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Decomposition of spectral covers

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DECOMPOSITION OF SPECTRAL COVERS

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1 Introduction

To a vector bundle $E \to X$ and an endomorphism $\varphi : E \to E$ one associates a spectral cover $\pi : \widetilde{X} \to X$, whose fibers $\pi^{-1}(x)$, $x \in X$, are given by the eigenvalues of φ_x. If φ is, more generally, a K-valued endomorphism (a "Higgs bundle") $\varphi : E \to E \otimes K$, where K is a line bundle on X, we still get a cover $\pi : \widetilde{X} \to X$, but now $\widetilde{X}$ is contained in the total space $|K|$ of K, since the eigenvalues live in K. The eigenspaces of φ give a sheaf L on $\widetilde{X}$, which is a line bundle if φ is regular [BNR,B]. (Even more generally, K can be allowed to be a vector bundle on X, as long as a symmetry condition (trivial in case K is a line bundle) is imposed on φ: the case where K is the cotangent bundle of X arises in [S]. In this work we will consider only the case of a line bundle K.) One way to construct these objects is to let $\pi_K : |K| \to X$ denote the natural projection, and let τ be the tautological section of $\pi_K^* K$. Then $\pi_K^*\varphi - \tau$ is a π_K^*K-valued endomorphism of $\pi_K^* E$. Now L is the cokernel of $\pi_K^*\varphi - \tau$, considered as a sheaf on its support $\widetilde{X} := Supp(L) \subset |K|$.

This situation arises frequently in the study of completely integrable Hamiltonian systems on a manifold M which can be written as a Lax equation depending on parameters [AvM,B,G,H,K]; here X is the parameter space, often the affine line or $\mathbf{P}^1$, and the flow of the system is linearized on the Picard variety $Pic\widetilde{X}$ (or the Jacobian, when X is a curve). The linearization map typically gives an isogeny from the Liouville tori of the completely integrable system to (an Abelian subvariety of) $Pic\widetilde{X}$, by sending a point of M where the Lax equation is regular to the eigen line bundle computed at that point.

The vector bundle $E \to X$ often has G-structure, where G is some reductive Lie group. In other words, E is associated to a principal G-bundle $\mathcal{V} \to X$ via a representation $\rho : G \to GL(V)$ of G. The endomorphism φ then becomes a section of $\mathbf{ad}\mathcal{V} \otimes K$, where $\mathbf{ad}\mathcal{V}$ is the associated bundle of Lie algebras $\mathcal{V} \times_G \mathbf{g}$. In [AvM], Adler and van Moerbeke raised the question of the dependence of the resulting cover $\widetilde{X}_\rho$ on the representation ρ. If the situation comes from a completely integrable system as above, then the Liouville torus, which depends on the differential equation but not on the particular Lax equations or on the representation ρ, should occur, up to isogeny, as a subvariety of $Pic\widetilde{X}_\rho$, for all ρ. One may therefore expect to find a natural, Prym-type subvariety of each $Pic\widetilde{X}_\rho$, together with correspondences between pairs $\widetilde{X}_\rho$, $\widetilde{X}_{\rho'}$ whose images in the Picard varieties should be isogenous to this generalized Prym. More generally, one may wish to describe all correspondences acting on each $\widetilde{X}_\rho$ (or between pairs) over the base X, and to find the isogeny decomposition of $Pic\widetilde{X}_\rho$

into isotypic pieces under this action. One of these isotypic pieces should be common to all $\widetilde{X}_\rho$, and this would be the generalized Prym.

Several special cases of this situation, arising from orthogonal groups, are well known in Prym theory, e.g. Recillas' trigonal construction [R], my tetragonal construction [D1,D2], and Pantazis' bigonal construction [P]. The case of the exceptional group G_2 is discussed in [KP]. Other examples, related to the geometry of families of Del Pezzo surfaces, are given in [K]. In that work, Kanev gives a solution of Adler-van Moerbeke's question, under a few hypotheses: the base X is $\mathbf{P}^1$, the principal bundle $\mathcal{V}$ is trivial, the Lie algebra $\mathbf{g}$ is simple of type A_n, D_n, or E_n. Under these assumptions he constructs, for each $\widetilde{X}_\rho$, a Prym-Tyurin variety $Prym(\widetilde{X}_\rho/X) \subset Jac(\widetilde{X}_\rho)$ and a correspondence whose image is $Prym(\widetilde{X}_\rho/X)$. The Prym-Tyurin varieties for different representations are isogenous, and even isomorphic if both representations are minuscule.

The purpose of this work is to analyze the decomposition of the Picard varieties of general spectral covers for a reductive group G. We will show (Theorem 8.1) that there is a distinguished isotypic component of $Pic\widetilde{X}_\rho$, corresponding to the reflection representation Λ of the Weyl group W. When G is one of the classical simple groups, this is the unique piece common to $Pic\widetilde{X}_\rho$ for all non-trivial representations ρ of G. For some exceptional groups the uniqueness fails, as we see in Sections 10,11.

Our approach throughout is based on the observation that the geometry of the spectral covers reflects not so much the representations of G as those of its Weyl group W. Various questions about a spectral cover $\widetilde{X}_\rho$ simplify considerably when the emphasis is placed on the action of W rather than on the way $\widetilde{X}_\rho$ sits inside K. Here is what we do in more detail:

The spectral covers $\widetilde{X}_\rho$ decompose into subcovers $\widetilde{X}_\lambda$, indexed by W-orbits of weights λ. There are infinitely many distinct covers $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$, but they fall into only a finite number (2^r, where $r = rank_{ss}(G)$) of birational classes, cf. lemma (3.3). In section 2 we construct an abstract W-Galois cover $\widetilde{X} \to X$ which dominates all $\widetilde{X}_\lambda$. In good cases, points of $\widetilde{X}$ over $x \in X$ parametrize chambers in the dual of the unique Cartan subalgebra $\mathbf{t}(\varphi(x))$ containing $\varphi(x)$, so we call $\widetilde{X} \to X$ the *cameral* cover. With very few exceptions (listed in (4.3)), the spectral covers $\widetilde{X}_\lambda$ are forced to be singular as soon as X contains a compact curve, while the cameral cover $\widetilde{X}$ and its quotients by the parabolic subgroups serve as natural desingularizations, as long as the endomorphism φ remains regular. For example, this happens for $\mathbf{g} = \mathbf{so}(2n)$ and *any* non-trivial representation. (For the standard, $2n$-dimensional representation of $\mathbf{so}(2n)$, Hitchin notes these accidental singularities in [H], and attributes them to the vanishing of the Pfaffian.) In particular, it is unrealistic to hope that the eigensheaf L will "generically" be a line bundle on $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$: In typical situations we get torsion free sheaves on $\widetilde{X}_\lambda$, which come from line bundles on the cameral $\widetilde{X}$. (The original situation, where $G = GL(n)$ and ρ is the standard representation, is thus quite atypical!)

The ring of natural correspondences on $\widetilde{X}_\lambda$ is described in §6 in terms of the Weyl group W and the parabolic subgroup W_P determined by λ. The question of decomposing the spectral Picards is translated to decomposition of the permutation representation $\mathbf{Z}[W/W_p]$ as W-module. Some general results, based on Springer's representation and the work of [BM], are reviewed in Section 9. These results clarify the general form of the decomposition, but do not seem to imply the uniqueness of the common component. We thus work out the uniqueness for classical groups, and the non-uniqueness for some exceptional groups, by direct computations, in Sections 8, 10 and 11.

In this group-theoretic context, actually writing down the decomposition in any given case is very easy. In §12, we write down some formulas for the projection of a spectral Picard onto any generalized Prym. In the case of the projection to the distinguished Prym we recover Kanev's formulas (with minor modifications, which we explain). Kanev's construction, which is very geometric, is motivated by the interpretation of certain Weyl groups as symmetries of line configurations on rational surfaces. Our point is that similar formulas work much more generally, and require only elementary group theory. J.Y. Merindol informed me, during the Orsay conference, that he has also obtained projection formulas (onto the distinguished Prym) for arbitrary reductive groups, removing Kanev's restriction to "simply laced" groups, of types A_n, D_n, E_n.

For our purpose in this paper, we can take G to be any complex reductive group, but the resulting spectral and cameral covers depend only on the semisimple part G_{ss} of G, as does the distinguished Prym. There is however a more natural subvariety of $Pic\widetilde{X}$, consisting up to isogeny of $Prym(\widetilde{X})$ together with a number (equal to the dimension of the center of G) of copies of $PicX$. This corresponds to the reflection representation of W on the weights of G, which decomposes up to isogeny into the weights of G_{ss} and a trivial representation. In a sequel to this work [D3] we will describe this enlarged Prym in terms of W-equivariant bundles on $\widetilde{X}$, and interpret it as a moduli space of generalized Higgs bundles on X with given spectral invariants. Combined with work of Markman on the existence of Poisson structures [M], this leads to an algebraically completely integrable Hamiltonian system, generalizing those of Hitchin, Jacobi-Mumford-Beauville [B], and so on. The construction extends to Higgs bundles with values in a vector bundle K, as in [S], where K is the cotangent bundle of X.

