(\cdot, \omega)^3$:

$$(2.78) \quad \begin{aligned} \widehat{\chi}(\cdot, \omega)^3 &= \chi_2(\cdot, \nu)^3 + \chi_2(\cdot, \nu)^2 F_1^j + \chi_2(\cdot, \nu)^2 F_2^j + \chi_2(\cdot, \nu) F_3^j \\ &\quad + \chi_2(\cdot, \nu) F_4^j + \chi_2(\cdot, \nu) F_5^j + F_6^j + F_7^j + F_8^j + F_9^j, \end{aligned}$$

where F_1^j , F_3^j and F_6^j do not depend on ω and satisfy:

$$\|F_1^j\|_{L_{u,\nu}^\infty L_t^2 L^\infty(P_{t,u\nu})} + \|F_3^j\|_{L_{u,\nu}^\infty L_t^2 L^\infty(P_{t,u\nu})} + \|F_6^j\|_{L_{u,\nu}^\infty L_t^2 L^\infty(P_{t,u\nu})} \lesssim \varepsilon,$$

where F_2^j , F_4^j and F_7^j satisfy:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|F_4^j\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|F_7^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} \varepsilon,$$

where F_5^j and F_8^j satisfy

$$\|F_5^j\|_{L^2(\mathcal{M})} + \|F_8^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}$$

and where F_9^j satisfies

$$\|F_9^j\|_{L^{2-}(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{3j}{2}}.$$

— We have the following decomposition for $b(\cdot, \omega) - b(\cdot, \nu)$:

$$(2.79) \quad 2^{\frac{j}{2}}(b(\cdot, \omega) - b(\cdot, \nu)) = f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon$$

and where the scalar f_2^j satisfies:

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}} \varepsilon.$$

— We have following decomposition for ζ and $\nabla(b)$:

$$(2.80) \quad \zeta(\cdot, \omega), \nabla(b)(\cdot, \omega) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L_{u,\nu}^\infty L_t^2 L_{x_t'}^p} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{4}}.$$

Remark 2.14. — Let us give some insight on these decompositions by considering the particular example of the decomposition for $\text{tr}\chi$ (2.72). A naive approach consists in writing the following decomposition

$$\text{tr}\chi(t, x, \omega) = \text{tr}\chi(t, x, \nu) + (\text{tr}\chi(t, x, \omega) - \text{tr}\chi(t, x, \nu)) = f_1^j + f_2^j.$$

f_1^j does not depend on ω and satisfies, in view of the estimate (2.39)

$$\|f_1^j\|_{L^\infty} \lesssim \|\text{tr}\chi(\cdot, \nu)\|_{L^\infty} \lesssim \varepsilon.$$

Also, we have

$$f_2^j = (\omega - \nu) \int_0^1 \partial_\omega \text{tr}\chi(t, x, \omega_\sigma) d\sigma,$$

which together with the fact that $|\omega - \nu| \lesssim 2^{-\frac{j}{2}}$ yields

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{E}\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} \left\| \int_0^1 \partial_\omega \text{tr}\chi(t, x, \omega_\sigma) d\sigma \right\|_{L_u^\infty L^2(\mathcal{E}\mathcal{H}_u)}.$$

Unfortunately, we can not obtain the desired estimate for f_2^j since we have $\partial_\omega \text{tr}\chi(\cdot, \omega_\sigma) \in L_{u_\sigma}^\infty L^2(\mathcal{E}\mathcal{H}_{u_\sigma})$, and $L_u^\infty L^2(\mathcal{E}\mathcal{H}_u)$ and $L_{u_\sigma}^\infty L^2(\mathcal{E}\mathcal{H}_{u_\sigma})$ are not directly comparable. Nevertheless, in Section 8 of [17], we are able to improve on this naive approach, in order to obtain the above decompositions within the regularity assumptions of Section 2.4.

2.9. The boundedness of the error term

The main result of this paper is the following L^2 bound on the error term (2.29). It achieves step C4 and therefore, together with the results in [15], [16], [17] completes step C.

Theorem 2.15. — *Let u be a function on $\mathcal{M} \times \mathbb{S}^2$ satisfying Assumption 1, Assumption 2 and Assumption 3, as well as the assumptions of Sections 2.5-2.8. Let E the Fourier integral operator with phase $u(t, x, \omega)$ and symbol $b^{-1}(t, x, \omega)\text{tr}\chi(t, x, \omega)$:*

$$(2.81) \quad Ef(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} b^{-1}(t, x, \omega) \text{tr}\chi(t, x, \omega) f(\lambda\omega) \lambda^3 d\lambda d\omega.$$

Then, E satisfies the estimate:

$$(2.82) \quad \|Ef\|_{L^2(\mathcal{M})} \lesssim \varepsilon \|\lambda f\|_{L^2(\mathbb{R}^3)}.$$

CHAPTER 3

PROOF OF THEOREM 2.15 (CONTROL OF THE ERROR TERM)

3.1. The basic computation

We start the proof of Theorem 2.15 with the following instructive computation:

(3.1)

$$\begin{aligned} \|Ef\|_{L^2(\mathcal{M})} &\leq \int_{\mathbb{S}^2} \left\| b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega) \int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L^2(\mathcal{M})} d\omega \\ &\leq \int_{\mathbb{S}^2} \|b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega)\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L_u^2} d\omega \\ &\leq \varepsilon \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where we have used Plancherel with respect to λ , Cauchy-Schwarz with respect to ω , and the estimates (2.38) for b and (2.39) for $\operatorname{tr} \chi$. (3.1) misses the conclusion (2.82) of Theorem 2.15 by a power of λ . Now, assume for a moment that we may replace a power of λ by a derivative on $b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega)$. Then, the same computation yields:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \int_0^{+\infty} \mathbf{D}(b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega)) e^{i\lambda u} f(\lambda \omega) \lambda d\lambda d\omega \right\|_{L^2(\mathcal{M})} \\ (3.2) \quad &\leq \int_{\mathbb{S}^2} \|\mathbf{D}(b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega))\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L_u^2} d\omega \\ &\leq \varepsilon \|\lambda f\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

which is (2.82). This suggests a strategy which consists in making integrations by parts to trade powers of λ against derivatives of the symbol $b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega)$.

3.2. Structure of the proof of Theorem 2.15

The proof of Theorem 2.15 proceeds in three steps. We first localize in frequencies of size $\lambda \sim 2^j$. We then localize the angle ω in patches on the sphere \mathbb{S}^2 of diameter $2^{-j/2}$. Finally, we estimate the diagonal terms.

3.2.1. Step 1: decomposition in frequency. — For the first step, we introduce φ and ψ two smooth compactly supported functions on \mathbb{R} such that:

$$(3.3) \quad \varphi(\lambda) + \sum_{j \geq 0} \psi(2^{-j}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}.$$

We use (3.3) to decompose Ef as follows:

$$(3.4) \quad Ef(t, x) = \sum_{j \geq -1} E_j f(t, x),$$

where for $j \geq 0$:

$$(3.5) \quad E_j f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t, x, \omega)^{-1} \text{tr} \chi(t, x, \omega) \psi(2^{-j}\lambda) f(\lambda \omega) \lambda^2 d\lambda d\omega,$$

and

$$(3.6) \quad E_{-1} f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t, x, \omega)^{-1} \text{tr} \chi(t, x, \omega) \varphi(\lambda) f(\lambda \omega) \lambda^2 d\lambda d\omega.$$

This decomposition is classical and is known as the first dyadic decomposition (see [13]). The goal of this first step is to prove the following proposition:

Proposition 3.1. — *The decomposition (3.4) satisfies an almost orthogonality property:*

$$(3.7) \quad \|Ef\|_{L^2(\mathcal{M})}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.$$

The proof of Proposition 3.1 is postponed to Chapter 4.

3.2.2. Step 2: decomposition in angle. — Proposition 3.1 allows us to estimate $\|E_j f\|_{L^2(\mathcal{M})}$ instead of $\|Ef\|_{L^2(\mathcal{M})}$. The analog of computation (3.1) for $\|E_j f\|_{L^2(\mathcal{M})}$ yields:

$$(3.8) \quad \|E_j f\|_{L^2(\mathcal{M})} \leq \varepsilon \|\lambda \psi(2^{-j}\lambda) f\|_{L^2(\sigma)} \lesssim \varepsilon 2^j \|\psi(2^{-j}\lambda) f\|_{L^2(\mathbb{R}^3)},$$

which misses the wanted estimate by a power of 2^j . We thus need to perform a second dyadic decomposition (see [13]). We introduce a smooth partition of unity on the sphere \mathbb{S}^2 :

$$(3.9) \quad \sum_{\nu \in \Gamma} \eta_\nu^\nu(\omega) = 1 \text{ for all } \omega \in \mathbb{S}^2,$$

where the support of η_ν^ν is a patch on \mathbb{S}^2 of diameter $\sim 2^{-j/2}$. We use (3.9) to decompose $E_j f$ as follows:

$$(3.10) \quad E_j f(t, x) = \sum_{\nu \in \Gamma} E_j^\nu f(t, x),$$

where:

$$(3.11) \quad E_j^\nu f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t, x, \omega)^{-1} \text{tr} \chi(t, x, \omega) \psi(2^{-j}\lambda) \eta_\nu^\nu(\omega) f(\lambda \omega) \lambda^2 d\lambda d\omega.$$

We also define:

$$(3.12) \quad \begin{aligned} \gamma_{-1} &= \|\varphi(\lambda)f\|_{L^2(\mathbb{R}^3)}, & \gamma_j &= \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}, \quad j \geq 0, \\ \gamma_j^\nu &= \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)}, \quad j \geq 0, \nu \in \Gamma, \end{aligned}$$

which satisfy:

$$(3.13) \quad \|f\|_{L^2(\mathbb{R}^3)}^2 = \sum_{j \geq -1} \gamma_j^2 = \sum_{j \geq -1} \sum_{\nu \in \Gamma} (\gamma_j^\nu)^2.$$

The goal of this second step is to prove the following proposition:

Proposition 3.2. — *The decomposition (3.10) satisfies an almost orthogonality property:*

$$(3.14) \quad \|E_j f\|_{L^2(\mathcal{M})}^2 \lesssim \sum_{\nu \in \Gamma} \|E_j^\nu f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \gamma_j^2.$$

The proof of Proposition 3.2 is postponed to Chapter 6.

3.2.3. Step 3: control of the diagonal term. — Proposition 3.2 allows us to estimate $\|E_j^\nu f\|_{L^2(\mathcal{M})}$ instead of $\|E_j f\|_{L^2(\mathcal{M})}$. The analog of computation (3.1) for $\|E_j^\nu f\|_{L^2(\mathcal{M})}$ yields:

$$(3.15) \quad \begin{aligned} &\|E_j^\nu f\|_{L^2(\mathcal{M})} \\ &\leq \int_{\mathbb{S}^2} \|b(t, x, \omega)^{-1} \operatorname{tr} \chi(t, x, \omega)\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) \eta_j^\nu(\omega) f(\lambda \omega) \lambda^2 d\lambda \right\|_{L_u^2} d\omega \\ &\leq \varepsilon \sqrt{\operatorname{vol}(\operatorname{supp}(\eta_j^\nu))} \|\lambda \psi(2^{-j}\lambda) \eta_j^\nu(\omega) f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \varepsilon 2^{j/2} \gamma_j^\nu, \end{aligned}$$

where the term $\sqrt{\operatorname{vol}(\operatorname{supp}(\eta_j^\nu))}$ comes from the fact that we apply Cauchy-Schwarz in ω . Note that we have used in (3.15) the fact that the support of η_j^ν is 2 dimensional and has diameter $2^{-j/2}$ so that:

$$(3.16) \quad \sqrt{\operatorname{vol}(\operatorname{supp}(\eta_j^\nu))} \lesssim 2^{-j/2}.$$

Now, (3.15) still misses the wanted estimate by a power of $2^{j/2}$. Nevertheless, we are able to estimate the diagonal term:

Proposition 3.3. — *The diagonal term $E_j^\nu f$ satisfies the following estimate:*

$$(3.17) \quad \|E_j^\nu f\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu.$$

The proof of Proposition 3.3 is postponed to Chapter 5.

3.2.4. Proof of Theorem 2.15. — Proposition 3.1, 3.2 and 3.3 immediately yield the proof of Theorem 2.15. Indeed, (3.7), (3.13), (3.14) and (3.17) imply:

$$\begin{aligned}
 \|Ef\|_{L^2(\mathcal{M})}^2 &\lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
 (3.18) \quad &\lesssim \sum_{j \geq -1} \sum_{\nu \in \Gamma} \|E_j^\nu f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \sum_{j \geq -1} \gamma_j^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
 &\lesssim \varepsilon^2 \sum_{j \geq -1} \sum_{\nu \in \Gamma} (\gamma_j^\nu)^2 + \varepsilon^2 \sum_{j \geq -1} \gamma_j^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
 &\lesssim \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2,
 \end{aligned}$$

which is the conclusion of Theorem 2.15. □

The rest of the paper is dedicated to the proof of Proposition 3.1, 3.2 and 3.3. In Chapter 4, we prove Proposition 3.1. In Chapter 5, we prove Proposition 3.3. Finally, we turn to the proof of Proposition 3.2 which constitutes the most technical part of this paper and occupies Chapters 6 to 10.

CHAPTER 4

PROOF OF PROPOSITION 3.1 (ALMOST ORTHOGONALITY IN FREQUENCY)

We have to prove (3.7):

$$(4.1) \quad \|Ef\|_{L^2(\mathcal{M})}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This will result from the following inequality using Shur's Lemma:

$$(4.2) \quad \left| \int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d\mathcal{M} \right| \lesssim \varepsilon^2 2^{-\frac{|j-k|}{4}} \gamma_j \gamma_k \text{ for } |j - k| > 2.$$

Before we proceed with the proof of (4.2), let us recall that the volume element on \mathcal{M} expressed in the coordinate system (u, t, x') is given by:

$$d\mathcal{M} = nb \, du \, dt \, d\mu_{t,u},$$

where $d\mu_{t,u}$ denotes the volume element on $P_{t,u}$. Since we have:

$$\|n - 1\|_{L^\infty} \lesssim \varepsilon$$

from the estimates for n (2.36), the $L^2(\mathcal{M})$ norm of a tensor F on \mathcal{M} is equivalent to:

$$(4.3) \quad \int_{\mathcal{M}} |F(t, x)|^2 b \, du \, dt \, d\mu_{t,u},$$

where we have removed the lapse n in the definition of the volume element. To avoid unnecessary terms containing derivatives of n in the numerous integrations by parts of this paper, we will estimate the equivalent (4.3) of the $L^2(\mathcal{M})$ norm for Ef , $E_j f$ and $E_j' f$. By a slight abuse of notation which we shall do throughout the paper, this is equivalent to modifying the volume element by removing the time lapse n :

$$(4.4) \quad d\mathcal{M} = b \, du \, dt \, d\mu_{t,u}.$$

Remark 4.1. — One may want to further simplify the expression of the volume element (4.4) by removing the null lapse b . However, this is not possible since the decomposition (4.4) depends of the angle $\omega \in \mathbb{S}^2$ under consideration. Indeed, the coordinate system is (u, t, x') where $u = u(t, x, \omega)$ and thus $b = b(t, x, \omega)$ in (4.4), while the time lapse n is independent of ω .

Also, recall that we have for all integrable scalar functions f :

$$(4.5) \quad \frac{d}{du} \left(\int_{P_{t,u}} f d\mu_{t,u} \right) = \int_{P_{t,u}} b(\nabla_N f + \text{tr}\theta) d\mu_{t,u},$$

where θ is the second fundamental form of $P_{t,u}$ in Σ_t , i.e., $\theta_{ij} = \nabla_i N_j$. Note that from the definition of k , χ and θ , we have:

$$(4.6) \quad \chi_{AB} = \langle \mathbf{D}_A L, e_B \rangle = \langle \nabla_A T, e_B \rangle + \langle \nabla_A N, e_B \rangle = -k_{AB} + \theta_{AB}.$$

Together with the estimate (2.39) (2.40) for χ and (2.37) for k , we obtain:

$$(4.7) \quad \mathcal{N}_1(\theta) \lesssim \varepsilon.$$

Finally, we recall that we have:

$$(4.8) \quad \begin{cases} \nabla_A N = \theta_{AB} e_B, \\ \nabla_N N = -b^{-1} \nabla b. \end{cases}$$

4.1. A first integration by parts

From now on, we focus on proving (4.2). We may assume $j \geq k + 3$. We have:

$$(4.9) \quad \int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d_{\mathcal{M}} \\ = \int_{\mathbb{S}^2} \int_0^{+\infty} \int_{\mathbb{S}^2} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b(t, x, \omega)^{-1} \text{tr}\chi(t, x, \omega) \overline{b(x, \omega')^{-1} \text{tr}\chi(t, x, \omega')} d_{\mathcal{M}} \right) \\ \times \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 \psi(2^{-k}\lambda') \overline{f(\lambda'\omega')} (\lambda')^2 d\lambda d\omega d\lambda' d\omega'.$$

In view of the expression of the volume element (4.4) on \mathcal{M} , we integrate by parts with respect to ∂_u in

$$\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b(t, x, \omega)^{-1} \text{tr}\chi(t, x, \omega) \overline{b(x, \omega')^{-1} \text{tr}\chi(t, x, \omega')} d_{\mathcal{M}} \\ = \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b(t, x, \omega)^{-1} \text{tr}\chi(t, x, \omega) \overline{b(x, \omega')^{-1} \text{tr}\chi(t, x, \omega')} b du dt d\mu_{t,u}$$

using the fact that:

$$(4.10) \quad e^{i\lambda u - i\lambda' u'} = -\frac{i}{\lambda - \lambda' \frac{b}{b'}} g(N, N') \partial_u (e^{i\lambda u - i\lambda' u'}),$$

where we use the notation u for $u(t, x, \omega)$, b for $b(t, x, \omega)$, N for $N(t, x, \omega)$, u' for $u(t, x, \omega')$, b' for $b(t, x, \omega')$ and N' for $N(t, x, \omega')$. We will also use the notation $\text{tr}\chi$ for $\text{tr}\chi(t, x, \omega)$, $\text{tr}\chi'$ for $\text{tr}\chi(t, x, \omega')$, $\text{tr}\theta$ for $\text{tr}\theta(t, x, \omega)$, and $\text{tr}\theta'$ for $\text{tr}\theta(t, x, \omega')$. Using (4.10) and the expression for the volume element (4.4), we obtain:

$$(4.11) \quad \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b \overline{b'} d_{\mathcal{M}}$$

$$\begin{aligned}
&= i \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1} \partial_u \overline{\text{tr} \chi b'^{-1} \text{tr} \chi'}}{\lambda - \lambda' \frac{b}{b'} g(N, N')} d\mathcal{M} + i \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1} \text{tr} \chi \partial_u \overline{(b'^{-1} \text{tr} \chi')}}{\lambda - \lambda' \frac{b}{b'} g(N, N')} d\mathcal{M} \\
&\quad + i \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1} \text{tr} \chi \overline{b'^{-1} \text{tr} \chi' \text{tr} \theta}}{\lambda - \lambda' \frac{b}{b'} g(N, N')} d\mathcal{M} \\
&\quad + i \lambda' \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1} \text{tr} \chi \overline{b'^{-1} \text{tr} \chi' \left(\frac{\nabla_N b}{b'} g(N, N') - \frac{b \nabla_N b'}{b'^2} g(N, N') \right)}}{(\lambda - \lambda' \frac{b}{b'} g(N, N'))^2} d\mathcal{M} \\
&\quad + i \lambda' \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1} \text{tr} \chi \overline{b'^{-1} \text{tr} \chi' \frac{b}{b'} (g(\nabla_N N, N') + g(N, \nabla_N N'))}}{(\lambda - \lambda' \frac{b}{b'} g(N, N'))^2} d\mathcal{M},
\end{aligned}$$

where we have used (4.5) to obtain the third term in the right-hand side of (4.11). Since $|\lambda' \frac{b}{b'} g(N, N')| < \lambda$, we may expand the fractions in (4.11):

$$(4.12) \quad \frac{1}{\lambda - \lambda' \frac{b}{b'} g(N, N')} = \sum_{p \geq 0} \frac{1}{\lambda} \left(\frac{\lambda' \frac{b}{b'} g(N, N')}{\lambda} \right)^p$$

and

$$(4.13) \quad \frac{1}{(\lambda - \lambda' \frac{b}{b'} g(N, N'))^2} = \sum_{p \geq 0} \frac{p+1}{\lambda^2} \left(\frac{\lambda' \frac{b}{b'} g(N, N')}{\lambda} \right)^p.$$

For $p \in \mathbb{Z}$, We introduce the notation $F_{j,p}(u)$:

$$(4.14) \quad F_{j,p}(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda\omega) (2^{-j}\lambda)^p \lambda^2 d\lambda.$$

Together with (4.9), (4.11) and (4.12), this implies:

$$(4.15) \quad \int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d\mathcal{M} = \sum_{p \geq 0} A_p^1 + \sum_{p \geq 0} A_p^2 + \sum_{p \geq 0} A_p^3 + \sum_{p \geq 0} A_p^4,$$

where A_p^1 , A_p^2 , A_p^3 and A_p^4 are given by:

$$\begin{aligned}
(4.16) \quad A_p^1 &= 2^{-j-p(j-k)} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} (b^{-1} \nabla_N \text{tr} \chi + b^{-1} \text{tr} \chi \text{tr} \theta) b^{p+1} N^p F_{j,-p-1}(u) d\omega \right) \\
&\quad \cdot \overline{\left(\int_{\mathbb{S}^2} b'^{-1} \text{tr} \chi' b'^{-p} N'^p F_{k,p}(u') d\omega' \right)} d\mathcal{M}, \\
(4.17) \quad A_p^2 &= 2^{-j-p(j-k)} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} \text{tr} \chi b^{p+1} N^{p+1} F_{j,-p-1}(u) d\omega \right) \\
&\quad \cdot \overline{\left(\int_{\mathbb{S}^2} \nabla (b'^{-1} \text{tr} \chi') b'^{-p} N'^p F_{k,p}(u') d\omega' \right)} d\mathcal{M},
\end{aligned}$$

(4.18)

$$A_p^3 = (p+1)2^{-j-(p+1)(j-k)} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi (\nabla_N b N + b \nabla_N N) b^p N^p F_{j,-p-2}(u) d\omega \right) \cdot \overline{\left(\int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' b'^{-p-1} N'^{p+1} F_{k,p+1}(u') d\omega' \right)} d_{\mathcal{M}}$$

and

$$(4.19) \quad A_p^4 = (p+1)2^{-j-(p+1)(j-k)} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi b^{p+1} N^{p+2} F_{j,-p-2}(u) d\omega \right) \cdot \overline{\left(\int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' (\nabla \log(b') N' + \nabla N') b'^{-p-1} N'^p F_{k,p+1}(u') d\omega' \right)} d_{\mathcal{M}}.$$

Remark 4.2. — The expansion (4.12) allows us to rewrite $\int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d_{\mathcal{M}}$ in the form (4.15), i.e., as a sum of terms $A_p^1, A_p^2, A_p^3, A_p^4$. The key point is that in each of these terms—according to (4.16)–(4.19)—one may separate the terms depending of (λ, ω) from the terms depending on (λ', ω') .

4.2. Estimates for A_p^1 and A_p^2

Let $H(t, x, \omega)$ a tensor on \mathcal{M} such that $\|H\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon$. Then proceeding as in the basic computation (3.1), we have for any $p \in \mathbb{Z}$:

$$(4.20) \quad \left\| \int_{\mathbb{S}^2} H(x, \omega) F_{j,p}(u) d\omega \right\|_{L^2(\mathcal{M})} \leq \int_{\mathbb{S}^2} \|H\|_{L_u^\infty L^2(\mathcal{H}_u)} \|F_{j,p}(u)\|_{L_u^2} d\omega \\ \leq \|H\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\psi(2^{-j}\lambda) f(\lambda\omega) (2^{-j}\lambda)^p \lambda\|_{L^2(\mathbb{R}^3)} \\ \lesssim \varepsilon 2^{|p|+j} \gamma_j,$$

where we have used the fact that $1/2 \leq 2^{-j}\lambda \leq 2$ on the support of $\psi(2^{-j}\lambda)$. Now, the estimates (2.38) on b , (4.7) on θ , and the equation for ∇N (4.8) yield:

(4.21)

$$\|(b^{-1} \nabla_N \operatorname{tr} \chi + b^{-1} \operatorname{tr} \chi \operatorname{tr} \theta) b^{p+1} N^p\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\nabla(b'^{-1} \operatorname{tr} \chi') b'^{-p} N'^p\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \\ + \|b^{-1} \operatorname{tr} \chi (\nabla_N b N + b \nabla_N N) b^p N^p\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ + \|b'^{-1} \operatorname{tr} \chi' (\nabla \log(b') N' + \nabla N') b'^{-p-1} N'^p\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \varepsilon,$$

which together with (4.20) implies:

$$(4.22) \quad \left\| \int_{\mathbb{S}^2} (b^{-1} \nabla_N \operatorname{tr} \chi + b^{-1} \operatorname{tr} \chi \operatorname{tr} \theta) b^{p+1} N^p F_{j,-p-1}(u) d\omega \right\|_{L^2(\mathcal{M})} \\ + \left\| \int_{\mathbb{S}^2} \nabla(b'^{-1} \operatorname{tr} \chi') b'^{-p} N'^p F_{k,p}(u') d\omega' \right\|_{L^2(\mathcal{M})} \\ + \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi (\nabla_N b N + b \nabla_N N) b^p N^p F_{j,-p-2}(u) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned}
& + \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' (\nabla \log(b') N' + \nabla N') b'^{-p-1} N'^p F_{k,p+1}(u') d\omega' \right\|_{L^2(\mathcal{M})} \\
& \lesssim \varepsilon 2^{p+j} \gamma_j.
\end{aligned}$$

Note that Proposition 3.2 together with Proposition 3.3 yields the estimate:

$$(4.23) \quad \|E_j f\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j,$$

for any symbol satisfying the same regularity assumptions than $b^{-1} \operatorname{tr} \chi$ where b satisfies (2.38), and $\operatorname{tr} \chi$ satisfies (2.39). Now, the terms containing no derivative in (4.16)–(4.19) have a symbol given respectively by $b'^{-1} b'^{-p} N'^p$, $b^{-1} \operatorname{tr} \chi b^{p+1} N^{p+1}$, $b'^{-1} \operatorname{tr} \chi' b'^{-p-1} N'^{p+1}$ and $b^{-1} \operatorname{tr} \chi b^{p+1} N^{p+2}$. Since N satisfies regularity assumptions which are at least as good as $\operatorname{tr} \chi$, these symbols satisfies the same regularity assumptions than $b^{-1} \operatorname{tr} \chi$. Applying (4.23), we obtain:

$$\begin{aligned}
(4.24) \quad & \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' b'^{-p} N'^p F_{k,p}(u') d\omega' \right\|_{L^2(\mathcal{M})} \\
& + \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' b'^{-p-1} N'^{p+1} F_{k,p+1}(u') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^p \gamma_k,
\end{aligned}$$

and

$$\begin{aligned}
(4.25) \quad & \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi b^{p+1} N^{p+1} F_{j,-p-1}(u) d\omega \right\|_{L^2(\mathcal{M})} \\
& + \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi b^{p+1} N^{p+2} F_{j,-p-2}(u) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^p \gamma_j,
\end{aligned}$$

where we have used the fact that $1/2 \leq 2^{-j} \lambda \leq 2$ on the support of $\psi(2^{-j} \lambda)$.

Finally, the definition of $A_p^1 - A_p^4$ given by (4.16)–(4.19) and the estimates (4.22), (4.24) and (4.25) yield:

$$(4.26) \quad |A_p^1| \lesssim \varepsilon 2^{2p-p(j-k)} \gamma_j \gamma_k, \quad \forall p \geq 0,$$

and

$$(4.27) \quad |A_p^2| + |A_p^3| + |A_p^4| \lesssim \varepsilon 2^{2p-(p+1)(j-k)} \gamma_j \gamma_k, \quad \forall p \geq 0.$$

(4.26) and (4.27) imply:

(4.28)

$$\sum_{p \geq 1} |A_p^1| + \sum_{p \geq 0} (|A_p^2| + |A_p^3| + |A_p^4|) \lesssim \varepsilon 2^{-(j-k)} \left(\sum_{p \geq 0} 2^{-p(j-k-2)} \right) \gamma_j \gamma_k \lesssim \varepsilon 2^{-(j-k)} \gamma_j \gamma_k,$$

where we have used the assumption $j - k - 2 > 0$. (4.15) and (4.28) will yield (4.2) provided we obtain a similar estimate for A_0^1 . Now, the estimate of A_0^1 provided by (4.26) is not sufficient since it does not contain any decay in $j - k$. We will need to perform a second integration by parts for this term.

4.3. A more precise estimate for A_0^1

From (4.16) with $p = 0$, we have:

$$(4.29) \quad A_0^1 = 2^{-j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} (\nabla_N \text{tr} \chi + \text{tr} \chi \text{tr} \theta) F_{j,-1}(u) d\omega \right) \overline{E_k f(t, x)} d_{\mathcal{M}}.$$

Using the geometric Littlewood-Paley projections on the 2-surfaces $P_{t,u}$, we decompose $\nabla_N \text{tr} \chi$ as:

$$\nabla_N \text{tr} \chi = P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr} \chi) + P_{> \frac{j+k}{2}}(\nabla_N \text{tr} \chi).$$

In turn, this yields a decomposition for A_0^1 :

$$(4.30) \quad A_0^1 = A_{0,1}^1 + A_{0,2}^1,$$

where:

$$(4.31) \quad \begin{aligned} A_{0,1}^1 &= 2^{-j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} P_{> \frac{j+k}{2}}(\nabla_N \text{tr} \chi) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d_{\mathcal{M}}, \\ A_{0,2}^1 &= 2^{-j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} (P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr} \chi) + \text{tr} \chi \text{tr} \theta) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d_{\mathcal{M}}. \end{aligned}$$

We first estimate $A_{0,1}^1$. The finite band property yields:

$$\begin{aligned} P_{> \frac{j+k}{2}}(\nabla_N \text{tr} \chi) &= \sum_{l > \frac{j+k}{2}} P_l(\nabla_N \text{tr} \chi) \\ &= \sum_{l > \frac{j+k}{2}} 2^{-2l} \Delta P_l(\nabla_N \text{tr} \chi), \end{aligned}$$

which yields the following decomposition for $A_{0,1}^1$:

$$(4.32) \quad A_{0,1}^1 = \sum_{l > \frac{j+k}{2}} A_{0,1,l}^1,$$

where $A_{0,1,l}^1$ is given by:

$$A_{0,1,l}^1 = 2^{-j-2l} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} \Delta P_l(\nabla_N \text{tr} \chi) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d_{\mathcal{M}}.$$

Now, the decomposition of the volume element (4.4) yields:

$$A_{0,1,l}^1 = 2^{-j-2l} \int_{\mathbb{S}^2} \int_{t,u} \left(\int_{P_{t,u}} \Delta P_l(\nabla_N \text{tr} \chi) \overline{E_k f(t, x)} b d\mu_{t,u} \right) F_{j,0}(u) du dt d\omega.$$

Integrating by parts Δ on $P_{t,u}$, we obtain:

$$\begin{aligned} A_{0,1,l}^1 &= -2^{-j-2l} \int_{\mathbb{S}^2} \int_{t,u} \left(\int_{P_{t,u}} \nabla P_l(\nabla_N \text{tr} \chi) \nabla (\overline{E_k f(t, x)} b) d\mu_{t,u} \right) F_{j,0}(u) du dt d\omega \\ &= -2^{-j-2l} \int_{\mathbb{S}^2} \left(\int_{\mathcal{M}} \nabla P_l(\nabla_N \text{tr} \chi) F_{j,0}(u) \nabla (\overline{E_k f(t, x)} b) b^{-1} d_{\mathcal{M}} \right) d\omega, \end{aligned}$$

where we used again the decomposition of the volume element (4.4) in the last equality. We apply Cauchy-Schwartz to the integral on \mathcal{M} and obtain:

$$\begin{aligned}
 (4.33) \quad |A_{0,1,l}^1| &\leq 2^{-j-2l} \int_{\mathbb{S}^2} \|\nabla P_l(\nabla_N \text{tr}\chi) F_{j,0}(u)\|_{L^2(\mathcal{M})} \|\nabla(E_k b) b^{-1}\|_{L^2(\mathcal{M})} d\omega \\
 &\lesssim 2^{-j-2l} \int_{\mathbb{S}^2} \|\nabla P_l(\nabla_N \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \|F_{j,0}(u)\|_{L_u^2} \|\nabla(E_k b)\|_{L^2(\mathcal{M})} \|b^{-1}\|_{L^\infty} d\omega \\
 &\lesssim 2^{-j-l} \int_{\mathbb{S}^2} \|\nabla_N \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \|F_{j,0}(u)\|_{L_u^2} \|\nabla(E_k b)\|_{L^2(\mathcal{M})} d\omega \\
 &\lesssim \varepsilon 2^{-j-l} \int_{\mathbb{S}^2} \|F_{j,0}(u)\|_{L_u^2} \|\nabla(E_k b)\|_{L^2(\mathcal{M})} d\omega,
 \end{aligned}$$

where we used the finite band property for P_l , the estimates (2.38) for b , and the estimates (2.39) for $\text{tr}\chi$. Plancherel yields:

$$(4.34) \quad \|F_{j,0}\|_{L_{\omega,u}^2} \leq \|\psi(2^{-j}\lambda) f(\lambda\omega)\lambda\|_{L^2(\mathbb{R}^3)} \lesssim 2^j \gamma_j.$$

In view of (4.33), we also need to estimate $\|\nabla(E_k b)\|_{L^2(\mathcal{M})}$. We have:

$$\begin{aligned}
 (4.35) \quad \|\nabla(E_k b)\|_{L^2(\mathcal{M})} &\lesssim \|E_k \nabla b\|_{L^2(\mathcal{M})} + \|b \nabla E_k\|_{L^2(\mathcal{M})} \\
 &\lesssim \|\nabla b\|_{L^4(\mathcal{M})} \|E_k\|_{L^4(\mathcal{M})} + \|b\|_{L^\infty} \|\mathbf{D}E_k\|_{L^2(\mathcal{M})} \\
 &\lesssim \|\mathbf{D}E_k\|_{L^2(\mathcal{M})},
 \end{aligned}$$

where we used in the last inequality the estimates (2.38) for b , the Sobolev embedding on \mathcal{H}_u (2.50), and the Sobolev inequality on the 4-dimensional manifold \mathcal{M} (2.54). We still need to estimate $\|\mathbf{D}E_k\|_{L^2(\mathcal{M})}$. We have:

$$\begin{aligned}
 (4.36) \quad \mathbf{D}E_k f(t, x) &= \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} \mathbf{D}(b^{-1} \text{tr}\chi) \psi(2^{-k}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega \\
 &\quad + i 2^k \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} \text{tr}\chi L\psi(2^{-k}\lambda) (2^{-k}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega.
 \end{aligned}$$

Using the basic computation (3.1) for the first term together with the fact that $\mathbf{D}(b^{-1} \text{tr}\chi) \in L_u^\infty L^2(\mathcal{H}_u)$ from the estimates (2.38) for b and (2.39) for $\text{tr}\chi$, and (4.23) for the second term together with the fact that $\text{tr}\chi N$ satisfies the same regularity assumptions than $b^{-1} \text{tr}\chi$, we obtain:

$$(4.37) \quad \|\mathbf{D}E_k\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^k \gamma_k.$$

(4.33), (4.34), (4.35) and (4.37) yield:

$$|A_{0,1,l}^1| \lesssim \varepsilon^2 2^{-l+k} \varepsilon^2 \gamma_j \gamma_k.$$

Together with (4.32), this yields:

$$(4.38) \quad |A_{0,1}^1| \lesssim \left(\sum_{l > \frac{j+k}{2}} 2^{-l} \right) \varepsilon 2^k \varepsilon^2 \gamma_j \gamma_k \lesssim \varepsilon^2 2^{-\frac{j-k}{2}} \gamma_j \gamma_k.$$

4.4. A second integration by parts

We now estimate the term $A_{0,2}^1$ defined in (4.31).

We perform a second integration by parts relying again on (4.10). We obtain:

(4.39)

$$\begin{aligned} A_{0,2}^1 &= 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} \left(b(\nabla_N P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \nabla_N(\text{tr}\chi \text{tr}\theta)) + \nabla_N(b)(P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) \right. \right. \\ &\quad \left. \left. + \text{tr}\chi \text{tr}\theta) + b \text{tr}\theta(P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \text{tr}\chi \text{tr}\theta) \right) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d\mathcal{M} \\ &\quad + 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} (P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \text{tr}\chi \text{tr}\theta) b N F_{j,0}(u) d\omega \right) \cdot \overline{\nabla E_k f(t, x)} d\mathcal{M} + \dots, \end{aligned}$$

where we only mention the first term generated by the expansion (4.12). In fact, the other terms generated by (4.12) and the ones generated by (4.13) are estimated in the same way and generate more decay in $j - k$ similarly to the estimates (4.26) (4.27). In view of (4.39), we decompose the main part of $A_{0,2}^1$ as the sum of three terms:

$$(4.40) \quad A_{0,2}^1 = A_{0,2,1}^1 + A_{0,2,2}^1 + A_{0,2,3}^1 + \dots,$$

where $A_{0,2,1}^1$ is given by:

(4.41)

$$\begin{aligned} A_{0,2,1}^1 &= 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} \left(b \nabla_N P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) \right. \right. \\ &\quad \left. \left. + \nabla_N(b) P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + b \text{tr}\theta P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) \right) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d\mathcal{M}, \end{aligned}$$

where $A_{0,2,2}^1$ is given by:

$$(4.42) \quad \begin{aligned} A_{0,2,2}^1 &= 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} \left(b \nabla_N(\text{tr}\chi \text{tr}\theta) \right. \right. \\ &\quad \left. \left. + \nabla_N(b) \text{tr}\chi \text{tr}\theta + b \text{tr}\theta^2 \text{tr}\chi \right) F_{j,0}(u) d\omega \right) \overline{E_k f(t, x)} d\mathcal{M} \end{aligned}$$

and where $A_{0,2,3}^1$ is given by:

(4.43)

$$A_{0,2,3}^1 = 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} (P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \text{tr}\chi \text{tr}\theta) b N F_{j,0}(u) d\omega \right) \cdot \overline{\nabla E_k f(t, x)} d\mathcal{M}.$$

We first estimate $A_{0,2,1}^1$. We have:

$$\begin{aligned} &\left\| b \nabla_N P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \nabla_N(b) P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + b \text{tr}\theta P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) \right\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|b\|_{L^\infty} \left(\sum_{l \leq \frac{j+k}{2}} \|\nabla_N P_l(\nabla_N \text{tr}\chi)\|_{L^2(\mathcal{H}_u)} \right) \end{aligned}$$

$$\begin{aligned}
& + (\|\nabla_N b\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|b\|_{L^\infty} \|\operatorname{tr}\theta\|_{L_u^\infty L^2(\mathcal{H}_u)}) \|P_{\leq \frac{j+k}{2}}(\nabla_N \operatorname{tr}\chi)\|_{L^\infty} \\
& \lesssim \sum_{l \leq \frac{j+k}{2}} (2^l \varepsilon + 2^{\frac{l}{2}} \varepsilon \mu(u)) + \varepsilon 2^{\frac{j+k}{2}} \|\nabla_N \operatorname{tr}\chi\|_{L_t^\infty L_x^2}, \\
& \lesssim \varepsilon 2^{\frac{j+k}{2}} + 2^{\frac{j+k}{4}} \varepsilon \mu(u),
\end{aligned}$$

where we used the estimates (2.38) for b , the estimates (4.7) for θ , the estimate (2.48) for $\nabla_N P_l(\nabla_N \operatorname{tr}\chi)$, the strong Bernstein inequality (2.59), and the estimates (2.39) for $\operatorname{tr}\chi$, and where μ is a function satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \lesssim 1,$$

according to (2.48). In view of (4.41), this yields:

(4.44)

$$\begin{aligned}
|A_{0,2,1}^1| & \lesssim 2^{-2j} \|E_k\|_{L^2(\mathcal{M})} \int_{\mathbb{S}^2} \left\| b \nabla_N P_{\leq \frac{j+k}{2}}(\nabla_N \operatorname{tr}\chi) \right. \\
& \quad \left. + \nabla_N(b) P_{\leq \frac{j+k}{2}}(\nabla_N \operatorname{tr}\chi) + b \operatorname{tr}\theta P_{\leq \frac{j+k}{2}}(\nabla_N \operatorname{tr}\chi) \right\|_{L^2(\mathcal{H}_u)} F_{j,0}(u) \Big\|_{L_u^2} d\omega \\
& \lesssim 2^{-2j} \varepsilon \gamma_k \int_{\mathbb{S}^2} \left\| (\varepsilon 2^{\frac{j+k}{2}} + \varepsilon 2^{\frac{j+k}{4}} \mu(u)) F_{j,0}(u) \right\|_{L_u^2} d\omega \\
& \lesssim 2^{-2j} \varepsilon \gamma_k \left(\varepsilon 2^{\frac{j+k}{2}} \int_{\mathbb{S}^2} \|F_{j,0}(u)\|_{L_u^2} d\omega + 2^{\frac{j+k}{4}} \int_{\mathbb{S}^2} \|\mu\|_{L^2(\mathbb{R})} \|F_{j,0}(u)\|_{L_u^\infty} d\omega \right) \\
& \lesssim 2^{-\frac{j-k}{4}} \varepsilon^2 \gamma_k \gamma_j,
\end{aligned}$$

where we used (4.23) for $E_k f$, Plancherel with respect to λ for $\|F_{j,0}(u)\|_{L_u^2}$, Cauchy-Schwarz in λ for $\|F_{j,0}(u)\|_{L_u^\infty}$, and Cauchy-Schwarz in ω .

Next, we estimate $A_{0,2,2}^1$. We have:

$$\begin{aligned}
& \|b \nabla_N(\operatorname{tr}\chi \operatorname{tr}\theta) + \nabla_N(b) \operatorname{tr}\chi \operatorname{tr}\theta + b \operatorname{tr}\theta^2 \operatorname{tr}\chi\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \\
& \lesssim \|b\|_{L^\infty} (\|\nabla_N \operatorname{tr}\theta\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\operatorname{tr}\chi\|_{L_u^\infty L^6(\mathcal{H}_u)} + \|\nabla_N \operatorname{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\operatorname{tr}\theta\|_{L_u^\infty L^6(\mathcal{H}_u)}) \\
& \quad + \|\nabla_N b\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\operatorname{tr}\theta\|_{L_u^\infty L^6(\mathcal{H}_u)} \|\operatorname{tr}\chi\|_{L^\infty} + \|b\|_{L^\infty} \|\operatorname{tr}\theta\|_{L_u^\infty L^3(\mathcal{H}_u)}^2 \|\operatorname{tr}\chi\|_{L^\infty} \\
& \lesssim \varepsilon,
\end{aligned}$$

where we used the Sobolev embedding (2.50) on \mathcal{H}_u , the estimates (2.38) for b , the estimates (2.39) for $\operatorname{tr}\chi$ and the estimates (4.7) for θ . In view of (4.42), this yields:

$$\begin{aligned}
(4.45) \quad |A_{0,2,2}^1| & \lesssim 2^{-2j} \int_{\mathbb{S}^2} \|b \nabla_N(\operatorname{tr}\chi \operatorname{tr}\theta) + \nabla_N(b) \operatorname{tr}\chi \operatorname{tr}\theta \\
& \quad + b \operatorname{tr}\theta^2 \operatorname{tr}\chi\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \|F_{j,0}(u)\|_{L_u^2} \|E_k f\|_{L_u^2 L^3(\mathcal{H}_u)} d\omega \\
& \lesssim 2^{-2j} \varepsilon \left(\int_{\mathbb{S}^2} \|F_{j,0}(u)\|_{L_u^2} d\omega \right) \|E_k f\|_{L^4(\mathcal{M})} \\
& \lesssim 2^{-j} \varepsilon \gamma_j \|\mathbf{D}E_k f\|_{L^2(\mathcal{M})}
\end{aligned}$$

$$\lesssim 2^{-(j-k)} \varepsilon^2 \gamma_j \gamma_k,$$

where we used Plancherel with respect to λ for $\|F_{j,0}(u)\|_{L_u^2}$, Cauchy-Schwarz in ω , the Sobolev embedding on \mathcal{M} (2.54), and (4.37) for $\mathbf{D}E_k f$.

Finally, we estimate $A_{0,2,3}^1$. We have:

$$\begin{aligned} & \| (P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \text{tr}\chi \text{tr}\theta) b \|_{L_u^\infty L^2(\mathcal{H}_u)} \\ & \lesssim \|b\|_{L^\infty} (\|P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\text{tr}\chi\|_{L^\infty} \|\text{tr}\theta\|_{L_u^\infty L^2(\mathcal{H}_u)}) \\ & \lesssim \|\nabla_N \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} + \varepsilon \\ & \lesssim \varepsilon, \end{aligned}$$

where we used the estimates (2.38) for b , (2.39) for $\text{tr}\chi$ and (4.7) for θ . In view of (4.43), this yields:

(4.46)

$$\begin{aligned} & |A_{0,2,3}^1| \\ & \lesssim 2^{-2j} \left(\int_{\mathbb{S}^2} \| (P_{\leq \frac{j+k}{2}}(\nabla_N \text{tr}\chi) + \text{tr}\chi \text{tr}\theta) b N \|_{L_u^\infty L^2(\mathcal{H}_u)} \|F_{j,0}(u)\|_{L_u^2} d\omega \right) \|\nabla E_k f\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \varepsilon^2 2^{k-2j} \gamma_k \left(\int_{\mathbb{S}^2} \|F_{j,0}(u)\|_{L_u^2} d\omega \right) \\ & \lesssim \varepsilon^2 2^{-(j-k)} \gamma_k \gamma_j, \end{aligned}$$

where we used (4.23) for $E_k f$, Plancherel with respect to λ for $\|F_{j,0}(u)\|_{L_u^2}$, and Cauchy-Schwarz in ω . Finally, (4.40), (4.44), (4.45) and (4.46) imply:

$$(4.47) \quad |A_{0,2}^1| \lesssim \varepsilon^2 2^{-\frac{j-k}{4}} \gamma_j \gamma_k.$$

4.5. End of the proof of Proposition 3.1

Since $A_0^1 = A_1^1 + A_2^1$, the estimate (4.38) of $A_{0,1}^1$ and the estimate (4.47) of $A_{0,2}^1$ yield:

$$(4.48) \quad |A_0^1| \lesssim \varepsilon^2 2^{-\frac{j-k}{4}} \gamma_j \gamma_k.$$

Together with (4.15) and (4.28), this implies:

$$(4.49) \quad \left| \int_{\mathcal{O}\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d_{\mathcal{O}\mathcal{M}} \right| \lesssim \varepsilon^2 2^{-\frac{|j-k|}{4}} \gamma_j \gamma_k \text{ for } |j-k| > 2.$$

Finally, (4.49) together with Shur's Lemma yields:

$$(4.50) \quad \|E f\|_{L^2(\mathcal{O}\mathcal{M})}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(\mathcal{O}\mathcal{M})}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 3.1. \square

CHAPTER 5

PROOF OF PROPOSITION 3.3 (CONTROL OF THE DIAGONAL TERM)

Since the orthogonality argument in angle in the core of the paper, we choose to deal first with the control of the diagonal term in this chapter. We will then proceed with the orthogonality argument in angle in the rest of the paper.

In order to control the diagonal term, we have to prove (3.17):

$$(5.1) \quad \|E_j^\nu f\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu.$$

Recall that E_j^ν is given by:

$$(5.2) \quad E_j^\nu f(t, x) = \int_{\mathbb{S}^2} b^{-1}(t, x, \omega) \operatorname{tr} \chi(t, x, \omega) F_j(u) \eta_j^\nu(\omega) d\omega,$$

where $F_j(u)$ is defined by:

$$(5.3) \quad F_j(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 d\lambda.$$

In view of the decompositions (2.72), and the decomposition (2.73) with $p = -1$, we have the following decomposition for $b^{-1} \operatorname{tr} \chi$:

$$(5.4) \quad b^{-1} \operatorname{tr} \chi = f_1^j + f_2^j,$$

where f_1^j only depends on (t, x, ν) and satisfies:

$$(5.5) \quad \|f_1^j\|_{L^\infty} \lesssim \varepsilon$$

and where f_2^j satisfies

$$(5.6) \quad \|f_2^j\|_{L^\infty L^2(\mathcal{M}_u)} \lesssim 2^{-\frac{j}{2}} \varepsilon.$$

In view of (5.2), (5.4) yields the following decomposition for $E_j^\nu f$:

$$E_j^\nu f(t, x) = f_1^j(t, x, \nu) \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega + \int_{\mathbb{S}^2} F_j(u) f_2^j(t, x, \omega, \nu) \eta_j^\nu(\omega) d\omega,$$

which together with the estimates (5.5) and (5.6) implies:

$$(5.7) \quad \|E_j^\nu f\|_{L^2(\mathcal{M})}$$

$$\begin{aligned} &\lesssim \|f_1^j\|_{L^\infty(\mathcal{M})} \left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2} \|f_2^j\|_{L_u^\infty L^2(\mathcal{M}_u)} \eta_j^\nu(\omega) d\omega \\ &\lesssim \varepsilon \left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch.

The following proposition allows us to estimate the right-hand side of (5.7).

Proposition 5.1. — *We have the following bound:*

$$(5.8) \quad \left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu,x_\nu}^2 L_{t_\nu}^\infty} \lesssim \gamma_j^\nu.$$

The proof of Proposition 5.1 is postponed to the end of this chapter. (5.8) and (5.7) yield:

$$\|E_j^\nu f\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu,$$

which is the wanted estimate (5.1). This concludes the proof of Proposition 3.3. \square

Remark 5.2. — In order to control the diagonal term, it suffices to have a bound of the $L^2(\mathcal{M})$ norm for the left-hand side of (5.8). The improvement to a bound for the $L_{u,\nu,x_\nu}^2 L_{t_\nu}^\infty$ norm will be crucial when proving the almost orthogonality in angle.

We still need to prove Proposition 5.1. Note that it suffices to show:

$$(5.9) \quad \left\| L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \lesssim \gamma_j^\nu.$$

Now, since the space-time gradient of u is given by $b^{-1}L$, we have:

$$(5.10) \quad L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) = \int_{\mathbb{S}^2} b^{-1} \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega,$$

where F_j^1 is given by:

$$F_j^1(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^3 d\lambda.$$

We have:

$$(5.11) \quad \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) = \mathbf{g}(N(t, x, \omega) - N(t, x, \nu), N(t, x, \omega) - N(t, x, \nu)).$$

Thus, the estimate (2.42) for $\partial_\omega N$ and the size of the patch yields:

$$(5.12) \quad \|\mathbf{g}(L(t, x, \omega), L(t, x, \nu))\|_{L^\infty(\mathcal{M}_u)} \lesssim 2^{-j},$$

which implies:

$$(5.13) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} (b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu)) \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \int_{\mathbb{S}^2} \|b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu)\|_{L_u^\infty L^2(\mathcal{M}_u)} \|\mathbf{g}(L(t, x, \omega), L(t, x, \nu))\|_{L^\infty} \end{aligned}$$

$$\begin{aligned} & \times \|F_j^1(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ & \lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality (5.12), the estimate (2.44) for $\partial_\omega b$, Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch.

Now, in view of (5.10), we have:

$$\begin{aligned} L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) &= b^{-1}(t, x, \nu) \int_{\mathbb{S}^2} b^{-1} \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \\ &+ \int_{\mathbb{S}^2} (b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu)) \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega, \end{aligned}$$

which together with (5.13) and the estimate (2.44) for $\partial_\omega b$ yields:

$$\begin{aligned} (5.14) \quad & \left\| L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu. \end{aligned}$$

Next, we estimate the right-hand side of (5.14). Using the decomposition (2.71), we have, taking into account (5.11):

$$(5.15) \quad \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) = (f_1^j + f_2^j)(\omega - \nu)^2,$$

where f_1^j only depends on ν and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim 1$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{E}_u)} \lesssim 2^{-\frac{j}{2}},$$

where we took into account the size of the patch in the last inequality. Thus, we may rewrite the oscillatory integral in the right-hand side of (5.14) as:

$$\begin{aligned} \int_{\mathbb{S}^2} \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega &= f_1^j(t, x, \nu) \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \\ &+ \int_{\mathbb{S}^2} f_2^j(t, x, \omega, \nu) (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega, \end{aligned}$$

which yields:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} \mathbf{g}(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \|f_1^j\|_{L^\infty} \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & + \int_{\mathbb{S}^2} \|f_2^j\|_{L_u^\infty L^2(\mathcal{E}_u)} |\omega - \nu|^2 \|F_j^1(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ & \lesssim \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimates for f_j^1 and f_j^2 , Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch. Together with (5.14), this implies:

$$(5.16) \quad \left\| L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \gamma_j^\nu.$$

Finally, we need to estimate the first term in the right-hand side of (5.14). We will rely on the energy estimate for the wave equation⁽¹⁾. Recall from (2.28) that:

$$\square_{\mathbf{g}} u = b^{-1} \operatorname{tr} \chi.$$

Thus, we have:

$$(5.17) \quad \square_{\mathbf{g}} \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 F_j(u) \eta_j^\nu(\omega) d\omega \right) = \int_{\mathbb{S}^2} b^{-1}(t, x, \omega) \operatorname{tr} \chi(t, x, \omega) (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega.$$

Arguing as in (5.4)–(5.7), we have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} b^{-1}(t, x, \omega) \operatorname{tr} \chi(t, x, \omega) (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu, \end{aligned}$$

which together with (5.17) implies:

$$(5.18) \quad \begin{aligned} & \left\| \square_{\mathbf{g}} \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu. \end{aligned}$$

Let us now define the scalar function ϕ on \mathcal{M} as:

$$(5.19) \quad \phi(t, x) = \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j(u) \eta_j^\nu(\omega) d\omega.$$

Then, ϕ satisfies the following wave equation on \mathcal{M} :

$$(5.20) \quad \begin{cases} \square_{\mathbf{g}} \phi = F, \\ \phi|_{\Sigma_0} = \phi_0, \partial_0(\phi)|_{\Sigma_0} = \phi_1, \end{cases}$$

where in view of (5.18), F satisfies:

$$(5.21) \quad \|F\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu.$$

Note also that ϕ_0 and ϕ_1 correspond to the initial data of the half wave parametrix ϕ . The corresponding control is the subject of step **C2** and has been obtained in [16]:

$$(5.22) \quad \|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} \lesssim \gamma_j^\nu.$$

1. Let us note that in [12], the authors also rely on the energy estimate for the wave equation to estimate the diagonal term in their parametrix.

Next, we recall how to derive the energy estimate for the wave Equation (5.20). Recall that T , the future unit normal to the Σ_t foliation. Let π be the deformation tensor of T , that is the symmetric 2-tensor on \mathcal{M} defined as:

$$\pi_{\alpha\beta} = \mathbf{D}_\alpha T_\beta + \mathbf{D}_\beta T_\alpha.$$

In view of the definition of the second fundamental form k and the lapse n , we have:

$$(5.23) \quad \pi_{ij} = -2k_{ij}, \quad \pi_{iT} = \pi_{Ti} = n^{-1}\nabla_i n, \quad \pi_{TT} = 0.$$

We also introduce the energy momentum tensor $Q_{\alpha\beta}$ on \mathcal{M} given by:

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} \mathbf{g}_{\alpha\beta} (\mathbf{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi).$$

We have the following energy estimate for the scalar wave equation:

Lemma 5.3. — *Let F a scalar function on \mathcal{M} , and let ϕ_0 and ϕ_1 two scalar functions on Σ_0 . Let ϕ the solution of the wave Equation (5.20). Then, ϕ satisfies the following energy estimate:*

$$\|\mathbf{D}\phi\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})} + \left| \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} \right|^{\frac{1}{2}},$$

where $Q_{\alpha\beta}$ is the energy momentum tensor of ϕ , and where π is the deformation tensor of T .

Proof. — In view of the Equation (5.20) satisfied by ϕ , we have:

$$\mathbf{D}^\alpha Q_{\alpha\beta} = F \partial_b \phi.$$

Now, we form the 1-tensor P :

$$P_\alpha = Q_{\alpha 0}$$

and we obtain:

$$\mathbf{D}^\alpha P_\alpha = \mathbf{D}^\alpha Q_{\alpha 0} + Q_{\alpha\beta} \mathbf{D}^\alpha T^\beta = F \partial_0 \phi + \frac{1}{2} Q_{\alpha\beta} \pi^{\alpha\beta},$$

where π is the deformation tensor of e_0 . Integrating over the region $0 \leq t \leq 1$, we obtain:

$$(5.24) \quad \begin{aligned} & \|\mathbf{D}\phi\|_{L_t^\infty L^2(\Sigma_t)}^2 \\ & \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)}^2 + \|\phi_1\|_{L^2(\Sigma_0)}^2 + \left| \int_{\mathcal{M}} FT(\phi) d\mathcal{M} \right| + \left| \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} \right| \\ & \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)}^2 + \|\phi_1\|_{L^2(\Sigma_0)}^2 + \|F\|_{L^2(\mathcal{M})} \|T(\phi)\|_{L^2(\mathcal{M})} + \left| \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} \right|, \end{aligned}$$

which concludes the proof of the lemma. \square

We are now in position to estimate ϕ given by (5.19). In view of the estimate (5.21) for F and (5.22) for ϕ_0, ϕ_1 , Lemma 5.3 implies:

$$(5.25) \quad \|\mathbf{D}\phi\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \left| \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} \right|^{\frac{1}{2}} + \gamma_j^\nu.$$

Then, note from the decomposition of π (5.23) and the maximal foliation assumption (2.2) that:

$$g_{\alpha\beta} \pi^{\alpha\beta} = 0,$$

which together with the definition of the energy momentum tensor Q yields:

$$Q_{\alpha\beta} \pi^{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi \pi^{\alpha\beta}.$$

Together with the definition of ϕ (5.19), we obtain:

$$(5.26) \quad \begin{aligned} \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} &= \pi^{\alpha\beta} \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) L_\alpha(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) L_\beta(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \\ &= 2n^{-1} \nabla_i n \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_i(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad - 2k_{ij} \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_i(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_j(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega \right), \end{aligned}$$

where we used in the last equality the decomposition of π (5.23) and the fact that $\mathbf{g}(T, L) = -1$ and $L_i = N_i$. Now, we have:

$$\begin{aligned} &\left\| k_{ij} \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 (b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \\ &\lesssim \int_{\mathbb{S}^2} (\omega - \nu)^2 \|b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu)\|_{L^\infty} \|k\|_{L_u^\infty L^2(\mathcal{M}_u)} \|F_j^1(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ &\lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimates (2.37) for k , (2.38) for b , (2.44) for $\partial_\omega b$ and (2.42) for $\partial_\omega N$, Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch. Treating the other terms in the right-hand side of (5.26) similarly, we obtain:

$$\begin{aligned} \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d\mathcal{M} &= 2n^{-1}(t, x) \nabla_{N_\nu} n(t, x) b^{-2}(t, x, \nu) \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right)^2 \\ &\quad - 2\delta(t, x, \nu) b^{-2}(t, x, \nu) \left(\int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right)^2 + O(\varepsilon \gamma_j^\nu), \end{aligned}$$

which yields:

$$\begin{aligned}
\left| \int_{\partial\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} d_{\mathcal{O}\mathcal{M}} \right| &\lesssim \|n^{-1} \nabla_{N_\nu} n b^{-2}\|_{L^\infty} \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})}^2 \\
&\quad + \|\delta\|_{L_{x_\nu}^\infty L_{t_\nu}^2} \|b^{-2}\|_{L^\infty} \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u_\nu, x_\nu}^2 L_{t_\nu}^\infty} \\
&\quad \times \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})} + \varepsilon \gamma_j^\nu \\
&\lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})}^2 \\
&\quad + \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u_\nu, x_\nu}^2 L_{t_\nu}^\infty}^2 + \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimates (2.37) for δ , (2.36) for n , and (2.38) for b . Together with (5.25), this yields:

$$(5.27) \quad \|\mathbf{D}\phi\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u_\nu, x_\nu}^2 L_{t_\nu}^\infty} + \gamma_j^\nu.$$

In view of the definition of ϕ (5.19), we have:

$$\mathbf{D}\phi(t, x) = \int_{\mathbb{S}^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) L(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega.$$

Also:

$$\begin{aligned}
&\left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 (b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})} \\
&\lesssim \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimates (2.38) for b , (2.44) for $\partial_\omega b$ and (2.42) for $\partial_\omega L = \partial_\omega N$, Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch. Thus, we obtain:

$$\left\| \mathbf{D}\phi(t, x) - b^{-1}(t, x, \nu) L(t, x, \nu) \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})} \lesssim \gamma_j^\nu,$$

which together with the estimate (2.38) for b implies:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})} \lesssim \|\mathbf{D}\phi\|_{L^2(\partial\mathcal{M})} + \gamma_j^\nu.$$

Together with (5.27), we obtain:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\partial\mathcal{M})} \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u_\nu, x_\nu}^2 L_{t_\nu}^\infty} + \gamma_j^\nu.$$

Together with (5.16), this implies:

$$\left\| L_\nu \left(\int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} + \gamma_j^\nu$$

and thus:

$$\left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \leq C\varepsilon \left\| \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} + C\gamma_j^\nu$$

for some universal constant $C > 0$. Iterating, we obtain for any $q \geq 0$:

$$(5.28) \quad \left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \leq C^q \varepsilon^q \left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} + C \left(\sum_{l=0}^{q-1} C^l \varepsilon^l \right) \gamma_j^\nu,$$

where $F_j^q(u)$ is defined as:

$$F_j^q(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^{2+q} d\lambda.$$

We have:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \lesssim \left\| L_\nu \left(\int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})},$$

which together with the analog of (5.16) yields:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \lesssim \left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2(q+1)} F_j^{q+1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \gamma_j^\nu.$$

This implies the non sharp estimate:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \lesssim 2^{\frac{j}{2}} \gamma_j^\nu,$$

where we used Plancherel in λ , Cauchy-Schwarz in ω , and the size of the patch. Thus, we have:

$$(5.29) \quad C^q \varepsilon^q \left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \lesssim C^q \varepsilon^q 2^{\frac{j}{2}} \gamma_j^\nu \rightarrow 0 \text{ as } q \rightarrow +\infty,$$

since $\varepsilon > 0$ is small and may be chosen to ensure $0 < C\varepsilon < 1$. Finally, letting $q \rightarrow +\infty$ in (5.28) and taking (5.29) into account yields:

$$\left\| \int_{\mathbb{S}^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x\nu} L^{\infty}_{t\nu}} \lesssim \gamma_j^\nu.$$

This concludes the proof of Proposition 5.1.

CHAPTER 6

PROOF OF PROPOSITION 3.2 (ALMOST ORTHOGONALITY IN ANGLE)

We have to prove (3.14):

$$(6.1) \quad \|E_j f\|_{L^2(\mathcal{M})}^2 \lesssim \sum_{\nu \in \Gamma} \|E_j^\nu f\|_{L^2(\mathcal{M})}^2 + \varepsilon^2 \gamma_j^2.$$

This will result from an estimate for:

$$(6.2) \quad \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d\mathcal{M} \right|.$$

Remark 6.1. — In [12], the authors rely on a partial Fourier transform with respect to a coordinate system on $P_{t,u}$ to prove almost orthogonality in angle for their parametrix. In our case, coordinate systems on $P_{t,u}$ are not regular enough, which forces us to work invariantly. More precisely, we will use geometric integrations by parts tied to the u -foliation on \mathcal{M} in order to estimate (6.2).

Let us first explain why proceeding directly by integration by parts in (6.2) results in a log-loss.

6.1. Presence of a log-loss

Let us first introduce integrations by parts with respect to tangential derivatives. By definition of ∇ , we have $\nabla h = \nabla h - (\nabla_N h)N$ for any function h on σ . In particular, we have $\nabla(u) = 0$ and $\nabla(u') = b'^{-1}N' - b'^{-1}\mathbf{g}(N', N)N$. Now, since $|N' - \mathbf{g}(N', N)N|^2 = 1 - \mathbf{g}(N', N)^2$, this yields:

$$(6.3) \quad e^{i\lambda u - i\lambda' u'} = \frac{ib'}{\lambda'(1 - \mathbf{g}(N', N)^2)} \nabla_{N' - \mathbf{g}(N', N)N} (e^{i\lambda u - i\lambda' u'}),$$

where we have used the fact that $N' - \mathbf{g}(N', N)N$ is a tangent vector with respect of the level surfaces of u . Similarly, we have:

$$(6.4) \quad e^{i\lambda u - i\lambda' u'} = -\frac{ib}{\lambda(1 - \mathbf{g}(N, N')^2)} \nabla_{N - \mathbf{g}(N, N')N'} (e^{i\lambda u - i\lambda' u'}),$$

where we have used the fact that $N - \mathbf{g}(N, N')N'$ is a tangent vector with respect of the level surfaces of u' .

Next, we also introduce integrations by parts with respect to L . Since $L(u) = 0$ and $L(u') = b'^{-1}\mathbf{g}(L, L')$, we have:

$$(6.5) \quad e^{i\lambda u - i\lambda' u'} = \frac{ib'}{\lambda' \mathbf{g}(L, L')} L(e^{i\lambda u - i\lambda' u'}).$$

Similarly, we have:

$$(6.6) \quad e^{i\lambda u - i\lambda' u'} = \frac{ib}{\lambda \mathbf{g}(L, L')} L'(e^{i\lambda u - i\lambda' u'}).$$

We have:

$$\begin{aligned} \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d\mathcal{M} &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} \text{tr} \chi b'^{-1} \text{tr} \chi' d\mathcal{M} \right) \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'. \end{aligned}$$

We integrate by parts tangentially using (6.3). Since $\lambda' \sim 2^j$, and

$$(6.7) \quad 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2} \sim |\omega - \omega'|^2 \sim |\nu - \nu'|^2$$

in view of (2.43), we see that integrating by parts using (6.3) gains roughly $2^j|\nu - \nu'|$ at the expense of a tangential derivative. Consider the term where the tangential derivative falls on $\text{tr} \chi$, which is roughly of the form:

$$\begin{aligned} \frac{1}{2^j|\nu - \nu'|} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} \nabla \text{tr} \chi b'^{-1} \text{tr} \chi' d\mathcal{M} \right) \\ \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'. \end{aligned}$$

Since $L \nabla \text{tr} \chi$ is the only derivative of $\nabla \text{tr} \chi$ for which we have an estimate, our next integration by parts must be with respect to L , that is we use (6.5). Since $\lambda' \sim 2^j$, and since

$$(6.8) \quad \mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \sim |\nu - \nu'|^2,$$

in view of (6.7), we see that integrating by parts using (6.5) gains roughly $2^j|\nu - \nu'|^2$ at the expense of an L derivative. Consider the term where the L derivative falls on $\text{tr} \chi'$, which is roughly of the form:

$$\begin{aligned} \frac{1}{2^{2j}|\nu - \nu'|^3} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} \nabla \text{tr} \chi b'^{-1} L(\text{tr} \chi') d\mathcal{M} \right) \\ \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'. \end{aligned}$$

Now, note in view of (6.8) and the estimate (2.42) for $\partial_\omega N$, that:

$$\mathbf{g}(L, L') \sim |\nu - \nu'|^2, \quad \mathbf{g}(L, e'_A) = \mathbf{g}(L - L', e'_A) \sim |\nu - \nu'| \quad \text{and} \quad \mathbf{g}(L, \underline{L}') = -2 + \mathbf{g}(L, L') \sim 1.$$

Thus, decomposing L on the frame L', \underline{L}', e'_A , we obtain:

$$(6.9) \quad L \sim L' + |\nu - \nu'| \nabla' + |\nu - \nu'|^2 \underline{L}'.$$

We finally consider the term $|\nu - \nu'| \nabla' \text{tr} \chi'$ in the expansion of $L(\text{tr} \chi')$, and we obtain a term which is roughly of the form:

$$\frac{1}{2^{2j} |\nu - \nu'|^2} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} \nabla' \text{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \cdot \left(\int_{\mathbb{S}^2} b'^{-1} \nabla' \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}}.$$

We claim that such a term leads to a log-loss. Indeed, we have:

$$\begin{aligned} & \frac{1}{2^{2j} |\nu - \nu'|^2} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} \nabla' \text{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \cdot \left(\int_{\mathbb{S}^2} b'^{-1} \nabla' \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}} \\ & \lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \left\| \int_{\mathbb{S}^2} b^{-1} \nabla' \text{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \left\| \int_{\mathbb{S}^2} b'^{-1} \nabla' \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \left(\int_{\mathbb{S}^2} \|b^{-1} \nabla' \text{tr} \chi F_j(u)\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \|b'^{-1} \nabla' \text{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right) \\ & \lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \left(\int_{\mathbb{S}^2} \|b^{-1}\|_{L^\infty} \|\nabla' \text{tr} \chi\|_{L_u^\infty L^2(\mathcal{E}_u)} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \|b'^{-1}\|_{L^\infty} \|\nabla' \text{tr} \chi'\|_{L_u^\infty L^2(\mathcal{E}_u)} \|F_j(u')\|_{L_u^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ (6.10) \quad & \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}, \end{aligned}$$

where we used in the last inequality Plancherel in λ and λ' , Cauchy-Schwartz in ω and ω' which gains the square root of the volume of the patch, the estimates (2.38) for b , and the estimates (2.39) for $\text{tr} \chi$. This corresponds to a log-loss since we have ⁽¹⁾:

$$(6.11) \quad \sup_{\nu} \sum_{\nu' / 1 \leq 2^{j/2} |\nu - \nu'| \leq 2^{j/2}} \frac{1}{(2^{j/2} |\nu - \nu'|)^2} \sim j.$$

Indeed, note that ν' runs on a lattice on \mathbb{S}^2 of basic size $2^{-j/2}$ so that (6.11) corresponds to the sum

$$\sum_{l \in \mathbb{Z}^2, 1 \leq |l| \leq 2^{j/2}} \frac{1}{|l|^2} \sim j.$$

6.2. A physical space decomposition for $E_j^\nu f$

To remove the log-loss exhibited in (6.10) (6.11), we need a further decomposition. The same problem was present when dealing with the parametrix at initial time

1. The log divergence in (6.11) is due to the fact that we are working at the level of H^2 solutions for Einstein equations. Indeed, summations similar to (6.11) appear in particular in [12] in the context of $H^{2+\varepsilon}$ solutions of quasilinear wave equations, albeit with a power strictly larger than 2, and hence without log divergence.

in [16]. In that case, we introduced a second decomposition in λ and exploited the corresponding gain of the size of the patch in λ when estimating $F_j(u)$ in L^∞ and taking Cauchy-Schwartz in λ . In the present situation, all norms for $\text{tr}\chi$ are estimated by taking the L^∞ norm in u . In turn, we always estimate $F_j(u)$ in L^2 using Plancherel and can therefore not exploit the size of the patch in λ .

Instead, we rely here on a decomposition of $\text{tr}\chi$ using the geometric Littlewood-Paley projections P_j . We have:

$$\text{tr}\chi = P_{\leq j/2}(\text{tr}\chi) + \sum_{l>j/2} P_l \text{tr}\chi,$$

which in turn yields the following decomposition for $E_j^\nu f$:

$$(6.12) \quad E_j^\nu f(t, x) = \sum_{l \geq j/2} E_j^{\nu, l} f(t, x),$$

where:

$$(6.13) \quad E_j^{\nu, l} f(t, x) = \int_{\mathbb{S}^2} b(t, x, \omega)^{-1} P_l \text{tr}\chi(t, x, \omega) F_j(u) \eta_j^\nu(\omega) d\omega \quad \forall l > \frac{j}{2}$$

and:

$$(6.14) \quad E_j^{\nu, j/2} f(t, x) = \int_{\mathbb{S}^2} b(t, x, \omega)^{-1} P_{\leq j/2} \text{tr}\chi(t, x, \omega) F_j(u) \eta_j^\nu(\omega) d\omega.$$

6.3. The mechanism to remove the log-loss

In order to prove almost orthogonality in angle, i.e., (6.1), we will estimate:

$$(6.15) \quad \left| \sum_{l, m} \int_{\mathcal{M}} E_j^{\nu, l} f(t, x) \overline{E_j^{\nu', m} f(t, x)} d\mathcal{O}\mathcal{M} \right|.$$

Let us assume for convenience that $m \leq l$ in (6.15). In order to remove the log-loss, our goal will be to always put more tangential derivatives on the lowest frequency, i.e., $P_m \text{tr}\chi'$ (as opposed to the higher frequency $P_l \text{tr}\chi$). This will be achieved as follows:

1. Integrate by parts with respect to L using (6.5).
2. One term corresponds to the case where the L derivative falls on the largest frequency $P_l \text{tr}\chi$, while the other term corresponds to the case where L falls on the lowest frequency $P_m \text{tr}\chi'$. For the second term, decompose the L derivative on the frame L', N', e'_A as in (6.9).
3. We claim the the terms involving L and L' do not contain any log-loss. Indeed, instead of the sum (6.11) containing the log-loss, they will ultimately yield

$$\sup_{\nu} \sum_{\nu' / 1 \leq 2^{j/2} |\nu - \nu'| \leq 2^{j/2}} \frac{1}{(2^{j/2} |\nu - \nu'|)^3} \leq 1.$$

4. We claim the the term involving N' does not contain any log-loss. Indeed, instead of the sum (6.11) containing the log-loss, it will ultimately yield

$$\sup_{\nu} \sum_{\nu' / 1 \leq 2^{j/2} |\nu - \nu'| \leq 2^{j/2}} \frac{1}{2^{j/2} (2^{j/2} |\nu - \nu'|)} \leq 1.$$

5. Finally, the last term is the one containing the ∇' derivative. This term is the only one which contains the log-loss exhibited in (6.11). Now, we have achieved our goal since after integration by parts, the tangential derivative fell on $P_m \text{tr} \chi'$ which is the lowest frequency.

Remark 6.2. — Due to the decomposition (6.12), we now not only need to obtain summability in (ν, ν') , but also in (l, m) .

6.4. The main estimates

Recall that in order to prove almost orthogonality in angle, i.e., (6.1), we will estimate:

$$\left| \sum_{l,m} \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} \right|.$$

We will distinguish the following two regions:

$$2^{\min(l,m)} > 2^j |\nu - \nu'| \text{ and } 2^{\min(l,m)} \leq 2^j |\nu - \nu'|.$$

We start with the estimate in the first region.

Proposition 6.3. — *If $\nu \neq \nu'$ and $2^{\min(l,m)} > 2^j |\nu - \nu'|$, we have the following estimate:*

$$(6.16) \quad \left| \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} \right| \lesssim 2^{-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Proof. — We have:

$$(6.17) \quad \begin{aligned} & \left| \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} \right| \\ & \lesssim \|E_j^{\nu,l} f\|_{L^2(\mathcal{M})} \|E_j^{\nu',m} f\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\int_{\mathbb{S}^2} \|b(t, x, \omega)^{-1} P_l \text{tr} \chi(t, x, \omega) F_j(u)\|_{L^2(\mathcal{M})} \eta_j^{\nu}(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \|b(t, x, \omega')^{-1} P'_m \text{tr} \chi(t, x, \omega') F_j(u')\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

$$\begin{aligned}
&\lesssim \|b^{-1}\|_{L^\infty}^2 \left(\int_{\mathbb{S}^2} \left\| \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)} F_j(u) \right\|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} \left\| \|P_m' \operatorname{tr} \chi'\|_{L^2(\mathcal{M}_{u'})} F_j(u') \right\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\
&\lesssim 2^{-j} \left\| \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)} F_j(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L_{\omega,u}^2} \left\| \|P_m' \operatorname{tr} \chi'\|_{L^2(\mathcal{M}_{u'})} F_j(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L_{\omega',u'}^2},
\end{aligned}$$

where we used in the last inequality the estimates (2.38) for b , Cauchy Schwarz in ω and ω' , and the size of the patch.

