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A Finiteness Theorem for Isogeny Correspondences

Alexandru Buium

0. Introduction

Let $A_{g,n}$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$ of dimension $g \geq 2$ with level n structure, $n \geq 3$; we will view $A_{g,n}$ as an algebraic variety over $\mathbb{C}$. Moreover, let $Y \subset A_{g,n}$ be a curve (by which we will understand an irreducible, closed, possibly singular subvariety of dimension 1). By an isogeny correspondence on Y we will understand an (irreducible, closed, possibly singular) curve $Z \subset Y \times Y$ for which there exists a quasi-finite map $Z' \to Z$ from an irreducible curve Z' with the property that the two abelian schemes over Z' deduced by base change via

$$Z' \to Z \subset Y \times Y \xrightarrow{p_i} Y \qquad i = 1, 2$$

(p_i = i-th projection) are isogenous. Note that two abelian schemes over Z' are called isogenous if there exists a surjective homomorphism between them with kernel finite over Z'; so we do not require our isogenies preserve, say, polarizations.

The question which we address in this paper is: how many isogeny correspondences can exist on a "sufficiently general" curve $Y \subset A_{g,n}$?

It is easy to see that there exist "lots" of curves $Y \subset A_{g,n}$ carrying infinitely many isogeny correspondences: more precisely, the union of all such

Y's in $A_{g,n}(\mathbb{C})$ is dense in the complex topology of $A_{g,n}(\mathbb{C})$ (see the Proposition from Section 1). Nevertheless, our main result here will imply in particular that "most" curves $Y \subset A_{g,n}$ carry at most finitely many isogeny correspondences (see Theorem 1 below).

Indeed, let $C(A_{g,n})$ be the set of all (irreducible, closed, possibly singular) curves in $A_{g,n}$; we will put a natural topology on $C(A_{g,n})$ which we call the Kolchin topology such that $C(A_{g,n})$ becomes an irreducible Noetherian topological space and then we will prove in particular the following:

Theorem 1. *There exists a dense Kolchin open subset C_0 of $C(A_{g,n})$ such that any curve Y belonging to C_0 carries at most finitely many isogeny correspondences.*

Remark. If a curve $Y \subset A_{g,n}$ carries at most finitely many isogeny correspondences Z then any such Z must have only finite orbits.

Let's define in what follows the Kolchin topology on $C(A_{g,n})$. More generally one can define the Kolchin topology on the set $C(A)$ of all (irreducible, closed, possibly singular) curves embedded in a given (irreducible, possibly singular) algebraic variety A over $\mathbb{C}$. Indeed, we consider first the "jet scheme" $\text{jet}(A)$, cf. [B1]; recall that this is by definition an A-scheme with a $\mathbb{C}$-derivation δ of its structure sheaf, characterized by the fact that for any pair (Z, d) consisting of an A-scheme Z and a $\mathbb{C}$-derivation d on $\mathcal{O}_Z$ there is a unique horizontal morphism of A-schemes $Z \to \text{jet}(A)$; "horizontal" here means "commuting with δ and d". For instance, if $A = \mathbb{A}^n = \operatorname{Spec}\mathbb{C}[y_1, \dots, y_n]$ then $\text{jet}(A) = \operatorname{Spec}\mathbb{C}\{y_1, \dots, h_n\}$ where $\mathbb{C}\{y_1, \dots, y_n\}$ is the ring of δ-polynomials in $y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ (which by definition is the ring of polynomials with coefficients in $\mathbb{C}$ in the infinite family of variables $y_j^{(i)}$, $i \geq 0$, $1 \leq j \leq n$, with $\mathbb{C}$-derivation δ sending $y_j^{(i)}$ into $y_j^{(i+1)}$). Now for any Zariski closed subset H of $\mathrm{jet}\,(A)$ we denote by $C_H(A)$ the set of all curves $Y \in C(A)$ such that the image of the natural horizontal closed immersion $\mathrm{jet}\,(Y) \to \mathrm{jet}\,(A)$ is contained in H. One easily checks that the sets $C_H(A)$ are the closed sets of a topology which we call the Kolchin topology (one has to use the non-obvious fact that $\mathrm{jet}\,(Y)$ is an irreducible scheme which follows from correctly interpreting a theorem of Kolchin, [K] p. 200). We will check in Section 2 below that $C(A)$ with the Kolchin topology is an irreducible Noetherian topological space.

Remark. Intuitively a subset of $C(A)$ is Kolchin closed if it consists of all curves $Y \in C(A)$ which "satisfy a certain system of algebraic differential equations on A". As the proof of Theorem 1 will show, the "system defining" $C(A_{g,n}) \setminus C_0$ has "order 6" (i.e. "comes from jets of order 6") and is highly nonlinear.

Actually we can do much better than in Theorem 1, namely we can "bound asymptotically" (for $Y \in C_0$) the number of isogeny correspondences on Y "counted with certain natural multiplicities" (see Theorem 1′ below). We need more notations. For any curve $Y \subset A_{g,n}$ we denote by $p(Y)$ the genus of a smooth projective model of Y. Moreover, for any isogeny correspondence $Z \subset Y \times Y$ we let $[Z : Y]_i$ denote the degree of the map $Z \subset Y \times Y \xrightarrow{p_i} Y$, $i = 1, 2$ and put $i(Y) = \sum [Z : Y]_1 = \sum [Z : Y]_2 \in \mathbb{N} \cup \{\infty\}$, where Z runs through the set of all isogeny correspondences on Y (we put $i(Y) = 0$ if this set is empty). This $i(Y)$ is the "number of isogeny correspondences counted with multiplicities"; for alternative descriptions of $i(Y)$ we refer to Lemmas 1 and 2 from Section 1. Finally, we shall fix a smooth projective compactification

$\bar{A}_{g,n}$ of $A_{g,n}$ and a very ample line bundle $\mathcal{O}(1)$ on $\bar{A}_{g,n}$; then for any curve $Y \subset A_{g,n}$ we shall denote by $\deg(Y)$ the degree of the Zariski closure of Y in $\bar{A}_{g,n}$ with respect to $\mathcal{O}(1)$.

We can state the following strengthening of Theorem 1:

Theorem 1′. *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and two positive integers m_1, m_2 such that for all $Y \in C_0$ we have*

$$i(Y) \leq m_1 \deg(Y) + m_2 p(Y)$$

Remark. A careful examination of the proof leads to an explicit value for m_2. But determining such a value for m_1 seems much harder.