My interest in these questions arose from conversations with M. Adler and P. van Moerbeke, P. Griffiths, and V. Kanev, about their works [AvM,G,K], followed by discussions with L. Katzarkov and T. Pantev about the G_2 case, which they analyzed in [KP], and with E. Markman about the general version of Hitchin's system. Conversations with C. Curtis and N. Spaltenstein provided valuable information about Weyl group representations. I also enjoyed and benefitted from discussions with A. Beauville, A. Kouvidakis, R. Lazarsfeld and E. Previato.

2 Cameral covers.

Given a principal Higgs bundle $(\mathcal{V}, \varphi)$ on X, we are going to construct a W-Galois cover $\widetilde{X} \to X$, which we call the cameral cover of $(\mathcal{V}, \varphi)$. It is independent of the choice of a representation. For each representation $\rho : G \to GL(V)$, the spectral cover $\widetilde{X}_\rho$ will break into pieces indexed by W-orbits of weights of ρ. Each of these pieces will be the image, under an appropriate morphism, of the cameral cover $\widetilde{X}$, in fact of a certain parabolic quotient $\widetilde{X}/W_P$. If the Higgs bundle is regular, we also have for each weight λ a line bundle L_λ on $\widetilde{X}$. It descends to a line bundle on the quotient $\widetilde{X}/W_P$, but only to torsion-free sheaves on $\widetilde{X}_\lambda, \widetilde{X}_\rho$, "usually" of rank 1 on $\widetilde{X}_\lambda$.

We start with some elementary observations on components of the spectral covers $\widetilde{X}_\rho$. First, if ρ is reducible:

$$(V, \rho) = \oplus(V_i, \rho_i),$$

then the spectral cover $\widetilde{X}_\rho \to X$ is just the union of the covers $\widetilde{X}_{\rho_i} \to X$. We may thus restrict attention to irreducible ρ. Next, consider the weight decomposition of V with respect to a maximal torus $T \subset G$:

$$V = \oplus_{\lambda \in D} V_\lambda = \oplus_{\lambda \in D \cap C} \oplus_{\mu \in W\lambda} V_\mu, \tag{2.1}$$

where Λ_G is the lattice of weights of G, $D \subset \Lambda_G$ is the set of weights of ρ, and C is the closed Weyl chamber (determined by a Borel $B \supset T$). There is a corresponding decomposition of the spectral cover:

$$\widetilde{X}_\rho = \sum_{\lambda \in D \cap C} m_\lambda \widetilde{X}_\lambda. \tag{2.2}$$

Here $m_\lambda = dim V_\lambda$ are the multiplicities, and the $\widetilde{X}_\lambda$ are constructed as follows:

Recall that Chevalley's theorem [Hu1] says that the restriction map

$$\mathbf{C}[\mathbf{g}]^G \to \mathbf{C}[\mathbf{t}]^W,$$

from **ad**-invariant polynomial functions on the Lie algebra $\mathbf{g}$ to W-invariant polynomial functions on the Cartan subalgebra $\mathbf{t}$, is an isomorphism. This implies that for any weight λ, there is a unique **ad**-invariant polynomial function

$$P_\lambda : \mathbf{g} \to \mathbf{C}[x]$$

(the values are polynomials in one variable x), whose restriction to the Cartan $\mathbf{t}$ is the W-invariant function

$$\prod_{\mu \in W\lambda} (x - \mu) : \ \mathbf{t} \to \mathbf{C}[x].$$

The **ad**-invariance implies that P_λ makes sense on the bundle of algebras $\mathbf{ad}(\mathcal{V})$. The quantity $P_\lambda(\varphi)$ then gives a morphism between the total spaces of the line bundles:

$$P_\lambda(\varphi) : |K| \to |K^N|,$$

where $N = \#(W\lambda)$.

Definition 2.3. *The spectral cover $\widetilde{X}_\lambda$ determined by the Higgs bundle $(\mathcal{V}, \varphi)$ and the weight λ is the inverse image by $P_\lambda(\varphi)$ of the 0-section.*

By construction, $\widetilde{X}_\lambda$ is a subscheme of $|K|$, finite of degree N over X. The decomposition (2.2) now follows from (2.1) and the definitions of $\widetilde{X}_\rho$, $\widetilde{X}_\lambda$; in fact,

$$\prod_{\lambda \in D \cap C} (P_\lambda(\varphi))^{m_\lambda} = char(\rho(\varphi)).$$

From now on, instead of the (usually reducible) cover $\widetilde{X}_\rho$, we consider the collection of spectral covers $\widetilde{X}_\lambda$. We note that an irreducible ρ determines an "extremal" $\widetilde{X}_\lambda$, corresponding to the W-orbit of extremal weights for ρ. It occurs with multiplicity 1 in $\widetilde{X}_\rho$. Equality $\widetilde{X}_\rho = \widetilde{X}_\lambda$ holds if and only if the representation ρ is minuscule. In general, Weyl's character formula gives an explicit way of reconstructing $\widetilde{X}_\rho$ from this extremal piece $\widetilde{X}_\lambda$.

Although in general there are infinitely many non-isomorphic covers $\widetilde{X}_\lambda$, they fall into only a finite number of birational equivalence classes. Next we construct an object $\widetilde{X}$ which dominates all of them.

Using Chevalley's theorem again, we have an injective ring homomorphism

$$\mathbf{C}[\mathbf{t}]^W \approx \mathbf{C}[\mathbf{g}]^G \hookrightarrow \mathbf{C}[\mathbf{g}].$$

Taking Spec, we find a surjective, G-invariant morphism of affine varieties

$$\mathbf{g} \twoheadrightarrow \mathbf{t}/W. \tag{2.4}$$

We can then form the fiber product

$$\tilde{\mathbf{g}} = \mathbf{g} \times_{\mathbf{t}/W} \mathbf{t}. \tag{2.5}$$

The projection $\pi : \tilde{\mathbf{g}} \to \mathbf{g}$ is a finite morphism which is W-Galois; we call it the cameral cover of the Lie algebra $\mathbf{g}$. A regular semisimple element $g \in \mathbf{g}$ is contained in a unique Cartan subalgebra $\mathbf{t}$. The fiber $\pi^{-1}(g)$ can be identified with the set of Borels containing $\mathbf{t}$, or equivalently with the set of chambers in $\mathbf{t}^*$.

Given a Higgs bundle $(\mathcal{V}, \varphi)$ on X, we relativize the previous construction to define $\widetilde{X}$: Since (2.4) is G-invariant and $\mathbf{C}^*$-equivariant, it extends to a morphism

$$|\mathbf{ad}(\mathcal{V}) \otimes K| \to |(\mathbf{t} \otimes K)|/W$$

so we can form the fiber product with $\mathbf{t} \otimes K$ as in (2.5), then pull back to X via φ:

Definition (2.6). *The cameral cover determined by the principal Higgs bundle $(\mathcal{V}, \varphi)$ is given by $\pi : \widetilde{X} \to X$, where*

$$\widetilde{X} = \varphi^*(|\mathbf{ad}(\mathcal{V}) \otimes K| \times_{|\mathbf{t} \otimes K|/W} |\mathbf{t} \otimes K|)$$

and π is the projection on the first factor.

We can describe the fiber $\pi^{-1}(x)$ over $x \in X$ in several ways, e.g. as the set of prints (in $\mathbf{t} \otimes K_x$) which are conjugate (via elements of $\mathcal{V}_x$) to the semisimple part of $\varphi(x) \in \mathbf{ad}(\mathcal{V}_x) \otimes K_x$. If $\varphi(x)$ is regular semisimple, hence contained in a unique Cartan $\mathbf{t}_x \subset \mathbf{ad}(\mathcal{V}_x) \otimes K_x$, the fiber can be more simply described as the set of Borels through $\mathbf{t}_x$, or chambers in $\mathbf{t}_x^*$. When $\mathbf{g} = \mathbf{gl}(n)$ on $\mathbf{sl}(n)$, a point of the fiber is given by an ordering of the eigenvalues of $\varphi(x)$.

For each λ, consider the morphism

$$j_\lambda : \tilde{\mathbf{g}} = \mathbf{g} \times_{\mathbf{t}/W} \mathbf{t} \to \mathbf{g} \times \mathbf{C}$$

sending

$$(g, t) \mapsto (g, \lambda(t)).$$

Clearly $P_\lambda \circ j_\lambda = 0$, so j_λ factors through a morphism

$$i_\lambda : \tilde{\mathbf{g}} \to \tilde{\mathbf{g}}_\lambda = \{P_\lambda = 0\} \subset \mathbf{g} \times \mathbf{C}.$$

Since the covers $\widetilde{X}$, $\widetilde{X}_\lambda$ are (locally) obtained as pullbacks via φ of $\tilde{\mathbf{g}}, \tilde{\mathbf{g}}_\lambda$ respectively, this globalizes to a morphism

$$i_\lambda : \widetilde{X} \to \widetilde{X}_\lambda. \tag{2.7}$$

Finally, for each λ, the line bundle L_λ on $\widetilde{X}$ is induced from the corresponding Borel-Weil-Bott line bundle on the flag variety G/B. These bundles are the main object of study in [D3], and will not be further discussed here.