Now, we have:

$$\begin{aligned}
(6.18) \quad &\sum_\nu \sum_{l>j/2} 2^{2l} \left\| \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)} F_j(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L_{\omega,u}^2}^2 \\
&= \sum_\nu \int_{\mathbb{S}^2} \left(\int_u \left(\sum_{l>j/2} 2^{2l} \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)}^2 \right) |F_j(u)|^2 du \right) \eta_j^\nu(\omega) d\omega \\
&\lesssim \sum_\nu \int_{\mathbb{S}^2} \left(\int_u \|\nabla \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)}^2 |F_j(u)|^2 du \right) \eta_j^\nu(\omega) d\omega \\
&\lesssim \sum_\nu \int_{\mathbb{S}^2} \|\nabla \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{M}_u)}^2 \|F_j(u)\|_{L_u^2}^2 \eta_j^\nu(\omega) d\omega \\
&\lesssim \varepsilon^2 2^{2j} \sum_\nu (\gamma_j^\nu)^2 \\
&\lesssim 2^{2j} \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned}$$

where we used the finite band property for P_l , the estimates (2.39) for $\operatorname{tr} \chi$ and Plancherel in λ . (6.17) and (6.18) yield the proof of the proposition. \square

Remark 6.4. — In (6.18), we used the estimate:

$$\sup_{\omega,u} \left(\sum_{l>j/2} 2^{2l} \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)}^2 \right) \lesssim \sup_{\omega,u} \|\nabla \operatorname{tr} \chi\|_{L^2(\mathcal{M}_u)}^2 \lesssim \varepsilon^2,$$

which is true in view of the finite band property for P_l and the estimates (2.39) for $\operatorname{tr} \chi$. Note that the sum in l has to be taken *before* the sup in ω and u for the estimate to hold. This explains why the L^2 norm on $\mathbb{R} \times \mathbb{S}^2$ is present in (6.16) and estimated only after letting the sum in l enter the integral as in (6.18).

Next, we consider the second region. We have the following decomposition:

Proposition 6.5. — *If $\nu \neq \nu'$ and $2^{\min(l,m)} \leq 2^j |\nu - \nu'|$, we have the following decomposition:*

$$(6.19) \quad \int_{\mathcal{M}} E_j^{\nu,l} f(t,x) \overline{E_j^{\nu',m} f(t,x)} d\mathcal{M} = A_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m},$$

where $B_{j,\nu,\nu',l,m}$ satisfies:

$$(6.20) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m} + B_{j,\nu',\nu,l,m}) \right| \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $A_{j,\nu,\nu',l,m}$ for (l, m) such that $2^{\min(l,m)} \leq 2^j |\nu - \nu'|$. We consider the following two subregions:

$$2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)} \quad \text{and} \quad 2^{\max(l,m)} \leq 2^j |\nu - \nu'|,$$

starting with the first one:

Proposition 6.6. — *If $\nu \neq \nu'$ and $2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}$, we have the following estimate:*

$$(6.21) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \right| \lesssim \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} \frac{2^{-2j} 2^{2\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|) 2^{\frac{j}{2}}} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l > j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l > j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Finally, we estimate $A_{j,\nu,\nu',l,m}$ for (l, m) such that $2^{\max(l,m)} \leq 2^j |\nu - \nu'|$.

Proposition 6.7. — *If $\nu \neq \nu'$ and $2^{\max(l,m)} \leq 2^j |\nu - \nu'|$, we have the following estimate:*

$$(6.22) \quad \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}$$

$$+ \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

The proof of Proposition 6.5 is postponed to Chapter 8, the proof of Proposition 6.6 is postponed to Chapter 9, and the proof of Proposition 6.7 is postponed to Chapter 10.

6.5. End of the proof of Proposition 3.2

We conclude the proof of Proposition 3.2 by using Proposition 6.3, Proposition 6.5, Proposition 6.6 and Proposition 6.7. In view of (6.12), we have:

$$\begin{aligned} & \sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d_{\mathcal{M}} \right| \\ & \lesssim \sum_{\nu \neq \nu'} \left| \sum_{l,m} \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d_{\mathcal{M}} \right| \\ & \lesssim \sum_{\nu \neq \nu'} \sum_{2^{\min(l,m)} > 2^j |\nu - \nu'|} \left| \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d_{\mathcal{M}} \right| \\ & \quad + \sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^{\nu, \leq j/2} f(t, x) \overline{E_j^{\nu', \leq j/2} f(t, x)} d_{\mathcal{M}} \right| \\ & \quad + \sum_{\nu \neq \nu'} \left| \sum_{2^{\min(l,m)} \leq 2^j |\nu - \nu'|} \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d_{\mathcal{M}} \right|. \end{aligned}$$

In view of Proposition 6.3 and Proposition 6.5, we obtain:

$$\begin{aligned} (6.23) \quad & \sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d_{\mathcal{M}} \right| \\ & \lesssim \sum_{\nu \neq \nu'} \sum_{2^{\min(l,m)} > 2^j |\nu - \nu'|} 2^{-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & \quad + \sum_{\nu \neq \nu'} \left| \sum_{2^{\min(l,m)} \leq 2^j |\nu - \nu'|} \int_{\mathcal{M}} A_{j,\nu,\nu',l,m} \right| \\ & \quad + \sum_{\nu \neq \nu'} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} \right] \end{aligned}$$

$$+ \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \left] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \right.$$

Now, we have:

$$\left(\sum_{2^{\frac{j}{2}}|\nu - \nu'| < 2^{\min(l,m) - \frac{j}{2}}} 1 \right)^{\frac{1}{2}} \lesssim 2^{2\min(l,m) - j}$$

and:

$$\begin{aligned} & \sum_{\nu \neq \nu'} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{3}{2}}} \right. \\ & \quad \left. + \frac{2^{-(\frac{1}{2}) - j}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \lesssim \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

Together with (6.23), we obtain:

$$\begin{aligned} & \sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d_{\mathcal{C}} \mathcal{M} \right| \\ & \lesssim 2^{-2j} \sum_{l, m} 2^{-|l-m|} \left(\sum_{\nu} 2^{2l} \|\mu_{j, \nu, l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{\nu'} 2^{2m} \|\mu_{j, \nu', m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \\ & \quad + \sum_{\nu \neq \nu'} \left| \sum_{2^{\min(l,m)} \leq 2^j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} \right| + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\ & \lesssim 2^{-2j} \left(\sum_{l, \nu} 2^{2l} \|\mu_{j, \nu, l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{m, \nu'} 2^{2m} \|\mu_{j, \nu', m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \\ & \quad + \sum_{\nu \neq \nu'} \left| \sum_{2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} \right| \\ & \quad + \sum_{\nu \neq \nu'} \left| \sum_{2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} \right| \\ & \quad + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

Together with Proposition 6.6 and Proposition 6.7, this yields:

(6.24)

$$\sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d_{\mathcal{C}} \mathcal{M} \right|$$

$$\begin{aligned}
&\lesssim 2^{-2j} \left(\sum_{l,\nu} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{m,\nu'} 2^{2m} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \\
&+ \sum_{\nu \neq \nu'} \sum_{2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} \frac{2^{-2j} 2^{2\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\
&+ \sum_{\nu \neq \nu'} \sum_{2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \\
&+ \sum_{\nu \neq \nu'} \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
&+ \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

Now, we have:

$$\begin{aligned}
\left(\sum_{2^{\min(l,m)-\frac{j}{2}} \leq 2^{\frac{j}{2}} |\nu - \nu'| < 2^{\max(l,m)-\frac{j}{2}}} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \right)^{\frac{1}{2}} &\lesssim \log(2^{\max(l,m)-\frac{j}{2}}) - \log(2^{\min(l,m)-\frac{j}{2}}) \\
&\lesssim \max(l,m) - \min(m,l), \\
\left(\sum_{2^{\frac{j}{2}} |\nu - \nu'| > 2^{\max(l,m)-\frac{j}{2}}} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \right)^{\frac{1}{2}} &\lesssim 2^{-\max(l,m)+\frac{j}{2}}
\end{aligned}$$

and:

$$\begin{aligned}
\sum_{\nu \neq \nu'} \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
\lesssim \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

Together with (6.24), we obtain:

$$\begin{aligned}
&\sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t,x) \overline{E_j^{\nu'} f(t,x)} d\mathcal{M} \right| \\
&\lesssim 2^{-2j} \sum_{l,m} (1+|l-m|) 2^{-|l-m|} \left(\sum_{\nu} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{\nu'} 2^{2m} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \\
&+ 2^{-2j} \left(\sum_{l,\nu} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{m,\nu'} 2^{2m} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
&\lesssim 2^{-2j} \left(\sum_{l,\nu} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left(\sum_{m,\nu'} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \right)^{\frac{1}{2}} + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

Since the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2,$$

we finally obtain:

$$\sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^{\nu'} f(t, x)} d\mathcal{M} \right| \lesssim \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 3.2.

The rest of the paper is as follows. In Chapter 7, we derive estimates for oscillatory integrals in various norms, as well as integrations by parts formulas tied to the u -foliation on \mathcal{M} . In Chapter 8, we prove Proposition 6.5. In Chapter 9, we prove Proposition 6.6. Finally, we prove Proposition 6.7 in Chapter 10.

CHAPTER 7

THE KEY ESTIMATES

7.1. Estimate of the $L^p(\mathcal{M})$ norm of oscillatory integrals

Lemma 7.1. — *Let H a tensor on \mathcal{M} . Then, we have the following estimate:*

$$(7.1) \quad \left\| \int_{\mathbb{S}^2} HF_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu.$$

More generally, for $2 \leq p \leq +\infty$, we have:

$$(7.2) \quad \left\| \int_{\mathbb{S}^2} HF_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L_u^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu.$$

Proof. — We have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} HF_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \|HF_j(u)\|_{L^p(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|H\|_{L_u^\infty L^p(\mathcal{H}_u)} \|F_j(u)\|_{L_u^p} \eta_j^\nu(\omega) d\omega \\ & \lesssim \left(\sup_{\omega} \|H\|_{L_u^\infty L^p(\mathcal{H}_u)} \right) \int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2}^{\frac{2}{p}} \|F_j(u)\|_{L_u^\infty}^{1-\frac{2}{p}} \eta_j^\nu(\omega) d\omega. \end{aligned}$$

Using Plancherel to estimate $\|F_j(u)\|_{L_u^2}$, Cauchy-Schwartz in λ to estimate $\|F_j(u)\|_{L_u^\infty}$, Cauchy-Schwarz in ω and the size of the patch, we obtain:

$$\left\| \int_{\mathbb{S}^2} HF_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L_u^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu,$$

which concludes the proof of the lemma. □

Corollary 7.2. — *Let H a tensor on \mathcal{M} . Then, we have the following estimate:*

$$(7.3) \quad \left\| \int_{\mathbb{S}^2} HP_m(\text{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m} 2^{\frac{j}{2}} \gamma_j^\nu.$$

More generally, for $2 \leq p \leq +\infty$, we have:

$$(7.4) \quad \left\| \int_{\mathbb{S}^2} HP_m(\operatorname{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m\frac{2}{p}} 2^{j(1-\frac{1}{p})} \gamma_j^\nu.$$

Proof. — In view of (7.2), we have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} HP_m(\operatorname{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \|HP_m(\operatorname{tr}\chi)\|_{L_u^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu \\ & \lesssim \left(\sup_{\omega} \|H\|_{L^\infty} \right) \left(\sup_{\omega} \|P_m(\operatorname{tr}\chi)\|_{L_u^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu. \end{aligned}$$

Using Bernstein on $P_{t,u}$ and the finite band property for P_m , we obtain:

$$\left\| \int_{\mathbb{S}^2} HP_m(\operatorname{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L^\infty} \right) 2^{-m\frac{2}{p}} \left(\sup_{\omega} \|\nabla \operatorname{tr}\chi\|_{L_t^\infty L_{x'}^2} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu.$$

Together with the estimates (2.39) for $\operatorname{tr}\chi$, we obtain:

$$\left\| \int_{\mathbb{S}^2} HP_m(\operatorname{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left(\sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m\frac{2}{p}} 2^{j(1-\frac{1}{p})} \gamma_j^\nu,$$

which concludes the proof of the corollary. \square

7.2. Estimates of the $L^1(\mathcal{M})$ norm of oscillatory integrals

Lemma 7.3. — *Let ν, ν' in \mathbb{S}^2 such that $\nu \neq \nu'$. Recall the decomposition $\widehat{\chi} = \chi_1 + \chi_2$ in (2.45). Let H a tensor on \mathcal{M} . Then, we have the following estimate:*

$$(7.5) \quad \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HD(L(\operatorname{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} \lesssim \left(\sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} (\|H\|_{L_u^2 L_t^4 L_{x'}^2} + |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H\|_{L^2(\mathcal{M})}) \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu.$$

Proof. — We have:

$$(7.6) \quad \begin{aligned} \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HD(L(\operatorname{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} & \lesssim \int_{\mathbb{S}^2} \|HD(L(\operatorname{tr}\chi))F_j(u)\|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|HD(L(\operatorname{tr}\chi))\|_{L_u^2 L^1(\mathcal{H}_u)} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega. \end{aligned}$$

Differentiating the Raychaudhuri Equation (2.22), we obtain:

$$\mathbf{D}(L(\operatorname{tr}\chi)) = -(\operatorname{tr}\chi + \bar{\delta})\mathbf{D}\operatorname{tr}\chi - 2\widehat{\chi}\mathbf{D}\widehat{\chi} - \mathbf{D}(\bar{\delta})\operatorname{tr}\chi.$$

Using the decomposition $\widehat{\chi} = \chi_1 + \chi_2$ in (2.45), we obtain:

$$(7.7) \quad \mathbf{D}(L(\operatorname{tr}\chi)) = G_1 + \chi_2 G_2,$$

where

$$G_1 = -(\operatorname{tr}\chi + \bar{\delta})\mathbf{D}\operatorname{tr}\chi - 2\chi_1\mathbf{D}\widehat{\chi} - \mathbf{D}(\bar{\delta})\operatorname{tr}\chi$$

and

$$G_2 = -2\mathbf{D}\widehat{\chi}.$$

In particular, we have:

$$(7.8) \quad \|G_1\|_{L_u^\infty L_t^{\frac{4}{3}} L_{x'}^2} + \|G_2\|_{L_u^\infty L^2(\mathcal{E}u)} \lesssim \varepsilon,$$

where we used the estimate (2.39) for $\operatorname{tr}\chi$, the estimate $\widehat{\chi}$ for $\widehat{\chi}$, the estimate (2.37) and (2.36) for $\bar{\delta}$, and the estimate (2.47) for χ_1 . In view of (7.6) and (7.7), we have:

$$(7.9) \quad \begin{aligned} & \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H\mathbf{D}(L(\operatorname{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d_{\mathcal{M}} \\ & \lesssim \int_{\mathbb{S}^2} \|HG_1\|_{L_u^2 L^1(\mathcal{E}u)} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2} \|\chi_2 HG_2\|_{L_u^2 L^1(\mathcal{E}u)} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|H\|_{L_u^2 L_t^4 L_{x'}^2} \|G_1\|_{L_u^\infty L_t^{\frac{4}{3}} L_{x'}^2} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega \\ & \quad + \int_{\mathbb{S}^2} \|\chi_2 H\|_{L^2(\mathcal{M})} \|G_2\|_{L_u^\infty L^2(\mathcal{E}u)} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega \\ & \lesssim \sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} (\|H\|_{L_u^2 L_t^4 L_{x'}^2}) \varepsilon \int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega \\ & \quad + \varepsilon \int_{\mathbb{S}^2} \|\chi_2 H\|_{L^2(\mathcal{M})} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega, \end{aligned}$$

where we used in the last inequality the estimate (7.8). In view of the estimate (2.47) for χ_2 , we have:

$$\|\chi_2 - \chi_{2\nu'}\|_{L^{6-}(\mathcal{M})} \lesssim |\nu - \nu'| \|\partial_\omega \chi_2\|_{L^{6-}(\mathcal{M})} \lesssim |\nu - \nu'| \varepsilon,$$

which yields:

$$\begin{aligned} \|\chi_2 H\|_{L^2(\mathcal{M})} & \lesssim \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} + \|(\chi_2 - \chi_{2\nu})H\|_{L^2(\mathcal{M})} \\ & \lesssim \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} + \|\chi_2 - \chi_{2\nu}\|_{L^{6-}(\mathcal{M})} \|H\|_{L^{3+}(\mathcal{M})} \\ & \lesssim \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} + \varepsilon |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})}. \end{aligned}$$

Together with (7.9), we obtain:

$$\begin{aligned} & \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H\nabla L(\operatorname{tr}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right| d_{\mathcal{M}} \\ & \lesssim \left(\sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} (\|H\|_{L_u^2 L_t^4 L_{x'}^2}) + |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H\|_{L^2(\mathcal{M})} \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega)d\omega \right) \end{aligned}$$

$$\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H\|_{L_u^2 L_t^4 L_x^2}) + |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H\|_{L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in λ for $\|F_j(u)\|_{L_u^2}$, Cauchy-Schwarz in ω , and the size of the patch. This concludes the proof of the lemma. \square

Lemma 7.4. — *Let ν, ν' in \mathbb{S}^2 such that $\nu \neq \nu'$. Let l an integer. Recall the decomposition $\widehat{\chi} = \chi_1 + \chi_2$ in (2.45). Let H a tensor on \mathcal{M} . Then, we have the following estimate:*

$$(7.10) \quad \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| d\mathcal{M} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L_{u,x}^2 L_t^\infty} \right) 2^{\frac{j}{2}-l} \varepsilon \gamma_j^\nu.$$

Proof. — We have:

$$\begin{aligned} \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| d\mathcal{M} &\lesssim \int_{\mathbb{S}^2} \|HL(P_l \text{tr} \chi) F_j(u)\|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|HL(P_l \text{tr} \chi)\|_{L_u^2 L^1(\mathcal{E}_{u_x})} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|nL(P_l \text{tr} \chi)\|_{L_u^2 L^1(\mathcal{E}_{u_x})} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used the estimate (2.36) on n in the last inequality. This yields:

$$(7.11) \quad \begin{aligned} \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| d\mathcal{M} \\ \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L_{u,x}^2 L_t^\infty} \right) \int_{\mathbb{S}^2} \|nL(P_l \text{tr} \chi)\|_{L_x^2 L_t^1} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega. \end{aligned}$$

Next, we estimate $nL(P_l \text{tr} \chi)$. We have:

$$nL(P_l \text{tr} \chi) = [nL, P_l](\text{tr} \chi) + P_l(nL(\text{tr} \chi)),$$

which yields:

$$(7.12) \quad \begin{aligned} \|nL(P_l \text{tr} \chi)\|_{L_x^2 L_t^1} &\lesssim \|[nL, P_l] \text{tr} \chi\|_{L_t^1 L_x^2} + \|P_l(nL \text{tr} \chi)\|_{L_x^2 L_t^1} \\ &\lesssim 2^{-l} \varepsilon, \end{aligned}$$

where we used in the last inequality the commutator estimate (2.66) and the estimate (2.69). Now, (7.11) and (7.12) imply:

$$\begin{aligned} \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| d\mathcal{M} \\ \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L_{u,x}^2 L_t^\infty} \right) 2^{-l} \varepsilon \left(\int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \\ \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L_{u,x}^2 L_t^\infty} \right) 2^{\frac{j}{2}-l} \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality Plancherel in u , Cauchy-Schwarz in ω and the size of the patch. This concludes the proof of the lemma. \square

Lemma 7.5. — *Let ν, ν' in \mathbb{S}^2 such that $\nu \neq \nu'$. Let l an integer. Recall the decomposition $\widehat{\chi} = \chi_1 + \chi_2$ in (2.45). Let H a tensor on \mathcal{M} . Then, we have the following estimate:*

(7.13)

$$\int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H \nabla(L(P_{\leq l} \text{tr} \chi)) F_j(u) \eta_j^{\nu'}(\omega) d\omega \right| d\mathcal{M} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^{\nu'})} \|H\|_{L_{u,x}^2, L_t^\infty} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Proof. — We have:

$$\begin{aligned} & \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H \nabla(L(P_{\leq l} \text{tr} \chi)) F_j(u) \eta_j^{\nu'}(\omega) d\omega \right| d\mathcal{M} \\ & \lesssim \int_{\mathbb{S}^2} \|H \nabla(L(P_{\leq l} \text{tr} \chi)) F_j(u)\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|H \nabla(L(P_{\leq l} \text{tr} \chi))\|_{L_u^2 L^1(\mathcal{E}_{\mu_u})} \|F_j(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|H n \nabla(L(P_{\leq l} \text{tr} \chi))\|_{L_u^2 L^1(\mathcal{E}_{\mu_u})} \|F_j(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega, \end{aligned}$$

where we used the estimate (2.36) for n in the last inequality. This yields:

$$(7.14) \quad \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H \nabla(L(P_{\leq l} \text{tr} \chi)) F_j(u) \eta_j^{\nu'}(\omega) d\omega \right| d\mathcal{M} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^{\nu'})} \|H\|_{L_{u,x}^2, L_t^\infty} \right) \int_{\mathbb{S}^2} \|n \nabla(L(P_{\leq l} \text{tr} \chi))\|_{L_x^2, L_t^1} \|F_j(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega.$$

Next, we estimate $\nabla(nL(P_{\leq l} \text{tr} \chi))$. We have:

$$n \nabla(L(P_{\leq l} \text{tr} \chi)) = -\nabla n L(P_{\leq l} \text{tr} \chi) + \nabla[nL, P_{\leq l}](\text{tr} \chi) + \nabla(P_{\leq l}(nL(\text{tr} \chi))),$$

which yields:

$$(7.15) \quad \begin{aligned} \|n \nabla(L(P_{\leq l} \text{tr} \chi))\|_{L_x^2, L_t^1} & \lesssim \|n^{-1} \nabla n\|_{L^\infty} \|nL(P_{\leq l} \text{tr} \chi)\|_{L_t^1 L_x^2} \\ & \quad + \|\nabla[nL, P_{\leq l} \text{tr} \chi]\|_{L_t^1 L_x^2} + \|\nabla(P_{\leq l}(nL(\text{tr} \chi)))\|_{L_x^2, L_t^1} \\ & \lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.36) for n , the commutator estimate (2.66) and the estimate (2.70). Now, (7.14) and (7.15) imply:

$$\begin{aligned} & \int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H \nabla(L(P_{\leq l} \text{tr} \chi)) F_j(u) \eta_j^{\nu'}(\omega) d\omega \right| d\mathcal{M} \\ & \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^{\nu'})} \|H\|_{L_{u,x}^2, L_t^\infty} \right) \varepsilon \left(\int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega \right) \end{aligned}$$

$$\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L_{u,x'}^2, L_t^\infty} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in u , Cauchy-Schwarz in ω and the size of the patch. This concludes the proof of the lemma. \square

7.3. Estimate of the $L_{u,x'}^2, L_t^\infty$ norm of oscillatory integrals

Lemma 7.6. — *Let $p \in \mathbb{N}$. We have:*

$$(7.16) \quad \left\| \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,x',\nu}^2, L_t^\infty} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

Proof. — Note that it suffices to show:

$$(7.17) \quad \left\| L_\nu \left(\int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L_{u,x',\nu}^2, L_t^1} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

We have:

$$(7.18) \quad \begin{aligned} L_\nu \left(\int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ = i \int_{\mathbb{S}^2} b^{-2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p 2^j \mathbf{g}(L, L_\nu) F_{j,1}(u) \eta_j^\nu(\omega) d\omega \\ + \int_{\mathbb{S}^2} L_\nu (b^{-1} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \\ + p \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \left(2^{\frac{j}{2}} (\mathbf{D}_{L_\nu} L - \mathbf{D}_{L_\nu} L_\nu) \right) F_j(u) \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used the fact that $N - N_\nu = L - L_\nu$ since T does not depend on ω , and the fact that $b^{-1}L$ is the space-time gradient of u so that:

$$L_\nu(u) = b^{-1} \mathbf{g}(L_\nu, L).$$

Next, we evaluate the various terms in the right-hand side of (7.18). First, recall the identity (5.11):

$$(7.19) \quad \mathbf{g}(L, L_\nu) = \mathbf{g}(N - N_\nu, N - N_\nu).$$

Next, decompose L_ν on the frame $L, \underline{L}, e_A, A = 1, 2$ which yields:

$$(7.20) \quad L_\nu = \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) L + (N_\nu - \mathbf{g}(N, N_\nu) N) + \frac{1}{2} (1 - \mathbf{g}(N, N_\nu)) \underline{L},$$

which yields:

$$(7.21) \quad \begin{aligned} L_\nu (b^{-1} \text{tr} \chi) &= \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) L (b^{-1} \text{tr} \chi) + (N_\nu - \mathbf{g}(N, N_\nu) N) (b^{-1} \text{tr} \chi) \\ &\quad + \frac{1}{2} (1 - \mathbf{g}(N, N_\nu)) \underline{L} (b^{-1} \text{tr} \chi). \end{aligned}$$

Also, in view of the decomposition (7.19) and the Ricci Equations (2.17), we have:

(7.22)

$$\begin{aligned} & \mathbf{D}_{L_\nu} L - \mathbf{D}_{L_\nu} L_\nu \\ &= \frac{1}{2}(1 + \mathbf{g}(N, N_\nu))\mathbf{D}_L L + \mathbf{D}_{N_\nu - \mathbf{g}(N, N_\nu)N} L + \frac{1}{2}(1 - \mathbf{g}(N, N_\nu))\mathbf{D}_L L - \mathbf{D}_{L_\nu} L_\nu \\ &= -\frac{1}{2}(1 + \mathbf{g}(N, N_\nu))\bar{\delta}L + \chi(N_\nu - \mathbf{g}(N, N_\nu)N, e_A)e_A - \epsilon_{N_\nu - \mathbf{g}(N, N_\nu)N} L \\ & \quad + \frac{1}{2}(1 - \mathbf{g}(N, N_\nu))(\zeta_A e_A + (\delta + n^{-1}\nabla_N n)L) - \bar{\delta}_\nu L_\nu. \end{aligned}$$

Now, (7.18), (7.19), (7.21) and (7.22) yield:

$$(7.23) \quad \begin{aligned} & L_\nu \left(\int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7, \end{aligned}$$

where $A_1, A_2, A_3, A_4, A_5, A_6$ and A_7 are respectively given by:

$$(7.24) \quad A_1 = \int_{\mathbb{S}^2} b^{-2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p+2} F_{j,1}(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.25) \quad A_2 = \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) L (b^{-1} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.26) \quad A_3 = \int_{\mathbb{S}^2} (N_\nu - \mathbf{g}(N, N_\nu)N) (b^{-1} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.27) \quad A_4 = \int_{\mathbb{S}^2} \frac{1}{2} (1 - \mathbf{g}(N, N_\nu)) \underline{L} (b^{-1} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.28) \quad \begin{aligned} A_5 &= p \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} (2^{\frac{j}{2}} (-\bar{\delta}L + \bar{\delta}_\nu L_\nu)) \\ & \quad \times F_j(u) \eta_j^\nu(\omega) d\omega, \end{aligned}$$

$$(7.29) \quad \begin{aligned} A_6 &= p \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \\ & \quad \times (2^{\frac{j}{2}} (\chi(N_\nu - \mathbf{g}(N, N_\nu)N, e_A)e_A - \epsilon_{N_\nu - \mathbf{g}(N, N_\nu)N} L)) F_j(u) \eta_j^\nu(\omega) d\omega \end{aligned}$$

and:

$$(7.30) \quad \begin{aligned} A_7 &= p \int_{\mathbb{S}^2} \frac{1}{2} (1 - \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \\ & \quad \times (2^{\frac{j}{2}} (\zeta_A e_A + (\delta + n^{-1}\nabla_N n)L)) F_j(u) \eta_j^\nu(\omega) d\omega. \end{aligned}$$

We estimate $A_1, A_2, A_3, A_4, A_5, A_6$ and A_7 starting with A_1 . Recall the decomposition (2.71) for $2^{\frac{j}{2}}(N - N_\nu)$:

$$(7.31) \quad 2^{\frac{j}{2}}(N - N_\nu) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$(7.32) \quad \|F_1^j\|_{L^\infty} \lesssim 1$$

and where the tensor F_2^j satisfies:

$$(7.33) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{E}u)} \lesssim 2^{-\frac{j}{2}}.$$

This yields:

$$\left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p+2} = \sum_{m=0}^{p+1} F_{1,j}^m \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} F_{2,j} + F_{1,j}^{p+2}$$

and thus:

$$\begin{aligned} A_1 &= \sum_{m=0}^{p+1} F_{1,j}^m \left(\int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad + F_{1,j}^{p+2} \left(\int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right), \end{aligned}$$

where we used the fact that F_1^j does not depend on ω . We obtain:

$$\begin{aligned} \|A_1\|_{L^2(\mathcal{O}\mathcal{M})} &\lesssim \sum_{m=0}^{p+1} \|F_{1,j}\|_{L^\infty}^m \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad + \|F_{1,j}\|_{L^\infty}^{p+2} \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \sum_{m=0}^{p+1} \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used (7.32) in the last inequality. The estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) for oscillatory integrals yields:

$$\begin{aligned} \|A_1\|_{L^2(\mathcal{O}\mathcal{M})} &\lesssim \sum_{m=0}^{p+1} \left(\sup_{\omega} \left\| b^{-2} \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} F_{2,j} \right\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\quad + \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \sum_{m=0}^{p+1} \left(\sup_{\omega} \left\| b^{-2} \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-m+1} \right\|_{L^\infty} \|F_{2,j}\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\quad + \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ (7.34) \quad &\lesssim (1+p) \varepsilon \gamma_j^\nu + \left\| \int_{\mathbb{S}^2} b^{-2} \operatorname{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (7.33) and the estimates (2.38) for b and (2.39) for $\operatorname{tr} \chi$.

Next, we estimate the second term in the right-hand side of (7.34). In view of the decomposition (2.72) for $\text{tr}\chi$, and the decomposition (2.73) for b , we have:

$$(7.35) \quad b^{-2}\text{tr}\chi = f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$(7.36) \quad \|f_1^j\|_{L^\infty} \lesssim \varepsilon$$

and where the scalar f_2^j satisfies:

$$(7.37) \quad \|f_2^j\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

This yields:

$$\int_{\mathbb{S}^2} b^{-2}\text{tr}\chi F_{j,1}(u)\eta_j^\nu(\omega)d\omega = f_1^j \int_{\mathbb{S}^2} F_{j,1}(u)\eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2} f_2^j F_{j,1}(u)\eta_j^\nu(\omega)d\omega,$$

which together with (7.36) implies:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} b^{-2}\text{tr}\chi F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \|f_1^j\|_{L^\infty} \left\| \int_{\mathbb{S}^2} F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{\mathbb{S}^2} f_2^j F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{\mathbb{S}^2} f_2^j F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Using the estimate (5.8) and the estimate in $L^2(\mathcal{M})$ (7.1) for oscillatory integrals, we finally obtain:

$$\begin{aligned} \left\| \int_{\mathbb{S}^2} b^{-2}\text{tr}\chi F_{j,1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} & \lesssim \varepsilon\gamma_j^\nu + \varepsilon\gamma_j^\nu 2^{\frac{j}{2}} \left(\sup_{\omega} \|f_2^j\|_{L^\infty L^2(\mathcal{H}_u)} \right) \\ & \lesssim \varepsilon\gamma_j^\nu, \end{aligned}$$

where we used (7.37) in the last estimate. Together with (7.34), we obtain:

$$(7.38) \quad \|A_1\|_{L^2(\mathcal{M})} \lesssim (1+p)\varepsilon\gamma_j^\nu.$$

Next, we estimate A_2 defined by (7.25). In view of the Raychaudhuri Equation (2.22) satisfied by $\text{tr}\chi$ and the transport Equation (2.23) satisfied by b , we have:

$$L(b^{-1}\text{tr}\chi) = -b^{-1}\frac{1}{2}(\text{tr}\chi)^2 - b^{-1}|\widehat{\chi}|^2.$$

Together with the decomposition (2.72) for $\text{tr}\chi$, (2.77) for $|\widehat{\chi}|^2$ and (2.73) for b^{-1} , and with the L^∞ estimates for b and $\text{tr}\chi$ provided respectively by (2.38) and (2.39), we obtain the following decomposition for $L(b^{-1}\text{tr}\chi)$:

$$(7.39) \quad L(b^{-1}\text{tr}\chi) = |\chi_{2\nu}|^2 + \chi_{2\nu} \cdot F_1^j + \chi_{2\nu} \cdot F_2^j + f_3^j + f_4^j + f_5^j,$$

where the tensor F_1^j and the scalar f_3^j only depends on ν and satisfy:

$$(7.40) \quad \|F_1^j\|_{L^\infty_{u,\nu} L^2_t L^\infty(P_{t,u,\nu})} + \|f_3^j\|_{L^\infty_{u,\nu} L^2_t L^\infty(P_{t,u,\nu})} \lesssim \varepsilon,$$

where the tensor F_2^j , and the scalar f_4^j satisfy:

$$(7.41) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{S}^2_{u'})} + \|f_4^j\|_{L_u^\infty L^2(\mathcal{S}^2_{u'})} \lesssim \varepsilon 2^{-\frac{j}{2}}$$

and where the scalar f_5^j satisfies:

$$(7.42) \quad \|f_5^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}.$$

Together with the Definition (7.25) for A_2 , this yields the following decomposition for A_2

$$\begin{aligned} A_2 &= (|\chi_{2\nu}|^2 + \chi_{2\nu} \cdot F_1^j + f_3^j) \left(\int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad + \chi_{2\nu} \cdot \left(\int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_2^j F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad + \left(\int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_4^j F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad + \left(\int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_5^j F_j(u) \eta_j^\nu(\omega) d\omega \right). \end{aligned}$$

We may now estimate A_2 . We have:

$$\begin{aligned} &\|A_2\|_{L^2_{u\nu, x'\nu} L^1_t} \\ &\lesssim (\|\chi_{2\nu}\|_{L^\infty_{u\nu, x'\nu} L^2_t}^2 + \|F_1^j\|_{L^\infty_{u\nu} L^2_t L^\infty(P_{t, u\nu})}^2 + \|f_3^j\|_{L^\infty_{u\nu} L^2_t L^\infty(P_{t, u\nu})}) \\ &\quad \times \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x'\nu} L^\infty_t} \\ &\quad + \|\chi_{2\nu}\|_{L^\infty_{u\nu, x'\nu} L^2_t} \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_2^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_4^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_5^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu, x'\nu} L^\infty_t} \\ &\quad + \varepsilon \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_2^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_4^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P f_5^j F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (7.40) for F_1^j and F_3^j and the estimate (2.46) for χ_2 . Using the estimate in $L^2(\mathcal{M})$ (7.1) for oscillatory integrals we obtain:

$$\begin{aligned}
& \|A_2\|_{L^2_{u\nu x'}L^1_t} \\
& \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'}L^\infty_t} \\
& \quad + \sup_\omega \left(\|F_2^j\|_{L^\infty_u L^2(\mathcal{E}^u)} + \|f_4^j\|_{L^\infty_u L^2(\mathcal{E}^u)} \right) \left\| (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^\nu \\
& \quad + \int_{\mathbb{S}^2} \left\| (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} \|f_5^j\|_{L^2(\mathcal{M})} \|F_j(u)\|_{L^\infty_u} \eta_j^\nu(\omega) d\omega \\
& \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'}L^\infty_t} \\
& \quad + \varepsilon \gamma_j^\nu + \varepsilon 2^{-j} \int_{\mathbb{S}^2} \|F_j(u)\|_{L^\infty_u} \eta_j^\nu(\omega) d\omega \\
& \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'}L^\infty_t} + \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used the estimates (7.41) and (7.42), Cauchy-Schwarz in λ to estimate $\|F_j(u)\|_{L^\infty_u}$, Cauchy-Schwarz in ω and the size of the patch, and the fact that:

$$\|2^{\frac{j}{2}}(N - N_\nu)\|_{L^\infty} \lesssim 1,$$

in view of the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Next, we estimate A_3 and A_4 defined respectively by (7.26) and (7.27). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we obtain:

$$\begin{aligned}
(7.44) \quad & \|A_3\|_{L^2(\mathcal{M})} + \|A_4\|_{L^2(\mathcal{M})} \\
& \lesssim \sup_\omega \left(\|\mathbf{Dtr}\chi\|_{L^\infty_u L^2(\mathcal{E}^u)} \left(\left\| (N_\nu - \mathbf{g}(N, N_\nu)N) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{2}(1 - \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} \right) \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \\
& \lesssim \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Next, we estimate A_5 defined in (7.28). We first decompose $-\bar{\delta}L + \bar{\delta}_\nu L_\nu$. We have:

$$\begin{aligned}
(7.45) \quad & 2^{\frac{j}{2}}(-\bar{\delta}L + \bar{\delta}_\nu L_\nu) = 2^{\frac{j}{2}} \left(-\bar{\delta}(L - L_\nu) + (-\bar{\delta} + \bar{\delta}_\nu)L_\nu \right) \\
& = 2^{\frac{j}{2}} \left(-\bar{\delta}_\nu(N - N_\nu) + (-\bar{\delta} + \bar{\delta}_\nu)(N - N_\nu + L_\nu) \right).
\end{aligned}$$

Furthermore:

$$\begin{aligned}
(7.46) \quad -\bar{\delta} + \bar{\delta}_\nu &= -k_{NN} + n^{-1}n\nabla_N n + k_{N_\nu N_\nu} - n^{-1}\nabla_{N_\nu} n \\
&= n^{-1}\nabla n \cdot (N - N_\nu) - k_{N_\nu \cdot} (N - N_\nu) + k_{N_\nu N} - k_{NN} \\
&= \left(n^{-1}\nabla n - 2k_{N_\nu \cdot} \right) \cdot (N - N_\nu) - k(N - N_\nu, N - N_\nu) \\
&= \left(n^{-1}\nabla n - 2\delta_\nu N_\nu - 2\epsilon_\nu \right) \cdot (N - N_\nu) - k(N - N_\nu, N - N_\nu).
\end{aligned}$$

(7.45), (7.46) and the Definition (7.28) of A_5 yield:

$$\begin{aligned}
A_5 &= -p\bar{\delta}_\nu \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \\
&\quad + p \left(n^{-1}\nabla n - 2\delta_\nu N_\nu - 2\epsilon_\nu \right) \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \\
&\quad \quad \times \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \\
&\quad + p 2^{\frac{j}{2}} \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) k(N - N_\nu, N - N_\nu) \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \\
&\quad \quad \times \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega,
\end{aligned}$$

which implies:

(7.47)

$$\begin{aligned}
&\|A_5\|_{L^2_{u_\nu x'_\nu} L^1_t} \\
&\lesssim p \|\bar{\delta}_\nu\|_{L^\infty_{u_\nu x'_\nu} L^2_t} \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p \left(\|n^{-1}\nabla n\|_{L^\infty(\mathcal{M})} + \|\delta_\nu\|_{L^\infty_{u_\nu x'_\nu}} + \|\epsilon_\nu\|_{L^\infty_{u_\nu x'_\nu}} \right) \\
&\quad \quad \times \left\| \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p 2^{\frac{j}{2}} \left\| \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) k(N - N_\nu, N - N_\nu) (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \right. \\
&\quad \quad \times \left. \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\lesssim p\varepsilon \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p\varepsilon \left\| \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p 2^{\frac{j}{2}} \left\| \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) k(N - N_\nu, N - N_\nu) (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \right.
\end{aligned}$$

$$\times \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \Big\|_{L^2(\mathcal{M})},$$

where we used in the last inequality the estimates (2.36) for n and the estimates (2.37) for δ and ϵ . The first two terms in the right-hand side of (7.47) are similar to A_1 and can be estimated in the same way. In view of (7.38), we obtain:

$$\|A_5\|_{L^2_{u\nu x'}, L^1_t} \lesssim (1 + p^2) \gamma_j^\nu + p 2^{\frac{j}{2}} \left\| \int_{\mathbb{S}^2} (N - N_\nu + L_\nu) k(N - N_\nu, N - N_\nu) \right. \\ \left. \times (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

Using the basic estimate in $L^2(\mathcal{M})$ (7.1), this yields:

(7.48)

$$\|A_5\|_{L^2_{u\nu x'}, L^1_t} \\ \lesssim (1 + p^2) \gamma_j^\nu + p 2^{\frac{j}{2}} \sup_\omega \left(\|k\|_{L^\infty_u L^2(\mathcal{E}h_u)} \right. \\ \left. \left\| (N - N_\nu + L_\nu)(N - N_\nu)^2 (1 + \mathbf{g}(N, N_\nu)) b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ \lesssim (1 + p^2) \gamma_j^\nu,$$

where we used in the last inequality the estimates (2.37) for k , (2.38) for b , (2.39) for $\text{tr} \chi$ and (2.42) for $\partial_\omega N$, and the size of the patch.

Next, we estimate A_6 defined in (7.29). We first decompose ϵ . We have, schematically:

$$(7.49) \quad \epsilon = k_{N_\nu} + k(N - N_\nu, \cdot) = \delta_\nu N_\nu + \varepsilon_\nu + k(N - N_\nu, \cdot).$$

Together with the decompositions (2.72) for $\text{tr} \chi$ and (2.74) $\widehat{\chi}$, this yields:

$$2^{\frac{j}{2}} (\chi(N_\nu - \mathbf{g}(N, N_\nu)N, e_A) e_A - \epsilon_{N_\nu - \mathbf{g}(N, N_\nu)N} L) \\ = F_1^j 2^{\frac{j}{2}} (N - N_\nu) + F_1^j 2^{\frac{j}{2}} (N - N_\nu),$$

where the tensor F_1^j only depends on ν and satisfies:

$$(7.50) \quad \|F_1^j\|_{L^2_{u\nu}, x', L^2_t} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(7.51) \quad \|F_2^j\|_{L^\infty_u L^2(\mathcal{E}h_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

In view of the Definition (7.29) of A_6 , we obtain:

$$A_6 = p F_1^j \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \\ + p \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi F_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega.$$

This yields:

(7.52)

$$\begin{aligned}
\|A_6\|_{L^2_{u_\nu x'_\nu} L^1_t} &\lesssim p \|F_1^j\|_{L^\infty_{u_\nu x'_\nu} L^2_t} \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi F_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\lesssim p \varepsilon \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad + p \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi F_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},
\end{aligned}$$

where we used the estimate (7.50) in the last inequality. The first term in the right-hand side of (7.52) are similar to A_1 and can be estimated in the same way. In view of (7.38), we obtain:

$$\|A_6\|_{L^2_{u_\nu x'_\nu} L^1_t} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu + p \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi F_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

Using the basic estimate in $L^2(\mathcal{M})$ (7.1), this yields:

$$\begin{aligned}
(7.53) \quad \|A_6\|_{L^2_{u_\nu x'_\nu} L^1_t} &\lesssim (1 + p^2) \varepsilon \gamma_j^\nu + p \sup_{\omega} \left(\|F_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \left\| b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
&\lesssim (1 + p^2) \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (7.51) for F_2^j , the estimate (2.38) for b , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch.

Finally, we estimate A_7 . In view of the Definition (7.30) for A_7 and the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\begin{aligned}
(7.54) \quad \|A_7\|_{L^2(\mathcal{M})} &\lesssim p 2^{\frac{j}{2}} \sup_{\omega} \left(\|\zeta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\delta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|n^{-1} \nabla n\|_{L^\infty_u L^2(\mathcal{H}_u)} \right) \\
&\quad \times \left\| (1 - \mathbf{g}(N, N_\nu)) b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^\nu \\
&\lesssim p \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (2.41) for ζ , the estimate (2.37) for δ , the estimate (2.36) for n , the estimate (2.38) for b , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, and the fact that:

$$(7.55) \quad 1 - \mathbf{g}(N, N_\nu) = \frac{\mathbf{g}(N - N_\nu, N - N_\nu)}{2}.$$

We have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} \\ & \lesssim \|L_\nu \left(\int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right)\|_{L^2_{u\nu x'} L_t^1} \\ & \lesssim \|A_1\|_{L^2_{u\nu x'} L_t^1} + \|A_2\|_{L^2_{u\nu x'} L_t^1} + \|A_3\|_{L^2_{u\nu x'} L_t^1} + \|A_4\|_{L^2_{u\nu x'} L_t^1} + \|A_5\|_{L^2_{u\nu x'} L_t^1} \\ & \quad + \|A_6\|_{L^2_{u\nu x'} L_t^1} + \|A_7\|_{L^2_{u\nu x'} L_t^1}, \end{aligned}$$

where we used (7.23) in the last inequality. Together with (7.38), (7.43), (7.44), (7.48), (7.53) and (7.54), we obtain:

$$(7.56) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} \\ & \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} + (1 + p^2) \varepsilon \gamma_j^\nu. \end{aligned}$$

Now, we have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (1 - \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} \\ & \lesssim \int_{\mathbb{S}^2} \left\| (1 - \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \right\| \eta_j^\nu(\omega) d\omega \Big|_{L^2_{u\nu x'} L_t^\infty} \\ & \lesssim \sup_\omega \left(\|1 - \mathbf{g}(N, N_\nu)\|_{L^2_{u\nu x'} L_t^\infty} \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty} \right) \int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega \\ & \lesssim 2^{-j} \int_{\mathbb{S}^2} \|F_j(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega \\ & \lesssim \gamma_j^\nu, \end{aligned}$$

where we used (7.55), the estimate (2.42) for $\partial_\omega N$, the size of the patch, Cauchy-Schwarz in λ to estimate $\|F_j(u)\|_{L_u^\infty}$, and Cauchy-Schwarz in ω . Together with (7.56), this yields:

$$(7.57) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} b^{-1} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} \\ & \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u\nu x'} L_t^\infty} + (1 + p^2) \varepsilon \gamma_j^\nu. \end{aligned}$$

Note that the first term in the right-hand side corresponds to the left-hand side where $b^{-1} \operatorname{tr} \chi$ has been replaced by 1. In particular, we have the analog of (7.23):

$$(7.58) \quad L_\nu \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right)$$

$$= A'_1 + A'_5 + A'_6 + A'_7,$$

where A'_1, A'_5, A'_6 and A'_7 are respectively given by:

$$(7.59) \quad A'_1 = \int_{\mathbb{S}^2} b^{-1} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p+2} F_{j,1}(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.60) \quad A'_5 = p \int_{\mathbb{S}^2} \frac{1}{2} (1 + \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} (2^{\frac{j}{2}} (-\bar{\delta}L + \bar{\delta}_\nu L_\nu)) \\ \times F_j(u) \eta_j^\nu(\omega) d\omega,$$

$$(7.61) \quad A'_6 = p \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \\ \times (2^{\frac{j}{2}} (\chi(N_\nu - \mathbf{g}(N, N_\nu)N, e_A) e_A - \epsilon_{N_\nu - \mathbf{g}(N, N_\nu)N} L)) F_j(u) \eta_j^\nu(\omega) d\omega$$

and:

$$(7.62) \quad A'_7 = p \int_{\mathbb{S}^2} \frac{1}{2} (1 - \mathbf{g}(N, N_\nu)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \\ \times (2^{\frac{j}{2}} (\zeta_A e_A + (\delta + n^{-1} \nabla_N n) L)) F_j(u) \eta_j^\nu(\omega) d\omega.$$

The analog of the estimates (7.38), (7.48), (7.53) and (7.54) for A'_1, A'_5, A'_6 and A'_7 yield:

$$\left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu x'_\nu} L^1_{t_i}} \\ \lesssim \|A'_1\|_{L^2_{u_\nu x'_\nu} L^1_{t_i}} + \|A'_5\|_{L^2_{u_\nu x'_\nu} L^1_{t_i}} + \|A'_6\|_{L^2_{u_\nu x'_\nu} L^1_{t_i}} + \|A'_7\|_{L^2_{u_\nu x'_\nu} L^1_{t_i}} \\ \lesssim (1 + p^2) \gamma_j^\nu.$$

Together with (7.57), we obtain:

$$\left\| \int_{\mathbb{S}^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu x'_\nu} L^\infty_{t_i}} \\ \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

This concludes the proof of the lemma. \square

Lemma 7.7. — *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. We have:*

$$(7.63) \quad \left\| \int_{\mathbb{S}^2} b^q \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu x'_\nu} L^\infty_{t_i}} \lesssim (1 + p^2) \gamma_j^\nu$$

and:

$$(7.64) \quad \left\| \int_{\mathbb{S}^2} b^q \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu x'_\nu} L^\infty_{t_i}} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

The proof of Lemma 7.7 is completely analogous to the proof of Lemma 7.6 and is left to the reader.

Next, we obtain estimates evaluating the L^2_{u,x',L_t^∞} of H where $u = u(t, x, \omega)$, and where H is naturally defined with respect to the foliation of $u' = u(t, x, \omega')$. We start with a basic lemma.

Lemma 7.8. — *Let H a tensor on \mathcal{M} , and ω, ω' two angles in \mathbb{S}^2 . Let $u = u(t, x, \omega)$, and L corresponding to u . Let $u' = u(t, x, \omega')$, and $L', \underline{L}', \nabla'$ corresponding to u' . Then, we have the following estimate for the $L^2_u L_t^\infty L^2(P_{t,u})$ of H :*

$$(7.65) \quad \begin{aligned} \|H\|_{L^2_{u,x',L_t^\infty}}^2 &\lesssim \|H\|_{L^2(\mathcal{M})} \|\mathbf{D}_{\underline{L}'} H\|_{L^2(\mathcal{M})} |\omega - \omega'|^2 + \|H\|_{L^2(\mathcal{M})} \|\nabla' H\|_{L^2(\mathcal{M})} |\omega - \omega'| \\ &+ \int_{\mathcal{M}} |H| |\mathbf{D}_{L'} H| d\mathcal{M} + \|H\|_{L^2(\mathcal{M})}^2. \end{aligned}$$

Proof. — Recall the estimate (2.52) on \mathcal{H}_u :

$$\|H\|_{L^2_x, L_t^\infty}^2 \lesssim \int_{\mathcal{H}_u} |H| |\mathbf{D}_L H| dt d\mu_{t,u} + \|H\|_{L^2(\mathcal{H}_u)}^2.$$

Integrating in u , and using the expression of the volume element $d\mathcal{M}$ in the coordinate system (u, t, x') (4.4) and the control of b in L^∞ given by (2.38), we obtain:

$$(7.66) \quad \|H\|_{L^2_{u,x',L_t^\infty}}^2 \lesssim \int_{\mathcal{M}} |H| |\mathbf{D}_L H| d\mathcal{M} + \|H\|_{L^2(\mathcal{M})}^2.$$

Next, we decompose L on the frame $L', \underline{L}', e'_A, A = 1, 2$. We have:

$$(7.67) \quad L = -\frac{1}{2} \mathbf{g}(L, \underline{L}') L' - \frac{1}{2} \mathbf{g}(L, L') \underline{L}' + \mathbf{g}(L, e'_A) e'_A.$$

Now, we have:

$$(7.68) \quad 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2} \sim |\omega - \omega'|^2,$$

where we used (2.43). (7.68) and the estimate (2.42) for $\partial_\omega N$ yield:

$$(7.69) \quad \mathbf{g}(L, L') \sim |\omega - \omega'|^2, \mathbf{g}(L, e'_A) = \mathbf{g}(L - L', e'_A) \sim |\omega - \omega'| \text{ and } \mathbf{g}(L, \underline{L}') = -2 + \mathbf{g}(L, L').$$

Together with (7.67), this yields:

$$(7.70) \quad L = (-2 + O(|\omega - \omega'|^2)) L' + O(|\omega - \omega'|) \nabla' + O(|\omega - \omega'|^2) \underline{L}'.$$

Finally, plugging (7.70) in (7.66) and using Cauchy-Schwartz yields (7.65). This concludes the proof of the lemma. \square

We have the following corollary of Lemma 7.8

Corollary 7.9. — *Let $\omega, \nu, \omega', \nu'$ four angles in \mathbb{S}^2 such that ω belongs to the patch of center ν and ω' belongs to the patch of center ν' . Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', \underline{L}', \nabla'$ corresponding to u' . Let a tensor G on \mathcal{M} . Then, we have the following estimate:*

$$(7.71) \quad \left\| \int_{\mathbb{S}^2} G P_l' \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x',L_t^\infty}}$$

$$\lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l + \frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{j}{4}}) \gamma_j^{\nu'}.$$

Proof. — We have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} G P'_l \operatorname{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'} L^\infty_{t'}} \\ & \lesssim \int_{\mathbb{S}^2} \|G\|_{L^\infty} \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2_u L^\infty_{x'} L^2_{t'}} \eta_j^{\nu'}(\omega') d\omega' \\ (7.72) \quad & \lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) 2^{-\frac{j}{2}} \left\| \sqrt{\eta_j^{\nu'}} \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2_u L^\infty_{x'} L^2_{t'}} \right\|_{L^2_{\omega'}}, \end{aligned}$$

where we used in the last inequality the estimate (2.38) for b , Cauchy-Schwartz in ω' and the size of the patch.

Next, we apply (7.65) with the choice $H = P'_l \operatorname{tr} \chi' F_j(u')$:

$$\begin{aligned} \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2_{u,x'} L^\infty_{t'}}^2 & \lesssim \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} \|\underline{L}'(P'_l \operatorname{tr} \chi' F_j(u'))\|_{L^2(\mathcal{M})} |\nu - \nu'|^2 \\ & \quad + \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} \|\nabla'(P'_l \operatorname{tr} \chi' F_j(u'))\|_{L^2(\mathcal{M})} |\nu - \nu'| \\ & \quad + \int_{\mathcal{M}} |P'_l \operatorname{tr} \chi' F_j(u')| |L'(P'_l \operatorname{tr} \chi' F_j(u'))| d\mathcal{M} \\ & \quad + \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})}^2, \end{aligned}$$

where we used the fact that $|\omega - \omega'| \sim |\nu - \nu'|$ and $L'(u') = \nabla'(u') = 0$. Also, since $\underline{L}'(u') = -2b'^{-1}$ and $\lambda' \sim 2^j$, we have:

$$\underline{L}'(F_j(u')) \sim 2^j b'^{-1} F_j(u')$$

and we obtain:

(7.73)

$$\begin{aligned} \|P'_l \operatorname{tr} \chi' F_j(u')\|_{L^2_{u,x'} L^\infty_{t'}}^2 & \lesssim \left(\|P'_l \operatorname{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} (2^j \|P'_l \operatorname{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 \right. \\ & \quad \left. + \|\underline{L}'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 + \|\nabla' P'_l \operatorname{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'| \right) \\ & \quad \left. + \int_{\mathcal{H}_u} |P'_l \operatorname{tr} \chi'| |L'(P'_l \operatorname{tr} \chi')| d\mathcal{H}_u + \|P'_l \operatorname{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)}^2 \right) \|F_j(u')\|_{L^2_u}^2 \\ & \lesssim \left(2^{-l} \varepsilon (2^{j-l} \varepsilon |\nu - \nu'|^2 + \|\underline{L}'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 + \varepsilon |\nu - \nu'|) \right. \\ & \quad \left. + \int_{\mathcal{H}_u} |P'_l \operatorname{tr} \chi'| |L'(P'_l \operatorname{tr} \chi')| d\mathcal{H}_u + 2^{-2l} \varepsilon^2 \right) \|F_j(u')\|_{L^2_u}^2, \end{aligned}$$

where we used in the last inequality the finite band property for P'_l and the estimate (2.39) for $\operatorname{tr} \chi$.

Next, we evaluate $\|\underline{L}'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)}$. We have:

$$\begin{aligned} \|\underline{L}'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} & \lesssim \|L'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|N'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim \|nL'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b'N'(P'_l \operatorname{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \end{aligned}$$

$$\begin{aligned} &\lesssim \|P'_l(nL'(\text{tr}\chi'))\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[nL', P'_l]\text{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ &\quad + \|P'_l(b'N'(\text{tr}\chi'))\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[b'N', P'_l]\text{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_u)}, \end{aligned}$$

where we used the fact that $\underline{L}' = L' - 2N'$, the estimate (2.38) for b and the estimate (2.36) for n . Together with the estimates (2.39) for $\text{tr}\chi$, (2.38) for b and (2.36) for n , and the commutator estimates (2.64) and (2.65), we obtain:

$$(7.74) \quad \|\underline{L}'(P'_l \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|nL'(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b'N'(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \mathcal{N}_1(\text{tr}\chi') \lesssim \varepsilon.$$

Next, we estimate the integral over \mathcal{H}_u in the right-hand side of (7.73). We have:

$$(7.75) \quad \begin{aligned} \int_{\mathcal{H}_u} |P'_l \text{tr}\chi'| |L'(P'_l \text{tr}\chi')| d\mathcal{H}_u &\lesssim \int_{\mathcal{H}_u} |P'_l \text{tr}\chi'| |nL'(P'_l \text{tr}\chi')| d\mathcal{H}_u \\ &\lesssim \|P'_l \text{tr}\chi'\|_{L^2_{x'}, L^\infty_t} \|P'_l(n\text{tr}\chi')\|_{L^2_{x'}, L^1_t} \\ &\quad + \|P'_l \text{tr}\chi'\|_{L^2_{x'}, L^\infty_t} \|[nL', P'_l](\text{tr}\chi')\|_{L^2_{x'}, L^1_t} \\ &\lesssim 2^{-2l} \varepsilon^2, \end{aligned}$$

where we used in the last inequality the finite band property for P'_l , the commutator estimate (2.66), the estimate (2.39) for $\text{tr}\chi$, and the estimate (2.69) for $P_l \text{tr}\chi$ and $P_l(nL \text{tr}\chi)$.

Finally, (7.73), (7.74) and (7.75) imply:

$$\|P'_l \text{tr}\chi' F_j(u')\|_{L^2_{u,x'}, L^\infty_t}^2 \lesssim (2^j |\nu - \nu'|^2 + 2^l |\nu - \nu'| + 1) \varepsilon^2 2^{-2l} \|F_j(u')\|_{L^2_u}^2,$$

which together with (7.72) implies:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} G P'_l \text{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'}, L^\infty_t} \\ &\lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \left(2^{\frac{j}{2}} |\nu - \nu'| + 2^{\frac{l}{2}} |\nu - \nu'|^{\frac{1}{2}} + 1 \right) \varepsilon 2^{-l - \frac{j}{2}} \left\| \sqrt{\eta_j^{\nu'}(\omega')} \|F_j(u')\|_{L^2_u} \right\|_{L^2_{\omega'}} \\ &\lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-l + \frac{j}{2}} + 2^{-\frac{l}{2} + \frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \right) \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality Plancherel in u' and the fact that

$$2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1,$$

since $\nu \neq \nu'$. This concludes the proof of the corollary. \square

Lemma 7.8 yields also a second corollary.

Corollary 7.10. — *Let $\omega, \nu, \omega', \nu'$ four angles in \mathbb{S}^2 such that ω belongs to the patch of center ν and ω' belongs to the patch of center ν' . Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', \underline{L}', \nabla'$ corresponding to u' . Let a tensor G on \mathcal{M} . Then, we have the following*

estimate:

$$(7.76) \quad \left\| \int_{\mathbb{S}^2} G \nabla' P'_{\leq l} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'} L^{\infty}_t} \\ \lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{\frac{1}{2} + \frac{j}{4}}) \gamma_j^{\nu'}.$$

Proof. — We have:

$$(7.77) \quad \left\| \int_{\mathbb{S}^2} G \nabla' P'_{\leq l} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'} L^{\infty}_t} \\ \lesssim \int_{\mathbb{S}^2} \|G\|_{L^\infty} \|\nabla' P'_{\leq l} \text{tr} \chi' F_j(u')\|_{L^2_u L^{\infty}_t L^2_x} \eta_j^{\nu'}(\omega') d\omega' \\ \lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) 2^{-\frac{j}{2}} \left\| \sqrt{\eta_j^{\nu'}(\omega')} \|\nabla' P'_{\leq l} \text{tr} \chi' F_j(u')\|_{L^2_u L^{\infty}_t L^2_x} \right\|_{L^2_{\omega'}},$$

where we used in the last inequality the estimate (2.38) for b , Cauchy-Schwartz in ω' and the size of the patch.

Next, we apply (7.65) with the choice $H = \nabla' P'_{\leq l} \text{tr} \chi' F_j(u')$, and we obtain the analogous estimate to (7.73):

(7.78)

$$\|\nabla' P'_{\leq l} \text{tr} \chi' F_j(u')\|_{L^2_{u,x'} L^{\infty}_t}^2 \\ \lesssim \left(\|\nabla' P'_{\leq l} \text{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} (2^j \|\nabla' P'_{\leq l} \text{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 \right. \\ \left. + \|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 + \|\nabla'^2 P'_{\leq l} \text{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'| \right) \\ \left. + \int_{\mathcal{H}_u} |\nabla' P'_{\leq l} \text{tr} \chi'| \|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\| d\mathcal{H}_u + \|\nabla' P'_{\leq l} \text{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)}^2 \right) \|F_j(u')\|_{L^2_u}^2 \\ \lesssim \left(\varepsilon (2^j \varepsilon |\nu - \nu'|^2 + \|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} |\nu - \nu'|^2 + \varepsilon 2^l |\nu - \nu'|) \right) \\ \left. + \int_{\mathcal{H}_u} |\nabla' P'_{\leq l} \text{tr} \chi'| \|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\| d\mathcal{H}_u + \varepsilon^2 \right) \|F_j(u')\|_{L^2_u}^2,$$

where we used in the last inequality the finite band property for $P'_{\leq l}$, the Bochner inequality (2.61), and the estimate (2.39) for $\text{tr} \chi$.

Next, we evaluate $\|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)}$. We have:

$$\|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim \|\underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|N' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim \|n \underline{L}' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|b' N' \nabla' (P'_{\leq l} \text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \lesssim \|\nabla' P'_{\leq l} (n \underline{L}' (\text{tr} \chi'))\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[n \underline{L}', \nabla'] P'_{\leq l} (\text{tr} \chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ \quad + \|\nabla' [n \underline{L}', P'_{\leq l}] \text{tr} \chi'\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla' P'_{\leq l} (b' N' (\text{tr} \chi'))\|_{L^\infty_u L^2(\mathcal{H}_u)}$$

$$+ \|[b'N', \nabla'] (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla' [b'N', P'_{\leq l}] \text{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_u)},$$

where we used the fact that $\underline{L}' = L' - 2N'$, the estimate (2.38) for b and the estimate (2.36) for n . Together with the finite band property for $P'_{\leq l}$, the estimates (2.39) for $\text{tr}\chi$, (2.38) for b and (2.36) for n , and the commutator estimates (2.64) and (2.65), we obtain:

$$\begin{aligned} & \|\underline{L}' \nabla' (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim 2^l \|nL'(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + 2^l \|b'N'(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \mathcal{N}_1(\text{tr}\chi') \\ & \quad + \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[b'N', \nabla'] (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ (7.79) \quad & \lesssim 2^l \varepsilon + \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[b'N', \nabla'] (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)}. \end{aligned}$$

To estimate the commutator terms in the right-hand side of (7.79), we use the commutator Formulas (2.34) and (2.35):

$$\begin{aligned} & \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[b'N', \nabla'] (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim \|\chi' \nabla' (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|k \nabla' P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim (\|\chi'\|_{L^\infty_t L^4_{x'}} + \|k\|_{L^\infty_t L^4_{x'}}) \|\nabla' (P'_{\leq l} \text{tr}\chi')\|_{L^4_t L^4_{x'}}. \end{aligned}$$

Together with the estimate (2.51) and the Gagliardo-Nirenberg inequality (2.49), we obtain:

$$\begin{aligned} & \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|[b'N', \nabla'] (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim (\mathcal{N}_1(\chi') + \mathcal{N}_1(k)) \|\nabla'^2 (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \\ & \lesssim 2^l \varepsilon, \end{aligned}$$

where we used in the last inequality the Bochner inequality for scalars (2.61), the finite band property for $P'_{\leq l}$, the estimates (2.39) and (2.40) for χ' and the estimates (2.37) for k . Together with (7.79), this yields:

$$(7.80) \quad \|\underline{L}' \nabla' (P'_{\leq l} \text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^l \varepsilon.$$

Next, we estimate the integral over \mathcal{H}_u in the right-hand side of (7.78). We have:

(7.81)

$$\begin{aligned} & \int_{\mathcal{H}_u} |\nabla' P'_{\leq l} \text{tr}\chi'| |L' \nabla' (P'_{\leq l} \text{tr}\chi')| d\mathcal{H}_u \\ & \lesssim \int_{\mathcal{H}_u} |\nabla' P'_{\leq l} \text{tr}\chi'| |nL' \nabla' (P'_{\leq l} \text{tr}\chi')| d\mathcal{H}_u \\ & \lesssim \|\nabla' P'_{\leq l} \text{tr}\chi'\|_{L^2_{x'}, L^\infty_t} \|\nabla' P'_{\leq l} (nL'(\text{tr}\chi'))\|_{L^2_{x'}, L^1_t} + \|\nabla' P'_{\leq l} \text{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_{u'})} \\ & \quad \times \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_{u'})} + \|\nabla' P'_{\leq l} \text{tr}\chi'\|_{L^2_{x'}, L^\infty_t} \|\nabla' [nL', P'_{\leq l}](\text{tr}\chi')\|_{L^2_{x'}, L^1_t} \\ & \lesssim \varepsilon \|[nL', \nabla'] P'_{\leq l}(\text{tr}\chi')\|_{L^\infty_u L^2(\mathcal{H}_{u'})} + \varepsilon^2, \end{aligned}$$

where we used in the last inequality the finite band property for $P'_{\leq l}$, the commutator estimate (2.66), the estimate (2.39) for $\text{tr}\chi$, and the estimate (2.70) for $P_{\leq l}\text{tr}\chi$ and $P_{\leq l}(nL\text{tr}\chi)$.

Next, we estimate the right-hand side of (7.81):

$$\begin{aligned} \|[nL', \nabla']P'_{\leq l}(\text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim \|\chi' \nabla'(P'_{\leq l}\text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \|\chi'\|_{L_{x', L_t}^\infty} \|\nabla'(P'_{\leq l}\text{tr}\chi')\|_{L_{x', L_t}^2} \\ &\lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ' and the estimate (2.70) for $\nabla'(P'_{\leq l}\text{tr}\chi')$. Together with (7.81), we obtain:

$$(7.82) \quad \int_{\mathcal{H}_u} |\nabla' P'_{\leq l}\text{tr}\chi'| |L' \nabla'(P'_{\leq l}\text{tr}\chi')| d\mathcal{H}_u \lesssim \varepsilon^2.$$

Finally, (7.78), (7.80) and (7.82) imply:

$$\|P'_{\leq l}\text{tr}\chi' F_j(u')\|_{L_{u, x', L_t}^2}^2 \lesssim (2^j |\nu - \nu'|^2 + 2^l |\nu - \nu'| + 1) \varepsilon^2 \|F_j(u')\|_{L_u^2}^2,$$

which together with (7.77) implies:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} G P'_{\leq l}\text{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u, x', L_t}^2} \\ &\lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \left(2^{\frac{j}{2}} |\nu - \nu'| + 2^{\frac{l}{2}} |\nu - \nu'|^{\frac{1}{2}} + 1 \right) \varepsilon 2^{-\frac{j}{2}} \left\| \sqrt{\eta_j^{\nu'}(\omega')} \|F_j(u')\|_{L_u^2} \right\|_{L_{\omega'}^2}, \\ &\lesssim \left(\sup_{\omega'} \|G\|_{L^\infty} \right) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{\frac{j}{2}} + 2^{\frac{l}{2} + \frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \right) \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality Plancherel in u' and the fact that

$$2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1,$$

since $\nu \neq \nu'$. This concludes the proof of the corollary. \square

Lemma 7.8 also yields a third corollary.

Corollary 7.11. — *Let $\omega, \nu, \omega', \nu'$ four angles in \mathbb{S}^2 such that ω belongs to the patch of center ν and ω' belongs to the patch of center ν' . Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', \underline{L}', \nabla'$ corresponding to u' , and let $L_{\nu'}, \underline{L}_{\nu'}, \nabla_{\nu'}$ corresponding to $u(t, x, \nu')$. Let $q \in \mathbb{N}$. Then, we have the following estimate:*

$$(7.83) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} b'^{-1} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u, x', L_t}^2} \\ &\lesssim (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| + 1) \gamma_j^{\nu'}. \end{aligned}$$

Proof. — We apply (7.65) with the choice

$$(7.84) \quad H = \int_{\mathbb{S}^2} b'^{-1} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega'.$$

We have:

$$\begin{aligned}
\|H\|_{L^2_{u,x'}L^\infty_{L_t}}^2 &\lesssim \|H\|_{L^2(\mathcal{M})}\|\underline{L}_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'|^2 + \|H\|_{L^2(\mathcal{M})}\|\nabla_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'| \\
&\quad + \int_{\mathcal{M}} |H|\|\underline{L}_{\nu'}(H)\|d\mathcal{M} + \|H\|_{L^2(\mathcal{M})}^2 \\
&\lesssim \|H\|_{L^2(\mathcal{M})}\|\underline{L}_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'|^2 + \|H\|_{L^2(\mathcal{M})}\|\nabla_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'| \\
(7.85) \quad &\quad + \|H\|_{L^2_{u\nu',x'\nu'}L^\infty_{L_t}}\|\underline{L}_{\nu'}(H)\|_{L^2_{u\nu',x'\nu'}L^1_{L_t}} + \|H\|_{L^2(\mathcal{M})}^2,
\end{aligned}$$

where we used the fact that $|\omega - \nu'| \sim |\nu - \nu'|$. Now, the estimate of the $L^2_{u,x'}L^\infty_{L_t}$ norm of oscillatory integrals (7.16) yields:

$$(7.86) \quad \|H\|_{L^2_{u\nu',x'\nu'}L^\infty_{L_t}} \lesssim (1 + q^2)\varepsilon\gamma_j^{\nu'}.$$

Furthermore in order to prove the estimate (7.16), we actually obtain the following estimate (see (7.17)):

$$(7.87) \quad \|\underline{L}_{\nu'}(H)\|_{L^2_{u\nu',x'\nu'}L^1_{L_t}} \lesssim (1 + q^2)\varepsilon\gamma_j^{\nu'}.$$

Together with (7.85) and (7.86), we obtain:

$$\begin{aligned}
\|H\|_{L^2_{u,x'}L^\infty_{L_t}}^2 &\lesssim (1 + q^2)\varepsilon\gamma_j^{\nu'} \left(\|\underline{L}_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'|^2 + \|\nabla_{\nu'}(H)\|_{L^2(\mathcal{M})}|\nu - \nu'| \right) \\
(7.88) \quad &\quad + (1 + q^2)\varepsilon^2(\gamma_j^{\nu'})^2.
\end{aligned}$$

Next, we estimate the various term in the right-hand side of (7.88) starting with the one involving $\underline{L}_{\nu'}(H)$. In view of (7.84), we have:

$$\begin{aligned}
\underline{L}_{\nu'}(H) &= i2^j \int_{\mathbb{S}^2} \mathbf{g}(\underline{L}_{\nu'}, L')b'^{-1}\mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,1}(u')\eta_j^{\nu'}(\omega')d\omega' \\
&\quad + \int_{\mathbb{S}^2} \underline{L}_{\nu'}(b'^{-1}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j^{\nu'}(\omega')d\omega' \\
&\quad + q2^{\frac{j}{2}} \int_{\mathbb{S}^2} b'^{-1}\mathrm{tr}\chi'(\mathbf{D}_{\underline{L}_{\nu'}}L' - \mathbf{D}_{\underline{L}_{\nu'}}L_{\nu'}) \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^{q-1} F_j(u')\eta_j^{\nu'}(\omega')d\omega' \\
&= i2^j \int_{\mathbb{S}^2} \mathbf{g}(\underline{L}_{\nu'}, L')b'^{-1}\mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,1}(u')\eta_j^{\nu'}(\omega')d\omega' \\
&\quad + \int_{\mathbb{S}^2} \underline{L}_{\nu'}(b'^{-1}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j^{\nu'}(\omega')d\omega' \\
&\quad + q2^{\frac{j}{2}} \int_{\mathbb{S}^2} b'^{-1}\mathrm{tr}\chi'\mathbf{D}_{\underline{L}_{\nu'}}L' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^{q-1} F_j(u')\eta_j^{\nu'}(\omega')d\omega' \\
&\quad - q2^{\frac{j}{2}}\mathbf{D}_{\underline{L}_{\nu'}}L_{\nu'} \int_{\mathbb{S}^2} b'^{-1}\mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^{q-1} F_j(u')\eta_j^{\nu'}(\omega')d\omega',
\end{aligned}$$

where we used the fact that $\underline{L}_{\nu'}(u) = \mathbf{g}(\underline{L}_{\nu'}, L')$ and $N' - N_{\nu'} = L' - L_{\nu'}$. This yields:

$$\|\underline{L}_{\nu'}(H)\|_{L^2(\mathcal{M})}$$

$$\begin{aligned}
&\lesssim 2^j \left\| \int_{\mathbb{S}^2} \mathbf{g}(L_{\nu'}, L') b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\quad + \left\| \int_{\mathbb{S}^2} L_{\nu'} (b'^{-1} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\quad + q 2^{\frac{j}{2}} \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \mathbf{D}_{L_{\nu'}, L'} \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\quad + q 2^{\frac{j}{2}} \|\mathbf{D}_{L_{\nu'}, L_{\nu'}}\|_{L_{u', x', \nu'}^\infty, L_t^2} \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u', x', \nu'}^2, L_t^\infty}.
\end{aligned}$$

Using the estimate of the $L_{u', x', \nu'}^\infty, L_t^\infty$ norm of oscillatory integrals (7.16) for the first and the last term in the right-hand side, and the basic estimate in $L^2(\mathcal{M})$ (7.1) for the second and the third term in the right-hand side, we obtain:

(7.89)

$$\begin{aligned}
&\|L_{\nu'}(H)\|_{L^2(\mathcal{M})} \\
&\lesssim 2^j (1 + q^2) \varepsilon \gamma_j^{\nu'} + \sup_{\omega} \left(\|\mathbf{D}(b^{-1} \operatorname{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
&\quad + q 2^{\frac{j}{2}} \sup_{\omega} \left(\|\mathbf{D}L'\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
&\quad + q 2^{\frac{j}{2}} \|\mathbf{D}L_{\nu'}\|_{L_{u', x', \nu'}^\infty, L_t^2} (1 + q^2) \varepsilon \gamma_j^{\nu'} \\
&\lesssim 2^j (1 + q^3) \varepsilon \gamma_j^{\nu'},
\end{aligned}$$

where we used in the last inequality the Ricci Equations (2.17) for $\mathbf{D}L_{\nu'}$ and $\mathbf{D}L'$, the estimate (2.38) for b , the estimate (2.36) for n , the estimate (2.37) for k , the estimates (2.39)–(2.40) for χ , the estimate (2.41) for ζ , and the estimate (2.42) for $\partial_\omega N$.