We close this introduction by giving a consequence of Theorem 1′. To state it note that the set $A_{g,n}(\mathbb{C})$ of $\mathbb{C}$-points of $A_{g,n}$ has a natural equivalence relation on it given by isogeny: two points in $A_{g,n}(\mathbb{C})$ will be called isogenous if the corresponding abelian $\mathbb{C}$-varieties are isogenous. Each isogeny class in $A_{g,n}(\mathbb{C})$ is dense in the complex topology because it contains the image of a $\mathrm{Sp}(2g, \mathbb{Q})$-orbit on the Siegel upper half space. For any $y \in A_{g,n}(\mathbb{C})$ we denote by $I_y \subset A_{g,n}(\mathbb{C})$ the isogeny class of y. Then Theorem 1′ will imply the following:

Theorem 2 *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and two positive integers m_1, m_2 such that for all $Y \in C_0$ and for any point $y \in Y(\mathbb{C})$ outside a certain countable subset of $Y(\mathbb{C})$, the set $Y(\mathbb{C}) \cap I_y$ is finite of cardinality at most $m_1 \deg(Y) + m_2 p(Y)$.*

Remark. As the proof will show, the countable subset of $Y(\mathbb{C})$ appearing in the above statement can be taken simply to be the set of all points in $Y(\mathbb{C})$

whose coordinates lie in the algebraic closure of the smallest field of definition of the embedding $Y \subset A_{g,n}$.

The paper is organized as follows. In Section 1 we make some remarks on isogeny correspondences and we deduce Theorem 2 from Theorem 1'. In Sections 2 – 4 we introduce and review a series of concepts from [B$_1$, B$_2$, B$_3$] and provide complements to that material; a rough sketch of the strategy of the proof of Theorem 1' is given at the end of Section 2. The main body of the proof of Theorem 1' is contained in Sections 5 – 7.

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1. Some easy remarks on isogeny correspondences

Let $k \subset F$ be an extension of algebraically closed fields of characteristic zero, $k \neq F$; in applications we shall be interested in both situations when $F = \mathbb{C}$ and $k = \mathbb{C}$.

Let A_k denote the moduli k-scheme of principally polarized abelian varieties over k of dimension $g \geq 2$ with level n structure, $n \geq 3$. For any curve $Y_k \subset A_k$ (i.e. irreducible closed k-subvariety of A_k of dimension 1) we may introduce exactly as in Section 0 the notion of isogeny correspondence $Z_k \subset Y_k \times Y_k$ on Y_k and we may define $i(Y_k)$ similarly. On the other hand, we may consider on the set $A_k(F)$ of F-points of A_k the equivalence relation given by isogeny: two points in $A_k(F)$ are called isogenous if the corresponding abelian F-varieties are isogenous (over F). For $y \in A_k(F)$ we denote by $I_{y,F} \subset A_k(F)$ the isogeny class of y.

Lemma 1. *Let $Y_k \subset A_k$ be a curve and $y \in Y_k(F) \setminus Y_k(k)$. Then we have:*

$$i(Y_k) = \mathrm{card}(Y_k(F) \cap I_{y,F})$$

Proof: Let $L = k(Y_k)$ be the field of rational functions on Y_k and let $\varepsilon_0 \colon L \to F$ be the k-embedding corresponding to y. Since $Y_k(F) \cap I_{y,F} \subset Y_k(F) \setminus Y_k(k)$ each point in $Y_k(F) \cap I_{y,F}$ identifies with a k-embedding $\varepsilon \colon L \to F$; note that the compositum of the fields $\varepsilon_0 L$ and εL in F is algebraic over both $\varepsilon_0 L$ and εL (because the abelian F-variety corresponding to ε, being isogenous to the one corresponding to ε_0, must be defined over an algebraic extension of $\varepsilon_0 L$). Therefore the ideal $\ker(\varepsilon_0 \otimes \varepsilon \colon L \otimes_k L \to F)$ in $L \otimes_k L$ is non-zero so it corresponds to a curve $Z_k(\varepsilon) \subset Y_k \times Y_k$ which clearly is an isogeny correspondence. We have constructed a map $\varepsilon \longmapsto Z_k(\varepsilon)$ from the set $Y_k(F) \cap I_{y,F}$ to the set of all isogeny correspondences on Y_k which is clearly surjective and whose fiber at an isogeny correspondence $Z_k \subset Y_k \times Y_k$ has precisely $[Z_k : Y_k]_1$ elements. This closes the proof of the Lemma.

Lemma 2. *Let $Y_k \subset A_k$ be a curve, fix a k-embedding $\varepsilon \colon L = k(Y_k) \to F$, let X be the abelian F-variety deduced via ε and for any $\sigma \in \mathrm{Aut}\,(F/k)$ denote* *by X^σ the abelian F-variety deduced via σ from X. Consider the groups*

$$G(X) = Aut(F/\varepsilon L)$$
$$G'(X) = \{\sigma \in Aut(F/k);\ X^\sigma \text{ is isogenous to } X\}$$

Then $i(Y_k)$ equals the index $[G'(X) : G(X)]$.

Proof: Let $y \in Y_k(F) \setminus Y_k(k)$ be defined by ε. Then clearly $Y_k(F) \cap I_{y,F}$ identifies with the coset set $G'(X)/G(X)$ and conclude by Lemma 1.

Let's show how Theorem 1′ from Section 0 implies Theorem 2. Denote $A_{g,n}$ simply by A and assume C_0, m_1, m_2 are as in Theorem 1′. For any $Y \in C_0$ let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition of the embedding $Y \subset A$ and let $Y_k \subset A_k$ be the embedding of k-varieties giving rise to $Y \subset A$; then $Y_k(k)$ is a countable subset of $Y(\mathbb{C})$. Let $y \in Y(\mathbb{C}) \setminus Y_k(k)$; by Lemma 1 (applied to $F = \mathbb{C}$) we have

$$\mathrm{card}\,(Y(\mathbb{C}) \cap I_y) = i(Y_k) \le i(Y) \le m_1 \deg(Y) + m_2 p(Y)$$

which proves Theorem 2.

We close this section by proving the following assertion (which was made in Section 0):

Proposition. *The union in $A_{g,n}(\mathbb{C})$ of all curves carrying infinitely many isogeny correspondences is dense in the complex topology of $A_{g,n}(\mathbb{C})$.*

Proof: Step 1. Note that there exists at least one curve $Y \subset A = A_{g,n}$ carrying infinitely many isogeny correspondences. Indeed, let $E \to S = \mathbb{A}^1_{\mathbb{C}} \setminus \{0, 1\}$ be the Weierstrass elliptic family, let $X = E \times_S \ldots \times_S E$ (g times) be viewed as a principally polarized abelian scheme over S and make a base change

$S' \to S$, S' some irreducible curve over $\mathbb{C}$, such that $X' := X \times_S S' \to S'$ has a level n structure. Then the closure Y in A of the image of the naturally induced map $S' \to A$ has $i(Y) = \infty$ (e.g. use Lemma 1).