3 Parabolic subgroups.

Fix a Cartan subalgebra $\mathbf{t} \subset \mathbf{g}$ and a Borel subalgebra $\mathbf{b} \supset \mathbf{t}$, with corresponding maximal torus T and Borel subgroup B in G. Let $R \subset \mathbf{t}^*$ be the root system, R^+ the positive roots with respect to $\mathbf{b}$, $S = \{\alpha_1, \cdots, \alpha_r\}$ the simple roots, and C the closed Weyl chamber. We recall (e.g. [Hu2], §30) that there is a natural bijection between the following sets,

(3.1)

(1) Parabolic subalgebras $\mathbf{p} \supset \mathbf{b}$

(2) Parabolic subgroups $P \supset B$.

(3) Subsets $S_P \subset S$

(4) Subgroups $W_P \subset W$ generated by reflections in simple roots.

(5) Faces C_P of C.

This goes as follows: to the subset S_P associate the reductive (Levi) subalgebra $\mathbf{l}_P$ spanned by the root spaces $\mathbf{g}_\alpha$ with α in

$$R_P := R \cap span(S_P),$$

and then

$$\mathbf{p} := \mathbf{l}_P \mathbf{b} = \mathbf{l}_P \mathbf{u}_P \subset \mathbf{g}$$

where $\mathbf{u}_P$ is spanned by $\mathbf{g}_\alpha$ with α in $R^+ - R_P$. We can also get the parabolic group directly as

$$P := BW_PB$$

where W_P is the subgroup of W generated by the reflections σ_α in simple roots $\alpha \in S_P$. Conversely, P determines W_P as its Weyl group, i.e. it determines the normalizer

$$N_P := N_P(T) = P \cap N_G(T),$$

hence also its image $W_P := N_P/T$ in $W = N_G(T)/T$. Now W_P determines

$$S_P := \{\alpha \in S | \sigma_\alpha \in W_P\}.$$

Finally, we define the face

$$C_P := (Span(S_P))^\perp = \{\text{fixed points of } W_P \text{ in } C\}.$$

The subgroup W_P is recovered as the stabilizer of (all, or any one of) the points in the interior C_P^0 of C_P.

Thus $\mathbf{g}, G$ correspond to the subset S and the group W, and the face C_G is the vertex; $\mathbf{b}, B$ correspond to $\varnothing$, (1), C; minimal parabolics correspond to singletons $S_{P_i} = \{\alpha_i\}$, subgroups $W_{P_i} = (\sigma_{\alpha_i})$, and to walls of C; maximal parabolics to $S \setminus \{\alpha_i\}$ and to edges of C (if $\mathbf{g}$ is semisimple; otherwise, C_P modulo the center is an edge).

Returning to the cameral cover, we define the intermediate cover $\widetilde{X}_P \to X$ corresponding to a parabolic $P \supset B$ (or subset $S_P \subset S$) as the quotient

$$\widetilde{X}_P = \widetilde{X}/W_P. \tag{3.2}$$

Thus $\widetilde{X}_G = X$ and $\widetilde{X}_B = \widetilde{X}$. We see immediately:

Lemma (3.3). *The map $i_\lambda : \widetilde{X} \to \widetilde{X}_\lambda$ of (2.7) factors through $\widetilde{X}_P$ if (and generically, only if) λ is in the face C_P.*

(One interpretation of the generic statement is that over the whole $\mathbf{g}$, $\tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$ is a birational morphism whenever λ is in the interior of face C_P.)

4 Accidental singularities.

Fix a parabolic P and a weight λ in the interior C_P^0 of the corresponding face. What is the expected behavior of the birational morphism $i_\lambda : \widetilde{X}_P \to \widetilde{X}_\lambda$ of (3.3)? Consider the simplest case: $\mathbf{g} = \mathbf{sl}(n)$, with its standard representation ρ, so λ is the fundamental weight ω_1, and $\mathbf{p}$ the corresponding maximal parabolic. In this case both $\widetilde{X}_P$ and $\widetilde{X}_\lambda$ parameterize the eigenvalues of $\rho(\varphi(x))$, for $x \in X$, so i_λ is an isomorphism. (These are the standard spectral covers considered, e.g. in [BNR].)

We would like to point out that this situation is quite atypical. In fact, for any $\mathbf{g}$ and almost any λ, the birational map

$$i_\lambda : \tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$$

will fail to be an isomorphism over a non-empty divisor in $\mathbf{g}$, consisting of elements $g \in \mathbf{g}$ at which distinct weights λ, $w\lambda$ accidentally take the same value. Most points of this divisor will actually be regular semisimple. As a result, we expect $i_\lambda : \widetilde{X}_P \to \widetilde{X}_\lambda$ to fail to be an isomorphism as soon as X (contains, or) is a complete curve, and no regularity requirement on $\varphi(x)$, $x \in X$ will improve this situation.

Let $t \in \mathbf{t}$ be regular, i.e. $\alpha(t) \neq 0$ for each root $\alpha \in R$. Then i_λ has an accidental singularity at t iff $(\lambda - w\lambda)t = 0$ for some $w \in W \setminus W_P$. Hence:

$$i_\lambda \text{ has no accidental singularities at regular semisimple points } \Leftrightarrow$$

$$\forall w \in W \setminus W_p,\ \lambda - w\lambda \text{ vanishes only at singular points} \Leftrightarrow$$

$$\forall w \in W,\ \lambda - w\lambda \ \text{ is a multiple of some root.} \tag{4.1}$$

This is therefore a necessary condition for $i_\lambda = \tilde{\mathbf{g}}_P \to \tilde{\mathbf{g}}_\lambda$ to be an isomorphism.

Lemma (4.2). *Condition (4.1) implies that* $\mathbf{p}$ *is maximal parabolic, so* λ *equals a multiple of a fundamental weight, modulo the center.*

To see this, we may as well divide by the center and assume that $\mathbf{g}$ is semisimple. Let $\{\alpha_i\}$ be the simple roots, $\{\omega_i\}$ the fundamental weights, and $\sigma_i \in W$ the reflection perpendicular to α_i. Write $\lambda = \sum m_i\omega_i$, $m_i \geq 0$. Then $\lambda - \sigma_i\lambda = m_i\alpha_i$, so $m_i\alpha_i - m_j\alpha_j$ should be a root multiple for every i, j. By the definition of a simple set of roots, m_i and $-m_j$ must have the same sign.

We conclude that at most one m_i is $\neq 0$, i.e. λ is a multiple of ω_i as claimed. Q.E.D.

For each Dynkin diagram there are therefore, up to homothety, only finitely many possibilities for λ such that i_λ is an isomorphism (on regular semisimple elements, or equivalently, everywhere). For the classical algebras and for G_2, we see easily that the possibilities are:

$$\begin{array}{lcl} A_n & : & \omega_1, \omega_n \\ B_2 & : & \omega_1, \omega_2 \\ B_n & : & \omega_1 \quad (n \geq 3) \\ C_n & : & \omega_1 \quad (n \geq 3) \\ D_n & : & none \\ G_2 & : & \omega_1, \omega_2 \end{array}$$

We note that the spectral curves considered by Hitchin[H] are the $\widetilde{X}_\rho$ for the classical algebras and for the standard representation ρ, of highest weight ω_1. These are minuscule for types A, C, D but have the additional weight 0 for B_n, and i_λ is an isomorphism for types A, B, C but has accidental singularities for D_n. Accordingly, his spectral curves are (generically) non-singular for A_n, C_n; always singular for D_n; and reducible for B_n.

5 Isotypic decomposition of Pic.

Consider, in this and the next section, the general situation of a finite group W acting faithfully on a variety $\widetilde{X}$, with quotient X. We get actions of $\mathbf{Z}[W]$ on $\widetilde{X}$, hence on $H_*(\widetilde{X}, \mathbf{Z})$ and on $Pic(\widetilde{X})$. Given an irreducible $\mathbf{Z}[W]$-module Λ, we consider its equivariant maps to $Pic\widetilde{X}$:

$$(5.1) \qquad Prym_\Lambda(\widetilde{X}) := Hom_W(\Lambda, Pic\widetilde{X}).$$

this is an algebraic group, and an abelian variety if $Pic\widetilde{X}$ is, e.g. if $\widetilde{X}$ is non-singular and projective.