Next, we estimate the term in the right-hand side of (7.88) involving $\nabla_{(e_{\nu'})_A}(H)$. In view of (7.84), we have for $A = 1, 2$:

$$\begin{aligned}
\nabla_{(e_{\nu'})_A}(H) &= i 2^j \int_{\mathbb{S}^2} \mathbf{g}((e_{\nu'})_A, L') b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,1}(u') \eta_j^{\nu'}(\omega') d\omega' \\
&\quad + \int_{\mathbb{S}^2} (e_{\nu'})_A (b'^{-1} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \\
&\quad + q 2^{\frac{j}{2}} \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' (\mathbf{D}_{(e_{\nu'})_A} L' - \mathbf{D}_{(e_{\nu'})_A} L_{\nu'}) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \\
&= i 2^{\frac{j}{2}} (e_{\nu'})_A \cdot \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q+1} F_{j,1}(u') \eta_j^{\nu'}(\omega') d\omega' \\
&\quad + \int_{\mathbb{S}^2} (e_{\nu'})_A (b'^{-1} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \\
&\quad + q 2^{\frac{j}{2}} \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \mathbf{D}_{(e_{\nu'})_A} L' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega'
\end{aligned}$$

$$-q2^{\frac{j}{2}}\mathbf{D}_{(e_{\nu'})_A}L_{\nu'}\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q-1}F_j(u')\eta_j^{\nu'}(\omega')d\omega',$$

where we used the fact that:

$$(e_{\nu'})_A(u) = \mathbf{g}((e_{\nu'})_A, L') = \mathbf{g}((e_{\nu'})_A, L' - L_{\nu'}) = \mathbf{g}((e_{\nu'})_A, N' - N_{\nu'})$$

and $N' - N_{\nu'} = L' - L_{\nu'}$. This yields:

$$\begin{aligned} & \|\mathcal{V}_{(e_{\nu'})_A}(H)\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{\frac{j}{2}}\left\|\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q+1}F_{j,1}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})} \\ & \quad + \left\|\int_{\mathbb{S}^2}(e_{\nu'})_A(b'^{-1}\mathrm{tr}\chi')\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^qF_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})} \\ & \quad + q2^{\frac{j}{2}}\left\|\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\mathbf{D}_{(e_{\nu'})_A}L'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q-1}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})} \\ & \quad + q2^{\frac{j}{2}}\|\mathbf{D}_{(e_{\nu'})_A}L_{\nu'}\|_{L^\infty_{u,\nu'}L^2_t}\left\|\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q-1}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2_{u,\nu'}L^\infty_t}. \end{aligned}$$

Using the estimate of the $L^2_{u,x'}L^\infty_t$ norm of oscillatory integrals (7.16) for the first and the last term in the right-hand side, and the basic estimate in $L^2(\mathcal{M})$ (7.1) for the second term in the right-hand side, we obtain:

(7.90)

$$\begin{aligned} & \|\mathcal{V}_{(e_{\nu'})_A}(H)\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{\frac{j}{2}}(1+q^2)\varepsilon\gamma_j^{\nu'} + \sup_{\omega'}\left(\|\mathbf{D}(b^{-1}\mathrm{tr}\chi)\|_{L^\infty_{\omega'}L^2(\mathcal{H}_u)}\right)\left\|\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^q\right\|_{L^\infty}2^{\frac{j}{2}}\gamma_j^{\nu'} \\ & \quad + q2^{\frac{j}{2}}\left\|\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\mathbf{D}_{(e_{\nu'})_A}L'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q-1}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})} \\ & \quad + q2^{\frac{j}{2}}\|\mathbf{D}_{(e_{\nu'})_A}L_{\nu'}\|_{L^\infty_{u,\nu'}L^2_t}(1+q^2)\varepsilon\gamma_j^{\nu'} \\ & \lesssim 2^{\frac{j}{2}}(1+q^3)\varepsilon\gamma_j^{\nu'} \\ & \quad + q2^{\frac{j}{2}}\varepsilon\left\|\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi'\mathbf{D}_{(e_{\nu'})_A}L'\left(2^{\frac{j}{2}}(N'-N_{\nu'})\right)^{q-1}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the Ricci Equations (2.17) for $\mathbf{D}_{(e_{\nu'})_A}L_{\nu'}$, the estimate (2.38) for b , the estimate (2.37) for k , the estimates (2.39) (2.40) for χ , and the estimate (2.42) for $\partial_\omega N$. Next, we decompose the term $\mathbf{D}_{(e_{\nu'})_A}L'$ in the right-hand side of (7.90). First, we decompose $(e_{\nu'})_A$ on the frame L', \underline{L}', e'_B , $B = 1, 2$. We have:

$$(e_{\nu'})_A = -\frac{1}{2}\mathbf{g}((e_{\nu'})_A, \underline{L}')L' - \frac{1}{2}\mathbf{g}((e_{\nu'})_A, L')\underline{L}' + ((e_{\nu'})_A - \mathbf{g}((e_{\nu'})_A, N')N').$$

Together with the Ricci Equations (2.17), this yields, schematically:

$$\mathbf{D}_{(e_{\nu'})_A} L' = (\chi + \epsilon)(e_{\nu'})_A + (N' - N_{\nu'}) (\chi + \epsilon + \zeta + \delta + n^{-1} \nabla_N n).$$

In view of the decompositions (2.74)–(2.72) for χ , the fact that $\epsilon_A = k_{NA}$ with k independent of ω , the estimate (2.36) for n , the estimate (2.37) for $\bar{\delta}$, ϵ and k , the estimate (2.39)–(2.40) for χ , and the estimate (2.41) for ζ , we obtain the following decomposition:

$$(7.91) \quad \mathbf{D}_{(e_{\nu'})_A} L' = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν' and satisfies:

$$(7.92) \quad \|F_1^j\|_{L_{u_{\nu'}, x_{\nu'}, L_t^2}^\infty} \lesssim \epsilon$$

and the tensor F_2^j satisfies:

$$(7.93) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{S}_{u'})} \lesssim 2^{-\frac{j}{2}} \epsilon.$$

In view of (7.91), we obtain:

$$\begin{aligned} & \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \mathbf{D}_{(e_{\nu'})_A} L' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \\ &= F_1^j \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \\ & \quad + \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' F_2^j \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega', \end{aligned}$$

which yields:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \mathbf{D}_{(e_{\nu'})_A} L' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{C}\mathcal{M})} \\ & \lesssim \|F_1^j\|_{L_{u_{\nu'}, x_{\nu'}, L_t^2}^\infty} \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u_{\nu'}, x_{\nu'}, L_t^\infty}^2} \\ & \quad + \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' F_2^j \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{C}\mathcal{M})} \\ & \lesssim \epsilon \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u_{\nu'}, x_{\nu'}, L_t^\infty}^2} \\ & \quad + \left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' F_2^j \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{C}\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (7.92). Using the estimate of the $L_{u, x'}^2 L_t^\infty$ norm of oscillatory integrals (7.16) for the first term in the right-hand side, and the basic estimate in $L^2(\mathcal{C}\mathcal{M})$ (7.1) for the second term in the right-hand side, we obtain:

$$\left\| \int_{\mathbb{S}^2} b'^{-1} \operatorname{tr} \chi' \mathbf{D}_{(e_{\nu'})_A} L' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q-1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{C}\mathcal{M})}$$

$$\begin{aligned} &\lesssim \varepsilon(1+q^2)\gamma_j^{\nu'} + \sup_{\omega'} \left(\|F_2^j\|_{L^\infty_u L^2(\mathcal{E}u)} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^{q-1} \right\|_{L^\infty} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'} \\ &\lesssim \varepsilon(1+q^2)\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (7.93), the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Together with (7.90), we finally obtain:

$$(7.94) \quad \|\nabla_{(e_{\nu'})_A}(H)\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}}(1+q^3)\varepsilon\gamma_j^{\nu'}.$$

Together with (7.88) and (7.89), this yields:

$$\|H\|_{L^2_{u,x'}L^\infty_{t'}} \lesssim (1+q^5)\varepsilon^2(\gamma_j^{\nu'})^2 \left(2^j|\nu - \nu'|^2 + 2^{\frac{j}{2}}|\nu - \nu'| + 1\right).$$

This concludes the proof of the lemma. \square

We have finally a last corollary of Lemma 7.8.

Corollary 7.12. — *Let $\omega, \nu, \omega', \nu'$ four angles in \mathbb{S}^2 such that ω belongs to the patch of center ν and ω' belongs to the patch of center ν' . Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', \underline{L}', \nabla'$ corresponding to u' , and let $L_{\nu'}, \underline{L}_{\nu'}, \nabla_{\nu'}$ corresponding to $u(t, x, \nu')$. Let $q \in \mathbb{N}$. Then, we have the following estimate:*

$$(7.95) \quad \left\| \int_{\mathbb{S}^2} (b' - b_{\nu'}) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'}L^\infty_{t'}} \\ \lesssim 2^{-\frac{j}{4}}(1+q^{\frac{5}{2}})\varepsilon(2^{\frac{j}{2}}|\nu - \nu'| + 1)\gamma_j^{\nu'}.$$

Proof. — We apply (7.65) with the choice

$$(7.96) \quad H = \int_{\mathbb{S}^2} (b' - b_{\nu'}) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega'.$$

As in (7.85), we have:

$$(7.97) \quad \|H\|_{L^2_{u,x'}L^\infty_{t'}} \lesssim \|H\|_{L^2(\mathcal{M})} \|\underline{L}_{\nu'}(H)\|_{L^2(\mathcal{M})} |\nu - \nu'|^2 + \|H\|_{L^2(\mathcal{M})} \|\nabla_{\nu'}(H)\|_{L^2(\mathcal{M})} |\nu - \nu'| \\ + \|H\|_{L^2_{u,\nu'},x',\nu'} L^\infty_{t'} \|\underline{L}_{\nu'}(H)\|_{L^2_{u,\nu'},x',\nu'} L^\infty_{t'} + \|H\|_{L^2(\mathcal{M})}^2.$$

We first estimate $\|H\|_{L^2(\mathcal{M})}$. Recall the decomposition (2.79) for $b - b_\nu$. We have:

$$(7.98) \quad b - b_\nu = 2^{-\frac{j}{2}}(f_1^j + f_2^j),$$

where the tensor f_1^j only depends on ν and satisfies:

$$(7.99) \quad \|f_1^j\|_{L^\infty} \lesssim \varepsilon$$

and where the tensor f_2^j satisfies:

$$(7.100) \quad \|f_2^j\|_{L^\infty_u L^2(\mathcal{E}u)} \lesssim \varepsilon 2^{-\frac{j}{4}}.$$

This yields:

$$(7.101) \quad \left\| \int_{\mathbb{S}^2} (b' - b_{\nu'}) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned}
&\lesssim 2^{-\frac{j}{2}} \|f_1^j\|_{L^\infty(\mathcal{M})} \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\quad + 2^{-\frac{j}{2}} \left\| \int_{\mathbb{S}^2} f_2^j \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\lesssim 2^{-\frac{j}{2}} \varepsilon \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\quad + 2^{-\frac{j}{2}} \left\| \int_{\mathbb{S}^2} f_2^j \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\end{aligned}$$

where we used in the last inequality the estimate (7.99) for f_1^j . We control the first term in the right-hand side of (7.101) using the estimate (7.64), and the second term in the right-hand side of (7.101) using the basic estimate in $L^2(\mathcal{M})$ (7.1). We obtain:

$$\begin{aligned}
&\left\| \int_{\mathbb{S}^2} (b' - b_{\nu'}) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\lesssim 2^{-\frac{j}{2}} \varepsilon (1 + q^2) \gamma_j^{\nu'} + \left(\sup_{\omega'} \|f_2^j \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) \gamma_j^{\nu'} \\
&\lesssim 2^{-\frac{j}{2}} \varepsilon (1 + q^2) \gamma_j^{\nu'} + \left(\sup_{\omega'} \|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \|\text{tr}\chi'\|_{L^\infty} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \gamma_j^{\nu'} \\
(7.102) \quad &\lesssim 2^{-\frac{j}{4}} \varepsilon (1 + q^2) \gamma_j^{\nu'},
\end{aligned}$$

where we used in the last inequality the estimate (7.100) for f_2^j , the estimate (2.39) for $\text{tr}\chi'$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the Definition (7.96) of H , this yields:

$$(7.103) \quad \|H\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{j}{4}} \varepsilon (1 + q^2) \gamma_j^{\nu'}.$$

Together with (7.97), we obtain:

$$\begin{aligned}
(7.104) \quad &\|H\|_{L_{u,x'}^2 L_t^\infty}^2 \lesssim (\|L_{\nu'}(H)\|_{L^2(\mathcal{M})} |\nu - \nu'|^2 + \|\nabla_{\nu'}(H)\|_{L^2(\mathcal{M})} |\nu - \nu'|) 2^{-\frac{j}{2}} \varepsilon (1 + q^2) \gamma_j^{\nu'} \\
&\quad + \|H\|_{L_{u,\nu',x',\nu'}^2 L_t^\infty} \|L_{\nu'}(H)\|_{L_{u,\nu',x',\nu'}^2 L_t^1} + 2^{-\frac{j}{2}} \varepsilon^2 (1 + q^4) (\gamma_j^{\nu'})^2.
\end{aligned}$$

Next, we define:

$$(7.105) \quad H_1 = \int_{\mathbb{S}^2} (\underline{L}_{\nu'}(b') - \underline{L}_{\nu'}(b_{\nu'})) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega',$$

$$(7.106) \quad H_2 = \int_{\mathbb{S}^2} (\nabla_{\nu'}(b') - \nabla_{\nu'}(b_{\nu'})) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega',$$

$$(7.107) \quad H_3 = \int_{\mathbb{S}^2} (L_{\nu'}(b') - L_{\nu'}(b_{\nu'})) \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega'$$

and:

$$(7.108) \quad H'_1 = \underline{L}_{\nu'}(H) - H_1, \quad H'_2 = \nabla_{\nu'}(H) - H_2 \quad \text{and} \quad H'_3 = L_{\nu'}(H) - H_3.$$

The terms H_1, H_2, H_3 denote the contributions in the right-hand side of (7.110) where the derivatives $\underline{L}_{\nu'}, \nabla_{\nu'}, L_{\nu'}$ fall on b . The terms H'_1, H'_2, H'_3 are the ones already treated in the proof of Corollary 7.11 up to the presence of the extra term $b' - b_{\nu'}$ which is evaluated in L^∞ norm. In view of the estimate (2.44) for $\partial_\omega b$ and the size of the patch, we have:

$$(7.109) \quad \|b' - b_{\nu'}\|_{L^\infty} \lesssim 2^{-\frac{j}{2}}.$$

Thus, in view of the estimates (7.87), (7.89) and (7.94) of the proof of Corollary 7.11, and taking into account the extra $2^{-\frac{j}{2}}$ factor coming from (7.109), we obtain the analog of (7.87) (7.89) (7.94):

$$\begin{aligned} \|H'_3\|_{L^2_{u_{\nu'}, x'_{\nu'}}, L^1_t} &\lesssim 2^{-\frac{j}{2}}(1+q^2)\varepsilon\gamma_j^{\nu'}, \\ \|H'_1\|_{L^2(\mathcal{M})} &\lesssim 2^{\frac{j}{2}}(1+q^3)\varepsilon\gamma_j^{\nu'} \end{aligned}$$

and:

$$\|H'_2\|_{L^2(\mathcal{M})} \lesssim (1+q^3)\varepsilon\gamma_j^{\nu'}.$$

Together with (7.104) and in view of the decompositions (7.108), we get:

$$(7.110) \quad \begin{aligned} \|H\|_{L^2_{u,x'}, L^\infty_t} &\lesssim (\|H_1\|_{L^2(\mathcal{M})}|\nu - \nu'|^2 + \|H_2\|_{L^2(\mathcal{M})}|\nu - \nu'|)2^{-\frac{j}{2}}\varepsilon(1+q^2)\gamma_j^{\nu'} \\ &\quad + \|H\|_{L^2_{u_{\nu'}, x'_{\nu'}}, L^\infty_t} \|H_3\|_{L^2_{u_{\nu'}, x'_{\nu'}}, L^1_t} + (1+q^5)2^{-\frac{j}{2}}(1+(2^{\frac{j}{2}}|\nu - \nu'|)^2)\varepsilon^2(\gamma_j^{\nu'})^2. \end{aligned}$$

Next, we estimate the various term in the right-hand side of (7.110) starting with H_1 . In view of the Definition (7.105) for H_1 , we have:

$$(7.111) \quad \begin{aligned} \|H_1\|_{L^2(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \underline{L}_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \underline{L}_{\nu'}(b_{\nu'}) \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Next, we evaluate both terms in the right-hand side of (7.111) starting with the first one. Using the basic estimate in $L^2(\mathcal{M})$, we have:

$$(7.112) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \underline{L}_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega'} \left\| \underline{L}_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty_u L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}. \end{aligned}$$

Now, we have:

$$\begin{aligned} \left\| \underline{L}_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|\mathbf{D}b'\|_{L^\infty_u L^2(\mathcal{H}_u)} \|\operatorname{tr} \chi'\|_{L^\infty} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \\ &\lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.38) for b , the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (7.112), we obtain:

$$(7.113) \quad \left\| \int_{\mathbb{S}^2} \underline{L}_{\nu'}(b') \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

Next, we estimate the second term in the right-hand side of (7.111). We have:

$$(7.114) \quad \left\| \underline{L}_{\nu'}(b_{\nu'}) \left(\int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) \right\|_{L^2(\mathcal{M})} \\ \lesssim \|\underline{L}_{\nu'}(b_{\nu'})\|_{L^4(\mathcal{M})} \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^4(\mathcal{M})} \\ \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^4(\mathcal{M})},$$

where we used in the last inequality the estimate (2.38) for b . Now, we have:

$$(7.115) \quad \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u_{\nu'}, x'_{\nu'}, L^{\infty}_t}} \\ \lesssim \varepsilon \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (7.64) of the $L^2_{u_{\nu'}, x'_{\nu'}, L^{\infty}_t}$ of oscillatory integrals. Also, we have:

$$(7.116) \quad \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^{\infty}(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \left\| \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^{\infty}(\mathcal{M})} \|F_{j,-1}(u')\|_{L^{\infty}_{u'}} \eta_j^{\nu'}(\omega') d\omega' \\ \lesssim \varepsilon \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L^{\infty}_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ \lesssim \varepsilon 2^j \gamma_j^{\nu'},$$

where we used, the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$, Cauchy-Schwarz in λ' to estimate $\|F_{j,-1}(u')\|_{L^{\infty}_{u'}}$, Cauchy-Schwarz in ω' and the size of the patch. Interpolating (7.115) and (7.116), we obtain:

$$(7.117) \quad \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Finally, (7.111), (7.113) and (7.117) imply:

$$(7.118) \quad \|H_1\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Next, we estimate the term H_2 in the right-hand side of (7.110). In view of the Definition (7.106) for H_2 , we have:

(7.119)

$$\begin{aligned} \|H_2\|_{L^2(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \nabla_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + \left\| \nabla_{\nu'}(b_{\nu'}) \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Next, we evaluate both terms in the right-hand side of (7.119) starting with the last one. We have:

(7.120)

$$\begin{aligned} &\left\| \nabla_{\nu'}(b_{\nu'}) \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) \right\|_{L^2(\mathcal{M})} \\ &\lesssim \|\nabla_{\nu'}(b_{\nu'})\|_{L^\infty_{u,\nu'} L^2_{x'}, L^2_t} \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,\nu'} L^2_{x'}, L^\infty_t} \\ &\lesssim \varepsilon(1 + q^2) \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.38) for b and the estimate in $L^2_{u,\nu'} L^\infty_{x'} L^\infty_t$ (7.64). Next, we evaluate the first term in the right-hand side of (7.111).

Decomposing $\nabla_{\nu'}$ on the frame L', \underline{L}', e'_A and using the fact that:

$$|\omega' - \nu'| \lesssim 2^{\frac{j}{2}},$$

we have schematically:

$$\nabla_{\nu'}(b') = \nabla'(b') + 2^{-\frac{j}{2}} \mathbf{D}(b')$$

and thus:

$$\begin{aligned} (7.121) \quad &\left\| \int_{\mathbb{S}^2} \nabla_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} \nabla'(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + 2^{-\frac{j}{2}} \left\| \int_{\mathbb{S}^2} \mathbf{D}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} \nabla'(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality an estimate analog to (7.113). In order to estimate the right-hand side of (7.121), we use the decomposition (2.80) of $\nabla'(b')$. We have:

$$(7.122) \quad \nabla'(b) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$(7.123) \quad \|F_1^j\|_{L^\infty_{u,\nu'} L^2_t L^8_{x'}, \nu'} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(7.124) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{M}_{u'})} \lesssim \varepsilon 2^{-\frac{j}{4}}.$$

In view of (7.122), we have:

$$(7.125) \quad \left\| \int_{\mathbb{S}^2} \nabla'(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \|F_1^j\|_{L_{u'}^2, L_t^2, L_{x'}^8} \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u'}^2, L_t^\infty, L_{x'}^{\frac{8}{3}}} \\ + \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u'}^2, L_t^\infty, L_{x'}^{\frac{8}{3}}} \\ + \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

where we used the estimate (7.123) for F_1^j in the last inequality. Now, interpolating between the the estimate in $L_{u', x'}^2, L_t^\infty$ (7.64) and the L^∞ estimate (7.116), we obtain:

$$(7.126) \quad \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u', x'}^{\frac{3}{2}}, L_t^\infty} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^{\nu'}.$$

For the second term in the right-hand side of (7.125), we have:

$$\left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \|F_2^j \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u')\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ \lesssim \int_{\mathbb{S}^2} \|F_2^j\|_{L_u^\infty L^2(\mathcal{M}_{u'})} \|F_j(u')\|_{L_{u'}^2} \left\| \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega'.$$

Together with the estimate (7.124) for F_2^j , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$(7.127) \quad \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \varepsilon \int_{\mathbb{S}^2} 2^{-\frac{j}{4}} \varepsilon \|F_j(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \\ \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'},$$

where we used in the last inequality Plancherel in \mathcal{X}' for $\|F_j(u')\|_{L^2_{u'}}$, Cauchy Schwartz in ω' and the size of the patch. Finally, (7.121), (7.125), (7.126) and (7.127) imply:

$$(7.128) \quad \left\| \int_{\mathbb{S}^2} \nabla_{\nu'}(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'}.$$

Together with (7.119) and (7.120), this yields:

$$(7.129) \quad \|H_2\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'}.$$

Next, we estimate the term H_3 in the right-hand side of (7.110). Decomposing $L_{\nu'}$ on the frame L', \underline{L}', e'_A and using the fact that:

$$|\omega' - \nu'| \lesssim 2^{\frac{j}{2}},$$

we have schematically:

$$L_{\nu'}(b') = L'(b') + 2^{-\frac{j}{2}} \nabla'(b') + 2^{-j} \underline{L}'(b').$$

Together with the transport Equation (2.23) satisfied by b , we obtain:

$$L_{\nu'}(b') - L_{\nu'}(b_{\nu'}) = -\bar{\delta}' b' + \bar{\delta}_{\nu'} b_{\nu'} + 2^{-\frac{j}{2}} \nabla'(b') + 2^{-j} \underline{L}'(b').$$

In view of the Definition (7.107) for H_3 , this yields:

$$\begin{aligned} \|H_3\|_{L^2(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' b' + \bar{\delta}_{\nu'} b_{\nu'}) \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + 2^{-\frac{j}{2}} \left\| \int_{\mathbb{S}^2} \nabla'(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + 2^{-j} \left\| \int_{\mathbb{S}^2} \underline{L}'(b') \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Together with an estimate analog to (7.128) and an estimate analog to (7.113), we get:

$$(7.130) \quad \begin{aligned} \|H_3\|_{L^2(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' b' + \bar{\delta}_{\nu'} b_{\nu'}) \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + \varepsilon 2^{-\frac{j}{4}} \gamma_j^{\nu'} + \varepsilon 2^{-\frac{j}{2}} \gamma_j^{\nu'}. \end{aligned}$$

Next, we estimate the right-hand side of (7.130). We have:

$$(7.131) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} (-\bar{\delta}' b' + \bar{\delta}_{\nu'} b_{\nu'}) \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' + \bar{\delta}_{\nu'}) b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\quad + \|\bar{\delta}_{\nu'}\|_{L^\infty_{u',x'}, L^2_{t'}} \left\| \int_{\mathbb{S}^2} (-b' + b_{\nu'}) \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u',x'}, L^\infty_{t'}} \end{aligned}$$

$$\lesssim \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' + \bar{\delta}_{\nu'}) b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} + \varepsilon \|H\|_{L^2_{u, x', L_t^\infty}},$$

where we used in the last inequality the estimates (2.36) (2.37) for $\bar{\delta}_{\nu'}$ and the Definition (7.96) for H . Now, recall the decomposition (7.46):

$$(7.132) \quad -\bar{\delta}' + \bar{\delta}_{\nu'} = \left(n^{-1} \nabla n - 2\delta_{\nu'} N_{\nu'} - 2\epsilon_{\nu'} \right) \cdot (N' - N_{\nu'}) - k(N' - N_{\nu'}, N' - N_{\nu'}).$$

This yields:

$$(7.133) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' + \bar{\delta}_{\nu'}) b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim 2^{-\frac{j}{2}} \|n^{-1} \nabla n - 2\delta_{\nu'} N_{\nu'} - 2\epsilon_{\nu'}\|_{L^\infty_{u, x', L_t^\infty}} \\ & \quad \times \left\| \int_{\mathbb{S}^2} b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q+1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u, x', L_t^\infty}} \\ & \quad + 2^{-j} \|k\|_{L^6(\mathcal{O}\mathcal{M})} \left\| \int_{\mathbb{S}^2} b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q+2} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^3(\mathcal{O}\mathcal{M})} \\ & \lesssim 2^{-\frac{j}{2}} \varepsilon \left\| \int_{\mathbb{S}^2} b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q+1} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u, x', L_t^\infty}} \\ & \quad + 2^{-j} \varepsilon \left\| \int_{\mathbb{S}^2} b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q+2} F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^3(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (2.36) for n , the estimates (2.36) (2.37) for $\bar{\delta}$ and ϵ and the estimate (2.37) for k . (7.133) together with the estimate (7.115) and the interpolation of (7.115) with (7.116) implies:

$$(7.134) \quad \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' + \bar{\delta}_{\nu'}) b' \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim 2^{-\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Finally, (7.130), (7.131) and (7.134) imply:

$$(7.135) \quad \|H_3\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon \|H\|_{L^2_{u, x', L_t^\infty}} + \varepsilon 2^{-\frac{j}{4}} \gamma_j^{\nu'}.$$

Finally, (7.110), (7.118), (7.129) and (7.135) yield:

$$\begin{aligned} \|H\|_{L^2_{u, x', L_t^\infty}}^2 & \lesssim (2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)^2 + 2^{-\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)) 2^{-\frac{j}{2}} \varepsilon^2 (1 + q^2) (\gamma_j^{\nu'})^2 \\ & \quad + \varepsilon \|H\|_{L^2_{u, x', L_t^\infty}}^2 + \varepsilon 2^{-\frac{j}{4}} \gamma_j^{\nu'} \|H\|_{L^2_{u, x', L_t^\infty}} \\ & \quad + (1 + q^5) 2^{-\frac{j}{2}} (1 + (2^{\frac{j}{2}} |\nu - \nu'|)^2) \varepsilon^2 (\gamma_j^{\nu'})^2. \end{aligned}$$

This implies:

$$\|H\|_{L^2_{u, x', L_t^\infty}} \lesssim \left(2^{-\frac{j}{2}} + 2^{-\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|) + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)^2 \right) (1 + q^5) \varepsilon^2 (\gamma_j^{\nu'})^2.$$

This concludes the proof of the lemma. \square

7.4. Integration by parts

7.4.1. Integration by parts in tangential directions

Lemma 7.13. — *We consider an oscillatory integral of the following form:*

$$\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b'^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

where h is a scalar function on \mathcal{M} . Integrating by parts once using (6.3) yields:

(7.136)

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b'^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\ &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N' - \mathbf{g}(N, N')N)(h) + (\mathrm{tr}\theta' - \mathbf{g}(N, N')\mathrm{tr}\theta \right. \\ & \quad \left. - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') - \mathbf{g}(N, N')b'^{-1}(N - \mathbf{g}(N, N')N')(b') \right) h \\ & \quad + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right. \\ & \quad \left. - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right) h \Big) \\ & \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Also, integrating by parts once using (6.4) yields:

(7.137)

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b'^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\ &= i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + (\mathrm{tr}\theta - \mathbf{g}(N, N')\mathrm{tr}\theta' \right. \\ & \quad \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right) h \\ & \quad + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \\ & \quad \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) h \Big) \\ & \quad \times F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Proof. — We have:

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b'^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\ &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} b'^{-1} h d_{\mathcal{M}} \right) \end{aligned}$$

$$\times \eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)(2^{-j}\lambda')\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega'.$$

We integrate by parts in tangential directions using (6.3). We obtain:

(7.138)

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\ &= -i2^{-j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N' - \mathbf{g}(N, N')N)(h) \right. \right. \\ & \quad + (b(N' - \mathbf{g}(N, N')N)(b^{-1}) + \operatorname{div}_{\mathbf{g}}(N' - \mathbf{g}(N, N')N))h \\ & \quad \left. \left. + 2 \frac{\mathbf{g}(N, N')(N' - \mathbf{g}(N, N')N)(\mathbf{g}(N, N'))}{1 - \mathbf{g}(N, N')^2} h \right) d_{\mathcal{M}} \right) \\ & \quad \times \eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')(2^{-j}\lambda')^{-1}\psi(2^{-j}\lambda)(2^{-j}\lambda')\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega', \end{aligned}$$

where $\operatorname{div}_{\mathbf{g}}(N' - \mathbf{g}(N, N')N)$ denotes the space-time divergence of $N' - \mathbf{g}(N, N')N$.

Next, we consider the various terms in the right-hand side of (7.138). Using (2.21), we have:

$$\begin{aligned} (7.139) \quad & b(N' - \mathbf{g}(N, N')N)(b^{-1}) + \operatorname{div}_{\mathbf{g}}(N' - \mathbf{g}(N, N')N) \\ &= \operatorname{tr}\theta' - \mathbf{g}(N, N')\operatorname{tr}\theta - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \\ & \quad + \mathbf{g}(N, N')(N - \mathbf{g}(N, N')N')(b'), \end{aligned}$$

where we used the decomposition of N in the frame N', e'_A :

$$(7.140) \quad N = \mathbf{g}(N, N')N' + (N - \mathbf{g}(N, N')N')$$

and the decomposition of N' in the frame N, e_A :

$$(7.141) \quad N' = \mathbf{g}(N, N')N + (N' - \mathbf{g}(N, N')N)$$

and where θ is the second fundamental form of $P_{t,u}$ in Σ_t . We also have in view of (2.21), (7.140) and (7.141):

(7.142)

$$\begin{aligned} & (N' - \mathbf{g}(N, N')N)(\mathbf{g}(N, N')) \\ &= (\mathbf{g}(N, N')^2 - 1)b'^{-1}(N - \mathbf{g}(N, N')N')(b') + \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \\ & \quad - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N'). \end{aligned}$$

Using (7.138), (7.139) and (7.142), we obtain:

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\ &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N' - \mathbf{g}(N, N')N)(h) + \left(\operatorname{tr}\theta' - \mathbf{g}(N, N')\operatorname{tr}\theta \right. \right. \\ & \quad \left. \left. - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') - \mathbf{g}(N, N')b'^{-1}(N - \mathbf{g}(N, N')N')(b') \right) h \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{2\mathbf{g}(N, N')}{(1 - \mathbf{g}(N, N')^2)} \left(\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right. \\
& - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \Big) h \\
& \times F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}},
\end{aligned}$$

which concludes the proof of (7.136).

In order to obtain (7.137), we integrate by parts in tangential directions using (6.4) instead of (6.3). The proof is completely analogous by exchanging the role played by N and N' , so we omit it. This concludes the proof of the lemma. \square

7.4.2. Integration by parts in L

Lemma 7.14. — *We consider an oscillatory integral of the following form:*

$$\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1}b'^{-1}h(t, x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}},$$

where h is a scalar function on \mathcal{M} . Integrating by parts once using (6.5) yields:

$$\begin{aligned}
(7.143) \quad & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1}b'^{-1}h(t, x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}} \\
& = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(h) + \text{tr}\chi h - \bar{\delta}h - \bar{\delta}'h - (1 - \mathbf{g}(N, N'))\delta'h \right. \\
& \quad \left. - 2\zeta'_{N-\mathbf{g}(N, N')N'}h - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')}h \right) \\
& \quad F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Also, integrating by parts once using (6.6) yields:

$$\begin{aligned}
(7.144) \quad & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1}b'^{-1}h(t, x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}} \\
& = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{\mathbf{g}(L, L')} \left(L'(h) + \text{tr}\chi'h - \bar{\delta}h - \bar{\delta}'h - (1 - \mathbf{g}(N, N'))\delta h \right. \\
& \quad \left. - 2\zeta_{N'-\mathbf{g}(N, N')N}h - \frac{\chi(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N)}{\mathbf{g}(L, L')}h \right) \\
& \quad F_{j,-1}(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Proof. — We have:

$$\begin{aligned}
& \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1}b'^{-1}h(t, x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}} \\
& = \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1}b'^{-1}h d_{\mathcal{M}} \right) \\
& \quad \times \eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)(2^{-j}\lambda')\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega'.
\end{aligned}$$

We integrate by parts in L using (6.5). We obtain:

(7.145)

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}\mathcal{M}} \\ &= -i2^{-j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(h) + (bL(b^{-1}) + \operatorname{div}_{\mathbf{g}}(L))h \right. \right. \\ & \quad \left. \left. - \frac{L(\mathbf{g}(L, L'))}{g(L, L')} h \right) d_{\mathcal{C}\mathcal{M}} \right) \\ & \quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega', \end{aligned}$$

where $\operatorname{div}_{\mathbf{g}}(L)$ denotes the space-time divergence of L .

Next, we consider the various terms in the right-hand side of (7.145). Using the Ricci Equations (2.17), we have:

$$(7.146) \quad L(b^{-1}) + \operatorname{div}_{\mathbf{g}}(L) = b^{-1} \operatorname{tr} \chi$$

and:

$$(7.147) \quad L(\mathbf{g}(L, L')) = -\bar{\delta} \mathbf{g}(L, L') + \mathbf{g}(L, \mathbf{D}_L L').$$

We decompose L on the frame $(L', \underline{L}', e'_A)$:

$$(7.148) \quad L = \frac{1}{2}(1 + \mathbf{g}(N, N'))L' + \frac{1}{2}(1 - \mathbf{g}(N, N'))\underline{L}' + N - \mathbf{g}(N, N')N',$$

where the vector $N - \mathbf{g}(N, N')N'$ is tangent to $P_{t, u'}$. (7.147), (7.148) and the Ricci Equations (2.17) yields:

(7.149)

$$\begin{aligned} & L(\mathbf{g}(L, L')) \\ &= -\bar{\delta} \mathbf{g}(L, L') - \frac{1}{2}(1 + \mathbf{g}(N, N'))\bar{\delta}' \mathbf{g}(L, L') + (1 - \mathbf{g}(N, N'))\zeta'_{N - \mathbf{g}(N, N')N'} \\ & \quad + \frac{1}{2}(1 - \mathbf{g}(N, N'))(\delta' + n^{-1} \nabla_{N'} n) \mathbf{g}(L, L') + \chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \\ & \quad - \zeta'_{N - \mathbf{g}(N, N')N'} \mathbf{g}(L, L'). \end{aligned}$$

Using (7.145), (7.146) and (7.149), we obtain:

$$\begin{aligned} & \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}\mathcal{M}} \\ &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(h) + \operatorname{tr} \chi h - \bar{\delta} h - \bar{\delta}' h - (1 - \mathbf{g}(N, N'))\delta' h \right. \\ & \quad \left. - 2\zeta'_{N - \mathbf{g}(N, N')N'} h - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} h \right) \\ & \quad F_j(u) F_{j, -1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}\mathcal{M}}, \end{aligned}$$

where we also used the identity:

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N').$$

This concludes the proof of (7.143).

In order to obtain (7.144), we integrate by parts in L' using (6.6) instead of (6.5). The proof is completely analogous by exchanging the role played by L and L' , so we omit it. This concludes the proof of the lemma. \square

CHAPTER 8

PROOF OF PROPOSITION 6.5

Since $2^{\min(l,m)} \leq 2^j |\nu - \nu'|$, we may assume that $l > m$ and thus:

$$(8.1) \quad m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.$$

In order to prove Proposition 6.5, recall that we need to exhibit a decomposition:

$$(8.2) \quad \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} = A_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m},$$

where $B_{j,\nu,\nu',l,m}$ satisfies:

$$(8.3) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m} + B_{j,\nu',\nu,l,m}) \right| \\ \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{2})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right. \\ \left. + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

We have:

$$\int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} = \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} P_l \text{tr} \chi b'^{-1} P_m \text{tr} \chi' \\ \times F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

We first integrate by parts in L using (7.143) with the choice $h = P_l \text{tr} \chi P_m \text{tr} \chi'$. We obtain:

$$(8.4) \quad \int_{\mathcal{M}} E_j^{\nu,l} f(t, x) \overline{E_j^{\nu',m} f(t, x)} d\mathcal{M} \\ = -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) \text{tr} \chi P_m \text{tr} \chi' + P_l \text{tr} \chi L(P_m \text{tr} \chi') \right) + \left(\text{tr} \chi - \bar{\delta} - \bar{\delta}' \right)$$

$$\begin{aligned}
& - (1 - \mathbf{g}(N, N'))\delta' - 2\zeta'_{N-\mathbf{g}(N, N')N'} - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \\
& \times P_l \operatorname{tr} \chi P_m \operatorname{tr} \chi') F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Next, we decompose L on the frame (L', N', e'_A) :

$$(8.5) \quad L = L' + (\mathbf{g}(N, N') - 1)N' + N - \mathbf{g}(N, N')N',$$

which yields:

$$(8.6) \quad L(P'_m \operatorname{tr} \chi') = L'(P'_m \operatorname{tr} \chi') + (\mathbf{g}(N, N') - 1)N'(P'_m \operatorname{tr} \chi') + (N - \mathbf{g}(N, N')N')(P'_m \operatorname{tr} \chi').$$

Now, (8.4) and (8.6) yield:

$$(8.7) \quad \int_{\mathcal{M}} E_j^{\nu, l} f(t, x) \overline{E_j^{\nu', m} f(t, x)} d_{\mathcal{M}} = A_{j, \nu, \nu', l, m} + B_{j, \nu, \nu', l, m},$$

where $A_{j, \nu, \nu', l, m}$ is given by:

$$(8.8) \quad A_{j, \nu, \nu', l, m} = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \operatorname{tr} \chi (N - \mathbf{g}(N, N')N')(P_m \operatorname{tr} \chi')}{\mathbf{g}(L, L')} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}$$

and $B_{j, \nu, \nu', l, m}$ may be decomposed as:

$$(8.9) \quad B_{j, \nu, \nu', l, m} = B_{j, \nu, \nu', l, m}^1 + B_{j, \nu, \nu', l, m}^2,$$

where $B_{j, \nu, \nu', l, m}^1$ and $B_{j, \nu, \nu', l, m}^2$ are given by:

$$(8.10) \quad B_{j, \nu, \nu', l, m}^1 = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \operatorname{tr} \chi) P_m \operatorname{tr} \chi' + P_l \operatorname{tr} \chi L'(P_m \operatorname{tr} \chi') \right) \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}$$

and:

$$(8.11) \quad B_{j, \nu, \nu', l, m}^2 = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left((\mathbf{g}(N, N') - 1) P_l \operatorname{tr} \chi N'(P_m \operatorname{tr} \chi') \right. \\ \left. + \left(\operatorname{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N'))\delta' - 2\zeta'_{N-\mathbf{g}(N, N')N'} \right. \right. \\ \left. \left. - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) \right) \\ P_l \operatorname{tr} \chi P_m \operatorname{tr} \chi') F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

The estimates satisfied by $B_{j, \nu, \nu', l, m}^1$ and $B_{j, \nu, \nu', l, m}^2$ are provided by the following propositions.

Proposition 8.1. — *Let $B_{j,\nu,\nu',l,m}^1$ be given by (8.10). Then, we have the following estimate:*

$$(8.12) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^1 + B_{j,\nu',\nu,l,m}^1) \right| \\ \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right. \\ \left. + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Proposition 8.2. — *Let $B_{j,\nu,\nu',l,m}^2$ be given by (8.11). Then, we have the following estimate:*

$$(8.13) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu',\nu,l,m}^2) \right| \\ \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right. \\ \left. + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, in view of the decomposition (8.9), we have:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m} + B_{j,\nu',\nu,l,m}) \right| \\ \lesssim \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^1 + B_{j,\nu',\nu,l,m}^1) \right| \\ + \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu',\nu,l,m}^2) \right|.$$

Together with (8.12) and (8.13), this yields the estimate (8.3) and thus concludes the proof of Proposition 6.5. The rest of this chapter is devoted to the proof of Proposition 8.1 and Proposition 8.2.

We start with the proof of Proposition 8.1. We rewrite $B_{j,\nu,\nu',l,m}^1$ as:

$$(8.14) \quad B_{j,\nu,\nu',l,m}^1 = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') (2^{-j} \lambda')^{-1} \psi(2^{-j} \lambda) (2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \lambda^2 \lambda'^2 d\lambda d\lambda' \\ = B_{j,\nu,\nu',l,m}^{1,1} + B_{j,\nu,\nu',l,m}^{1,2},$$

where $B_{j,\nu,\nu',l,m}^{1,1}$ and $B_{j,\nu,\nu',l,m}^{1,2}$ are given by:

$$(8.15) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,1} &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \frac{(2^{-j} \lambda')^{-1} + (2^{-j} \lambda)^{-1}}{2} \psi(2^{-j} \lambda) (2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \\ &\quad \times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' d\mathcal{M} \end{aligned}$$

and:

$$(8.16) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,2} &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \frac{(2^{-j} \lambda')^{-1} - (2^{-j} \lambda)^{-1}}{2} \psi(2^{-j} \lambda) (2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \\ &\quad \times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' d\mathcal{M}. \end{aligned}$$

$B_{j,\nu,\nu',l,m}^{1,1}$ and $B_{j,\nu,\nu',l,m}^{1,2}$ satisfy the following estimates:

Proposition 8.3. — *Let $B_{j,\nu,\nu',l,m}^{1,1}$ be given by (8.15). Then, we have the following estimate:*

$$(8.17) \quad \begin{aligned} &\left| \sum_{(m,l)} \sum_{2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} + \sum_{(m,l)} \sum_{2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right| \\ &\lesssim |B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu',\nu}^{1,1,1}| + |B_{j,\nu,\nu'}^{1,1,2} + B_{j,\nu',\nu}^{1,1,2}| + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{3-}} \\ &\lesssim \left[\frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} \right] \varepsilon^2 \gamma_j^\nu \gamma_{\nu'}^j. \end{aligned}$$

Proposition 8.4. — *Let $B_{j,\nu,\nu',l,m}^{1,2}$ be given by (8.16). Then, we have the following estimate:*

$$(8.18) \quad \begin{aligned} &\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2} + B_{j,\nu',\nu,l,m}^{1,2}) \right| \\ &\lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{j2^{-\frac{j}{12}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Now, the decomposition (8.14) of $B_{j,\nu,\nu',l,m}^1$ yields:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^1 \right| \lesssim \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right|$$

$$+ \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2} \right|.$$

Together with the estimates (8.17) and (8.18), we obtain:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^1 + B_{j,\nu',\nu,l,m}^1) \right| \\ & \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right. \\ & \quad \left. + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

This concludes the proof of Proposition 8.1.

The rest of this chapter is organized as follows. Proposition 8.3 is proved in Section 8.1, Proposition 8.4 is proved in Section 8.2, and Proposition 8.2 is proved in Section 8.3.

8.1. Proof of Proposition 8.5 (Control of $B_{j,\nu,\nu',l,m}^{1,1}$)

Recall the Definition (8.15) of $B_{j,\nu,\nu',l,m}^{1,1}$:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,1} &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ & \quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \frac{(2^{-j} \lambda')^{-1} + (2^{-j} \lambda)^{-1}}{2} \psi(2^{-j} \lambda) (2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \\ & \quad \times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We have:

(8.19)

$$\begin{aligned} & \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ &= L(\text{tr} \chi) \text{tr} \chi' + \text{tr} \chi L(\text{tr} \chi') - \sum_{(l,m)/2^{\min(l,m)} > 2^j |\nu - \nu'|} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right). \end{aligned}$$

Now, the difference between $B_{j,\nu,\nu',l,m}^1$ and $B_{j,\nu,\nu',l,m}^{1,1}$ is the fact that the term $(2^{-j} \lambda')^{-1}$ has been replaced by:

$$\frac{(2^{-j} \lambda')^{-1} + (2^{-j} \lambda)^{-1}}{2},$$

such as to obtain an expression which is totally symmetric in (λ, λ') and (ω, ω') . In turn, we may sum over l, m belonging to the region $2^m \leq 2^j |\nu - \nu'|$. Together with

(8.19) we obtain:

$$(8.20) \quad \sum_{(m,l)} 2^{m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \\ = B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu,\nu'}^{1,1,2} + \sum_{(m,l)/2^m > 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1,3} + \sum_{(m,l)/2^m > 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1,4},$$

where $B_{j,\nu,\nu'}^{1,1,1}$, $B_{j,\nu,\nu'}^{1,1,2}$, $B_{j,\nu,\nu',l,m}^{1,1,3}$ and $B_{j,\nu,\nu',l,m}^{1,1,4}$ are given by:

$$(8.21) \quad B_{j,\nu,\nu'}^{1,1,1} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi') \right) \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}}\mathcal{M},$$

$$(8.22) \quad B_{j,\nu,\nu'}^{1,1,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi') \right) \\ \times F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}}\mathcal{M},$$

$$(8.23) \quad B_{j,\nu,\nu',l,m}^{1,1,3} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}}\mathcal{M}$$

and:

$$(8.24) \quad B_{j,\nu,\nu',l,m}^{1,1,4} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ \times F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}}\mathcal{M}.$$

We have the following propositions:

Proposition 8.5. — Let $B_{j,\nu,\nu',l,m}^{1,1,3}$ be given by (8.23), and let $B_{j,\nu,\nu',l,m}^{1,1,4}$ be given by (8.24). Then, we have the following estimate:

$$(8.25) \quad \sum_{(m,l)/2^{\min(m,l)} > 2^j |\nu - \nu'|} (|B_{j,\nu,\nu',l,m}^{1,1,3}| + |B_{j,\nu,\nu',l,m}^{1,1,4}|) \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_{j'}^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{3-}}.$$

Proposition 8.6. — Let $B_{j,\nu,\nu'}^{1,1,1}$ be given by (8.21), and let $B_{j,\nu,\nu'}^{1,1,2}$ be given by (8.22). Then, we have the following estimate:

$$(8.26) \quad |B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu,\nu'}^{1,1,1}| + |B_{j,\nu,\nu'}^{1,1,2} + B_{j,\nu,\nu'}^{1,1,2}| \lesssim \left[\frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} \right] \varepsilon^2 \gamma_j^\nu \gamma_{j'}^j.$$

In view of the decomposition (8.20), we have:

$$\begin{aligned} & \left| \sum_{(m,l)/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} - (B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu,\nu'}^{1,1,2}) \right| \\ & \lesssim \sum_{(m,l)/2^m > 2^j |\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,3}| + \sum_{(m,l)/2^m > 2^j |\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,4}| \\ & \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{3-}}, \end{aligned}$$

where we used the estimate (8.25) in the last inequality. Together with (8.26), this yields:

$$\begin{aligned} & \left| \sum_{(m,l)/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} + \sum_{(m,l)/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right| \\ & \lesssim |B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu',\nu}^{1,1,1}| + |B_{j,\nu,\nu'}^{1,1,2} + B_{j,\nu',\nu}^{1,1,2}| + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{3-}} \\ & \lesssim \left[\frac{2^{-(\frac{1}{2})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} \right] \varepsilon^2 \gamma_j^\nu \gamma_{\nu'}^j. \end{aligned}$$

This concludes the proof of Proposition 8.3.

The rest of this section is as follows. In Section 8.1.1, we give a proof of Proposition 8.5, and In Section 8.1.2, we give a proof of Proposition 8.6.

8.1.1. Proof of Proposition 8.5 (Control of $B_{j,\nu,\nu',l,m}^{1,1,3}$ and $B_{j,\nu,\nu',l,m}^{1,1,4}$). — We further decompose. We have:

$$(8.27) \quad B_{j,\nu,\nu',l,m}^{1,1,3} = B_{j,\nu,\nu',l,m}^{1,1,3,1} + B_{j,\nu,\nu',l,m}^{1,1,3,2}$$

where $B_{j,\nu,\nu',l,m}^{1,1,3,1}$ and $B_{j,\nu,\nu',l,m}^{1,1,3,2}$ are given by:

$$(8.28) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,1,3,1} &= i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} L(P_l \text{tr} \chi) P_m \text{tr} \chi' F_j(u) F_{j,-1}(u') \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M} \end{aligned}$$

and:

$$(8.29) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,1,3,2} &= i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} P_l \text{tr} \chi L'(P_m \text{tr} \chi') F_j(u) F_{j,-1}(u') \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \end{aligned}$$

The terms $B_{j,\nu,\nu',l,m}^{1,1,3,1}$ and $B_{j,\nu,\nu',l,m}^{1,1,3,2}$ are estimated in the same way, so we focus on $B_{j,\nu,\nu',l,m}^{1,1,3,1}$. We first deal with $\mathbf{g}(L, L')$. We have the identities:

$$(8.30) \quad \mathbf{g}(L, L') = -1 + \mathbf{g}(N, N')$$

and

$$(8.31) \quad 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

Furthermore, the estimates on N (2.42) and (2.43) yield:

$$(8.32) \quad |N - N_\nu| \lesssim |\omega - \nu|, \quad |N' - N_{\nu'}| \lesssim |\omega' - \nu'| \quad \text{and} \quad |N_\nu - N_{\nu'}| \gtrsim |\nu - \nu'|,$$

where we have used the following notation for any vectorfield tangent to Σ_t :

$$|X| = \mathbf{g}(X, X)^{\frac{1}{2}}.$$

Since ω belongs to the patch of center ν , ω' belongs to the patch of center ν' , and $\nu \neq \nu'$, we obtain in view of (8.30), (8.31) and (8.32):

$$(8.33) \quad \frac{1}{\mathbf{g}(L, L')} = \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p, q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right),$$

for some explicit real coefficients c_{pq} such that the series

$$\sum_{p, q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

In view of (8.28) and (8.33), we may rewrite $B_{j, \nu, \nu', l, m}^{1, 1, 3, 1}$ as:

$$\begin{aligned} B_{j, \nu, \nu', l, m}^{1, 1, 3, 1} &= i2^{-j-1} \sum_{p, q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{|N_\nu - N_{\nu'}|^2} \\ &\quad \times \left(\int_{\mathbb{S}^2} b^{-1} L(P_l \text{tr} \chi) \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q F_{j, -1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}}. \end{aligned}$$

Using the estimate (7.10) with the choice:

$$(8.34) \quad H_{pq} = \frac{b^{-1}}{|N_\nu - N_{\nu'}|^2} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q F_{j, -1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

we obtain:

$$(8.35) \quad |B_{j, \nu, \nu', l, m}^{1, 1, 3, 1}| \lesssim \left(\sum_{p, q \geq 0} c_{pq} \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H_{pq}\|_{L^2_{u, x'} L^\infty_{t'}}) \right) \right) 2^{-\frac{j}{2} - l} \varepsilon \gamma_j^\nu.$$

Next, we evaluate the right-hand side of (8.35). In view of (8.32), we have:

$$(8.36) \quad \begin{aligned} \left\| \frac{b^{-1}}{|N_\nu - N_{\nu'}|^2} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \right\|_{L^\infty(\mathcal{M})} &\lesssim \frac{\|b^{-1}\|_{L^\infty(\mathcal{M})}}{|\nu - \nu'|^2} \left(\frac{|\omega - \nu|}{|\nu - \nu'|} \right)^p \\ &\lesssim \frac{1}{|\nu - \nu'|^2} \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \right)^p, \end{aligned}$$

where we used in the last inequality the estimate (2.38) for b , and the fact that ω is in the patch centered around ν of diameter $\sim 2^{\frac{j}{2}}$. Let:

$$H_{pq}^1 = \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(\frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|} \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

Then, (8.34), (8.35) and (8.36) yield:

$$(8.37) \quad |B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \right)^p \left(\sup_{\omega \in \operatorname{supp}(\eta_j^{\nu'})} (\|H_{pq}^1\|_{L_{u,x'}^2 L_t^\infty}) \right) \right) \frac{2^{\frac{j}{2}-l} \varepsilon \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}.$$

Next, we evaluate H_{pq}^1 . In view of (8.32), we have:

$$(8.38) \quad \left\| \left(\frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|} \right)^q \right\|_{L^\infty(\mathcal{M})} \lesssim \left(\frac{|\omega' - \nu'|}{|\nu - \nu'|} \right)^q \lesssim \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \right)^q,$$

where we used in the last inequality the fact that ω' is in the patch centered around ν' of diameter $\sim 2^{\frac{j}{2}}$. Now, (8.38) together with Corollary 7.9 yields:

$$(8.39) \quad \|H_{pq}^1\|_{L_{u,x'}^2 L_t^\infty} \lesssim \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \right)^q \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}}) \gamma_j^{\nu'}.$$

Finally, (8.37) and (8.39) imply:

$$(8.40) \quad |B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \right)^{p+q} \right) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \frac{2^{\frac{j}{2}-l} \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \\ \lesssim \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \frac{2^{\frac{j}{2}-l} \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}.$$

(8.40) implies:

$$\sum_{(m,l)/2^{\min(m,l)} > 2^j |\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \frac{\varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3}.$$

The term $B_{j,\nu,\nu',l,m}^{1,1,3,2}$ is completely analogous, so we obtain in view of (8.27):

$$\sum_{(m,l)/2^{\min(m,l)} > 2^j |\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,3}| \lesssim \frac{\varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3}.$$

The term $B_{j,\nu,\nu',l,m}^{1,1,4}$ is completely analogous to $B_{j,\nu,\nu',l,m}^{1,1,3}$. This concludes the proof of Proposition 8.6.

8.1.2. Proof of Proposition 8.6 (Control of $B_{j,\nu,\nu'}^{1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,2}$). — We need to estimate $B_{j,\nu,\nu'}^{1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,2}$. These terms are estimated in the same way, so we focus on $B_{j,\nu,\nu'}^{1,1,1}$. We further decompose:

$$(8.41) \quad B_{j,\nu,\nu'}^{1,1,1} = B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2},$$

where $B_{j,\nu,\nu'}^{1,1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,1,2}$ are given by:

$$(8.42) \quad B_{j,\nu,\nu'}^{1,1,1,1} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} L(\mathrm{tr}\chi) \mathrm{tr}\chi' \\ \times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}$$

and:

$$(8.43) \quad B_{j,\nu,\nu'}^{1,1,1,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \mathrm{tr}\chi L'(\mathrm{tr}\chi') \\ \times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

The terms $B_{j,\nu,\nu',l,m}^{1,1,1,1}$ and $B_{j,\nu,\nu',l,m}^{1,1,1,2}$ are estimated in the same way, so we focus on $B_{j,\nu,\nu',l,m}^{1,1,1,1}$. We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,1}$ using (7.137).

Lemma 8.7. — *Let $B_{j,\nu,\nu'}^{1,1,1,1}$ be defined by (8.42). Integrating by parts using (7.137) yields:*

$$(8.44)$$

$$B_{j,\nu,\nu'}^{1,1,1,1} \\ = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \\ \times \left[\frac{1}{|N_\nu - N_{\nu'}|^2} h_{1,p,q} + \frac{1}{|N_\nu - N_{\nu'}|^3} (h_{2,p,q} + h_{3,p,q} + h_{4,p,q}) \right] d_{\mathcal{M}} \\ + 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi - \chi') L(\mathrm{tr}\chi) \mathrm{tr}\chi'}{\mathbf{g}(L, L')^2} F_{j-1}(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}$ on \mathcal{M} are given by:

$$(8.45) \quad h_{1,p,q} = \left(\int_{\mathbb{S}^2} N(L(\mathrm{tr}\chi)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \mathrm{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.46) \quad h_{2,p,q} = \left(\int_{\mathbb{S}^2} \nabla L(\mathrm{tr}\chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$(8.47) \quad h_{3,p,q} = \left(\int_{\mathbb{S}^2} L(\operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} \operatorname{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.48) \quad h_{4,p,q} = \left(\int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} \operatorname{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

where the tensor H_1 on \mathcal{M} involved in the definition of $h_{3,p,q}$ is a linear combination of terms in the following list:

$$(8.49) \quad b'^{-1}\nabla'(b'\operatorname{tr}\chi'), \theta'\operatorname{tr}\chi'$$

and where the tensor H_2 on \mathcal{M} involved in the definition of $h_{4,p,q}$ is a linear combination of terms in the following list:

$$(8.50) \quad \theta L(\operatorname{tr}\chi), b^{-1}\nabla(b)L(\operatorname{tr}\chi).$$

The proof of Lemma 8.7 is postponed to Appendix A. In the rest of this chapter, we use Lemma 8.7 to obtain the control of $B_{j,\nu,\nu'}^{1,1,1,1}$.

We first estimate $h_{1,p,q}$. We have:

$$(8.51) \quad h_{1,p,q} = \int_{\mathbb{S}^2} HN(L(\operatorname{tr}\chi))F_{j,-1}(u)\eta_j^{\nu'}(\omega) d\omega,$$

where the tensor H is given by:

$$(8.52) \quad H = \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^p \left(\int_{\mathbb{S}^2} \operatorname{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right).$$

In view of (8.51), the estimate (7.5) in $L^1(\mathcal{M})$ yields:

$$(8.53) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega \in \operatorname{supp}(\eta_j^{\nu'})} (\|H\|_{L_u^2 L_t^4 L_x^2}) + |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H\|_{L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Let the tensor H_1 be defined by:

$$(8.54) \quad H_1 = \int_{\mathbb{S}^2} \operatorname{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega'.$$

Then, we have:

$$H = \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^p H_1,$$

which together with (8.53) yields:

(8.55)

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H_1\|_{L_u^2 L_t^4 L_{x'}^2}) + |\nu - \nu'| \|H_1\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H_1\|_{L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used the estimate (2.42) for $\partial_\omega N$, and the size of the patch.

Next, we estimate the various terms in the right-hand side of (8.55) starting with the last one. In view of (8.54), we have:

$$\begin{aligned} \|\chi_{2\nu'} H_1\|_{L^2(\mathcal{M})} &\lesssim \|\chi_{2\nu'}\|_{L_{u,\nu',x'}^\infty L_t^2} \left\| \int_{\mathbb{S}^2} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u) \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x'}^2 L_t^\infty} \\ (8.56) \quad &\lesssim \varepsilon (1 + q^2) \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (7.64) of the $L_{u,\nu',x'}^2 L_t^\infty$ of oscillatory integrals together with the estimate (2.46) for χ_2 .

Next, we estimate the second term in the right-hand side of (8.55). In view of the Definition (8.54) of H_1 , and in view of the estimates (7.115) and (7.116), we have:

$$\|H_1\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^{\nu'}$$

and

$$\|H_1\|_{L^\infty(\mathcal{M})} \lesssim 2^j \varepsilon \gamma_j^{\nu'}.$$

Interpolating between these two estimates, we obtain:

$$(8.57) \quad \|H_1\|_{L^{3+}(\mathcal{M})} \lesssim 2^{(\frac{1}{3})+j} \varepsilon \gamma_j^{\nu'}.$$

Next, we estimate the first term in the right-hand side of (8.55). The estimate (7.83) applied to H yields:

$$\sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H_1\|_{L_u^2 L_t^\infty L_{x'}^2}) \lesssim (1 + q^{\frac{5}{2}}) \varepsilon 2^{\frac{j}{2}} |\nu - \nu'| \gamma_j^{\nu'}.$$

Interpolating with (7.115), we obtain:

$$(8.58) \quad \sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H_1\|_{L_u^2 L_t^4 L_{x'}^2}) \lesssim (1 + q^2) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \gamma_j^{\nu'}.$$

Finally, (8.55), (8.56), (8.57) and (8.58) imply:

$$(8.59) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-(\frac{1}{6})-j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_{\nu'}^j.$$

Next, we estimate $h_{2,p,q}$ defined in (8.46). We have:

$$(8.60) \quad h_{2,p,q} = \int_{\mathbb{S}^2} H \nabla L(\text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega,$$

where the tensor H is given by:

$$(8.61) \quad H = \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \left(\int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u')\eta_j^{\nu'}(\omega)d\omega'\right).$$

In view of (8.60), the estimate (7.5) in $L^1(\mathcal{M})$ yields:

$$(8.62) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} (\|H\|_{L_u^2 L_t^4 L_x^2}) + |\nu - \nu'| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu'} H\|_{L^2(\mathcal{M})}\right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu.$$

In view, of (8.52), (8.53), (8.61) and (8.62), $h_{2,p,q}$ satisfies the same estimate as $h_{1,p,q}$. Thus, we have in view of (8.59):

$$(8.63) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-(\frac{1}{6})-j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}\right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_{\nu'}^j.$$

Next, we estimate $h_{3,p,q}$. In view of the Raychaudhuri Equation (2.22) satisfied by $\text{tr}\chi$, the decomposition (2.72) for $\text{tr}\chi$, the decomposition (2.77) for $|\widehat{\chi}|^2$, and with the L^∞ estimates for b and $\text{tr}\chi$ provided respectively by (2.38) and (2.39), we obtain the following decomposition for $L(\text{tr}\chi)$:

$$(8.64) \quad L(\text{tr}\chi) = \chi_{2\nu} \cdot (2\chi_1 + \widehat{\chi}) + f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$(8.65) \quad \|f_1^j\|_{L_{u,\nu}^\infty L_t^2 L^\infty(P_{t,u,\nu})} \lesssim \varepsilon,$$

where the scalar f_2^j satisfies:

$$(8.66) \quad \|f_2^j\|_{L_u^\infty L^2(\mathcal{M}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

This implies the following decomposition:

$$(8.67) \quad \begin{aligned} & \int_{\mathbb{S}^2} L(\text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \\ &= -\chi_{2\nu} \cdot \left(\int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) \\ & \quad + f_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) \\ & \quad + \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \\ &= -\chi_2' \cdot \left(\int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) \\ & \quad - (\chi_{2\nu} - \chi_2') \cdot \left(\int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) \\ & \quad + f_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) \end{aligned}$$

$$+ \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega.$$

We obtain the following estimate for $h_{3,p,q}$:

$$\begin{aligned} & \|h_{3,p,q}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} \chi'_2 H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & + \int_{\mathbb{S}^2} \|\chi_{2\nu} - \chi'_{2\nu}\|_{L^{6-}(\mathcal{M})} \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{3+}(\mathcal{M})} \\ & \quad \times \left\| H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ & + \left(\|f_1^j\|_{L_{u\nu, x'_\nu}^\infty L_t^\infty}^2 \left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu, x'_\nu}^2 L_t^\infty} \right. \\ & \quad \left. + \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \right) \\ & \quad \times \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

which together with the estimate (8.65) for f_1^j , the estimate in $L^2(\mathcal{M})$ (8.56), the estimates (2.46) and (2.47) for χ_2 , and the estimate (7.63) of the $L_{u\nu, x'_\nu}^2 L_t^\infty$ of oscillatory integrals yields:

$$\begin{aligned} (8.68) \quad & \|h_{3,p,q}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} \chi'_2 H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & + \varepsilon |\nu - \nu'| \left(\int_{\mathbb{S}^2} \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{3+}(\mathcal{M})} \right. \\ & \quad \left. \times \|H_1\|_{L_u^\infty L^2(\mathcal{M}_{u'})} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \|F_{j,-1}(u')\|_{L_u^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ & + \varepsilon \gamma_\nu^j \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Next, we estimate the $L^{3+}(\mathcal{M})$ norm in the right-hand side of (8.68). Using the estimate for the $L^p(\mathcal{M})$ norm (7.2) with $p = 6$, we have:

$$(8.69) \quad \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^6(\mathcal{M})}$$

$$\begin{aligned} &\lesssim \sup_{\omega} \left((\|\chi_1\|_{L_u^\infty L^6(\mathcal{H}_u)} + \|\widehat{\chi}\|_{L_u^\infty L^6(\mathcal{H}_u)}) \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P \right\|_{L^\infty} \right) 2^{\frac{5j}{6}} \gamma_j^\nu \\ &\lesssim 2^{\frac{5j}{6}} \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.40) for $\widehat{\chi}$, the estimate (2.42) for $\partial_\omega N$, and the estimate (2.46) for χ_1 . Next, recall the decomposition (2.74) for $\widehat{\chi}$ and the decomposition (2.75) for χ_2 which yield:

$$2\chi_1 + \widehat{\chi} = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$(8.70) \quad \|F_1^j\|_{L_{u,\nu,x',\nu}^\infty L_t^2} \lesssim \varepsilon,$$

where the scalar F_2^j satisfies:

$$(8.71) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

This yields:

$$\begin{aligned} &\int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \\ &= F_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right) + \int_{\mathbb{S}^2} F_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \end{aligned}$$

and thus:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \|F_1^j\|_{L_{u,\nu,x',\nu}^\infty L_t^2} \left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu,x',\nu}^2 L_t^\infty} \\ &\quad + \left\| \int_{\mathbb{S}^2} F_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon \gamma_j^\nu + \left\| \int_{\mathbb{S}^2} F_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.70) and the estimate (7.63) of the $L_{u,\nu,x',\nu}^2 L_t^\infty$ of oscillatory integrals. Then, using the basic estimate in $L^2(\mathcal{M})$ (7.1), we obtain:

$$\begin{aligned} (8.72) \quad &\left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) b^{-1} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon \gamma_j^\nu + \sup_{\omega} \left(\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| b^{-1} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.71). Next, interpolating (8.69) and (8.72), we obtain:

$$\left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi})b^{-1} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^{3+}(\mathcal{M})} \lesssim 2^{(\frac{5}{12})+j}\varepsilon\gamma_j^\nu.$$

Together with (8.68), the estimate (2.42) for $\partial_\omega N$, the size of the patch, and the estimate in $L^2(\mathcal{O}\mathcal{M})$ (8.72), we obtain:

(8.73)

$$\begin{aligned} \|h_{3,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \varepsilon\gamma_j^\nu \left\| \int_{\mathbb{S}^2} \chi_2' H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad + \varepsilon|\nu - \nu'|2^{(\frac{5}{12})+j}\gamma_j^\nu \left(\int_{\mathbb{S}^2} \|H_1\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega')d\omega' \right) \\ &\quad + \varepsilon\gamma_\nu^j \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Next, using the Definition (8.49) of H_1 , the estimate (2.38) for b , the estimate (2.39) and (2.40) for χ , and the estimate (2.46) for χ_2 , we have:

$$(8.74) \quad \|H_1\|_{L_u^\infty L^2(\mathcal{H}_{u'})} + \|\chi_2' H_1\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \varepsilon.$$

Using the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1), we have:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \chi_2' H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad + \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \sup_\omega \left(\left(\|\chi_2' H_1\|_{L_u^\infty L^2(\mathcal{H}_{u'})} + \|H_1\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}}\gamma_j^\nu \\ &\lesssim 2^{\frac{j}{2}}\varepsilon\gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.74), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.73) and (8.74), this yields:

(8.75)

$$\begin{aligned} \|h_{3,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim 2^{\frac{j}{2}}\varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j + \varepsilon^2|\nu - \nu'|2^{(\frac{5}{12})+j}\gamma_j^\nu \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega')d\omega' \right) \\ &\lesssim 2^{\frac{j}{2}} \left(1 + 2^{\frac{j}{2}}|\nu - \nu'|2^{-(\frac{1}{12})-j} \right) \varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j, \end{aligned}$$

where we used in the last inequality Cauchy-Schwarz in λ' to evaluate $\|F_{j,-1}(u')\|_{L_{u'}^2}$, Cauchy-Schwarz in ω' , and the size of the patch.

Next, we estimate $h_{4,p,q}$. We have:

$$(8.76) \quad \|h_{4,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}$$

$$\times \left\| \int_{\mathbb{S}^2} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.77) \quad \left\| \int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \sup_{\omega} \left(\|H_2\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \right) \left\| \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P \right\|_{L^{\infty}} 2^{\frac{j}{2}} \gamma_j^{\nu} \\ \lesssim \sup_{\omega} \left(\|H_2\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu},$$

where we used in the last inequality the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. In view of (8.50), the estimate (2.38) for b , the estimates (2.36) (2.37) for $\bar{\delta}$, the estimates (2.39) (2.40) for χ , and the Raychaudhuri Equation (2.22) satisfied by $\text{tr} \chi$, we have:

$$\|H_2\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \lesssim \varepsilon,$$

which together with (8.77) yields:

$$\left\| \int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu}.$$

Together with (8.76) and the estimate (7.63) of the $L_{u,\nu,x',\nu'}^2 L_t^{\infty}$ of oscillatory integrals, we obtain:

$$(8.78) \quad \|h_{4,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^{\nu} \gamma_{\nu'}^j.$$

Finally, in view of (8.44), (8.59), (8.63), (8.75) and (8.78), we obtain for $B_{j,\nu,\nu'}^{1,1,1,1}$ the following decomposition:

$$(8.79) \quad B_{j,\nu,\nu'}^{1,1,1,1} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_{\nu} - N_{\nu'}|)^{p+q}} \\ \times \left[\frac{1}{|N_{\nu} - N_{\nu'}|^2} h_{1,p,q} + \frac{1}{|N_{\nu} - N_{\nu'}|^3} (h_{2,p,q} + h_{3,p,q} + h_{4,p,q}) \right] d_{\mathcal{M}} \\ + 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi - \chi') L(\text{tr} \chi) \text{tr} \chi'}{\mathbf{g}(L, L')^2} F_{j,-1}(u) F_{j,-1}(u') \eta_j^{\nu}(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, and where $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$ and $h_{4,p,q}$ satisfy the following estimate:

$$(8.80) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h_{2,p,q}\|_{L^1(\mathcal{M})} + \|h_{3,p,q}\|_{L^1(\mathcal{M})} + \|h_{4,p,q}\|_{L^1(\mathcal{M})} \\ \lesssim (1 + q^2) \left(1 + 2^{\frac{j}{2}} |\nu - \nu'| 2^{-\left(\frac{1}{12}\right) - j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^{\nu} \gamma_{\nu'}^j.$$

The term $B_{j,\nu,\nu'}^{1,1,1,2}$ defined by (8.43) is estimated in the same way. Indeed, proceeding as for $B_{j,\nu,\nu'}^{1,1,1,1}$, we integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,2}$ using (7.137).

Lemma 8.8. — *Let $B_{j,\nu,\nu'}^{1,1,1,2}$ be defined by (8.43). Integrating by parts using (7.137) yields:*

(8.81)

$$\begin{aligned} B_{j,\nu,\nu'}^{1,1,1,2} &= 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \\ &\quad \times \left[\frac{1}{|N_\nu - N_{\nu'}|^2} h'_{1,p,q} + \frac{1}{|N_\nu - N_{\nu'}|^3} (h'_{2,p,q} + h'_{3,p,q} + h'_{4,p,q}) \right] d\mathcal{M} \\ &\quad + 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi - \chi') \operatorname{tr} \chi L'(\operatorname{tr} \chi')}{\mathbf{g}(L, L')^2} F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h'_{1,p,q}, h'_{2,p,q}, h'_{3,p,q}, h'_{4,p,q}$ on \mathcal{M} are given by:

$$\begin{aligned} (8.82) \quad h'_{1,p,q} &= \left(\int_{\mathbb{S}^2} N(\operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} L'(\operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

$$\begin{aligned} (8.83) \quad h'_{2,p,q} &= \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \nabla' L'(\operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

$$\begin{aligned} (8.84) \quad h'_{3,p,q} &= \left(\int_{\mathbb{S}^2} H'_1 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} L'(\operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

$$\begin{aligned} (8.85) \quad h'_{4,p,q} &= \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} H'_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

where the tensor H'_1 on \mathcal{M} involved in the definition of $h'_{3,p,q}$ is a linear combination of terms in the following list:

$$(8.86) \quad b^{-1} \nabla'(b \operatorname{tr} \chi), \theta \operatorname{tr} \chi$$

and where the tensor H_2' on \mathcal{M} involved in the definition of $h_{4,p,q}$ is a linear combination of terms in the following list:

$$(8.87) \quad \theta' L'(\mathrm{tr}\chi'), b'^{-1} \nabla'(b') L'(\mathrm{tr}\chi').$$

The proof of Lemma 8.8 is postponed to Appendix B. Next, we use Lemma 8.8 to obtain the control of $B_{j,\nu,\nu'}^{1,1,1,2}$. Now, note that exchanging the role of ω and ω' , we obtain that $h'_{2,p,q}$ corresponds to $h_{2,p,q}$, $h'_{3,p,q}$ corresponds to $h_{3,p,q}$, and $h'_{4,p,q}$ corresponds to $h_{4,p,q}$. Also, exchanging the role of ω and ω' , we obtain that $h'_{1,p,q}$ corresponds to $h_{3,p,q}$ where H_1 has been replaced with $N(\mathrm{tr}\chi)$ which satisfies (8.74) in view of the estimate (2.39) for $\mathrm{tr}\chi$, and the estimate (2.46) for χ_2 . Thus, since $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$ and $h_{4,p,q}$ satisfy the estimate (8.80), we obtain that $h'_{1,p,q}$, $h'_{2,p,q}$, $h'_{3,p,q}$ and $h'_{4,p,q}$ satisfy the following estimate:

$$(8.88) \quad \begin{aligned} & \|h'_{1,p,q}\|_{L^1(\mathcal{M})} + \|h'_{2,p,q}\|_{L^1(\mathcal{M})} + \|h'_{3,p,q}\|_{L^1(\mathcal{M})} + \|h'_{4,p,q}\|_{L^1(\mathcal{M})} \\ & \lesssim (1+q^2) \left(1 + 2^{\frac{j}{2}} |\nu - \nu'| 2^{-(\frac{1}{2})-j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}\right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_\nu^\nu \gamma_{\nu'}^j. \end{aligned}$$

Summing (8.79) and (8.81), we obtain:

$$(8.89) \quad \begin{aligned} & B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} \\ & = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \left[\frac{1}{|N_\nu - N_{\nu'}|^2} (h_{1,p,q} + h'_{1,p,q}) \right. \\ & \quad \left. + \frac{1}{|N_\nu - N_{\nu'}|^3} (h_{2,p,q} + h'_{2,p,q} + h_{3,p,q} + h'_{3,p,q} + h_{4,p,q} + h'_{4,p,q}) \right] d\mathcal{M} \\ & \quad + 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi - \chi')(L(\mathrm{tr}\chi)\mathrm{tr}\chi' + \mathrm{tr}\chi L'(\mathrm{tr}\chi'))}{\mathbf{g}(L, L')^2} \\ & \quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \end{aligned}$$

Note that the last term in the right-hand side is antisymmetric in (ν, ν') and thus vanishes when considering the sum:

$$(8.90) \quad B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}.$$

This cancelation together with (8.89) yields:

$$\begin{aligned} & |B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}| \\ & \lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|^2} \right\|_{L^\infty(\mathcal{M})} \right. \\ & \quad \times (\|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h'_{1,p,q}\|_{L^1(\mathcal{M})}) \\ & \quad + \left\| \frac{1}{|N_\nu - N_{\nu'}|^3} \right\|_{L^\infty(\mathcal{M})} (\|h_{2,p,q}\|_{L^1(\mathcal{M})} + \|h'_{2,p,q}\|_{L^1(\mathcal{M})}) \\ & \quad \left. + \|h_{3,p,q}\|_{L^1(\mathcal{M})} + \|h'_{3,p,q}\|_{L^1(\mathcal{M})} + \|h_{4,p,q}\|_{L^1(\mathcal{M})} + \|h'_{4,p,q}\|_{L^1(\mathcal{M})} \right]. \end{aligned}$$

Together with the estimate (8.32) for $N_\nu - N_{\nu'}$, and the estimates (8.80) and (8.88), we obtain:

$$\begin{aligned} & |B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}| \\ & \lesssim \left(\sum_{p,q \geq 0} c_{pq} \frac{1+q^2}{(2^{\frac{j}{2}}|\nu-\nu'|)^{p+q}} \right) \left[\frac{2^{-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} \right] \\ & \quad \times \left(1 + 2^{\frac{j}{2}}|\nu-\nu'|2^{-(\frac{1}{12})-j} + (2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}} \right) 2^{\frac{j}{2}}\varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j, \\ & \lesssim \left[\frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} \right] \varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j. \end{aligned}$$

Since we have:

$$B_{j,\nu,\nu'}^{1,1,1} = B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2},$$

in view of the decomposition (8.41), this yields:

(8.91)

$$\begin{aligned} |B_{j,\nu,\nu'}^{1,1,1} + B_{j,\nu',\nu}^{1,1,1}| & \lesssim \left(\sum_{p,q \geq 0} c_{pq} \frac{1+q^2}{(2^{\frac{j}{2}}|\nu-\nu'|)^{p+q}} \right) \left[\frac{2^{-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} \right] \\ & \quad \times \left(1 + 2^{\frac{j}{2}}|\nu-\nu'|2^{-(\frac{1}{12})-j} + (2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}} \right) 2^{\frac{j}{2}}\varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j, \\ & \lesssim \left[\frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} \right] \varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j. \end{aligned}$$

Remark 8.9. — The cancelation of the last term of (8.89) when considering the sum (8.90) in view of the antisymmetry in (ν, ν') is crucial. Indeed, we would not be able to estimate this term directly.

