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REGULAR LINEAR SYSTEMS ON CP^1 AND THEIR MONODROMY GROUPS

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1. INTRODUCTION

1.1

A meromorphic linear system of differential equations on $\mathbf{C}P^1$ can be presented in the form

$$\dot{X} = A(t)X \tag{1}$$

where $A(t)$ is a meromorphic on $\mathbf{C}P^1$ $n \times n$ matrix function, " $\cdot$ " $\equiv d/dt$. Denote its poles $a_1, \ldots, a_{p+1}$, $p \geq 1$. We consider the dependent variable X to be also $n \times n$-matrix.

Definition. System (1) is called *fuchsian* if all the poles of the matrix-function $A(t)$ are of first order.

Definition. System (1) is called *regular* at the pole a_j if in its neighbourhood the solutions of the system are of moderate growth rate, i.e.

$$||X(t - a_j)|| = 0\left(|t - a_j|^{N_j}\right), \quad N_j \in \mathbf{R}, \; j = 1, \ldots, p+1$$

Here $|| \cdot ||$ denotes an arbitrary norm in $gl(n, \mathbf{C})$ and we consider a restriction of the solution to a sector with vertex at a_j and of a sufficiently small radius, i.e. not containing other poles of $A(t)$. Every fuchsian system is regular, see [1]. The restriction to a sector is essential, if we approach the pole along a spiral encircling it sufficiently fast, then we can obtain an exponential growth rate for $||X||$.

Two systems (1) with the same set of poles are called *equivalent* if there exists a meromorphic transformation (equivalency) on $\mathbf{C}P^1$

$$X \mapsto W(t)X \tag{2}$$

with $W \in \mathcal{O}(CP^1\backslash\{a_1,...,a_{p+1}\})$ and $\det W(t) \neq 0$ for $t \in CP^1\backslash\{a_1,...,a_{p+1}\}$ which brings the first system to the second one. A transformation (2) changes system (1) according to the rule

$$A(t) \rightarrow -W^{-1}(t)\dot{W}(t) + W^{-1}(t)A(t)W(t) \tag{3}$$

1.2

The monodromy group of system (1) is defined as follows: fix a point $a \neq a_j$ for $j = 1,...,p+1$, fix a matrix $B \in GL(n, C)$ and fix p closed contours on CP^1 beginning at the point a each of which contains exactly one of the poles a_j of system (1), see Fig. 1. The *monodromy operator* corresponding to such a contour is the linear operator mapping the matrix B onto the value of the analytic continuation of the solution of system (1) which equals B for $t = a$ along the contour encircling a_j; we assume that all the contours are positively orientated. Monodromy operators act on the right, i.e. we have $B \mapsto BM_j$. The monodromy operators $M_1,...,M_p$ corresponding to $a_1,...,a_p$ generate the *monodromy group* of system (1) which is a presentation of the fundamental group $\pi_1(CP^1\backslash(a_1,\ldots,a_{p+1})$ into GL(n, C); we have

$$M_{p+1} = (M_1....M_p)^{-1} \tag{4}$$

for a suitable ordering of the points a_j and the contours, see Fig. 1.

It is clear that

1. the monodromy group is defined up to conjugacy due to the freedom in choosing the point a and the matrix B.
2. the monodromy groups of equivalent systems are the same.

The monodromy group of a regular system is its only invariant under meromorphic equivalence.

Capital Latin letters (in most cases) denote matrices or their blocks; by I we denote diag$(1,\ldots,1)$.

1.3

It is natural to consider $GL(n, C)^p$ as the space of monodromy groups of regular systems on CP^1 with $p+1$ prescribed poles (because the operators $M_1,\ldots,M_p$ define the monodromy group of system (1)). Condition (4) allows one to consider M_{p+1} as an analytic matrix-function defined on $GL(n, C)^p$. Of course, in a certain sense, $M_1,\ldots,M_{p+1}$ are 'equal', i.e. anyone of them can play the role of