For each $e \in \Lambda$ we get an evaluation map

$$eval_e : Prym_\Lambda(\widetilde{X}) \to Pic\widetilde{X}.$$

The kernel of $eval_e$ is finite, for any $e \neq 0$, since $\mathbf{Z}[W]e$ has finite index in the irreducible module Λ. Up to isogeny we may therefore think of $Prym_\Lambda(\widetilde{X})$ as an algebraic subgroup of $Pic\widetilde{X}$; but this is usually unnatural, since changing e can result in a different (isogenous) copy of $Prym_\Lambda(\widetilde{X})$ inside $Pic\widetilde{X}$.

A W-submodule $\Lambda' \subset \Lambda$ of finite index determines a restriction map

$$Res: \ Prym_\Lambda \widetilde{X} \to Prym_{\Lambda'} \widetilde{X}$$

of finite kernel and cokernel. Therefore, $Prym_\Lambda \widetilde{X}$ makes sense up to isogeny if Λ is only an irreducible $\mathbf{Q[W]}$-module.

From now on, we will assume that all irreducible representations of W are defined over $\mathbf{Q}$. We can then choose a set $\{\Lambda_i\}$, $i \in W$, of irreducible $\mathbf{Z}[W]$-modules whose complexifications give all the irreducible representations. $\mathbf{Z}[W]$ then decomposes as a two-sided W module, up to isogeny:

$$\mathbf{Z}[W] \sim \oplus_i \Lambda_i \otimes \Lambda_i^*, \tag{5.2}$$

where W acts on the left on Λ_i and on the right on Λ_i^*. We obtain the corresponding isotypic isogeny decomposition:

$$Pic\widetilde{X} \sim \oplus_i \Lambda_i \otimes_{\mathbf{Z}} Prym_{\Lambda_i} \widetilde{X}. \tag{5.3}$$

6 Subgroups, subcovers, correspondences.

Fix a subgroup W_P of W. The action of W on $\widetilde{X}$ restricts to an action of W_P, giving an intermediate cover $\widetilde{X}_P$:

$$\widetilde{X} \xrightarrow{\pi_P} \widetilde{X}_P \xrightarrow{\pi^P} X.$$

There is no natural action of W on $\widetilde{X}_P$ or on $Pic(\widetilde{X}_P)$, but $Pic(\widetilde{X}_P)$ can still be decomposed into its isotypic components with respect to the action of the ("Hecke") ring $\mathcal{C}_P$ of correspondences on $\widetilde{X}_P$ over X. This has the following elementary description:

(6.1) A correspondence on $\widetilde{X}_P$ over X is a top dimensional cycle, or linear combination of components, of

$$\widetilde{X}_P \times_X \widetilde{X}_P = (\widetilde{X}/W_P) \times_X (\widetilde{X}/W_P).$$

The set of components is given by the quotient

$$W\backslash(W \times W)/W_P \times W_P \approx W_P\backslash W/W_P,$$

i.e. by double W_P-cosets in W. Each double-coset $C = W_PwW_P$, $w \in W$, gives an effective correspondence:

$$I_C = I_w := \{(xW_P, xw'W_P)|w' \in C/W_P,\ x \in \widetilde{X}\}.$$

These correspondences are independent (as long as the action of W is faithful), and if $\widetilde{X}$ is irreducible, these I_w form a $\mathbf{Z}$-basis for all correspondences. So, as a group, $\mathcal{C}_P \approx \mathbf{Z}[W_P\backslash W/W_P]$.

(6.2) When $W_P = (1)$, the ring of correspondences is just the integral group ring $\mathcal{C}_1 \approx \mathbf{Z}[W]$, acting naturally on $\widetilde{X}$ and hence on $Pic(\widetilde{X})$. In general, there is an injective pullback map

$$\begin{array}{rcl} \pi_P^* : \mathcal{C}_P & \to & \mathcal{C}_1 \approx \mathbf{Z}[W] \\ I_C & \mapsto & \sum_{w \in C} I_w \end{array}$$

satisfying

$$\pi_P^* I_{c_1} \cdot \pi_P^* I_{c_2} := \#(W_P) \cdot \pi_P^*(I_{c_1} \cdot I_{c_2}).$$

We can thus identify $\mathcal{C}_P$ with the subring of $\mathbf{Q}[W]$ generated (as abelian group) by

$$i_C := \frac{1}{\#(W_P)} \sum_{w \in C} I_w$$

as C runs over the double W_P cosets in W. (The image will usually not contain the identity element.)

We can now describe several ways of decomposing $Pic(\widetilde{X}_P)$ into natural components:

(6.3) The ring of correspondences $\mathcal{C}_P$ acts naturally on $Pic(\widetilde{X}_P)$, so every integral representation of $\mathcal{C}_P$ determines a generalized Prym variety as in (5.1), and these can be grouped into isotypic components as in (5.3).

(6.4) We can map $Pic(\widetilde{X}_P)$ to $Pic(\widetilde{X})$ via π_P^*, and intersect the image with the isotypic components $\Lambda_i \otimes Prym_{\Lambda_i}\widetilde{X}$ (with respect to the W action) in $Pic(\widetilde{X})$, defined in (5.3).

(6.5) The direct image sheaf $\pi_*^P\mathbf{Z}$ on X is associated to the W-cover $\widetilde{X}$ by the permutation representation $\mathbf{Z}[W_P\backslash W]$:

$$\pi_*^P\mathbf{Z} \approx \mathbf{Z}[W_P\backslash W] \times_W \widetilde{X}.$$

The decomposition of $\mathbf{Z}[W_P\backslash W]$ into irreducible W-representations (over $\mathbf{Q}$) determines a decomposition of $\pi_*^P\mathbf{Z}$, hence of $H^1(\widetilde{X},\mathbf{Z})$, and hence an isogeny decomposition of $Pic(\widetilde{X}_P)$.

We see that all these decompositions are essentially the same (up to isogeny, i.e. the connected components agree). In terms of the decomposition (5.3) of $Pic\widetilde{X}$, the image of π_P^* in (6.4) is given up to isogeny by

$$\oplus_i M_i \otimes Prym_{\Lambda_i}\widetilde{X} \subset \oplus_i\ \Lambda_i \otimes Prym_{\Lambda_i}\widetilde{X}, \tag{6.6}$$

Where the multiplicity space M_i is given by the W_P-invariants $(\Lambda_i)^{W_P}$. By Frobenius reciprocity, this corresponds exactly to the decomposition of $\mathbf{Q}[W_P\backslash W]$:

$$\mathbf{Q}[W_P\backslash W] \approx \oplus_i\ \Lambda_i^{W_P} \otimes \Lambda_i^* \otimes \mathbf{Q},$$

so (6.4) and (6.5) agree.

To compare with the action of the ring of correspondences, we note that

$$\begin{aligned} \mathbf{Q}\otimes \mathcal{C}_P &= \mathbf{Q}[W_P\backslash W/W_P] = End_{\mathbf{Q}[W]}\mathbf{Q}[W_P\backslash W] \\ &= End_{\mathbf{Q}[W]}(\oplus_i M_i \otimes \Lambda_i^* \otimes \mathbf{Q}) \\ &= \oplus_i End_{\mathbf{Q}}(M_i \otimes \mathbf{Q}) = \oplus_i (End M_i) \otimes \mathbf{Q}, \end{aligned}$$

and the action on $Pic\widetilde{X}_P \sim \oplus_i M_i \otimes Prym_{\Lambda_i}(\widetilde{X})$ is consistent with this decomposition, i.e. $End M_i$ acts as 0 on $M_j \otimes Prym_{\Lambda_j}\widetilde{X}$ if $j \neq i$, and through its action on M_i if $j = i$. So the generalized Pryms obtainable from (6.3) are precisely those $Prym_{\Lambda_i}(\widetilde{X})$ for which $M_i = \Lambda_i^{W_P}$ is non-zero, and the isotypic decomposition is the same as the one obtained from (6.4) or (6.5).

Let $\mathbf{1}_{W_i}^W$ denote the permutation representation of W on W_i cosets, or its character. We note for subsequent use the following corollary of Frobenius reciprocity:

Lemma (6.7). *If W_i, W_j are subgroups of W, then*

$$(\mathbf{1}_{W_i}^W, \mathbf{1}_{W_j}^W) = dim(\mathbf{1}_{W_j}^W)^{W_i} = \#(W_j\backslash W/W_i),$$

where the left side denotes the inner product of characters, and the right side is the number of two sided cosets.