Note that exchanging the role of ω and ω' , we obtain that the term $B_{j,\nu,\nu',l,m}^{1,1,2}$ corresponds to $B_{j,\nu,\nu',l,m}^{1,1,1}$. Thus, we obtain in view of (8.91):

$$\begin{aligned} |B_{j,\nu,\nu'}^{1,1,2} + B_{j,\nu',\nu}^{1,1,2}| & \lesssim \left(\sum_{p,q \geq 0} c_{pq} \frac{1+q^2}{(2^{\frac{j}{2}}|\nu-\nu'|)^{p+q}} \right) \left[\frac{2^{-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} \right] \\ & \quad \times \left(1 + 2^{\frac{j}{2}}|\nu-\nu'|2^{-(\frac{1}{12})-j} + (2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}} \right) 2^{\frac{j}{2}}\varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j, \\ & \lesssim \left[\frac{2^{-(\frac{1}{12})-j}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} \right] \varepsilon^2\gamma_j^\nu\gamma_{\nu'}^j. \end{aligned}$$

Together with (8.91), this concludes the proof of Proposition 8.6.

8.2. Proof of Proposition 8.4 (Control of $B_{j,\nu,\nu',l,m}^{1,2}$)

Recall from (8.16) that $B_{j,\nu,\nu',l,m}^{1,2}$ is given by:

$$(8.92) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,2} &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') \frac{(2^{-j}\lambda')^{-1} - (2^{-j}\lambda)^{-1}}{2} \psi(2^{-j}\lambda)(2^{-j}\lambda') \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \\ &\quad \times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Since $\nabla u = b^{-1}N$ and $\nabla u' = b'^{-1}u'$, we have:

$$\begin{aligned} -iN(e^{i\lambda u - i\lambda' u'}) &= e^{i\lambda u - i\lambda' u'} \left(b^{-1}\lambda - b'^{-1}\mathbf{g}(N, N')\lambda' \right) \\ &= e^{i\lambda u - i\lambda' u'} b^{-1}(\lambda - \lambda') + e^{i\lambda u - i\lambda' u'} (b^{-1} - b'^{-1}\mathbf{g}(N, N'))\lambda'. \end{aligned}$$

This yields:

$$\begin{aligned} &((2^{-j}\lambda')^{-1} - (2^{-j}\lambda)^{-1}) e^{i\lambda u - i\lambda' u'} b^{-1} \\ &= -i \frac{2^j}{\lambda\lambda'} N(e^{i\lambda u - i\lambda' u'}) + \frac{2^j}{\lambda} e^{i\lambda u - i\lambda' u'} (b^{-1} - b'^{-1}\mathbf{g}(N, N')) \\ &= -i \frac{2^j}{\lambda\lambda'} N(e^{i\lambda u - i\lambda' u'}) + \frac{2^j}{\lambda} e^{i\lambda u - i\lambda' u'} (b^{-1} - b'^{-1}) + \frac{2^j}{\lambda} e^{i\lambda u - i\lambda' u'} b'^{-1} (1 - \mathbf{g}(N, N')). \end{aligned}$$

In view of (8.92), this implies the following decomposition for $B_{j,\nu,\nu',l,m}^{1,2}$:

$$(8.93) \quad B_{j,\nu,\nu',l,m}^{1,2} = B_{j,\nu,\nu',l,m}^{1,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu,\nu',l,m}^{1,2,3}$$

where $B_{j,\nu,\nu',l,m}^{1,2,1}$, $B_{j,\nu,\nu',l,m}^{1,2,2}$ and $B_{j,\nu,\nu',l,m}^{1,2,3}$ are respectively given by:

$$(8.94)$$

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,1} &= -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^\infty \int_0^\infty \frac{N(e^{i\lambda u - i\lambda' u'})}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ &\quad \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') (2^{-j}\lambda')^{-1} (2^{-j}\lambda)^{-1} \psi(2^{-j}\lambda)(2^{-j}\lambda') \psi(2^{-j}\lambda') f(\lambda\omega) f(\lambda'\omega') \\ &\quad \times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' d_{\mathcal{M}}, \end{aligned}$$

$$(8.95) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,2} &= -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ &\quad \times (b^{-1} - b'^{-1}) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \end{aligned}$$

and:

$$(8.96) \quad \begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,3} &= -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ &\quad \times b'^{-1} (1 - \mathbf{g}(N, N')) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We have the following propositions:

Proposition 8.10. — Let $B_{j,\nu,\nu',l,m}^{1,2,1}$ be given by (8.94). Then, we have the following estimate:

$$(8.97) \quad \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1} \right| \\ \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^j |\nu - \nu'|} + \frac{j 2^{-\frac{j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3}.$$

Proposition 8.11. — Let $B_{j,\nu,\nu',l,m}^{1,2,2}$ be given by (8.95). Then, we have the following estimate:

$$(8.98) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu',\nu,l,m}^{1,2,2}) \right| \\ \lesssim \left[2^{-j} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{1}{2^{\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{4})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Proposition 8.12. — Let $B_{j,\nu,\nu',l,m}^{1,2,3}$ be given by (8.96). Then, we have the following estimate:

$$(8.99) \quad \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \right| \\ \lesssim \left[\frac{j^2 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

In view of the decomposition (8.93) of $B_{j,\nu,\nu',l,m}^{1,2}$, we have:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2} \right| \\ \lesssim \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1} \right| + \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2} \right| \\ + \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \right|.$$

Together with the estimates (8.97), (8.98) and (8.99), we obtain:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2} + B_{j,\nu',\nu,l,m}^{1,2}) \right|$$

$$\lesssim \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{j2^{-\frac{j}{12}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

This concludes the proof of Proposition 8.4.

The rest of this section is as follows. In Section 8.2.1, we give a proof of Proposition 8.10, in Section 8.2.2, we give a proof of Proposition 8.11, and in Section 8.2.3, we give a proof of Proposition 8.12.

8.2.1. Proof of Proposition 8.10 (Control of $B_{j,\nu,\nu',l,m}^{1,2,1}$). — Integrating by parts the N derivative in (8.94), we obtain:

(8.100)

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,1} &= 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(N(L(P_l \text{tr}\chi)) P_m \text{tr}\chi' \right. \\ &\quad + P_l \text{tr}\chi N(L'(P_m \text{tr}\chi')) + L(P_l \text{tr}\chi) N(P_m \text{tr}\chi') + N(P_l \text{tr}\chi) L'(P_m \text{tr}\chi') \\ &\quad \left. + \left(-\frac{N(\mathbf{g}(L, L'))}{\mathbf{g}(L, L')} + \text{tr}\theta \right) \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \right) \\ &\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{O}}\mathcal{M}. \end{aligned}$$

Recall the decomposition of N in the frame N', e'_A :

$$(8.101) \quad N = \mathbf{g}(N, N') N' + (N - \mathbf{g}(N, N') N')$$

and the decomposition of N' in the frame N, e_A :

$$(8.102) \quad N' = \mathbf{g}(N, N') N + (N' - \mathbf{g}(N, N') N).$$

(8.101) yields:

$$(8.103) \quad \begin{aligned} &P_l \text{tr}\chi N(L'(P_m \text{tr}\chi')) + L(P_l \text{tr}\chi) N(P_m \text{tr}\chi') \\ &= \mathbf{g}(N, N') P_l \text{tr}\chi N'(L'(P_m \text{tr}\chi')) + P_l \text{tr}\chi (N - \mathbf{g}(N, N') N')(L'(P_m \text{tr}\chi')) \\ &\quad + \mathbf{g}(N, N') L(P_l \text{tr}\chi) N'(P_m \text{tr}\chi') + L(P_l \text{tr}\chi) (N - \mathbf{g}(N, N') N')(P_m \text{tr}\chi'). \end{aligned}$$

Also, recall that:

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N'),$$

which together with (2.21), (8.101) and (8.102) yields:

$$(8.104) \quad \begin{aligned} N(\mathbf{g}(L, L')) &= -\mathbf{g}(\nabla_N N, N') - \mathbf{g}(N, \nabla_N N') \\ &= b^{-1} \nabla_{N' - \mathbf{g}(N, N') N}(b) + \mathbf{g}(N, N') b'^{-1} \nabla_{N - \mathbf{g}(N, N') N'}(b') \\ &\quad - \theta'(N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N'). \end{aligned}$$

In view of (8.100), (8.103) and (8.104), we obtain:

$$(8.105) \quad B_{j,\nu,\nu',l,m}^{1,2,1} = B_{j,\nu,\nu',l,m}^{1,2,1,1} + B_{j,\nu,\nu',l,m}^{1,2,1,2} + B_{j,\nu,\nu',l,m}^{1,2,1,3}$$

where $B_{j,\nu,\nu',l,m}^{1,2,1,1}$ is given by:

$$(8.106) \quad B_{j,\nu,\nu',l,m}^{1,2,1,1} = 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{N(L(P_l \text{tr}\chi)) P_m \text{tr}\chi' + P_l \text{tr}\chi N'(L'(P_m \text{tr}\chi'))}{\mathbf{g}(L, L')}$$

$$\times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}},$$

where $B_{j,\nu,\nu',l,m}^{1,2,1,2}$ is given by:

$$(8.107) \quad B_{j,\nu,\nu',l,m}^{1,2,1,2} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$(8.108) \quad \begin{aligned} H = & \frac{1}{\mathbf{g}(L, L')} \left(P_l \text{tr} \chi \nabla' (L' (P_m \text{tr} \chi')) (N - N') + L (P_l \text{tr} \chi) N' (P_m \text{tr} \chi') \right. \\ & + L (P_l \text{tr} \chi) \nabla' (P_m \text{tr} \chi') (N - N') + N (P_l \text{tr} \chi) L' (P_m \text{tr} \chi') \\ & \left. + \left(\frac{(b'^{-1} \nabla(b') + \theta')(N - N')^2}{\mathbf{g}(L, L')} + \text{tr} \theta \right) (L (P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L' (P_m \text{tr} \chi')) \right) \end{aligned}$$

and where $B_{j,\nu,\nu',l,m}^{1,2,1,3}$ is given by:

$$(8.109) \quad \begin{aligned} & B_{j,\nu,\nu',l,m}^{1,2,1,3} \\ &= -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b) + b'^{-1} \nabla_{N - \mathbf{g}(N, N')N'}(b')}{\mathbf{g}(L, L')^2} \\ & \quad \times (L (P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L' (P_m \text{tr} \chi')) F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Next, we estimate the three terms in the right-hand side of (8.105) starting with $B_{j,\nu,\nu',l,m}^{1,2,1,1}$. Recall from (8.1) that (l, m) satisfy:

$$m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.$$

Summing in (l, m) , we obtain:

$$\begin{aligned} & \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} \frac{N(L(P_l \text{tr} \chi)) P_m \text{tr} \chi' + P_l \text{tr} \chi N'(L'(P_m \text{tr} \chi'))}{\mathbf{g}(L, L')} \\ & + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} \frac{N(L(P_m \text{tr} \chi)) P_l \text{tr} \chi' + P_m \text{tr} \chi N'(L'(P_l \text{tr} \chi'))}{\mathbf{g}(L, L')} \\ & = \frac{N(L(\text{tr} \chi)) \text{tr} \chi' + \text{tr} \chi N'(L'(\text{tr} \chi'))}{\mathbf{g}(L, L')} \\ & - \frac{N(L(P_{>2^j |\nu - \nu'|} \text{tr} \chi)) P_{>2^j |\nu - \nu'|} \text{tr} \chi' + P_{>2^j |\nu - \nu'|} \text{tr} \chi N'(L'(P_{>2^j |\nu - \nu'|} \text{tr} \chi'))}{\mathbf{g}(L, L')}. \end{aligned}$$

Thus, using the symmetry in (ω, ω') of the integrant in $B_{j,\nu,\nu',l,m}^{1,2,1,1}$, we obtain:

(8.110)

$$\begin{aligned}
& \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \\
&= 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{N(L(\text{tr}\chi)) \text{tr}\chi' + \text{tr}\chi N'(L'(P_m \text{tr}\chi'))}{\mathbf{g}(L, L')} \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} - 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \\
&\quad \times \frac{N(L(P_{>2^j|\nu-\nu'} \text{tr}\chi)) P_{>2^j|\nu-\nu'} \text{tr}\chi' + P_{>2^j|\nu-\nu'} \text{tr}\chi N'(L'(P_{>2^j|\nu-\nu'} \text{tr}\chi'))}{\mathbf{g}(L, L')} \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Estimating the terms $N(L(P_{>2^j|\nu-\nu'} \text{tr}\chi))$ and $N'(L'(P_{>2^j|\nu-\nu'} \text{tr}\chi'))$ would involve commutator terms which are difficult to handle. To avoid this issue, we commute L with N and L' with N' , and then integrate the L and the L' derivative by parts. We obtain schematically in view of (8.110):

(8.111)

$$\begin{aligned}
& \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \\
&= 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{N(L(\text{tr}\chi)) \text{tr}\chi'}{\mathbf{g}(L, L')} \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\
&\quad - 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{[N, L](P_{>2^j|\nu-\nu'} \text{tr}\chi) P_{>2^j|\nu-\nu'} \text{tr}\chi'}{\mathbf{g}(L, L')} \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\
&\quad - 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{N(P_{>2^j|\nu-\nu'} \text{tr}\chi) L(P_{>2^j|\nu-\nu'} \text{tr}\chi')}{\mathbf{g}(L, L')} \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\
&\quad - 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left(\frac{\text{div}_{\mathbf{g}}(L)}{\mathbf{g}(L, L')} - \frac{L(\mathbf{g}(L, L'))}{\mathbf{g}(L, L')^2} \right) N(P_{>2^j|\nu-\nu'} \text{tr}\chi) P_{>2^j|\nu-\nu'} \text{tr}\chi' \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\
&\quad - 2^{-j-1} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'} \text{tr}\chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} b^{\nu-1} P_{>2^j|\nu-\nu'} \text{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}} + \text{terms interverting } (\nu, \nu'),
\end{aligned}$$

where the last term in the right-hand side of (8.111) appears when the L derivative falls on the phase in view of (6.5), and where we chose to ignore the terms which are obtained by interverting ν and ν' since they are treated in the exact same way.

We decompose L in the frame L', N', e'_A :

$$(8.112) \quad L = L' + (N - \mathbf{g}(N, N')N') + (\mathbf{g}(N, N') - 1)N',$$

which yields the following decomposition:

$$(8.113) \quad L(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') = L'(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') + (N - \mathbf{g}(N, N')N')(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') \\ + (\mathbf{g}(N, N') - 1)N'(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi').$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')} \text{ and } \frac{1}{\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.111), (8.113), the Formula (7.146) for $\mathrm{div}_{\mathbf{g}}(L)$ and the Formula (7.149) for $L(\mathbf{g}(L, L'))$, we obtain, schematically:

$$(8.114) \quad \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu-\nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu-\nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \\ = 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}}|N_{\nu} - N_{\nu'}|)^{p+q}} \left[\frac{1}{(2^{\frac{j}{2}}|N_{\nu} - N_{\nu'}|)^2} (h_{1,p,q} + h_{2,p,q} + h_{3,p,q}) \right. \\ \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|N_{\nu} - N_{\nu'}|)} h_{4,p,q} + 2^{-j} h_{5,p,q} \right] d_{\mathcal{M}} \\ - 2^{-j-1} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'}\mathrm{tr}\chi) F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} b'^{-1} P_{>2^j|\nu-\nu'}\mathrm{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}} + \text{terms interverting } (\nu, \nu'),$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}, h_{5,p,q}$ on \mathcal{M} are given by:

$$(8.115) \quad h_{1,p,q} = \left(\int_{\mathbb{S}^2} N(L(\mathrm{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.116) \quad h_{2,p,q} = \left(\int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'}\mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.117) \quad h_{3,p,q} = \left(\int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right),$$

$$(8.118) \quad h_{4,p,q} = \left(\int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right)$$

and:

$$(8.119) \quad h_{5,p,q} = \left(\int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} N'(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right),$$

where the tensors G_1 and G_2 are schematically given by:

$$(8.120) \quad G_1 = [N, L](P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi) + \bar{\delta}N(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi)$$

and:

$$(8.121) \quad G_2 = L'(P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi') + (\bar{\delta}' + \chi' + \zeta')P_{>2^j|\nu-\nu'|}\mathrm{tr}\chi'$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}, h_{5,p,q}$ starting with $h_{1,p,q}$. We have:

$$(8.122) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} N(L(\mathrm{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ \left\| \int_{\mathbb{S}^2} \mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^3(\mathcal{M})}.$$

We estimate the $L^\infty(\mathcal{M})$ norm of the last term:

$$(8.123) \quad \left\| \int_{\mathbb{S}^2} \mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^\infty(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \left\| \mathrm{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \right\|_{L^\infty} \eta_j^{\nu'}(\omega')d\omega' \\ \lesssim \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L_w^\infty} \eta_j^{\nu'}(\omega')d\omega' \\ \lesssim 2^j \varepsilon \gamma_j^{\nu'},$$

where we used the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, Cauchy Schwartz in λ' for $\|F_{j,-1}(u')\|_{L_u^\infty}$, and Cauchy Schwartz in ω' . On the other hand, the estimate (7.64) yields:

$$\left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1+q^2) \varepsilon \gamma_j^{\nu'}$$

Interpolating these two estimates, we obtain:

$$(8.124) \quad \left\| \int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^3(\mathcal{M})} \lesssim 2^{\frac{j}{3}} (1+q) \varepsilon \gamma_j^{\nu'}$$

Next, we estimate the first term in the right-hand side of (8.122). We have:

$$(8.125) \quad \left\| \int_{\mathbb{S}^2} N(L(\text{tr}\chi)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \|N(L(\text{tr}\chi))\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty} \|F_{j,-1}(u)\|_{L_u^{\frac{3}{2}}} \eta_j^\nu(\omega) d\omega \\ \lesssim \int_{\mathbb{S}^2} \|N(L(\text{tr}\chi))\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega,$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next, we estimate $N(L(\text{tr}\chi))$. In view of the Raychaudhuri Equation (2.22), we have:

$$N(L(\text{tr}\chi)) = -\text{tr}\chi N(\text{tr}\chi) - 2\widehat{\chi} \cdot \mathbf{D}_N \widehat{\chi} - N(\bar{\delta})\text{tr}\chi - \bar{\delta}N(\text{tr}\chi),$$

which together with the Sobolev embedding (2.50), and the estimates (2.39) for $\text{tr}\chi$, (2.40) for $\widehat{\chi}$, and (2.36) (2.37) for $\bar{\delta}$ yields:

$$\|N(L(\text{tr}\chi))\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \\ \lesssim (\|\mathbf{D}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\mathbf{D}\bar{\delta}\|_{L_u^\infty L^2(\mathcal{H}_u)}) (\|\chi\|_{L_u^\infty L^6(\mathcal{H}_u)} + \|\bar{\delta}\|_{L_u^\infty L^6(\mathcal{H}_u)}) \lesssim \varepsilon.$$

Together with (8.125), we obtain:

$$(8.126) \quad \left\| \int_{\mathbb{S}^2} N(L(\text{tr}\chi)) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ \lesssim \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in u , Cauchy Schwarz in ω and the size of the patch. Finally, (8.122), (8.124) and (8.126) imply:

$$(8.127) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1+q) 2^{\frac{5j}{6}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}$$

Next, we estimate $h_{2,p,q}$. We have:

(8.128)

$$\|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^3(\mathcal{M})}.$$

Arguing as in (8.123), we have:

$$\left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^\infty(\mathcal{M})} \lesssim 2^j \varepsilon \gamma_j^{\nu'}.$$

On the other hand, we have in view of the L^2 estimate (7.3):

$$\begin{aligned} (8.129) \quad & \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \frac{2^{\frac{j}{2}}}{2^j |\nu - \nu'|} \varepsilon \gamma_j^{\nu'} \\ & \lesssim \frac{\varepsilon \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Interpolating these two estimates, and using the fact that:

$$2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1,$$

we obtain:

$$(8.130) \quad \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^3(\mathcal{M})} \lesssim 2^{\frac{j}{3}} \varepsilon \gamma_j^{\nu'}.$$

Next, we estimate the first term in the right-hand side of (8.128). Arguing as in (8.125), we have:

$$\begin{aligned} (8.131) \quad & \left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \|G_1\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{M}_u)} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega. \end{aligned}$$

Next, we estimate G_1 . In view of the definition of G_1 (8.120), the commutator Formulas (2.32) for $[L, \underline{L}]$ and the fact that $2N = L - \underline{L}$, we have schematically:

$$G_1 = \bar{\delta}N(P_{>2^j|\nu-\nu'|} \text{tr}\chi) + n^{-1} \nabla_N n L P_{>2^j|\nu-\nu'|} \text{tr}\chi + (\zeta - \underline{\zeta}) \cdot \nabla(P_{>2^j|\nu-\nu'|} \text{tr}\chi).$$

This yields:

$$\begin{aligned} \|G_1\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{M}_u)} & \lesssim (\|\bar{\delta}\|_{L_u^\infty L^6(\mathcal{M}_u)} + \|n^{-1} \nabla_N n\|_{L_u^\infty L^6(\mathcal{M}_u)} + \|\zeta\|_{L_u^\infty L^6(\mathcal{M}_u)} + \|\underline{\zeta}\|_{L_u^\infty L^6(\mathcal{M}_u)}) \\ & \quad \times \|\mathbf{D}P_{>2^j|\nu-\nu'|} \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{M}_u)} \\ & \lesssim \varepsilon \|\mathbf{D}P_{>2^j|\nu-\nu'|} \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{M}_u)}, \end{aligned}$$

where we used in the last inequality the Sobolev embedding (2.50), and the estimates (2.37)–(2.36) for n , $\bar{\delta}$ and $\underline{\zeta}$, and the estimate (2.41) for ζ . Together with the basic properties of $P_{>2^j|\nu-\nu'|}$, the commutator estimates (2.64) and (2.65), and the estimate (2.39) for $\text{tr}\chi$, this implies:

$$(8.132) \quad \|G_1\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{H}_u)} \lesssim \varepsilon.$$

Together with (8.131), and arguing as in (8.126), we obtain:

$$(8.133) \quad \left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu.$$

Finally, (8.128), (8.130) and (8.133) imply:

$$(8.134) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{5j}{6}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $h_{3,p,q}$. We have:

$$(8.135) \quad \|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

We estimate the first term in the right-hand side of (8.135). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\left\| \int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_\omega \left\| N(P_{>2^j|\nu-\nu'} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ \lesssim \left(\sup_\omega \|N(P_{>2^j|\nu-\nu'} \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^\nu.$$

Together with the estimate (2.39) for $\text{tr}\chi$, the commutator estimate (2.64), the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$(8.136) \quad \left\| \int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.$$

Next, we estimate the second term in the right-hand side of (8.135). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$(8.137) \quad \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_\omega \left\| G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ \lesssim \left(\sup_\omega \|G_2\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

In view of the definition of G_2 (8.121), we have:

$$(8.138) \quad \begin{aligned} \|G_2\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim (\|\bar{\delta}\|_{L_t^\infty L_x^4} + \|\chi\|_{L_t^\infty L_x^4} + \|\zeta\|_{L_t^\infty L_x^4}) \|P_{>2^j|\nu-\nu'|} \text{tr}\chi\|_{L_t^2 L_x^4} \\ &\quad + \|L(P_{>2^j|\nu-\nu'|} \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \varepsilon \|P_{>2^j|\nu-\nu'|} \text{tr}\chi\|_{L_t^2 L_x^4} + \|L(P_{>2^j|\nu-\nu'|} \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)}, \end{aligned}$$

where we used in the last inequality the embedding (2.51), and the estimates (2.36) (2.37) for $\bar{\delta}$, the estimates (2.39) (2.40) for χ , and the estimate (2.41) for ζ . Using the Bernstein inequality and the finite band property for P_l , we have:

$$(8.139) \quad \begin{aligned} \|P_{>2^j|\nu-\nu'|} \text{tr}\chi\|_{L_t^2 L_x^4} &\lesssim \sum_{l>2^j|\nu-\nu'|} \|P_l \text{tr}\chi\|_{L_t^2 L_x^4} \\ &\lesssim \sum_{l>2^j|\nu-\nu'|} 2^{\frac{l}{2}} \|P_l \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \sum_{l>2^j|\nu-\nu'|} 2^{-\frac{l}{2}} \|\nabla \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}}, \end{aligned}$$

where we used the estimate (2.39) for $\text{tr}\chi$ in the last inequality. In view of (8.138), we also need to estimate $L(P_{>2^j|\nu-\nu'|} \text{tr}\chi)$. Using the estimate (2.36) for n , we have:

$$(8.140) \quad \begin{aligned} \|L(P_l \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim \|nL(P_l \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \|P_l(nL \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|[nL, P_l] \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \|P_l(nL \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} + 2^{-\frac{l}{2}} \varepsilon, \end{aligned}$$

where we used in the last inequality the commutator estimate (2.67). Now, in view of the Raychaudhuri Equation (2.22), the worst term in $P_l(nL \text{tr}\chi)$ is $P_l(n|\widehat{\chi}|^2)$. In view of the finite band property, we have:

$$\begin{aligned} \|P_l(n|\widehat{\chi}|^2)\|_{L_u^\infty L^2(\mathcal{H}_u)} &= 2^{-2l} \|P_l(\text{div}(\nabla(n|\widehat{\chi}|^2)))\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim 2^{-2l} 2^{\frac{3l}{2}} \|\nabla(n|\widehat{\chi}|^2)\|_{L_t^2 L_x^{\frac{4}{3}}}, \end{aligned}$$

where we used (2.63) with $p = \frac{4}{3}$ in the last inequality. This yields:

$$\begin{aligned} \|P_l(n|\widehat{\chi}|^2)\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim 2^{-\frac{l}{2}} \|\widehat{\chi}\|_{L_t^\infty L_x^4} (\|n\|_{L^\infty(\mathcal{M})} \|\widehat{\chi}\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\quad + \|\nabla n\|_{L_u^\infty L^3(\mathcal{H}_u)} \|\widehat{\chi}\|_{L_u^\infty L^6(\mathcal{H}_u)}) \\ &\lesssim 2^{-\frac{l}{2}} \varepsilon, \end{aligned}$$

where we used in the last inequality the Sobolev embedding (2.50) and the embedding (2.51), and the estimates (2.36) for n and (2.40) for $\widehat{\chi}$. Since $P_l(n|\widehat{\chi}|^2)$ is the worst term in $P_l(nL \text{tr}\chi)$ in view of the Raychaudhuri Equation (2.22), we obtain:

$$(8.141) \quad \|P_l(nL \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{l}{2}} \varepsilon,$$

which together with (8.140) yields:

$$(8.142) \quad \|L(P_t \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{1}{2}} \varepsilon.$$

Now, in view of (8.138), (8.139) and (8.142), and since $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$, we obtain:

$$\|G_2\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{4}} \varepsilon.$$

Together with (8.137), the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we deduce:

$$(8.143) \quad \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'},$$

where we also used the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.135), (8.136) and (8.143) imply:

$$(8.144) \quad \|h_{3,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{3j}{4}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{4,p,q}$. In view of the definition of $h_{4,p,q}$ (8.118), we have:

$$\begin{aligned} \|h_{4,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'|} \text{tr}\chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Using the estimate (8.136) for the first term, and the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) for the second term, we obtain:

$$(8.145) \quad \begin{aligned} \|h_{4,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \left(\sup_{\omega'} \left\| \nabla'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) \varepsilon 2^j \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \left(\sup_{\omega'} \left\| \nabla'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) \varepsilon 2^j \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Using the finite band property and the estimate (2.39) for $\text{tr}\chi$, we obtain:

$$\left\| \nabla'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \varepsilon.$$

Together with (8.145), we finally obtain:

$$(8.146) \quad \|h_{4,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim 2^j \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{5,p,q}$. In view of the definition of $h_{5,p,q}$ (8.119), we have:

$$\begin{aligned} \|h_{5,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} N(P_{>2^j|\nu-\nu'|} \text{tr}\chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Then, using the estimate (8.136) for both terms, we obtain:

$$(8.147) \quad \|h_{5,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^j \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, we have in view of (8.114), we have:

$$\begin{aligned} & \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \right| \\ & \lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left[\left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^2} \right\|_{L^\infty(\mathcal{M})} \right. \\ & \quad \times (\|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h_{2,p,q}\|_{L^1(\mathcal{M})} + \|h_{3,p,q}\|_{L^1(\mathcal{M})}) \\ & \quad \left. + \left\| \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)} \right\|_{L^\infty(\mathcal{M})} (\|h_{4,p,q}\|_{L^1(\mathcal{M})} + 2^{-j} \|h_{5,p,q}\|_{L^1(\mathcal{M})}) \right] \\ & \quad + 2^{-j} \left\| \int_{\mathbb{S}^2} N(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} b'^{-1} P_{>2^j |\nu - \nu'|} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

which together with (8.32), (8.127), (8.134), (8.144), (8.146) and (8.147) yields:

$$\begin{aligned} & \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \right| \\ & \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q}} \left[\frac{(1+q)2^{\frac{5j}{6}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \quad + 2^{-j} \left\| \int_{\mathbb{S}^2} N(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} b'^{-1} P_{>2^j |\nu - \nu'|} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \frac{2^{-\frac{j}{6}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^j |\nu - \nu'|} + 2^{-j} \left\| \int_{\mathbb{S}^2} N(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} b'^{-1} P_{>2^j |\nu - \nu'|} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Using the corresponding analog of (8.136) and the corresponding analog of (8.129) to estimate the last term in the right-hand side, we deduce:

$$(8.148) \quad \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} \right|$$

$$\lesssim \frac{2^{-\frac{j}{6}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^j |\nu - \nu'|}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,1,2}$. Recall from (8.107) and (8.108) that $B_{j,\nu,\nu',l,m}^{1,2,1,2}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,1,2} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned} H = & \frac{1}{\mathbf{g}(L, L')} \left(P_l \text{tr} \chi \nabla' (L' (P_m \text{tr} \chi')) (N - N') + L (P_l \text{tr} \chi) N' (P_m \text{tr} \chi') \right. \\ & + L (P_l \text{tr} \chi) \nabla' (P_m \text{tr} \chi') (N - N') + N (P_l \text{tr} \chi) L' (P_m \text{tr} \chi') \\ & \left. + \left(\frac{(b'^{-1} \nabla(b') + \theta')(N - N')^2}{\mathbf{g}(L, L')} + \text{tr} \theta \right) (L (P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L' (P_m \text{tr} \chi')) \right). \end{aligned}$$

Expanding

$$\frac{1}{\mathbf{g}(L, L')} \text{ and } \frac{1}{\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.108), we obtain, schematically:

$$\begin{aligned} (8.149) \quad H = & \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\ & \times (L (P_l \text{tr} \chi) H_1 + H_2 L' (P_m \text{tr} \chi') + P_l \text{tr} \chi (\nabla' (L' (P_m \text{tr} \chi'))) \\ & + (b'^{-1} \nabla(b') + \theta') L' (P_m \text{tr} \chi') + \text{tr} \theta L (P_l \text{tr} \chi) P_m \text{tr} \chi'), \end{aligned}$$

where the tensors H_1, H_2 on \mathcal{M} are schematically given by:

$$(8.150) \quad H_1 = N' (P_m \text{tr} \chi') + \nabla' (P_m \text{tr} \chi') + \theta' P_m \text{tr} \chi'$$

and:

$$(8.151) \quad H_2 = N (P_l \text{tr} \chi) + \text{tr} \theta P_l \text{tr} \chi$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In turn, this yields in view of (8.107) and (8.108) the following decomposition for $B_{j,\nu,\nu',l,m}^{1,2,1,2}$:

$$\begin{aligned} (8.152) \quad B_{j,\nu,\nu',l,m}^{1,2,1,2} = & 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+2}} \\ & \times [h_{1,p,q,l,m} + h_{2,p,q,l,m} + h_{3,p,q,l,m} + h_{4,p,q,l,m}] d\mathcal{M}, \end{aligned}$$

where the scalar functions $h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}$ on \mathcal{M} are schematically given by:

(8.153)

$$h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.154)

$$h_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.155)

$$h_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} (b'^{-1} \nabla(b') + \theta') L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) + \left(\int_{\mathbb{S}^2} \text{tr} \theta L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

(8.156)

$$h_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} \nabla'(L'(P_m \text{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}$, starting with $h_{1,p,q,l,m}$. We have:

$$(8.157) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

Next, we evaluate both terms in the right-hand side of (8.157) starting with the first one. Assume first that $l > j/2$. Then, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.158) \quad \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned} &\lesssim \left(\sup_{\omega} \left\| L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P \right\|_{L_u^{\infty} L^2(\mathcal{M}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu} \\ &\lesssim \left(\sup_{\omega} \left\| L(P_l \text{tr} \chi) \right\|_{L_u^{\infty} L^2(\mathcal{M}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. In view of (8.158) and the estimate (8.142) for $L(P_l \text{tr} \chi)$, we obtain, in the case $l > j/2$:

$$(8.159) \quad \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} - \frac{l}{2}} \gamma_j^{\nu}.$$

Next, we evaluate the first term in the right-hand side of (8.157) in the case $l = j/2$, which is given by:

$$\left\| \int_{\mathbb{S}^2} L(P_{\leq j/2} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

We first decompose $L(P_{\leq j/2} \text{tr} \chi)$ as:

$$L(P_{\leq j/2} \text{tr} \chi) = L(\text{tr} \chi) - \sum_{l > \frac{j}{2}} L(P_l \text{tr} \chi),$$

which together with (8.159) yields:

$$\begin{aligned} (8.160) \quad &\left\| \int_{\mathbb{S}^2} L(P_{\leq \frac{j}{2}} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} L(\text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \sum_{l > \frac{j}{2}} \varepsilon 2^{\frac{j}{2} - \frac{l}{2}} \gamma_j^{\nu} \\ &\lesssim \left\| \int_{\mathbb{S}^2} L(\text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu}. \end{aligned}$$

Now, recall the decomposition (8.64) (8.65) (8.66) for $L(\text{tr} \chi)$. We have:

$$(8.161) \quad L(\text{tr} \chi) = \chi_{2\nu} \cdot (2\chi_1 + \widehat{\chi}) + f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$(8.162) \quad \|f_1^j\|_{L_{u\nu}^{\infty} L_t^2 L^{\infty}(P_{t,u\nu})} \lesssim \varepsilon,$$

where the scalar f_2^j satisfies:

$$(8.163) \quad \|f_2^j\|_{L_u^{\infty} L^2(\mathcal{M}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

Together with (8.159), this yields:

$$\begin{aligned} (8.164) \quad &\left\| \int_{\mathbb{S}^2} L(P_{\leq \frac{j}{2}} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \chi_{2\nu} \cdot \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$\begin{aligned}
& + \left\| f_1^j \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& + \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon 2^{\frac{j}{4}} \gamma_j^\nu.
\end{aligned}$$

Next, we evaluate the various terms in the right-hand side of (8.164) starting with the first one. We have:

$$\begin{aligned}
(8.165) \quad & \left\| \chi_{2\nu} \cdot \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \lesssim \|\chi_2\|_{L^6(\mathcal{M})} \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^3(\mathcal{M})} \\
& \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^3(\mathcal{M})},
\end{aligned}$$

where we used in the last inequality the Sobolev embedding (2.50) and the estimate (2.46) for χ_2 . Interpolating (8.69) and (8.72), we obtain:

$$\left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^3(\mathcal{M})} \lesssim 2^{\frac{5j}{12}} \varepsilon \gamma_j^\nu.$$

Together with (8.165), this yields:

$$(8.166) \quad \left\| \chi_{2\nu} \cdot \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{5j}{12}} \varepsilon \gamma_j^\nu.$$

Next, we evaluate the second term in the right-hand side of (8.164). We have:

$$\begin{aligned}
(8.167) \quad & \left\| f_1^j \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu, x'_\nu}^2 L_t^\infty} \\
& \lesssim (1 + p^2) \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used the estimate (8.162) for f_1^j , and the estimate (7.63) to bound the $L_{u\nu, x'_\nu}^2 L_t^\infty$ norm.

Next, we evaluate the third term in the right-hand side of (8.164). In view of the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\begin{aligned}
(8.168) \quad & \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \lesssim \left(\sup_\omega \left\| f_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
& \lesssim \varepsilon \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (8.163) for f_2^j , the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Finally, (8.164), (8.166), (8.167) and (8.168) imply:

$$(8.169) \quad \left\| \int_{\mathbb{S}^2} L(P_{\leq \frac{j}{2}} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim (1+p^2)2^{\frac{5j}{12}}\varepsilon\gamma_j^\nu.$$

Now, in view of (8.159) in the case $l > j/2$ and (8.169) in the case $l = j/2$, we finally obtain for any $l \geq j/2$:

$$(8.170) \quad \left\| \int_{\mathbb{S}^2} L(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim (1+p^2)2^{\frac{5j}{12}}\varepsilon\gamma_j^\nu.$$

Next, we evaluate the second term in the right-hand side of (8.157). In view of the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1), we have:

$$(8.171) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{E}h_u)} \right) 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|H_1\|_{L_u^\infty L^2(\mathcal{E}h_u)} \right) 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the Definition (8.150) of H_1 , we have:

$$(8.172) \quad \begin{aligned} \|H_1\|_{L_u^\infty L^2(\mathcal{E}h_u)} & \lesssim \|N'(P_m \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}h_u)} + \|\nabla'(P_m \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}h_u)} + \|\theta' P_m \text{tr}\chi'\|_{L_u^\infty L^2(\mathcal{E}h_u)} \\ & \lesssim \|P_m(b'N'(\text{tr}\chi'))\|_{L_u^\infty L^2(\mathcal{E}h_u)} + \|[b'N', P_m]\text{tr}\chi'\|_{L_u^\infty L^2(\mathcal{E}h_u)} \\ & \quad + \|\nabla' \text{tr}\chi'\|_{L_u^\infty L^2(\mathcal{E}h_u)} + \|\theta'\|_{L_u^\infty L^2(\mathcal{E}h_u)} \|P_m \text{tr}\chi'\|_{L^\infty} \\ & \lesssim \varepsilon, \end{aligned}$$

where we used the finite band property and the boundedness on $L^p(P_{t,u})$ for P_m , the commutator estimate (2.64) for $[b'N', P_m]$, and the estimates (2.38) for b , (2.37) (2.39) (2.40) for θ , and (2.39) for $\text{tr}\chi$. In view of (8.171), this yields:

$$(8.173) \quad \left\| \int_{\mathbb{S}^2} H_1 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'}.$$

Finally, (8.157), (8.170) and (8.173) imply:

$$(8.174) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim (1+p^2)2^{\frac{11j}{12}}\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of (8.154), we have:

$$(8.175) \quad \begin{aligned} \|h_{2,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} & \lesssim \left\| \int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} L'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Next, we evaluate both terms in the right-hand side of (8.157) starting with the first one. In view of the Definition (8.151) of H_2 , and proceeding as in (8.172), we have:

$$\|H_2\|_{L_u^\infty L^2(\mathcal{E}u)} \lesssim \|N(P_l \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} + \|\text{tr}\theta P_l \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)} \lesssim \varepsilon.$$

Thus, proceeding as in (8.173), we obtain:

$$(8.176) \quad \left\| \int_{\mathbb{S}^2} H_2 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu.$$

Also, the analog of (8.170) yields:

$$\left\| \int_{\mathbb{S}^2} L'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{5j}{12}} \varepsilon \gamma_j^{\nu'}.$$

Together with (8.175) and (8.176), we deduce:

$$(8.177) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{11j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{3,p,q,l,m}$. In view of (8.155), we have:

(8.178)

$$\begin{aligned} & \|h_{3,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} (b'^{-1} \nabla(b') + \theta') L'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & + \left\| \int_{\mathbb{S}^2} \text{tr}\theta L(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Next, we estimate the various terms in the right-hand side of (8.178) starting with the first one. Assume first that $l > j/2$. Then, the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) yields:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_\omega \left\| P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ & \lesssim \left(\sup_\omega \|P_l \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the finite band property for P_l , this yields:

$$(8.179) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}$$

$$\begin{aligned} &\lesssim 2^{-l} \left(\sup_{\omega} \|\nabla \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{E}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu, \end{aligned}$$

where we used the estimate (2.39) for $\operatorname{tr} \chi$ in the last inequality.

Next, we evaluate the first term in the right-hand side of (8.178) in the case $l = j/2$, which is given by:

$$\left\| \int_{\mathbb{S}^2} P_{\leq j/2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

We first decompose $P_{\leq j/2} \operatorname{tr} \chi$ as:

$$P_{\leq j/2} \operatorname{tr} \chi = \operatorname{tr} \chi - \sum_{l > \frac{j}{2}} P_l \operatorname{tr} \chi,$$

which together with (8.179) yields:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} P_{\leq \frac{j}{2}} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \sum_{l > \frac{j}{2}} \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \\ &\lesssim \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu. \end{aligned}$$

Together with the estimate (7.64), this yields:

$$(8.180) \quad \left\| \int_{\mathbb{S}^2} P_{\leq \frac{j}{2}} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

Now, in view of (8.179) in the case $l > j/2$ and (8.180) in the case $l = j/2$, we finally obtain for any $l \geq l/2$:

$$(8.181) \quad \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

Arguing similarly for the third term in the right-hand side of (8.178), we obtain:

$$\left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'},$$

which together with (8.178) and (8.181) yields:

$$(8.182)$$

$$\begin{aligned} &\|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\lesssim (1 + p^2) \varepsilon \gamma_j^\nu \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla(b')) L'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$+ (1+q^2)\varepsilon\gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \operatorname{tr}\theta L(P_l \operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}.$$

We estimate the two terms in the right-hand side of (8.182) starting with the first one. Using the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1), we have:

(8.183)

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (b'^{-1}\nabla(b') + \theta')L'(P_m \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| (\nabla'(L'(P_m \operatorname{tr}\chi')) + (b'^{-1}\nabla(b') + \theta')L'(P_m \operatorname{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega} \left\| (\theta' + b'^{-1}\nabla(b'))L'(P_m \operatorname{tr}\chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, using the estimate (2.36) for n , we have for any tensor G and any integer r :

$$\begin{aligned} & \|GLP_r(\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ & \lesssim \|GnLP_r(\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ & \lesssim \|GP_r(nL\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + \|G[nL, P_r](\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ & \lesssim \|G\|_{L_u^\infty L^6(\mathcal{E}_{\mathcal{H}_u})} \|P_r(nL\operatorname{tr}\chi)\|_{L_u^\infty L^3(\mathcal{E}_{\mathcal{H}_u})} + \|G\|_{L_t^\infty L_x^4} \| [nL, P_r](\operatorname{tr}\chi) \|_{L_t^2 L_x^4}. \end{aligned}$$

Together with the embeddings (2.50) and (2.51), the L^p boundedness of P_r , the Gagliardo-Nirenberg inequality (2.49) and the estimate (2.36) for n , we obtain:

$$\begin{aligned} & \|GLP_r(\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ & \lesssim \mathcal{N}_1(G) (\|L\operatorname{tr}\chi\|_{L_u^\infty L^3(\mathcal{E}_{\mathcal{H}_u})} + \|[nL, P_r](\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})}^{\frac{1}{2}} \|\nabla[nL, P_r](\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})}^{\frac{1}{2}}). \end{aligned}$$

Together with the commutator estimate (2.67), we deduce:

$$(8.184) \quad \|GLP_r(\operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim (\|L\operatorname{tr}\chi\|_{L_u^\infty L^3(\mathcal{E}_{\mathcal{H}_u})} + \varepsilon)\varepsilon\mathcal{N}_1(G) \lesssim \varepsilon\varepsilon\mathcal{N}_1(G),$$

where we used the fact that:

$$(8.185) \quad \|L\operatorname{tr}\chi\|_{L_u^\infty L^3(\mathcal{E}_{\mathcal{H}_u})} \lesssim \varepsilon,$$

in view of the Raychaudhuri Equation (2.22), and the estimates (2.36) (2.37) for $\bar{\delta}$ and (2.39) (2.40) for χ . Choosing $G = \theta' + b'^{-1}\nabla(b')$, we obtain:

$$(8.186) \quad \left\| (\theta' + b'^{-1}\nabla(b'))L'(P_m \operatorname{tr}\chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim \varepsilon\mathcal{N}_1(\theta') \lesssim \varepsilon,$$

where we used in the last inequality the estimates (2.36) (2.37) (2.39) (2.40) for θ' . Together with (8.183), we obtain:

(8.187)

$$\left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1}\nabla(b'))L'(P_m \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}}\gamma_j^{\nu'}.$$

Arguing similarly, we obtain:

$$(8.188) \quad \left\| \int_{\mathbb{S}^2} \operatorname{tr} \theta L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.$$

Finally, (8.178), (8.187), and (8.188) imply:

$$(8.189) \quad \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2 + q^2) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$. In view of (8.155), we have:

$$(8.190) \quad \begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla' L'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Let us first estimate the last term in the right-hand side of (8.190). Using the basic estimate in $L^2(\mathcal{C}\mathcal{M})$ (7.1), we have:

$$(8.191) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \nabla' L'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| (\nabla'(L'(P_m \operatorname{tr} \chi')) + \theta' L'(P_m \operatorname{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ &\lesssim \left(\sup_{\omega} \left\| \nabla' L'(P_m \operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Using the estimate (2.36) for n , we estimate the right-hand side of (8.191):

$$\begin{aligned} \left\| \nabla'(L'(P_m \operatorname{tr} \chi')) \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} &\lesssim \left\| \nabla' P_m(nL' \operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + \left\| \nabla'[nL', P_m](\operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ &\quad + \left\| n^{-1} \nabla' nL' P_m(\operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})}. \end{aligned}$$

Applying (8.184) with the choice $G = n^{-1} \nabla' n$, we obtain:

$$(8.192) \quad \begin{aligned} \left\| \nabla'(L'(P_m \operatorname{tr} \chi')) \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} &\lesssim \left\| \nabla' P_m(nL' \operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + \left\| \nabla'[nL', P_m](\operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ &\quad + \varepsilon_{\mathcal{C}\mathcal{N}_1} (n^{-1} \nabla' n) \\ &\lesssim \left\| \nabla' P_m(nL' \operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} + 2^{\frac{m}{2}} \varepsilon, \end{aligned}$$

where we used in the last inequality the estimate (2.36) for n and the commutator estimate (2.67). Next, using the finite band property for P_m , we have:

$$\begin{aligned} \left\| \nabla'(P_m(nL' \operatorname{tr} \chi')) \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} &\lesssim 2^m \left\| P_m(nL' \operatorname{tr} \chi') \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \\ &\lesssim 2^{\frac{m}{2}} \varepsilon, \end{aligned}$$

where we used (8.141) in the last inequality. Together with (8.192), we obtain:

$$(8.193) \quad \left\| \nabla'(L'(P_m \operatorname{tr} \chi')) \right\|_{L_u^\infty L^2(\mathcal{E}_{\mathcal{H}_u})} \lesssim 2^{\frac{m}{2}} \varepsilon,$$

which in view of (8.191) yields:

$$\left\| \int_{\mathbb{S}^2} \nabla' L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{m}{2} + \frac{j}{2}} \gamma_j^{\nu'}.$$

Together with (8.190), we obtain:

(8.194)

$$\|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{\frac{m}{2} + \frac{j}{2}} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

Assume first that $l > j/2$. Then, (8.179) and (8.194) yield:

$$\|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j + \frac{m}{2} - l} \gamma_j^\nu \gamma_j^{\nu'},$$

which together with the fact that $l > m$ from (8.1) and the assumption $l > j/2$ yields:

$$(8.195) \quad \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{3j}{4}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, assume that $l = j/2$. Then, (8.180) and (8.194) yield:

$$\|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon^2 2^{\frac{m}{2} + \frac{j}{2}} \gamma_j^\nu \gamma_j^{\nu'},$$

which together with the fact that $l \geq m$ from (8.1) and the assumption $l = j/2$ yields:

$$(8.196) \quad \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon^2 2^{\frac{3j}{4}} \gamma_j^\nu \gamma_j^{\nu'}.$$

In view of (8.195) and (8.196), we finally obtain in all cases:

$$(8.197) \quad \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon^2 2^{\frac{3j}{4}} \gamma_j^\nu \gamma_j^{\nu'}.$$

We are now ready to estimate $B_{j,\nu,\nu',l,m}^{1,2,1,2}$. In view of (8.152), we have:

$$|B_{j,\nu,\nu',l,m}^{1,2,1,2}| \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+2}} \right\|_{L^\infty(\mathcal{M})} \\ \times \left[\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \right],$$

which together with (8.32), (8.174), (8.177), (8.189) and (8.197) yields:

$$(8.198) \quad |B_{j,\nu,\nu',l,m}^{1,2,1,2}| \lesssim \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+2}} (1 + p^2 + q^2) 2^{-\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ \lesssim \frac{2^{-\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}.$$

Note that summing the estimate (8.198) in m is not a problem. Indeed, we have from (8.1):

$$2^m \leq 2^j |\nu - \nu'|.$$

Now, we have:

$$(8.199) \quad \#\{m / 2^m \leq 2^j |\nu - \nu'|\} \lesssim j,$$

so that the sum in m generates a $O(j)$ term which is absorbed by the extra gain $2^{-\frac{j}{12}}$ in (8.198). On the other hand, there is no a priori bound on l so that summing the estimate (8.198) in l is problematic. To fix this issue, it suffices, since $l \geq m$ in view of (8.1), to obtain an upper bound for

$$\left| \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right|.$$

To this end, it suffices to replace P_l with $P_{\geq m}$ in the definition and the estimate of $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$. The estimates are completely analogous, and we obtain as in (8.198):

$$(8.200) \quad \left| \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \frac{2^{-\frac{j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}.$$

Now, we have:

$$\left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \sum_{2^m \leq 2^j |\nu - \nu'|} \left| \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right|,$$

which together with (8.199) and (8.200) implies:

$$(8.201) \quad \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \frac{j 2^{-\frac{j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,1,3}$ defined in (8.109). Recall from (8.1) that (l, m) satisfy:

$$m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.$$

Summing in (l, m) , we obtain:

$$\begin{aligned} & \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \\ & + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \\ & = L(\text{tr} \chi) \text{tr} \chi' + \text{tr} \chi L'(\text{tr} \chi') \\ & - L(P_{> 2^j |\nu - \nu'|} \text{tr} \chi) P_{> 2^j |\nu - \nu'|} \text{tr} \chi' + P_{> 2^j |\nu - \nu'|} \text{tr} \chi L'(P_{> 2^j |\nu - \nu'|} \text{tr} \chi'). \end{aligned}$$

Thus, using the symmetry in (ω, ω') of the integrant in $B_{j,\nu,\nu',l,m}^{1,2,1,3}$ defined in (8.109), we obtain:

$$(8.202) \quad \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,3} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,3}$$

$$\begin{aligned}
&= 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b)}{\mathbf{g}(L, L')^2} (L(\mathrm{tr}\chi) \mathrm{tr}\chi' + \mathrm{tr}\chi L'(\mathrm{tr}\chi')) \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \\
&- 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b)}{\mathbf{g}(L, L')^2} \\
&\quad \times (L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi) P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi' + P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi L'(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi')) \\
&\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} + \text{terms interverting } (\nu, \nu').
\end{aligned}$$

We estimate the two terms in the right-hand side of (8.202) starting with the last one. We have:

$$\begin{aligned}
&\left| 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b)}{\mathbf{g}(L, L')^2} (L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi) P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi' \right. \\
&\quad \left. + P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi L'(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi')) F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \right| \\
&\lesssim 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left\| \frac{N' - \mathbf{g}(N, N')N}{\mathbf{g}(L, L')^2} \right\|_{L^\infty(\mathcal{M})} \left(\|b^{-1} \nabla(b) L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi) F_{j,-1}(u)\|_{L^2(\mathcal{M})} \right. \\
&\quad \times \|P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi' F_{j,-1}(u')\|_{L^2(\mathcal{M})} + \|b^{-1} \nabla(b) P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi F_{j,-1}(u)\|_{L^2(\mathcal{M})} \\
&\quad \left. \times \|L'(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi') F_{j,-1}(u')\|_{L^2(\mathcal{M})} \right) \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega'.
\end{aligned}$$

In view of the identities (8.30)–(8.31) for $\mathbf{g}(L, L')$ and $\mathbf{g}(N, N')$, and in view of the estimate (8.32), we obtain:

$$\begin{aligned}
(8.203) \quad &\left| 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b)}{\mathbf{g}(L, L')^2} (L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi) P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi' \right. \\
&\quad \left. + P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi L'(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi')) F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}} \right| \\
&\lesssim \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)^3} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left(\|b^{-1} \nabla(b) L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \|F_{j,-1}(u)\|_{L_u^2} \right. \\
&\quad \times \|P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi'\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \|F_{j,-1}(u')\|_{L_{u'}^2} + \|b^{-1} \nabla(b) P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \\
&\quad \left. \times \|F_{j,-1}(u)\|_{L_u^2} \|L'(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \|F_{j,-1}(u')\|_{L_{u'}^2} \right) \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega'.
\end{aligned}$$

Next, we evaluate the various terms in the right-hand side of (8.203). Choosing $G = b^{-1} \nabla(b)$ in (8.184), we have:

$$(8.204) \quad \|b^{-1} \nabla(b) L(P_{>2^j|\nu-\nu'|} \mathrm{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \lesssim \varepsilon_{\mathcal{M}} \mathcal{N}_1(b^{-1} \nabla(b)) \lesssim \varepsilon,$$

where we used the estimate (2.38) for b in the last inequality. Also, (8.142) together with the estimate (2.36) for n yields:

$$(8.205) \quad \left\| L'(P_{>2^j|\nu-\nu'|} \text{tr}\chi') \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \sum_{2^m > 2^j|\nu-\nu'|} 2^{-\frac{m}{2}} \varepsilon \lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}}.$$

Using the finite band property for P_m , we have:

$$(8.206) \quad \begin{aligned} \left\| P_{>2^j|\nu-\nu'|} \text{tr}\chi' \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} &\lesssim \sum_{2^m > 2^j|\nu-\nu'|} \left\| P_m \text{tr}\chi' \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \\ &\lesssim \sum_{2^m > 2^j|\nu-\nu'|} 2^{-m} \left\| \nabla' \text{tr}\chi' \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \\ &\lesssim \frac{\varepsilon}{2^j|\nu-\nu'|}, \end{aligned}$$

where we used the estimate (2.39) for $\text{tr}\chi$ in the last inequality. Also, we have:

$$\begin{aligned} \left\| b^{-1} \nabla(b) P_{>2^j|\nu-\nu'|} \text{tr}\chi \right\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim \left\| b^{-1} \nabla(b) \right\|_{L_t^\infty L_x^4} \left\| P_{>2^j|\nu-\nu'|} \text{tr}\chi \right\|_{L_t^2 L_x^4} \\ &\lesssim \mathcal{N}_1(b^{-1} \nabla(b)) \left(\sum_{2^m > 2^j|\nu-\nu'|} \left\| P_m \text{tr}\chi \right\|_{L_t^2 L_x^4} \right) \\ &\lesssim \varepsilon \left(\sum_{2^m > 2^j|\nu-\nu'|} 2^{\frac{m}{2}} \left\| P_m \text{tr}\chi \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right), \end{aligned}$$

where we used the embedding (2.51), the estimate (2.38) for b and the Bernstein inequality for P_m . Together with the finite band property for P_m , this yields:

$$(8.207) \quad \left\| b^{-1} \nabla(b) P_{>2^j|\nu-\nu'|} \text{tr}\chi \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon \left(\sum_{2^m > 2^j|\nu-\nu'|} 2^{-\frac{m}{2}} \left\| \nabla \text{tr}\chi \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}},$$

where we used the estimate (2.39) for $\text{tr}\chi$ in the last inequality.

Finally, (8.203) (8.204) (8.205) (8.206) (8.207) yield:

$$(8.208) \quad \begin{aligned} &\left| 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N'-\mathbf{g}(N,N')N}(b)}{\mathbf{g}(L, L')^2} (L(P_{>2^j|\nu-\nu'|} \text{tr}\chi) P_{>2^j|\nu-\nu'|} \text{tr}\chi' \right. \\ &\quad \left. + P_{>2^j|\nu-\nu'|} \text{tr}\chi L'(P_{>2^j|\nu-\nu'|} \text{tr}\chi')) F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M} \right| \\ &\lesssim \frac{\varepsilon^2}{2^j(2^{\frac{j}{2}}|\nu-\nu'|)^4} \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu-\nu'|)^4}, \end{aligned}$$

where we used in the last inequality Plancherel in λ for $\|F_{j,-1}(u)\|_{L_u^2}$, Plancherel in λ' for $\|F_{j,-1}(u')\|_{L_{u'}^2}$, Cauchy Schwarz in ω and ω' , and the size of the patches.

Next, we estimate the first term in the right-hand side of (8.202), which is given by:

$$(8.209) \quad 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N'-\mathbf{g}(N,N')N}(b)}{\mathbf{g}(L, L')^2} (L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) \\ \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.209), we obtain, schematically:

$$(8.210) \quad 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N'-\mathbf{g}(N,N')N}(b)}{\mathbf{g}(L, L')^2} (L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) \\ \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}} \\ = 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q+3}} [h_{1,p,q} + h_{2,p,q}] d_{\mathcal{M}},$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}$ on \mathcal{M} are given by:

$$(8.211) \quad h_{1,p,q} = \left(\int_{\mathbb{S}^2} b^{-1} \nabla(b)L(\text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right)$$

and:

$$(8.212) \quad h_{2,p,q} = \left(\int_{\mathbb{S}^2} b^{-1} \nabla(b)\text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} L'(\text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right)$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}, h_{2,p,q}$ starting with $h_{1,p,q}$. We have:

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2)\varepsilon\gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} b^{-1} \nabla(b)L(\text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})},$$

where we used the estimate (7.64) in the last inequality. Together with the basic estimate in $L^2(\mathcal{C}\mathcal{M})$ (7.1), we obtain:

$$(8.213) \quad \|h_{1,p,q}\|_{L^1(\mathcal{C}\mathcal{M})} \lesssim (1+q^2)\varepsilon\gamma_j^{\nu'} \left(\sup_{\omega} \|b^{-1}\nabla(b)L(\mathrm{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'}.$$

Next, we estimate $b^{-1}\nabla(b)L(\mathrm{tr}\chi)$. We have:

$$\begin{aligned} \|b^{-1}\nabla(b)L(\mathrm{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} &\lesssim \|b^{-1}\nabla(b)\|_{L_u^\infty L^6(\mathcal{E}u)} \|L(\mathrm{tr}\chi)\|_{L_u^\infty L^3(\mathcal{E}u)} \\ &\lesssim \mathcal{N}_1(b^{-1}\nabla(b)) \|L(\mathrm{tr}\chi)\|_{L_u^\infty L^3(\mathcal{E}u)} \\ &\lesssim \varepsilon, \end{aligned}$$

where we used the Sobolev embedding (2.50), the estimate (2.38) for b and the estimate (8.185) for $L(\mathrm{tr}\chi)$. Together with (8.213), we deduce:

$$(8.214) \quad \|h_{1,p,q}\|_{L^1(\mathcal{C}\mathcal{M})} \lesssim (1+q^2)\varepsilon^2 2^{\frac{j}{2}}\gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{C}\mathcal{M})$ norm of $h_{2,p,q}$. Recall the decomposition (8.67):

$$\begin{aligned} &\int_{\mathbb{S}^2} L'(\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \\ &= -\chi_2 \cdot \left(\int_{\mathbb{S}^2} (2\chi_{1'} + \widehat{\chi}') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad - (\chi_{2\nu'} - \chi_2) \cdot \left(\int_{\mathbb{S}^2} (2\chi_{1'} + \widehat{\chi}') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + f_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega', \end{aligned}$$

where the scalar f_1^j only depends on ν' and satisfies:

$$(8.215) \quad \|f_1^j\|_{L_{u,\nu'}^\infty L_t^2 L^\infty(P_t, u_{\nu'})} \lesssim \varepsilon,$$

where the scalar f_2^j satisfies:

$$(8.216) \quad \|f_2^j\|_{L_{u'}^\infty L^2(\mathcal{E}u')} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

Together with the Definition (8.212) of $h_{2,p,q}$, this yields:

$$\begin{aligned} \|h_{2,p,q}\|_{L^1(\mathcal{C}\mathcal{M})} &\lesssim \left(\left\| \int_{\mathbb{S}^2} \chi_2 b^{-1}\nabla(b)\mathrm{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{C}\mathcal{M})} \right. \\ &\quad \left. + \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1}\nabla(b)\mathrm{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{C}\mathcal{M})} \right) \\ &\quad \times \left\| \int_{\mathbb{S}^2} (2\chi_{1'} + \widehat{\chi}') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{C}\mathcal{M})} \end{aligned}$$

$$\begin{aligned}
& + \left\| \int_{\mathbb{S}^2} b^{-1} \nabla(b) \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \quad \times \left(\left\| f_1^j \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \right. \\
& \quad \left. + \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \right).
\end{aligned}$$

Together with the estimates (7.63) and (8.72), and the estimate (8.215) for f_1^j , we obtain:

(8.217)

$$\begin{aligned}
& \|h_{2,p,q}\|_{L^1(\mathcal{M})} \\
& \lesssim \left(\left\| \int_{\mathbb{S}^2} \chi_2 b^{-1} \nabla(b) \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \right. \\
& \quad + \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \varepsilon \gamma_j^{\nu'} \\
& \quad + \left\| \int_{\mathbb{S}^2} b^{-1} \nabla(b) \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \quad \left. \times \left((1 + q^2) \varepsilon \gamma_j^{\nu'} + \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \right).
\end{aligned}$$

Next, we we apply the basic estimate in $L^2(\mathcal{M})$ to the first, the third and the last term in the right-hand side of (8.217). We obtain:

(8.218)

$$\begin{aligned}
& \|h_{2,p,q}\|_{L^1(\mathcal{M})} \\
& \lesssim \left(\left(\sup_{\omega} \|\chi_2 b^{-1} \nabla(b) \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \right. \\
& \quad + \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \gamma_j^{\nu'} \\
& \quad + \left(\sup_{\omega} \|b^{-1} \nabla(b) \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \\
& \quad \left. \times \left((1 + q^2) \varepsilon \gamma_j^{\nu'} + \left(\sup_{\omega} \|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \right).
\end{aligned}$$

Now, the Sobolev embedding (2.50) :

$$\begin{aligned}
& \|\chi_2 b^{-1} \nabla(b) \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|b^{-1} \nabla(b) \operatorname{tr} \chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \\
& \lesssim (\|\chi_2\|_{L_u^\infty L^4(\mathcal{H}_u)} \|b^{-1} \nabla(b)\|_{L_u^\infty L^4(\mathcal{H}_u)} + \|b^{-1} \nabla(b)\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\operatorname{tr} \chi\|_{L^\infty(\mathcal{M})}) \\
& \lesssim (\mathcal{M}_1(\chi_2) + 1) \mathcal{M}_1(b^{-1} \nabla(b)) \|\operatorname{tr} \chi\|_{L^\infty(\mathcal{M})} \\
& \lesssim \varepsilon,
\end{aligned}$$

where we used in the last inequality the estimate (2.38) for b , the estimate (2.39) for $\text{tr}\chi$ and the estimate (2.46) for χ_2 . Together with (8.218) and the estimate (8.216) for f_2^j , this yields:

(8.219)

$$\begin{aligned} & \|h_{2,p,q}\|_{L^1(\mathcal{M})} \\ & \lesssim \left(2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'} + \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \text{tr}\chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \right) \varepsilon \gamma_j^{\nu'} \\ & \quad + 2^{\frac{j}{2}} (1 + q^2) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Next, we estimate the right-hand side of (8.219). We have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \text{tr}\chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \left\| (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \text{tr}\chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|\chi_{2\nu'} - \chi_2\|_{L^3(\mathcal{M})} \|b^{-1} \nabla(b)\|_{L_u^\infty L^6(\mathcal{H}_u)} \|\text{tr}\chi\|_{L^\infty(\mathcal{M})} \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty(\mathcal{M})} \\ & \quad \|F_{j,-1}(u)\|_{L_u^6} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

which together with the Sobolev embedding (2.50) yields:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \text{tr}\chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} |\omega - \nu'| \|\partial_\omega \chi_2\|_{L^3(\mathcal{M})} \mathcal{N}_1(b^{-1} \nabla(b)) \|\text{tr}\chi\|_{L^\infty(\mathcal{M})} \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty(\mathcal{M})} \\ & \quad \|F_{j,-1}(u)\|_{L_u^\infty}^{\frac{2}{3}} \|F_{j,-1}(u)\|_{L_u^2}^{\frac{1}{3}} \eta_j^\nu(\omega) d\omega \\ & \lesssim |\nu - \nu'| \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^\infty}^{\frac{2}{3}} \|F_{j,-1}(u)\|_{L_u^2}^{\frac{1}{3}} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimate (2.47) for χ_2 , the estimate (2.38) for b , the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Using Cauchy Schwartz in λ for $\|F_{j,-1}(u)\|_{L_u^\infty}^{\frac{1}{2}}$, Plancherel in u for $\|F_{j,-1}(u)\|_{L_u^2}^{\frac{1}{2}}$, Cauchy Schwarz in ω and the volume of the patch, we finally obtain:

$$\left\| \int_{\mathbb{S}^2} (\chi_{2\nu'} - \chi_2) b^{-1} \nabla(b) \text{tr}\chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim |\nu - \nu'| \varepsilon 2^{\frac{5j}{6}} \gamma_j^{\nu'}.$$

In view of (8.219), we deduce:

$$(8.220) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) (1 + |\nu - \nu'| 2^{\frac{j}{3}}) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

In view of (8.210), we have:

$$\left| 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N') N}(b)}{\mathbf{g}(L, L')^2} (L(\text{tr}\chi) \text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) \right|$$

$$\begin{aligned} & \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'd\mathcal{M} \Big| \\ & \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q+3}} \right\|_{L^\infty(\mathcal{M})} \left[\|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h_{2,p,q}\|_{L^1(\mathcal{M})} \right], \end{aligned}$$

which together with (8.32), (8.214) and (8.215) yields:

$$\begin{aligned} (8.221) \quad & \left| 2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1} \nabla_{N' - \mathbf{g}(N, N')N}(b)}{\mathbf{g}(L, L')^2} (L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) \right. \\ & \left. \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'd\mathcal{M} \right| \\ & \lesssim \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+3}} (1+q^2)(1+|\nu - \nu'|2^{\frac{j}{3}})\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^{\frac{j}{6}}(2^{\frac{j}{2}}|\nu - \nu'|)^2}. \end{aligned}$$

Finally, (8.202), (8.208) and (8.221) imply:

$$\begin{aligned} (8.222) \quad & \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,3} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,3} \right| \\ & \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^{\frac{j}{6}}(2^{\frac{j}{2}}|\nu - \nu'|)^2}. \end{aligned}$$

Now, recall (8.105):

$$B_{j,\nu,\nu',l,m}^{1,2,1} = B_{j,\nu,\nu',l,m}^{1,2,1,1} + B_{j,\nu,\nu',l,m}^{1,2,1,2} + B_{j,\nu,\nu',l,m}^{1,2,1,3}.$$

Together with the estimates (8.148), (8.201) and (8.222), we finally obtain:

$$\begin{aligned} & \left| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1} \right| \\ & \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^j |\nu - \nu'|} + \frac{j 2^{-\frac{j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3}. \end{aligned}$$

This concludes the proof of Proposition 8.11.

8.2.2. Proof of Proposition 8.11 (Control of $B_{j,\nu,\nu',l,m}^{1,2,2}$). — Recall that $B_{j,\nu,\nu',l,m}^{1,2,2}$ is defined by (8.95):

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,2} & = -i 2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ & \quad \times (b^{-1} - b'^{-1}) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \end{aligned}$$

Recall (8.1):

$$m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.$$

We first consider the range of (l, m) such that:

$$2^m \leq 2^j |\nu - \nu'| < 2^l.$$

This yields:

(8.223)

$$\begin{aligned} & \sum_{m/2^m \leq 2^j |\nu - \nu'| < 2^l} B_{j, \nu, \nu', l, m}^{1, 2, 2} \\ &= -i 2^{-j-1} \sum_{2^l > 2^j |\nu - \nu'|} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' + P_l \text{tr} \chi L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \right) \\ & \quad \times (b^{-1} - b'^{-1}) F_{j, -1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Next, recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \quad \text{and} \quad 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')}$$

in the same fashion than (8.33), and in view of (8.223), we obtain, schematically:

(8.224)

$$\begin{aligned} & \sum_{m/2^m \leq 2^j |\nu - \nu'| < 2^l} B_{j, \nu, \nu', l, m}^{1, 2, 2} \\ &= \sum_{p, q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+2}} [h_{1,p,q,l} + h_{2,p,q,l} + h_{3,p,q,l} + h_{4,p,q,l}] d_{\mathcal{M}}, \end{aligned}$$

where the scalar functions $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}, h_{4,p,q,l}$ on \mathcal{M} are given by:

(8.225)

$$\begin{aligned} h_{1,p,q,l} &= \left(\int_{\mathbb{S}^2} (b^{-1} - b_{\nu'}^{-1}) L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j, -1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j, -1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.226)

$$\begin{aligned} h_{2,p,q,l} &= \left(\int_{\mathbb{S}^2} (b^{-1} - b_{\nu'}^{-1}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j, -1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j, -1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.227)

$$h_{3,p,q,l} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j, -1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left(\int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

(8.228)

$$\begin{aligned} h_{4,p,q,l} = & - \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}, h_{4,p,q,l}$ starting with $h_{1,p,q,l}$. We first estimate $L(P_l \text{tr} \chi)$. We have:

$$nL(P_l \text{tr} \chi) = P_l(nL \text{tr} \chi) + [nL, P_l] \text{tr} \chi,$$

which together with the estimate (2.36) for n yields:

$$\|L(P_l \text{tr} \chi)\|_{L_x^2, L_t^1} \lesssim \|[nL, P_l] \text{tr} \chi\|_{L_t^1 L_x^2} + \|P_l(nL \text{tr} \chi)\|_{L_x^2, L_t^1}.$$

Together with the commutator estimate (2.66) for $[nL, P_l] \text{tr} \chi$ and the estimate (2.69) for $P_l(nL \text{tr} \chi)$, we obtain:

$$(8.229) \quad \|L(P_l \text{tr} \chi)\|_{L_x^2, L_t^1} \lesssim 2^{-l} \varepsilon.$$

Now, in view of the definition of $h_{1,p,q,l}$ (8.225), we have:

$$\begin{aligned} & \|h_{1,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \left\| (b^{-1} - b_{\nu'}^{-1}) L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \right. \\ & \quad \left. \times \left(\int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \right\|_{L^1(\mathcal{O}\mathcal{M})} \eta_j^{\nu}(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|b^{-1} - b_{\nu'}^{-1}\|_{L^\infty(\mathcal{O}\mathcal{M})} \|L(P_l \text{tr} \chi)\|_{L_x^2, L_t^1} \left\| \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L^\infty(\mathcal{O}\mathcal{M})} \|F_{j,-1}(u)\|_{L_u^2} \\ & \quad \times \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,x'}^2, L_t^\infty} \eta_j^{\nu}(\omega) d\omega. \end{aligned}$$

Together with the estimate (2.44) for $\partial_\omega b$, the estimate (8.229) for $L(P_l \text{tr} \chi)$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch, we obtain:

(8.230)

$$\|h_{1,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim 2^{-l} |\nu - \nu'| \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2}$$

$$\times \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,x'}^2 L_t^\infty} \eta_j^{\nu'}(\omega) d\omega.$$

Next, we estimate the last term in the right-hand side of (8.230). We have:

$$P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' = \operatorname{tr} \chi' - \sum_{2^m > 2^j |\nu - \nu'|} P_m \operatorname{tr} \chi',$$

which yields the decomposition:

$$\begin{aligned} & \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \\ &= \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \\ & \quad - \sum_{2^m > 2^j |\nu - \nu'|} \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'. \end{aligned}$$

Together with (7.71) and (7.83), we obtain:

$$\begin{aligned} (8.231) \quad & \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,x'}^2 L_t^\infty} \\ & \lesssim (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'|) \gamma_j^{\nu'} + \left(\sup_{\omega'} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty(\partial \mathcal{M})} \right) \\ & \quad \times \sum_{2^m > 2^j |\nu - \nu'|} \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-m + \frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2} + \frac{j}{4}}) \gamma_j^{\nu'} \\ & \lesssim (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'|) \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.230) and (8.231) imply:

(8.232)

$$\begin{aligned} \|h_{1,p,q,l}\|_{L^1(\partial \mathcal{M})} & \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 2^{-l} |\nu - \nu'| (2^{\frac{j}{2}} |\nu - \nu'|) \gamma_j^{\nu'} \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega \\ & \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 2^{-l} (2^{\frac{j}{2}} |\nu - \nu'|)^2 \gamma_j^{\nu'} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality Plancherel in λ for $\|F_{j,-1}(u)\|_{L_u^2}$, Cauchy Swartz in ω and the size of the patch.