Step 2. Consider any curve $Y \subset A$ with $i(Y) = \infty$ (which exists by Step 1). Then, starting from Y, we shall produce a family of curves Y^z with $i(Y^z) = \infty$ and whose union is dense in the complex topology of $A(\mathbb{C})$. Indeed, let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition for the embedding $Y \subset A$ and let $Y_k \subset A_k$ be the embedding of k-varieties from which $Y \subset A$ is deduced; upon enlarging k we may assume $i(Y_k) = \infty$. Take any point $y \in Y(\mathbb{C}) \setminus Y_k(k)$ and consider the isogeny class $I_y = I_{y,\mathbb{C}}$ of y in $A(\mathbb{C})$. For any point $z \in I_y$ let Y_k^z denote the Zariski closure in A_k of the image of the morphism $\operatorname{Spec} \mathbb{C} \to A_k$ defined by z and let $Y^z \subset A$ be the curve over $\mathbb{C}$ obtained from Y_k^z by base change $k \subset \mathbb{C}$; clearly $z \in Y^z(\mathbb{C})$. We claim that $i(Y_k^z) = \infty$. This will close the proof of the Proposition, for then $i(Y^z) = \infty$ and $I_y \subset \cup Y^z(\mathbb{C})$ the union being taken for all $z \in I_y$; but I_y is already dense in the complex topology of $A(\mathbb{C})$. To check the claim let X_y, X_z be the abelian $\mathbb{C}$-varieties corresponding to y, z; since they are isogenous, $G'(X_y) = G'(X_z)$ (notations as in Lemma 2). Now let L, L_z be the fields of rational funcitons on Y_k, Y_k^z and let $\varepsilon\colon L \to \mathbb{C}$, $\varepsilon_z\colon L_z \to \mathbb{C}$ be the k-embeddings defined by y and z, respectively. Since X_y, X_z are isogenous the compositum of the fields εL and $\varepsilon_z L_z$ in $\mathbb{C}$ is finite over both εL and $\varepsilon_z L_z$. In particular, one of the indices $[G'(X_y)\colon G(X_y)]$ and $[G'(X_z)\colon G(X_z)]$ is finite if and only if the other is so. Now our claim follows from Lemma 2 and our Proposition is proved.

2. Introducing the δ-field U

The most economic way of presenting the proof of Theorem $1'$ is to use the setting of δ-fields and the theory of Ritt-Kolchin which goes with them [K] (a δ-field is by definition a field F of characteristic zero with a fixed derivation on it always to be denoted by $\delta: F \to F$).

Instead of dealing with many δ-fields it is still better to deal with one universal δ-field in Kolchin's sense; for convenience we recall the definition of this concept. First there is an obvious notion of morphism of δ-fields and of δ-subfield (morphisms of δ-fields are by definition field homomorphisms which commute with the fixed derivations). If $F_1 \to F_2$ is a morphism of δ-fields we say that F_2 is δ-finitely generated over F_1 if there exist $x_1, \dots, x_n \in F_2$ such that F_2 is generated as a field by F_1 and the elements $\delta^i x_j$, $i \geq 0, 1 \leq j \leq n$. A δ-field U is called universal if for any δ-subfield F_1 of it which is δ-finitely generated over $\mathbb{Q}$ and for any morphism of δ-fields $F_1 \to F_2$ with F_2 δ-finitely generated over F_1 there is a morphism of δ-fields $F_2 \to U$ over F_1. By [K] p. 134 there exists a universal δ-field U whose constant field $\{x \in U; \delta x = 0\}$ has the same cardinality as $\mathbb{C}$ hence is isomorphic to $\mathbb{C}$. From now on we fix such an U, identify its constant field with $\mathbb{C}$ and write x', x'', x''', ... instead of δx, $\delta^2 x$, $\delta^3 x$, ... for $x \in U$. For any $\mathbb{C}$-variety A the set $A(U)$ of its U-points has a natural topology called the Kolchin topology defined as follows (cf. [B_1]): for any Zariski closed subset H of $\operatorname{jet}(A)$ let $A_H(U)$ denote the set of all points $\operatorname{Spec} U \to A$ in $A(U)$ whose unique horizontal lifting $\operatorname{Spec} U \to \operatorname{jet}(A)$ has the image contained in H. Then the sets $A_H(U)$ are by definition the closed sets of the Kolchin topology on $A(U)$ (note that this is what Kolchin calls in [K] the δ-$\mathbb{C}$-topology and is slightly different from the topology considered in [B_1] where we do not assume the open sets are "defined over constants"). By [K] p. 200 if A is irreducible in the Zariski topology then $A(U)$ is irreducible in the Kolchin topology. Moreover, by a theorem of Ritt [R] p. 10 the Kolchin topology on $A(U)$ is Noetherian. Note that if we are given a morphism of algebraic $\mathbb{C}$-varieties $A \to B$ then the induced map $A(U) \to B(U)$ is continuous in the Kolchin topology of $A(U)$ and $B(U)$. Moreover, any map $\mathbb{A}^m(U) = U^m \to \mathbb{A}^n(U) = U^n$ whose components are defined by δ-polynomials $f_1, \ldots, f_n \in \mathbb{C}\{y_1, \ldots, y_m\}$ is continuous in the Kolchin topology. Coming back to the set $C(A)$ of curves in A, it is easy to see that the Kolchin open sets of $C(A)$ defined in Section 0 are precisely the sets of the form $C_\Omega(A) = \{Y \in C(A); Y(U) \cap \Omega \neq \emptyset\}$ where Ω is Kolchin open in $A(U)$ (to check this just apply the "δ-Nullstellensatz" [K] p. 148). Noetherianity of Kolchin's topology on $A(U)$ already implies Noetherianity of the Kolchin topology on $C(A)$. Let's check that $C(A)$ is irreducible in the Kolchin topology. It is sufficient to check that $C_\Omega(A) \neq \emptyset$ whenever $\Omega \neq \emptyset$. We may assume A is affine. Then by Noether normalization we may easily assume A is the affine space $\mathbb{A}^n$. Now if $C_\Omega(A)$ is empty for some non-empty $\Omega \subset A(U) = U^n$ there exists a non-zero δ-polynomial $P \in \mathbb{C}\{y_1, \ldots, h_n\}$ such that for any choice of polynomials $f_1, \ldots, f_n \in \mathbb{C}[t]$ we have the equality $P(f_1(t), \ldots, f_n(t)) = 0$. By [K] p. 99 this implies $P = 0$, a contradiction.