7 An example.

Consider the symmetric group $W = S_3$. It has 3 subgroups of order 2, generated by (23), (13), and (12), and a normal subgroup A_3, of order 3. The W-cover $\widetilde{X} \to X$ has corresponding intermediate covers $X_1, X_2, X_3, \overline{X}$. W has character table:

	1	C_2	C_3
$\mathbf{1}$	1	1	1
ε	1	-1	1
ρ	2	0	-1

where C_i is the conjugacy class of elements of order i; 1 is the trivial character, ε the sign character, and ρ the character of the 2-dimensional reflection representation Λ:

$$\Lambda = \mathbf{Z}e \oplus \mathbf{Z}f$$

$$\begin{array}{ll} (12)\ e = e & (12)\ f = -e - f \\ (23)\ e = -e - f & (23)\ f = f \end{array}$$

We obtain the decompositions,

$$\begin{array}{lclclccl}
Pic\widetilde{X} & \sim & PicX & + & Prym_{\varepsilon}\widetilde{X} & + & \Lambda & \otimes & Prym_{\Lambda}\widetilde{X} \\
Pic\overline{X} & \sim & PicX & + & Prym_{\varepsilon}\widetilde{X} & & & & \\
PicX_1 & \sim & PicX & + & & & e & \otimes & Prym_{\Lambda}\widetilde{X} \\
PicX_2 & \sim & PicX & + & & & (-e-f) & \otimes & Prym_{\Lambda}\widetilde{X} \\
PicX_3 & \sim & PicX & + & & & f & \otimes & Prym_{\Lambda}\widetilde{X}
\end{array}$$

8 The distinguished Prym.

As explained in the introduction, the linearization of algebraically completely integrable systems via spectral covers suggests that there should be a unique, or at least a distinguished, nontrivial irreducible representation $\Lambda \neq 1$ of W such that the Prym variety $Prym_{\Lambda}$ occurs in $\widetilde{X}_P$ for all proper subgroups $W_P \neq W$. As the example in §7 shows, this is not true for all finite groups W, in fact not even for Weyl groups if we allow W_P to be an arbitrary subgroup. Restricting attention only to the Weyl subgroups, we are left in the above example with

$\widetilde{X}$, X_1 and X_3, each of which contains the trivial piece $PicX$ and at least one copy of $Prym_\Lambda \widetilde{X}$.

This picture generalizes as follows. For any Weyl group W of a reductive Lie group G, all representations are defined over $\mathbf{Q}$. We can describe three natural irreducible representations: the trivial representation $\mathbf{1}$, the sign representation ε, and the reflection representation of W acting on the weight lattice Λ of the semisimple part of G. Of these, $\mathbf{1}$ occurs in $\mathbf{1}_{W_P}^W$ for any $W_P \subset W$, and ε does not occur in $\mathbf{1}_{W_P}^W$ for any proper Weyl subgroup W_P.

Theorem (8.1). *(1) The Prym variety $Prym_\Lambda \widetilde{X}$ corresponding to the reflection representation Λ occurs with positive multiplicity in $Pic(\widetilde{X}_P)$ for any proper Weyl subgroup $W_p \neq W$.*

(2) For the classical groups, Λ is the only nontrivial irreducible representation of W with this property.

Proof.

$$\begin{aligned} (1)\ mult(Prym_\Lambda \widetilde{X}, Pic\widetilde{X}_P) &= mult(\Lambda, \mathbf{1}_{W_P}^W) = && \text{(by Frobenius)} \\ &= dim(\Lambda)^{W_P} = && \text{(compare (3.1))} \\ &= dim(C_P/\text{center}) > 0. \end{aligned}$$

(2) For a given Weyl group W, the question is: Find all irreducible W-representations V such that $V^{W_P} \neq (0)$ for each proper Weyl subgroup $W_P \neq W$.

For type A_n, i.e. $G = SL(n+1)$ and $W = S_{n+1}$, take $W_{P_0} = S_n$ corresponding to a Dynkin subdiagram of type A_{n-1}. Then over $\mathbf{Q}$:

$$\mathbf{1}_{W_{P_0}}^W = \mathbf{1} \oplus \Lambda,$$

so no representation other then ($\mathbf{1}$ and) Λ is common to all $\mathbf{1}_{W_P}^W$, as required.

Consider the Weyl group W of type B_n, with the nodes labeled as in [Bo] so that $\alpha_1, \cdots, \alpha_{n-1}$ are long and α_n is short. Let W_i denote the Weyl subgroup obtained by deleting α_i. In the standard permutation representation of W on the $2n$ vectors $\pm\varepsilon_i$, $1 \leq i \leq n$, the stabilizer of ε_1 is W_1, so we get the decomposition

$$\mathbf{1}_{W_P}^W = \mathbf{1} \oplus \Lambda \oplus \Lambda'$$

where Λ is the n-dimensional reflection representation of W, and Λ' is the $(n-1)$-dimensional (reflection) representation of S_n, pulled back to W.

More generally, W_i is the stabilizer in W of $\varepsilon_1 + \cdots + \varepsilon_i$. We see from (6.7) that

$$(\mathbf{1}_{W_1}^W, \mathbf{1}_{W_i}^W) = \#(W_1 \backslash W / W_i) = \begin{cases} 3 & i = 1, 2, \cdots, n-1 \\ 2 & i = n \end{cases}.$$

In particular, for $i = n$ we have by part (1) a decomposition

$$\mathbf{1}_{W_n}^W = \mathbf{1} \oplus \Lambda \oplus V$$

for some representation V satisfying $0 = (V, \mathbf{1}_{W_1}^W)$, and in particular $0 = (V, \Lambda')$. So again, $\mathbf{1}$ and Λ are the only irreducible representations common to the $\mathbf{1}_{W_i}^W$ for all i. The same argument works with no change for type C_n. For D_n, the only change is that W_{n-1} is the stabilizer of $\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n$; but W_n is still the stabilizer of $\varepsilon_1 + \cdots + \varepsilon_n$, so we still have

$$(\mathbf{1}_{W_1}^W, \mathbf{1}_{W_n}^W) = 2,$$

and this case follows as well. Q.E.D.

9 Remarks on Springer's correspondence.

In fact, we can say much more about the decompositon of $\mathbf{1}_{W_i}^W$, or more generally of $\mathbf{1}_{W_P}^W$, into irreducibles. The picture is clearest for type A_n; in this case, the irreducible representations V_λ of $W = S_{n+1}$ are parametrized by partitions λ of $n+1$, as are the conjugacy classes (in $GL(n+1)$) of Levi subgroups, the conjugacy classes in W of Weyl sugbroups W_λ, and the unipotent conjugacy classes in $GL(n+1)$ (or the nilpotent classes in $\mathbf{gl}(n+1)$). These classes are partially ordered (e.g. by inclusion of unipotent class closures), and Young's rule says that the decomposition matrix $(m_{\lambda,\mu})$, giving the multiplicity of V_λ in $\mathbf{1}_{W_\mu}^W$, is a triangular matrix with $1's$ on the diagonal.

In fact, $m_{\lambda\mu}$ is the Kostka number [J], defined as the number of semistandard tableaux on μ of type λ. The uniqueness of Λ of course follows from the triangularity of the decomposition matrix. Explicitly, the representations $\mathbf{1}$, Λ and ε correspond to the partitions (n), $(n-1, 1)$, and (1^n). The partitions which occur in the $\mathbf{1}_{W_i}^W$ are those with at most two parts:

(9.1) $$\mathbf{1}_{W_i}^W = \oplus_{j=0}^i V_{n-j,j} \qquad \text{if} \quad 2i \leq n.$$

For other groups, the picture is somewhat more complicated. To a Levi subgroup $L \subset G$ (e.g. to the Levi $L(P)$ of a parabolic P) we can associate the (unipotent) conjugacy class $\mathcal{O}_u$ of a regular unipotent element $u \in L$. In general, this correspondence may no longer be bijective. We say that a conjugacy class is of parabolic type if it comes from some $L(P)$. For a unipotent u, consider the Springer fiber

$$\mathcal{B}_u := \{\text{Borel subgroups of } \text{ G } \text{ containing } u\}.$$

Springer and others construct an action of W on $\mathcal{B}_u$, hence on its cohomology $H^*(\mathcal{B}_u)$. Alvis and Lusztig [AL] identify $H^*(\mathcal{B}_u)$ with $\mathbf{1}_{W_p}^W$, in case u is regular unipotent in $L(P)$ as above. On the other hand, the top cohomology $H^{top}(\mathcal{B}_u)$ decomposes into irreducible W- representations $S_{u,\ell}$ indexed by the irreducible local systems ℓ on $\mathcal{O}_u$. (In case A_n, $\mathcal{O}_u$ is simply connected so ℓ is trivial.) The Springer correspondence

$$(\mathcal{O}_u, \ell) \mapsto S_{u,\ell}$$

gives all irreducible W-representations.

The triangularity part of Young's rule has an analogue for arbitrary W, due to Borho and MacPherson[BM]: any component of $H^*(\mathcal{B}_u)$ (i.e. of $\mathbf{1}_{W_p}^W$, by [AL]) is of the form $S_{v,\ell}$ for some unipotent v (and local system ℓ on $\mathcal{O}_v$) such that $\overline{\mathcal{O}}_v \supset \mathcal{O}_u$.

Since the Springer correspondence has been completely determined (see, e.g. [Ca] §13.3), the triangularity result provides a powerful tool for analyzing the decomposition of permutation representations of W. Yet we do not see how to use it for our purposes, since it only provides block-triangularity. Two things can go wrong:

- The largest non regular (="subregular") unipotent class corresponds to the reflection representation. But it may fail to be simply connected, and may thus contribute to more than a single irreducible W-representation which cannot be excluded; or
- There may be unipotent classes, strictly smaller (= in the closure of) the subregular, but containing in their closure all unipotent classes of parabolic type.