Next, we evaluate the $L^1(\partial \mathcal{M})$ norm of $h_{2,p,q,l}$. In view of the Definition (8.226) of $h_{2,p,q,l}$, we have:

(8.233)

$$\begin{aligned} \|h_{2,p,q,l}\|_{L^1(\partial \mathcal{M})} & \lesssim \left\| \int_{\mathbb{S}^2} (b^{-1} - b_{\nu'}^{-1}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L_{u,\nu',x_{\nu'}}^2 L_t^\infty} \\ & \quad \times \left\| \int_{\mathbb{S}^2} L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x_{\nu'}}^2 L_t^1}. \end{aligned}$$

Next, we estimate the two terms in the right-hand side of (8.233) starting with the first one. Using the estimate (7.71) with $G = b^{-1} - b_{\nu'}^{-1}$, we have:

$$(8.234) \quad \left\| \int_{\mathbb{S}^2} (b^{-1} - b_{\nu'}^{-1}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L_{u,\nu',x'}^2, L_t^\infty} \\ \lesssim \left(\sup_{\omega} \|b^{-1} - b_{\nu'}^{-1}\|_{L^\infty} \right) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2}+\frac{j}{4}}) \gamma_j^{\nu} \\ \lesssim |\nu - \nu'| \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2}+\frac{j}{4}}) \gamma_j^{\nu},$$

where we used in the last inequality the estimate (2.38) for b and the estimate (2.44) for $\partial_\omega b$. Next, we estimate the second term in the right-hand side of (8.233). We have:

$$(8.235) \quad \left\| \int_{\mathbb{S}^2} L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x'}^2, L_t^1} \\ \lesssim \left\| \int_{\mathbb{S}^2} L'(\operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x'}^2, L_t^1} \\ + \left\| \int_{\mathbb{S}^2} L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

In view of (7.17), we have:

$$(8.236) \quad \left\| \int_{\mathbb{S}^2} L'(\operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x'}^2, L_t^1} \lesssim (1+q^2) \varepsilon \gamma_j^{\nu'}.$$

Also, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.237) \quad \left\| \int_{\mathbb{S}^2} L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega'} \left\| L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

Now, we have:

$$\left\| L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \\ \lesssim \|L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi')\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty(\mathcal{M})} \\ \lesssim \frac{\varepsilon}{(2^j |\nu - \nu'|)^{\frac{1}{2}}},$$

where we used in the last inequality the estimate (8.205), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.237), we obtain:

$$(8.238) \quad \left\| \int_{\mathbb{S}^2} L'(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \frac{\varepsilon 2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} \gamma_j^{\nu'}.$$

(8.235), (8.236) and (8.238) imply:

$$(8.239) \quad \left\| \int_{\mathbb{S}^2} L'(P_{\leq 2^j|\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u', \nu', \nu'} L^1_t}$$

$$\lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'} + \frac{\varepsilon 2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} \gamma_j^{\nu'}.$$

Finally, (8.233), (8.234) and (8.239) yield:

$$(8.240)$$

$$\|h_{2,p,q,l}\|_{L^1(\mathcal{M})}$$

$$\lesssim \left((2^{\frac{j}{2}}|\nu - \nu'|)^2 2^{-l} + (2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{3}{2}} 2^{-\frac{l}{2} - \frac{j}{4}} \right) \left((1 + q^2) + \frac{2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} \right) \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l}$. We decompose

$$\int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) P_{\leq 2^j|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'$$

$$= H + \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) P_{> 2^j|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

where H is given by

$$H = \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

Together with the Definition (8.227) of $h_{3,p,q,l}$, we obtain:

$$(8.241)$$

$$\|h_{3,p,q,l}\|_{L^1(\mathcal{M})}$$

$$\lesssim \left\| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^1(\mathcal{M})}$$

$$+ \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\times \left\| \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) P_{> 2^j|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

Next, we evaluate the three terms in the right-hand side of (8.241) starting with the first one. In view of the estimate (7.10), we have

$$(8.242) \quad \left\| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^1(\mathcal{M})}$$

$$\begin{aligned}
&\lesssim \sup_{\omega \in \text{supp}(\eta_j^\nu)} \left(\left\| H \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_{u,x'}^2 L_t^\infty} \right) \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \\
&\lesssim \sup_{\omega \in \text{supp}(\eta_j^\nu)} \left(\|H\|_{L_{u,x'}^2 L_t^\infty} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty} \right) \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \\
&\lesssim \sup_{\omega \in \text{supp}(\eta_j^\nu)} \left(\|H\|_{L_{u,x'}^2 L_t^\infty} \right) \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the estimate (7.95) and the definition of H , we have:

$$\|H\|_{L_{u,x'}^2 L_t^\infty} \lesssim 2^{-\frac{j}{4}} (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| + 1) \gamma_j^{\nu'},$$

which together with (8.242) implies

$$\begin{aligned}
(8.243) \quad &\left\| \int_{\mathbb{S}^2} HL(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^1(\mathcal{M})} \\
&\lesssim (1 + q^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| + 1) \varepsilon^2 2^{\frac{j}{4}-l} \gamma_j^\nu \gamma_j^{\nu'}.
\end{aligned}$$

Next, we evaluate the second term in the right-hand side of (8.241). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\begin{aligned}
(8.244) \quad &\left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\lesssim \left(\sup_{\omega} \left\| L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
&\lesssim \left(\sup_{\omega} \|L(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Also, (8.142) together with the estimate (2.36) for n yields:

$$(8.245) \quad \|L(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{l}{2}} \varepsilon,$$

which together with (8.244) yields:

$$(8.246) \quad \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}-\frac{l}{2}} \varepsilon \gamma_j^\nu.$$

Next, we evaluate the third term in the right-hand side of (8.241). Using the basic estimate in $L^2(\mathcal{M})$ (7.3), we have:

$$\begin{aligned}
(8.247) \quad &\left\| \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1}) P_{>2^j|\nu-\nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\lesssim \left(\sup_{\omega'} \left\| (b_{\nu'}^{-1} - b'^{-1}) \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \left(\sum_{2^m > 2^j |\nu - \nu'|} 2^{-m} \right) \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}
\end{aligned}$$

$$\lesssim \frac{\varepsilon 2^{-\frac{j}{2}} \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|},$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.241), (8.243), (8.246) and (8.247) imply:

$$(8.248) \quad \|h_{3,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| + 1) \varepsilon^2 2^{\frac{j}{4}-l} \gamma_j^\nu \gamma_j^{\nu'} + \frac{\varepsilon^2 2^{-\frac{j}{2}} \gamma_j^\nu \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{4,p,q,l}$. In view of the Definition (8.228) of $h_{4,p,q,l}$, we have:

$$(8.249)$$

$$\begin{aligned} & \|h_{4,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b^{-1}) L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}. \end{aligned}$$

Next, we estimate the two terms in the right-hand side of (8.249) starting with the first one. Using the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.3), we have:

$$(8.250) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_\omega \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty} \right) \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \\ & \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next, we estimate the second term in the right-hand side of (8.249). Using the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1), we have:

$$(8.251)$$

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b^{-1}) L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| (b_{\nu'}^{-1} - b^{-1}) L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}. \end{aligned}$$

Now, we have:

$$(8.252) \quad \begin{aligned} & \left\| (b_{\nu'}^{-1} - b^{-1}) L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \\ & \lesssim \left\| b_{\nu'}^{-1} - b^{-1} \right\|_{L^\infty} \|L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi')\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \\ & \lesssim 2^{-\frac{j}{2}} \varepsilon \|L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi')\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})}, \end{aligned}$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Furthermore, we have:

$$\begin{aligned} \|L'(P_{\leq 2^j|\nu-\nu'|}\mathrm{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}_{u'})} &\lesssim \|L'(\mathrm{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}_{u'})} + \|L'(P_{> 2^j|\nu-\nu'|}\mathrm{tr}\chi')\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \\ &\lesssim \varepsilon + \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}}, \end{aligned}$$

where we used in the last inequality the estimate (2.39) for $\mathrm{tr}\chi'$ and the estimate (8.205) for $L'(P_{> 2^j|\nu-\nu'|}\mathrm{tr}\chi')$. Together with (8.252) and the fact that $2^{\frac{j}{2}}|\nu-\nu'| \gtrsim 1$, we obtain:

$$\left\| (b_{\nu'}^{-1} - b'^{-1})L'(P_{\leq 2^j|\nu-\nu'|}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \lesssim 2^{-\frac{j}{2}}\varepsilon.$$

Together with (8.251), this yields:

$$(8.253) \quad \left\| \int_{\mathbb{S}^2} (b_{\nu'}^{-1} - b'^{-1})L'(P_{\leq 2^j|\nu-\nu'|}\mathrm{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma_j^{\nu'}.$$

Finally, (8.249), (8.250) and (8.253) yield:

$$(8.254) \quad \|h_{4,p,q,l}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{j}{2}-l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Finally, in view of (8.224), we have:

$$\begin{aligned} \left| \sum_{m/2^m \leq 2^j|\nu-\nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,2} \right| &\lesssim \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q+2}} \right\|_{L^\infty(\mathcal{M})} \left[\|h_{1,p,q,l}\|_{L^1(\mathcal{M})} \right. \\ &\quad \left. + \|h_{2,p,q,l}\|_{L^1(\mathcal{M})} + \|h_{3,p,q,l}\|_{L^1(\mathcal{M})} + \|h_{4,p,q,l}\|_{L^1(\mathcal{M})} \right], \end{aligned}$$

which together with (8.32), (8.232), (8.240), (8.248) and (8.254) yields:

$$\begin{aligned} &\left| \sum_{m/2^m \leq 2^j|\nu-\nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,2} \right| \\ &\lesssim \sum_{p,q \geq 0} c_{pq} \frac{(1+q^{\frac{5}{2}})\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu-\nu'|)^{p+q+2}} \left[2^{-l}(2^{\frac{j}{2}}|\nu-\nu'|)^2 \right. \\ &\quad \left. + \left((2^{\frac{j}{2}}|\nu-\nu'|)^2 2^{-l} + (2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{3}{2}} 2^{-\frac{l}{2}-\frac{j}{4}} \right) \left(1 + \frac{2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}}} \right) \right. \\ &\quad \left. + (1+q^{\frac{5}{2}})(2^{\frac{j}{2}}|\nu-\nu'|+1)2^{\frac{j}{4}-l} + \frac{2^{-\frac{l}{2}}}{2^{\frac{j}{2}}|\nu-\nu'|} + 2^{\frac{j}{2}-l} \right] \\ &\lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} \left[2^{-l}(2^{\frac{j}{2}}|\nu-\nu'|)^2 \right. \\ &\quad \left. + \left((2^{\frac{j}{2}}|\nu-\nu'|)^2 2^{-l} + (2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{3}{2}} 2^{-\frac{l}{2}-\frac{j}{4}} \right) \left(1 + \frac{2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}}} \right) \right] \end{aligned}$$

$$+ \left(2^{\frac{j}{2}} |\nu - \nu'| + 1 \right) 2^{\frac{j}{4} - l} + \frac{2^{-\frac{l}{2}}}{2^{\frac{j}{2}} |\nu - \nu'|} + 2^{\frac{j}{2} - l} \Big].$$

Summing in l , we obtain:

$$(8.255) \quad \left| \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,2} \right| \lesssim \left[\frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

In view of (8.1), we still need to estimate $B_{j,\nu,\nu',l,m}^{1,2,2}$ in the range of (l, m) such that:

$$2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

Recall the Definition (8.95) of $B_{j,\nu,\nu',l,m}^{1,2,2}$:

$$B_{j,\nu,\nu',l,m}^{1,2,2} = -i 2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \times (b^{-1} - b'^{-1}) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

We integrate by parts tangentially using (7.136).

Lemma 8.13. — *Let $B_{j,\nu,\nu',l,m}^{1,2,2}$ defined by (8.95). Integrating by parts using (7.136) yields:*

$$(8.256) \quad B_{j,\nu,\nu',l,m}^{1,2,2} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \left[\frac{1}{|N_\nu - N_{\nu'}|^2} h_{1,p,q,l,m} + \frac{1}{|N_\nu - N_{\nu'}|^3} h_{2,p,q,l,m} \right] d\mathcal{M} + B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,3},$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q,l,m}, h_{2,p,q,l,m}$ on \mathcal{M} are given by:

$$(8.257) \quad h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} N'(P_m \text{tr} \chi')(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) + \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi)(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left(\int_{\mathbb{S}^2} N'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

(8.258)

$$\begin{aligned} h_{2,p,q,l,m} = & \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi'(b' - b_{\nu'}) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \\ & + \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} L'(P_m \text{tr} \chi')(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \\ & + \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) L(P_l \text{tr} \chi)(b_{\nu'} - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \\ & + \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \text{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and where $B_{j,\nu,\nu',l,m}^{1,2,2,1}$, $B_{j,\nu,\nu',l,m}^{1,2,2,2}$, $B_{j,\nu,\nu',l,m}^{1,2,2,3}$ are given, schematically, by:

(8.259)

$$\begin{aligned} & B_{j,\nu,\nu',l,m}^{1,2,2,1} \\ & = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - N)(b' - b)}{\mathbf{g}(L, L')^2} \left(\nabla(L(P_l \text{tr} \chi)) P_m \text{tr} \chi' + \nabla(P_l \text{tr} \chi) L'(P_m \text{tr} \chi') \right. \\ & \quad \left. + L(P_l \text{tr} \chi) \nabla'(P_m \text{tr} \chi') + P_l \text{tr} \chi \nabla'(L'(P_m \text{tr} \chi')) \right) F_{j,-1}(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega', \end{aligned}$$

(8.260)

$$B_{j,\nu,\nu',l,m}^{1,2,2,2} = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi \nabla_{N'}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')^2} F_{j,-1}(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega'$$

and:

$$\begin{aligned} & B_{j,\nu,\nu',l,m}^{1,2,2,3} = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi' - \chi)(b' - b)}{\mathbf{g}(L, L')^2} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ & \quad \times F_{j,-1}(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega'. \end{aligned}$$

The proof of Lemma 8.13 is postponed to Appendix C. In the rest of this section, we use Lemma 8.13 to obtain the control of $B_{j,\nu,\nu',l,m}^{1,2,2}$.

We first estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}, h_{2,p,q,l,m}$ starting with $h_{1,p,q,l,m}$. In view of the Definition (8.257) of $h_{1,p,q,l,m}$, we have:

$$\begin{aligned} & \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} L(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} N'(P_m \text{tr}\chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & + \left\| \int_{\mathbb{S}^2} L(P_l \text{tr}\chi)(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} N'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Using the basic estimate in $L^2(\mathcal{M})$ (7.1) and the estimate (8.170), we obtain:

(8.262)

$$\begin{aligned} & \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| N'(P_m \text{tr}\chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) (1 + p^2) 2^{\frac{11j}{12}} \varepsilon \gamma_j^\nu \gamma_j^{\nu'} \\ & + \left(\sup_{\omega} \left\| L(P_l \text{tr}\chi)(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \\ & \quad \times \left(\sup_{\omega'} \left\| N'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Using the estimate (2.38) for b' , we have:

$$\|N'(P_m \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \|P_m(b' N' \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_{u'})} + \|[b' N', P_m] \text{tr}\chi'\|_{L_u^\infty L^2(\mathcal{H}_{u'})},$$

which together with the estimate (2.39) for $\text{tr}\chi$, the commutator estimate (2.64), and the boundedness of P_m on $L^2(P_{t,u'})$ yields:

$$(8.263) \quad \|N'(P_m \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \varepsilon.$$

Now, we have:

$$\begin{aligned} & \left\| N'(P_m \text{tr}\chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \\ & \lesssim \|N'(P_m \text{tr}\chi')\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \|b' - b_\nu\|_{L^\infty} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \\ & \lesssim \varepsilon |\nu - \nu'|, \end{aligned}$$

where we used in the last inequality the estimate (8.263), the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Using the same estimates, we obtain similar estimates for the last two terms in the right-hand side of (8.262):

$$\left\| L(P_l \text{tr}\chi)(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}} \varepsilon$$

and:

$$\left\| N'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \lesssim \varepsilon.$$

In the end, we obtain:

$$(8.264) \quad \begin{aligned} \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim ((1+p^2)2^{\frac{j}{11}}|\nu-\nu'|+2^{\frac{j}{2}})\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'} \\ &\lesssim (1+p^2)2^{\frac{j}{11}}\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the fact that $|\nu-\nu'| \lesssim 1$.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of the Definition (8.258) of $h_{2,p,q,l,m}$, we have:

$$(8.265)$$

$$\begin{aligned} &\|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b))L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} P_m \text{tr} \chi' (b' - b_{\nu'}) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &+ \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b))P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} L'(P_m \text{tr} \chi')(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &+ \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b))L(P_l \text{tr} \chi)(b_{\nu'} - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &+ \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b))P_l \text{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Using the basic estimate in $L^2(\mathcal{M})$ (7.1) for the first term, the fourth term, the fifth term and the eighth term in the right-hand side of (8.265), the estimate (7.102) for the second term in the right-hand side of (8.265), and the estimate (8.181) for the sixth term in the right-hand side of (8.265), we obtain:

$$\begin{aligned} &\|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\lesssim \left(\sup_\omega \left\| (\theta + b^{-1}\nabla(b))L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^\nu 2^{-\frac{j}{4}} \varepsilon (1+q^2) \gamma_j^{\nu'} \\ &\quad + \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b))P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$\begin{aligned}
& \times \left(\sup_{\omega'} \left\| L'(P_m \operatorname{tr} \chi')(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^\infty L^2(\mathcal{E}_{\mathcal{H}_{\omega'}})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
& + \left(\sup_{\omega} \left\| (\theta + b^{-1} \nabla(b)) L(P_l \operatorname{tr} \chi)(b_{\nu'} - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{H}_\omega})} \right) 2^{\frac{j}{2}} \gamma_j^\nu \varepsilon (1 + q^2) \gamma_j^{\nu'} \\
& + \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\
& \times \left(\sup_{\omega'} \left\| L'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^\infty L^2(\mathcal{E}_{\mathcal{H}_{\omega'}})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}.
\end{aligned}$$

Together with the estimate (2.38) for \bar{b} , the estimates (2.37) (2.39) (2.40) for θ , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (8.186) for $L(P_l \operatorname{tr} \chi)$ and $L'(P_m \operatorname{tr} \chi')$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

(8.266)

$$\begin{aligned}
& \|h_{2,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \\
& \lesssim (1 + q^2) 2^{\frac{j}{4}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
& + \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} 2^{\frac{j}{2}} |\nu - \nu'| \gamma_j^{\nu'} \\
& + 2^{\frac{j}{2}} |\nu - \nu'| (1 + q^2) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
& + \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}.
\end{aligned}$$

Next, we estimate the two $L^2(\mathcal{O}\mathcal{M})$ norms in the right-hand side of (8.266). Recall the definition of θ (2.20):

$$\theta = \chi + k.$$

Now, since k does not depend on ω , and in view of the decomposition (2.72) (2.74) for χ , and the decomposition (2.80) for $b^{-1} \nabla(b)$, we have the following decomposition for $\theta + b^{-1} \nabla(b)$:

$$(8.267) \quad \theta + b^{-1} \nabla(b) = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$(8.268) \quad \|F_1^j\|_{L_{\nu}^\infty L_t^2 L_{x'}^8} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(8.269) \quad \|F_2^j\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{H}_\omega})} \lesssim \varepsilon 2^{-\frac{1}{4}}.$$

We estimate the first term in the right-hand side of (8.266). In view of (8.267), we have:

$$(8.270) \quad \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}$$

$$\begin{aligned} &\lesssim \|F_1^j\|_{L_{u\nu}^2 L_t^2 L_{x'_\nu}^8} \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu}^2 L_{x'_\nu}^{\frac{8}{3}} L_t^\infty} \\ &\quad + \left\| \int_{\mathbb{S}^2} F_2^j P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu}^2 L_{x'_\nu}^{\frac{8}{3}} L_t^\infty} \\ &\quad + \left\| \int_{\mathbb{S}^2} F_2^j P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used the estimate (8.268) for F_1^j in the last inequality. Now, using (7.71) in the case $l > j/2$, and:

$$P_{\leq j/2} \text{tr}\chi = \text{tr}\chi - \sum_{l > j/2} P_l \text{tr}\chi,$$

together with (7.71) and (7.16) in the case $l = j/2$, we obtain:

$$(8.271) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu}^2 L_{x'_\nu}^{\frac{8}{3}} L_t^\infty} \lesssim (1 + p^2)\varepsilon\gamma_j^\nu.$$

Also, we have:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L^\infty(\mathcal{M})} \\ &\quad \lesssim \int_{\mathbb{S}^2} \left\| P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u) \right\|_{L^\infty} \eta_j^\nu(\omega) d\omega \\ &\quad \lesssim \varepsilon \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega' \right) \\ (8.272) \quad &\lesssim \varepsilon 2^j \gamma_j^\nu, \end{aligned}$$

where we used, the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$, Cauchy-Schwarz in λ to estimate $\|F_{j,-1}(u)\|_{L_u^\infty}$, Cauchy-Schwarz in ω and the size of the patch. Interpolating between (8.271) and (8.272), we obtain:

$$(8.273) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu}^2 L_{x'_\nu}^{\frac{8}{3}} L_t^\infty} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^\nu.$$

For the second term in the right-hand side of (8.270), we have:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} F_2^j P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \int_{\mathbb{S}^2} 2^{-\frac{j}{4}} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ (8.274) \quad &\lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality Plancherel in λ , Cauchy Schwartz in ω and the size of the patch. Finally, (8.270), (8.273) and (8.274) imply:

$$(8.275) \quad \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^\nu.$$

Next, we estimate the second term in the right-hand side of (8.266). In view of (8.267), we have:

$$(8.276) \quad \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi (b - b_\nu) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \|F_1^j\|_{L_{u,\nu}^2, L_t^2, L_{x'_\nu}^8} \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi (b - b_\nu) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu}^2, L_{x'_\nu}^{\frac{8}{3}}, L_t^\infty} \\ + \left\| \int_{\mathbb{S}^2} F_2^j P_l \operatorname{tr} \chi (b - b_\nu) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi (b - b_\nu) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu}^2, L_{x'_\nu}^{\frac{8}{3}}, L_t^\infty} \\ + \left\| \int_{\mathbb{S}^2} F_2^j P_l \operatorname{tr} \chi (b - b_\nu) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},$$

where we used the estimate (8.268) for F_1^j in the last inequality. Now, using (7.71) in the case $l > j/2$, and:

$$P_{\leq j/2} \operatorname{tr} \chi = \operatorname{tr} \chi - \sum_{l > j/2} P_l \operatorname{tr} \chi,$$

together with (7.71) and (7.95) in the case $l = j/2$, we obtain:

$$(8.277) \quad \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi (b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu,x'_\nu}^2, L_t^\infty} \lesssim 2^{-\frac{j}{4}} (1 + p^{\frac{5}{2}}) \varepsilon \gamma_j^\nu.$$

Also, we have:

$$\left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi (b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^\infty(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \left\| P_l \operatorname{tr} \chi (b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_j(u) \right\|_{L^\infty} \eta_j^\nu(\omega) d\omega \\ \lesssim \varepsilon 2^{-\frac{j}{2}} \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega' \right) \\ (8.278) \quad \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu,$$

where we used, the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, Cauchy-Schwarz in λ to estimate $\|F_{j,-1}(u)\|_{L_u^\infty}$, Cauchy-Schwarz in ω

and the size of the patch. Interpolating between (8.277) and (8.278), we obtain:

$$(8.279) \quad \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u,\nu}^2 L_{x',\nu'}^{\frac{3}{8}} L_t^\infty} \lesssim 2^{-\frac{j}{16}} \varepsilon \gamma_j^\nu.$$

For the second term in the right-hand side of (8.276), we have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \|F_2^j P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u)\|_{L^2(\mathcal{O}\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|F_2^j\|_{L_u^\infty L^2(\mathcal{S}^2_{\mathcal{U}})} \|F_j(u)\|_{L_u^2} \left\| P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \right\|_{L^\infty(\mathcal{O}\mathcal{M})} \eta_j^\nu(\omega) d\omega. \end{aligned}$$

Together with the estimate (8.269) for F_2^j , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.44) for b , the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$(8.280) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \varepsilon 2^{-\frac{j}{2}} \int_{\mathbb{S}^2} 2^{-\frac{j}{4}} \|F_j(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ & \lesssim \varepsilon 2^{-\frac{j}{4}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality Plancherel in λ , Cauchy Schwartz in ω and the size of the patch. Finally, (8.276), (8.279) and (8.280) imply:

$$(8.281) \quad \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon 2^{-\frac{j}{16}} \varepsilon \gamma_j^\nu.$$

Finally, (8.266), (8.281) and (8.275) yield:

$$(8.282) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim (1 + q^2) (2^{\frac{j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|) + 2^{\frac{7}{16}}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,2,1}$. Recall that we are considering the range of (l, m) :

$$2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

Summing in (l, m) , we have:

$$\begin{aligned} & \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} \left(\nabla(L(P_l \operatorname{tr} \chi)) P_m \operatorname{tr} \chi' + \nabla(P_l \operatorname{tr} \chi) L'(P_m \operatorname{tr} \chi') \right) \\ & \quad + L(P_l \operatorname{tr} \chi) \nabla'(P_m \operatorname{tr} \chi') + P_l \operatorname{tr} \chi \nabla'(L'(P_m \operatorname{tr} \chi')) \\ & = \nabla(L(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi)) P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' + \nabla(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi) L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \\ & \quad + L(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi) \nabla'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') + P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi \nabla'(L'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi')). \end{aligned}$$

Thus, using the symmetry in (ω, ω') of the integrant in $B_{j,\nu,\nu',l,m}^{1,2,2,1}$, we obtain in view of the Definition (8.259) of $B_{j,\nu,\nu',l,m}^{1,2,2,1}$:

(8.283)

$$\begin{aligned} & \sum_{(l,m)/2^m \leq 2^l \leq 2^{j|\nu-\nu'|}} (B_{j,\nu',\nu,l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,1}) \\ &= 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - N)(b' - b)}{\mathbf{g}(L, L')^2} \left(\nabla(L(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi)) P_{\leq 2^j|\nu-\nu'} \text{tr}\chi' \right. \\ & \quad + \nabla(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi) L'(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi') + L(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi) \nabla'(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi') \\ & \quad \left. + P_{\leq 2^j|\nu-\nu'} \text{tr}\chi \nabla'(L'(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi')) \right) F_{j,-1}(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega'. \end{aligned}$$

Now, note that the right-hand side of (8.283) is the analog of the right-hand side of (8.223) provided one replaces

$$P_l \text{tr}\chi, \quad 2^l > 2^j |\nu - \nu'|$$

with:

$$\frac{2^{-j}(N' - N)}{\mathbf{g}(L, L')} \nabla(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi).$$

We obtain the analog of (8.255):

$$(8.284) \quad \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^{j|\nu-\nu'|}} (B_{j,\nu',\nu,l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,1}) \right| \lesssim \left[\frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu-\nu'|)} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{3}{2}}} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

The proof of (8.284) is essentially the same as the proof of the estimate (8.255) and is left to the reader. The similarity in these proofs originates from the fact that

$$\sum_{2^l > 2^j |\nu-\nu'|} P_l \text{tr}\chi$$

and

$$\frac{2^{-j}(N' - N)}{\mathbf{g}(L, L')} \nabla(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi)$$

satisfy the same estimates. For instance, in view of the finite band property, the identities (8.30) (8.31), and the estimate (8.32), we have:

$$\sum_{2^l > 2^j |\nu-\nu'|} P_l \text{tr}\chi \sim \frac{\nabla \text{tr}\chi}{2^j |\nu-\nu'|} \quad \text{and} \quad \frac{2^{-j}(N' - N)}{\mathbf{g}(L, L')} \nabla(P_{\leq 2^j|\nu-\nu'} \text{tr}\chi) \sim \frac{\nabla \text{tr}\chi}{2^j |\nu-\nu'|}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2}$. Recall the Definition (8.260) of $B_{j,\nu,\nu',l,m}^{1,2,2,2}$:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2} = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr}\chi N'(L'(P_m \text{tr}\chi'))(b' - b)}{\mathbf{g}(L, L')} F_{j,-1}(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega'.$$

Estimating the term $N'(L'(P_m \text{tr}\chi'))$ would involve commutator terms which are difficult to handle. To avoid this issue, we commute L' with N' , and then integrate the L' derivative by parts. We obtain schematically in view of the Definition (8.260) of $B_{j,\nu,\nu',l,m}^{1,2,2,2}$:

(8.285)

$$\begin{aligned} & B_{j,\nu,\nu',l,m}^{1,2,2,2} \\ &= 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr}\chi[N', L'](P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega' \\ &\quad - 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left(\frac{\text{div}_{\mathbf{g}}(L')}{\mathbf{g}(L, L')} - \frac{L'(\mathbf{g}(L, L'))}{\mathbf{g}(L, L')^2} \right) \frac{P_l \text{tr}\chi N'(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} \\ &\quad \times F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega' \\ &\quad - 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{L'(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega' \\ &\quad - 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr}\chi N'(P_m \text{tr}\chi')(L'(b') - L'(b))}{\mathbf{g}(L, L')} F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega' \\ &\quad - i2^{-j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} P_l \text{tr}\chi N'(P_m \text{tr}\chi')(b' - b) F_j(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega', \end{aligned}$$

where the last term in the right-hand side of (8.285) appears when the L' derivative falls on the phase in view of (6.6).

We decompose L' in the frame L, N, e_A :

$$(8.286) \quad L' = L + (N' - \mathbf{g}(N, N')N) + (\mathbf{g}(N, N') - 1)N,$$

which yields the following decompositions:

$$(8.287) \quad L'(P_l \text{tr}\chi) = L(P_l \text{tr}\chi) + (N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi) + (\mathbf{g}(N, N') - 1)N(P_l \text{tr}\chi)$$

and:

$$(8.288) \quad L'(b) = L(b) + (N' - \mathbf{g}(N, N')N)(b) + (\mathbf{g}(N, N') - 1)N(b).$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')} \text{ and } \frac{1}{\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.285), (8.287), (8.288), the Formula (7.146) for $\text{div}_{\mathbf{g}}(L')$ and the Formula (7.149) for $L'(\mathbf{g}(L, L'))$, we obtain, schematically:

$$(8.289) \quad \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2}$$

$$\begin{aligned}
&= 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \left[\frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^2} (h_{1,p,q} + h_{2,p,q}) \right. \\
&\quad \left. + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)} h_{3,p,q} + 2^{-j} h_{4,p,q} \right] d_{\mathcal{M}} \\
&\quad + \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}),
\end{aligned}$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}$ on \mathcal{M} are given by:

(8.290)

$$\begin{aligned}
h_{1,p,q} &= \left(\int_{\mathbb{S}^2} G_1 (b - b_\nu)^r \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi')(b_\nu - b')^s \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),
\end{aligned}$$

(8.291)

$$\begin{aligned}
h_{2,p,q} &= \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),
\end{aligned}$$

(8.292)

$$\begin{aligned}
h_{3,p,q} &= \left(\int_{\mathbb{S}^2} \nabla(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)
\end{aligned}$$

and:

(8.293)

$$\begin{aligned}
h_{4,p,q} &= \left(\int_{\mathbb{S}^2} N(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
&\quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),
\end{aligned}$$

where the integer r, s satisfy:

$$r + s = 1,$$

where the tensors G_1 and G_2 are schematically given by:

$$(8.294) \quad G_1 = L(P_l \text{tr} \chi) + (\bar{\delta} + \chi + \zeta + L(b)) P_l \text{tr} \chi$$

and:

$$(8.295) \quad G_2 = [N', L'](P_{\leq l} \text{tr} \chi') + (\bar{\delta}' + L'(b')) N'(P_{\leq l} \text{tr} \chi'),$$

where $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}, B_{j,\nu,\nu',l,m}^{1,2,2,2,2}, B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$ are given by:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,1} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')}$$

$$(8.296) \quad \times F_{j,-1}(u)\eta_j^{\nu'}(\omega)F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega'dc_{\mathcal{M}},$$

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,2} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} N(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')(b' - b)$$

$$(8.297) \quad \times F_{j,-1}(u)\eta_j^{\nu'}(\omega)F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega'dc_{\mathcal{M}},$$

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,3} = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1}P_l \text{tr}\chi N'(P_m \text{tr}\chi')(b' - b)F_j(u)\eta_j^{\nu'}(\omega)$$

$$(8.298) \quad \times F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega'dc_{\mathcal{M}}$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. Note that the term $b - b'$ present in all terms of (8.285) is present in the definition of $h_{1,p,q}$, but absent from the definition of $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$. Indeed, we do not need to exploit the gain $b - b'$ in $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$, and we just separate b and b' and estimate them in L^∞ using the estimate (2.38) for b . To simplify the notations, we chose not to specify these factors of b and b' .

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$ starting with $h_{1,p,q}$. We have:

$$(8.299)$$

$$\begin{aligned} \|h_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} G_1(b - b_\nu)^r \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr}\chi')(b_\nu - b')^s \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

We estimate the second term in the right-hand side of (8.299). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr}\chi')(b_\nu - b')^s \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega'} \left\| N'(P_{\leq l} \text{tr}\chi')(b_\nu - b')^s \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_w^\infty L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ &\lesssim \left(\sup_{\omega} \|N'(P_{\leq l} \text{tr}\chi')\|_{L_w^\infty L^2(\mathcal{H}_{u'})} \right) |\nu - \nu'|^s 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the estimate (2.39) for $\text{tr}\chi$ and the commutator estimate (2.64), we obtain:

$$(8.300)$$

$$\left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr}\chi')(b_\nu - b')^s \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon |\nu - \nu'|^s 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

Next, we estimate the first term in the right-hand side of (8.299). Using the basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1), we have:

$$\begin{aligned}
 (8.301) \quad & \left\| \int_{\mathbb{S}^2} G_1(b - b_\nu)^r \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\
 & \lesssim \left(\sup_{\omega} \left\| G_1(b - b_\nu)^r \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
 & \lesssim \left(\sup_{\omega} \|G_1\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{-\frac{rj}{2}} 2^{\frac{j}{2}} \gamma_j^\nu,
 \end{aligned}$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the definition of G_1 (8.294), we have:

$$\begin{aligned}
 (8.302) \quad & \|G_1\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim (\|\bar{\delta}\|_{L_t^\infty L_x^4} + \|\chi\|_{L_t^\infty L_x^4} + \|\zeta\|_{L_t^\infty L_x^4} + \|L(b)\|_{L_t^\infty L_x^4}) \|P_l \text{tr} \chi\|_{L_t^2 L_x^4} \\
 & \quad + \|L(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \\
 & \lesssim \varepsilon \|P_l \text{tr} \chi\|_{L_t^2 L_x^4} + \|L(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)},
 \end{aligned}$$

where we used in the last inequality the embedding (2.51), and the estimates (2.36) (2.37) for $\bar{\delta}$, the estimates (2.39) (2.40) for χ , the estimate (2.38) for b , and the estimate (2.41) for ζ . If $l > j/2$, we have the analog of (8.139):

$$(8.303) \quad \|P_l \text{tr} \chi\|_{L_t^2 L_x^4} \lesssim 2^{-\frac{l}{2}} \varepsilon,$$

while in the case $l = j/2$, the boundedness of $P_{\leq j/2}$ on $L^4(P_{t,u})$ and the estimate (2.39) for $\text{tr} \chi$ yields:

$$(8.304) \quad \|P_{\leq j/2} \text{tr} \chi\|_{L_t^2 L_x^4} \lesssim \varepsilon.$$

In view of (8.302), we also need to estimate $L(P_l \text{tr} \chi)$. In the case $l > j/2$, the estimate (8.142) yields:

$$(8.305) \quad \|L(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{l}{2}} \varepsilon,$$

which together with the estimate (2.39) for $\text{tr} \chi$ and the decomposition:

$$P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > \frac{j}{2}} P_l \text{tr} \chi$$

implies in the case $l = j/2$:

$$(8.306) \quad \|L(P_{\leq j/2} \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon.$$

Now, in view of (8.302), (8.303), (8.304), (8.305) and (8.306), we obtain:

$$\|G_1\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-l(1-\delta_{l,j/2})},$$

where we defined:

$$\delta_{l,j/2} = 1 \text{ if } l = j/2 \text{ and } \delta_{l,j/2} = 0 \text{ otherwise.}$$

Together with (8.301), we deduce:

$$(8.307) \quad \left\| \int_{\mathbb{S}^2} G_1(b - b_\nu)^r \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{rj}{2}} 2^{\frac{j}{2}} 2^{-l(1-\delta_{l,j/2})} \gamma_j^{\nu'}.$$

Finally, (8.299), (8.300) and (8.307) imply:

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 |\nu - \nu'|^s 2^{-\frac{rj}{2}} 2^{j-l(1-\delta_{l,j/2})} \gamma_j^\nu \gamma_j^{\nu'}.$$

Since we have:

$$2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1 \text{ and } r + s = 1,$$

this yields:

$$(8.308) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 |\nu - \nu'| 2^{j-l(1-\delta_{l,j/2})} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q}$. In view of the Definition (8.291) of $h_{2,p,q}$, we have:

$$(8.309) \quad \|h_{2,p,q}\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^3(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^{\frac{3}{2}}(\mathcal{M})}.$$

Recall (8.272):

$$(8.310) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^\infty(\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j^\nu.$$

On the other hand, (8.179) and (8.180) imply:

$$(8.311) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) 2^{\frac{j}{2}-l} \varepsilon \gamma_j^\nu.$$

Interpolating between (8.310) and (8.311), we obtain:

$$(8.312) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^3(\mathcal{M})} \lesssim \varepsilon 2^{\frac{2j}{3}} 2^{-\frac{2l}{3}} \gamma_j^\nu.$$

Next, we estimate the second term in the right-hand side of (8.309). We have:

$$(8.313) \quad \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ \lesssim \int_{\mathbb{S}^2} \|G_2\|_{L_{u'}^\infty L^{\frac{3}{2}}(\mathcal{M}_{u'})} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \\ \lesssim \int_{\mathbb{S}^2} \|G_2\|_{L_{u'}^\infty L^{\frac{3}{2}}(\mathcal{M}_{u'})} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega',$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next, we estimate G_2 . In view of the definition of G_2 (8.295), the commutator

formulas (2.32) for $[L, \underline{L}]$ and the fact that $2N = L - \underline{L}$, we have schematically:

$$G_2 = n^{-1} \nabla_{N'} n L' (P_{\leq l} \text{tr} \chi') + (\zeta' - \underline{\zeta}') \cdot \nabla' (P_{\leq l} \text{tr} \chi') + (\delta' + L'(b')) N' (P_{\leq l} \text{tr} \chi').$$

This yields:

$$\begin{aligned} \|G_2\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{E}_{u'})} &\lesssim (\|\delta'\|_{L_u^\infty L^6(\mathcal{E}_{u'})} + \|n^{-1} \nabla_{N'} n\|_{L_u^\infty L^6(\mathcal{E}_{u'})} + \|\zeta'\|_{L_u^\infty L^6(\mathcal{E}_{u'})}) \\ &\quad + \|\underline{\zeta}'\|_{L_u^\infty L^6(\mathcal{E}_{u'})} + \|L'(b')\|_{L_u^\infty L^6(\mathcal{E}_{u'})} \| \mathbf{D} P_{\leq l} \text{tr} \chi' \|_{L_u^\infty L^2(\mathcal{E}_{u'})} \\ &\lesssim \varepsilon \| \mathbf{D} P_{\leq l} \text{tr} \chi' \|_{L_u^\infty L^2(\mathcal{E}_{u'})}, \end{aligned}$$

where we used in the last inequality the Sobolev embedding (2.50), and the estimates (2.37)–(2.36) for n , δ and ζ , the estimate (2.38) for b , and the estimate (2.41) for ζ . Together with the basic properties of $P_{\leq l}$, the commutator estimates (2.64) and (2.65), and the estimate (2.39) for $\text{tr} \chi$, this implies:

$$\|G_2\|_{L_u^\infty L^{\frac{3}{2}}(\mathcal{E}_{u'})} \lesssim \varepsilon.$$

Together with (8.313), we obtain:

$$\begin{aligned} (8.314) \quad &\left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^{\frac{3}{2}}(\mathcal{M})} \\ &\lesssim \varepsilon \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality Plancherel in λ for $\|F_{j,-1}(u')\|_{L_u^2}$, Cauchy Schwarz in ω' and the size of the patch. Finally, (8.309), (8.312) and (8.314) imply:

$$(8.315) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{7j}{6}} 2^{-\frac{2j}{3}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{3,p,q}$. In view of the definition of $h_{3,p,q}$ (8.292), we have:

$$\begin{aligned} (8.316) \quad &\|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} \nabla(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Arguing as for the proof of (8.136), we have:

$$(8.317) \quad \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

Also, using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \nabla(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_\omega \left\| \nabla(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^\nu \end{aligned}$$

$$\lesssim \left(\sup_{\omega} \|\nabla(b)\|_{L_t^\infty L_x^4}, \|P_l \text{tr} \chi\|_{L_t^2 L_x^4} \right) 2^{\frac{j}{2}} \gamma_j^\nu,$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the estimate (2.38) for b , the embedding (2.51), and the estimates (8.303) and (8.304) for $P_l \text{tr} \chi$, we obtain:

$$(8.318) \quad \left\| \int_{\mathbb{S}^2} \nabla(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^{\frac{j}{2}} \gamma_j^\nu.$$

Finally, (8.316), (8.317) and (8.318) imply:

$$(8.319) \quad \|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{4,p,q}$. In view of the definition of $h_{4,p,q}$ (8.293), we have:

$$(8.320) \quad \begin{aligned} \|h_{4,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} N(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} N(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.317). In view of the estimate (2.38) for b , we have:

$$\|N(b)\|_{L_t^\infty L_x^4} \lesssim \varepsilon.$$

Thus, arguing as for the proof of (8.318), we obtain:

$$\left\| \int_{\mathbb{S}^2} N(b) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^{\frac{j}{2}} \gamma_j^\nu.$$

Together with (8.320), this yields:

$$(8.321) \quad \|h_{4,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, in view of the decomposition (8.289) of $B_{j,\nu,\nu',l,m}^{1,2,2,2}$, we have:

$$\begin{aligned} &\left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \right| \\ &\lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left[\left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^2} \right\|_{L^\infty(\mathcal{M})} (\|h_{1,p,q}\|_{L^1(\mathcal{M})}) \right. \\ &\quad \left. + \|h_{2,p,q}\|_{L^1(\mathcal{M})} + \left\| \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)} \right\|_{L^\infty(\mathcal{M})} \|h_{3,p,q}\|_{L^1(\mathcal{M})} + 2^{-j} \|h_{4,p,q}\|_{L^1(\mathcal{M})} \right], \end{aligned}$$

which together with (8.32), (8.308), (8.315), (8.319) and (8.321) yields:

$$\begin{aligned}
& \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \right| \\
& \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q}} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} (|\nu - \nu'| 2^{j-l(1-\delta_{l,j/2})} + 2^{\frac{7j}{6}} 2^{-\frac{2l}{3}}) \right. \\
& \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j + 2^{-j} 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
& \lesssim 2^{-j} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} (|\nu - \nu'| 2^{j-l(1-\delta_{l,j/2})} + 2^{\frac{7j}{6}} 2^{-\frac{2l}{3}}) \right. \\
& \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j + 2^{-j} 2^{-\frac{1}{2}(1-\delta_{j/2,l})} 2^j \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.
\end{aligned}$$

Summing in l , we obtain:

$$\begin{aligned}
& \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \right| \\
& (8.322) \\
& \lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.
\end{aligned}$$

In view of (8.322), we need to estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$, $B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$, $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$. We start with $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$ which is defined in (8.298) as:

$$\begin{aligned}
B_{j,\nu,\nu',l,m}^{1,2,2,2,3} &= -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi N' (P_m \operatorname{tr} \chi') (b' - b) \\
& \quad \times F_j(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

We integrate by parts tangentially using (7.137).

Lemma 8.14. — *Let $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$ defined in (8.298). Integrating by parts using (7.137) yields:*

$$\begin{aligned}
& (8.323) \\
& \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \\
& = 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q}} \left[\frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^2} (h'_{1,p,q} + h'_{2,p,q}) \right. \\
& \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)} (h_{3,p,q} + h'_{5,p,q}) + 2^{-j} h_{4,p,q} \right] d_{\mathcal{M}} + \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}),
\end{aligned}$$

where the scalar functions $h_{3,p,q}, h_{4,p,q}$ are given respectively by (8.292) and (8.293), where the scalar functions $h'_{1,p,q}, h'_{2,p,q}, h'_{5,p,q}$ on \mathcal{M} are given by:

(8.324)

$$h'_{1,p,q} = \left(\int_{\mathbb{S}^2} G'_1 (b - b_\nu)^r \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi')(b_\nu - b')^s \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.325)

$$h'_{2,p,q} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} G'_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.326)

$$h'_{5,p,q} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} G'_3 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

where the integer r, s satisfy:

$$r + s = 1,$$

where the tensors G'_1, G'_2 and G'_3 are schematically given by:

(8.327) $G'_1 = (\chi + \theta) P_l \text{tr} \chi,$

(8.328) $G'_2 = (\chi' + L'(b') + \theta' + \nabla'(b)) N'(P_{\leq l} \text{tr} \chi')$

and:

(8.329) $G'_3 = \nabla' N'(P_{\leq l} \text{tr} \chi'),$

where $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}, B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$ are given respectively by (8.296) and (8.297), and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

The proof of Lemma 8.14 is postponed to Appendix D. We now use this lemma to estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$.

We estimate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q}, h'_{2,p,q}, h'_{5,p,q}$. The estimate of $h'_{1,p,q}$ is completely analogous to the one of $h_{1,p,q}$ defined in (8.290). Thus, we obtain in view of (8.308):

(8.330) $\|h'_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 |\nu - \nu'| 2^{j-l(1-\delta_{l,j/2})} \gamma_j^\nu \gamma_j^{\nu'}.$

Also, the estimate of $h'_{2,p,q}$ is completely analogous to the one of $h_{2,p,q}$ defined in (8.291). Thus, we obtain in view of (8.315):

$$(8.331) \quad \|h'_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{7j}{6}} 2^{-\frac{2l}{3}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h'_{5,p,q}$. In view of (8.326), we have:

$$\begin{aligned} & \|h'_{5,p,q}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} G'_3 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \left\| P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \quad \times \int_{\mathbb{S}^2} \left\| G'_3 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ & \lesssim \left(\int_{\mathbb{S}^2} \|P_l \operatorname{tr} \chi F_{j,-1}(u)\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \left(\int_{\mathbb{S}^2} \|G'_3 F_{j,-1}(u')\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the definition of G'_3 (8.329), this yields:

$$(8.332) \quad \|h'_{5,p,q}\|_{L^1(\mathcal{M})} \lesssim \sum_{m/l \leq l} \left(\int_{\mathbb{S}^2} \left\| \|P_l \operatorname{tr} \chi\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \left\| \|\nabla' N'(P_m \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right).$$

Next, we estimate $\nabla' N'(P_m \operatorname{tr} \chi')$. Using the estimate (2.38) for b' , we have:

$$\begin{aligned} & \|\nabla' N'(P_m \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} \\ & \lesssim \|b'^{-1} \nabla'(b') N'(P_m \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} + \|b'^{-1} \nabla'(b' N'(P_m \operatorname{tr} \chi'))\|_{L^2(\mathcal{H}_{u'})} \\ & \lesssim \|b'^{-1} \nabla'(b')\|_{L_t^\infty L_{x'}^4} \|b' N'(P_m \operatorname{tr} \chi')\|_{L_t^2 L_{x'}^4} + \|\nabla' P_m(b' N' \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} \\ & \quad + \|\nabla'[b' N', P_m] \operatorname{tr} \chi'\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \\ & \lesssim \|P_m(b' N' \operatorname{tr} \chi')\|_{L_t^2 L_{x'}^4} + \|[b' N', P_m] \operatorname{tr} \chi'\|_{L_t^2 L_{x'}^4} + \|\nabla' P_m(b' N' \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} \\ & \quad + \|\nabla'[b' N', P_m] \operatorname{tr} \chi'\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \\ & \lesssim \|P_m(b' N' \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} + \|\nabla' P_m(b' N' \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} \\ & \quad + \|[b' N', P_m] \operatorname{tr} \chi'\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} + \|\nabla'[b' N', P_m] \operatorname{tr} \chi'\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})}, \end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (2.49). Together with the finite band property for P_m and the commutator estimate (2.68), we obtain:

$$(8.333) \quad \|\nabla' N'(P_m \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} \lesssim 2^m \|P_m(b' N' \operatorname{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} + 2^{\frac{m}{2}} \varepsilon.$$

Together with (8.332), this yields:

(8.334)

$$\begin{aligned} \|h'_{5,p,q}\|_{L^1(\mathcal{M})} &\lesssim \sum_{m/m \leq l} 2^{-|l-m|} \left(\int_{\mathbb{S}^2} 2^l \left\| \|P_l \text{tr} \chi\|_{L^2(\mathcal{M}_u)} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| (\|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{M}_{u'})} + 2^{-\frac{m}{2}} \varepsilon) F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim 2^{-j} \sum_{m/m \leq l} 2^{-|l-m|} \left\| 2^l \|P_l \text{tr} \chi\|_{L^2(\mathcal{M}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{\omega,u}} \\ &\quad \times \left\| (\|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{M}_{u'})} + 2^{-\frac{m}{2}} \varepsilon) F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L^2_{\omega',u'}}, \end{aligned}$$

where we used in the last inequality Cauchy Schwarz in ω and ω' , and the size of the patch. Now, we have:

$$\begin{aligned} (8.335) \quad &\sum_l \left\| 2^l \|P_l \text{tr} \chi\|_{L^2(\mathcal{M}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{\omega,u}}^2 \\ &= \int_{\mathbb{S}^2} \left(\int_u \left(\sum_l 2^{2l} \|P_l \text{tr} \chi\|_{L^2(\mathcal{M}_u)}^2 \right) |F_j(u)|^2 du \right) \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|\nabla \text{tr} \chi\|_{L^\infty_u L^2(\mathcal{M}_u)}^2 \|F_j(u)\|_{L^2_u}^2 \eta_j^\nu(\omega) d\omega \\ &\lesssim \varepsilon^2 2^{2j} (\gamma_j^\nu)^2, \end{aligned}$$

where we used the finite band property for P_l , the estimates (2.39) for $\text{tr} \chi$ and Plancherel in λ . Also, we have:

(8.336)

$$\begin{aligned} &\sum_m \left\| (\|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{M}_{u'})} + 2^{-\frac{m}{2}} \varepsilon) F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L^2_{\omega',u'}}^2 \\ &= \int_{\mathbb{S}^2} \left(\int_{u'} \left(\sum_m (\|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{M}_{u'})} + 2^{-\frac{m}{2}} \varepsilon)^2 \right) |F_j(u')|^2 du' \right) \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \int_{\mathbb{S}^2} (\|b' N' \text{tr} \chi'\|_{L^\infty_{u'} L^2(\mathcal{M}_{u'})}^2 + \varepsilon^2) \|F_j(u')\|_{L^2_{u'}}^2 \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \varepsilon^2 2^{2j} (\gamma_j^{\nu'})^2, \end{aligned}$$

where we used the finite band property for P_m , the estimates (2.39) for $\text{tr} \chi'$ and Plancherel in λ' . Finally, (8.334), (8.335) and (8.336) yield:

$$\begin{aligned} (8.337) \quad &\sum_{(l,m)/m \leq l} \|h'_{5,p,q}\|_{L^1(\mathcal{M})} \\ &\lesssim 2^{-j} \left(\sum_l \left\| 2^l \|P_l \text{tr} \chi\|_{L^2(\mathcal{M}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{\omega,u}}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} & \times \left(\sum_m \left\| \left(\|P_m(b'N'\text{tr}\chi')\|_{L^2(\mathcal{E}u')} + 2^{-\frac{m}{2}}\varepsilon \right) F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L_{\omega',u'}^2} \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon^2 2^j \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Now, in view of the decomposition (8.323) of $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$, we have:

$$\begin{aligned} & \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}) \right| \\ & \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left[\left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^2} \right\|_{L^\infty(\mathcal{M})} (\|h'_{1,p,q}\|_{L^1(\mathcal{M})} \right. \\ & \quad \left. + \|h'_{2,p,q}\|_{L^1(\mathcal{M})}) + \left\| \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)} \right\|_{L^\infty(\mathcal{M})} (\|h_{3,p,q}\|_{L^1(\mathcal{M})} + \|h'_{5,p,q}\|_{L^1(\mathcal{M})}) \right. \\ & \quad \left. + 2^{-j} \|h_{4,p,q}\|_{L^1(\mathcal{M})} \right], \end{aligned}$$

which together with (8.32), (8.319), (8.321), (8.330) and (8.331) yields:

$$\begin{aligned} & \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}) \right| \\ & \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q}} \left[\left(\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} (|\nu - \nu'| 2^{j-l(1-\delta_{i,j/2})} + 2^{\frac{7j}{6}} 2^{-\frac{2l}{3}}) \right) \right. \\ & \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^{-\frac{1}{2}(1-\delta_{j/2,i})} 2^j + 2^{-j} 2^{-\frac{1}{2}(1-\delta_{j/2,i})} 2^j \right) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} \|h'_{5,p,q}\|_{L^1(\mathcal{M})} \right] \\ & \lesssim 2^{-j} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} (|\nu - \nu'| 2^{j-l(1-\delta_{i,j/2})} + 2^{\frac{7j}{6}} 2^{-\frac{2l}{3}}) \right. \\ & \quad \left. + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^{-\frac{1}{2}(1-\delta_{j/2,i})} 2^j + 2^{-j} 2^{-\frac{1}{2}(1-\delta_{j/2,i})} 2^j \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \quad + 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+1}} \|h'_{5,p,q}\|_{L^1(\mathcal{M})}. \end{aligned}$$

Summing in l , we obtain:

$$\left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} - \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}) \right|$$

$$\begin{aligned} &\lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'} \\ &\quad + 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+1}} \left(\sum_{(l,m)/m \leq l} \|h'_{5,p,q}\|_{L^1(\mathcal{M})} \right). \end{aligned}$$

Together with (8.337), we get:

$$\begin{aligned} &\left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} - \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}) \right| \\ &\lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'} \\ &\quad + 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+1}} \varepsilon^2 2^j \gamma_j^{\nu'} \gamma_j^{\nu'} \\ (8.338) \quad &\lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}. \end{aligned}$$

Finally, (8.322) and (8.338) imply:

$$\begin{aligned} &\left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}) \right| \\ (8.339) \quad &\lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}. \end{aligned}$$

In view of (8.339), we still need to estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$ and $B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$. We start with $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$ which is defined by (8.296) as:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,2,2,1} &= -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)N'(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\ &\quad \times F_{j,-1}(u)\eta_j^{\nu'}(\omega)F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts tangentially using (7.137).

Lemma 8.15. — *Let $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$ defined in (8.296). Integrating by parts using (7.137) yields:*

$$\begin{aligned} (8.340) \quad &\sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,1} \\ &= 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}}|N_{\nu} - N_{\nu'}|)^{p+q+1}} \left[\frac{1}{(2^{\frac{j}{2}}|N_{\nu} - N_{\nu'}|)^2} (h''_{1,p,q} + h''_{2,p,q}) \right] \end{aligned}$$

$$+ \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)}(h''_{3,p,q} + h''_{4,p,q}) + 2^{-j}h''_{5,p,q} \Big] d_{\mathcal{C}\mathcal{M}},$$

where the scalar functions $h''_{1,p,q}, h''_{2,p,q}, h''_{3,p,q}, h''_{4,p,q}, h''_{5,p,q}$ on $\mathcal{C}\mathcal{M}$ are given by:

$$(8.341) \quad h''_{1,p,q} = \left(\int_{\mathbb{S}^2} G_1'' \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.342) \quad h''_{2,p,q} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} G_2'' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.343) \quad h''_{3,p,q} = \left(\int_{\mathbb{S}^2} \nabla^2(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.344) \quad h''_{4,p,q} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(N'(P_{\leq l} \text{tr} \chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

$$(8.345) \quad h''_{5,p,q} = \left(\int_{\mathbb{S}^2} \nabla(N(P_l \text{tr} \chi)) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

where the tensors G_1'' and G_2'' are schematically given by:

$$(8.346) \quad G_1'' = (\chi + \theta + \nabla(b)) \nabla(P_l \text{tr} \chi)$$

and:

$$(8.347) \quad G_2'' = (\chi' + \theta' + \nabla'(b')) N'(P_{\leq l} \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

The proof of Lemma 8.15 is postponed to Appendix E. We now use this lemma to estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$.

We estimate the $L^1(\mathcal{M})$ norm of $h''_{1,p,q}, h''_{2,p,q}, h''_{3,p,q}, h''_{4,p,q}, h''_{5,p,q}$ starting with $h''_{1,p,q}$. In view of (8.341), we have:

$$\begin{aligned} \|h''_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} G''_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Together with the basic estimate in $L^2(\mathcal{M})$ (7.1), this yields:

(8.348)

$$\begin{aligned} \|h''_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left(\sup_{\omega} \left\| G''_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \\ &\quad \times \left(\sup_{\omega'} \left\| N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \left(\sup_{\omega} \|G''_1\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \left(\sup_{\omega'} \|N'(P_{\leq l} \text{tr} \chi')\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, the estimate (2.39) for $\text{tr} \chi$, the boundedness of P_m on $L^2(P_{t,u})$ and the commutator estimate (2.68) yields:

$$(8.349) \quad \|N'(P_{\leq l} \text{tr} \chi')\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \lesssim \varepsilon.$$

Also, we have in view of the Definition (8.346) of G''_1 :

$$\begin{aligned} \|G''_1\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim (\|\chi\|_{L_t^\infty L_x^4} + \|\theta\|_{L_t^\infty L_x^4} + \|\nabla(b)\|_{L_t^\infty L_x^4}) \|\nabla(P_l \text{tr} \chi)\|_{L_t^2 L_x^4} \\ &\lesssim (\mathcal{N}_1(\chi) + \mathcal{N}_1(\theta) + \mathcal{N}_1(\nabla(b))) \|\nabla(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)}^{\frac{1}{2}} \|\nabla^2(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)}^{\frac{1}{2}}, \end{aligned}$$

where we used in the last inequality the embedding (2.51) and the Gagliardo-Nirenberg inequality (2.49). Together with the Bochner inequality (2.61) and the finite band property for P_l , we obtain:

$$(8.350) \quad \begin{aligned} \|G''_1\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim (\mathcal{N}_1(\chi) + \mathcal{N}_1(\theta) + \mathcal{N}_1(\nabla(b))) 2^{\frac{l}{2}} \|\nabla \text{tr} \chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim 2^{\frac{l}{2}} \varepsilon, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ , the estimate (2.38) for b and the estimates (2.37) (2.39) (2.40) for θ . Finally, (8.348), (8.349) and (8.350) imply:

$$(8.351) \quad \|h''_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{l}{2}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $h''_{2,p,q}$. In view of its Definition (8.342), we have the analog of the estimate (8.348):

$$(8.352) \quad \|h''_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega} \|\nabla(P_l \text{tr} \chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \left(\sup_{\omega'} \|G''_2\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

The estimate (2.39) for $\text{tr}\chi$ together with the finite band property for P_l yields:

$$(8.353) \quad \|\nabla(P_l \text{tr}\chi)\|_{L^\infty L^2(\mathcal{E}_{u'})} \lesssim \varepsilon.$$

Also, in view of the Definition (8.347) of G_2'' and the estimate (8.349), the analog of the estimate (8.350) yields:

$$\|G_2''\|_{L^\infty L^2(\mathcal{E}_{u'})} \lesssim 2^{\frac{1}{2}} \varepsilon,$$

which together with (8.352) and (8.353) implies:

$$(8.354) \quad \|h_{3,p,q}''\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{1}{2}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $h_{3,p,q}''$. In view of its Definition (8.343), we have the analog of the estimate (8.348):

$$(8.355) \quad \|h_{3,p,q}''\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega} \|\nabla^2(P_l \text{tr}\chi)\|_{L^\infty L^2(\mathcal{E}_u)} \right) \left(\sup_{\omega'} \|N'(P_{\leq l} \text{tr}\chi')\|_{L^\infty L^2(\mathcal{E}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, the Bochner inequality (2.61), the finite band property for P_l , and the estimate (2.39) for $\text{tr}\chi$ yield:

$$(8.356) \quad \|\nabla^2(P_l \text{tr}\chi)\|_{L^\infty L^2(\mathcal{E}_u)} \lesssim 2^l \varepsilon.$$

Finally, (8.355), (8.356) and (8.349) yield:

$$(8.357) \quad \|h_{3,p,q}''\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $h_{4,p,q}''$. In view of its Definition (8.344), we have the analog of the estimate (8.348):

$$(8.358) \quad \|h_{4,p,q}''\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega} \|\nabla(P_l \text{tr}\chi)\|_{L^\infty L^2(\mathcal{E}_u)} \right) \left(\sup_{\omega'} \|\nabla'(N'(P_{\leq l} \text{tr}\chi'))\|_{L^\infty L^2(\mathcal{E}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, we have in view of the estimate (8.333):

$$(8.359) \quad \begin{aligned} \|\nabla' N'(P_{\leq l} \text{tr}\chi')\|_{L^2(\mathcal{E}_{u'})} &\lesssim 2^l \|P_{\leq l}(b' N' \text{tr}\chi')\|_{L^2(\mathcal{E}_{u'})} + 2^{\frac{1}{2}} \varepsilon \\ &\lesssim 2^l \varepsilon, \end{aligned}$$

where we used in the last inequality the boundedness of P_m on $L^2(P_{t,u})$, the estimate (2.38) for b and the estimate (2.39) for $\text{tr}\chi$. Finally, (8.358), (8.359) and (8.353) yield:

$$(8.360) \quad \|h_{4,p,q}''\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $h_{5,p,q}''$. In view of its Definition (8.345), we have the analog of the estimate (8.348):

$$\|h_{5,p,q}''\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega} \|\nabla(N(P_l \text{tr}\chi))\|_{L^\infty L^2(\mathcal{E}_u)} \right) \left(\sup_{\omega'} \|N'(P_{\leq l} \text{tr}\chi')\|_{L^\infty L^2(\mathcal{E}_{u'})} \right) 2^j \gamma_j^\nu \gamma_j^{\nu'}.$$

Together with the estimate (8.349) and the estimate (8.359), we obtain:

$$(8.361) \quad \|h_{5,p,q}''\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, in view of the decomposition (8.340) of $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$, we have:

$$\begin{aligned} & \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,1} \right| \\ & \lesssim 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q+1}} \right\|_{L^\infty(\mathcal{M})} \left[\left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^2} \right\|_{L^\infty(\mathcal{M})} (\|h''_{1,p,q}\|_{L^1(\mathcal{M})}) \right. \\ & \quad + \|h''_{2,p,q}\|_{L^1(\mathcal{M})} + \left\| \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)} \right\|_{L^\infty(\mathcal{M})} (\|h''_{3,p,q}\|_{L^1(\mathcal{M})} + \|h''_{4,p,q}\|_{L^1(\mathcal{M})}) \\ & \quad \left. + 2^{-j} \|h''_{5,p,q}\|_{L^1(\mathcal{M})} \right], \end{aligned}$$

which together with (8.32), (8.351), (8.354), (8.357), (8.360) and (8.361) implies:

$$\begin{aligned} & \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,1} \right| \\ & \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+1}} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} 2^{\frac{j}{2}} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^l + 2^{-j+l} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \lesssim 2^{-\frac{j}{2}} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)} \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} 2^{\frac{j}{2}} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} 2^l + 2^{-j+l} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Summing in l , we obtain:

$$(8.362) \quad \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2,1} \right| \lesssim \left[\frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, (8.339) and (8.362), together with the fact that

$$2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$$

implies:

$$(8.363) \quad \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2,2} \right| \lesssim \left[\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

In view of (8.363), we still need to estimate $B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$. Recall the Definition (8.297) of $B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,2} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} N(P_l \text{tr} \chi) N'(P_m \text{tr} \chi') (b' - b) F_{j,-1}(u) \eta_j^\nu(\omega)$$

$$\times F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{C}\mathcal{M}}.$$

Note that the integrant in the definition of $B_{j,\nu,\nu',l,m}^{1,2,2,2,2}$ is antisymmetric in $((l, \omega, \nu), (m, \omega', \nu'))$, and thus we have the following cancelation:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu',\nu,m,l}^{1,2,2,2,2} = 0.$$

This yields:

$$\sum_{(l,m)/2^{\max(m,l)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu',\nu,m,l}^{1,2,2,2,2}) = 0,$$

which together with (8.363) implies:

(8.364)

$$\left| \sum_{(l,m)/2^{\max(m,l)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2,2} \right| \lesssim \left[\frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,2,3}$. Recall the Definition (8.261) of $B_{j,\nu,\nu',l,m}^{1,2,2,3}$:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,2,3} &= 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi' - \chi)(b' - b)}{\mathbf{g}(L, L')^2} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ &\quad \times F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega'. \end{aligned}$$

Recall also that we are considering the range of (l, m) :

$$2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

Summing in (l, m) , we have:

$$\begin{aligned} \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) \\ = L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' + P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi'). \end{aligned}$$

Thus, using the symmetry in (ω, ω') of the integrant in $B_{j,\nu,\nu',l,m}^{1,2,2,3}$, we obtain in view of the Definition (8.261) of $B_{j,\nu,\nu',l,m}^{1,2,2,3}$:

$$\begin{aligned} (8.365) \quad \sum_{(l,m)/2^{\max(l,m)} l \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{1,2,2,3} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \\ = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi' - \chi)(b' - b)}{\mathbf{g}(L, L')^2} L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \\ \quad \times F_{j,-1}(u)\eta_j^\nu(\omega) F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega d\omega' + \text{terms interverting } (\nu, \nu'), \end{aligned}$$

where we chose to ignore the terms which are obtained by interverting ν and ν' since they are treated in the exact same way.

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \quad \text{and} \quad 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.365), we obtain, schematically:

(8.366)

$$\begin{aligned} & \sum_{(l,m)/2^{\max(l,m)} l \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{1,2,2,3} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \\ &= \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+4}} [h_{1,p,q} + h_{2,p,q}] d\mathcal{M} + \text{terms interverting } (\nu, \nu'), \end{aligned}$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}$ on \mathcal{M} are given by:

(8.367)

$$\begin{aligned} h_{1,p,q} &= \left(\int_{\mathbb{S}^2} \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'})^r \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi'(b_{\nu'} - b')^s \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and:

(8.368)

$$\begin{aligned} h_{2,p,q} &= \left(\int_{\mathbb{S}^2} L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'})^r \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \chi' P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi'(b_{\nu'} - b')^s \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

where the integer r, s satisfy:

$$r + s = 1$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}, h_{2,p,q}$ starting with $h_{1,p,q}$. We consider first the case $r = 1$ and $s = 0$ which is easier. In view of the Definition (8.367) of $h_{1,p,q}$ in the case $r = 1$ and $s = 0$, we have:

$$\begin{aligned} (8.369) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

We estimate the two terms in the right-hand side of (8.369) starting with the first one. The basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) yields:

$$(8.370) \quad \left\| \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ \lesssim \left(\sup_{\omega} \left\| \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P \right\|_{L_u^\infty L^2(\mathcal{E}^u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ \lesssim |\nu - \nu'| \left(\sup_{\omega} \left\| \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) \right\|_{L_u^\infty L^2(\mathcal{E}^u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Now, the estimate (8.184) yields:

$$\left\| \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) \right\|_{L_u^\infty L^2(\mathcal{E}^u)} \lesssim \varepsilon \mathcal{N}_1(\chi) \lesssim \varepsilon,$$

where we used in the last inequality the estimates (2.39) (2.40) for χ . Together with (8.370), we obtain:

$$(8.371) \quad \left\| \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon |\nu - \nu'| 2^{\frac{j}{2}} \gamma_j^{\nu'}.$$

Next, we estimate the second term in the right-hand side of (8.369). We have:

$$\left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ \lesssim \left\| \int_{\mathbb{S}^2} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ + \left\| \int_{\mathbb{S}^2} P_{> 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})},$$

which together with the estimate (8.129), the estimate (7.64) and the fact that $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$ yields:

$$(8.372) \quad \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon (1 + q^2) \varepsilon \gamma_j^{\nu'}.$$

Finally, (8.369), (8.371) and (8.372) imply in the case $r = 1$ and $s = 0$:

$$(8.373) \quad \|h_{1,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim (1 + q^2) |\nu - \nu'| 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we consider the case $r = 0$ and $s = 1$. We decompose $h_{1,p,q}$ as:

$$(8.374) \quad h_{1,p,q} = h_{1,p,q,1} + h_{1,p,q,2} + h_{1,p,q,3},$$

where $h_{1,p,q,1}$, $h_{1,p,q,2}$ and $h_{1,p,q,3}$ are given respectively by

$$(8.375) \quad h_{1,p,q,1} = \left(\int_{\mathbb{S}^2} \chi L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right)$$

$$\times \left(\int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \quad (8.376)$$

$$h_{1,p,q,2} = \left(\int_{\mathbb{S}^2} \chi L(P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and

$$(8.377) \quad h_{1,p,q,3} = \left(\int_{\mathbb{S}^2} \chi L(\operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

Next, we estimate the $L^1(\mathcal{CM})$ norm of $h_{1,p,q,1}$, $h_{1,p,q,2}$ and $h_{1,p,q,3}$ starting with $h_{1,p,q,1}$. In view of the Definition (8.375) of $h_{1,p,q,1}$, we have

$$(8.378) \quad \|h_{1,p,q,1}\|_{L^1(\mathcal{CM})} \lesssim \left\| \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{CM})} \\ \times \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{CM})}.$$

We estimate the two terms in the right-hand side of (8.369) starting with the first one. Proceeding as for the proof of (8.371), and noticing that the only difference is the missing factor of $b - b_{\nu'}$, we obtain:

$$(8.379) \quad \left\| \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{CM})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu}.$$

Next, we estimate the second term in the right-hand side of (8.369). The analog of (8.247) yields:

$$(8.380) \quad \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{CM})} \lesssim \frac{\varepsilon 2^{-\frac{j}{2}} \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|}.$$

(8.378), (8.379) and (8.380) imply:

$$(8.381) \quad \|h_{1,p,q,1}\|_{L^1(\mathcal{CM})} \lesssim \frac{\varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|}.$$

Next, we estimate the $L^1(\mathcal{CM})$ norm of $h_{1,p,q,2}$. In view of the Definition (8.376) of $h_{1,p,q,2}$, we have

$$h_{1,p,q,2} = \int_{\mathbb{S}^2} \chi H L(P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega,$$

where H is given by

$$H = \int_{\mathbb{S}^2} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

This yields

(8.382)

$$\begin{aligned} & \|h_{1,p,q,2}\|_{L^1(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \left\| \chi H L(P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|\chi\|_{L_{u,x'}^{\infty} L_t^2} \|H\|_{L_{u,x'}^2 L_t^{\infty}} \|L(P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi)\|_{L_u^{\infty} L^2(\mathcal{M}_u)} \left\| \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L^{\infty}} \\ & \quad \times \|F_{j,-1}(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega \\ & \lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}} \int_{\mathbb{S}^2} \|H\|_{L_{u,x'}^2 L_t^{\infty}} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ , the estimate (2.42) for $\partial_{\omega} N$, the estimate (8.142) for $L(P_{>2^j|\nu-\nu'|} \operatorname{tr} \chi)$, and the size of the patch. In view of the definition of H and the estimate (7.95), we have

$$\|H\|_{L_{u,x'}^2 L_t^{\infty}} \lesssim 2^{-\frac{j}{4}} (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| + 1) \gamma_j^{\nu'},$$

which together with (8.382) implies

$$\|h_{1,p,q,2}\|_{L^1(\mathcal{M})} \lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}} 2^{-\frac{j}{4}} (1 + q^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| + 1) \gamma_j^{\nu'} \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^{\nu'}(\omega) d\omega.$$

Taking Plancherel in λ , Cauchy-Schwarz in ω and using the size of the patch, we finally obtain:

$$(8.383) \quad \|h_{1,p,q,2}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,3}$. In view of the Raychaudhuri Equation (2.22) satisfied by $\operatorname{tr} \chi$, the worst term in $\chi L(\operatorname{tr} \chi)$ is of the form $\widehat{\chi}^3$. In view of the decomposition (2.78) for $\widehat{\chi}^3$, we obtain the following decomposition for $\chi L(\operatorname{tr} \chi)$:

$$(8.384) \quad \begin{aligned} \chi L(\operatorname{tr} \chi) &= \chi_{2\nu}^3 + \chi_{2\nu}^2 F_1^j + \chi_{2\nu}^2 F_2^j + \chi_{2\nu} F_3^j + \chi_{2\nu} F_4^j \\ &\quad + \chi_{2\nu} F_5^j + F_6^j + F_7^j + F_8^j + F_9^j, \end{aligned}$$

where F_1^j , F_3^j and F_6^j only depend on (t, x) and ν and satisfy:

$$(8.385) \quad \|F_1^j\|_{L_{u\nu}^{\infty} L_t^2 L^{\infty}(P_{t,u\nu})} + \|F_3^j\|_{L_{u\nu}^{\infty} L_t^2 L^{\infty}(P_{t,u\nu})} + \|F_6^j\|_{L_{u\nu}^{\infty} L_t^2 L^{\infty}(P_{t,u\nu})} \lesssim \varepsilon,$$

where F_2^j , F_4^j and F_7^j satisfy:

$$(8.386) \quad \|F_2^j\|_{L_u^{\infty} L^2(\mathcal{M}_u)} + \|F_4^j\|_{L_u^{\infty} L^2(\mathcal{M}_u)} + \|F_7^j\|_{L_u^{\infty} L^2(\mathcal{M}_u)} \lesssim 2^{-\frac{j}{2}} \varepsilon,$$

where F_5^j and F_8^j satisfy

$$(8.387) \quad \|F_5^j\|_{L^2(\mathcal{M})} + \|F_8^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}$$

and where F_9^j satisfies

$$(8.388) \quad \|F_9^j\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{3j}{2}}.$$

In view of the Definition (8.377) of $h_{1,p,q,3}$, this yields the following decomposition

$$(8.389)$$

$$h_{1,p,q,3} = h_{1,p,q,3,1} + h_{1,p,q,3,2} + h_{1,p,q,3,3} + h_{1,p,q,3,4} + h_{1,p,q,3,5} + h_{1,p,q,3,6} + h_{1,p,q,3,7},$$

where $h_{1,p,q,3,1}$, $h_{1,p,q,3,2}$, $h_{1,p,q,3,3}$, $h_{1,p,q,3,4}$, $h_{1,p,q,3,5}$, $h_{1,p,q,3,6}$ and $h_{1,p,q,3,7}$ are given by:

$$(8.390)$$

$$h_{1,p,q,3,1} = (\chi_{2\nu}^3 + \chi_{2\nu}^2 F_1^j + \chi_{2\nu} F_3^j + F_6^j) \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.391)$$

$$h_{1,p,q,3,2} = \chi_{2\nu}^2 \left(\int_{\mathbb{S}^2} F_2^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.392)$$

$$h_{1,p,q,3,3} = \chi_{2\nu} \left(\int_{\mathbb{S}^2} F_4^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.393)$$

$$h_{1,p,q,3,4} = \chi_{2\nu} \left(\int_{\mathbb{S}^2} F_5^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.394)$$

$$h_{1,p,q,3,5} = \left(\int_{\mathbb{S}^2} F_7^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.395)$$

$$h_{1,p,q,3,6} = \left(\int_{\mathbb{S}^2} F_8^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H$$

and

$$(8.396)$$

$$h_{1,p,q,3,7} = \left(\int_{\mathbb{S}^2} F_9^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

with H given by:

$$(8.397)$$

$$H = \int_{\mathbb{S}^2} \text{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_{j'}^{\nu'}(\omega') d\omega'.$$

Using the basic estimate (7.1), we have for $n = 2, 4, 7$

$$(8.398)$$

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} F_n^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| F_n^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{M}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \end{aligned}$$

$$\begin{aligned} &\lesssim \left(\sup_{\omega} \|F_n^j\|_{L_u^\infty L^2(\mathcal{E}_u)} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.386), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Also, we have for $n = 5, 8$

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} F_n^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \int_{\mathbb{S}^2} \left\| F_n^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \right\|_{L^2(\mathcal{O}\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|F_n^j\|_{L^2(\mathcal{O}\mathcal{M})} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P \right\|_{L^\infty} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega \\ &\lesssim 2^{-j} \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimate (8.387), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Taking Cauchy-Schwarz both in λ and ω and using the size of the patch, we obtain for $n = 5, 8$:

$$(8.399) \quad \left\| \int_{\mathbb{S}^2} F_n^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu.$$

Also, we have

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} F_9^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2-}(\mathcal{O}\mathcal{M})} \\ &\lesssim \int_{\mathbb{S}^2} \left\| F_9^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \right\|_{L^{2-}(\mathcal{O}\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|F_9^j\|_{L^{2-}(\mathcal{O}\mathcal{M})} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P \right\|_{L^\infty} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega \\ &\lesssim 2^{-\frac{3j}{2}} \varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L_u^\infty} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimate (8.388), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Taking Cauchy-Schwarz both in λ and ω and using the size of the patch, we obtain:

$$(8.400) \quad \left\| \int_{\mathbb{S}^2} F_9^j \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2-}(\mathcal{O}\mathcal{M})} \lesssim 2^{-\frac{j}{2}} \varepsilon \gamma_j^\nu.$$