Now Lemma 1 (applied to $k = \mathbb{C}$ and $F = U$) shows that Theorem $1'$ is implied by the following:

Theorem $1''$. *There exist a dense Kolchin open subset C_0 of $C(A_{g,n})$ and positive integers m_1, m_2 such that for any curve $Y \in C_0$ and for any isogeny* *class* $I \subset A_{g,n}(U) \setminus A_{g,n}(\mathbb{C})$ *we have:*

$$card(Y(U) \cap I) \leq m_1 \deg(Y) + m_2 p(Y)$$

From now on we concentrate ourselves on Theorem 1″. The very rough idea of its proof is the following. We will find Kolchin open sets Ω_1, Ω_2 of $A_{g,n}(U)$, positive integers m_1, m_2 and a map $b: \Omega_1 \to U$ which is "constant on isogeny classes" such that for any curve $Y \subset A_{g,n}$ with $Y(U) \cap \Omega_1 \cap \Omega_2 \neq \emptyset$ we have that $Y(U) \cap \Omega_1 = Y(U) \setminus Y(\mathbb{C})$ and the restriction of b to $Y(U) \setminus Y(\mathbb{C})$ is given by a rational function on Y of degree at most $m_1 \deg(Y) + m_2 p(Y)$. A moment's reflection shows that this implies Theorem 1″ (see the first lines of Section 6 for a few more details). The map b will be constructed in Section 6 as a (quite explicit) differential algebraic invariant of "δ-Hodge structures" of abelian U-varieties. The latter structures will be introduced in Section 3 and morally they are a differential algebraic (simplified) version of usual variations of Hodge structure. An argument different from ours for the existence of the map b was given by P. Deligne in a letter to the author [D].

3. Review of some δ-linear algebra

We shall "recall" and complete some discussion made in [B$_2$] on "δ-Hodge structures". Let $D = U[\delta] = \Sigma U \delta^i$ be the ring of linear differential operators on U generated by U and δ. By a δ-Hodge structure (of weight 1 and dimension g) we understand a pair (V, W) consisting of a D-module V of dimension $2g$ over U and of a U-linear subspace W of V of dimension g. We have an obvious notion of isomoprhism of δ-Hodge structures and we denote by H_g the set of isomorphism classes of such objects. We say that (V, W) has δ-rank g if the U-linear map $W \subset V \xrightarrow{\delta} V \to V/W$ is an isomorphism (where $V \xrightarrow{\delta} V$is the multiplication by δ in the D-module V) and we denote by $H_g^{(g)}$ the set of isomorphism classes of δ-Hodge structures of δ-rank g. There is a natural map $\Phi: H_g^{(g)} \to \mathbb{A}^g(U) = U^g$ defined as follows. For any δ-Hodge structure (V, W) of δ-rank g choose a U-basis $w_1, \dots, w_g$ of W; then $w_1, \dots, w_g$, $\delta w_1, \dots, \delta w_g$ will be a U-linear basis for V hence one can write

$$\delta^2 w + \alpha \delta w + \beta w = 0$$

where $\alpha, \beta \in \mathrm{gl}_g(U)$ are suitable $g \times g$ matrices and w is the transpose of $(w_1, \dots, w_g)$. Then we define Φ by attaching to (V, W) the characteristic polynomial

$$\det(xI_g - \gamma) = x^g + v_1 x^{g-1} + \cdots + v_g$$

of the matrix $\gamma = \beta - \alpha^2/4 - \alpha'/2$ (where we identify polynomials as above with vectors $(v_1, \dots, v_g) \in U^g$). It is trivial to check that changing the basis $w_1, \dots, w_g$ of W amounts to replacing the matrix γ by a matrix conjugated to it so Φ is well defined. Now let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathrm{gl}_{2g}(U), \qquad M_{ij} \in \mathrm{gl}_g(U)$$

By [K] pp. 420–421, there exists a matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathrm{GL}_{2g}(U)$$

such that $B' = MB$ and B is unique up to right multiplication by a matrix in $\mathrm{GL}_{2g}(\mathbb{C})$. It is trivial to check that $\deg M_{12} \neq 0$ if and only if

$$\det \begin{bmatrix} B_{11} & B_{12} \\ B'_{11} & B'_{12} \end{bmatrix} \neq 0$$

Let $\mathrm{gl}_{2g}(U)^{(g)}$ be the set of all matrices $M \in \mathrm{gl}_{2g}(U)$ with $\det M_{12} \neq 0$. There is a natural map $\Gamma: \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)}$ defined as follows. For any $M \in \mathrm{gl}_{2g}(U)^{(g)}$ let $V = U^{2g}$ (viewed as a D-module via the D-module structure of each factor) and let W be the U-linear subspace of U^{2g} spanned by the rows of the $g \times 2g$ matrix (B_{11}, B_{12}) where $B \in \mathrm{GL}_{2g}(U)$ is such that $B' = MB$. By the above discussion on determinants (V, W) has δ-rank g. By uniqueness of B up to $\mathrm{GL}_{2g}(\mathbb{C})$-action the isomoprhism class of the δ-Hodge structure (V, W) depends only on M and not on the choice of B; so we got a well defined map Γ as desired. It is an easy exercise of linear algebra to compute explicitly the composed map $\Phi \circ \Gamma: \mathrm{gl}_{2g}(U)^{(g)} \to \mathbb{A}^g(U) = U^g$; the result is

$$\begin{aligned}
\Phi(\Gamma(M)) &= \det(xI_g - \gamma) \quad \text{where} \\
\gamma &= \beta - \alpha^2/4 - \alpha'/2 \\
\alpha &= -M'_{12}M_{12}^{-1} - M_{11} - M_{12}M_{22}M_{12}^{-1} \\
\beta &= -M'_{11} + M'_{12}M_{12}^{-1}M_{11} + M_{12}M_{22}M_{12}^{-1}M_{11} - M_{12}M_{21}
\end{aligned}$$

So we see that $\Phi \circ \Gamma$ is defined by g rational fractions whose denominators are powers of $\det M_{12}$ and whose numerators are δ-polynomials with coefficients in $\mathbb{Q}$ in $4g^2$ variables; in particular $\Phi \circ \Gamma$ is continuous in the Kolchin topology. Intuitively it should be viewed as a (highly) non-linear differential operator of order 2.

4. Review of internal versus external Gauss-Manin connection

In this section we review some material from [B_2] and [B_3], chapter 5; we refer to loc. cit. for details of proof.

Let Y be a smooth $\mathbb{C}$-variety and $TY = \mathrm{Spec}\,(\Omega_Y)$ its tangent bundle;

then there is a natural map $\nabla: Y(U) \to (TY)(U)$ continuous in the Kolchin topology defined as follows.