We will see below that for the exceptional groups both problems do occur.

10 Decomposition for G_2.

In this section and the next we show that the uniqueness (part (2) of Theorem (8.1))

can fail for some exceptional groups. We do this by explicit calculation.

The Weyl group $W(G_2)$ is isomorphic to the dihedral group $Dihed_6$, of order 12, generated by a rotation r, of order 6, and a reflection s. The character table is, in the notation of [Se]:

	ψ_1	ψ_2	ψ_3	ψ_4	χ_1	χ_2
1	1	1	1	1	2	2
r, r^{-1}	1	1	-1	-1	1	-1
r^2, r^{-2}	1	1	1	1	-1	-1
r^3	1	1	-1	-1	-2	2
s, sr^2, sr^{-2}	1	-1	1	-1	0	0
sr, sr^3, sr^{-1}	1	-1	-1	1	0	0

Here $\mathbf{1} = \psi_1$, $\Lambda = \chi_1$, $\varepsilon = \psi_2$. There are only two non-trivial Weyl subgroups, say $W_1 = (s)$ and $W_2 = (sr)$. We see then that the characters of $\mathbf{1}_{W_1}^W$ and $\mathbf{1}_{W_2}^W$ are, respectively, $(6,0,0,0,2,0)$ and $(6,0,0,0,0,2)$. The decomposition is thus:

$$(10.1) \qquad \mathbf{1}_{W_1}^W = \psi_1 + \psi_3 + \chi_1 + \chi_2, \quad \mathbf{1}_{W_2}^W = \psi_1 + \psi_4 + \chi_1 + \chi_2.$$

This can also be seen very explicitly: let H denote $W_1 = (1, s)$, then $\mathbf{Q}[W/H] =:$ $\mathcal{U}_1$ decomposes into:

$$\begin{array}{lcl}
\psi_1 & \sim & \mathbf{Q}[H + rH + r^2H + r^3H + r^{-2}H + r^{-1}H] \\
\psi_3 & \sim & \mathbf{Q}[H - rH + r^2H - r^3H + r^{-2}H - r^{-1}H] \\
\chi_1 & \sim & \mathbf{Q}[H - rH - r^3H + r^{-2}H, H - r^2H - r^3H + r^{-1}H] \\
\chi_2 & \sim & \mathbf{Q}[H - rH + r^3H - r^{-2}H, H - r^2H + r^3H - r^{-1}H].
\end{array}$$

The decomposition for $\mathcal{U}_2 := \mathbf{1}_{W_2}^W$ is obtained similarly.

Theorem (8.1) is thus <u>false</u> for G_2: there are two non-trivial pieces common to all $Pic(\widetilde{X_P})$, namely $\Lambda = \chi_1$ and χ_2.

These Pryms can be described more explicitly as follows: The cover $\widetilde{X}^1 = \widetilde{X}/W_1$, of degree 6 over X, is the fiber product of its two intermediate covers:

$$X' = \widetilde{X}^1/(r^2), \quad X'' = \widetilde{X}^1/(r^3),$$

of degrees $2,3$ respectively over X. From the decomposition of permutation representations:

$$\begin{aligned} \mathbf{1}^W_{(s,r^2)} &= \psi_1 + \psi_3 \\ \mathbf{1}^W_{(s,r^3)} &= \psi_1 + \chi_2 \end{aligned}$$

We find

$$\begin{aligned} PicX' &\sim PicX + Prym_{\psi_3}\widetilde{X} \\ PicX'' &\sim PicX + Prym_{\chi_2}\widetilde{X}, \end{aligned}$$

hence

$$\begin{aligned} Prym_{\chi_2}\widetilde{X} &\sim Prym(X''/X) \\ Prym_{\chi_1}\widetilde{X} + Prym_{\chi_2}\widetilde{X} &\sim Prym(\widetilde{X}^1/X') \\ Prym_{\chi_1}\widetilde{X} &\sim Prym(\widetilde{X}^1/(X',X'')). \end{aligned}$$

We note that $W(G_2)$ has the outer automorphism $s \mapsto sr$, $r \mapsto r$, which exchanges W_1 and W_2, ψ_3 and ψ_4, and fixes $\psi_1, \psi_2, \chi_1, \chi_2$; so the above decomposition of $Pic\widetilde{X}^1$ is transformed to the corresponding decomposition for $Pic\widetilde{X}^2$.

Starting with arbitrary X and (branched) covers X', X'' of degrees $2,3$, we construct $X''!$, of degree 6, and then set

$$\widetilde{X}^1 = X' \times_X X'', \quad \widetilde{X} = X' \times_X X''!,$$

recovering the previous situation. If X is a curve of genus g, and X', X'' have respectively $2n, 2m$ simple ramification points over disjoint branch loci in X, we find for the Pryms of types $\psi_1, \psi_3, \chi_1, \chi_2$ the dimensions

$$g, g-1+n, 2g-2+m+2n, 2g-2+m.$$

In particular, we see that the three components $Prym_{\psi_1}$, $Prym_{\chi_1}$, $Prym_{\chi_2}$ common to $Pic\widetilde{X}^1$ and $Pic\widetilde{X}^2$ have different dimensions, and in general there will be no nontrivial maps between them. If we take $X = \mathbf{P}^1$, the common piece $Prym_{\chi_2}$ becomes the Jacobian of the trigonal curve X''. But the distinguished piece, $Prym_{\chi_1}$, still seems to be neither a Jacobian nor a classical Prym.

It is also interesting to compare this with the explicit description of the Springer correspondence, in [Ca]. There are 5 unipotent conjugacy classes in G_2, denoted there G_2 (the regular unipotents), $G_2(a_1)$ (the subregulars), $\tilde{A}_1$, A_1 and 1. These are in descending order (the partial order is a total order for G_2). The Springer correspondence then sends:

$$\begin{array}{rcl} G_2 & \mapsto & \psi_1 \\ G_2(a_1) & \mapsto & \psi_4, \chi_1 \\ \tilde{A}_1 & \mapsto & \chi_2 \\ A_1 & \mapsto & \psi_3 \\ 1 & \mapsto & \psi_2, \end{array}$$

where χ_1 comes from the trivial local system on the subregular orbit, and ψ_4 from the nontrivial rank-2 local system (the fundamental group is S_3). In the terminology of §9, the unipotent class associated to (the Levi of) W_2 is $\tilde{A}_1$, and all characters allowed by Borho-MacPherson's triangularity do occur. But the unipotent class associated to W_1 is A_1, and one character (ψ_4) allowed by [BM] is missing.

11 The decomposition for E_6.

We use Schläfli's description of the exceptional Weyl group $W := W(E_6)$, as incidence-preserving permutations of the 27 lines on a general cubic surface (cf.[CCNPW], p.26). For the "lines" we take the $27 = 6 + 6 + 15$ objects:

$$a_i \quad , \quad b_j \quad , \quad c_{ij} = c_{ji} \qquad (i, j = 1, \cdots, 6, \ i \neq j).$$

Two lines are incident if they lie in one of the $45 = 30 + 15$ "tritangent planes":

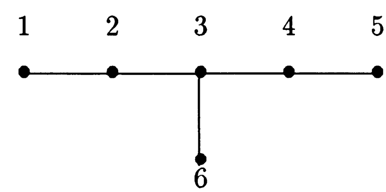
$$(a_i, b_j, c_{ij}) \quad , \quad (c_{ij}, c_{k\ell}, c_{mn}) \qquad (i, \cdots, n \text{ distinct } \in \{1, \cdots, 6\}).$$

There are 72 "sixes" of pairwise non-incident lines, arranged in $36 = 1+20+15$ "double-sixes" in which each line of one of the sixes meets all but one of the lines in the other six:

$$\begin{array}{lcllllll} s & = & \{a_1, & a_2, & a_3, & a_4, & a_5, & a_6\} \\ s' & = & \{b_1, & b_2, & b_3, & b_4, & b_5, & b_6\} \\ & & & & & & & \\ s_{ijk} & = & \{a_i, & a_j, & a_k, & c_{\ell m}, & c_{\ell n}, & c_{mn}\} \\ s'_{ijk} & = & \{c_{jk}, & c_{ik}, & c_{ij}, & b_n, & b_m, & b_\ell\ \} \\ & & & & & & & \\ s_{ij} & = & \{a_i, & b_i, & c_{jk}, & c_{j\ell}, & c_{jm}, & c_{jn}\} \\ s'_{ij} & = & \{a_j, & b_j, & c_{ik}, & c_{i\ell}, & c_{im}, & c_{in}\} \quad = s_{ji} \end{array}$$

Each "six" s_I determines an involution $\sigma_I \in W$ which exchanges s_I with s'_I and fixes the remaining 15 lines.