M_{p+1}. We define *an analytic stratification* of $(GL(n,\mathbf{C}))^p$ by the Jordan normal forms of the operators $M_1,\ldots,M_{p+1}$ and the possible reducibility of the group $\{M_1,\ldots,M_p\}$. Fixing the Jordan normal form of $M_1,\ldots,M_p$ is equivalent to restricting the matrix-function $M_{p+1} = (M_1\ldots M_p)^{-1}$ to a smooth analytic subvariety of $GL(n,\mathbf{C})^p$, but if we want to fix the one of M_{p+1} as well, then we a priori can say nothing about the smoothness of the subset of $GL(n,\mathbf{C})^p$ (called *superstratum*) obtained in this way. The basic aim of this paper is to begin the study of the stratification of $GL(n,\mathbf{C})^p$ and the smoothness of the strata and superstrata.

Throughout the paper 'to fix the Jordan normal form' means 'to define the multiplicities of the eigenvalues and the sizes and numbers of Jordan blocks corresponding to each of them', but *not* to fix the eigenvalues as well; this is called 'to fix the orbit'.

2 The stratification of the space of monodromy groups

Definition. Let the group $\{M_1,\ldots,M_p\}\subset GL(n,\mathbf{C})$ be conjugate to one in block-diagonal form, the diagonal blocks (called *big blocks*) being themselves block upper-triangular; their block structure is defined by their diagonal blocks (called *small blocks*). The restriction of the group to everyone of the small blocks is assumed to be an irreducible matrix group of the corresponding size. The sizes of the big and small blocks are correctly defined modulo permutation of the big blocks (if we require that the sizes of the big blocks are the minimal possible) and define the *reducibility type* of the group.

Example : The reducibility type $\begin{pmatrix} A & B & 0\\ 0 & C & 0\\ 0 & 0 & Q\end{pmatrix}$ has two big ($\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}$ and Q) and three small blocks (A, C and Q).

Definition. A *stratum* of $GL(n,\mathbf{C})$ is its subset of matrices with one and the same Jordan normal form. A group $\{M_1,\ldots,M_p\}\subset GL(n,\mathbf{C})$ defines a *stratum* of $GL(n,\mathbf{C})^p$: the stratum is defined by

1) the reducibility type of the group;

2) the Jordan normal forms of the small and big blocks of the matrices $M_1,\ldots,M_{p+1}$ and the ones of the matrices M_j themselves;

3) two groups whose matrices $M_1,\ldots,M_p$ are blocked as their reducibility type belong to the same stratum if and only if the corresponding M_j are conjugate to each other by matrices (in general, different for the different j) blocked as the reducibility type.

A stratum is called *irreducible* if its reducibility type is one big and at the same time small block.

A reducible stratum is called *special* if there exists a pair of small blocks of the same size, belonging to one and the same big block, such that the restrictions of the matrices M_j to them have the same Jordan normal form for all $j = 1, \ldots, p+1$.

Remark: Suppose that the definition of a stratum doesn't contain 3). Then some of the reducible strata defined in this way will turn out to be reducible analytic varieties (see the example below; note the double sense of 'reducible'). The good definition of a stratum is obtained when the strata defined above are decomposed into irreducible components if this is possible. After such a decomposition we obtain again a finite number of strata.

Example: Let the reducibility type be $\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, P, Q and R being 3×3.

Let $M_2, \ldots, M_{p+1}$ have distinct eigenvalues. Let $M_1 = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & a \\ 0 & 0 & \lambda & b & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$.

For $a = 0, b \neq 0$ and for $a \neq 0, b = 0$ the Jordan normal forms of the $P-$ and $Q-$block of M_1 and of M_1 itself are the same (M_1 has one eigenvalue - λ - and three Jordan blocks, of sizes 3, 2 and 1 respectively). In the first case the dimension of the intersection of the subspace invariant for M_1 upon which M_1 acts as one Jordan block of size 3 with the subspace invariant for all operators M_j is equal to 1, in the second case it is equal to 2. It can be checked directly that the two matrices (corresponding to $(a, b) = (*, 0)$ and $(a, b) = (0, *), * \neq 0$) aren't conjugate to each other by a matrix blocked in the same way.