Finally, let G be given by

$$G = \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}}(N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega.$$

Then, we have in view of Lemma 7.6,

$$(8.401) \quad \|G\|_{L^2_{u,\omega'} L^{\infty}_t} \lesssim (1 + p^2) \gamma_j^\nu.$$

(8.389)–(8.400) imply

(8.402)

$$\begin{aligned} & \|h_{1,p,q,3}\|_{L^1(\mathcal{M})} \\ & \lesssim \|(\chi_{2\nu}^2, \chi_{2\nu}, 1)H\|_{L^2(\mathcal{M})} (\|(\chi_{2\nu}, F_1^j, F_3^j, F_6^j)G\|_{L^2(\mathcal{M})} + \varepsilon\gamma_j^{\nu'}) + \|H\|_{L^2+(\mathcal{M})} 2^{-\frac{j}{2}}\varepsilon\gamma_j^{\nu'} \\ & \lesssim (\|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})} + \|\chi_{2\nu}^2\|_{L^6(\mathcal{M})} \|H\|_{L^3(\mathcal{M})} + \|H\|_{L^2(\mathcal{M})}) \\ & \quad \times (\|\chi_{2\nu}, F_1^j, F_3^j, F_6^j\|_{L_{u\nu, x', \nu}^\infty L_{\tilde{t}}^\infty} \|G\|_{L_{u\nu, x', \nu}^2 L_{\tilde{t}}^\infty} + \varepsilon\gamma_j^{\nu'}) + \|H\|_{L^2+(\mathcal{M})} 2^{-\frac{j}{2}}\varepsilon\gamma_j^{\nu'} \\ & \lesssim (\|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})} + \|H\|_{L^3(\mathcal{M})} + \|H\|_{L^2(\mathcal{M})})(1 + p^2)\varepsilon\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.46) for $\chi_{2\nu}$, the estimate (8.385) for F_1^j , F_3^j and F_6^j , and the estimate (8.401) for G . Next, we estimate H . In view of its Definition (8.397), we have from (7.103)

$$(8.403) \quad \|H\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{j}{4}}\varepsilon(1 + q^2)\gamma_j^{\nu'}.$$

Also, we have

$$\begin{aligned} \|H\|_{L^\infty(\mathcal{M})} & \lesssim \int_{\mathbb{S}^2} \left\| \text{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^\infty} \eta_j^{\nu'}(\omega') d\omega' \\ & \lesssim 2^{-\frac{j}{2}}\varepsilon \int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L_u^\infty} \eta_j^{\nu'}(\omega') d\omega', \end{aligned}$$

where we used in the last inequality the estimate (2.39) for $\text{tr}\chi'$, the estimate (2.44) for $\partial_\omega b'$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Taking Cauchy-Schwarz in λ' and ω' , and using the size of the patch, we obtain

$$(8.404) \quad \|H\|_{L^\infty(\mathcal{M})} \lesssim 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'}.$$

In particular, interpolating (8.403) and (8.404), we obtain

$$\|H\|_{L^3(\mathcal{M})} \lesssim (1 + q^2)^{\frac{2}{3}}\varepsilon\gamma_j^{\nu'},$$

which together with (8.402) and (8.403) implies

$$(8.405) \quad \|h_{1,p,q,3}\|_{L^1(\mathcal{M})} \lesssim \left(\|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})} + (1 + q^2)^{\frac{2}{3}}\varepsilon\gamma_j^{\nu'} \right) (1 + p^2)\varepsilon\gamma_j^{\nu'}.$$

Thus, in view of (8.405), it remains to estimate $\|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})}$. We decompose $\chi_{2\nu}^2 H$:

$$(8.406) \quad \chi_{2\nu}^2 H = \chi_{2\nu} H_1 + H_2 + H_3,$$

where H_1 , H_2 and H_3 are given by

$$(8.407) \quad H_1 = \int_{\mathbb{S}^2} (\chi_{2\nu} - \chi_{2\nu}') \text{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

$$(8.408) \quad H_2 = \int_{\mathbb{S}^2} (\chi_{2\nu} - \chi_{2\nu}') \chi_{2\nu}' \text{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'$$

and

$$(8.409) \quad H_3 = \int_{\mathbb{S}^2} \chi_{2\nu}'^2 \text{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

In view of (8.406), we have

$$\begin{aligned}
 (8.410) \quad \|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})} &\lesssim \|\chi_{2\nu} H_1\|_{L^2(\mathcal{M})} + \|H_2\|_{L^2(\mathcal{M})} + \|H_3\|_{L^2(\mathcal{M})} \\
 &\lesssim \|\chi_{2\nu}\|_{L^6(\mathcal{M})} \|H_1\|_{L^3(\mathcal{M})} + \|H_2\|_{L^2(\mathcal{M})} + \|H_3\|_{L^2(\mathcal{M})} \\
 &\lesssim \varepsilon \|H_1\|_{L^3(\mathcal{M})} + \|H_2\|_{L^2(\mathcal{M})} + \|H_3\|_{L^2(\mathcal{M})},
 \end{aligned}$$

where we used in the last inequality the estimate (2.46) for $\chi_{2\nu}$. Next, we estimate each term in the right-hand side of (8.410) starting with the first one. In view of the Definition (8.407) of H_1 and the estimate (7.2), we have

$$\begin{aligned}
 \|H_1\|_{L^3(\mathcal{M})} &\lesssim \left(\sup_{\omega'} \left\| (\chi_{2\nu} - \chi_{2\nu}') \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^{\infty} L^3(\mathcal{E}_{\omega'})} \right) 2^{\frac{2j}{3}} \gamma_j^{\nu'} \\
 &\lesssim \left(\sup_{\omega'} \|\chi_{2\nu} - \chi_{2\nu}'\|_{L_{\omega'}^{\infty} L^3(\mathcal{E}_{\omega'})} \left\| \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^{\infty}} \right) 2^{\frac{2j}{3}} \gamma_j^{\nu'} \\
 (8.411) \quad &\lesssim |\nu - \nu'| 2^{\frac{j}{6}} \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate (2.75) for $\chi_{2\nu} - \chi_{2\nu}'$, the estimate (2.39) for $\operatorname{tr} \chi'$, the estimate (2.44) for $\partial_{\omega} b$, the estimate (2.42) for $\partial_{\omega} N$, and the size of the patch. Next, we estimate H_2 . In view of the Definition (8.408) of H_2 and the estimate (7.1), we have

$$\begin{aligned}
 (8.412) \quad \|H_2\|_{L^2(\mathcal{M})} &\lesssim \left(\sup_{\omega'} \left\| (\chi_{2\nu} - \chi_{2\nu}') \chi_{2\nu}' \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^{\infty} L^2(\mathcal{E}_{\omega'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
 &\lesssim \left(\sup_{\omega'} \|\chi_{2\nu} - \chi_{2\nu}'\|_{L_{\omega'}^{\infty} L^3(\mathcal{E}_{\omega'})} \|\chi_{2\nu}'\|_{L_{\omega'}^{\infty} L^6(\mathcal{E}_{\omega'})} \left\| \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^{\infty}} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
 &\lesssim |\nu - \nu'| \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate (2.75) for $\chi_{2\nu} - \chi_{2\nu}'$, the estimate (2.46) for $\chi_{2\nu}'$, the estimate (2.39) for $\operatorname{tr} \chi'$, the estimate (2.44) for $\partial_{\omega} b$, the estimate (2.42) for $\partial_{\omega} N$, and the size of the patch. Next, we estimate H_3 . In view of the Definition (8.409) of H_3 and the estimate (7.1), we have

$$\begin{aligned}
 (8.413) \quad \|H_3\|_{L^2(\mathcal{M})} &\lesssim \left(\sup_{\omega'} \left\| \chi_{2\nu}'^2 \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^{\infty} L^2(\mathcal{E}_{\omega'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
 &\lesssim \left(\sup_{\omega'} \|\chi_{2\nu}'\|_{L_{\omega'}^{\infty} L^4(\mathcal{E}_{\omega'})}^2 \left\| \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^{\infty}} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
 &\lesssim \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate (2.46) for $\chi_{2\nu}'$, the estimate (2.39) for $\operatorname{tr} \chi'$, the estimate (2.44) for $\partial_{\omega} b$, the estimate (2.42) for $\partial_{\omega} N$, and the size of the patch. Finally, (8.410)–(8.413) yield

$$\|\chi_{2\nu}^2 H\|_{L^2(\mathcal{M})} \lesssim (1 + |\nu - \nu'| 2^{\frac{j}{6}}) \varepsilon \gamma_j^{\nu'}.$$

In view (8.405), we obtain:

$$\|h_{1,p,q,3}\|_{L^1(\mathcal{M})} \lesssim (|\nu - \nu'|2^{\frac{j}{2}} + (1 + q^2)^{\frac{2}{3}}) (1 + p^2)\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'},$$

which together with (8.374), (8.381), (8.383), and the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, implies in the case $r = 0$ and $s = 1$

$$(8.414) \quad \begin{aligned} \|h_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \|h_{1,p,q,1}\|_{L^1(\mathcal{M})} + \|h_{1,p,q,2}\|_{L^1(\mathcal{M})} + \|h_{1,p,q,3}\|_{L^1(\mathcal{M})} \\ &\lesssim (1 + p^2)(1 + q^{\frac{5}{2}})(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}. \end{aligned}$$

Using (8.373) in the case $r = 1$ and $s = 0$, and (8.414) in the case $r = 0$ and $s = 1$, together with the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, we finally obtain

$$(8.415) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2)(1 + q^{\frac{5}{2}})2^{\frac{j}{2}}|\nu - \nu'|\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q}$. We start with the case $r = 1$ and $s = 0$ which is easier. In view of Definition (8.368), we have:

$$(8.416) \quad \begin{aligned} \|h_{2,p,q}\|_{L^2(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \chi' P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

We estimate the two terms in the right-hand side of (8.369) starting with the first one. The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.417) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}}\gamma_j^\nu \\ &\lesssim |\nu - \nu'| \left(\sup_{\omega} \|L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}}\gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Now, the estimate (8.184) yields:

$$(8.418) \quad \|L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} \lesssim \varepsilon.$$

Together with (8.417), we obtain:

$$(8.419) \quad \left\| \int_{\mathbb{S}^2} L(P_{\leq 2^j|\nu - \nu'|}\text{tr}\chi)(b - b_{\nu'}) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon|\nu - \nu'|2^{\frac{j}{2}}\gamma_j^\nu.$$

Next, we estimate the second term in the right-hand side of (8.416). Recall the decomposition (2.76) for χ' :

$$(8.420) \quad \chi' = \chi_{2\nu'} + F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν' and satisfies for any $2 \leq p < +\infty$:

$$(8.421) \quad \|F_1^j\|_{L_{u,\nu'}^\infty, L_t^p L_{x,\nu'}^\infty} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(8.422) \quad \|F_2^j\|_{L_{u,\nu'}^\infty L^2(\mathcal{M}_{u,\nu'})} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

The decomposition (8.420) implies:

$$(8.423) \quad \left\| \int_{\mathbb{S}^2} \chi' P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\|\chi_{2\nu'}\|_{L_{u,\nu'}^\infty, x', L_t^2} + \|F_1^j\|_{L_{u,\nu'}^\infty, x', L_t^2} \right) \\ \times \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu'}^2, x', L_t^\infty} \\ + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \varepsilon \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu'}^2, x', L_t^\infty} \\ + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

where we used in the last inequality the estimate (8.421) for F_1^j and the estimate (2.46) for $\chi_{2\nu'}$. Now, in view of the estimate (7.71), we have for $m > j/2$:

$$(8.424) \quad \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu'}^2, x', L_t^\infty} \\ \lesssim \left(\sup_{\omega'} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \varepsilon (2^{-l+\frac{j}{2}} + 2^{-\frac{l}{2}+\frac{j}{4}}) \gamma_j^{\nu'} \\ \lesssim \varepsilon (2^{-l+\frac{j}{2}} + 2^{-\frac{l}{2}+\frac{j}{4}}) \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Also, in view of the estimate (7.83), we have:

$$(8.425) \quad \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu'}^2, x', L_t^\infty} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'}.$$

(8.424), (8.425), the fact that $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$, and the fact that:

$$P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' = \operatorname{tr} \chi' - \sum_{m/2^m > 2^j |\nu - \nu'|} P_m \operatorname{tr} \chi'$$

implies:

$$(8.426) \quad \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x',L_t}^\infty} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'}.$$

On the other hand, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.427) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|F_2^j\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \left\| P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty(\mathcal{M})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.422) for F_2^j , the boundedness of $P_{\leq 2^j |\nu - \nu'|}$ on $L^\infty(P_{t,u})$, the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Now, (8.423), (8.426) and (8.427) imply:

$$(8.428) \quad \left\| \int_{\mathbb{S}^2} \chi' P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'}.$$

Finally, (8.416), (8.419) and (8.428) imply in the case $r = 1$ and $s = 0$:

$$(8.429) \quad \|h_{2,p,q}\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}} |\nu - \nu'| (1 + q^{\frac{5}{2}}) \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we estimate $h_{2,p,q}$ in the case $r = 0$ and $s = 1$. In view of the Definition (8.368), we may decompose $h_{2,p,q}$ as:

$$(8.430) \quad h_{2,p,q} = h_{2,p,q,1} + h_{2,p,q,2} + h_{2,p,q,3},$$

where $h_{2,p,q,1}$, $h_{2,p,q,2}$ and $h_{2,p,q,3}$ are given respectively by

$$(8.431) \quad \begin{aligned} h_{2,p,q,1} &= \left(\int_{\mathbb{S}^2} L(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \chi' P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

$$(8.432) \quad \begin{aligned} h_{2,p,q,2} &= \left(\int_{\mathbb{S}^2} L(P_{> 2^j |\nu - \nu'|} \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \chi' \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and

(8.433)

$$h_{2,p,q,3} = \left(\int_{\mathbb{S}^2} L(\operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \chi' \operatorname{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,1}$, $h_{2,p,q,2}$ and $h_{2,p,q,3}$ starting with $h_{2,p,q,1}$. In view of the Definition (8.431) of $h_{2,p,q,1}$, we have

(8.434)

$$\|h_{2,p,q,1}\|_{L^1(\mathcal{M})} \\ \lesssim \left\| \int_{\mathbb{S}^2} L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} \chi' P_{> 2^j|\nu-\nu'|} \operatorname{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

We estimate the two terms in the right-hand side of (8.434) starting with the first one. The basic estimate (7.1) yields

$$(8.435) \quad \left\| \int_{\mathbb{S}^2} L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega'} \left\| L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ \lesssim \left(\sup_{\omega'} \left\| L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi) \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$, the size of the patch, and the following estimate

$$\|L(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon,$$

which follows from the estimates (8.305) and (8.306). Next, we estimate the second term in the right-hand side of (8.434). Using the basic estimate (7.1), we have

(8.436)

$$\left\| \int_{\mathbb{S}^2} \chi' P_{> 2^j|\nu-\nu'|} \operatorname{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega'} \left\| \chi' P_{> 2^j|\nu-\nu'|} \operatorname{tr}\chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ \lesssim \sum_{2^{j'} > 2^j \nu - \nu'} \left(\sup_{\omega'} \|\chi'\|_{L_{u',x',t'}^\infty L_{t'}^2} \|P_{t'} \operatorname{tr}\chi'\|_{L_{u',x',t'}^\infty L_{t'}^\infty} \left\| (b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}$$

$$\begin{aligned} &\lesssim \left(\sum_{2^l > 2^j |\nu - \nu'|} 2^{-l} \right) \varepsilon \gamma_j^{\nu'} \\ &\lesssim \frac{2^{-\frac{j}{2}} \varepsilon}{(2^j |\nu - \nu'|)^{\frac{1}{2}}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ' , the estimate (2.69) for $P_l \text{tr} \chi'$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. (8.434), (8.435) and (8.436) imply:

$$(8.437) \quad \|h_{2,p,q,1}\|_{L^1(\mathcal{M})} \lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,2}$. In view of the Definition (8.432) of $h_{2,p,q,2}$, we have

$$(8.438) \quad \begin{aligned} \|h_{2,p,q,2}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \chi' \text{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

We estimate both terms in the right-hand side of (8.438) starting with the first one. In view of (8.159), we have

$$(8.439) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sum_{2^l > 2^j |\nu - \nu'|} 2^{\frac{j}{2} - \frac{l}{2}} \right) \varepsilon \gamma_j^\nu \\ &\lesssim \frac{2^{\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} \varepsilon \gamma_j^\nu. \end{aligned}$$

For the second term in the right-hand side of (8.438), we use the decomposition (2.76) for χ' which yields

$$(8.440) \quad \chi' = F_1^j + F_j^2,$$

where the tensor F_1^j depends only on (t, x) and ν' and satisfies

$$(8.441) \quad \|F_1^j\|_{L_{u,\nu',x',\nu'}^\infty L_t^2} \lesssim \varepsilon$$

and where the tensor F_j^2 satisfies

$$(8.442) \quad \|F_j^2\|_{L_{u'}^\infty L^2(\mathcal{M}_{u'})} \lesssim 2^{-\frac{j}{2}} \varepsilon.$$

Using the decomposition (8.440), we have

$$\begin{aligned} & \int_{\mathbb{S}^2} \chi' \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \\ &= F_1^j \left(\int_{\mathbb{S}^2} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \\ & \quad + \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'. \end{aligned}$$

This yields

(8.443)

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} \chi' \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \|F_1^j\|_{L^\infty_{u',x',\nu'}, L^2_t} \left\| \int_{\mathbb{S}^2} \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u',x',\nu'}, L^\infty_t} \\ & \quad + \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{-\frac{j}{4}} (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'} + \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.441) for F_1^j and the estimate (7.95). The basic estimate (7.1) yields

$$\begin{aligned} (8.444) \quad & \left\| \int_{\mathbb{S}^2} F_2^j \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| F_2^j \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty_{\omega'} L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|F_2^j\|_{L^\infty_{\omega'} L^2(\mathcal{H}_{u'})} \left\| \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim 2^{-\frac{j}{2}} \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.442) for F_2^j , the estimate (2.39) for $\operatorname{tr} \chi'$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Finally, (8.443) and (8.444) yield

(8.445)

$$\left\| \int_{\mathbb{S}^2} \chi' \operatorname{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{j}{4}} (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'}.$$

(8.438), (8.439) and (8.445), together with the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, imply

$$(8.446) \quad \|h_{2,p,q,2}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,3}$. Recall the decomposition (8.64) (8.65) (8.66) for $L(\text{tr}\chi)$. We have:

$$(8.447) \quad L(\text{tr}\chi) = \chi_{2\nu} \cdot (2\chi_1 + \widehat{\chi}) + f_1^j + f_2^j,$$

where the scalar f_1^j only depends on ν and satisfies:

$$(8.448) \quad \|f_1^j\|_{L_{u\nu}^\infty L_t^2 L^\infty(P_{t,u\nu})} \lesssim \varepsilon,$$

where the scalar f_2^j satisfies:

$$(8.449) \quad \|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

In view of the Definition (8.433) of $h_{2,p,q,3}$, this yields the following decomposition

$$(8.450) \quad h_{2,p,q,3} = h_{2,p,q,3,1} + h_{2,p,q,3,2} + h_{2,p,q,3,3},$$

where $h_{2,p,q,3,1}$, $h_{2,p,q,3,2}$ and $h_{2,p,q,3,3}$ are given by:

$$(8.451) \quad h_{2,p,q,3,1} = \chi_{2\nu} \left(\int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

$$(8.452) \quad h_{2,p,q,3,2} = f_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H$$

and

$$(8.453) \quad h_{2,p,q,3,3} = \left(\int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) H,$$

with H given by:

$$(8.454) \quad H = \int_{\mathbb{S}^2} \chi' \text{tr} \chi'(b_{\nu'} - b') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

Using the basic estimate (7.1), we have

$$(8.455) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} f_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| f_2^j \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ & \lesssim \left(\sup_{\omega} \left\| f_2^j \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^\nu \\ & \lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.449), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Also, we have

$$(8.456) \quad \begin{aligned} & \left\| f_1^j \left(\int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \\ & \lesssim \|f_1^j\|_{L_{u\nu}^\infty L_t^2 L^\infty(P_{t,u\nu})} \left\| \int_{\mathbb{S}^2} \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L_{u\nu, \nu'}^2 L_t^2} \\ & \lesssim (1 + p^2) \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.448) for f_1^j , and Lemma 7.6. Also, recall (8.72):

$$(8.457) \quad \left\| \int_{\mathbb{S}^2} (2\chi_1 + \widehat{\chi}) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu.$$

(8.450)–(8.457) imply

$$(8.458) \quad \|h_{2,p,q,3}\|_{L^1(\mathcal{M})} \lesssim (\|\chi_{2\nu} H\|_{L^2(\mathcal{M})} + \|H\|_{L^2(\mathcal{M})}) (1 + p^2) \gamma_j^{\nu'}.$$

Next, we estimate H . In view of the Definition (8.454) of H , the analog of (8.72) yields

$$(8.459) \quad \|H\|_{L^2(\mathcal{M})} \lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'}.$$

Also, in view of the Definition (8.454) of H , we have

$$\chi_{2\nu} H = H_2 + H_3,$$

where H_2 and H_3 have been defined respectively in (8.408) and (8.409). We deduce

$$(8.460) \quad \begin{aligned} \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} &\lesssim \|H_2\|_{L^2(\mathcal{M})} + \|H_3\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.412) for H_2 and the estimate (8.413) for H_3 . (8.458), (8.459) and (8.460) imply

$$\|h_{2,p,q,3}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) (1 + q^2) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

which together with (8.430), (8.437), (8.446), and the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, implies in the case $r = 0$ and $s = 1$

$$(8.461) \quad \begin{aligned} \|h_{2,p,q}\|_{L^1(\mathcal{M})} &\lesssim \|h_{2,p,q,1}\|_{L^1(\mathcal{M})} + \|h_{2,p,q,2}\|_{L^1(\mathcal{M})} + \|h_{2,p,q,3}\|_{L^1(\mathcal{M})} \\ &\lesssim (1 + p^2) (1 + q^{\frac{5}{2}}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Using (8.429) in the case $r = 1$ and $s = 0$, and (8.461) in the case $r = 0$ and $s = 1$, together with the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, we finally obtain

$$(8.462) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) (1 + q^{\frac{5}{2}}) 2^{\frac{j}{2}} |\nu - \nu'| \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, in view of the decomposition (8.366), we have:

$$\begin{aligned} &\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{1,2,2,3} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \right| \\ &\lesssim \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+4}} \right\|_{L^\infty(\mathcal{M})} [\|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h_{2,p,q}\|_{L^1(\mathcal{M})}], \end{aligned}$$

which together with (8.32), (8.415) and (8.462) implies:

$$(8.463) \quad \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{1,2,2,3} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \right|$$

$$\begin{aligned} &\lesssim \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q+4}} (1+p^2)(1+q^{\frac{5}{2}}) 2^{\frac{j}{2}} |\nu - \nu'| \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \frac{\varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3}. \end{aligned}$$

In view of (8.256), we have in the range $2^{\max(l,m)} \leq 2^j |\nu - \nu'|$:

$$\begin{aligned} &\left| B_{j,\nu,\nu',l,m}^{1,2,2} - (B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \right| \\ &\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \\ &\quad \times \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|^2} \right\|_{L^\infty(\mathcal{M})} \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} + \left\| \frac{1}{|N_\nu - N_{\nu'}|^3} \right\|_{L^\infty(\mathcal{M})} \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \right]. \end{aligned}$$

Together with (8.32), (8.264) and (8.282), we obtain:

$$\begin{aligned} &\left| B_{j,\nu,\nu',l,m}^{1,2,2} - (B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,3}) \right| \\ &\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q}} \left[\frac{1}{|\nu - \nu'|^2} ((1+p^2)2^{\frac{j}{11}}|\nu - \nu'| + 2^{\frac{j}{2}}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \right. \\ &\quad \left. + \frac{1}{|\nu - \nu'|^3} (1+q^2)(2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|) + 2^{\frac{7j}{16}}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \right] \\ &\lesssim \left[\frac{2^{-\frac{10j}{11}}}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{2^{-\frac{j}{16}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Together with (8.284), (8.364) and (8.463), we finally obtain:

$$\begin{aligned} &\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu',\nu,l,m}^{1,2,2}) \right| \\ &\lesssim \left| \sum_{(l,m)/2^{\max(l,m)}} (B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu',\nu,l,m}^{1,2,2,1}) \right| + \left| \sum_{(l,m)/2^{\max(l,m)}} (B_{j,\nu,\nu',l,m}^{1,2,2,2} + B_{j,\nu',\nu,l,m}^{1,2,2,2}) \right| \\ &\quad + \left| \sum_{(l,m)/2^{\max(l,m)}} (B_{j,\nu,\nu',l,m}^{1,2,2,3} + B_{j,\nu',\nu,l,m}^{1,2,2,3}) \right| + \left[\frac{j2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{j2^{-\frac{j}{16}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \left[2^{-j} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{4})-j}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Together with (8.255), we obtain the estimate on the whole range $2^{\min(l,m)} \leq 2^j |\nu - \nu'|$:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu',\nu,l,m}^{1,2,2}) \right|$$

$$\lesssim \left[2^{-j} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{1}{4})-j}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

This concludes the proof of Proposition 8.11.

8.2.3. Proof of Proposition 8.12 (Control of $B_{j,\nu,\nu',l,m}^{1,2,3}$). — Recall that $B_{j,\nu,\nu',l,m}^{1,2,3}$ is defined by (8.96):

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{1,2,3} &= -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ &\quad \times b'^{-1} (1 - \mathbf{g}(N, N')) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \end{aligned}$$

Together with the identity (8.30):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N'),$$

this yields:

$$(8.464) \quad B_{j,\nu,\nu',l,m}^{1,2,3} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b'^{-1} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ \times F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Recall (8.1):

$$m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.$$

We first consider the range of (l, m) such that:

$$2^m \leq 2^j |\nu - \nu'| < 2^l.$$

This yields:

$$(8.465) \quad \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,3} = -i2^{-j-1} \int_{\mathcal{M}} (h_1 + h_2) d\mathcal{M},$$

where the scalar functions h_1, h_2 on \mathcal{M} are given by:

(8.466)

$$h_1 = \left(\int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr}\chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \left(\int_{\mathbb{S}^2} b'^{-1} P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

(8.467)

$$h_2 = \left(\int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr}\chi F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \left(\int_{\mathbb{S}^2} b'^{-1} L'(P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi') F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

Next, we estimate the $L^1(\mathcal{M})$ norm of h_1 and h_2 starting with h_1 . In view of the definition of h_1 (8.466), we have:

$$(8.468) \quad \|h_1\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr}\chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} b'^{-1} P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \frac{2^{\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where we used in the last inequality the estimates (8.238) and (8.372).

Next, we estimate the $L^1(\mathcal{M})$ norm of h_2 . In view of the definition of h_2 (8.467), we have:

$$(8.469) \quad \begin{aligned} \|h_2\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \text{tr} \chi F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} b'^{-1} L'(P_{\leq 2^j|\nu-\nu'|} \text{tr} \chi') F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \frac{2^{\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu - \nu'|)} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimates (8.129) and (8.435).

Finally, (8.465), (8.468) and (8.469) imply the following estimate in the range of (l, m) such that $2^m \leq 2^j|\nu - \nu'| < 2^l$:

$$(8.470) \quad \left| \sum_{(l,m)/2^m \leq 2^j|\nu-\nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,3} \right| \lesssim 2^{-j} (\|h_1\|_{L^1(\mathcal{M})} + \|h_2\|_{L^1(\mathcal{M})}) \\ \lesssim \left[\frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}}|\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,3}$ in the range of (l, m) such that:

$$2^m \leq 2^l \leq 2^j|\nu - \nu'|.$$

We have the following decomposition for $B_{j,\nu,\nu',l,m}^{1,2,3}$:

$$(8.471) \quad B_{j,\nu,\nu',l,m}^{1,2,3} = B_{j,\nu,\nu',l,m}^{1,2,3,1} + B_{j,\nu,\nu',l,m}^{1,2,3,2}$$

where $B_{j,\nu,\nu',l,m}^{1,2,3,1}$ and $B_{j,\nu,\nu',l,m}^{1,2,3,2}$ are given by:

$$(8.472) \quad B_{j,\nu,\nu',l,m}^{1,2,3,1} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b'^{-1} L(P_l \text{tr} \chi) P_m \text{tr} \chi' F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}$$

and

$$(8.473) \quad B_{j,\nu,\nu',l,m}^{1,2,3,2} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b'^{-1} P_l \text{tr} \chi L'(P_m \text{tr} \chi') F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

We first estimate $B_{j,\nu,\nu',l,m}^{1,2,3,1}$. We integrate by parts tangentially using (7.136).

Lemma 8.16. — Let $B_{j,\nu,\nu',l,m}^{1,2,3,1}$ defined by (8.472). Integrating by parts using (7.136) yields:

$$(8.474) \quad B_{j,\nu,\nu',l,m}^{1,2,3,1} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_{\nu} - N_{\nu'}|)^{p+q}} \left[\frac{1}{|N_{\nu} - N_{\nu'}|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) + \frac{1}{|N_{\nu} - N_{\nu'}|} (h_{3,p,q,l,m} + h_{4,p,q,l,m}) + h_{5,p,q,l,m} \right] d\mathcal{M},$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$ on \mathcal{M} are given by:

$$(8.475) \quad h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} \chi L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.476) \quad h_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} \chi' P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.477) \quad h_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.478) \quad h_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

$$(8.479) \quad h_{5,p,q,l,m} = \left(\int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \times \left(\int_{\mathbb{S}^2} N' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and where the tensors G_1 and G_2 on \mathcal{M} are given by:

$$(8.480) \quad G_1 = \nabla(L(P_l \text{tr} \chi)) + (\theta + b^{-1} \nabla(b)) L(P_l \text{tr} \chi)$$

and:

$$(8.481) \quad G_2 = \nabla'(P_m \operatorname{tr} \chi) + (\theta' + b'^{-1} \nabla(b')) P_m \operatorname{tr} \chi'.$$

The proof of Lemma 8.16 is postponed to Appendix F. In the rest of this section, we use Lemma 8.16 to obtain the control of $B_{j,\nu,\nu',l,m}^{1,2,3,1}$.

We estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}$ starting with $h_{1,p,q,l,m}$. In view of the Definition (8.475) of $h_{1,p,q,l,m}$, we have:

$$(8.482) \quad \begin{aligned} \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \chi L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \chi L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.181). Now, the analog of the estimate (8.371) with $r = 0$ yields:

$$(8.483) \quad \left\| \int_{\mathbb{S}^2} \chi L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu,$$

which together with (8.482) implies:

$$(8.484) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of the Definition (8.476) of $h_{2,p,q,l,m}$, we have:

$$(8.485) \quad \begin{aligned} \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \chi' P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.486) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_u^\infty L^2(\mathcal{M}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.418), the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Also, the analog of the estimate (8.428)

yields:

$$(8.487) \quad \left\| \int_{\mathbb{S}^2} \chi' P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon (1 + q^{\frac{5}{2}}) \gamma_j^{\nu'}.$$

Finally, (8.485), (8.486) and (8.487) imply:

$$(8.488) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l,m}$. In view of the Definition (8.477) of $h_{3,p,q,l,m}$, we have:

$$\begin{aligned} \sum_{m \leq l} h_{3,p,q,l,m} &= \left(\int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} P_{\leq l} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

which yields:

$$(8.489) \quad \left\| \sum_{m \leq l} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{\mathbb{S}^2} P_{\leq l} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

In view of the estimates (8.179) and (8.180), we have:

$$(8.490) \quad \left\| \int_{\mathbb{S}^2} P_{\leq l} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'}.$$

Also, using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$(8.491) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| G_1 \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu} \\ &\lesssim \left(\sup_{\omega} \|G_1\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. In view of the Definition (8.480) of G_1 , we have:

$$(8.492) \quad \begin{aligned} \|G_1\|_{L_u^{\infty} L^2(\mathcal{H}_u)} &\lesssim \|\nabla(L(P_l \operatorname{tr} \chi))\|_{L_u^{\infty} L^2(\mathcal{H}_u)} + \|(\theta + b^{-1} \nabla(b))L(P_l \operatorname{tr} \chi)\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \\ &\lesssim 2^{\frac{l}{2}} \varepsilon, \end{aligned}$$

where we used the estimate (8.193) for the first term and the estimate (8.186) for the second term. Now, (8.491) and (8.492) imply:

$$\left\| \int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} + \frac{1}{2}} \gamma_j^\nu,$$

which together with (8.489) and (8.490) yields:

$$(8.493) \quad \left\| \sum_{m \leq l} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{j}{2} + \frac{1}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$. In view of the Definition (8.478) of $h_{4,p,q,l,m}$, we have:

$$(8.494) \quad \begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} L(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim (1 + p^2) 2^{\frac{5j}{12}} \varepsilon \gamma_j^\nu \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.170). Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.495) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega'} \|G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ &\lesssim \left(\sup_{\omega'} \|G_2\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, in view of the Definition (8.481) for G_2 , we have:

$$\begin{aligned} \|G_2\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} &\lesssim \|\nabla'(P_{\leq l} \text{tr} \chi)\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} + \|\theta' + b'^{-1} \nabla(b')\|_{L_{u'}^\infty L^2(\mathcal{E}_{u'})} \|P_{\leq l} \text{tr} \chi'\|_{L^\infty(\mathcal{M})} \\ &\lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the finite band property and the boundedness on $L^\infty(P_{l,u'})$ for $P_{\leq l}$, the estimate (2.39) for $\text{tr} \chi'$, the estimate (2.38) for b' , and the estimates (2.37) (2.39) (2.40) for θ' . Together with (8.495), this yields:

$$\left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'},$$

which in view of (8.494) implies:

$$(8.496) \quad \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) 2^{\frac{11j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{5,p,q,l,m}$. In view of the Definition (8.479) of $h_{5,p,q,l,m}$, we have:

(8.497)

$$\begin{aligned} \|h_{5,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} L(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim (1 + p^2) 2^{\frac{11j}{12}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimates (8.170) and (8.317).

Finally, we have in view of (8.474):

$$\begin{aligned} \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| &\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}}|N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \\ &\quad \times \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|^2} \right\|_{L^\infty(\mathcal{M})} \sum_{m/m \leq l} (\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})}) \right. \\ &\quad \left. + \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} + \left\| \frac{1}{|N_\nu - N_{\nu'}|} \right\|_{L^1(\mathcal{M})} \right. \\ &\quad \left. \times \left(\left\| \sum_{m/m \leq l} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} + \sum_{m/m \leq l} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \right) \right. \\ &\quad \left. + \sum_{m/m \leq l} \|h_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \right] d\mathcal{M}. \end{aligned}$$

Together with the estimates (8.32), (8.484), (8.488), (8.493), (8.496) and (8.497), and the fact that we are in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$, we obtain:

$$\begin{aligned} \left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| &\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{p+q}} \\ &\quad \times \left[\frac{1}{|\nu - \nu'|^2} (1 + q^2) j 2^{\frac{j}{2}} + \frac{1}{|\nu - \nu'|} ((1 + q^2) 2^{\frac{j}{2} + \frac{l}{2}} + (1 + p^2) j 2^{\frac{11j}{12}}) \right. \\ &\quad \left. + (1 + p^2) j 2^{\frac{11j}{12}} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \left[\frac{j 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}}|\nu - \nu'|)^2} + \frac{2^{-\frac{j}{2} + \frac{l}{2}} + j 2^{-\frac{j}{12}}}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} + j 2^{-\frac{j}{12}} 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Summing in l , we finally obtain:

$$(8.498) \quad \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| \lesssim \left[\frac{j^2 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,3,2}$ in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$. We obtain the analog of the estimate (8.498):

$$(8.499) \quad \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,2} \right| \lesssim \left[\frac{j^2 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

To this end, we proceed exactly as for $B_{j,\nu,\nu',l,m}^{1,2,3,1}$, the only difference being that we integrate by parts tangentially using (7.137) instead of (7.136) to obtain the analog of Lemma 8.16. This is left to the reader.

Finally, the decomposition (8.471) of $B_{j,\nu,\nu',l,m}^{1,2,3}$ together with the estimates (8.498) and (8.499) imply:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \right| \\ & \lesssim \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| + \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,2} \right| \\ & \lesssim \left[\frac{j^2 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Together with the estimate (8.470) in the range of (l, m) such that $2^m \leq 2^j |\nu - \nu'| < 2^l$, we obtain:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \right| \\ & \lesssim \left[\frac{j^2 2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + \frac{1}{2^{\frac{3j}{4}} (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

This concludes the proof of Proposition 8.12.

8.3. Proof of Proposition 8.2 (Control of $B_{j,\nu,\nu',l,m}^2$)

Recall the definition of $B_{j,\nu,\nu',l,m}^2$ (8.11):

$$B_{j,\nu,\nu',l,m}^2 = -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left((\mathbf{g}(N, N') - 1) P_l \operatorname{tr} \chi N' (P_m \operatorname{tr} \chi') \right)$$

$$\begin{aligned}
& + \left(\operatorname{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' - 2\zeta'_{N - \mathbf{g}(N, N')N'} \right. \\
& \quad \left. - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) \\
& \quad \times P_l \operatorname{tr} \chi P_m \operatorname{tr} \chi' \Big) F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

We first consider the range of (l, m) such that:

$$2^m \leq 2^j |\nu - \nu'| < 2^l.$$

This yields:

$$\begin{aligned}
& \sum_{m/2^m \leq 2^j |\nu - \nu'|} B_{j, \nu, \nu', l, m}^2 \\
& = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left((\mathbf{g}(N, N') - 1) P_l \operatorname{tr} \chi N' (P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') + (\operatorname{tr} \chi - \bar{\delta} \right. \\
& \quad \left. - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' - 2\zeta'_{N - \mathbf{g}(N, N')N'} - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) \\
& \quad \times P_l \operatorname{tr} \chi P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \Big) F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Together with the identity (8.30):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N'),$$

we obtain:

(8.500)

$$\begin{aligned}
& \sum_{m/2^m \leq 2^j |\nu - \nu'|} B_{j, \nu, \nu', l, m}^2 \\
& = -i2^{-j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \left(N' (P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}} \\
& \quad - i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(\operatorname{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' - 2\zeta'_{N - \mathbf{g}(N, N')N'} \right. \\
& \quad \left. - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) P_l \operatorname{tr} \chi P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \\
& \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Next, recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{\mathbf{g}(L, L')}$$

in the same fashion than (8.33), and in view of (8.500), we obtain, schematically:

(8.501)

$$\begin{aligned} & \sum_{m/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^2 \\ &= 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \left[\frac{1}{|N_\nu - N_{\nu'}|} (h_{1,p,q,l} + h_{2,p,q,l}) + h_{3,p,q,l} \right] d\mathcal{M} \\ & \quad - i 2^{-j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} N' (P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{M}, \end{aligned}$$

where the scalar functions $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}$ on \mathcal{M} are given by:

(8.502)

$$\begin{aligned} h_{1,p,q,l} &= \left(\int_{\mathbb{S}^2} (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.503)

$$\begin{aligned} h_{2,p,q,l} &= \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} (-\bar{\delta}' - \chi') P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and:

(8.504)

$$\begin{aligned} h_{3,p,q,l} &= \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} \zeta' P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}$ starting with $h_{1,p,q,l}$. We have:

(8.505)

$$\|h_{1,p,q,l}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned} & \times \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim (1 + q^2) \varepsilon \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used (8.372) in the last inequality. The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.506) \quad \left\| \int_{\mathbb{S}^2} (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega} \left\| (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu.$$

Now, for any tensor G , we have:

$$\|G P_m \operatorname{tr} \chi'\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim \|G\|_{L_x^\infty L_t^2} \|P_m \operatorname{tr} \chi'\|_{L_x^2 L_t^\infty},$$

which together with the estimate (2.69) for $P_m \operatorname{tr} \chi'$ yields:

$$(8.507) \quad \|G P_m \operatorname{tr} \chi'\|_{L_u^\infty L^2(\mathcal{H}_{u'})} \lesssim 2^{-m} \varepsilon \|G\|_{L_x^\infty L_t^2}.$$

Using the estimate (8.507) with $G = \operatorname{tr} \chi - \bar{\delta}$, the estimate (2.39) for $\operatorname{tr} \chi$, the estimates (2.36) (2.37) for $\bar{\delta}$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we have:

$$\left\| (\operatorname{tr} \chi - \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-l}.$$

Together with (8.505) and (8.506), we obtain:

$$(8.508) \quad \|h_{1,p,q,l}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) 2^{-l + \frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l}$. In view of the Definition (8.503), we have:

$$(8.509) \quad \|h_{2,p,q,l}\|_{L^1(\mathcal{M})} \\ \lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' - \chi') P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

The basic estimate in $L^2(\mathcal{M})$ yields:

$$(8.510) \quad \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega} \left\| P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ \lesssim 2^{-l + \frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality the finite band property for P_l , the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. On the other hand, the decomposition (2.76) for χ' , the decomposition (7.132) for $\bar{\delta}'$, and the estimate (2.42) for $\partial_\omega N$ yield the following decomposition for $\bar{\delta}' + \chi'$:

$$(8.511) \quad \bar{\delta}' + \chi' = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν' and satisfies:

$$(8.512) \quad \|F_1^j\|_{L_{u,\nu',x',L_t}^\infty} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(8.513) \quad \|F_2^j\|_{L_u^\infty L^2(\mathcal{U}_{u'})} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

In view of (8.511), we obtain:

$$(8.514) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} (-\bar{\delta}' - \chi') P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \|F_1^j\|_{L_{u,\nu',x',L_t}^\infty} \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x',L_t}^\infty} \\ & \quad + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'} + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.512) for F_1^j and the estimate (8.426). Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{u'}^\infty L^2(\mathcal{U}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.513) for F_2^j , the boundedness of $P_{\leq 2^j |\nu - \nu'|}$ on $L^\infty(P_{t,u'})$, the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.514), this yields:

$$\left\| \int_{\mathbb{S}^2} (-\bar{\delta}' - \chi') P_{\leq 2^j |\nu - \nu'|} \text{tr}\chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j^{\nu'}.$$

Together with (8.509) and (8.510), we finally obtain:

$$(8.515) \quad \|h_{2,p,q,l}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) 2^{-l + \frac{j}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{3,p,q,l}$. In view of the Definition (8.504), we have:

$$(8.516) \quad \begin{aligned} \|h_{3,p,q,l}\|_{L^2(\mathcal{O}\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \zeta' P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim 2^{-l+\frac{j}{2}} \varepsilon \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \zeta' P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used (8.510) in the last inequality. In order to estimate the right-hand side of (8.516), we use the decomposition (2.80) of ζ' . We have:

$$(8.517) \quad \zeta' = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν' and satisfies:

$$(8.518) \quad \|F_1^j\|_{L_{u,\nu'}^\infty, L_t^2, L_{x,\nu'}^8} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(8.519) \quad \|F_2^j\|_{L_{u,\nu'}^\infty, L^2(\mathcal{O}\mathcal{M}_{\nu'})} \lesssim \varepsilon 2^{-\frac{j}{4}}.$$

In view of (8.517), we have:

$$(8.520) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \zeta' P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \|F_1^j\|_{L_{u,\nu'}^2, L_t^2, L_{x,\nu'}^8} \left\| \int_{\mathbb{S}^2} P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x,\nu'}^{\frac{8}{3}}, L_t^\infty} \\ &\quad + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \varepsilon \left\| \int_{\mathbb{S}^2} P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x,\nu'}^{\frac{8}{3}}, L_t^\infty} \\ &\quad + \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used the estimate (8.518) for F_1^j in the last inequality. We have the analog of (8.272):

$$(8.521) \quad \left\| \int_{\mathbb{S}^2} P_{\leq 2j|\nu-\nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N - N_{\nu'}) \right)^P F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^\infty(\mathcal{O}\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j^{\nu'}.$$

Now, interpolating between the estimate in $L_{u,\nu',x'}^2 L_t^\infty$ (8.426) and the L^∞ estimate (8.521), we obtain:

$$(8.522) \quad \left\| \int_{\mathbb{S}^2} P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,\nu',x'}^4 L_t^\infty} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^{\nu'}.$$

For the second term in the right-hand side of (8.520), we have:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \int_{\mathbb{S}^2} \|F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u')\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ & \lesssim \int_{\mathbb{S}^2} \|F_2^j\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \|F_j(u')\|_{L_u^2} \left\| P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega'. \end{aligned}$$

Together with the estimate (7.124) for F_2^j , the boundedness of $P_{\leq 2^j |\nu - \nu'|}$ on $L^\infty(P_{t,u'})$, the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$(8.523) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{-\frac{j}{4}} \varepsilon \int_{\mathbb{S}^2} \|F_j(u')\|_{L_u^2} \eta_j^{\nu'}(\omega') d\omega' \\ & \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality Plancherel in λ' , Cauchy Schwartz in ω' and the size of the patch. Finally, (8.520), (8.522) and (8.523) imply:

$$(8.524) \quad \left\| \int_{\mathbb{S}^2} \zeta' P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^{\nu'}.$$

Together with (8.516), this yields:

$$(8.525) \quad \|h_{3,p,q,l}\|_{L^2(\mathcal{M})} \lesssim 2^{-l+\frac{j}{2}} 2^{\frac{j}{4}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we estimate the last term in the right-hand side of (8.501):

$$\int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \left(N'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{M}.$$

We have:

$$(8.526) \quad \begin{aligned} & \left| \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \left(N'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{M} \right| \\ & \lesssim \left\| \int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \left\| \int_{\mathbb{S}^2} N'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{-l+j} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.510) and the estimate (8.317).

Now, we have in view of the decomposition (8.501) of $B_{j,\nu,\nu',l,m}^2$:

$$\begin{aligned} & \left| \sum_{m/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^2 \right| \\ & \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \right\|_{L^\infty(\mathcal{O}\mathcal{M})} \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|} \right\|_{L^\infty(\mathcal{O}\mathcal{M})} (\|h_{1,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} \right. \\ & \quad \left. + \|h_{2,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})} + \|h_{3,p,q,l}\|_{L^1(\mathcal{O}\mathcal{M})}) \right] \\ & \quad + 2^{-j} \left| \int_{\mathcal{O}\mathcal{M}} \left(\int_{\mathbb{S}^2} b^{-1} P_l \operatorname{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \left(N'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{O}\mathcal{M} \right|. \end{aligned}$$

Together with (8.32), (8.508), (8.515), (8.525) and (8.526), we obtain:

$$\begin{aligned} & \left| \sum_{m/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^2 \right| \\ & \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+1}} \left[\frac{1}{|\nu - \nu'|} (1 + q^{\frac{5}{2}}) 2^{-l+\frac{j}{2}} + 2^{-l+\frac{j}{2}} 2^{\frac{j}{4}} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \quad + 2^{-l} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \lesssim \left[\frac{2^{-l+\frac{j}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{2^{-l} 2^{\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-l} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Summing in l , we finally obtain in the range $2^m \leq 2^j |\nu - \nu'| < 2^l$:

$$(8.527) \quad \left| \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^l} B_{j,\nu,\nu',l,m}^2 \right| \lesssim \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we consider the range of (l, m) such that:

$$2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

We integrate by parts in tangential directions using (7.137).

Lemma 8.17. — *Let $B_{j,\nu,\nu',l,m}^2$ be defined by (8.11). Integrating by parts using (7.137) yields:*

$$\begin{aligned} B_{j,\nu,\nu',l,m}^2 & = 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{O}\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \left[\frac{1}{|N_\nu - N_{\nu'}|^3} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) \right. \\ & \quad \left. + \frac{1}{|N_\nu - N_{\nu'}|^2} (h_{3,p,q,l,m} + h_{4,p,q,l,m} + h_{5,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|} (h_{6,p,q,l,m}) \right] \end{aligned}$$

$$(8.528) \quad \left. \begin{aligned} &+ h_{7,p,q,l,m} + h_{8,p,q,l,m} + h_{9,p,q,l,m} + h_{10,p,q,l,m} \end{aligned} \right] d_{\mathcal{M}} \\ + \text{terms interverting } (\nu, \nu') + B_{j,\nu,\nu',l,m}^{2,1} + B_{j,\nu,\nu',l,m}^{2,2},$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$, $h_{8,p,q,l,m}$, $h_{9,p,q,l,m}$, $h_{10,p,q,l,m}$ on \mathcal{M} are given by:

(8.529)

$$h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} \chi(\chi + \bar{\delta}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.530)

$$h_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\chi' + \bar{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.531)

$$h_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} G_1 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.532)

$$h_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\chi' + \bar{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.533)

$$h_{5,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\theta' + \zeta') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.534)

$$h_{6,p,q,l,m} = \left(\int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left(\int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.535)

$$h_{7,p,q,l,m} = \left(\int_{\mathbb{S}^2} (N(P_l \operatorname{tr} \chi) + \nabla(P_l \operatorname{tr} \chi)) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\chi' + \bar{\delta}' + \zeta') P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.536)

$$h_{8,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \zeta' P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(8.537)

$$h_{9,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(N'(P_m \operatorname{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

(8.538)

$$h_{10,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

where the tensors G_1 and G_2 on \mathcal{M} are given by:

$$(8.539) \quad G_1 = (\chi + \bar{\delta}) \nabla P_l \operatorname{tr} \chi + (\nabla(\chi) + \nabla(\bar{\delta}) + (\chi + \bar{\delta})(\theta + b^{-1} \nabla(b))) P_l \operatorname{tr} \chi$$

and:

$$(8.540) \quad G_2 = (\chi + \bar{\delta}) N(P_l \operatorname{tr} \chi) + \zeta \nabla P_l \operatorname{tr} \chi + (\mathbf{D}_N(\chi) + N(\bar{\delta}) + \nabla(\zeta)\theta) P_l \operatorname{tr} \chi$$

and where $B_{j,\nu,\nu',l,m}^{2,1}$ and $B_{j,\nu,\nu',l,m}^{2,2}$ are defined by:

$$(8.541) \quad B_{j,\nu,\nu',l,m}^{2,1} = 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} N(P_l \operatorname{tr} \chi) F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} N'(P_m \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{M}$$

and:

$$(8.542) \quad B_{j,\nu,\nu',l,m}^{2,2} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \operatorname{tr} \chi) N'(P_m \operatorname{tr} \chi')}{1 - \mathbf{g}(N, N')^2} \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^{\nu}(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

The proof of Lemma 8.17 is postponed to Appendix G. In the rest of this section, we use Lemma 8.17 to control $B_{j,\nu,\nu',l,m}^2$ over the range of (l, m) such that $2^m \leq 2^l \leq 2^j |\nu - \nu'|$.

8.3.1. Control of the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}$. — We estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$, $h_{8,p,q,l,m}$, $h_{9,p,q,l,m}$, $h_{10,p,q,l,m}$ starting with $h_{1,p,q,l,m}$. Consider first the case $l > j/2$. Let H be defined by:

$$(8.543) \quad H = \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

Then, we have in view of the Definition (8.529) of $h_{1,p,q,l,m}$:

$$(8.544) \quad \begin{aligned} \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \chi H (\chi + \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^1(\mathcal{M})} \\ &\lesssim \int_{\mathbb{S}^2} \|\chi H\|_{L^2(\mathcal{M})} \left\| (\chi + \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|\chi\|_{L_x^\infty L_t^2} \|H\|_{L_{u,x}^2 L_t^\infty} \|(\chi + \bar{\delta}) P_l \operatorname{tr} \chi\|_{L_{u,x}^\infty L^2(\mathcal{E}^u)} \\ &\quad \times \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty(\mathcal{M})} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ &\lesssim \varepsilon 2^{-l} \int_{\mathbb{S}^2} \|H\|_{L_{u,x}^2 L_t^\infty} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimate (8.507) with the choice $G = \chi + \bar{\delta}$, the estimates (2.39) (2.40) for χ , the estimates (2.36) (2.37) for $\bar{\delta}$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next we estimate the term in H in the right-hand side of (8.544). Using the estimate (7.71) in the case $m > j/2$, we have:

$$(8.545) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,x}^2 L_t^\infty} \\ &\lesssim \varepsilon \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m + \frac{j}{2}} + \left(2^{\frac{j}{2}} |\nu - \nu'| \right)^{\frac{1}{2}} 2^{-\frac{m}{2} + \frac{j}{4}} \right) \gamma_j^{\nu'}. \end{aligned}$$

Also, using the decomposition:

$$P_{\leq j/2} \operatorname{tr} \chi' = \operatorname{tr} \chi' - \sum_{m/m > j/2} P_m \operatorname{tr} \chi',$$

together with the estimate (7.71) and the estimate (7.83), we obtain in the case $m = j/2$:

$$\left\| \int_{\mathbb{S}^2} P_{\leq j/2} \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u,x}^2 L_t^\infty} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \left(2^{\frac{j}{2}} |\nu - \nu'| + 1 \right) \gamma_j^{\nu'}.$$

Together with (8.545), we obtain for all $m \geq j/2$:

$$(8.546) \quad \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2_{u,x'}, L^\infty_t} \\ \lesssim \varepsilon \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \gamma_j^{\nu'}.$$

In view of the Definition (8.543) of H , this yields:

$$\|H\|_{L^2_{u,\nu',x'}, L^\infty_t} \lesssim \varepsilon \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \gamma_j^{\nu'}.$$

Together with (8.544), we obtain in the case $l > j/2$:

$$(8.547) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) 2^{-l} \gamma_j^{\nu'} \\ \times \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u)\|_{L^2_u} \eta_j^{\nu'}(\omega) d\omega \right) \\ \lesssim \varepsilon^2 \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) 2^{\frac{j}{2}-l} \gamma_j^{\nu'} \gamma_j^{\nu'},$$

where we used in the last inequality Plancherel in λ for $\|F_{j,-1}(u)\|_{L^2_u}$, Cauchy Schwartz in ω and the size of the patch.

Next, we consider the case $l = j/2$. Recall that in view of the decomposition for χ (2.76), we have:

$$(8.548) \quad \chi = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν and satisfies:

$$(8.549) \quad \|F_1^j\|_{L^\infty_{u,\nu,x'}, L^2_t} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$(8.550) \quad \|F_2^j\|_{L^\infty_{u,\nu} L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

In view of the Definition (8.529) of $h_{1,p,q,l,m}$, the decomposition (8.548) and the definition of H (8.543), we have in the case $l = j/2$:

$$(8.551)$$

$$\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \\ \lesssim \|F_1^j\|_{L^\infty_{u,\nu,x'}, L^2_t} \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_{\leq j/2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \|H\|_{L^2_{u,\nu,x'}, L^\infty_t} \\ + \left\| \int_{\mathbb{S}^2} F_2^j (\chi + \bar{\delta}) H P_{\leq j/2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^1(\mathcal{M})} \\ \lesssim \varepsilon \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \gamma_j^{\nu'} \\ \times \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_{\leq j/2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ + \int_{\mathbb{S}^2} \left\| F_2^j (\chi + \bar{\delta}) H P_{\leq j/2} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega) d\omega,$$

where we used in the last inequality the estimate (8.544) for F_1^j and the estimate (8.546) for H . Next, we estimate the two terms in the right-hand side of (8.551) starting with the first one. In view of the decomposition (7.132) for $\bar{\delta}$, $\bar{\delta}$ also has a decomposition of the form (8.543) (8.544) (8.545), and thus so has $\chi + \bar{\delta}$. Proceeding as in (8.423), (8.426) and (8.427), we obtain the analog of (8.428):

$$(8.552) \quad \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_{\leq j/2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon (1 + q^{\frac{5}{2}}) \gamma_j^\nu.$$

On the other hand, we have:

$$\begin{aligned} & \int_{\mathbb{S}^2} \left\| F_2^j (\chi + \bar{\delta}) H P_{\leq j/2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)} \|\chi + \bar{\delta}\|_{L_{x'}^\infty L_t^2} \|H\|_{L_{u,x'}^2 L_t^\infty} \left\| P_{\leq j/2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty(\mathcal{M})} \\ & \quad \times \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega \\ & \lesssim \varepsilon 2^{-\frac{j}{2}} \int_{\mathbb{S}^2} \|H\|_{L_{u,x'}^2 L_t^\infty} \|F_{j,-1}(u)\|_{L_u^2} \eta_j^\nu(\omega) d\omega, \end{aligned}$$

where we used in the last inequality the estimate (8.550) for F_2^j , the estimates (2.39) (2.40) for χ , the estimates (2.36) (2.37) for $\bar{\delta}$, the boundedness of $P_{\leq j/2}$ on $L^\infty(P_{t,u})$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the estimate (8.546) for H , Plancherel in λ for $\|F_{j,-1}(u)\|_{L_u^2}$, Cauchy Schwartz in ω and the size of the patch, we obtain:

$$(8.553) \quad \int_{\mathbb{S}^2} \left\| F_2^j (\chi + \bar{\delta}) H P_{\leq j/2} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega \\ \lesssim \varepsilon^2 \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, (8.551), (8.552) and (8.553) yield in the case $l = j/2$:

$$\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) \gamma_j^\nu \gamma_j^{\nu'}.$$

Together with (8.547), we finally obtain for all $l \geq j/2$:

$$(8.554) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-m+\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{m}{2}+\frac{j}{4}} \right) 2^{\frac{j}{2}-l} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.2. Control of the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. — Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of the Definition (8.530) of $h_{2,p,q,l,m}$, we have:

$$(8.555) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} (\chi' + \bar{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

In the case $l > j/2$, we use the basic estimate in $L^2(\mathcal{M})$:

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| (\chi + \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ & \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.507) with the choice $G = \chi + \bar{\delta}$, the estimates (2.39) (2.40) for χ , the estimates (2.36) (2.37) for $\bar{\delta}$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.549), we obtain for all $l \geq j/2$:

$$(8.556) \quad \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu.$$

Finally, (8.555), (8.556) and the analog of (8.556) for the second term in the right-hand side of (8.555) implies:

$$(8.557) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j-l-m} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.3. Control of the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l,m}$. — In view of the Definition (8.531) of $h_{3,p,q,l,m}$ and in view of the Definition (8.539) of G_1 , we have:

$$\begin{aligned} \sum_{l/l \geq m} h_{3,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \tilde{G}_1 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

where \tilde{G}_1 is given by:

$$(8.558) \quad \tilde{G}_1 = (\chi + \bar{\delta}) \nabla P_{\geq m} \operatorname{tr} \chi + (\nabla(\chi) + \nabla(\bar{\delta}) + (\chi + \bar{\delta})(\theta + b^{-1} \nabla(b))) P_{\geq m} \operatorname{tr} \chi.$$

This yields the following estimate:

$$(8.559) \quad \left\| \sum_{l/l \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} \tilde{G}_1 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

Now, using (7.71) in the case $m > j/2$, and (8.271) in the case $m = j/2$, we obtain for all $m \geq j/2$:

$$(8.560) \quad \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L_{u',x'}^2 L_t^\infty} \lesssim (1 + q^2) \varepsilon 2^{\frac{j}{2}-m} \gamma_j^{\nu'}.$$

Together with (8.559), this yields:

$$(8.561) \quad \left\| \sum_{l/l \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim (1+q^2)\varepsilon 2^{\frac{j}{2}-m} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \tilde{G}_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$

Next, we estimate the right-hand side of (8.561). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.562) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} \tilde{G}_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega} \left\| \tilde{G}_1 \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ & \lesssim \left(\sup_{\omega} \left\| \tilde{G}_1 \right\|_{L_u^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of (8.558), we have:

$$(8.563) \quad \begin{aligned} \left\| \tilde{G}_1 \right\|_{L_u^\infty L^2(\mathcal{H}_u)} & \lesssim (\|\chi\|_{L_x^\infty L_t^2} + \|\bar{\delta}\|_{L_x^\infty L_t^2}) (\|\nabla \text{tr} \chi\|_{L_x^2, L_t^\infty} + \|\nabla P_{\leq m} \text{tr} \chi\|_{L_x^2, L_t^\infty}) \\ & \quad + (\|\nabla(\chi)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\nabla(\bar{\delta})\|_{L_u^\infty L^2(\mathcal{H}_u)}) \\ & \quad + \|(\chi + \bar{\delta})(\theta + b^{-1} \nabla(b))\|_{L_u^\infty L^2(\mathcal{H}_u)} \|P_{\geq m} \text{tr} \chi\|_{L^\infty} \\ & \lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ , the estimates (2.37) (2.36) for $\bar{\delta}$, the estimate (2.38) for b , the estimates (2.37) (2.39) (2.40) for θ , the decomposition:

$$\text{tr} \chi = P_{\leq m} \text{tr} \chi + P_{> m} \text{tr} \chi,$$

the estimate (2.70) for $\nabla P_{\leq m} \text{tr} \chi$, and the fact that $P_{\geq m}$ is bounded on $L^\infty(P_{t,u})$. Finally, (8.561), (8.562) and (8.563) imply:

$$(8.564) \quad \left\| \sum_{l/l \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim (1+q^2)\varepsilon^2 2^{j-m} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.4. Control of the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$. — In view of the Definition (8.532) of $h_{4,p,q,l,m}$, we have:

$$\begin{aligned} \sum_{l/l \geq m} h_{4,p,q,l,m} & = \left(\int_{\mathbb{S}^2} \nabla(P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} (\chi' + \bar{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

which yields:

$$\begin{aligned}
 (8.565) \quad & \left\| \sum_{l/l \geq m} h_{4,p,q,l,m} \right\|_{L^1(\mathcal{M})} \\
 & \leq \left\| \int_{\mathbb{S}^2} \mathcal{V}(P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 & \quad \times \left\| \int_{\mathbb{S}^2} (\chi' + \bar{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 & \lesssim \varepsilon 2^{\frac{j}{2}-m} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \mathcal{V}(P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},
 \end{aligned}$$

where we used in the last inequality the estimate (8.556).

Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\begin{aligned}
 (8.566) \quad & \left\| \int_{\mathbb{S}^2} \mathcal{V}(P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 & \lesssim \left(\sup_{\omega} \left\| \mathcal{V}(P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_u^\infty L^2(\mathcal{M}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
 & \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu,
 \end{aligned}$$

where we used in the last inequality the finite band property for $P_{\geq m}$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.565) and (8.566) yield:

$$(8.567) \quad \left\| \sum_{l/l \geq m} h_{4,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j-m} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.5. Control of the $L^1(\mathcal{M})$ norm of $h_{5,p,q,l,m}$. — In view of the Definition (8.533) of $h_{5,p,q,l,m}$, we have:

$$\begin{aligned}
 (8.568) \quad & \|h_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} (\chi + \bar{\delta}) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 & \quad \times \left\| \int_{\mathbb{S}^2} (\theta' + \zeta') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 & \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \left\| \int_{\mathbb{S}^2} (\theta' + \zeta') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
 \end{aligned}$$

where we used in the last inequality the estimate (8.556).

Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.569) \quad \left\| \int_{\mathbb{S}^2} (\theta' + \zeta') P_m \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned} &\lesssim \left(\sup_{\omega'} \left\| (\theta' + \zeta') P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{\omega'}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega'}})} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega'} \left\| (\theta' + \zeta') P_m \operatorname{tr} \chi' \right\|_{L_{\omega'}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega'}})} \right) 2^{\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. We have:

$$(8.570) \quad \begin{aligned} \left\| (\theta' + \zeta') P_m \operatorname{tr} \chi' \right\|_{L_{\omega'}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega'}})} &\lesssim (\|\theta'\|_{L_t^\infty L_x^4} + \|\zeta'\|_{L_t^\infty L_x^4}) \|P_m \operatorname{tr} \chi'\|_{L_{\omega'}^\infty L^4(\mathcal{E}_{\mathcal{U}_{\omega'}})} \\ &\lesssim 2^{-\frac{m}{2}(1-\delta_{m,j/2})} \varepsilon, \end{aligned}$$

where we used in the last inequality the embedding (2.51), the estimates (2.37)–(2.39)–(2.40) for θ' , the estimates (2.41) for ζ' , the Bernstein inequality and the finite band property for P_m , and the estimate (2.39) for $\operatorname{tr} \chi'$. Note that the factor $1 - \delta_{j/2,m}$ comes from the fact that we use the finite band property for P_m in the case $m > j/2$, but only the boundedness of $P_{\leq j/2}$ in the case $m = j/2$. Finally, (8.568), (8.569) and (8.570) yield:

$$(8.571) \quad \|h_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j-l-\frac{m}{2}(1-\delta_{m,j/2})} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.6. Control of the $L^1(\mathcal{M})$ norm of $h_{6,p,q,l,m}$. — In view of the Definition (8.534) of $h_{6,p,q,l,m}$, we have:

$$(8.572) \quad \begin{aligned} \|h_{6,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} P_m \operatorname{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon 2^{\frac{j}{2}-m} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.556).

Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.573) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| G_2 \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega}})} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega} \|G_2\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega}})} \right) 2^{\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next, we estimate G_2 . In view of the Definition (8.540) of G_2 , we have:

$$(8.574) \quad \begin{aligned} \|G_2\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega}})} &\lesssim (\|\chi\|_{L_t^\infty L_x^4} + \|\bar{\delta}\|_{L_t^\infty L_x^4}) \|N(P_l \operatorname{tr} \chi)\|_{L_t^2 L_x^4} \\ &\quad + \|\zeta\|_{L_t^\infty L_x^4} \|\nabla P_l \operatorname{tr} \chi\|_{L_t^2 L_x^4} + (\|\mathbf{D}N(\chi)\|_{L_{\omega}^\infty L^2(\mathcal{E}_{\mathcal{U}_{\omega}})}) \end{aligned}$$

$$\begin{aligned}
& + \|N(\bar{\delta})\|_{L_u^\infty L^2(\mathcal{E}u)} + \|\nabla(\zeta)\theta\|_{L_u^\infty L^2(\mathcal{E}u)} \|P_l \text{tr}\chi\|_{L^\infty} \\
& \lesssim \varepsilon (\|N(P_l \text{tr}\chi)\|_{L_t^2 L_x^4} + \|\nabla P_l \text{tr}\chi\|_{L_t^2 L_x^4}) + \varepsilon,
\end{aligned}$$

where we used in the last inequality the embedding (2.51), the estimates (2.39)–(2.40) for χ , the estimates (2.36)–(2.37) for $\bar{\delta}$, the estimate (2.41) for ζ , the estimates (2.37)–(2.39)–(2.40) for θ , the estimate (2.39) for $\text{tr}\chi$ and the boundedness of P_l on $L^\infty(P_{t,u})$. Now, the estimate (2.38) for b and the Gagliardo-Nirenberg inequality (2.49), yields:

$$\begin{aligned}
& \|N(P_l \text{tr}\chi)\|_{L_t^2 L_x^4} + \|\nabla P_l \text{tr}\chi\|_{L_t^2 L_x^4} \\
& \lesssim \|P_l(bN \text{tr}\chi)\|_{L_t^2 L_x^4} + \|[bN, P_l] \text{tr}\chi\|_{L_t^2 L_x^4} + \|\nabla^2 P_l \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)}^{1/2} \|\nabla P_l \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)}^{1/2} \\
& \lesssim 2^{1/2} \|bN \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)} + \|\nabla[bN, P_l] \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)}^{1/2} \|[bN, P_l] \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)}^{1/2} + 2^{1/2} \|\nabla \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{E}u)},
\end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (2.49), the Bernstein inequality for P_l , the Bochner inequality (2.61), and the finite band property for P_l . Together with the estimate (2.39) for $\text{tr}\chi$ and the commutator estimate (2.68) for $[bN, P_l] \text{tr}\chi$, we obtain:

$$(8.575) \quad \|N(P_l \text{tr}\chi)\|_{L_t^2 L_x^4} + \|\nabla P_l \text{tr}\chi\|_{L_t^2 L_x^4} \lesssim 2^{1/2} \varepsilon.$$

Finally, (8.572), (8.573), (8.574) and (8.575) imply:

$$(8.576) \quad \|h_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{1}{2}-m} \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.7. Control of the $L^1(\mathcal{M})$ norm of $h_{7,p,q,l,m}$. — In view of the Definition (8.535) of $h_{7,p,q,l,m}$, we have:

$$\begin{aligned}
(8.577) \quad \|h_{7,p,q,l,m}\|_{L^1(\mathcal{M})} & \leq \left\| \int_{\mathbb{S}^2} (N(P_l \text{tr}\chi) + \nabla(P_l \text{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \quad \times \left\| \int_{\mathbb{S}^2} (\chi' + \bar{\delta}' + \zeta') P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\end{aligned}$$

The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\begin{aligned}
(8.578) \quad & \left\| \int_{\mathbb{S}^2} (N(P_l \text{tr}\chi) + \nabla(P_l \text{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
& \lesssim \left(\sup_\omega \left\| (N(P_l \text{tr}\chi) + \nabla(P_l \text{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_u^\infty L^2(\mathcal{E}u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
& \lesssim \left(\sup_\omega (\|N(P_l \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)} + \|\nabla(P_l \text{tr}\chi)\|_{L_u^\infty L^2(\mathcal{E}u)}) \left\| \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
& \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimate (8.263) for $N(P_l \text{tr}\chi)$, the finite band property for P_l , the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Now, since k does not depend on ω , and in view of the decomposition (2.72)–(2.74) for χ' , and the decomposition (2.80) for z' , we have the following decomposition for $\chi' + \bar{\delta}' + \zeta'$:

$$\chi' + \bar{\delta}' + \zeta' = F_1^j + F_2^j,$$

where the tensor F_1^j only depends on ν' and satisfies:

$$\|F_1^j\|_{L_{u,\nu'}^\infty L_t^2 L_{x,\nu'}^8} \lesssim \varepsilon$$

and where the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_{u'}^\infty L^2(\mathcal{M}_{u'})} \lesssim \varepsilon 2^{-\frac{j}{4}}.$$

Note that this decomposition has the same properties as the decomposition (8.267)–(8.268)–(8.269) for $\theta + b^{-1}\nabla(b)$. Thus, arguing as in (8.270)–(8.275), we obtain:

$$(8.579) \quad \left\| \int_{\mathbb{S}^2} (\chi' + \bar{\delta}' + \zeta') P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^{\nu'}.$$

Finally, (8.577), (8.578) and (8.579) imply:

$$(8.580) \quad \|h_{7,p,q,l,m}\|_{L^1(\mathcal{M})} \leq \varepsilon^2 2^{\frac{3j}{4}} \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

8.3.8. Control of the $L^1(\mathcal{M})$ norm of $h_{8,p,q,l,m}$. — In view of the Definition (8.536) of $h_{8,p,q,l,m}$, we have:

$$(8.581) \quad \begin{aligned} \|h_{8,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} (\theta + b^{-1}\nabla(b)) P_l \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \zeta' P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^\nu \left\| \int_{\mathbb{S}^2} \zeta' P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used the estimate (8.275) in the last inequality. Also, we have the analog of (8.579):

$$(8.582) \quad \left\| \int_{\mathbb{S}^2} \zeta' P_m \text{tr}\chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j^{\nu'}.$$

Finally, (8.581) and (8.582) imply:

$$(8.583) \quad \|h_{8,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.9. Control of the $L^1(\mathcal{M})$ norm of $h_{9,p,q,l,m}$. — In view of the Definition (8.537) of $h_{9,p,q,l,m}$, we have:

$$\begin{aligned} \|h_{9,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla' (N' (P_m \operatorname{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\int_{\mathbb{S}^2} \left\| P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| \nabla' (N' (P_m \operatorname{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right). \end{aligned}$$

Together with the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$\begin{aligned} \|h_{9,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left(\int_{\mathbb{S}^2} \left\| P_l \operatorname{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| \nabla' (N' (P_m \operatorname{tr} \chi')) \right\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim 2^{-j} \left\| P_l \operatorname{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \Big\|_{L^2_{u,\omega}} \\ &\quad \times \left\| \nabla' (N' (P_m \operatorname{tr} \chi')) \right\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \Big\|_{L^2_{u',\omega'}}, \end{aligned}$$

where we used in the last inequality Cauchy Schwarz in ω and ω' , and the size of the patch. This yields:

$$\begin{aligned} (8.584) \quad &\sum_{(l,m)/m \leq l} \|h_{9,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\lesssim 2^{-j} \sum_{(l,m)} 2^{-|l-m|} \left(2^l \left\| P_l \operatorname{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{u,\omega}} \right) \\ &\quad \times \left(2^{-m} \left\| \nabla' (N' (P_m \operatorname{tr} \chi')) \right\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L^2_{u',\omega'}} \right) \\ &\lesssim 2^{-j} \left(\sum_l 2^{2l} \left\| P_l \operatorname{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{u,\omega}}^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_m 2^{-2m} \left\| \nabla' (N' (P_m \operatorname{tr} \chi')) \right\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L^2_{u',\omega'}}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now, we have:

$$(8.585) \quad \sum_l \left\| 2^l P_l \operatorname{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \Big\|_{L^2_{\omega,u}}^2$$

$$\begin{aligned} &= \int_{\mathbb{S}^2} \left(\int_u \left(\sum_l 2^{2l} \|P_l \text{tr} \chi\|_{L^2(\mathcal{E}_{u'})}^2 \right) |F_j(u)|^2 du \right) \eta_j^{\nu'}(\omega) d\omega \\ &\lesssim \int_{\mathbb{S}^2} \|\nabla \text{tr} \chi\|_{L_u^\infty L^2(\mathcal{E}_{u'})}^2 \|F_j(u)\|_{L_u^2}^2 \eta_j^{\nu'}(\omega) d\omega \\ &\lesssim \varepsilon^2 2^{2j} (\gamma_j^\nu)^2, \end{aligned}$$

where we used the finite band property for P_l , the estimates (2.39) for $\text{tr} \chi$ and Plancherel in λ . Also, in view of the estimate (2.38) for b' , we have:

$$\begin{aligned} &\|\nabla'(N'(P_m \text{tr} \chi'))\|_{L^2(\mathcal{E}_{u'})} \\ &\lesssim \|b' \nabla'(N'(P_m \text{tr} \chi'))\|_{L^2(\mathcal{E}_{u'})} \\ &\lesssim \|\nabla'(b')\|_{L_t^\infty L_x^4} \|N'(P_m \text{tr} \chi')\|_{L_t^2 L_x^4} + \|\nabla'(P_m(b' N' \text{tr} \chi'))\|_{L^2(\mathcal{E}_{u'})} \\ &\quad + \|\nabla'([b' N', P_m] \text{tr} \chi')\|_{L_u^\infty L^2(\mathcal{E}_{u'})}. \end{aligned}$$

Together with the estimate (2.38) for b , the estimate (8.575) for $N'(P_m \text{tr} \chi')$, the finite band property for P_m , and the commutator estimate (2.68), we obtain:

$$(8.586) \quad \|\nabla'(N'(P_m \text{tr} \chi'))\|_{L^2(\mathcal{E}_{u'})} \lesssim 2^{\frac{m}{2}} \varepsilon + 2^m \|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{E}_{u'})}.$$

This yields:

$$\begin{aligned} (8.587) \quad &\sum_m 2^{-2m} \left\| \|\nabla'(N'(P_m \text{tr} \chi'))\|_{L^2(\mathcal{E}_{u'})} F_{j,-1}(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L_{\omega',u'}^2}^2 \\ &= \int_{\mathbb{S}^2} \left(\int_{u'} \left(\sum_m (\|P_m(b' N' \text{tr} \chi')\|_{L^2(\mathcal{E}_{u'})} + 2^{-\frac{m}{2}} \varepsilon)^2 \right) |F_j(u')|^2 du' \right) \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \int_{\mathbb{S}^2} \left(\|b' N' \text{tr} \chi\|_{L_u^\infty L^2(\mathcal{E}_{u'})}^2 + \varepsilon^2 \right) \|F_j(u')\|_{L_{u'}^2}^2 \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \varepsilon^2 2^{2j} (\gamma_j^{\nu'})^2, \end{aligned}$$

where we used the finite band property for P_m , the estimates (2.39) for $\text{tr} \chi'$ and Plancherel in λ' .