We label the six fundamental weights ω_i as follows:



(This seems simpler than the notation in [Bou].) The corresponding reflections σ_i, $1 \leq i \leq 6$, generate W. Explicitly, for $1 \leq i \leq 5$, σ_i corresponds to the double-six $s_{i,i+1}, s'_{i,i+1}$; it preserves the partition of the lines into $a's$, $b's$ and $c's$, and exchanges indices $i, i+1$. The last reflection, σ_6, corresponds to s_{123}, s'_{123}.

The Weyl subgroup W_i is generated by $\{\sigma_j | j \neq i\}$, and is the stabilizer in W of ω_i. It has a simple description as stablilizer of a set of lines:

W_1 : the line b_1

W_2 : the disjoint pair $\{b_1, b_2\}$

W_3 : the disjoint triple $\{b_1, b_2, b_3\}$ (or $\{a_4, a_5, a_6\}$)

W_4 : the disjoint pair $\{a_5, a_6\}$

W_5 : the line a_6

W_6 : the disjoint "six" s (or s').

The graph automorphism takes the permutation representation $U_i := \mathbf{1}_{W_i}^{W}$ to U_{6-i}, $1 \leq i \leq 5$. We are thus concerned with the four fundamental representations U_1, U_2, U_3, U_6, of dimensions $27, 216, 720, 72$ respectively. We note that U_1 is the permutation representation of W on the 27 lines, and U_6 is the permutation representation on the 72 roots.

The character inner products $U_i \cdot U_j$ are given in the following table:

(11.1)

	U_1	U_2	U_3	U_6
U_1	3	4	5	3
U_2	4	10	17	6
U_3	5	17	36	9
U_6	3	6	9	5

The computation can be done by a straightforward application of lemma (6.7). For example, $U_1 \cdot U_i$ is given by the number of orbits of W_i on the 27 lines. These orbits are:

$$
\begin{array}{rcll}
U_1 & = & b_1 & \\
 & & \text{10 lines incident to } b_1 & (= a_i,\ c_{1i},\ i \neq 1) \\
 & & \text{16 non-incident lines} & (= a_1, b_i, c_{ij},\ i,j \neq 1) \\
U_2 & = & b_1, b_2 & \\
 & & \text{5 lines meeting both} & (= a_i, i \neq 1, 2, \text{and } c_{12}) \\
 & & \text{10 lines meeting one of } b_1, b_2 & (= a_1, a_2, c_{1i}, c_{2i}, i \neq 1, 2) \\
 & & \text{10 lines meeting neither} & (= b_i, c_{ij}, i, j \neq 1, 2) \\
U_3 & = & b_1, b_2, b_3 & \\
 & & \text{3 lines meeting all 3} & (= a_4, a_5, a_6) \\
 & & \text{6 lines meeting 2} & (= a_1, a_2, a_3, c_{12}, c_{13}, c_{23}) \\
 & & \text{9 lines meeting 1} & (= c_{ij}, 1 \leq i \leq 3, 4 \leq j \leq 6) \\
 & & \text{6 lines meeting none} & (= b_4, b_5, b_6, c_{45}, c_{46}, c_{56}) \\
U_6 & = & 6a's & \\
 & & 6b's & \\
 & & 15c's &
\end{array}
$$

Other rows are similarly computed as numbers of W_i-orbits on pairs, triples and sixes of disjoint lines. Here are the orbits on sixes:

Orbits of	Orbit size	Orbit description			
W_1	16	$s_I \ni b_1$			
	16	$s'_I \ni b_1$			
	40	neither			
W_2		$\#(s_I \cap \{b_1, b_2\})$	$\#(s'_I \cap \{b_1, b_2\})$		
	5	2	0		
	2	1	1		
	20	1	0		
	5	0	2		
	20	0	1		
	20	0	0		
W_3		$\#(s_I \cap \{b_1, b_2, b_3\})$	$\#(s_I \cap \{a_4, a_5, a_6\})$	$\#(s'_I \cap \{b_1, b_2, b_3\})$	$\#(s'_I \cap \{a_4, a_5, a_6\})$
	2	0	3	3	0
	2	3	0	0	3
	2	0	0	0	0
	18	0	1	1	0
	18	1	0	0	1
	9	0	2	2	0
	9	2	0	0	2
	6	1	0	1	0
	6	0	1	0	1
W_6	1	s			
	20	s_{ijk}			
	30	s_{ij}			
	20	s'_{ijk}			
	1	s'			

The analogue of (8.1) in this case is:

Proposition (11.2).

(i) $U_1 = \mathbf{1} \oplus \Lambda \oplus \Xi$, *where* Λ *is the* 6*-dimensional reflection representation of* $W(E_6)$, *and* Ξ *is an irreducible* 20*-dimensional representation.*

(ii) *Each* U_i *contains* $\mathbf{1}$ *and* Λ *with multiplicity* 1. *The multiplicity of* Ξ *in* U_1, U_2, U_3, U_6 *is, respectively,* $1, 2, 3, 1$.

Proof. The multiplicities of $\mathbf{1}, \Lambda$ in U_i are always 1. The irreducibility of the remaining piece Ξ in U_1 and its multiplicity in the U_i then follow from (the first row of) (11.1). Q.E.D.

In particular, we again have two different non-trivial Pryms, P_Λ and P_Ξ, which occur in all spectral Picards $Pic(\widetilde{X}_\rho)$. Using the rest of the information in (11.1), we can work out the complete decomposition of the U_i.

We find in the Atlas [CCNPW] that the simple group

$$W^+ := ker(\varepsilon : W \to \mathbf{Z}/2)$$

has 20 irreducible characters. Of these, 10 merge in pairs to give 5 charcters of W which we denote by their dimensions: $\mathbf{10}, \mathbf{20}, \mathbf{60}, \mathbf{80}, \mathbf{90}$. (These are the characters which vanish on $W \setminus W^+$.) Each of the 10 remaining irreducible characters of W^+ splits into a pair of irreducible characters of W; the dimensions are $1, 6, 15, 15, 20, 24, 30, 60, 64, 81$. We denote each of these 20 characters by its dimension followed by a + or − according to its sign on the reflections σ_i (which are all in the same conjugacy class in W, class $2c$ in Atlas notation). The two 15-dimensional pairs are separated by their values on products (in W^+) of two commuting σ_i, σ_j: we write $\mathbf{15}^\pm$ (respectively $\mathbf{15}'^\pm$) for the pair where these values are positive (respectively negative), lifting the character χ_8 (respectively χ_7) of W^+, in Atlas notation.

Proposition (11.3). *The decomposition of the permutation representations U_i into irreducibles is given by:*

$$\begin{aligned}
U_1 &= \mathbf{1} \oplus \Lambda \oplus \Xi \qquad (\Lambda = \mathbf{6}^+, \quad \Xi = \mathbf{20}^+) \\
U_6 &= \mathbf{1} \oplus \Lambda \oplus \Xi \oplus \mathbf{15}^+ \oplus \mathbf{30}^+ \\
U_2 &= \mathbf{1} \oplus \Lambda \oplus 2\Xi \oplus \mathbf{15}^+ \oplus \mathbf{30}^+ \oplus \mathbf{60}^+ \oplus \mathbf{64}^+ \\
U_3 &= \mathbf{1} \oplus \Lambda \oplus 3\Xi \oplus 2\cdot\mathbf{15}^+ \oplus 2\cdot\mathbf{30}^+ \oplus 3\cdot\mathbf{60}^+ \oplus 2\cdot\mathbf{64}^+ \oplus \mathbf{60} \oplus \mathbf{90} \oplus \\
&\quad\ \mathbf{24}^+ \oplus \mathbf{81}^+
\end{aligned}$$

Proof.

We claim that the decomposition above is the only one consistent with Table (11.1), with the known dimensions of the U_i, and with the values of their characters χ_i on a simple reflection σ, say the one which exchanges $a's$ and $b's$ (i.e. corresponding to the double six s, s', or the root α_6):

$$\chi_i(\sigma) = \#(W_i\text{–cosets fixed by } \sigma)$$

$$= \ 15, 30, 66, 140 \ \text{ for } \ i = 1, 6, 2, 3 \text{ respectively.}$$

These numbers count the i-tuples of disjoint lines which are preserved by σ. For example, the 140 triples, for $i = 3$, are of the form:

$$\begin{array}{ll} < a_i, b_i, c_{jk} > & (60) \\ < c_{ij}, c_{ik}, c_{i\ell} > & (60) \\ < c_{ij}, c_{ik}, c_{jk} > & (20) \end{array}$$

Here are the main steps:

- We must have $\Lambda = \mathbf{6}^+$ or $\mathbf{6}^-$ and $\Xi = \mathbf{20}^+$, $\mathbf{20}^-$ or $\mathbf{20}$; since $\chi_{\mathbf{1}}(\sigma) = 15$, $\mathbf{6}^{\pm}(\sigma) = \pm 4$, $\mathbf{20}^{\pm}(\sigma) = \pm 10$, $\mathbf{20}(\sigma) = 0$, we must have $\Lambda = \mathbf{6}^+$, $\Xi = \mathbf{20}^+$.