One first defines it for $Y = \mathbb{A}^N$; here we identify

$$\mathbb{A}^N(U) = U^N, \ (T\mathbb{A}^N)(U) = U^{2N}$$

and put $\nabla(u_1, \ldots, u_N) = (u_1, \ldots, u_N, u_1', \ldots, u_N')$. Next if $Y \subset A^N$ is a closed subvariety and we embed TY naturally into $T\mathbb{A}^N$ then we define $\nabla: Y(U) \to (TY)(U)$ to be the restriction of the analogous map for $\mathbb{A}^N$. Finally if Y is arbitrary we define ∇ be gluing the ∇'s of its affine pieces. Note that if $y \in Y(U)$ then ∇y can be veiwed as vector in the Zariski tangent space $T_y Y_U$ of $Y_U := Y \otimes_{\mathbb{C}} U$ at y. Of course, there is an intrinsic definition of ∇ but we won't need it here.

Now let $X \to Y$ be a smooth projective morphism of smooth $\mathbb{C}$-varieties, let $y \in Y(U)$ be a point and X_y the fiber at y. Then we dispose of an "internal" Kodaira-Spencer map

$$\rho_y^{\mathrm{int}}: \mathrm{Der}_{\mathbb{C}} U \to H^1(X_y, T)$$

associated to the morphism $X_y \to \operatorname{Spec} U$ (here T denotes the tangent sheaf) and also of an "external" Kodaira-Spencer map

$$\rho_y^{\mathrm{ext}}: T_y Y_U \to H^1(X_y, T)$$

associated to the morphism $X_U \to Y_U$. One can easily prove the following formula:

$$(*) \qquad \rho_y^{\mathrm{int}}(\delta) = \rho_y^{\mathrm{ext}}(\nabla y)$$

Similarly we dispose of an "internal" Gauss-Manin connection

$$\nabla_y^{\mathrm{int}}: \mathrm{Der}_{\mathbb{C}} U \to \mathrm{End}_{\mathbb{C}}(H^1_{\mathrm{DR}}(X_y))$$

and (assuming for simplicity that Y is affine) of an "external" Gauss-Manin connection

$$\nabla^{\mathrm{ext}}: \mathrm{Der}_U \mathcal{O}(Y_U) \to \mathrm{End}_U(H^1_{\mathrm{DR}}(X_U/Y_U))$$

See [Ka] for background on Gauss-Manin connection. On the other hand the trivial lifting of δ from U to X_U, Y_U induces an endomorphism $\delta^* \in \mathrm{End}_{\mathbb{C}}(H^1_{\mathrm{DR}}(X_U/Y_U))$. For any $\mathcal{O}(Y_U)$-module E and any $\varphi \in E$ let's agree to denote by $\varphi(y)$ the image of φ in E/m_yE where m_y is the maximal ideal of the local ring of Y_U at y. For instance, if $\varphi \in H^0(Y_U, T)$ is a vector field then $\varphi(y) \in T_yY_U$ is the corresponding tangent vector; if $\varphi \in H^0(Y_U, \Omega)$ is a global 1-form then $\varphi(y)$ is an element in the dual of T_yY_U while if $\varphi \in H^1_{\mathrm{DR}}(X_U/Y_U)$ is a relative de Rham class then $\varphi(y) \in H^1_{\mathrm{DR}}(X_y)$ is the corresponding de Rham class on the fiber. With this convention let $\omega \in H^1_{\mathrm{DR}}(X_U/Y_U)$ and let $\theta_y \in H^0(Y_U, T)$ be such that $\theta_y(y) = \nabla y$. Then one can prove the following formula ([B3] Chapter 5):

$$(**) \qquad \nabla_y^{\mathrm{int}}(\delta)(\omega(y)) = (\delta^*\omega + \nabla^{\mathrm{ext}}(\theta_y)\omega)(y)$$

From now on let X/Y be an abelian scheme of relative dimension $g \geq 1$. The space $H^1(X_y, T)$ naturally identifies with $\mathrm{Hom}_U(H^0(X_y, \Omega), H^1(X_y, \mathcal{O}))$ so for each element of this space we may speak about its determinant which will be a U-linear map between the g-th exterior powers of $H^0(X_y, \Omega)$ and $H^1(X_y, \mathcal{O})$. Then formula $(*)$ easily implies that the set $Y^{(g)}(U)$ of all $y \in Y(U)$ such that $\det \rho_y^{\mathrm{int}}(\delta) \neq 0$ is a Kolchin open subset of $Y(U)$ (which may of course be empty but which is certainly non-empty when $Y = A_{g,n}$ and $X \to Y$ is the universal abelian scheme because in this case any g-fold product of an elliptic curve with j-invariant belonging to $U \setminus \mathbb{C}$ provides an example of y for which $\det \rho_y^{\mathrm{int}}(\delta) \neq 0$).

A key role in what follows will be played by the map $h: Y(U) \to H_g$ defined to attaching to each $y \in Y(U)$ the δ-Hodge structure represented by (V, W) where $V = H^1_{\mathrm{DR}}(X_y)$ (viewed as a D-module via ∇_y^{int}) and $W = H^0(X_y, \Omega)$. Using the relation between "Kodaira-Spencer" and "Gauss-Manin" as explained in [Ka] we see that $h^{-1}(H_g^{(g)}) = Y^{(g)}(U)$ so if the latter is non-empty we dispose of an induced map

$$h: Y^{(g)}(U) \to H_g^{(g)}$$

The map h has the remarkable (easily checked) property that if $y, z \in Y(U)$ are such that X_y and X_z are isogenous then $h(y) = h(z)$. In particular $Y^{(g)}(U)$ is saturated with respect to the isogeny equivalence relation.

Assume in addition that $H^1_{\mathrm{DR}}(X/Y)$ and $H^0(X, \Omega_{X/Y})$ are free $\mathcal{O}(Y)$-modules (this is anyway the case if we replace Y by the Zariski open sets of a covering of it which will be allowed later). Then take an $\mathcal{O}(Y)$-module basis ω of the first module having the form $\omega_1, \dots, \omega_g$, $\omega_{g+1}, \dots, \omega_{2g}$ where the first g elements form a basis of the second module. For any vector field $\tau \in \mathrm{Der}_U \mathcal{O}(Y_U)$ on Y_U we may write $\nabla^{\mathrm{ext}}(\tau)\omega = \langle N, \tau \rangle \omega$ where N is a $2g \times 2g$ matrix of 1-forms on Y. The latter defines a morphism of $\mathbb{C}$-varieties $TY \to \mathrm{gl}_{2g}(\mathbb{C})$ which at the level of U-points gives the map still denoted by

$$N: (TY)(U) \to \mathrm{gl}_{2g}(U)$$

sending each tangent vector $t \in T_y Y_U$ into the matrix $\langle N(y), t \rangle$. Denote by $(TY)^{(g)}(U)$ the preimage via N of $\mathrm{gl}_{2g}(U)^{(g)}$; it is a Zariski open subset of $(TY)(U)$. Then one checks using $(**)$ that the map $h: Y^{(g)}(U) \to H_g^{(g)}$ coincides with the composition:

$$\Gamma \circ N \circ \nabla: Y^{(g)}(U) \to (TY)^{(g)}(U) \to \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)}$$

In particular the composition

$$\chi = \Phi \circ h: Y^{(g)}(U) \to H_g^{(g)} \to \mathbb{A}^g(U) = U^g$$

is continuous in the Kolchin topology. Intuitively χ should be viewed as a "third order non-linear differential operator".