Finally, (8.584), (8.585) and (8.587) yield:

$$(8.588) \quad \sum_{(l,m)/m \leq l} \|h_{9,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^j \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.10. Control of the $L^1(\mathcal{M})$ norm of $h_{10,p,q,l,m}$. — In view of the Definition (8.538) of $h_{10,p,q,l,m}$, we have:

$$(8.589) \quad \begin{aligned} &\|h_{10,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\leq \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$\times \left\| \int_{\mathbb{S}^2} b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

We estimate the first term in the right-hand side of (8.589). Using (8.510) for $l > j/2$ and (8.180) for $l = j/2$, we obtain for all $l \geq j/2$:

$$(8.590) \quad \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) 2^{-l + \frac{j}{2}} \varepsilon \gamma_j^{\nu}.$$

Next, we estimate the second term in the right-hand side of (8.589). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.591) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_u^{\infty} L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \left\| b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \right\|_{L_u^{\infty} L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'}. \end{aligned}$$

Now, we have:

$$\begin{aligned} \left\| b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \right\|_{L_u^{\infty} L^2(\mathcal{H}_{u'})} & \lesssim \left\| b'^{-1} \nabla(b') \right\|_{L_t^{\infty} L_x^4} \left\| N'(P_m \operatorname{tr} \chi') \right\|_{L_t^2 L_x^4} \\ & \lesssim \varepsilon 2^{\frac{m}{2}}, \end{aligned}$$

where we used in the last inequality the estimate (2.38) for b and the estimate (8.575) for $N'(P_m \operatorname{tr} \chi')$. Together with (8.591), we obtain:

$$(8.592) \quad \left\| \int_{\mathbb{S}^2} b'^{-1} \nabla(b') N'(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} + \frac{m}{2}} \gamma_j^{\nu'}.$$

Finally, (8.589), (8.590) and (8.592) imply:

$$(8.593) \quad \|h_{10,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2) 2^{j-l + \frac{m}{2}} \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'}.$$

8.3.11. Control of $B_{j,\nu,\nu',l,m}^{2,1}$. — Recall the Definition (8.541) of $B_{j,\nu,\nu',l,m}^{2,1}$:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{2,1} & = 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} N(P_l \operatorname{tr} \chi) F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} N'(P_m \operatorname{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d\mathcal{M}. \end{aligned}$$

Recall that we are considering the range of (l, m) :

$$2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

Summing in (l, m) , we have:

$$\sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} N(P_l \operatorname{tr} \chi) N'(P_m \operatorname{tr} \chi') = N(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi) N'(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi').$$

Thus, using the symmetry in (ω, ω') of the integrant in $B_{j,\nu,\nu',l,m}^{2,1}$, we obtain in view of the definition $B_{j,\nu,\nu',l,m}^{2,1}$:

$$\begin{aligned} & \sum_{(l,m)/2^{\max(l,m)} l \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{2,1} + B_{j,\nu,\nu',l,m}^{2,1}) \\ &= 2^{-2j} \int_{\mathcal{M}} \left(\int_{\mathbb{S}^2} N(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) d_{\mathcal{M}}, \end{aligned}$$

which yields:

$$\begin{aligned} (8.594) \quad & \left| \sum_{(l,m)/2^{\max(l,m)} l \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{2,1} + B_{j,\nu,\nu',l,m}^{2,1}) \right| \\ & \leq 2^{-2j} \left\| \int_{\mathbb{S}^2} N(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \end{aligned}$$

Together with the estimate (8.317) applied to both terms in the right-hand side of (8.594), we finally obtain:

$$(8.595) \quad \left| \sum_{(l,m)/2^{\max(l,m)} l \leq 2^j |\nu - \nu'|} (B_{j,\nu',\nu,l,m}^{2,1} + B_{j,\nu,\nu',l,m}^{2,1}) \right| \lesssim 2^{-j} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

8.3.12. Control of $B_{j,\nu,\nu',l,m}^{2,2}$. — Recall the Definition (8.542) of $B_{j,\nu,\nu',l,m}^{2,2}$:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^{2,2} &= 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2} \\ & \quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We sum for $m \leq l$, and we obtain:

$$\begin{aligned} (8.596) \quad & \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2} \\ & \quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We can not estimate $\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2}$ directly due to a lack of summability in l . Instead, we integrate by parts in tangential directions using (7.137).

Lemma 8.18. — *Let $\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2}$ be defined by (8.596). Integrating by parts using (7.137) yields:*

(8.597)

$$\begin{aligned} \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} &= 2^{-\frac{5j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \\ &\quad \times \left[\frac{1}{|N_\nu - N_{\nu'}|^2} (h'_{1,p,q,l,m} + h'_{2,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|} (h'_{3,p,q,l,m} \right. \\ &\quad \left. + h'_{4,p,q,l,m} + h'_{5,p,q,l,m} + h'_{6,p,q,l,m}) + h'_{7,p,q,l,m} + h'_{8,p,q,l,m} \right] d\mathcal{M}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$, $h'_{8,p,q,l,m}$ on \mathcal{M} are given by:

(8.598)

$$\begin{aligned} h'_{1,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \chi \nabla (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} N' (P_{\leq l} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.599)

$$\begin{aligned} h'_{2,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \nabla (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \chi' N' (P_{\leq l} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.600)

$$\begin{aligned} h'_{3,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \nabla^2 (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} N' (P_{\leq l} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.601)

$$\begin{aligned} h'_{4,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \nabla (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \nabla' (N' (P_{\leq l} \operatorname{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(8.602)

$$h'_{5,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) \nabla (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$(8.603) \quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(8.604) \quad \begin{aligned} h'_{6,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} (b'^{-1} \nabla'(b') + \theta') N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

$$(8.605) \quad \begin{aligned} h'_{7,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \nabla(N(P_l \text{tr} \chi)) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) \end{aligned}$$

and:

$$(8.605) \quad \begin{aligned} h'_{8,p,q,l,m} &= \left(\int_{\mathbb{S}^2} (\theta + N(b)) \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} N'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right). \end{aligned}$$

The proof of Lemma 8.18 is postponed to Appendix H. In the rest of this section, we use Lemma 8.18 to control $\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2}$ over the range of (l, m) such that $2^m \leq 2^l \leq 2^j |\nu - \nu'|$.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q,l,m}, h'_{2,p,q,l,m}, h'_{3,p,q,l,m}, h'_{4,p,q,l,m}, h'_{5,p,q,l,m}, h'_{6,p,q,l,m}, h'_{7,p,q,l,m}, h'_{8,p,q,l,m}$ starting with $h'_{1,p,q,l,m}$. In view of the Definition (8.598), we have:

$$(8.606) \quad \begin{aligned} \|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} \chi \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \chi \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.317). Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(8.607) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \chi \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| \chi \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L_u^{\infty} L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^{\nu} \end{aligned}$$

$$\begin{aligned} &\lesssim \left(\sup_{\omega} \|\chi\|_{L_t^\infty L_{x'}^4} \|\nabla(P_l \text{tr}\chi)\|_{L_t^2 L_{x'}^4} \left\| \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon 2^{\frac{l}{2} + \frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimates (2.39)–(2.40) for χ , the estimate (8.575) for $\nabla(P_l \text{tr}\chi)$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of (8.606) and (8.607), we obtain:

$$(8.608) \quad \|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{l}{2}} \gamma_j^\nu \gamma_{j'}^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{2,p,q,l,m}$. In view of the Definition (8.599) of $h'_{2,p,q,l,m}$, we have:

$$(8.609) \quad \begin{aligned} \|h'_{2,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \chi' N'(P_{\leq l} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_{j'}^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon^2 2^{j+\frac{l}{2}} \gamma_j^\nu \gamma_{j'}^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317) for the first term and the analog of the estimate (8.607) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{3,p,q,l,m}$. In view of the Definition (8.600) of $h'_{3,p,q,l,m}$, we have:

$$(8.610) \quad \begin{aligned} \|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} \nabla^2(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_{j'}^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} \nabla^2(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.317). Now, the basic estimate in $L^2(\mathcal{M})$ yields:

$$(8.611) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \nabla^2(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| \nabla^2(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L_w^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \varepsilon 2^{l+\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.356) for $\nabla^2(P_l \text{tr}\chi)$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.610) and (8.611) yield:

$$(8.612) \quad \|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{4,p,q,l,m}$. In view of the Definition (8.601) of $h'_{4,p,q,l,m}$, we have:

$$(8.613) \quad \begin{aligned} \|h'_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\leq \left\| \int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'(N'(P_{\leq l} \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu \left\| \int_{\mathbb{S}^2} \nabla'(N'(P_{\leq l} \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the analog of estimate (8.317). Now, the basic estimate in $L^2(\mathcal{M})$ yields:

$$(8.614) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} \nabla'(N'(P_{\leq l} \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega'} \left\| \nabla'(N'(P_{\leq l} \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{M}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ &\lesssim \varepsilon 2^{l+\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.359) for $\nabla'(N'(P_{\leq l} \text{tr}\chi'))$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.613) and (8.614) yield:

$$(8.615) \quad \|h'_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{5,p,q,l,m}$. In view of the Definition (8.602) of $h'_{5,p,q,l,m}$, we have:

$$(8.616) \quad \begin{aligned} &\|h'_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \\ &\leq \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.317). Proceeding as for the estimate of (8.607) and using the estimates (2.38) for b and (2.37) (2.39) (2.40) for θ , we

obtain the following estimate:

$$(8.617) \quad \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) \nabla(P_t \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} + \frac{j}{2}} \gamma_j^\nu.$$

Finally, (8.616) and (8.617) imply:

$$(8.618) \quad \|h'_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{1}{2}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{6,p,q,l,m}$. In view of the Definition (8.603) of $h'_{6,p,q,l,m}$, we have:

$$(8.619) \quad \begin{aligned} & \|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \leq \left\| \int_{\mathbb{S}^2} \nabla(P_t \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} (b'^{-1} \nabla'(b') + \theta') N'(P_{\leq t} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon^2 2^{j+\frac{1}{2}} \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of the estimate (8.617) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{7,p,q,l,m}$. In view of the Definition (8.604) of $h'_{7,p,q,l,m}$, we have:

$$(8.620) \quad \begin{aligned} \|h'_{7,p,q,l,m}\|_{L^1(\mathcal{M})} & \leq \left\| \int_{\mathbb{S}^2} \nabla(N(P_t \text{tr} \chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} N'(P_{\leq t} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon^2 2^{j+l} \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.614) for the first term, and the estimate (8.317) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{8,p,q,l,m}$. In view of the Definition (8.605) of $h'_{8,p,q,l,m}$, we have:

$$(8.621) \quad \begin{aligned} & \|h'_{8,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \leq \left\| \int_{\mathbb{S}^2} (\theta + N(b)) \nabla(P_t \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} N'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$\lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} (\theta + N(b)) \nabla (P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},$$

where we used in the last inequality the analog of the estimate (8.317). Proceeding as for the estimate of (8.607) and using the estimates (2.38) for b and (2.37) (2.39) (2.40) for θ , we obtain the following estimate:

$$(8.622) \quad \left\| \int_{\mathbb{S}^2} (\theta + N(b)) \nabla (P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} + \frac{j}{2}} \gamma_j^{\nu'}.$$

Finally, (8.621) and (8.622) imply:

$$(8.623) \quad \|h'_{8,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+\frac{l}{2}} \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Now, we have in view of the decomposition (8.597) of $\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2}$:

$$\begin{aligned} & \left| \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} \right| \\ & \lesssim 2^{-\frac{5j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \right\|_{L^\infty} \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|^2} \right\|_{L^\infty} (\|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})}) \right. \\ & \quad + \|h'_{2,p,q,l,m}\|_{L^2(\mathcal{M})} + \left\| \frac{1}{|N_\nu - N_{\nu'}|} \right\|_{L^\infty} (\|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h'_{4,p,q,l,m}\|_{L^1(\mathcal{M})}) \\ & \quad \left. + \|h'_{5,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h'_{7,p,q,l,m}\|_{L^1(\mathcal{M})} + \|h'_{8,p,q,l,m}\|_{L^1(\mathcal{M})} \right]. \end{aligned}$$

Together with (8.32), (8.608), (8.609), (8.612), (8.615), (8.618), (8.619), (8.620) and (8.623), we obtain:

$$\begin{aligned} & \left| \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} \right| \\ & \lesssim 2^{-\frac{5j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+1}} \left[\frac{2^{j+\frac{l}{2}}}{|\nu - \nu'|^2} + \frac{2^{j+l}}{|\nu - \nu'|} + 2^{j+l} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'} \\ & \lesssim \left[\frac{2^{-\frac{j}{2} + \frac{l}{2}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-j+l}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{2^{-\frac{3j}{2} + l}}{2^{\frac{j}{2}} |\nu - \nu'|} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}. \end{aligned}$$

Summing in l , we finally obtain in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$:

$$(8.624) \quad \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{2,2} \right| \lesssim \left[\frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

8.4. End of the proof of Proposition 8.2

In view of the decomposition (8.528) of $B_{j,\nu,\nu',l,m}^2$, the estimate (8.32), the estimates (8.547) (8.557) (8.564) (8.567) (8.571) (8.576) (8.580) (8.583) (8.588) (8.593) for $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$, $h_{8,p,q,l,m}$, $h_{9,p,q,l,m}$, $h_{10,p,q,l,m}$, the estimate (8.595) for $B_{j,\nu,\nu',l,m}^{2,1}$ and the estimate (8.624) for $B_{j,\nu,\nu',l,m}^{2,2}$, we obtain:

$$\begin{aligned} & \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^{j|\nu-\nu'|}} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu',\nu,l,m}^2) \right| \\ & \lesssim 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{p+q+1}} \\ & \quad \times \left[\frac{(1+q^2)2^{\frac{j}{2}}}{|\nu-\nu'|^2} + \frac{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}} + 2^{\frac{3j}{4}}}{|\nu-\nu'|} + 2^j + 2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \quad + 2^{-j} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} + \left[\frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu-\nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ & \lesssim \left[\frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{3}{2}}} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu-\nu'|)} \right. \\ & \quad \left. + \frac{1}{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Together with the estimate (8.527) for $B_{j,\nu,\nu',l,m}^2$ in $2^m \leq 2^j|\nu-\nu'| < 2^l$, we finally obtain the following control for $B_{j,\nu,\nu',l,m}^2$:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j|\nu-\nu'|} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu',\nu,l,m}^2) \right| \\ & \lesssim \left[\frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{3}{2}}} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu-\nu'|)} \right. \\ & \quad \left. + \frac{1}{2^{\frac{3j}{4}}(2^{\frac{j}{2}}|\nu-\nu'|)^{\frac{1}{2}}} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

This concludes the proof of Proposition 8.2.

CHAPTER 9

PROOF OF PROPOSITION 6.6

Since $2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}$, we may assume that $l > m$ and thus:

$$(9.1) \quad 2^m \leq 2^j |\nu - \nu'| < 2^l.$$

In order to prove Proposition 6.6, recall that we need to show:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \right| \\ & \lesssim \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} \frac{2^{-2j} 2^{2\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|) 2^{\frac{j}{2}}} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l \geq 0}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l \geq 0} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2$$

and where $A_{j,\nu,\nu',l,m}$ is given by (8.8):

$$\begin{aligned} A_{j,\nu,\nu',l,m} &= -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \operatorname{tr} \chi(N - \mathbf{g}(N, N') N') (P_m \operatorname{tr} \chi')}{\mathbf{g}(L, L')} \\ & \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We may sum over the region (9.1), and we obtain:

$$\begin{aligned} (9.2) \quad & \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \\ &= -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_{>2^j |\nu - \nu'|} \operatorname{tr} \chi(N - \mathbf{g}(N, N') N') (P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi')}{\mathbf{g}(L, L')} \\ & \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts using (7.143).

Lemma 9.1. — *Let $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}$ $A_{j,\nu,\nu',l,m}$ be defined by (9.2). Integrating by parts using (7.143) yields:*

(9.3)

$$\begin{aligned} & \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \\ = & 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \left[\frac{1}{|N_\nu - N_{\nu'}|^2} (h_{1,p,q} + h_{2,p,q} + h_{3,p,q} + h_{4,p,q}) \right. \\ & \left. + \frac{1}{|N_\nu - N_{\nu'}|} (h_{5,p,q} + h_{6,p,q} + h_{7,p,q}) + h_{8,p,q} \right] d\mathcal{M}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$, $h_{5,p,q}$, $h_{6,p,q}$, $h_{7,p,q}$, $h_{8,p,q}$ on \mathcal{M} are given by:

(9.4)

$$\begin{aligned} h_{1,p,q} = & \left(\int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} \nabla' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(9.5)

$$\begin{aligned} h_{2,p,q} = & \left(\int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} \nabla' (L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(9.6)

$$\begin{aligned} h_{3,p,q} = & \left(\int_{\mathbb{S}^2} H_1 P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} \nabla' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(9.7)

$$\begin{aligned} h_{4,p,q} = & \left(\int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ & \times \left(\int_{\mathbb{S}^2} H_2 \nabla' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

(9.8)

$$h_{5,p,q} = \left(\int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$(9.9) \quad \times \left(\int_{\mathbb{S}^2} \nabla'^2 (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(9.10) \quad h_{6,p,q} = \left(\int_{\mathbb{S}^2} H_3 P_{> 2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(9.11) \quad h_{7,p,q} = \left(\int_{\mathbb{S}^2} P_{> 2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} H_4 \nabla (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

$$(9.11) \quad h_{8,p,q} = \left(\int_{\mathbb{S}^2} P_{> 2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla' (N' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

where the tensor H_1 on \mathcal{M} involved in the definition of $h_{3,p,q}$ is given by:

$$(9.12) \quad H_1 = \chi + \epsilon + \delta + n^{-1} \nabla n + L(b),$$

where the tensor H_2 on \mathcal{M} involved in the definition of $h_{4,p,q}$ is given by:

$$(9.13) \quad H_2 = \chi' + \epsilon' + \delta' + n^{-1} \nabla n + L'(b'),$$

where the tensor H_3 on \mathcal{M} involved in the definition of $h_{6,p,q}$ is given by:

$$(9.14) \quad H_3 = k + n^{-1} \nabla n + \theta + b^{-1} \nabla(b) + \chi + \zeta,$$

and where the tensor H_4 on \mathcal{M} involved in the definition of $h_{7,p,q}$ is given by:

$$(9.15) \quad H_4 = k + n^{-1} \nabla n + \theta' + b'^{-1} \nabla'(b') + \zeta' + \nabla_{N'}(b').$$

The proof of Lemma 9.1 is postponed to Appendix I. In the rest of this chapter, we use Lemma 9.1 to obtain the control of $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)} A_{j,\nu,\nu',l,m}$.

We evaluate the $L^1(\mathcal{CM})$ norm of $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}, h_{5,p,q}, h_{6,p,q}, h_{7,p,q}, h_{8,p,q}$ starting with $h_{1,p,q}$. In view of the Definition (9.4) of $h_{1,p,q}$, we have:

$$h_{1,p,q} = \sum_{l > 2^j |\nu - \nu'|} \int_{\mathbb{S}^2} G_1 L(P_{> 2^j |\nu - \nu'|} \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega,$$

where the tensor G_1 on \mathcal{M} is given by:

$$(9.16) \quad G_1 = \int_{\mathbb{S}^2} \nabla' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

In view of the estimate (7.10), this yields:

$$(9.17) \quad \begin{aligned} \|h_{1,p,q}\|_{L^1(\mathcal{M})} &\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u,x'} L_t^\infty} \right) \left(\sum_{l > 2^j |\nu - \nu'|} 2^{\frac{j}{2} - l} \right) \varepsilon \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u,x'} L_t^\infty} \right) \frac{\varepsilon \gamma_j^\nu}{2^{\frac{j}{2} |\nu - \nu'|}}. \end{aligned}$$

Now, in view of the Definition (9.16) of G_1 and the estimate (7.76), we have:

$$(9.18) \quad \begin{aligned} \sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u,x'} L_t^\infty} &\lesssim \left(\sup_{\omega'} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \varepsilon 2^j |\nu - \nu'| \gamma_j^{\nu'} \\ &\lesssim \varepsilon 2^j |\nu - \nu'| \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (9.17) and (9.18) imply:

$$(9.19) \quad \|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{j}{2}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{2,p,q}$. In view of the Definition (9.5) of $h_{2,p,q}$, we have:

$$h_{2,p,q} = \sum_{l > 2^j |\nu - \nu'|} \int_{\mathbb{S}^2} G_2 \nabla' (L' (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

where the tensor G_2 on \mathcal{M} is given by:

$$(9.20) \quad G_2 = \int_{\mathbb{S}^2} P_{> 2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega.$$

In view of the estimate (7.13), this yields:

$$(9.21) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega' \in \text{supp}(\eta_j^{\nu'})} \|G_2\|_{L^2_{u,x'} L_t^\infty} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Now, in view of the Definition (9.20) of G_2 and the estimate (7.71), we have:

$$(9.22) \quad \begin{aligned} \sup_{\omega' \in \text{supp}(\eta_j^{\nu'})} \|G_2\|_{L^2_{u,x'} L_t^\infty} &\lesssim \left(\sup_{\omega} \left\| \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty} \right) \varepsilon \gamma_j^\nu \\ &\quad \times \sum_{2^l > 2^j |\nu - \nu'|} \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-l + \frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{j}{4}} \right) \\ &\lesssim \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (9.21) and (9.22) imply:

$$(9.23) \quad \|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{j}{2}} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{3,p,q}$. In view of the Definition (9.6) of $h_{3,p,q}$, we have:

$$(9.24) \quad \begin{aligned} \|h_{3,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} H_1 P_{>2^j|\nu-\nu'|} \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla' (P_{\leq 2^j|\nu-\nu'|} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} H_1 P_{>2^j|\nu-\nu'|} \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317). The basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) yields:

$$(9.25) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} H_1 P_t \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| H_1 P_t \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L_x^\infty L_t^2(\mathcal{E}h_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega} \|H_1\|_{L_x^\infty L_t^2} \|P_t \text{tr}\chi\|_{L_x^2 L_t^\infty} \left\| \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega} \|H_1\|_{L_x^\infty L_t^2} \right) 2^{\frac{j}{2}-l} \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.69) for $P_t \text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, the definition of H_1 (9.12), the estimates (2.39) (2.40) for χ , the estimate (2.37) for ϵ and δ , the estimate (2.36) for n and the estimate (2.38) for b imply:

$$(9.26) \quad \begin{aligned} \|H_1\|_{L_x^\infty L_t^2} &\lesssim \|\chi\|_{L_x^\infty L_t^2} + \|\epsilon\|_{L_x^\infty L_t^2} + \|\delta\|_{L_x^\infty L_t^2} + \|n^{-1} \nabla n\|_{L_x^\infty L_t^2} + \|L(b)\|_{L_x^\infty L_t^2} \\ &\lesssim \varepsilon, \end{aligned}$$

which together with (9.25) yields:

$$(9.27) \quad \left\| \int_{\mathbb{S}^2} H_1 P_t \text{tr}\chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim 2^{\frac{j}{2}-l} \varepsilon \gamma_j^\nu.$$

Finally, (9.24) and (9.27) imply:

$$(9.28) \quad \begin{aligned} \|h_{3,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim 2^j \left(\sum_{2^l > 2^j |\nu - \nu'|} 2^{-l} \right) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \frac{2^{\frac{j}{2}}}{2^{\frac{j}{2}} |\nu - \nu'|} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{4,p,q}$. In view of the Definition (9.7) of $h_{4,p,q}$, we have:

(9.29)

$$\begin{aligned} & \|h_{4,p,q}\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \operatorname{tr}\chi \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} H_2 \nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \frac{\varepsilon\gamma_j^{\nu'}}{2^{\frac{j}{2}}|\nu-\nu'|} \left\| \int_{\mathbb{S}^2} H_2 \nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (8.129). The basic estimate in $L^2(\mathcal{O}\mathcal{M})$ (7.1) yields:

$$\begin{aligned} (9.30) \quad & \left\| \int_{\mathbb{S}^2} H_2 \nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| H_2 \nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_x^\infty L_t^2(\mathcal{E}H_u)} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|H_2\|_{L_x^\infty L_t^2} \|\nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi')\|_{L_x^2 L_t^\infty} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}}\gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|H_2\|_{L_x^\infty L_t^2} \right) 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (2.70) for $\nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi')$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, in view of the definition of H_2 (9.13), and proceeding as for the proof of (9.26), we have:

$$\|H_2\|_{L_x^\infty L_t^2} \lesssim \varepsilon,$$

which together with (9.30) yields:

$$(9.31) \quad \left\| \int_{\mathbb{S}^2} H_2 \nabla'(P_{\leq 2^j|\nu-\nu'|} \operatorname{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \lesssim 2^{\frac{j}{2}}\varepsilon\gamma_j^{\nu'}.$$

Finally, (9.29) and (9.31) imply:

$$(9.32) \quad \|h_{4,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim \frac{2^{\frac{j}{2}}}{2^{\frac{j}{2}}|\nu-\nu'|} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ norm of $h_{5,p,q}$. In view of the Definition (9.8) of $h_{5,p,q}$, we have:

$$\|h_{5,p,q}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} P_{>2^j|\nu-\nu'|} \operatorname{tr}\chi \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^P F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})}$$

$$\begin{aligned}
 & \times \left\| \int_{\mathbb{S}^2} \nabla'^2 (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 & \lesssim \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} \left(\int_{\mathbb{S}^2} \| P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \\
 & \quad \times \left(\int_{\mathbb{S}^2} \| \nabla'^2 (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right) \\
 & \lesssim \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} \left(\int_{\mathbb{S}^2} \| \| P_l \text{tr} \chi \|_{L^2(\mathcal{M}_u)} F_{j,-1}(u) \|_{L_u^2} \eta_j^\nu(\omega) d\omega \right) \\
 & \quad \times \left(\int_{\mathbb{S}^2} \| \| \nabla'^2 (P_m \text{tr} \chi') \|_{L^2(\mathcal{M}_{u'})} F_{j,-1}(u') \|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right),
 \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Taking Cauchy Schwartz in ω and ω' , using the size of the patches, and using the Bochner inequality (2.61), we obtain:

$$\begin{aligned}
 (9.33) \quad \| h_{5,p,q} \|_{L^1(\mathcal{M})} & \lesssim \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} 2^{2m-j} \left\| \| P_l \text{tr} \chi \|_{L^2(\mathcal{M}_u)} F_j(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L_{\omega,u}^2} \\
 & \quad \times \left\| \| P'_m \text{tr} \chi \|_{L^2(\mathcal{M}_{u'})} F_j(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L_{\omega',u'}^2}.
 \end{aligned}$$

In view of (9.33) and the estimate (6.18), we finally obtain:

$$(9.34) \quad \| h_{5,p,q} \|_{L^1(\mathcal{M})} \lesssim \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} 2^{2m-j} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_\nu \sum_{l>j/2} 2^{2l} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \| f \|_{L^2(\mathbb{R}^3)}^2.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{6,p,q}$. In view of the Definition (9.9) of $h_{6,p,q}$, we have:

$$\begin{aligned}
 (9.35) \quad \| h_{6,p,q} \|_{L^1(\mathcal{M})} & \leq \left\| \int_{\mathbb{S}^2} H_3 P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 & \quad \times \left\| \int_{\mathbb{S}^2} \nabla (P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 & \lesssim \left\| \int_{\mathbb{S}^2} H_3 P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(9.36) \quad \left\| \int_{\mathbb{S}^2} H_3 P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\begin{aligned}
&\lesssim \left(\sup_{\omega} \left\| H_3 P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P \right\|_{L_t^{\infty} L^2(\mathcal{E}_{u,t})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
&\lesssim \left(\sup_{\omega} \|H_3\|_{L_t^{\infty} L_{x'}^4} \|P_l \operatorname{tr} \chi\|_{L_t^2 L_{x'}^4} \left\| \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P \right\|_{L^{\infty}} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
&\lesssim \left(\sup_{\omega} \|H_3\|_{L_t^{\infty} L_{x'}^4} \right) \varepsilon 2^{\frac{j}{2} - \frac{l}{2}} \gamma_j^{\nu'},
\end{aligned}$$

where we used in the last inequality the estimate (8.303) for $P_l \operatorname{tr} \chi$ and the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. Now, in view of the Definition (9.14) for H_3 , we have:

$$\begin{aligned}
(9.37) \quad \|H_3\|_{L_t^{\infty} L_{x'}^4} &\lesssim \|k\|_{L_t^{\infty} L_{x'}^4} + \|n^{-1} \nabla n\|_{L_t^{\infty} L_{x'}^4} + \|\theta\|_{L_t^{\infty} L_{x'}^4} \\
&\quad + \|b^{-1} \nabla(b)\|_{L_t^{\infty} L_{x'}^4} + \|\chi\|_{L_t^{\infty} L_{x'}^4} + \|\zeta\|_{L_t^{\infty} L_{x'}^4} \\
&\lesssim \varepsilon,
\end{aligned}$$

where we used in the last inequality the embedding (2.51), the estimate (2.37) for k , the estimate (2.36) for n , the estimates (2.37) (2.39) (2.40) for θ , the estimate (2.38) for b , the estimate (2.39) (2.40) for χ and the estimate (2.41) for ζ . (9.36) and (9.37) yield:

$$(9.38) \quad \left\| \int_{\mathbb{S}^2} H_3 P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} - \frac{l}{2}} \gamma_j^{\nu'}.$$

Finally, (9.35) and (9.38) imply:

$$\begin{aligned}
(9.39) \quad \|h_{6,p,q}\|_{L^1(\mathcal{M})} &\lesssim \varepsilon^2 \left(\sum_{2^l > 2^j |\nu - \nu'|} 2^{j - \frac{l}{2}} \right) \gamma_j^{\nu'} \gamma_j^{\nu'} \\
&\lesssim \frac{\varepsilon^2 2^{\frac{3j}{4}} \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}}.
\end{aligned}$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{7,p,q}$. In view of the Definition (9.10) of $h_{7,p,q}$, we have:

$$(9.40)$$

$$\begin{aligned}
&\|h_{7,p,q}\|_{L^1(\mathcal{M})} \\
&\leq \left\| \int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^P F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
&\quad \times \left\| \int_{\mathbb{S}^2} H_4 \nabla(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
&\lesssim \frac{\varepsilon \gamma_j^{\nu'}}{2^{\frac{j}{2}} |\nu - \nu'|} \left\| \int_{\mathbb{S}^2} H_4 \nabla(P_{\leq 2^j |\nu - \nu'|} \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\end{aligned}$$

where we used in the last inequality the estimate (8.129). Proceeding as for the estimate of (8.607), we have:

$$(9.41) \quad \left\| \int_{\mathbb{S}^2} H_4 \nabla(P_l \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim \left(\sup_{\omega'} \|H_4\|_{L_t^\infty L_{x'}^4} \right) 2^{\frac{j}{2} + \frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Now, in view of the Definition (9.15) of H_4 , we have:

$$(9.42) \quad \|H_4\|_{L_t^\infty L_{x'}^4} \lesssim \|k\|_{L_t^\infty L_{x'}^4} + \|n^{-1} \nabla n\|_{L_t^\infty L_{x'}^4} + \|\theta'\|_{L_t^\infty L_{x'}^4} \\ + \|b'^{-1} \nabla'(b')\|_{L_t^\infty L_{x'}^4} + \|\zeta'\|_{L_t^\infty L_{x'}^4} + \|\nabla N'(b')\|_{L_t^\infty L_{x'}^4} \\ \lesssim \varepsilon,$$

where we used in the last inequality the embedding (2.51), the estimate (2.37) for k , the estimate (2.36) for n , the estimates (2.37) (2.39) (2.40) for θ' , the estimate (2.38) for b' , and the estimate (2.41) for ζ' . (9.41) and (9.42) yield:

$$(9.43) \quad \left\| \int_{\mathbb{S}^2} H_4 \nabla(P_l \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2} + \frac{j}{2}} \varepsilon \gamma_j^{\nu'}.$$

Finally, (9.40) and (9.43) imply:

$$(9.44) \quad \|h_{7,p,q}\|_{L^1(\mathcal{M})} \lesssim \frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} \left(\sum_{2^l > 2^j |\nu - \nu'|} 2^{\frac{l}{2}} \right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'} \\ \lesssim \frac{2^{\frac{3j}{4}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{8,p,q}$. In view of the Definition (9.11) of $h_{8,p,q}$, we have:

$$(9.45) \quad \|h_{8,p,q}\|_{L^1(\mathcal{M})} \leq \left\| \int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} \nabla'(N'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ \lesssim 2^j \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (8.129) for the first term, and the analog of the estimate (8.614) for the second term.

Now, we have in view of the decomposition (9.3) of $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m}$:

$$\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \right|$$

$$\begin{aligned} &\leq 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+1}} \right\|_{L^\infty(\mathcal{M})} \\ &\quad \times \left[\left\| \frac{1}{|N_\nu - N_{\nu'}|^2} \right\|_{L^\infty(\mathcal{M})} (\|h_{1,p,q}\|_{L^1(\mathcal{M})} + \|h_{2,p,q}\|_{L^1(\mathcal{M})} + \|h_{3,p,q}\|_{L^1(\mathcal{M})} + \|h_{4,p,q}\|_{L^1(\mathcal{M})}) \right. \\ &\quad \left. + \left\| \frac{1}{|N_\nu - N_{\nu'}|} \right\|_{L^\infty(\mathcal{M})} (\|h_{5,p,q}\|_{L^1(\mathcal{M})} + \|h_{6,p,q}\|_{L^1(\mathcal{M})} + \|h_{7,p,q}\|_{L^1(\mathcal{M})} + \|h_{8,p,q}\|_{L^1(\mathcal{M})}) \right]. \end{aligned}$$

Together with (8.32), (9.19), (9.23), (9.28), (9.32), (9.34), (9.39), (9.44), and (9.45), we obtain:

$$\begin{aligned} &\left| \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \right| \\ &\lesssim 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+1}} \left[\frac{2^{\frac{j}{2}}}{|\nu - \nu'|^2} \right. \\ &\quad \left. + \frac{1}{|\nu - \nu'|} \left(\sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} 2^{2m-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} + 2^{\frac{3j}{4}} \right) + 2^j \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} \frac{2^{2m-2j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ &\quad + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l > j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l > j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 6.6.

CHAPTER 10

PROOF OF PROPOSITION 6.7

Since $2^{\max(l,m)} \leq 2^j |\nu - \nu'|$, we may assume that $l \geq m$ and thus:

$$(10.1) \quad 2^m \leq 2^l \leq 2^j |\nu - \nu'|.$$

In order to prove Proposition 6.7, recall that we need to show:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \\ & \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l > j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l > j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2$$

and where $A_{j,\nu,\nu',l,m}$ is given by (8.8):

$$(10.2) \quad \begin{aligned} A_{j,\nu,\nu',l,m} = & -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \operatorname{tr} \chi(N - \mathbf{g}(N, N') N')(P_m \operatorname{tr} \chi')}{\mathbf{g}(L, L')} \\ & \times F_j(u) F_{j,-1}(u') \eta_j^{\nu}(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts using (7.137).

Lemma 10.1. — *Let $A_{j,\nu,\nu',l,m}$ be defined by (10.2). Integrating by parts using (7.137) yields:*

$$(10.3) \quad \begin{aligned} A_{j,\nu,\nu',l,m} = & A_{j,\nu,\nu',l,m}^1 + A_{j,\nu,\nu',l,m}^2 + A_{j,\nu,\nu',l,m}^3 + 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_{\nu} - N_{\nu'}|)^{p+q+2}} \\ & \times \left[\frac{1}{|N_{\nu} - N_{\nu'}|} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) + h_{3,p,q,l,m} + h_{4,p,q,l,m} \right] d_{\mathcal{M}}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $A_{j,\nu,\nu',l,m}^1$, $A_{j,\nu,\nu',l,m}^2$, $A_{j,\nu,\nu',l,m}^3$ on \mathcal{M} are given by:

$$(10.4) \quad A_{j,\nu,\nu',l,m}^1 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N')}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

$$(10.5) \quad A_{j,\nu,\nu',l,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N') N) (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi')}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}$$

(10.6)

$$\text{and: } A_{j,\nu,\nu',l,m}^3 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{N (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi')}{\mathbf{g}(L, L')} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

and where the scalar functions $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$ on \mathcal{M} are given by:

(10.7)

$$h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.8)

$$h_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \chi' \nabla' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.9)

$$h_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

(10.10)

$$\text{and: } h_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

The proof of Lemma 10.1 is postponed to Appendix J. In the rest of this chapter, we use Lemma 10.1 to obtain the control of $A_{j,\nu,\nu',l,m}$.

We evaluate the $L^1(\mathcal{CM})$ norm of $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$ starting with $h_{1,p,q,l,m}$. In view of the Definition (10.7) of $h_{1,p,q,l,m}$, we have:

$$\begin{aligned} \sum_{m \leq l} h_{1,p,q,l,m} &= \left(\int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \nabla'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right). \end{aligned}$$

This yields:

$$\begin{aligned} (10.11) \quad \left\| \sum_{m \leq l} h_{1,p,q,l,m} \right\|_{L^1(\mathcal{CM})} &\lesssim \left\| \int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{CM})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{CM})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{CM})} \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317). Now, the analog of (8.556) implies:

$$(10.12) \quad \left\| \int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{CM})} \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^{\nu'}.$$

Finally, (10.11) and (10.12) imply:

$$(10.13) \quad \left\| \sum_{m \leq l} h_{1,p,q,l,m} \right\|_{L^1(\mathcal{CM})} \lesssim \varepsilon^2 2^{j-l} \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{CM})$ norm of $h_{2,p,q,l,m}$. In view of the Definition (10.8) of $h_{2,p,q,l,m}$, we have:

$$\begin{aligned} \sum_{m \leq l} h_{2,p,q,l,m} &= \left(\int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \chi' \nabla'(P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right). \end{aligned}$$

This yields:

$$(10.14) \quad \left\| \sum_{m \leq l} h_{2,p,q,l,m} \right\|_{L^1(\mathcal{CM})} \lesssim \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{CM})}$$

$$\times \left\| \int_{\mathbb{S}^2} \chi' \nabla' (P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$

Using (8.179) for $l > j/2$ and (8.180) for $l = j/2$, we obtain for all $l \geq j/2$:

$$(10.15) \quad \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}-l} \varepsilon \gamma_j^{\nu}.$$

Also, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(10.16) \quad \begin{aligned} & \left\| \int_{\mathbb{S}^2} \chi' \nabla' (P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| \chi' \nabla' (P_{\leq l} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_x^{\infty} L_t^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|\chi'\|_{L_x^{\infty} L_t^2} \|\nabla' (P_{\leq l} \text{tr} \chi')\|_{L_x^2, L_t^{\infty}} \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^{\infty}} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ , the estimate (2.70) for $\nabla' (P_{\leq l} \text{tr} \chi')$, the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. Finally, (10.14), (10.15) and (10.16) imply:

$$(10.17) \quad \left\| \sum_{m \leq l} h_{2,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j-l} \gamma_j^{\nu} \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l,m}$. In view of the Definition (10.9) of $h_{3,p,q,l,m}$, we have:

$$(10.18) \quad \begin{aligned} \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} & \lesssim \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} \nabla' (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon^2 2^{(\frac{3}{4}+j)} \gamma_j^{\nu} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.275) for the first term, and the analog of the estimate (8.317) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$. In view of the Definition (10.10) of $h_{4,p,q,l,m}$, we have:

$$(10.19) \quad \begin{aligned} & \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \end{aligned}$$

$$\begin{aligned} & \times \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \varepsilon^2 2^{j+\frac{m}{2}-l} \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (10.15) for the first term, and the analog of the estimate (8.622) for the second term.

Next, we estimate $A_{j,\nu,\nu',l,m}^3$. In view of the Definition (10.6) of $A_{j,\nu,\nu',l,m}^3$ and the Definition (8.542) of $B_{j,\nu,\nu',l,m}^{2,2}$, and in view of the fact that $\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N')$, we see that $A_{j,\nu,\nu',l,m}^3$ is essentially obtained from $B_{j,\nu,\nu',l,m}^{2,2}$ by exchanging the role of ν and ν' . Proceeding for $A_{j,\nu,\nu',l,m}^3$ as we did for $B_{j,\nu,\nu',l,m}^{2,2}$, using the integration by parts (7.136) instead of (7.137), we obtain the analog of the estimate (8.624):

$$(10.20) \quad \left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^3 \right| \lesssim \left[\frac{2^{-\frac{j}{4}}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we consider $A_{j,\nu,\nu',l,m}^1$. We have the following proposition.

Proposition 10.2. — *Let $A_{j,\nu,\nu',l,m}^1$ be given by (10.4). Then, $A_{j,\nu,\nu',l,m}^1$ satisfies the following estimate:*

$$(10.21) \quad \begin{aligned} & \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^1 \right| \\ & \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & \quad + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

The proof of Proposition 10.2 is postponed to Section 10.1.

Next, we consider $A_{j,\nu,\nu',l,m}^2$. We have the following proposition.

Proposition 10.3. — *Let $A_{j,\nu,\nu',l,m}^2$ be given by (10.5). Then, $A_{j,\nu,\nu',l,m}^2$ satisfies the following estimate:*

$$(10.22) \quad \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^2 \right|$$

$$\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

The proof of Proposition 10.3 is postponed to Section 10.2.

Finally, we estimate $A_{j,\nu,\nu',l,m}$. In view of the decomposition (10.3) of $A_{j,\nu,\nu',l,m}$, the estimate (8.32), the estimates (10.13) (10.17) (10.18) (10.19) for $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, and the estimates (10.20) (10.21) (10.22) for $A_{j,\nu,\nu',l,m}^1$, $A_{j,\nu,\nu',l,m}^2$ and $A_{j,\nu,\nu',l,m}^3$, we obtain:

$$\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \\ \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ + 2^{-j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+2}} \left[\frac{2^{\frac{j}{2}}}{|\nu - \nu'|} + 2^{(\frac{3}{4})+j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 6.7.

10.1. Proof of Proposition 10.2 (Control of $A_{j,\nu,\nu',l,m}^1$)

In order to prove Proposition 10.2, recall that we need to show:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^1 \right| \\ & \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & \quad + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2$$

and where $A_{j,\nu,\nu',l,m}^1$ is given by (10.4):

$$\begin{aligned} (10.23) \quad A_{j,\nu,\nu',l,m}^1 &= 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N')}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)} \\ &\quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts using (7.137).

Lemma 10.4. — *Let $A_{j,\nu,\nu',l,m}^1$ be defined by (10.23). Integrating by parts using (7.137) yields:*

(10.24)

$$\begin{aligned} A_{j,\nu,\nu',l,m}^1 &= 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+2}} \left[\frac{1}{|N_\nu - N_{\nu'}|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) \right. \\ &\quad \left. + \frac{1}{|N_\nu - N_{\nu'}|} (h_{3,p,q,l,m} + h_{4,p,q,l,m} + h_{5,p,q,l,m} + h_{6,p,q,l,m}) + h_{7,p,q,l,m} \right] d_{\mathcal{M}}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, and where the scalar functions $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$ on \mathcal{M} are given by:

(10.25)

$$h_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left(\int_{\mathbb{S}^2} \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.26)

$$h_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \chi' \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.27)

$$h_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla (P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.28)

$$h_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'^3 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.29)

$$h_{5,p,q,l,m} = \left(\int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

(10.30)

$$h_{6,p,q,l,m} = \left(\int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

$$h_{7,p,q,l,m} = \left(\int_{\mathbb{S}^2} N(P_l \operatorname{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).$$

(10.31)

The proof of Lemma 10.4 is postponed to Appendix K. In the rest of this section, we use Lemma 10.4 to obtain the control of $A_{j,\nu,\nu',l,m}^1$.

We evaluate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$ starting with $h_{1,p,q,l,m}$. In view of the Definition (10.25)

of $h_{1,p,q,l,m}$, we have:

$$\begin{aligned}
 (10.32) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \chi P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 &\lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^\nu \left\| \int_{\mathbb{S}^2} \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
 \end{aligned}$$

where we used in the last inequality the estimate (10.12). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\begin{aligned}
 (10.33) \quad &\left\| \int_{\mathbb{S}^2} \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 &\lesssim \left(\sup_{\omega'} \left\| \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L_u^\infty L^2(\mathcal{E}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\
 &\lesssim 2^{\frac{j}{2}+m} \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the Bochner inequality (2.61), the finite band property for P_m , the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Finally, (10.32) and (10.33) imply:

$$(10.34) \quad \|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{j+m-l} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ of $h_{2,p,q,l,m}$. In view of the Definition (10.26) of $h_{2,p,q,l,m}$, we have:

$$h_{2,p,q,l,m} = \int_{\mathbb{S}^2} H \chi' \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

where H is given by:

$$(10.35) \quad H = \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega.$$

This yields:

$$\begin{aligned}
 (10.36) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \int_{\mathbb{S}^2} \left\| H \chi' \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 &\lesssim \int_{\mathbb{S}^2} \left\| H \chi'_1 \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 &\quad + \int_{\mathbb{S}^2} \left\| H \chi'_2 \nabla'^2(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega',
 \end{aligned}$$

where we used in the last inequality the decomposition (2.45) of χ' , and where we neglected the $\text{tr} \chi'$ contribution to χ' since it satisfies better estimates than χ'_1 . We

start with the first term in the right-hand side of (10.36). We have:

$$\begin{aligned}
 (10.37) \quad & \int_{\mathbb{S}^2} \left\| H \chi'_1 \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 & \lesssim \|H\|_{L^2(\mathcal{M})} \left(\int_{\mathbb{S}^2} \|\chi'_1\|_{L_u^\infty L_t^2 L_{x'}^\infty} \|\nabla'^2 (P_m \operatorname{tr} \chi')\|_{L_t^\infty L_{x'}^2} \right. \\
 & \quad \left. \times \left\| \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^\infty} \eta_j^{\nu'}(\omega') d\omega' \right) \\
 & \lesssim \|H\|_{L^2(\mathcal{M})} \varepsilon 2^{\frac{j}{2}+m} \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate (2.47) for χ'_1 , the Bochner inequality (2.61), the finite band property for P_m , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, Plancherel in λ' for $F_{j,-1}(u')$, and Cauchy Schwarz in ω' . Next, we estimate the second term in the right-hand side of (10.36). We have:

$$\begin{aligned}
 (10.38) \quad & \int_{\mathbb{S}^2} \left\| H \chi'_2 \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 & \lesssim \int_{\mathbb{S}^2} \left\| H (\chi'_2 - \chi_{2\nu}) \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 & \quad + \int_{\mathbb{S}^2} \left\| H \chi_{2\nu} \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\
 & \lesssim \left(\left(\sup_{\omega'} \|\chi'_2 - \chi_{2\nu}\|_{L^{6-}(\mathcal{M})} \right) \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} \right) \\
 & \quad \times \left(\int_{\mathbb{S}^2} \left\| \nabla'^2 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right) \\
 & \lesssim (\|\nu - \nu'\| \|H\|_{L^{3+}(\mathcal{M})} + \|\chi_{2\nu} H\|_{L^2(\mathcal{M})}) \varepsilon 2^{\frac{j}{2}+m} \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate (2.47) for $\partial_\omega \chi'_2$, the Bochner inequality (2.61), the finite band property for P_m , the estimate (2.39) for $\operatorname{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, Plancherel in λ' for $F_{j,-1}(u')$, and Cauchy Schwarz in ω' . Next, we estimate H . In view of the Definition (10.35) of H and the estimate (10.15), we have:

$$(10.39) \quad \|H\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}-l} \gamma_j^{\nu}.$$

Also, in view of the estimate (8.272), we have:

$$\|H\|_{L^\infty(\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j^{\nu},$$

which by interpolation with (10.39) implies:

$$(10.40) \quad \|H\|_{L^{3+}(\mathcal{M})} \lesssim \varepsilon 2^{\left(\frac{1}{3}\right)+j} \gamma_j^{\nu}.$$

Finally, we have:

$$(10.41) \quad \|\chi_{2\nu} H\|_{L^2(\mathcal{M})} \lesssim \|\chi_{2\nu}\|_{L_{x'\nu}^\infty L_t^2} \|H\|_{L_{u\nu, x'\nu}^\infty L_t^2}$$

$$\lesssim (1 + p^2)\varepsilon 2^{\frac{j}{4} - \frac{l}{2}} \gamma_j^\nu,$$

where we used in the last inequality the estimate (2.46) for $\chi_{2\nu}$, the estimate (7.71) for H in the case $l > j/2$, and the estimate (8.271) for H in the case $l = j/2$. Finally, (10.36), (10.37), (10.38), (10.39), (10.40) and (10.41) imply:

$$(10.42) \quad \|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + p^2)\varepsilon^2 (2^{\frac{j}{4} - \frac{l}{2}} + |\nu - \nu'| 2^{(\frac{1}{3}+j)}) 2^{\frac{j}{2}+m} \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ of $h_{3,p,q,l,m}$. We first consider the case where $m = j/2$. In view of the Definition (10.27) of $h_{3,p,q,l,m}$ with $m = j/2$, we have:

$$\begin{aligned} \|h_{3,p,q,l,j/2}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'^2(P_{\leq j/2} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{\frac{3j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of (10.33) for the second term. Since $l \geq j/2$, this yields in the case $m = j/2$:

$$(10.43) \quad \|h_{3,p,q,l,j/2}\|_{L^1(\mathcal{M})} \lesssim 2^{\varepsilon + \frac{l}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we consider the case where $m > j/2$. Since $l \geq m$, we also have $l > j/2$. In view of the Definition (10.27) of $h_{3,p,q,l,m}$, we have:

$$\begin{aligned} \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left(\int_{\mathbb{S}^2} \left\| \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| \nabla'^2(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim \left(\int_{\mathbb{S}^2} \left\| \nabla(P_l \text{tr}\chi) \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| \nabla'^2(P_m \text{tr}\chi') \right\|_{L^2(\mathcal{H}_{u'})} F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right), \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Taking Cauchy Schwartz in ω and ω' , using the size of the patches, and using the Bochner inequality (2.61) and the finite band property for P_l and P_m , we obtain:

$$(10.44) \quad \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{2m+l-j} \left\| \|P_l \text{tr}\chi\|_{L^2(\mathcal{H}_u)} F_j(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{\omega,u}} \\ \times \left\| \|P'_m \text{tr}\chi'\|_{L^2(\mathcal{H}_{u'})} F_j(u') \sqrt{\eta_j^{\nu'}(\omega')} \right\|_{L^2_{\omega',u'}}.$$

In view of (10.44) and the estimate (6.18), we finally obtain in the case $m > j/2$:

$$(10.45) \quad \|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{m+l+\min(l,m)-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ of $h_{4,p,q,l,m}$. We first consider the case $m = j/2$. In view of the Definition (10.28) of $h_{4,p,q,l,m}$ with $m = j/2$, we have:

$$h_{4,p,q,l,j/2} = \int_{\mathbb{S}^2} H \nabla'^3 (P_{\leq j/2} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

where H is given by:

$$(10.46) \quad H = \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega.$$

This yields:

$$(10.47) \quad \begin{aligned} \|h_{4,p,q,l,j/2}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim \int_{\mathbb{S}^2} \left\| H \nabla'^3 (P_{\leq j/2} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{O}\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \int_{\mathbb{S}^2} \left\| \|H\|_{L^2(P_{t,u'})} \|\nabla'^3 (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{t,u'})} F_{j,-1}(u') \right\|_{L^1_{u',t}} \eta_j^{\nu'}(\omega') d\omega', \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. Now, in view of the estimate (2.62), we have:

$$(10.48) \quad \begin{aligned} &\|\nabla'^3 (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{t,u'})} \\ &\lesssim \|\nabla \Delta (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{t,u'})} + \mu(t) \|\Delta (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{t,u'})} + \mu^2(t) \|\nabla (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{t,u'})} \\ &\lesssim (2^j + \mu(t) 2^{\frac{j}{2}} + \mu(t)^2) \varepsilon, \end{aligned}$$

where we used in the last inequality the finite band property for $P_{\leq j/2}$ and the estimate (2.39) for $\text{tr} \chi$, and where μ in a function in $L^2(\mathbb{R})$ satisfying:

$$(10.49) \quad \|\mu\|_{L^2(\mathbb{R})} \lesssim 1.$$

(10.47), (10.48) and (10.49) yield:

$$(10.50) \quad \begin{aligned} \|h_{4,p,q,l,j/2}\|_{L^1(\mathcal{O}\mathcal{M})} &\lesssim 2^j \|H\|_{L^2(\mathcal{O}\mathcal{M})} \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + \left(\sup_{\omega'} \|H\|_{L^2_{u'} L^{\infty}_t L^2_{x'}} \right) \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim 2^{2j-l} \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'} + \left(\sup_{\omega'} \|H\|_{L^2_{u'} L^{\infty}_t L^2_{x'}} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (10.15) for H which holds in view of the Definition (10.46) of H , Plancherel in λ' for $\|F_{j,-1}(u')\|_{L^2_{u'}}$, Cauchy Schwarz in ω' and the size of the patch. Using the estimate (7.71) in the case $l > j/2$, and the

estimates (7.71) (7.83) in the case $l = j/2$ together with the decomposition

$$P_{\leq j/2} \operatorname{tr} \chi = \operatorname{tr} \chi - \sum_{l > \frac{j}{2}} P_l \operatorname{tr} \chi,$$

we obtain:

$$(10.51) \quad \sup_{\omega'} \|H\|_{L_{u',x'}^2, L_t^\infty} \lesssim (1 + p^{\frac{5}{2}}) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l + \frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{j}{4}}) \gamma_j^{\nu'}.$$

Together with (10.50), this yields in the case $m = j/2$:

$$(10.52) \quad \|h_{4,p,q,l,j/2}\|_{L^1(\mathcal{M})} \lesssim 2^{2j-l} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'} + (1 + p^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{3j}{4}}) \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we consider the case where $m > j/2$. Since $l \geq m$, we also have $l > j/2$. In view of the Definition (10.28) of $h_{4,p,q,l,m}$, we have:

$$h_{4,p,q,l,m} = \int_{\mathbb{S}^2} H \nabla'^3 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega',$$

where H is given by (10.46). This yields:

$$(10.53) \quad \begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \int_{\mathbb{S}^2} \|H \nabla'^3 (P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u')\|_{L^1(\mathcal{C}\mathcal{M})} \eta_j^{\nu'}(\omega') d\omega' \\ &\lesssim \int_{\mathbb{S}^2} \| \|H\|_{L^2(P_{t,u'})} \|\nabla'^3 (P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})} F_{j,-1}(u')\|_{L_{u',t}^1} \eta_j^{\nu'}(\omega') d\omega', \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, in view of the estimate (2.62), we have:

$$(10.54) \quad \begin{aligned} \|\nabla'^3 (P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})} &\lesssim \|\nabla \Delta (P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})} + \mu(t) \|\Delta (P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})} + \mu^2(t) \|\nabla (P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})} \\ &\lesssim 2^{3m} \|P_m \operatorname{tr} \chi'\|_{L^2(P_{t,u'})} + 2^m \mu(t)^2 \varepsilon, \end{aligned}$$

where we used in the last inequality the finite band property for P_m and the estimate (2.39) for $\operatorname{tr} \chi$, and where μ in a function in $L^2(\mathbb{R})$ satisfying (10.49). (10.53), (10.54) and (10.49) yield:

$$(10.55) \quad \begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim 2^{3m} \|H\|_{L^2(\mathcal{C}\mathcal{M})} \left(\int_{\mathbb{S}^2} \| \|P_m \operatorname{tr} \chi'\|_{L^2(\mathcal{E}\mathcal{U}_{u'})} F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + 2^m \varepsilon \left(\sup_{\omega'} \|H\|_{L_{u',L_t^\infty}^2, L_x^2} \right) \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\lesssim 2^{3m} \|H\|_{L^2(\mathcal{C}\mathcal{M})} \left(\int_{\mathbb{S}^2} \| \|P_m \operatorname{tr} \chi'\|_{L^2(\mathcal{E}\mathcal{U}_{u'})} F_{j,-1}(u')\|_{L_{u'}^2} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + (1 + p^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j+m} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{3j}{4} + m}) \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (10.51) for H , Plancherel in λ' for $\|F_{j,-1}(u')\|_{L^2_{u'}}$, Cauchy Schwarz in ω' and the size of the patch. In view of the Definition (10.46) of H , and using the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we have:

$$\|H\|_{L^2(\mathcal{M})} \lesssim \int_{\mathbb{S}^2} \left\| \|P_l \text{tr} \chi\|_{L^2(\mathcal{E}_{\mathcal{H}_u})} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega.$$

Together with (10.55), this yields:

$$\begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim 2^{3m} \left(\int_{\mathbb{S}^2} \left\| \|P_l \text{tr} \chi\|_{L^2(\mathcal{E}_{\mathcal{H}_u})} F_{j,-1}(u) \right\|_{L^2_u} \eta_j^\nu(\omega) d\omega \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} \left\| \|P_m \text{tr} \chi'\|_{L^2(\mathcal{E}_{\mathcal{H}_{u'}})} F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j^{\nu'}(\omega') d\omega' \right) \\ &\quad + (1+p^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j+m} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{1}{2} + \frac{3j}{4} + m}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

Taking Cauchy Schwarz in ω and ω' , using the size of the patches, and using the Bochner inequality (2.61) and the finite band property for P_l and P_m , we obtain:

(10.56)

$$\begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim 2^{3m-j} \left\| \|P_l \text{tr} \chi\|_{L^2(\mathcal{E}_{\mathcal{H}_u})} F_j(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_{\omega,u}} \\ &\quad \times \left\| \|P'_m \text{tr} \chi\|_{L^2(\mathcal{E}_{\mathcal{H}_u})} F_j(u) \sqrt{\eta_j^{\nu'}(\omega)} \right\|_{L^2_{\omega,u}} \\ &\quad + (1+p^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j+m} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{1}{2} + \frac{3j}{4} + m}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. \end{aligned}$$

In view of (10.56) and the estimate (6.18), we finally obtain in the case $m > j/2$:

(10.57)

$$\begin{aligned} \|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim 2^{m+l+\min(l,m)-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ &\quad + (1+p^{\frac{5}{2}}) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j+m} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{1}{2} + \frac{3j}{4} + m}) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Next, we evaluate the $L^1(\mathcal{M})$ of $h_{5,p,q,l,m}$. In view of the Definition (10.29) of $h_{5,p,q,l,m}$, we have:

(10.58)

$$\begin{aligned} \|h_{5,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} (\theta + b^{-1} \nabla(b)) P_l \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla^2 (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{\left(\frac{3}{4}\right)+j+m} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimate (8.275) for the first term and the estimate (10.33) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ of $h_{6,p,q,l,m}$. In view of the Definition (10.30) of $h_{6,p,q,l,m}$, we have:

$$(10.59)$$

$$\begin{aligned} & \|h_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \\ & \lesssim \left\| \int_{\mathbb{S}^2} P_l \operatorname{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ & \quad \times \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'^2(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim 2^{\frac{j}{2}-l} \varepsilon \gamma_j^{\nu'} \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'^2(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}, \end{aligned}$$

where we used in the last inequality the estimate (10.15). The basic estimate in $L^2(\mathcal{M})$ (7.1) implies:

$$(10.60)$$

$$\begin{aligned} & \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'^2(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left(\sup_{\omega'} \left\| (\theta' + b'^{-1} \nabla'(b')) \nabla'^2(P_m \operatorname{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L_t^\infty L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|\theta' + b'^{-1} \nabla'(b')\|_{L_t^\infty L_x^4}, \|\nabla'^2(P_m \operatorname{tr} \chi')\|_{L_t^2 L_x^4}, \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^{\nu'} \\ & \lesssim \left(\sup_{\omega'} \|\nabla'^2(P_m \operatorname{tr} \chi')\|_{L_t^2 L_x^4} \right) \varepsilon 2^{\frac{j}{2}} \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the estimates (2.37) (2.39) (2.40) for χ' , the estimate (2.38) for b' , the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, the Gagliardo-Nirenberg inequality (2.49) yields:

$$\begin{aligned} \|\nabla'^2(P_m \operatorname{tr} \chi')\|_{L^4(P_{t,u'})} & \lesssim \|\nabla'^2(P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})}^{\frac{1}{2}} \|\nabla'^3(P_m \operatorname{tr} \chi')\|_{L^2(P_{t,u'})}^{\frac{1}{2}} \\ & \lesssim 2^{\frac{3m}{2}} \varepsilon (1 + \mu(t)), \end{aligned}$$

where we used in the last inequality the Bochner inequality (2.61), the finite band property for P_m , the estimates (10.48) and (10.54) for $\nabla'^3(P_m \operatorname{tr} \chi')$, and the estimate (2.39) for $\operatorname{tr} \chi$. Together with the estimate (10.49) for μ , we obtain:

$$\|\nabla'^2(P_m \operatorname{tr} \chi')\|_{L_t^2 L_x^4} \lesssim 2^{\frac{3m}{2}} \varepsilon,$$

which together with (10.60) yields:

$$(10.61) \quad \left\| \int_{\mathbb{S}^2} (\theta' + b'^{-1} \nabla'(b')) \nabla'^2 (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ \lesssim 2^{\frac{3m}{2} + \frac{j}{2}} \varepsilon^2 \gamma_j^{\nu'}.$$

Finally, (10.59) and (10.61) imply:

$$(10.62) \quad \|h_{6,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim 2^{j-l+\frac{3m}{2}} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{O}\mathcal{M})$ of $h_{7,p,q,l,m}$. In view of the Definition (10.31) of $h_{7,p,q,l,m}$, we have:

$$(10.63) \quad \|h_{7,p,q,l,m}\|_{L^1(\mathcal{O}\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} N(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ \times \left\| \int_{\mathbb{S}^2} \nabla'^2 (P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{O}\mathcal{M})} \\ \lesssim 2^{j+m} \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'},$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the estimate (10.33) for the second term.

Finally, we estimate $A_{j,\nu,\nu',l,m}^1$. In view of the decomposition (10.24) of $A_{j,\nu,\nu',l,m}^1$, the estimate (8.32), and the estimates (10.34) (10.42) (10.43) (10.45) (10.52) (10.57) (10.58) (10.62) (10.63) for $h_{1,p,q,l,m}$, $h_{2,p,q,l,m}$, $h_{3,p,q,l,m}$, $h_{4,p,q,l,m}$, $h_{5,p,q,l,m}$, $h_{6,p,q,l,m}$, $h_{7,p,q,l,m}$, we obtain:

$$\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^1 \right| \\ \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \sum_{p,q \geq 0} c_{pq} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+3}} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ + \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+2}} \left[\frac{1+p^2}{2^{\frac{j}{2}} |\nu - \nu'|} + (1+p^{\frac{5}{2}}) 2^{-(\frac{1}{6})-j} + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \right] \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'} \\ \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{2^{-(\frac{1}{6})-j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^{\nu} \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 10.2.

10.2. Proof of Proposition 10.3 (Control of $A_{j,\nu,\nu',l,m}^2$)

In order to prove Proposition 10.3, recall that we need to show:

$$\begin{aligned} & \left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^2 \right| \\ & \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ & + \left[\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2$$

and where $A_{j,\nu,\nu',l,m}^2$ is given by (10.5):

$$\begin{aligned} A_{j,\nu,\nu',l,m}^2 &= 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m(\text{tr} \chi'))}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\ (10.64) \quad &\times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts using (7.143).

Lemma 10.5. — Let $A_{j,\nu,\nu',l,m}^2$ be defined by (10.64). Integrating by parts using (7.143) yields:

(10.65)

$$\begin{aligned} & A_{j,\nu,\nu',l,m}^2 \\ &= 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{\frac{j}{2}} |N_\nu - N_{\nu'}|)^{p+q+2}} \left[\frac{1}{|N_\nu - N_{\nu'}|^2} (h'_{1,p,q,l,m} + h'_{2,p,q,l,m} \right. \\ & \quad \left. + h'_{3,p,q,l,m} + h'_{4,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|} (h'_{5,p,q,l,m} + h'_{6,p,q,l,m} + h'_{7,p,q,l,m}) + h'_{8,p,q,l,m} \right] d_{\mathcal{M}}, \end{aligned}$$

where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$, $h'_{8,p,q,l,m}$ on \mathcal{M} are given by:

$$(10.66) \quad h'_{1,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(L(P_l \text{tr}\chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.67) \quad h'_{2,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(L'(P_m \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.68) \quad h'_{3,p,q,l,m} = \left(\int_{\mathbb{S}^2} H_1 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.69) \quad h'_{4,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} H_2 \nabla'(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.70) \quad h'_{5,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'^2(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.71) \quad h'_{6,p,q,l,m} = \left(\int_{\mathbb{S}^2} H_3 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right),$$

$$(10.72) \quad h'_{7,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} H_4 \nabla(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right)$$

and:

$$(10.73) \quad h'_{8,p,q,l,m} = \left(\int_{\mathbb{S}^2} \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right) \\ \times \left(\int_{\mathbb{S}^2} \nabla'(N'(P_m \text{tr}\chi')) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right)$$

and where the tensors H_1 , H_2 , H_3 and H_4 are given by:

$$(10.74) \quad H_1 = \chi + \epsilon + \delta + n^{-1} \nabla n + L(b),$$

$$(10.75) \quad H_2 = \chi' + \epsilon' + \delta' + n^{-1} \nabla n + L'(b'),$$

$$(10.76) \quad H_3 = k + n^{-1} \nabla n + \theta + b^{-1} \nabla(b) + \chi + \zeta$$

and:

$$(10.77) \quad H_4 = k + n^{-1} \nabla n + \theta' + b'^{-1} \nabla(b') + \zeta'.$$

The proof of Lemma 10.5 is postponed to Appendix L. In the rest of this section, we use Lemma 10.5 to obtain the control of $A_{j,\nu,\nu',l,m}^2$.

We evaluate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q,l,m}, h'_{2,p,q,l,m}, h'_{3,p,q,l,m}, h'_{4,p,q,l,m}, h'_{5,p,q,l,m}, h'_{6,p,q,l,m}, h'_{7,p,q,l,m}, h'_{8,p,q,l,m}$ starting with $h'_{1,p,q,l,m}$. In view of the Definition (10.66) of $h'_{1,p,q,l,m}$, we have:

$$h'_{1,p,q,l,m} = \int_{\mathbb{S}^2} G \nabla(L(P_1 \text{tr} \chi)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega,$$

where G is given by:

$$(10.78) \quad G = \int_{\mathbb{S}^2} \nabla'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

Using the analog of (7.13), this yields:

$$(10.79) \quad \begin{aligned} \|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left(\sup_{\omega' \in \text{supp}(\eta_j^{\nu'})} \left\| G \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^2_{u,x'} L^\infty_t} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G\|_{L^2_{u,x'} L^\infty_t} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, in view of the Definition (10.78) of G , the analog of the estimate (7.76) yields:

$$\begin{aligned} \|G\|_{L^2_{u,x'} L^\infty_t} &\lesssim \left(\sup_{\omega'} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \right) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{\frac{j}{2}} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{\frac{m}{2} + \frac{j}{4}}) \gamma_j^{\nu'} \\ &\lesssim \varepsilon 2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the fact that $2^m \leq 2^j |\nu - \nu'|$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (10.79), we obtain:

$$\|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^j (2^{\frac{j}{2}} |\nu - \nu'|) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, since $m \geq j/2$, we finally obtain:

$$(10.80) \quad \|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{3j}{4}} 2^{\frac{m}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{2,p,q,l,m}$. Comparing the Definition (10.67) of $h'_{2,p,q,l,m}$ and (10.66) of $h'_{1,p,q,l,m}$, we notice that these terms are similar. We proceed as for $h'_{1,p,q,l,m}$, and we obtain the analog of (10.80):

$$(10.81) \quad \|h'_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{3j}{4}} 2^{\frac{m}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{3,p,q,l,m}$. In view of the Definition (10.68) of $h'_{3,p,q,l,m}$, we have:

$$\begin{aligned}
 (10.82) \quad \|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} H_1 \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 &\quad \times \left\| \int_{\mathbb{S}^2} \nabla'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
 &\lesssim \left\| \int_{\mathbb{S}^2} H_1 \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'},
 \end{aligned}$$

where we used in the last inequality the estimate the analog of the estimate (8.317). Now, using the basic estimate (7.1) in $L^2(\mathcal{M})$, we have:

$$\begin{aligned}
 (10.83) \quad &\left\| \int_{\mathbb{S}^2} H_1 \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
 &\lesssim \left(\sup_{\omega} \left\| H_1 \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \right\|_{L_x^\infty L_t^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \\
 &\lesssim \left(\sup_{\omega} \|H_1\|_{L_x^\infty L_t^2} \|\nabla(P_l \text{tr} \chi)\|_{L_x^2, L_t^1} \left\| \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \\
 &\lesssim \left(\sup_{\omega} \|H_1\|_{L_x^\infty L_t^2} \right) \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu,
 \end{aligned}$$

where we used in the last inequality the estimate (2.70) for $\nabla'(P_{\leq l} \text{tr} \chi')$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the Definition (10.74) of H_1 , we have:

$$\begin{aligned}
 \|H_1\|_{L_x^\infty L_t^2} &\lesssim \|\chi\|_{L_x^\infty L_t^2} + \|\epsilon\|_{L_x^\infty L_t^2} + \|\delta\|_{L_x^\infty L_t^2} + \|n^{-1} \nabla n\|_{L_x^\infty L_t^2} + \|L(b)\|_{L_x^\infty L_t^2} \\
 &\lesssim \varepsilon,
 \end{aligned}$$

where we used in the last inequality the estimates (2.39) (2.40) for χ , the estimates (2.36) (2.37) for ϵ , the estimate (2.37) for δ , the estimate (2.36) for n and the estimate (2.38) for b . Together with (10.83), this yields:

$$(10.84) \quad \left\| \int_{\mathbb{S}^2} H_1 \nabla(P_l \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.$$

(10.82) and (10.84) imply:

$$\|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^j \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Now, since $m \geq j/2$, we finally obtain:

$$(10.85) \quad \|h'_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{j}{2}} 2^m \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{4,p,q,l,m}$. Comparing the Definition (10.69) of $h'_{4,p,q,l,m}$ and (10.68) of $h'_{3,p,q,l,m}$, we notice that these terms are similar. We proceed

as for $h'_{3,p,q,l,m}$, and we obtain the analog of (10.85):

$$(10.86) \quad \|h'_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{j}{2}} 2^m \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{5,p,q,l,m}$. Comparing the Definition (10.70) of $h'_{5,p,q,l,m}$ and (10.27) of $h_{3,p,q,l,m}$, we notice that these terms are the same. Thus, in view of the estimates (10.43) and (10.45), we have in the case $m = j/2$:

$$(10.87) \quad \|h'_{5,p,q,l,j/2}\|_{L^1(\mathcal{M})} \lesssim 2^{\alpha+\frac{1}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}$$

and in the case $m > j/2$:

$$(10.88) \quad \|h'_{5,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{m+l+\min(l,m)-j} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{6,p,q,l,m}$. In view of the Definition (10.71) of $h'_{6,p,q,l,m}$, we have:

$$(10.89) \quad \begin{aligned} \|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} H_3 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \nabla(P_m \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^Q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left\| \int_{\mathbb{S}^2} H_3 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} 2^{\frac{j}{2}} \varepsilon \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of (8.317). Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$(10.90) \quad \begin{aligned} &\left\| \int_{\mathbb{S}^2} H_3 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\lesssim \left(\sup_{\omega} \left\| H_3 \nabla(P_l \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \right\|_{L_t^\infty L_x^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega} \|H_3\|_{L_t^\infty L_x^4} \|\nabla(P_l \text{tr}\chi)\|_{L_t^2 L_x^4} \left\| \left(2^{\frac{j}{2}}(N - N_\nu)\right)^P \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\ &\lesssim \left(\sup_{\omega} \|H_3\|_{L_t^\infty L_x^4} \right) \varepsilon 2^{\frac{l}{2}+\frac{j}{2}} \gamma_j^\nu, \end{aligned}$$

where we used in the last inequality the estimate (8.575) for $\nabla(P_l \text{tr}\chi)$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the Definition (10.76) of H_3 , we have:

$$\begin{aligned} \|H_3\|_{L_t^\infty L_x^4} &\lesssim \|k\|_{L_t^\infty L_x^4} + \|n^{-1} \nabla n\|_{L_t^\infty L_x^4} + \|\theta\|_{L_t^\infty L_x^4} \\ &\quad + \|b^{-1} \nabla(b)\|_{L_t^\infty L_x^4} + \|\chi\|_{L_t^\infty L_x^4} + \|\zeta\|_{L_t^\infty L_x^4} \lesssim \varepsilon, \end{aligned}$$

where we used in the last inequality the embedding (2.51), the estimate (2.37) for k , the estimate (2.36) for n , the estimates (2.37) (2.39) (2.40) for θ , the estimate (2.38) for b , the estimates (2.39) (2.40) for χ , and the estimate (2.41) for ζ . Together with (10.90), this yields:

$$(10.91) \quad \left\| \int_{\mathbb{S}^2} H_3 \nabla (P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{1}{2} + \frac{j}{2}} \gamma_j^\nu.$$

Finally, (10.89) and (10.91) imply:

$$(10.92) \quad \|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{j+\frac{1}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{7,p,q,l,m}$. Comparing the Definition (10.72) of $h'_{7,p,q,l,m}$ and (10.71) of $h'_{6,p,q,l,m}$, we notice that these terms are similar. We proceed as for $h'_{6,p,q,l,m}$, and we obtain the analog of (10.92):

$$(10.93) \quad \|h'_{7,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{j+\frac{m}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}.$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{8,p,q,l,m}$. In view of the Definition (10.73) of $h'_{8,p,q,l,m}$, we have:

$$(10.94) \quad \begin{aligned} \|h'_{8,p,q,l,m}\|_{L^1(\mathcal{M})} &\lesssim \left\| \int_{\mathbb{S}^2} \mathfrak{V}(P_l \text{tr} \chi) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\ &\quad \times \left\| \int_{\mathbb{S}^2} \mathfrak{V}'(N'(P_m \text{tr} \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\ &\lesssim 2^{j+m} \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}, \end{aligned}$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of the estimate (8.614) for the second term.

Finally, we estimate $A_{j,\nu,\nu',l,m}^2$. In view of the decomposition (10.65) of $A_{j,\nu,\nu',l,m}^2$, the estimate (8.32), and the estimates (10.80) (10.81) (10.85) (10.86) (10.87) (10.88) (10.92) (10.93) (10.94) for $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$, $h'_{8,p,q,l,m}$, we obtain:

$$\begin{aligned} &\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} A_{j,\nu,\nu',l,m}^2 \right| \\ &\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \sum_{p,q \geq 0} c_{pq} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+3}} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\ &\quad + \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+2}} \left[\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}}} + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\ &\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \end{aligned}$$

$$+ \left[\frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{5}{2}}} + \frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu - \nu'|)} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'},$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 10.3.