Remark: The following example shows that the definition of a stratum of $GL(n, \mathbf{C})^p$ is still not good - there exist several connected components for irreducible strata in which every operator M_j, $j = 1, \ldots, p$ has one eigenvalue only. On the other hand-side, let there exist M_j with at least two different eigenvalues. Consider two systems belonging to the same stratum. One can deform continuously the sets of their eigenvalues, i.e. perform a homotopy from the first into the second set, keeping their product equal to 1 and their multiplicities unchanged, i.e. different (equal) eigenvalues remain such for every value of the homotopy parameter. Whether for any such homotopy there exists a homotopy of the monodromy group, irreducible for every value of the homotopy parameter - this is an open question.

Example: The monodromy groups of the following three systems are irreducible. Every monodromy operator M_1, M_2, M_3 is conjugate to one 3×3-Jordan block. The eigenvalues of M_1 and M_2 are equal to 1, the ones of M_3 in the first case are equal to 1, in the second case – to $e^{4\pi i/3}$, in the third case – to $e^{2\pi i/3}$. By t_j, $j = 1,2,3$ we denote $1/(t-a_j)$.

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} t_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} t_3 \right] X$$

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -26/27 & 0 \\ 0 & 0 & 0 \\ 1 & -1/3 & 1 \end{pmatrix} t_2 + \begin{pmatrix} 0 & -1/27 & 0 \\ 0 & 0 & -1 \\ -1 & 1/3 & -1 \end{pmatrix} t_3 \right] X$$

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -19/27 & 0 \\ 0 & 0 & 0 \\ 1 & -4/3 & 2 \end{pmatrix} t_2 + \begin{pmatrix} 0 & -8/27 & 0 \\ 0 & 0 & -1 \\ -1 & 4/3 & -2 \end{pmatrix} t_3 \right] X$$

Definition. Consider a subset Σ of $GL(n, \mathbf{C})$ consisting of matrices blocked as a given reducibility type. A *stratification* of this set is defined by

1) the Jordan normal forms of the small blocks, taking into account whether two small blocks have common eigenvalues or not

2) two matrices with the same reducibility type and orbits of the small blocks belong to the same stratum if and only if they can be conjugated with one another by a matrix blocked as the reducibility type.

Lemma 2.1. *Any stratum from this stratification is a connected smooth algebraic variety.*

The lemma is proved at the end of Section 3.

Definition. A *superstratum* of $GL(n, \mathbf{C})^p$ is defined by the Jordan normal forms of the matrices $M_1, \ldots, M_{p+1}$. Hence, every superstratum consists of a finite number of strata.

Evidently, every stratum and superstratum is locally an analytic subvariety of $(GL(n, \mathbf{C}))^p$.

Theorem 2.2.

1) All irreducible strata are locally smooth analytic subvarieties of $GL(n, \mathbf{C})^p$.

2) All strata and superstrata in which at least one of the matrices M_j, $j = 1, \ldots, p+1$ has n different eigenvalues are globally smooth irreducible semi-analytic subvarieties of $GL(n, \mathbf{C})^p$ ('semi-analytic' means 'defined by a finite number of equalities and by inequalities of the kind $P \neq 0$').

3) A reducible group $\{M_1, \ldots, M_p\}$ (in block upper-triangular form, same as the reducibility type) is a singular point of its superstratum only in case that *if* $Q_{11j}, \ldots, Q_{ssj}$ *are the restrictions of* M_j *to the small blocks of the reducibility type, then there exist linear relations of the kind*

$$\alpha_1 \text{var tr} Q_{11j} + \ldots + \alpha_s \text{var tr} Q_{ssj} = 0 \quad , \quad j = 1, \ldots, p+1 \quad , \quad \alpha_k \in \mathbf{Z}$$

Here 'var tr' denotes the possible variation of the trace when every eigenvalue varies independently (equal eigenvalues have equal variations) and the stratum to which $\{M_1, \ldots, M_p\}$ *belongs is fixed. Hence, the singular points of a superstratum are contained in one or more of its reducible strata – last equalities mean that for a fixed reducibility type the multiplicities of the eigenvalues of the small blocks* $Q_{kkj}, j = 1, \ldots p+1$ *for which* $\alpha_k \neq 0$ *remain the same.*

4) Every reducible stratum is locally a smooth analytic variety.