5. The case dim Y=1

Assume Y is an irreducible (possible singular) curve over $\mathbb{C}$; we will systematically apply the preparation made in Section 4 to the smooth locus of Y. Choose a non-zero $\mathbb{C}$-derivation τ of the function field $\mathbb{C}(Y)$; then τ induces a U-derivation of the function field $U(Y_U)$ of Y_U so we may (and will) also view τ as a rational vector field on the smooth locus of Y_U. Of course τ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$ and all singularities of the curve Y_U lie in $Y(\mathbb{C})$.

For any $y \in Y(U) \setminus Y(\mathbb{C})$ we denote by y' the unique element in $U^* = U \setminus \{0\}$ such that $\nabla y = y'\tau(y)$. Moreover we simply denote by $y'', y''', \ldots$ the usual derivatives of y' as an element of U. Any element φ in the function field $\mathbb{C}(Y)$ will be systematically viewed as a rational map $\varphi: Y(U) \dashrightarrow \mathbb{A}^1(U) = U$; if $\varphi \neq 0$ then obviously φ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$. For any $\varphi \in \mathbb{C}(Y)$ and any $y \in Y(U) \setminus Y(\mathbb{C})$ we have the following (easily checked) formula: $\varphi(y)' = y'(\tau\varphi)(y)$.

Assume we are given an abelian scheme X/Y of relative dimension $g \geq 1$. Formula $(*)$ from Section 4 (applied to the smooth locus of Y) says that for any $y \in Y(U) \setminus Y(\mathbb{C})$ we have

$$\rho_y^{\mathrm{int}}(\delta) = \rho_y^{\mathrm{ext}}(y'\tau(y)) = y'\rho^{\mathrm{ext}}(\tau)(y)$$

where $\rho^{\mathrm{ext}}: \mathrm{Der}_U \mathcal{O}(Y) \to H^1(X, T_{X/Y})$ is the natural "external" Kodaira-Spencer map of X/Y, $T_{X/Y}$ = relative tangent sheaf of X/Y. Exactly as in Section 4 we may consider the determinant $\det \rho^{\mathrm{ext}}(\tau)$ which identifies with an element in $\mathbb{C}(Y)$. (Recall we assumed Y small enough, so that the various vector bundles appearing in section 4 are trivial.) If we assume this rational function is non-zero then it has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$; so we get by the above equalities that if $Y^{(g)}(U)$ is non-empty then $Y^{(g)}(U) = Y(U) \setminus Y(\mathbb{C})$. Assume from now on that $Y^{(g)}(U) \neq \emptyset$ (equivalently that $\det \rho^{\mathrm{ext}}(\tau) \neq 0$ in $\mathbb{C}(Y)$). There is a finite set of closed points $S \subset Y$ containing all singular points of Y, all zeroes and poles of τ and of $\det \rho^{\mathrm{ext}}(\tau)$, such that if $Y_1 = Y \setminus S$, X_1 = inverse image of Y_1, we have a basis of the free $\mathcal{O}(Y_1)$-module $H^1_{\mathrm{DR}}(X_1/Y_1)$ of the form

$$\omega_1, \ldots, \omega_g, \nabla^{\mathrm{ext}}(\tau)\omega_1, \ldots, \nabla^{\mathrm{ext}}(\tau)\omega_g$$

where $\omega_1, \ldots, \omega_g$ is a basis of the free $\mathcal{O}(Y_1)$-module $H^0(X_1, \Omega_{X/Y})$. Note that $Y_1(U) \setminus Y_1(\mathbb{C}) = Y(U) \setminus Y(\mathbb{C})$. Then the map $N: (TY)(U) \to \mathrm{gl}_{2g}(U)$ from Section 4 has the form:

$$u\tau(y) \longmapsto \begin{bmatrix} 0 & uI_g \\ uN_{21}(y) & uN_{22}(y) \end{bmatrix} \qquad \text{for all } u \in U,\ y \in Y_1(U)$$

where N_{21}, N_{22} are $g \times g$ matrices with entries in $\mathcal{O}(Y_1)$. Now we compute the image of any $y \in Y(U) \setminus Y(\mathbb{C})$ via the map

$$\chi = \Phi \circ \Gamma \circ N \circ \nabla: Y^{(g)}(U) \to (TY)^{(g)}(U) \to \mathrm{gl}_{2g}(U)^{(g)} \to H_g^{(g)} \to U^g$$

Using the formula of $\Phi \circ \Gamma$ from Section 3 plus the above formula for N, we get that $\chi(y)$ equals the characteristic polynomial of $\gamma = \beta - \alpha^2/4 - \alpha'/2$ where

$$\alpha = -y''(y')^{-1}I_g - y'N_{22}(y)$$
$$\beta = -(y')^2 N_{21}(y)$$

Substituting the expressions of α and β in that of γ, we get

$$\gamma = \sigma(y)I_g + (y')^2 R(y)$$

where $\sigma(y) = y'''/2y' - (3/4)(y''/y')^2$ is the "Schwartzian" of y and R is some $g \times g$ matrix with entries in $\mathcal{O}(Y_1)$. In case $g = 1$ we simply get:

$$\chi(y) = \sigma(y) + (y')^2 R(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

for some regular function $R \in \mathcal{O}(Y_1)$. So if, moreover, $Y = \mathbb{A}^1 \setminus \{0, 1\}$ and X is the Weierstrass elliptic curve over Y, then by universality of U the map $\chi: U \setminus \mathbb{C} \to U$ is surjective. Coming back to arbitrary $g \geq 1$ and taking products of g elliptic curves with various j-invariants in $U \setminus \mathbb{C}$, one sees that the map $\chi: A_{g,n}^{(g)}(U) \to U^g$ is surjective too.