APPENDIX A

PROOF OF LEMMA 8.7

Recall from (8.42) that $B_{j,\nu,\nu'}^{1,1,1,1}$ is given by:

$$B_{j,\nu,\nu'}^{1,1,1,1} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} L(\mathrm{tr}\chi) \mathrm{tr}\chi' \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,1}$ using (7.137) with

$$(A.1) \quad h = \frac{L(\mathrm{tr}\chi) b' \mathrm{tr}\chi'}{\mathbf{g}(L, L')}.$$

We obtain:

(A.2)

$$B_{j,\nu,\nu'}^{1,1,1,1} \\ = -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\mathrm{tr}\theta - \mathbf{g}(N, N')\mathrm{tr}\theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. We have:

$$(A.3) \quad (N - \mathbf{g}(N, N')N')(h) \\ = \frac{\nabla_{N - \mathbf{g}(N, N')N'}(L(\mathrm{tr}\chi)) b' \mathrm{tr}\chi'}{\mathbf{g}(L, L')} + \frac{L(\mathrm{tr}\chi)(N - \mathbf{g}(N, N')N')(b' \mathrm{tr}\chi')}{\mathbf{g}(L, L')} \\ - \frac{(N - \mathbf{g}(N, N')N')(\mathbf{g}(L, L')) L(\mathrm{tr}\chi) b' \mathrm{tr}\chi'}{\mathbf{g}(L, L')^2}.$$

Decomposing $N - \mathbf{g}(N, N')N'$ on N and $N' - \mathbf{g}(N, N')N$, we have:

$$(A.4) \quad N - \mathbf{g}(N, N')N' = (1 - \mathbf{g}(N, N')^2)N - \mathbf{g}(N, N')(N' - \mathbf{g}(N, N')N),$$

which yields schematically for the first term in the right-hand side of (A.3):

$$(A.5) \quad \frac{\nabla_{N - \mathbf{g}(N, N')N'}(L(\text{tr}\chi))b'\text{tr}\chi'}{\mathbf{g}(L, L')} = \frac{N(L(\text{tr}\chi))b'\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla(L(\text{tr}\chi))b'\text{tr}\chi'(N' - N)}{\mathbf{g}(L, L')},$$

where we used the fact that:

$$\begin{aligned} 1 - \mathbf{g}(N, N')^2 &= (1 + \mathbf{g}(N, N'))(1 - \mathbf{g}(N, N')) \\ &= (1 + \mathbf{g}(N, N')) \frac{\mathbf{g}(N - N', N - N')}{2} \sim (N - N')^2. \end{aligned}$$

Finally, in order to estimate the third term in the right-hand side of (A.3), we need to compute $(N - \mathbf{g}(N, N')N')(\mathbf{g}(L, L'))$. Since $\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N')$, we need to compute $(N - \mathbf{g}(N, N')N')(\mathbf{g}(N, N'))$. Using the structure Equation (2.21) for N and the decomposition (A.4), we obtain:

$$(A.6) \quad \begin{aligned} \nabla_{N - \mathbf{g}(N, N')N'}(\mathbf{g}(N, N')) &= g(\nabla_{N - \mathbf{g}(N, N')N'}N, N') + g(N, \nabla_{N - \mathbf{g}(N, N')N'}N') \\ &= -(1 - \mathbf{g}(N, N')^2)g(b^{-1}\nabla b, N' - \mathbf{g}(N, N')N) \\ &\quad - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \\ &\quad + \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N'). \end{aligned}$$

Now, we have:

$$(A.7) \quad (N - \mathbf{g}(N, N')N') + (N' - \mathbf{g}(N, N')N) = (N + N')(1 - \mathbf{g}(N, N')) \sim (N - N')^2.$$

Also, since $\theta = \chi + k$ by definition, and since k does not depend on ω , we have:

$$(A.8) \quad \theta - \theta' = \chi - \chi'.$$

In view of (A.6), (A.7) and (A.8), we obtain schematically for the third term in the right-hand side of (A.3):

$$(A.9) \quad \begin{aligned} & - \frac{(N - \mathbf{g}(N, N')N')(\mathbf{g}(L, L'))L(\text{tr}\chi)b'\text{tr}\chi'}{\mathbf{g}(L, L')^2} \\ &= \frac{L(\text{tr}\chi)(\theta + b^{-1}\nabla(b))b'\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} + \frac{L(\text{tr}\chi)\theta'b'\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} \\ &\quad + \frac{(\chi - \chi')L(\text{tr}\chi)b'\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')^2}. \end{aligned}$$

Finally, (A.3), (A.5) and (A.9) imply, schematically:

$$(A.10) \quad \begin{aligned} & (N - \mathbf{g}(N, N')N')(h) \\ &= \frac{N(L(\text{tr}\chi))b'\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla(L(\text{tr}\chi))b'\text{tr}\chi'(N - N')}{\mathbf{g}(L, L')} \\ &\quad + \frac{L(\text{tr}\chi)\nabla'(b'\text{tr}\chi')(N - N')}{\mathbf{g}(L, L')} + \frac{L(\text{tr}\chi)(\theta + b^{-1}\nabla(b))b'\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} \end{aligned}$$

$$+ \frac{L(\text{tr}\chi)\theta'b'\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} + \frac{(\chi - \chi')L(\text{tr}\chi)b'\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')^2}.$$

We consider the term multiplied by h in the right-hand side of (A.2). Using (A.8), we have schematically:

$$\begin{aligned} \text{(A.11)} \quad & \text{tr}\theta - \mathbf{g}(N, N')\text{tr}\theta' - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \\ & - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \\ & + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \\ & \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \\ & = \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1}\nabla(b)(N - N'). \end{aligned}$$

Thus, in view of (A.1), (A.2), (A.10) and (A.11) we obtain:

$$\text{(A.12)} \quad B_{j,\nu,\nu'}^{1,1,1} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} HF_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(w)\eta_j^{\nu'}(w')dw dw' d_{\mathcal{M}},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned} \text{(A.13)} \quad H &= \frac{1}{1 - \mathbf{g}(N, N')^2} \left(\frac{N(L(\text{tr}\chi))\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla(L(\text{tr}\chi))\text{tr}\chi'(N - N')}{\mathbf{g}(L, L')} \right. \\ &+ \frac{L(\text{tr}\chi)b^{-1}\nabla'(b'\text{tr}\chi')(N - N')}{\mathbf{g}(L, L')} + \frac{L(\text{tr}\chi)(\theta + b^{-1}\nabla(b))\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} \\ &+ \frac{L(\text{tr}\chi)\theta'\text{tr}\chi'(N - N')^3}{\mathbf{g}(L, L')^2} + \frac{(\chi - \chi')L(\text{tr}\chi)\text{tr}\chi'(N - N')^2}{\mathbf{g}(L, L')^2} \\ &\left. + \left(\chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1}\nabla(b)(N - N') \right) \frac{L(\text{tr}\chi)\text{tr}\chi'}{\mathbf{g}(L, L')} \right). \end{aligned}$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand:

$$\frac{1}{(1 - \mathbf{g}(N, N')^2)\mathbf{g}(L, L')}, \frac{1}{\mathbf{g}(L, L')} \text{ and } \frac{1}{(1 - \mathbf{g}(N, N')^2)\mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and we obtain, schematically:

$$\begin{aligned} \text{(A.14)} \quad H &= \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\ &\times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right), \end{aligned}$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$\text{(A.15)} \quad H_1 = N(L(\text{tr}\chi))\text{tr}\chi',$$

$$(A.16) \quad H_2 = \nabla(L(\text{tr}\chi))\text{tr}\chi' + L(\text{tr}\chi)b'^{-1}\nabla'(b'\text{tr}\chi') + L(\text{tr}\chi)(\theta)\text{tr}\chi' \\ + \left(\theta + \theta' + b^{-1}\nabla(b)\right)L(\text{tr}\chi)\text{tr}\chi'$$

and:

$$(A.17) \quad H_3 = (\chi - \chi')L(\text{tr}\chi)\text{tr}\chi'$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (A.12), (A.14), (A.15), (A.16) and (A.17), we obtain the decomposition (8.44) (8.45) (8.46) (8.47) (8.48) (8.49) (8.50) of $B_{j,\nu,\nu'}^{1,1,1,1}$. This concludes the proof of Lemma 8.7.

APPENDIX B

PROOF OF LEMMA 8.8

Recall from (8.43) that $B_{j,\nu,\nu'}^{1,1,1,2}$ is given by:

$$B_{j,\nu,\nu'}^{1,1,1,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \operatorname{tr} \chi L' (\operatorname{tr} \chi') \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,2}$ using (7.137) with

$$(B.1) \quad h = \frac{\operatorname{tr} \chi b' L' (\operatorname{tr} \chi')}{\mathbf{g}(L, L')}.$$

We obtain:

(B.2)

$$B_{j,\nu,\nu'}^{1,1,1,2} \\ = -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N') N') (h) + \left(\operatorname{tr} \theta - \mathbf{g}(N, N') \operatorname{tr} \theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N') N, N' - \mathbf{g}(N, N') N) - \mathbf{g}(N, N') b^{-1} (N' - \mathbf{g}(N, N') N) (b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N') \theta(N' - \mathbf{g}(N, N') N, N' - \mathbf{g}(N, N') N) \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $(N - \mathbf{g}(N, N') N') (h)$. Proceeding as in (A.3), (A.5), (A.9) and (A.10), we obtain schematically:

$$(B.3) \quad (N - \mathbf{g}(N, N') N') (h) \\ = \frac{N(\operatorname{tr} \chi) b' L' (\operatorname{tr} \chi') (N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla'(\operatorname{tr} \chi) b' L' (\operatorname{tr} \chi') (N - N')}{\mathbf{g}(L, L')} \\ + \frac{\operatorname{tr} \chi \nabla' (b' L' (\operatorname{tr} \chi')) (N - N')}{\mathbf{g}(L, L')} + \frac{\operatorname{tr} \chi (\theta + b^{-1} \nabla'(b)) b' L' (\operatorname{tr} \chi') (N - N')^3}{\mathbf{g}(L, L')^2}$$

$$+ \frac{\operatorname{tr}\chi\theta'b'L'(\operatorname{tr}\chi')(N - N')^3}{\mathbf{g}(L, L')^2} + \frac{(\chi - \chi')\operatorname{tr}\chi b'L'(\operatorname{tr}\chi')(N - N')^2}{\mathbf{g}(L, L')^2}.$$

Thus, in view of (B.1), (B.2), (B.3) and (A.11) we obtain:

$$(B.4) \quad B_{j,\nu,\nu'}^{1,1,1,1} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^{\nu'}(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned} H = & \frac{1}{1 - \mathbf{g}(N, N')^2} \left(\frac{N(\operatorname{tr}\chi)L'(\operatorname{tr}\chi')(N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla'(\operatorname{tr}\chi)L'(\operatorname{tr}\chi')(N - N')}{\mathbf{g}(L, L')} \right. \\ & + \frac{\operatorname{tr}\chi b'^{-1}\nabla'(b'L'(\operatorname{tr}\chi'))(N - N')}{\mathbf{g}(L, L')} + \frac{\operatorname{tr}\chi(\theta + b^{-1}\nabla'(b))L'(\operatorname{tr}\chi')(N - N')^3}{\mathbf{g}(L, L')^2} \\ & + \frac{\operatorname{tr}\chi\theta'L'(\operatorname{tr}\chi')(N - N')^3}{\mathbf{g}(L, L')^2} + \frac{(\chi - \chi')\operatorname{tr}\chi L'(\operatorname{tr}\chi')(N - N')^2}{\mathbf{g}(L, L')^2} \\ & \left. + \left(\chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1}\nabla'(b)(N - N') \right) \frac{\operatorname{tr}\chi L'(\operatorname{tr}\chi')}{\mathbf{g}(L, L')} \right). \end{aligned}$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$(B.5) \quad H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\ \times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$(B.6) \quad H_1 = N(\operatorname{tr}\chi)L'(\operatorname{tr}\chi'),$$

$$(B.7) \quad H_2 = \nabla'(\operatorname{tr}\chi)L'(\operatorname{tr}\chi') + \operatorname{tr}\chi b'^{-1}\nabla'(b'L'(\operatorname{tr}\chi')) + \operatorname{tr}\chi(\theta)L'(\operatorname{tr}\chi') \\ + \left(\theta + \theta' + b^{-1}\nabla'(b) \right) \operatorname{tr}\chi L'(\operatorname{tr}\chi')$$

and:

$$(B.8) \quad H_3 = (\chi - \chi')\operatorname{tr}\chi L'(\operatorname{tr}\chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (B.4), (B.5), (B.6), (B.7) and (B.8), we obtain the decomposition (8.81) (8.82) (8.83) (8.84) (8.85) (8.86) (8.87) of $B_{j,\nu,\nu'}^{1,1,1,2}$. This concludes the proof of Lemma 8.8.

APPENDIX C

PROOF OF LEMMA 8.13

Recall from (8.95) that $B_{j,\nu,\nu',l,m}^{1,2,2}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{1}{\mathbf{g}(L, L')} \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) \\ \times (b^{-1} - b'^{-1} \mathbf{g}(N, N')) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,2,2}$ using (7.136) with

$$(C.1) \quad h = \frac{\left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) (b' - b)}{\mathbf{g}(L, L')}.$$

We obtain:

$$(C.2) \quad B_{j,\nu,\nu',l,m}^{1,2,2} \\ = -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N' - \mathbf{g}(N, N')N)(h) + \left(\text{tr}\theta' - \mathbf{g}(N, N')\text{tr}\theta \right. \right. \\ \left. \left. - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') - \mathbf{g}(N, N')b'^{-1}(N - \mathbf{g}(N, N')N')(b') \right) \right. \\ \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right. \right. \\ \left. \left. - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Next, we compute the term $(N' - \mathbf{g}(N, N')N)(h)$. We have:

$$(C.3) \quad (N' - \mathbf{g}(N, N')N)(h) \\ = \frac{\nabla_{N' - \mathbf{g}(N, N')N} (L(P_l \text{tr}\chi)) P_m \text{tr}\chi' (b' - b)}{\mathbf{g}(L, L')}$$

$$\begin{aligned}
 &+ \frac{L(P_l \text{tr} \chi)(N' - \mathbf{g}(N, N')N)(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)L'(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{P_l \text{tr} \chi \nabla_{N' - \mathbf{g}(N, N')N}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')} \\
 &- \frac{(N' - \mathbf{g}(N, N')N)(\mathbf{g}(L, L')) \left(L(P_l \text{tr} \chi)P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) (b' - b)}{\mathbf{g}(L, L')^2}.
 \end{aligned}$$

Decomposing $N' - \mathbf{g}(N, N')N$ on N' and $N - \mathbf{g}(N, N')N'$, we have:

$$(C.4) \quad N' - \mathbf{g}(N, N')N = (1 - \mathbf{g}(N, N')^2)N' - \mathbf{g}(N, N')(N - \mathbf{g}(N, N')N'),$$

which yields schematically for the second and the fourth term in the right-hand side of (C.3):

$$\begin{aligned}
 (C.5) \quad &\frac{L(P_l \text{tr} \chi)(N' - \mathbf{g}(N, N')N)(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} + \frac{P_l \text{tr} \chi \nabla_{N' - \mathbf{g}(N, N')N}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')} \\
 &= - \frac{L(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{L(P_l \text{tr} \chi)N'(P_m \text{tr} \chi')(b' - b)(N - N')^2}{\mathbf{g}(L, L')} \\
 &- \frac{P_l \text{tr} \chi \nabla_{N - \mathbf{g}(N, N')N'}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{P_l \text{tr} \chi \nabla_{N'}(L'(P_m \text{tr} \chi'))(b' - b)(N - N')^2}{\mathbf{g}(L, L')},
 \end{aligned}$$

where we used the fact that:

$$\begin{aligned}
 1 - \mathbf{g}(N, N')^2 &= (1 + \mathbf{g}(N, N'))(1 - \mathbf{g}(N, N')) \\
 &= (1 + \mathbf{g}(N, N')) \frac{\mathbf{g}(N - N', N - N')}{2} \sim (N - N')^2.
 \end{aligned}$$

Finally, in order to estimate the last term in the right-hand side of (A.3), we need to compute $(N' - \mathbf{g}(N, N')N)(\mathbf{g}(L, L'))$. We have the analog of (A.6):

$$\begin{aligned}
 (C.6) \quad \nabla_{N' - \mathbf{g}(N, N')N}(\mathbf{g}(N, N')) &= -(1 - \mathbf{g}(N, N')^2)g(b'^{-1} \nabla b', N - \mathbf{g}(N, N')N') \\
 &\quad - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \\
 &\quad + \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N).
 \end{aligned}$$

In view of (C.6), (A.7) and (A.8), we obtain schematically for the last term in the right-hand side of (C.3):

$$- \frac{(N' - \mathbf{g}(N, N')N)(\mathbf{g}(L, L')) \left(L(P_l \text{tr} \chi)P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) (b' - b)}{\mathbf{g}(L, L')^2}$$

$$\begin{aligned}
 &= \frac{\left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')\right)(b' - b)(\theta' + b'^{-1} \nabla'(b') + \theta)(N - N')^3}{\mathbf{g}(L, L')^2} \\
 \text{(C.7)} \quad &+ \frac{(\chi' - \chi) \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')\right)(b' - b)(N - N')^2}{\mathbf{g}(L, L')^2}.
 \end{aligned}$$

Finally, (C.3), (C.5) and (C.7) imply, schematically:

(C.8)

$$\begin{aligned}
 &(N' - \mathbf{g}(N, N')N)(h) \\
 &= \frac{\nabla_{N' - \mathbf{g}(N, N')N}(L(P_l \text{tr} \chi)) P_m \text{tr} \chi'(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) L'(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\
 &- \frac{L(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{L(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b)(N - N')^2}{\mathbf{g}(L, L')} \\
 &- \frac{P_l \text{tr} \chi \nabla_{N - \mathbf{g}(N, N')N'}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')} \\
 &+ \frac{P_l \text{tr} \chi \nabla_{N'}(L'(P_m \text{tr} \chi'))(b' - b)(N - N')^2}{\mathbf{g}(L, L')} \\
 &+ \frac{\left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')\right)(b' - b)(\theta' + b'^{-1} \nabla'(b') + \theta)(N - N')^3}{\mathbf{g}(L, L')^2} \\
 &+ \frac{(\chi' - \chi) \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')\right)(b' - b)(N - N')^2}{\mathbf{g}(L, L')^2}.
 \end{aligned}$$

We consider the term multiplied by h in the right-hand side of (C.2). Using (A.8), we have schematically:

$$\begin{aligned}
 \text{(C.9)} \quad &\text{tr} \theta - \mathbf{g}(N, N') \text{tr} \theta' - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \\
 &- \mathbf{g}(N, N') b^{-1} (N' - \mathbf{g}(N, N')N)(b) \\
 &+ \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \\
 &\quad \left. - \mathbf{g}(N, N') \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \\
 &= \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1} \nabla(b)(N - N').
 \end{aligned}$$

Thus, in view of (C.1), (C.2), (C.8) and (C.9) we obtain:

$$(C.10) \quad B_{j,\nu,\nu',l,m}^{1,2,2} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned} H = & \frac{1}{1 - \mathbf{g}(N, N')^2} \\ & \times \left[\frac{\nabla_{N' - \mathbf{g}(N, N')N} (L(P_l \text{tr}\chi)) P_m \text{tr}\chi'(b' - b)}{\mathbf{g}(L, L')} \right. \\ & + \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi) L'(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} \\ & - \frac{L(P_l \text{tr}\chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} \\ & + \frac{L(P_l \text{tr}\chi) N'(P_m \text{tr}\chi')(b' - b)(N - N')^2}{\mathbf{g}(L, L')} \\ & - \frac{P_l \text{tr}\chi \nabla_{N - \mathbf{g}(N, N')N'} (L'(P_m \text{tr}\chi'))(b' - b)}{\mathbf{g}(L, L')} \\ & + \frac{P_l \text{tr}\chi \nabla_{N'} (L'(P_m \text{tr}\chi'))(b' - b)(N - N')^2}{\mathbf{g}(L, L')} \\ & + \frac{\left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) (b' - b) (\theta' + b^{-1} \nabla'(b) + \theta) (N - N')^3}{\mathbf{g}(L, L')^2} \\ & + \frac{(\chi' - \chi) \left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) (b' - b) (N - N')^2}{\mathbf{g}(L, L')^2} \\ & + \left(\chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1} \nabla(b)(N - N') \right) \\ & \left. \times \frac{\left(L(P_l \text{tr}\chi) P_m \text{tr}\chi' + P_l \text{tr}\chi L'(P_m \text{tr}\chi') \right) (b' - b)}{\mathbf{g}(L, L')} \right]. \end{aligned}$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand:

$$\frac{1}{(1 - \mathbf{g}(N, N')^2) \mathbf{g}(L, L')}, \frac{1}{\mathbf{g}(L, L')} \text{ and } \frac{1}{(1 - \mathbf{g}(N, N')^2) \mathbf{g}(L, L')^2}$$

in the same fashion than (8.33), and we obtain, schematically:

(C.11)

$$\begin{aligned}
 H &= \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
 &\quad \times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 \right) \\
 &\quad + \frac{(N' - N)(b' - b)}{\mathbf{g}(L, L')^2} \left(\nabla(L(P_l \text{tr} \chi)) P_m \text{tr} \chi' + \nabla(P_l \text{tr} \chi) L'(P_m \text{tr} \chi') \right. \\
 &\quad \left. + L(P_l \text{tr} \chi) \nabla'(P_m \text{tr} \chi') + P_l \text{tr} \chi \nabla'(L'(P_m \text{tr} \chi')) \right) + \frac{P_l \text{tr} \chi \nabla_{N'}(L'(P_m \text{tr} \chi'))(b' - b)}{\mathbf{g}(L, L')} \\
 &\quad + \frac{(\chi' - \chi)(b' - b)}{\mathbf{g}(L, L')^2} \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right),
 \end{aligned}$$

where the tensors H_1, H_2 on \mathcal{M} are given by:

(C.12)

$$H_1 = L(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b)$$

and:

(C.13)

$$H_2 = \left(\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b') \right) \left(L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right) (b' - b)$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (C.10), (C.11), (C.12), and (C.13), we obtain the decomposition (8.256) (8.257) (8.258) (8.259) (8.260) (8.261) of $B_{j,\nu,\nu',l,m}^{1,2,2}$. This concludes the proof of Lemma 8.13.

APPENDIX D

PROOF OF LEMMA 8.14

Recall from (8.298) that $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,3} = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} P_l \text{tr} \chi N' (P_m \text{tr} \chi') (b' - b) \\ \times F_j(u) \eta_j^\nu(\omega) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$ using (7.137) with

$$(D.1) \quad h = P_l \text{tr} \chi b' N' (P_m \text{tr} \chi') (b' - b).$$

We obtain:

$$(D.2)$$

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \\ = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\text{tr} \theta - \mathbf{g}(N, N') \text{tr} \theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. Proceeding as in (A.3), (A.5) and (A.10), we obtain schematically:

$$(D.3)$$

$$(N - \mathbf{g}(N, N')N')(h) \\ = b'(1 - \mathbf{g}(N, N')^2)N(P_l \text{tr} \chi)N'(P_m \text{tr} \chi')(b' - b) \\ + b'(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)N'(P_m \text{tr} \chi')(b' - b) \\ + b'P_l \text{tr} \chi \nabla'(N'(P_m \text{tr} \chi'))(b' - b)(N - N') + b'b^{-1} \nabla(b)P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N')$$

$$+ \nabla(b')P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N') + b'N(b)P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N')^2.$$

Thus, in view of (D.1), (D.2), (D.3) and (A.11) we obtain:

$$(D.4) \quad B_{j,\nu,\nu',l,m}^{1,2,2,2,3} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned} H = & \frac{1}{1 - \mathbf{g}(N, N')^2} \left((1 - \mathbf{g}(N, N')^2) N(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b) \right. \\ & + (N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b) + P_l \text{tr} \chi \nabla'(N'(P_m \text{tr} \chi'))(b' - b)(N - N') \\ & + b^{-1} \nabla(b) P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N') + b'^{-1} \nabla(b') P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N') \\ & + N(b) P_l \text{tr} \chi' N'(P_m \text{tr} \chi')(N - N')^2 + \left(\chi - \chi' + \theta(N - N') + \theta'(N - N') \right. \\ & \left. + b^{-1} \nabla(b)(N - N') \right) P_l \text{tr} \chi N'(P_m \text{tr} \chi')(b' - b) \Big). \end{aligned}$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$(D.5) \quad \begin{aligned} H = & \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\ & \times \left(\frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + H_3 \right) \\ & + N(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b) + \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')}, \end{aligned}$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$(D.6) \quad H_1 = (\chi + \theta + \chi' + L'(b') + \theta') P_l \text{tr} \chi N'(P_m \text{tr} \chi')(b' - b),$$

$$(D.7) \quad H_2 = P_l \text{tr} \chi \nabla' N'(P_m \text{tr} \chi') + (\nabla'(b') + \nabla(b)) P_l \text{tr} \chi N'(P_m \text{tr} \chi')$$

and:

$$(D.8) \quad H_3 = N(b) P_l \text{tr} \chi N'(P_m \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (D.4), (D.5), (D.6), (D.7) and (D.8), we obtain the decomposition (8.323) (8.324) (8.325) (8.326) (8.327) (8.328) (8.329) of $B_{j,\nu,\nu',l,m}^{1,2,2,2,3}$. This concludes the proof of Lemma 8.14.

APPENDIX E

PROOF OF LEMMA 8.15

Recall from (8.296) that $B_{j,\nu,\nu',l,m}^{1,2,2,2,1}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,2,2,1} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')} \\ \times F_{j,-1}(u)\eta_j^\nu(\omega)F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,2}$ using (7.137) with

$$(E.1) \quad h = \frac{b(N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)b'N'(P_m \text{tr}\chi')(b' - b)}{\mathbf{g}(L, L')}.$$

We obtain:

$$(E.2) \quad B_{j,\nu,\nu'}^{1,1,1,2} = i2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\text{tr}\theta - \mathbf{g}(N, N')\text{tr}\theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) \\ \times F_{j,-2}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. Proceeding as in (A.3), (A.5), (A.9) and (A.10), we obtain schematically:

$$(E.3) \quad (N - \mathbf{g}(N, N')N')(h) \\ = \frac{N((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi))N'(P_m \text{tr}\chi')(N - N')}{\mathbf{g}(L, L')} \\ + \frac{(N' - \mathbf{g}(N, N')N)((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi))N'(P_m \text{tr}\chi')}{\mathbf{g}(L, L')}$$

$$\begin{aligned}
 & + \frac{\mathbb{V}(P_l \text{tr}\chi)\mathbb{V}(N'(P_m \text{tr}\chi'))(N - N')^2}{\mathbf{g}(L, L')} \\
 & + \left(\chi + \chi' + \theta + \theta' + b^{-1}\mathbb{V}(b) + b'^{-1}\mathbb{V}(b') \right) \frac{\mathbb{V}(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')(N - N')}{\mathbf{g}(L, L')}.
 \end{aligned}$$

Next, we evaluate the first two terms of (E.3) starting with the first one. We have, schematically:

$$\text{(E.4)} \quad N((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)) = \mathbb{V}(N'(P_l \text{tr}\chi))(N - N') + \nabla_{\nabla_N(N' - \mathbf{g}(N, N')N)}(P_l \text{tr}\chi).$$

Using the structure equations for N (2.21) together with the decomposition of N given by (7.140), we obtain, schematically:

$$\text{(E.5)} \quad \nabla_N(N' - \mathbf{g}(N, N')N) = \theta'(N - N') + b^{-1}\mathbb{V}(b) + b'^{-1}\mathbb{V}(b').$$

Together with (E.4), this yields, schematically:

$$\begin{aligned}
 \text{(E.6)} \quad N((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)) & = \mathbb{V}(N'(P_l \text{tr}\chi))(N - N') \\
 & + (\theta'(N - N') + b^{-1}\mathbb{V}(b) + b'^{-1}\mathbb{V}(b'))\mathbf{D}(P_l \text{tr}\chi).
 \end{aligned}$$

Next, we evaluate the second term in the right-hand side of (E.3). We have, schematically:

$$\begin{aligned}
 \text{(E.7)} \quad (N' - \mathbf{g}(N, N')N)((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)) & = \mathbb{V}^2 P_l \text{tr}\chi (N - N')^2 \\
 & + \nabla_{\nabla_{N' - \mathbf{g}(N, N')N}(N' - \mathbf{g}(N, N')N)}(P_l \text{tr}\chi).
 \end{aligned}$$

Using the structure equations for N (2.21) together with (A.6) and (A.7), we obtain, schematically:

$$\begin{aligned}
 \text{(E.8)} \quad & \nabla_{N' - \mathbf{g}(N, N')N}(N' - \mathbf{g}(N, N')N) \\
 & = \nabla_{N' - \mathbf{g}(N, N')N}N' - \mathbf{g}(N, N')\nabla_{N' - \mathbf{g}(N, N')N}N - \nabla_{N' - \mathbf{g}(N, N')N}(\mathbf{g}(N, N')N) \\
 & = (1 - \mathbf{g}(N, N')^2)\nabla_{N'}N' - \mathbf{g}(N, N')\nabla_{N - \mathbf{g}(N, N')N'}N' - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, e_A)e_A \\
 & \quad - ((1 - \mathbf{g}(N, N')^2)g(-\mathbb{V}' \log(a'), N) - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')) \\
 & \quad + \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N)N \\
 & = b'^{-1}\mathbb{V}(b')(N - N')^2 + (\theta + \theta')(N - N').
 \end{aligned}$$

Together with (E.7), this yields, schematically:

$$\begin{aligned}
 \text{(E.9)} \quad (N' - \mathbf{g}(N, N')N)((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)) \\
 = \mathbb{V}^2 P_l \text{tr}\chi (N - N')^2 + (b'^{-1}\mathbb{V}(b')(N - N')^2 + (\theta + \theta')(N - N'))\mathbb{V} P_l \text{tr}\chi.
 \end{aligned}$$

In view of (E.1), (E.2), (E.3), (E.6), (E.7), and (A.11) we obtain:

$$\text{(E.10)} \quad B_{j, \nu, \nu', l, m}^{1, 2, 2, 2, 1} = 2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j, -2}(u) F_{j, -1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$H = \frac{1}{1 - \mathbf{g}(N, N')^2} \left(\frac{\mathbb{V}^2(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')(N - N')^2}{\mathbf{g}(L, L')} + \frac{\mathbb{V}(N(P_l \text{tr}\chi))N'(P_m \text{tr}\chi')(N - N')^3}{\mathbf{g}(L, L')} \right. \\ \left. + \frac{\mathbb{V}'(P_l \text{tr}\chi')\mathbb{V}'(N'(P_m \text{tr}\chi'))(N - N')^2}{\mathbf{g}(L, L')} + \left(\chi + \chi' + \theta + \theta' + b^{-1}\mathbb{V}(b) + b'^{-1}\mathbb{V}(b') \right) \right. \\ \left. \times \frac{\mathbb{V}(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')}{\mathbf{g}(L, L')} \left((N - N') + \frac{(N - N')^3}{\mathbf{g}(L, L')} \right) \right).$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$(E.11) \quad H = \frac{1}{|N_\nu - N_{\nu'}|} \left(\sum_{p, q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\ \times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$(E.12) \quad H_1 = \mathbb{V}(N(P_l \text{tr}\chi))N'(P_m \text{tr}\chi'),$$

$$(E.13) \quad H_2 = \mathbb{V}^2(P_l \text{tr}\chi)N'(P_m \text{tr}\chi') + \mathbb{V}'(P_l \text{tr}\chi')\mathbb{V}'(N'(P_m \text{tr}\chi'))$$

and:

$$(E.14) \quad H_3 = \left(\chi + \chi' + \theta + \theta' + b^{-1}\mathbb{V}(b) + b'^{-1}\mathbb{V}(b') \right) \mathbb{V}(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p, q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (E.10), (E.11), (E.12), (E.13) and (E.14), we obtain the decomposition (8.340), (8.341), (8.342), (8.343), (8.344), (8.345), (8.346), (8.347) of $B_{j, \nu, \nu', l, m}^{1, 2, 2, 2, 1}$. This concludes the proof of Lemma 8.15.

APPENDIX F

PROOF OF LEMMA 8.16

Recall from (8.472) that $B_{j,\nu,\nu',l,m}^{1,2,3,1}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,3,1} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b'^{-1} L(P_l \text{tr} \chi) P_m \text{tr} \chi' F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,2,3,1}$ using (7.136) with

$$(F.1) \quad h = bL(P_l \text{tr} \chi) P_m \text{tr} \chi'.$$

We obtain:

(F.2)

$$\begin{aligned} & B_{j,\nu,\nu',l,m}^{1,2,3,1} \\ &= -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N' - \mathbf{g}(N, N')N)(h) + \left(\text{tr} \theta' - \mathbf{g}(N, N') \text{tr} \theta \right. \right. \\ & \quad \left. \left. - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') - \mathbf{g}(N, N')b'^{-1}(N - \mathbf{g}(N, N')N')(b') \right) \right. \\ & \quad \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right. \right. \\ & \quad \left. \left. - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right) \right) h \\ & \quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \end{aligned}$$

Next, we compute the term $(N' - \mathbf{g}(N, N')N)(h)$. Proceeding as in (C.3), (C.5), (C.7) and (C.8), we obtain schematically:

$$\begin{aligned} (F.3) \quad & (N' - \mathbf{g}(N, N')N)(h) \\ &= b\mathcal{V}(L(P_l \text{tr} \chi)) P_m \text{tr} \chi (N - N') + bL(P_l \text{tr} \chi) \mathcal{V}'(P_m \text{tr} \chi) (N - N') \\ & \quad + bL(P_l \text{tr} \chi) N' (P_m \text{tr} \chi) (N - N')^2 + (\chi - \chi') L(P_l \text{tr} \chi) P_m \text{tr} \chi \\ & \quad + (\theta + \theta' + b^{-1}\mathcal{V}(b) + b'^{-1}\mathcal{V}(b')) L(P_l \text{tr} \chi) P_m \text{tr} \chi (N - N'). \end{aligned}$$

Thus, in view of (F.1), (F.2), (F.3) and (C.9) we obtain:

$$(F.4) \quad B_{j,\nu,\nu',l,m}^{1,2,3,1} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$H = \frac{1}{1 - \mathbf{g}(N, N')^2} \times \left[\begin{aligned} &\mathcal{V}(L(P_l \text{tr}\chi)) P_m \text{tr}\chi (N - N') + L(P_l \text{tr}\chi) \mathcal{V}'(P_m \text{tr}\chi) (N - N') \\ &+ L(P_l \text{tr}\chi) N' (P_m \text{tr}\chi) (N - N')^2 + (\chi - \chi') L(P_l \text{tr}\chi) P_m \text{tr}\chi \\ &+ (\theta + \theta' + b^{-1} \mathcal{V}(b) + b'^{-1} \mathcal{V}(b')) L(P_l \text{tr}\chi) P_m \text{tr}\chi (N - N') \end{aligned} \right].$$

Proceeding in the same fashion than (C.11), we obtain, schematically:

$$(F.5) \quad H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \times \left(\frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + H_3 \right),$$

where the tensors H_1, H_2, H_3 on \mathcal{M} are given by:

$$(F.6) \quad H_1 = (\chi - \chi') L(P_l \text{tr}\chi) P_m \text{tr}\chi,$$

$$(F.7) \quad H_2 = \mathcal{V}(L(P_l \text{tr}\chi)) P_m \text{tr}\chi + L(P_l \text{tr}\chi) \mathcal{V}'(P_m \text{tr}\chi) + (\theta + \theta' + b^{-1} \mathcal{V}(b) + b'^{-1} \mathcal{V}(b')) L(P_l \text{tr}\chi) P_m \text{tr}\chi$$

and:

$$(F.8) \quad H_3 = L(P_l \text{tr}\chi) N' (P_m \text{tr}\chi)$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (F.4), (F.5), (F.6), (F.7) and (F.8), we obtain the decomposition (8.474) (8.475) (8.476) (8.477) (8.478) (8.479) (8.480) (8.481) of $B_{j,\nu,\nu',l,m}^{1,2,3,1}$. This concludes the proof of Lemma 8.16.

APPENDIX G

PROOF OF LEMMA 8.17

Recall from (8.11) that $B_{j,\nu,\nu',l,m}^2$ is given by:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^2 &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left((\mathbf{g}(N, N') - 1) P_l \text{tr} \chi N' (P_m \text{tr} \chi') \right. \\ &\quad + \left(\text{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' - 2\zeta'_{N - \mathbf{g}(N, N')N'} \right. \\ &\quad \left. \left. - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) \right) \\ &\quad \times P_l \text{tr} \chi P_m \text{tr} \chi' \Big) F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}} \mathcal{M}. \end{aligned}$$

Together with the identity (8.30):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N'),$$

we obtain:

$$\begin{aligned} B_{j,\nu,\nu',l,m}^2 &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left[\frac{b^{-1}}{\mathbf{g}(L, L')} \left(\text{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' \right. \right. \\ &\quad \left. \left. - 2\zeta'_{N - \mathbf{g}(N, N')N'} - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) P_l \text{tr} \chi P_m \text{tr} \chi' \right. \\ &\quad \left. + b^{-1} P_l \text{tr} \chi N' (P_m \text{tr} \chi') \right] F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}} \mathcal{M}. \end{aligned}$$

We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1,2}$ using (7.137) with

(G.1)

$$\begin{aligned} h &= \frac{b'}{\mathbf{g}(L, L')} \left(\text{tr} \chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N')) \delta' - 2\zeta'_{N - \mathbf{g}(N, N')N'} \right. \\ &\quad \left. - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \right) P_l \text{tr} \chi P_m \text{tr} \chi' + b' P_l \text{tr} \chi N' (P_m \text{tr} \chi'). \end{aligned}$$

We obtain:

(G.2)

$$\begin{aligned}
& B_{j,\nu,\nu',l,m}^2 \\
&= -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\text{tr}\theta - \mathbf{g}(N, N')\text{tr}\theta' \right. \right. \\
&\quad - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \\
&\quad + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \\
&\quad \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \\
&\quad \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.
\end{aligned}$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. Using the structure equation for N (2.21), we have, schematically:

$$\begin{aligned}
& \nabla_{N - \mathbf{g}(N, N')N'}(N - \mathbf{g}(N, N')N') \\
&= \nabla_{N - \mathbf{g}(N, N')N'}N - \mathbf{g}(N, N')\nabla_{N - \mathbf{g}(N, N')N'}N' - \nabla_{N - \mathbf{g}(N, N')N'}(\mathbf{g}(N, N'))N' \\
&= (1 - \mathbf{g}(N, N')^2)\nabla_N N - \mathbf{g}(N, N')\nabla_{N' - \mathbf{g}(N, N')N}N - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', e_{A'})e_{A'} \\
&\quad - ((1 - \mathbf{g}(N, N')^2)g(-\nabla \log(b), N') - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \\
&\quad + \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N'))N' \\
&= -(1 - \mathbf{g}(N, N')^2)\nabla \log(b) - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, e_A)e_A \\
&\quad - \mathbf{g}(N, N')\theta'(N - \mathbf{g}(N, N')N', e_{A'})e_{A'} + (1 - \mathbf{g}(N, N')^2)\nabla_{N' - \mathbf{g}(N, N')N} \log(b)N' \\
&\quad + \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N)N' \\
&\quad - \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')N',
\end{aligned}$$

which we rewrite schematically as:

(G.3)

$$\nabla_{N - \mathbf{g}(N, N')N'}(N - \mathbf{g}(N, N')N') = (\theta - \theta')(N - N') + (\theta + \theta' + b^{-1}\nabla(b))(N - N')^2.$$

Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3), we obtain schematically:

(G.4)

$$\begin{aligned}
& (N - \mathbf{g}(N, N')N')(h) \\
&= \left(\chi + \bar{\delta} + \chi' + \bar{\delta}' + \zeta'(N - N') \right) \frac{b'}{\mathbf{g}(L, L')} \left(\nabla P_l \text{tr}\chi P_m \text{tr}\chi'(N - N') \right. \\
&\quad + P_l \text{tr}\chi \nabla' P_m \text{tr}\chi'(N - N') + N(P_l \text{tr}\chi)P_m \text{tr}\chi'(N - N')^2 \left. \right) + \left(\nabla(\chi)(N - N') \right. \\
&\quad \left. + \nabla(\bar{\delta})(N - N') + \mathbf{D}_N(\chi)(N - N')^2 + N(\bar{\delta})(N - N')^2 + \nabla'(\chi')(N - N') \right)
\end{aligned}$$

$$\begin{aligned}
 & + \nabla'(\bar{\delta}')(N - N') + \nabla'(\zeta')(N - N')^2 + b^{-1}\nabla(b')(N - N') + (\chi + \bar{\delta} + \chi' + \bar{\delta}' \\
 & + \zeta'(N - N'))(\theta - \theta' + (\theta + \theta' + b^{-1}\nabla(b))(N - N')) \Big) \frac{b'}{\mathbf{g}(L, L')} P_l \text{tr} \chi P_m \text{tr} \chi' \\
 & + (1 - \mathbf{g}(N, N')^2) b' N (P_l \text{tr} \chi) N' (P_m \text{tr} \chi') + b'(N' - \mathbf{g}(N, N')N) (P_l \text{tr} \chi) N (P_m \text{tr} \chi') \\
 & + (N - N') b' P_l \text{tr} \chi \nabla'(N' (P_m \text{tr} \chi')) + (N - N') b'^{-1} \nabla'(b') P_l \text{tr} \chi N' (P_m \text{tr} \chi').
 \end{aligned}$$

Thus, in view of (G.1), (G.2), (G.4), (A.8) and (A.11) we obtain:

$$(G.5) \quad B_{j,\nu,\nu',l,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
 H = & \frac{1}{1 - \mathbf{g}(N, N')^2} \left((\chi + \bar{\delta} + \chi' + \bar{\delta}' + \zeta'(N - N')) \frac{1}{\mathbf{g}(L, L')} \left(\nabla P_l \text{tr} \chi P_m \text{tr} \chi' (N - N') \right. \right. \\
 & + P_l \text{tr} \chi \nabla' P_m \text{tr} \chi' (N - N') + N (P_l \text{tr} \chi) P_m \text{tr} \chi' (N - N')^2 \Big) + \left(\nabla(\chi)(N - N') \right. \\
 & + \nabla(\bar{\delta})(N - N') + \mathbf{D}_N(\chi)(N - N')^2 + N(\bar{\delta})(N - N')^2 + \nabla'(\chi')(N - N') \\
 & + \nabla'(\bar{\delta}')(N - N') + \nabla'(\zeta')(N - N')^2 + b^{-1}\nabla(b')(N - N') + (\chi + \bar{\delta} + \chi' + \bar{\delta}' \\
 & + \zeta'(N - N'))(\chi - \chi' + (\theta + \theta' + b^{-1}\nabla(b))(N - N')) \Big) \frac{1}{\mathbf{g}(L, L')} P_l \text{tr} \chi P_m \text{tr} \chi' \\
 & + (1 - \mathbf{g}(N, N')^2) b' N (P_l \text{tr} \chi) N' (P_m \text{tr} \chi') + (N' - \mathbf{g}(N, N')N) (P_l \text{tr} \chi) N (P_m \text{tr} \chi') \\
 & \left. + (N - N') P_l \text{tr} \chi \nabla'(N' (P_m \text{tr} \chi')) + (N - N') b'^{-1} \nabla'(b') P_l \text{tr} \chi N' (P_m \text{tr} \chi') \right).
 \end{aligned}$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$\begin{aligned}
 (G.6) \quad H = & \frac{1}{|N_\nu - N_{\nu'}|} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
 & \times \left(\frac{1}{|N_\nu - N_{\nu'}|^3} H_1 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_2 + \frac{1}{|N_\nu - N_{\nu'}|} H_3 + H_4 \right) \\
 & + N (P_l \text{tr} \chi) N' (P_m \text{tr} \chi') + \frac{(N' - \mathbf{g}(N, N')N) (P_l \text{tr} \chi) N (P_m \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2},
 \end{aligned}$$

where the tensors H_1, H_2, H_3 and H_4 on \mathcal{M} are given by:

$$(G.7) \quad H_1 = (\chi - \chi')(\chi + \bar{\delta} + \chi' + \bar{\delta}') P_l \text{tr} \chi P_m \text{tr} \chi',$$

$$\begin{aligned}
 (G.8) \quad H_2 = & (\chi + \bar{\delta} + \chi' + \bar{\delta}') (\nabla P_l \text{tr} \chi P_m \text{tr} \chi' + P_l \text{tr} \chi \nabla' P_m \text{tr} \chi') \\
 & + \left(\nabla(\chi) + \nabla(\bar{\delta}) + \nabla'(\chi') + \nabla'(\bar{\delta}') + b^{-1}\nabla(b') \right. \\
 & \left. + (\chi + \bar{\delta} + \chi' + \bar{\delta}')(\theta + \theta' + b^{-1}\nabla(b)) + (\chi - \chi')\zeta' \right) P_l \text{tr} \chi P_m \text{tr} \chi',
 \end{aligned}$$

(G.9)

$$H_3 = (\chi + \bar{\delta} + \chi' + \bar{\delta}')N(P_l \text{tr} \chi)P_m \text{tr} \chi' + \zeta'(\nabla P_l \text{tr} \chi P_m \text{tr} \chi' + P_l \text{tr} \chi \nabla' P_m \text{tr} \chi') \\ + \left(\mathbf{D}_N(\chi) + N(\bar{\delta}) + \nabla'(\zeta') + \zeta'(\theta + \theta' + b^{-1} \nabla(b)) \right) P_l \text{tr} \chi P_m \text{tr} \chi'$$

and:

(G.10)

$$H_4 = P_l \text{tr} \chi \nabla'(N'(P_m \text{tr} \chi')) + b'^{-1} \nabla'(b') P_l \text{tr} \chi N'(P_m \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (G.5), (G.6), (G.7), (G.8), (G.9) and (G.10), we obtain the decomposition (8.528) (8.529) (8.530) (8.531) (8.532) (8.533) (8.534) (8.535) (8.536) (8.537) (8.538) (8.539) (8.540) (8.541) (8.542) of $B_{j,\nu,\nu',l,m}^2$. This concludes the proof of Lemma 8.17.

APPENDIX H

PROOF OF LEMMA 8.18

Recall from (8.596) that $\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2}$ is given by:

$$\begin{aligned} \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} &= 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2} \\ &\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts using (7.137) with

$$(H.1) \quad h = \frac{bb'(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N(P_{\leq l} \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2}.$$

We obtain:

(H.2)

$$\begin{aligned} &\sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} \\ &= -i2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\text{tr} \theta - \mathbf{g}(N, N') \text{tr} \theta' \right. \right. \\ &\quad - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \\ &\quad + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \\ &\quad \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \\ &\quad \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. Using the structure equation for N (2.21), we have, schematically:

$$(H.3) \quad \nabla_N(N' - \mathbf{g}(N, N')N) = b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + (N - N')\theta'.$$

Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3) and (H.3), we obtain schematically:

(H.4)

$$\begin{aligned}
 & (N - \mathbf{g}(N, N')N')(h) \\
 &= \frac{bb'}{1 - \mathbf{g}(N, N')^2} \left((N - N')^2 \nabla^2(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi') \right. \\
 & \quad + (N - N')^3 \nabla(N(P_l \text{tr} \chi)) N'(P_{\leq l} \text{tr} \chi') \\
 & \quad + (N - N')^2 \nabla(P_l \text{tr} \chi) \nabla' N'(P_{\leq l} \text{tr} \chi') + \left((N - N')(\theta - \theta') \right. \\
 & \quad + (N - N')^2(\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \\
 & \quad \left. + (N - N')^3(\theta + \theta' + N(b)) \right) \nabla(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi') \Big) \\
 & \quad + \frac{bb'}{(1 - \mathbf{g}(N, N')^2)^2} \left((N - N')^3(\theta - \theta') + (N - N')^4 b^{-1} \nabla(b) \right) \nabla(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi').
 \end{aligned}$$

Thus, in view of (H.1), (H.2), (H.4), (A.8) and (A.11) we obtain:

$$\text{(H.5)} \quad \sum_{m \leq l} B_{j,\nu,\nu',l,m}^{2,2} = 2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d_{\mathcal{C}} \mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
 H &= \frac{1}{(1 - \mathbf{g}(N, N')^2)^2} \left((N - N')^2 \nabla^2(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi') \right. \\
 & \quad + (N - N')^3 \nabla(N(P_l \text{tr} \chi)) N'(P_{\leq l} \text{tr} \chi') \\
 & \quad + (N - N')^2 \nabla(P_l \text{tr} \chi) \nabla' N'(P_{\leq l} \text{tr} \chi') + \left((N - N')(\chi - \chi') \right. \\
 & \quad + (N - N')^2(\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \\
 & \quad \left. + (N - N')^3(\theta + \theta' + N(b)) \right) \nabla(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi') \Big) \\
 & \quad + \frac{1}{(1 - \mathbf{g}(N, N')^2)^3} \left((N - N')^3(\chi - \chi') + (N - N')^4 b^{-1} \nabla(b) \right) \nabla(P_l \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi').
 \end{aligned}$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$\begin{aligned}
 \text{(H.6)} \quad H &= \frac{1}{|N_\nu - N_{\nu'}|} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
 & \quad \times \left(\frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + H_3 \right) \\
 & \quad + N(P_l \text{tr} \chi) N'(P_m \text{tr} \chi') + \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi) N(P_m \text{tr} \chi')}{1 - \mathbf{g}(N, N')^2},
 \end{aligned}$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$(H.7) \quad H_1 = (\chi - \chi') \nabla(P_t \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi'),$$

$$(H.8) \quad H_2 = \nabla^2(P_t \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi') + \nabla(P_t \text{tr} \chi) \nabla' N'(P_{\leq l} \text{tr} \chi') \\ + (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \nabla(P_t \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi')$$

and:

$$(H.9) \quad H_3 = \nabla(N(P_t \text{tr} \chi)) N'(P_{\leq l} \text{tr} \chi') + (\theta + \theta' + N(b)) \nabla(P_t \text{tr} \chi) N'(P_{\leq l} \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p, q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (H.5), (H.6), (H.7), (H.8) and (H.9), we obtain the decomposition (8.597) (8.598) (8.599) (8.600) (8.601) (8.602) (8.603) (8.604) (8.605) of $\sum_{m \leq l} B_{j, \nu, \nu', l, m}^{2, 2}$. This concludes the proof of Lemma 8.18.

APPENDIX I

PROOF OF LEMMA 9.1

Recall from (9.2) that $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m}$ is given by:

$$\begin{aligned} & \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \\ &= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_{>2^j |\nu - \nu'|} \text{tr} \chi(N - \mathbf{g}(N, N')N')(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')}{\mathbf{g}(L, L')} \\ & \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

We integrate by parts in $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m}$ using (7.143) with

$$(I.1) \quad h = \frac{bb' P_{>2^j |\nu - \nu'|} \text{tr} \chi(N - \mathbf{g}(N, N')N')(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')}{\mathbf{g}(L, L')}.$$

We obtain:

$$\begin{aligned} (I.2) \quad & \sum_{(l,m)/2^{\min(l,m)} \leq 2^j |\nu - \nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \\ &= -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(h) + \text{tr} \chi h - \bar{\delta} h - \bar{\delta}' h - (1 - \mathbf{g}(N, N')) \delta' h \right. \\ & \quad \left. - 2\zeta'_{N - \mathbf{g}(N, N')N'} h - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} h \right) \\ & \quad \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}. \end{aligned}$$

Next, we compute the term $L(h)$. We have:

$$\begin{aligned} (I.3) \quad L(h) &= \frac{bb' L(P_{>2^j |\nu - \nu'|} \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')}{\mathbf{g}(L, L')} \\ & \quad + \frac{bb' P_{>2^j |\nu - \nu'|} \text{tr} \chi L((N - \mathbf{g}(N, N')N')(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi'))}{\mathbf{g}(L, L')} \\ & \quad + \frac{(L(b) + L(b')) P_{>2^j |\nu - \nu'|} \text{tr} \chi(N - \mathbf{g}(N, N')N')(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi')}{\mathbf{g}(L, L')} \end{aligned}$$

$$-\frac{L(\mathbf{g}(L, L'))bb'P_{>2^j|\nu-\nu'}\mathrm{tr}\chi(N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')}{\mathbf{g}(L, L')^2}.$$

Decomposing L on L', N' and $N - \mathbf{g}(N, N')N'$, we have:

$$(I.4) \quad L = L' + (N - \mathbf{g}(N, N')N') + (\mathbf{g}(N, N') - 1)N',$$

which yields schematically for the derivative in the second term in the right-hand side of (I.3):

(I.5)

$$\begin{aligned} & L((N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \\ &= L'((N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \\ & \quad + (N - \mathbf{g}(N, N')N')((N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \\ & \quad + (\mathbf{g}(N, N') - 1)N'((N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \\ &= (N - N')\nabla(L'(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) + [L', N - \mathbf{g}(N, N')N'](P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi') \\ & \quad + (N - N')^2\nabla'^2(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi') + \nabla_{N - \mathbf{g}(N, N')N'}(N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi') \\ & \quad + (N - N')^3\nabla'(N'(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) + (N - N')^2[N', N - \mathbf{g}(N, N')N'](P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi'), \end{aligned}$$

where we used in the last inequality the fact that, schematically, $1 - \mathbf{g}(N, N') = (N - N')^2$. Next, we compute the two commutators in the right-hand side of (I.5). Using the structure equation for N (2.21), we have, schematically:

$$(I.6) \quad [N', N - \mathbf{g}(N, N')N'] = b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + (N - N')(\theta + \theta').$$

Also, using the fact that $L = T + N$, $L' = T + N'$ and $\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N')$, we have:

$$\begin{aligned} [L', N - \mathbf{g}(N, N')N'] &= [L', L - \mathbf{g}(N, N')L' + (\mathbf{g}(N, N') - 1)T] \\ &= [L', L] - L'(\mathbf{g}(L, L'))N' + (\mathbf{g}(N, N') - 1)[L', T], \end{aligned}$$

which together with the Ricci Equations (2.17) implies, schematically:

$$\begin{aligned} [L', N - \mathbf{g}(N, N')N'] &= -\bar{\delta}L + \bar{\delta}'L' + (N - N')(\chi + \chi' + \epsilon + \epsilon') \\ & \quad + (N - N')^2(\zeta + \zeta + \delta + n^{-1}\nabla n + \bar{\delta}' + \chi). \end{aligned}$$

Using the analog of (7.45) (7.46) for $-\bar{\delta}L + \bar{\delta}'L'$, we finally obtain, schematically:

$$(I.7) \quad [L', N - \mathbf{g}(N, N')N'] = (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n) + (N - N')^2(k + n^{-1}\nabla n + \chi + \zeta).$$

Now, in view of (G.3), (I.5), (I.6) and (I.7), we obtain, schematically:

(I.8)

$$\begin{aligned} & L((N - \mathbf{g}(N, N')N')(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \\ &= (N - N')\nabla(L'(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) + (N - N')^2\nabla'^2(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi') \\ & \quad + (N - N')^3\nabla'(N'(P_{\leq 2^j|\nu-\nu'}\mathrm{tr}\chi')) \end{aligned}$$

$$\begin{aligned}
 &+ (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n)\nabla(P_{\leq 2^j|\nu-\nu'|}\text{tr}\chi') \\
 &+ (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + \chi + \zeta)\nabla(P_{\leq 2^j|\nu-\nu'|}\text{tr}\chi').
 \end{aligned}$$

Also, in view of (7.149), we have, schematically:

$$(I.9) \quad L(\mathbf{g}(L, L')) = (N - N')^2(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^3\zeta'.$$

Finally, (I.3), (I.8) and (I.9) yield, schematically:

$$\begin{aligned}
 (I.10) \quad L(h) &= \frac{1}{\mathbf{g}(L, L')} \left[bb'(N - N')L(P_{> 2^j|\nu-\nu'}\text{tr}\chi)\nabla'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right. \\
 &+ bb'P_{> 2^j|\nu-\nu'}\text{tr}\chi \left((N - N')\nabla(L'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')) + (N - N')^2\nabla'^2(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right. \\
 &+ (N - N')^3\nabla'(N'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')) \\
 &+ (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n + L(b) + L'(b'))\nabla(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \\
 &+ (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + \chi + \zeta \\
 &\left. \left. + \nabla_{N'}(b')\nabla(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right) \right] \\
 &- \frac{((N - N')^3(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^4\zeta') bb'P_{> 2^j|\nu-\nu'}\text{tr}\chi\nabla'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')}{\mathbf{g}(L, L')^2}.
 \end{aligned}$$

We consider the term multiplied by h in the right-hand side of (I.2). We have schematically:

$$\begin{aligned}
 (I.11) \quad \text{tr}\chi - \bar{\delta} - \bar{\delta}' - (1 - \mathbf{g}(N, N'))\delta' - 2\zeta'_{N-\mathbf{g}(N, N')} \\
 - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} \\
 = \chi + \bar{\delta} + \bar{\delta}' + (N - N')\zeta' + \frac{(N - N')^2\chi'}{\mathbf{g}(L, L')}.
 \end{aligned}$$

Thus, in view of (I.1), (I.2), (I.10) and (I.11) we obtain:

$$\begin{aligned}
 (I.12) \quad \sum_{(l,m)/2^{\min(l,m)} \leq 2^j|\nu-\nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m} \\
 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} HF_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},
 \end{aligned}$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
 H &= \frac{1}{\mathbf{g}(L, L')^2} \left[(N - N')L(P_{> 2^j|\nu-\nu'}\text{tr}\chi)\nabla'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right. \\
 &+ P_{> 2^j|\nu-\nu'}\text{tr}\chi \left((N - N')\nabla(L'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')) + (N - N')^2\nabla'^2(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right. \\
 &\left. \left. + (N - N')^3\nabla'(N'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')) \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n + L(b) + L'(b'))\nabla(P_{\leq 2^j|\nu-\nu'|}\text{tr}\chi') \\
 & + (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla(b') + \chi + \zeta + \zeta' \\
 & + \nabla_{N'}(b'))\nabla(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \Big] \\
 & - \frac{((N - N')^3(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^4\zeta') P_{> 2^j|\nu-\nu'}\text{tr}\chi \nabla'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')}{\mathbf{g}(L, L')^3}.
 \end{aligned}$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand:

$$\frac{1}{\mathbf{g}(L, L')^2} \text{ and } \frac{1}{\mathbf{g}(L, L')^3}$$

in the same fashion than (8.33), and we obtain, schematically:

$$\begin{aligned}
 \text{(I.13)} \quad H &= \frac{1}{|N_\nu - N_{\nu'}|} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
 &\times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),
 \end{aligned}$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$\begin{aligned}
 \text{(I.14)} \quad H_1 &= P_{> 2^j|\nu-\nu'}\text{tr}\chi \nabla'(N'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')), \\
 \text{(I.15)} \quad H_2 &= P_{> 2^j|\nu-\nu'}\text{tr}\chi \left(\nabla'^2(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') + (k + n^{-1}\nabla n + \theta + \theta' \right. \\
 &\quad \left. + b^{-1}\nabla(b) + b'^{-1}\nabla(b') + \chi + \zeta + \zeta' + \nabla_{N'}(b'))\nabla(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right)
 \end{aligned}$$

and:

$$\begin{aligned}
 \text{(I.16)} \quad H_3 &= L(P_{> 2^j|\nu-\nu'}\text{tr}\chi) \nabla'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') + P_{> 2^j|\nu-\nu'}\text{tr}\chi \left(\nabla(L'(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi')) \right. \\
 &\quad \left. + (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n + L(b) + L'(b'))\nabla(P_{\leq 2^j|\nu-\nu'}\text{tr}\chi') \right)
 \end{aligned}$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (I.12), (I.13), (I.14), (I.15) and (I.16), we obtain the decomposition (9.3) (9.4) (9.5) (9.6) (9.7) (9.8) (9.9) (9.10) (9.11) (9.12) (9.13) (9.14) (9.15) of $\sum_{(l,m)/2^{\min(l,m)} \leq 2^j|\nu-\nu'| < 2^{\max(l,m)}} A_{j,\nu,\nu',l,m}$. This concludes the proof of Lemma 9.1.

APPENDIX J

PROOF OF LEMMA 10.1

Recall from (10.2) that $A_{j,\nu,\nu',l,m}$ is given by:

$$A_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')}{\mathbf{g}(L, L')} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts using (7.137) with

$$(J.1) \quad h = \frac{bb' P_l \text{tr} \chi(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')}{\mathbf{g}(L, L')}.$$

We obtain:

(J.2)

$$A_{j,\nu,\nu',l,m} \\ = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N')N')(h) + \left(\text{tr} \theta - \mathbf{g}(N, N') \text{tr} \theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $(N - \mathbf{g}(N, N')N')(h)$. Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3), we obtain schematically:

(J.3)

$$(N - \mathbf{g}(N, N')N')(h) \\ = \frac{bb'}{\mathbf{g}(L, L')} \left((N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi') \right)$$

$$\begin{aligned}
 &+ P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N') \\
 &+ (1 - \mathbf{g}(N, N')^2) N (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi') \\
 &+ (N - N') (\theta - \theta') P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi') \\
 &+ (N - N')^2 (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi') \\
 &+ \frac{bb'}{\mathbf{g}(L, L')^2} \left((N - N')^3 (\theta - \theta') + (N - N')^4 (\theta + \theta' + b^{-1} \nabla(b)) \right) P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi').
 \end{aligned}$$

Thus, in view of (J.1), (J.2), (J.3), (A.8) and (A.11) we obtain:

$$(J.4) \quad A_{j,\nu,\nu',l,m} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d_{\mathcal{C}} \mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
 H &= \frac{1}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \left((N' - \mathbf{g}(N, N') N) (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi') \right. \\
 &\quad + P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N') \\
 &\quad + (1 - \mathbf{g}(N, N')^2) N (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi') \\
 &\quad + (N - N') (\chi - \chi') P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi') \\
 &\quad \left. + (N - N')^2 (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi') \right) \\
 &\quad + \frac{1}{\mathbf{g}(L, L')^2 (1 - \mathbf{g}(N, N')^2)} \left((N - N')^3 (\chi - \chi') + (N - N')^4 (\theta + \theta' + b^{-1} \nabla(b)) \right) \\
 &\quad \times P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi').
 \end{aligned}$$

Proceeding in the same fashion as in (A.14), we obtain, schematically:

$$\begin{aligned}
 (J.5) \quad H &= \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \left(\frac{1}{|N_\nu - N_{\nu'}|} H_1 + H_2 \right) \\
 &\quad + \frac{(N' - \mathbf{g}(N, N') N) (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\
 &\quad + \frac{P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\
 &\quad + \frac{N (P_l \text{tr} \chi) (N - \mathbf{g}(N, N') N') (P_m \text{tr} \chi')}{\mathbf{g}(L, L')},
 \end{aligned}$$

where the tensors H_1 and H_2 on \mathcal{M} are given by:

$$(J.6) \quad H_1 = (\chi - \chi') P_l \text{tr} \chi \nabla' (P_m \text{tr} \chi')$$

and:

$$(J.7) \quad H_2 = (\theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b'))P_l \text{tr} \chi \nabla'(P_m \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (J.4), (J.5), (J.6) and (J.7), we obtain the decomposition (10.3) (10.4) (10.5) (10.6) (10.7) (10.8) (10.9) (10.10) of $A_{j,\nu,\nu',l,m}$. This concludes the proof of Lemma 10.1.

APPENDIX K

PROOF OF LEMMA 10.4

Recall from (10.2) that $A_{j,\nu,\nu',l,m}^1$ is given by:

$$A_{j,\nu,\nu',l,m}^1 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N')}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)} \\ \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

We integrate by parts using (7.137) with

$$(K.1) \quad h = \frac{bb' P_l \text{tr} \chi \nabla'^2 P_m (\text{tr} \chi') (N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N')}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)}.$$

We obtain:

(K.2)

$$A_{j,\nu,\nu',l,m}^1 \\ = -i2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left((N - \mathbf{g}(N, N') N')(h) + \left(\text{tr} \theta - \mathbf{g}(N, N') \text{tr} \theta' \right. \right. \\ \left. \left. - \theta(N' - \mathbf{g}(N, N') N), N' - \mathbf{g}(N, N') N) - \mathbf{g}(N, N') b^{-1} (N' - \mathbf{g}(N, N') N)(b) \right. \right. \\ \left. \left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left(\theta'(N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N') \right. \right. \right. \\ \left. \left. \left. - \mathbf{g}(N, N') \theta(N' - \mathbf{g}(N, N') N), N' - \mathbf{g}(N, N') N) \right) \right) h \\ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Next, we compute the term $(N - \mathbf{g}(N, N') N')(h)$. Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3), we obtain schematically:

(K.3)

$$(N - \mathbf{g}(N, N') N')(h) \\ = \frac{bb'}{\mathbf{g}(L, L') (1 - \mathbf{g}(N, N')^2)} \left((N - N')^3 \nabla (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi') \right)$$

$$\begin{aligned}
& + (N - N')^4 N (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi') + (N - N')^3 P_l \text{tr} \chi \nabla'^3 (P_m \text{tr} \chi') \\
& + \left((N - N')^2 (\theta - \theta') + (N - N')^3 (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \right) P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi') \\
& + \frac{bb'}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \left(\frac{1}{\mathbf{g}(L, L')} + \frac{1}{1 - \mathbf{g}(N, N')^2} \right) \left((N - N')^4 (\theta - \theta') \right. \\
& \left. + (N - N')^5 (\theta + \theta' + b^{-1} \nabla(b)) \right) P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi').
\end{aligned}$$

Thus, in view of (K.1), (K.2), (K.3), (A.8) and (A.11) we obtain:

$$(K.4) \quad A_{j,\nu,\nu',l,m}^1 = 2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
H & = \frac{1}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)^2} \left((N - N')^3 \nabla (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi') \right. \\
& + (N - N')^4 N (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi') + (N - N')^3 P_l \text{tr} \chi \nabla'^3 (P_m \text{tr} \chi') \\
& + \left. \left((N - N')^2 (\chi - \chi') + (N - N')^3 (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \right) P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi') \right) \\
& + \frac{1}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)^2} \left(\frac{1}{\mathbf{g}(L, L')} + \frac{1}{1 - \mathbf{g}(N, N')^2} \right) \left((N - N')^4 (\chi - \chi') \right. \\
& \left. + (N - N')^5 (\theta + \theta' + b^{-1} \nabla(b)) \right) P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi').
\end{aligned}$$

Proceeding in the same fashion as in (A.14), we obtain, schematically:

$$\begin{aligned}
(K.5) \quad H & = \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p,q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
& \times \left(\frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + H_3 \right),
\end{aligned}$$

where the tensors H_1 , H_2 and H_3 on \mathcal{M} are given by:

$$(K.6) \quad H_1 = (\chi - \chi') P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi'),$$

$$\begin{aligned}
(K.7) \quad H_2 & = \nabla (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi') + P_l \text{tr} \chi \nabla'^3 (P_m \text{tr} \chi') \\
& + (\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) P_l \text{tr} \chi \nabla'^2 (P_m \text{tr} \chi')
\end{aligned}$$

and:

$$(K.8) \quad H_3 = N (P_l \text{tr} \chi) \nabla'^2 (P_m \text{tr} \chi')$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (K.4), (K.5), (K.6), (K.7) and (K.8), we obtain the decomposition (10.24) (10.25) (10.26) (10.27) (10.28) (10.29) (10.30) (10.31) of $A_{j,\nu,\nu',l,m}^1$. This concludes the proof of Lemma 10.4.

APPENDIX L

PROOF OF LEMMA 10.5

Recall from (10.64) that $A_{j,\nu,\nu',l,m}^2$ is given by:

$$A_{j,\nu,\nu',l,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m(\text{tr} \chi'))}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\ \times F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.$$

We integrate by parts in $A_{j,\nu,\nu',l,m}^2$ using (7.143) with

$$(L.1) \quad h = \frac{bb'(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m(\text{tr} \chi'))}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)}.$$

We obtain:

$$A_{j,\nu,\nu',l,m}^2 = -i2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} \left(L(h) + \text{tr} \chi h - \bar{\delta} h - \bar{\delta}' h - (1 - \mathbf{g}(N, N'))\delta' h \right. \\ \left. - 2\zeta'_{N - \mathbf{g}(N, N')N'} h - \frac{\chi'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N')}{\mathbf{g}(L, L')} h \right) \\ (L.2) \quad \times F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{M}}.$$

Next, we compute the term $L(h)$. We have:

$$(L.3) \quad L(h) = \frac{bb'L((N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi))(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\ + \frac{bb'(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)L((N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi'))}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\ + \frac{(L(b) + L(b'))(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} \\ - \left(\frac{1}{\mathbf{g}(L, L')} + \frac{1}{1 - \mathbf{g}(N, N')^2} \right) \\ \times \frac{L(\mathbf{g}(L, L'))bb'(N' - \mathbf{g}(N, N')N)(P_l \text{tr} \chi)(N - \mathbf{g}(N, N')N')(P_m \text{tr} \chi')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)}.$$

In view of (I.7), we have, schematically:

(L.4)

$$\begin{aligned} L((N' - \mathbf{g}(N, N')N)(P_l \text{tr}\chi)) \\ = (N - N')\nabla(L(P_l \text{tr}\chi)) + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n)\nabla(P_l \text{tr}\chi) \\ + (N - N')^2(k + n^{-1}\nabla n + \chi + \zeta)\nabla(P_l \text{tr}\chi). \end{aligned}$$

Next, recall from (I.8) that we have, schematically:

(L.5)

$$\begin{aligned} L((N - \mathbf{g}(N, N')N')(P_m \text{tr}\chi')) \\ = (N - N')\nabla(L'(P_m \text{tr}\chi')) + (N - N')^2\nabla'^2(P_m \text{tr}\chi') \\ + (N - N')^3\nabla'(N'(P_m \text{tr}\chi')) \\ + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n)\nabla(P_m \text{tr}\chi') \\ + (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + \chi + \zeta)\nabla(P_m \text{tr}\chi'). \end{aligned}$$

Also, in view of (7.149), we have, schematically:

$$(L.6) \quad L(\mathbf{g}(L, L')) = (N - N')^2(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^3\zeta'.$$

Finally, (L.3), (L.4), (L.5) and (L.6) yield, schematically:

(L.7)

$$\begin{aligned} L(h) = \frac{1}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)} & \left[bb'(N - N')^2\nabla(L(P_l \text{tr}\chi))\nabla'(P_m \text{tr}\chi') \right. \\ & + bb'\nabla(P_l \text{tr}\chi) \left((N - N')^2\nabla(L'(P_m \text{tr}\chi')) + (N - N')^3\nabla'^2(P_m \text{tr}\chi') \right. \\ & + (N - N')^4\nabla'(N'(P_m \text{tr}\chi')) \\ & + (N - N')^2(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n + L(b) + L'(b'))\nabla(P_m \text{tr}\chi') \\ & + (N - N')^3(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b') + \chi + \zeta \\ & \left. \left. + \nabla_{N'}(b')\right)\nabla(P_m \text{tr}\chi') \right] \\ & + bb'\nabla(P_l \text{tr}\chi)\nabla'(P_m \text{tr}\chi')(N - N')^3(k + n^{-1}\nabla n + \chi + \zeta) \\ & + \left(\frac{1}{\mathbf{g}(L, L')} + \frac{1}{1 - \mathbf{g}(N, N')^2} \right) \\ & \times \frac{((N - N')^4(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^5\zeta') bb'\nabla(P_l \text{tr}\chi)\nabla'(P_m \text{tr}\chi')}{\mathbf{g}(L, L')(1 - \mathbf{g}(N, N')^2)}. \end{aligned}$$

Thus, in view of (L.1), (L.2), (L.7) and (I.11) we obtain:

$$(L.8) \quad A_{j,\nu,\nu',l,m}^2 = 2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} HF_{j,-1}(u)F_{j,-1}(u)\eta_j^{\nu'}(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d_{\mathcal{C}}\mathcal{M},$$

with the tensor H on \mathcal{M} given, schematically, by:

$$\begin{aligned}
 H = & \frac{1}{\mathbf{g}(L, L')^2(1 - \mathbf{g}(N, N')^2)} \left[(N - N')^2 \nabla(L(P_l \text{tr}\chi)) \nabla'(P_m \text{tr}\chi') \right. \\
 & + \nabla(P_l \text{tr}\chi) \left((N - N')^2 \nabla(L'(P_m \text{tr}\chi')) + (N - N')^3 \nabla'^2(P_m \text{tr}\chi') \right. \\
 & + (N - N')^4 \nabla'(N'(P_m \text{tr}\chi')) \\
 & + (N - N')^2 (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla(P_m \text{tr}\chi') \\
 & + (N - N')^3 (k + n^{-1} \nabla n + \theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta' \\
 & \left. \left. + \nabla_{N'}(b') \nabla(P_m \text{tr}\chi') \right) \right] + \nabla(P_l \text{tr}\chi) \nabla'(P_m \text{tr}\chi') (N - N')^3 (k + n^{-1} \nabla n + \chi + \zeta) \\
 & + \left(\frac{1}{\mathbf{g}(L, L')} + \frac{1}{1 - \mathbf{g}(N, N')^2} \right) \\
 & \times \frac{((N - N')^4 (\delta + \delta' + n^{-1} \nabla n + \chi') + (N - N')^5 \zeta') \nabla(P_l \text{tr}\chi) \nabla'(P_m \text{tr}\chi')}{\mathbf{g}(L, L')^2(1 - \mathbf{g}(N, N')^2)}.
 \end{aligned}$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N') \text{ and } 1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$

We may thus expand:

$$\frac{1}{\mathbf{g}(L, L')^2(1 - \mathbf{g}(N, N')^2)}, \frac{1}{\mathbf{g}(L, L')^3(1 - \mathbf{g}(N, N')^2)} \text{ and } \frac{1}{\mathbf{g}(L, L')^2(1 - \mathbf{g}(N, N')^2)^2}$$

in the same fashion than (8.33), and we obtain, schematically:

$$\begin{aligned}
 \text{(L.9)} \quad H = & \frac{1}{|N_\nu - N_{\nu'}|^2} \left(\sum_{p, q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \\
 & \times \left(H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),
 \end{aligned}$$

where the tensors H_1, H_2 and H_3 on \mathcal{M} are given by:

$$\text{(L.10)} \quad H_1 = \nabla(P_l \text{tr}\chi) \nabla'(N'(P_m \text{tr}\chi')),$$

$$\begin{aligned}
 \text{(L.11)} \quad H_2 = & \nabla(P_l \text{tr}\chi) \nabla'^2(P_m \text{tr}\chi') + \left(k + n^{-1} \nabla n + \theta + \theta' \right. \\
 & \left. + b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta' + \nabla_{N'}(b') \right) \nabla(P_l \text{tr}\chi) \nabla(P_m \text{tr}\chi')
 \end{aligned}$$

and:

$$\begin{aligned}
 \text{(L.12)} \quad H_3 = & \nabla(L(P_l \text{tr}\chi)) \nabla'(P_m \text{tr}\chi') + \nabla(P_l \text{tr}\chi) \left(\nabla'(L'(P_m \text{tr}\chi')) \right. \\
 & \left. + (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla'(P_m \text{tr}\chi') \right)
 \end{aligned}$$

and where c_{pq} are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (L.8), (L.9), (L.10), (L.11) and (L.12), we obtain the decomposition (10.65) (10.66) (10.67) (10.68) (10.69) (10.70) (10.71) (10.72) (10.73) (10.74) (10.75) (10.76) (10.77) of $A_{j,\nu,\nu',l,m}^2$. This concludes the proof of Lemma 10.5.

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This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, where \mathbf{g} is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor \mathbf{R} of \mathbf{g} is a major step of the proof of the bounded L^2 curvature conjecture proposed in Klainerman (2000), and solved jointly in Klainerman, Rodnianski & Szeftel (2015). On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Cet ouvrage est dédié à la construction et au contrôle d'une paramétrix pour l'équation des ondes homogène $\square_{\mathbf{g}}\phi = 0$, où \mathbf{g} est une métrique peu régulière satisfaisant les équations d'Einstein dans le vide. Le contrôle d'une telle paramétrix ainsi que du terme d'erreur associé lorsque l'on suppose seulement des bornes L^2 sur le tenseur de courbure \mathbf{R} de \mathbf{g} est une étape cruciale de la preuve de la conjecture de courbure L^2 proposée dans Klainerman (2000), et résolue dans Klainerman, Rodnianski & Szeftel (2015). Plus généralement, cet ouvrage concerne le contrôle de l'équation eikonale sur un espace-temps peu régulier et la dérivation de bornes L^2 pour des opérateurs intégraux de Fourier sur des variétés avec une phase et un symbole peu réguliers, et possède de ce point de vue un intérêt propre.

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