5) An upper-triangular group with M_j *having one eigenvalue only ,* $j = 1, \ldots, p+1$*, is a singular point of its superstratum.*

3 Proof of Theorem 2.2.

0^0. We prove 1) in $1^0 - 3^0$, 2) and 3) in $4^0 - 7^0$, 4) in $8^0 - 13^0$ and 5) in 14^0. The proofs of the lemmas involved are given after the proof of the theorem.

1^0. Prove 1) (see $1^0 - 3^0$). Fix the Jordan normal forms of $M_1, \ldots, M_p$. This defines a smooth subvariety $\mathcal{T}$ of $(GL(n, \mathbf{C}))^p$. If we fix the Jordan normal form of M_{p+1} (or, equivalently, of M_{p+1}^{-1}), then this defines a smooth analytic subvariety $\mathcal{S}$ of $GL(n, \mathbf{C})$. Let the group $\mathbf{M} = \{M_1, \ldots, M_p\}$ be irreducible. We prove that the differential of the mapping

$$(M_1, \ldots, M_p) \mapsto M_{p+1} = (M_1 \ldots M_p)^{-1}$$

is non-degenerate at $\mathbf{M}$; in fact, we prove (what is equivalent) that the differential of the mapping

$$(M_1, \ldots, M_p) \mapsto M_{p+1}^{-1} = M_1 \ldots M_p \qquad (*)$$

is such, see $2^0 - 3^0$. Hence, the graph of the mapping (*) is a smooth analytic subvariety of $\mathcal{T} \times GL(n, \mathbf{C})$, transversal at $\mathbf{M}$ to the smooth analytic subvariety $\mathcal{U} = \mathcal{T} \times \mathcal{S}$; therefore their intersection is locally a smooth analytic subvariety.

2^0. The differential of (*) is the sum of two terms – the first (denoted by Φ_p) is obtained when M_j are conjugated by matrices of the kind $G_j = I + \varepsilon Y_j$, i.e. we move infinitesimally along the orbit without changing the eigenvalues. Hence, Φ_p is the coefficient before ε in the product $G_1^{-1} M_1 G_1 \ldots G_p^{-1} M_p G_p$. Note that for small values of ε the group $\{G_1^{-1} M_1 G_1, \ldots, G_p^{-1} M_p G_p\}$ is irreducible. The second term (denoted by Δ_p) is obtained when we change infinitesimally the eigenvalues (every eigenvalue changes independently, for every $M_j, j = 1, \dots, p$).

3^0. **Lemma 3.1.**

$$\Phi_p(\mathrm{M};\mathrm{Y}) \equiv \Phi_p(M_1,\dots,M_p;Y_1,\dots,Y_p) =$$

$$= [M_1,Y_1]M_2\dots M_p + M_1[M_2,Y_2]M_3\dots M_p + \dots + M_1\dots M_{p-1}[M_p,Y_p] =$$
$$= M_1\dots M_p\Psi_p(N_1,\dots,N_p;Z_1,\dots,Z_p)$$

where $\Psi_p = [N_1,Z_1] + \dots + [N_p,Z_p]$, $Z_j = S_j^{-1}M_j^{-1}Y_jS_j$, $N_j = S_j^{-1}M_jS_j$, $S_j = M_{j+1}\dots M_p$, $j = 1,\dots,p-1$, $S_p = I$. *Hence, the groups* $\{M_1,\dots,M_p\}$ *and* $\{N_1,\dots,N_p\}$ *coincide.*

The lemma is checked directly.

Lemma 3.2. *Let* $M_j = Q_j^{-1}J_jQ_j$ *where* $Q_j \in GL(n,\mathrm{C})$ *and* J_j *is the Jordan normal form of* M_j. *Then*

$$\Delta_p(\mathrm{M},\mathrm{V}) = V_1M_2\dots M_p + M_1V_2M_3\dots M_p + \dots + M_1M_2\dots M_{p-1}V_p$$

where $V_j = Q_j^{-1}D_jQ_j$, D_j *being a diagonal matrix whose diagonal entries are the variations of the eigenvalues of* M_j, *i.e. of the diagonal entries of* J_j *(equal eigenvalues have equal variations). We have*

$$\Delta_p = M_1\dots M_p\kappa_p \quad , \quad \kappa_p = \sum_{j=1}^{p} S_j^{-1}M_j^{-1}V_jS_j \quad , \quad S_j = M_{j+1}\dots M_p \quad , \quad S_p = I$$

The lemma is checked directly.