6. The basic "fifth order map"

The main idea in what follows is to construct non-empty Kolchin open sets Ω_D and Ω_P of $\Omega := A_{g,n}^{(g)}(U)$ and a map $b: \Omega_D \to \mathbb{A}^1(U) = U$ with the following properties: 1) for any points $y, z \in \Omega_D$ belonging to the same isogeny class in $A_{g,n}(U)$ we have $b(y) = b(z)$ and 2) for any curve $Y \subset A_{g,n}$ with $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$ we have $Y(U) \cap \Omega_D = Y(U) \setminus Y(\mathbb{C})$ and there exists a rational function $s \in \mathbb{C}(Y) \setminus \mathbb{C}$ such that the restrictions of b and s to $Y(U) \setminus Y(\mathbb{C})$ coincide (this makes sense because s viewed as a rational map $Y(U) \dashrightarrow \mathbb{A}^1(U) = U$ has all its poles contained in $Y(\mathbb{C})$). Then we will prove that there exist an effective divisor $W \subset A$ and two positive integers m_1, m_2 such that upon letting $\Omega_W = A_{g,n}(U) \setminus W(U)$ we have that $\deg s \leq m_1 \deg(Y) + m_2 p(Y)$ whenever $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$. This construction will end the proof of Theorem 1$''$. Indeed, let C_0 be the set of all $Y \in C(A_{g,n})$ such that $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$; if $I \subset A_{g,n}(U) \setminus A_{g,n}(\mathbb{C})$ is any isogeny class then for $Y \in C_0$ we have $I \cap Y(U) = I \cap (Y(U) \setminus Y(\mathbb{C}))$ hence $I \cap Y(U)$ will be contained in a fiber of the restriction of b to $Y(U) \setminus Y(\mathbb{C})$ hence in a fiber of the restriction of s to $Y(U) \setminus Y(\mathbb{C})$, consequently

$$\operatorname{card}(I \cap Y(U)) \leq \deg s \leq m_1 \deg(Y) + m_2 p(Y)$$

and Theorem 1$''$ will be proved. In this section we construct our "basic map" b; intuitively b will appear as a "non-linear differential operator of order 5". In the next section we will construct W and estimate $\deg s$.

Start by considering the maps $D, T, E \colon \mathbb{A}^g(U) \to \mathbb{A}^1(U)$ defined as follows: for $v = (v_1 \ldots, v_g) \in U^g$,

$$\begin{aligned}
D(v) &= D(x^g + v_1 x^{g-1} + \cdots + v_g) = \operatorname{disc}(x^g + v_1 x^{g-1} + \cdots + v_g) \\
T(v) &= -v_1 \\
E(v) &= 4g(g-1)D(v)(D(v)'' - 4(g-1)T(v)D(v)) - (2g-1)^2 (D(v)')^2
\end{aligned}$$

where "disc" means "discriminant". Note that D and T are regular maps of algebraic varieties while E is not; yet E is continuous in the Kolchin topology (intuitively E is a non-linear differential operator of order 2). Consider the

Kolchin open set $\Omega_D = \Omega \setminus \chi^{-1}(D^{-1}(0))$ of $A_{g,n}(U)$ (which is non-empty because the map $\chi: \Omega \to U^g$ appearing at the end of Section 4 is surjective) and also consider the Zariski open set $\mathbb{A}^g_D(U) = \mathbb{A}^g(U) \setminus D^{-1}(0)$. Then we may consider the maps $S: \mathbb{A}^g_D(U) \to \mathbb{A}^1(U)$, $P: \mathbb{A}^g(U) \to \mathbb{A}^1(U)$ defined by:

$$S(v) = E(v)^{g(g-1)}/D(v)^{2g(g-1)+1},$$
$$P(v) = (E(v)^{g(g-1)})' D(v)^{2g(g-1)+1} - E(v)^{g(g-1)}(D(v)^{2g(g-1)+1})'$$

The maps S and P are continuous in the Kolchin topology so we may consider the Kolchin open set $\Omega_P = \Omega \setminus \chi^{-1}(P^{-1}(0))$ which is non-empty also by surjectivity of $\chi: \Omega \to U^g$. Now we define our "basic map" to be the composition

$$b = S \circ \chi: \Omega_D \to \mathbb{A}^g_D(U) \to \mathbb{A}^1(U)$$

Clearly b maps each pair of points belonging to the same isogeny class into the same point (because χ has this property). Now let $Y \subset A_{g,n}$ be a curve with $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$; in particular, by Section 5, $Y^{(g)}(U) = Y(U) \setminus Y(\mathbb{C})$. Also by Section 5 we have

$$\chi(y) = \det(xI_g - \sigma(y)I_g - (y')^2 R(y)), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where R is some $g \times g$ matrix with entries in the function field $\mathbb{C}(Y)$. In particular we get:

$$(*) \quad T(\chi(y)) = \operatorname{tr}(\sigma(y)I_g + (y')^2 R(y)) = g\sigma(y) + (y')^2 t(y), \quad y \in Y(U) \setminus Y(\mathbb{C})$$

where $t = \operatorname{tr} R \in \mathbb{C}(Y)$ and "tr" denotes of course the trace of a matrix. Using the behavior of the discriminant of a polynomial with respect to linear changes of variable, we get

(**)
$$D(\chi(y)) = D(\det((x-\sigma(y))I_g - (y')^2R(y))) = D(\det(xI_g - (y')^2R(y))) =$$
$$= (y')^{2g(g-1)}d(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $d = D(\det(xI_g - R)) \in \mathbb{C}(Y)$. Since $Y(U) \cap \Omega_D \neq \emptyset$ there exists $y_0 \in Y(U) \setminus Y(\mathbb{C})$ such that $D(\chi(y_0)) \neq 0$; by (**) $d \neq 0$ in $\mathbb{C}(Y)$. Since d viewed as rational function on $Y(U)$ has neither zeroes nor poles in $Y(U) \setminus Y(\mathbb{C})$, it follows from (**) that $D(\chi(y)) \neq 0$ for all $y \in Y(U) \setminus Y(\mathbb{C})$ hence $Y(U) \cap \Omega_D = Y(U) \setminus Y(\mathbb{C})$. Now a tedious but straightforward computation with formulae (*) and (**) yields:

$$(***) \qquad E(\chi(y)) = (y')^{4g(g-1)+2}e(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $e = 4g(g-1)d(\tau^2 d - 4(g-1)td) - (2g-1)^2(\tau d)^2 \in \mathbb{C}(Y)$. From (**) and (***) we get:

$$(****) \qquad b(y) = S(\chi(y)) = s(y), \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

where $s = e^{g(g-1)}/d^{2g(g-1)+1} \in \mathbb{C}(Y)$. We claim that $s \notin \mathbb{C}$, equivalently, $\tau s \neq 0$. But, indeed, deriving (****), we get

$$y'(\tau s)(y) = P(\chi(y))/D(\chi(y))^{4g(g-1)+2}, \qquad y \in Y(U) \setminus Y(\mathbb{C})$$

Since $Y(U) \cap \Omega_D \cap \Omega_P \neq \emptyset$, there exists $y_1 \in Y(U) \setminus Y(\mathbb{C})$ such that $P(\chi(y_1)) \neq 0$ and $D(\chi(y_1)) \neq 0$ so we must have $(\tau s)(y_1) \neq 0$ hence $\tau s \neq 0$ as an element of $\mathbb{C}(Y)$ and our claim is proved.