- There are two new components in U_6, by (11.1). The dimensions should add up to 45 and the values on σ to 15, so one must be $\mathbf{30}^+$ and the other either $\mathbf{15}^+$ or $\mathbf{15}'^+$. The 36-dimensional representation $\mathbf{1}_{W_6}^{W+}$ of W^+ on double-sixes is decomposed in the Atlas as $\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$. Since $U_{\mathbf{6}}$ must contain a lift of these, it decomposes as stated.

- We have already seen the multiplicities of $1, \Lambda, \Xi$ in U_2, U_3. The difference $U_2 - 1 - \Lambda - 2\Xi$ has inner product 4 with itself and 9 with U_3, so it must be the sum of 4 distinct characters. Since the inner product with U_6 is 2, two of these must be $\mathbf{15}^+$, $\mathbf{30}^+$. Let γ, δ be the remaining two characters. We have

$$\gamma(1) + \delta(1) = 216 - 1 - 6 - 2 \cdot 20 - 15 - 30 = 124$$

$$\gamma(\sigma) + \delta(\sigma) = \ 66 - 1 - 4 - 2 \cdot 10 - 5 - 10 = \ 26$$

  so these characters must be $\mathbf{60}^+$ and $\mathbf{64}^+$, as claimed.

- Write U_3 as $\mathbf{1} \oplus \Lambda \oplus 3\Xi \oplus k \cdot \mathbf{15}^+ \oplus (4-k) \cdot \mathbf{30}^+ \oplus \ell \cdot \mathbf{64}^+ \oplus (5-\ell) \cdot \mathbf{60}^+ \oplus \sum m_a \varepsilon_a$, where the ε_a are new characters, the m_a non-negative integers, and the coefficients $4 - k$, $5 - \ell$ are determined by (11.1). Evaluating the self-product and values on $1, \sigma$, we find:

$$\begin{array}{lcl} \sum m_a^2 & = & 25 - (k^2 + (4-k)^2) - (\ell^2 + (5-\ell)^2) \\ \sum m_a \varepsilon_a(1) & = & 233 + 15k - 4\ell \\ \sum m_a \varepsilon_a(\sigma) & = & 15 + 5k - 6\ell. \end{array}$$

The first equation gives

$$\sum m_a^2 \leq 4,$$

the second gives, after some fiddling, that $k = 2$, there are exactly four ε_a's, with all $m_a = 1$, and that $\ell = 2$ or 3, which yields respectively

$$\begin{array}{lll} \sum_{a=1}^4 \varepsilon_a(1) = 255 & \text{or} & 251 \\ \sum_{a=1}^4 \varepsilon_a(\sigma) = 13 & \text{or} & 7. \end{array}$$

The only solution is $\ell = 3$ with the ε_a's equal $\mathbf{90}, \mathbf{81}^+, \mathbf{60}, \mathbf{24}^+$.

Q.E.D.

12 Projection formulas.

We conclude by writing down explicitly some correspondences on spectral covers which induce on the spectral Picards the projection to the spectral Pryms. The method is very general, so we return to the setting of §6. W is an arbitrary finite group, V an irreducible representation of W, $v_0 \in V$ a vector fixed by a subgroup $W_P \subset W$. We then have a natural projection

$$\begin{array}{rcl} pr: \ U_P := \mathbf{1}_{W_P}^W = \mathbf{C}[W/W_P] & \to & V \\ w & \mapsto & wv_0. \end{array}$$

Assume now that V is either real or quaternionic, so there is a W-invariant, non-degenerate bilinear form $<,>$ on V. We then get a W-equivariant transpose map:

$$\begin{array}{rcl} i := (pr)^t : \ V & \to & U_P \\ v & \mapsto & \displaystyle\sum_{w \in W/W_P} < v, wv_0 > w. \end{array}$$

The composite:

$$\begin{array}{rcl} c = c_{P,V} = i \circ pr : \ U_P & \to & U_P \\ w_0 & \mapsto & \sum_{w \in W/W_P} < w_0 v_0, wv_0 > w \end{array}$$

is then the desired correspondence on U_P giving the projection to the V-factor. It satisfies

$$c^2 = qc$$

for a constant q which depends on our choice of $<,>$. It can be computed directly:

$$q \cdot dimV = Trace(c) = \sum_{w \in W/W_P} < wv_0, wv_0 > = \frac{\#(W)}{\#(W_P)} |v_0|^2,$$

so

$$q = \frac{dimU_P}{dimV} |v_0|^2. \tag{12.1}$$

It can also be computed by considering

$$\begin{array}{rccl} c' := pr \circ i : & V & \rightarrow & V \\ & v & \mapsto & \sum_{w \in W/W_P} < v, wv_0 > wv_0. \end{array}$$

By Schur's lemma, c' is multiplication by the scalar q, which is determined by:

$$q < v_0, v_0 > = < v_0, c'v_0 > = \sum_{w \in W/W_P} < v_0, wv_0 >^2,$$

so

$$q = \sum_{w \in W/W_P} \frac{< v_0, wv_0 >^2}{< v_0, v_0 >}. \tag{12.2}$$

(The compatibility of (12.1) and (12.2) amounts to the identity:

$$\text{Average}_{w \in W} \frac{< v_0, wv_0 >^2}{< v_0, v_0 >^2} = \frac{1}{dimV}.)$$

In the ring $\mathcal{C}_P$ of correspondences (6.1), we have

$$c = \sum_{w \in W_P \backslash W/W_P} < v_0, wv_0 > I_w. \tag{12.3}$$

When the representation V is rational it is therefore natural to choose $<,>$ so that the coefficients $< v_0, wv_0 >$ will be integers.

When W is a Weyl group, all irreducible representations V are rational, so the above applies. The integral correspondence $c = c_{P,V}$ acts on the spectral Picard, $Pic(\widetilde{X}_P)$, projecting it to a copy of $Prym_V(\widetilde{X})$. Projections to different copies of $Prym_V(\widetilde{X})$ are obtained by varying the initial vector v_0 within the fixed subspace $(V)^{W_P}$. When V is the reflection representation Λ, these v_0 can be taken to be the fundamental weights ω_i in the face C_P of the Weyl chamber determined by the subgroup W_P, cf. (3.1).

In [K], Kanev obtains (essentially) formula (12.3) for the projection and analogues of (12.1), (12.2) for the eigenvalue q, in case the base X is $\mathbf{P}^1$, the Lie algebra is of type A_n, D_n or E_n, and the representation V of W is the reflection representation Λ. (But W_P is an arbitrary Weyl subgroup, which by our preliminary observations in §§2,3 is equivalent to considering spectral covers $\widetilde{X}_\rho$ for arbitrary representations ρ of G. In other words, Kanev considers the distinguished $Prym$, $Prym_\Lambda(\widetilde{X}_\rho)$, in $Pic(\widetilde{X}_\rho)$, arbitrary ρ.) His approach is based on the construction, for each $\lambda \in \Lambda$ (corresponding to our choice of $v_0 \in V$), of a lattice $N(\Lambda, \lambda)$ with bilinear pairing $(,)$. For $\mathbf{g}$ of type A_n, D_n, E_n, these lattices are interpreted as cohomology of an appropriate rational surface, with Λ recovered as the primitive cohomology. Kanev's correspondence differs from our c by (a sign, since the primitive cohomology is negative definite, and) a translation by a multiple of $\sum I_w$ (= projection onto $\mathbf{1}$). When $PicX$ is trivial (e.g. under his assumption that $X = \mathbf{P}^1$), this translation is immaterial, and yields an effective representative of the correspondence. (In general it would map $Pic(\widetilde{X}_P)$ to the sum of $PicX$ and $Prym_\Lambda(\widetilde{X})$.)

For example, when $G = GL(n)$, $W = S_n$, the fundamental spectral covers are $\widetilde{X}_i$, $1 \leq i \leq n-1$, of degree $\begin{pmatrix} n \\ i \end{pmatrix}$ over X. A $\mathbf{Z}$-basis for the correspondences on $\widetilde{X}_i$ is given by I_j, $0 \leq j \leq i$, sending an i-tuple to all other i-tuples intersecting it with cardinality j. Kanev's formula for the projection of $Pic(\widetilde{X}_i)$ to $Prym_\Lambda(\widetilde{X})$ is

$$c = \sum_{j=0}^{i} (i-1-j) I_j,$$

while ours gives:

$$c = \sum_{j=0}^{i} (j - \frac{i^2}{n}) I.$$

At the Orsay meeting, I was informed by J.Y. Merindol that he had also obtained extensions of Kanev's results, similar in spirit to the formulas in this section. It seems that he still considers only the distinguished $Prym$, $Prym_\Lambda \widetilde{X}$ (and takes $X = \mathbf{P}^1$), but removes the restrictions on the type of the reductive Lie algebra $\mathbf{g}$.

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