To conclude the proof of Theorem 1″ we need to bound $\deg s$ in terms of $\deg(Y)$ and $p(Y)$ which will be done in the next section.

Remark. The main point in the last step of the proof above was the "miracle" that, for $y \in Y(U) \setminus Y(\mathbb{C})$, both $D(\chi(y))$ and $E(\chi(y))$ were expressible in the form of a product of some power of y' with a sutiable rational function in $\mathbb{C}(Y)$. In case of $D(\chi(y))$, this "miracle" is the reflection of remarkable properties of the discriminant. In a similar way the "miracle" for $E(\chi(y))$ is the reflection of remarkable properties of what may be called the "differential resultant" of two δ-polynomials; this interpretation is of course irrelevant for the proof (but it was quite relevant for the way we were led to the somewhat tricky definition of E). Deligne's arguments in [D] avoid this "miraculous" point in our proof.

7. Bounding deg s

First let's recall various trivial facts related to degrees on curves. Let L be a function field of one variable over $\mathbb{C}$ of genus p. Then for any $f \in L$ we define $\deg f = \deg(f)_\infty$ (where $(f)_\infty$ is the negative part of the principal divisor associated to f of the smooth projective model of L) if $f \neq 0$ and $\deg f = 0$ if $f = 0$. This is of course nothing but the "usual height" of the point $(1 : f)$ on the projective line. Similarly if $\omega \in \Omega^1_{L/\mathbb{C}}$ is a 1-form we let $\deg \omega = \deg(\omega)_\infty$ if $\omega \neq 0$ and $\deg \omega = 0$ if $\omega = 0$. Finally, if $\tau \in \mathrm{Der}_\mathbb{C} L$, $\tau \neq 0$ we write $\deg \tau = \deg(\tau)_\infty$. Here $(\omega)_\infty$, $(\tau)_\infty$ have the obvious meaning analogue to $(f)_\infty$. It is trivial to check that:

(i) $\deg\langle \tau, \omega\rangle \leq \deg \tau + \deg \omega$

$\deg f\omega \leq \deg f + \deg \omega$

$\deg \tau f \leq \deg \tau + 2 \deg f$

$\deg(f_1 + f_2) \leq \deg f_1 + \deg f_2$

$$\deg f_1 f_2 \leq \deg f_1 + \deg f_2$$

Note that by Riemann-Roch there always exists $\tau \in \mathrm{Der}_{\mathbb{C}} L$, $\tau \neq 0$ with $\deg \tau \leq 2p$. For any matrix $M = (f_{ij}), f_{ij} \in L$ it will be convenient to denote by deg M the maximum of the numbers $\deg f_{ij}$.

Now let us come back to the set $C(A)$ of curves in $A = A_{g,n}$ and recall that we have fixed a projective compactification $\bar{A}$ of A and a very ample line bundle $\mathcal{O}(1)$ on $\bar{A}$. For any two functions $\varphi, \psi: C(A) \to \mathbb{N}$ and any subset C' of $C(A)$ we write $\varphi(Y) \ll \psi(Y)$, $Y \in C'$ if and only if there exists a constant $m > 0$ such that $\varphi(Y) \leq m\psi(Y)$ for all $Y \in C'$.

After these notational preparations we may proceed to proving the existence of the desired bound for deg s.

Let X/A be the universal abelian scheme over A, let R be the field of rational functions on A and let $X_R = X \times_A \mathrm{Spec}\, R$. Pick any R-basis $\omega_1, \ldots, \omega_{2g}$ of $H^1_{\mathrm{DR}}(X_R/R)$ such that $\omega_1, \ldots, \omega_g$ is an R-basis of $H^0(X_R, \Omega)$ and write

$$\nabla \omega_i = \sum \omega_{ij} \otimes \omega_j, \ \omega_{ij} \in \Omega^1_{R/\mathbb{C}}, \ 1 \leq i, j \leq 2g$$

where ∇ is the Gauss-Manin connection of X_R/R. There clearly exists an integer $N \geq 1$ and a divisor $\bar{W}$ on $\bar{A}$ whose associated line bundle is $\mathcal{O}(N)$ such that, upon letting V be $\bar{A} \setminus \bar{W}$, the following hold:

1) $V \subset A$,

2) $\omega_1, \ldots, \omega_{2g}$ is a basis of the $\mathcal{O}(V)$-module $H^1_{\mathrm{DR}}(X/A)_V$,

3) $\omega_1, \ldots, \omega_g$ is a basis of the $\mathcal{O}(V)$-module of relative regular 1-differentials of X/A over V,

4) ω_{ij} are regular on V,

5) $(\omega_{ij})_\infty \leq \bar{W}$ for all i, j (where $(\omega_{ij})_\infty$ is the divisor of poles of ω_{ij} on $\bar{A}$).

Put $W = \bar{W} \cap A$ and let's check that with this W we have

$$\deg s \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

which will close the proof of Theorem $1''$. Recall from Section 6 that we defined C_0 to be the set of all $Y \in C(A_{g,n})$ such that $Y(U) \cap \Omega_D \cap \Omega_P \cap \Omega_W \neq \emptyset$; moreover we defined s by the formula $s = e^{g(g-1)}/d^{2g(g-1)+1} \in \mathbb{C}(Y)$. Now if $Y \in C_0$ and if $\bar{\omega}_{ij}$ are the restrictions of ω_{ij} to Y, then obviously we have

$$\deg \bar{\omega}_{ij} \ll \deg(Y), \qquad Y \in C_0$$

Using formulae (i) we easily check that the matrices N_{21}, N_{22} in Section 5 may be chosen to satisfy

$$\deg N_{21}, \deg N_{22} \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

Clearly the matrix R in Section 5 can be expressed explicitly in terms of N_{21}, N_{22}; using (i) again we get

$$\deg R \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

Finally, if t, d, e, s are as in Section 6, we deduce step by step (using (i)) that

$$\deg t, \deg d, \deg e, \deg s \ll \deg(Y) + p(Y), \qquad Y \in C_0$$

and Theorem $1''$ is proved.

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