Lemma 3.3. *Let the group* $\{M_1,\dots,M_p\}$ *be irreducible. Then for every matrix* $L \in gl(n,\mathrm{C})$, $tr(M_{p+1}L) = 0$ *there exist matrices* $Y_1,\dots,Y_p$ *such that* $L = \Phi_p(\mathrm{M};\mathrm{Y})$, *see Lemma 3.1.*

Lemma 3.4. *For every* $d \in \mathrm{C}$ *there exist matrices* $V_1,\dots,V_p$, *see Lemma 3.2., such that* $tr(M_{p+1}\Delta_p(\mathrm{M};\mathrm{V})) = d$.

The first statement of the theorem follows from Lemmas 3.3. and 3.4. Really, for $L \in gl(n,\mathrm{C})$ choose $V_1,\dots,V_p$ such that $\mathrm{tr}(M_{p+1}\Delta_p(\mathrm{M};\mathrm{V})) = \mathrm{tr}(M_{p+1}L)$. Hence, $\mathrm{tr}(M_{p+1}^{-1}(L - \Delta_p)) = 0$ and we can choose $Y_1,\dots,Y_p$ such that $L - \Delta_p = \Phi_p(\mathrm{M};\mathrm{Y})$.

4^0. Prove 2) and 3). To this end we use a similar idea to the one of the proof of 1). We show for what groups M the tangent spaces to the graph of (*) (denoted by $T(*)$) and to the variety $\mathcal{U}$, see 1^0, are transversal. The space $T\mathcal{U}$ contains the tangent space to $\mathcal{T}$, therefore it suffices to find the cases when the sum of the projections of the spaces $T(*)$ and $T\mathcal{U}$ into $GL(n,\mathbf{C})$, the space of M_{p+1}^{-1}, is the whole space $gl(n,\mathbf{C})$. Similarly to Lemmas 3.1. and 3.2. we find that the projection of $T\mathcal{U}$ into $GL(n,\mathbf{C})$ is equal to

$$\Lambda_{p+1}(M_{p+1},Y_{p+1},V_{p+1})=[M_{p+1}^{-1},Y_{p+1}]+V_{p+1}$$

where $Y_{p+1}\in gl(n,\mathbf{C})$ and if $M_{p+1}^{-1}=Q_{p+1}^{-1}J_{p+1}Q_{p+1}$, J_{p+1} being the Jordan normal form of M_{p+1}^{-1}, $Q_{p+1}\in GL(n,\mathbf{C})$, then $V_{p+1}=Q_{p+1}^{-1}D_{p+1}Q_{p+1}$, D_{p+1} being diagonal, whose diagonal entries are the variations of the eigenvalues of M_{p+1}^{-1}. Hence, the two varieties $\mathcal{U}$ and the graph of $(*)$ are transversal if and only if every matrix $L\in gl(n,\mathbf{C})$ can be presented as

$$L=\mathcal{L}_p(\mathbf{M};\mathbf{Y},\mathbf{V},Y_{p+1},V_{p+1})=$$

$$\Phi_p(\mathbf{M};\mathbf{Y})+\Delta_p(\mathbf{M};\mathbf{V})+\Lambda_{p+1}(M_{p+1},Y_{p+1},V_{p+1}) \qquad (**)$$

Present (**) with $Y_{p+1}=V_{p+1}=0$, i.e. with $\Lambda_{p+1}=0$, in the form

$$L'\equiv(M_1\dots M_p)^{-1}L=\mathcal{L}'_p(\mathbf{M};\mathbf{Y};\mathbf{V})\equiv\Psi_p(\mathbf{N};\mathbf{Z})+\kappa_p(\mathbf{M};\mathbf{V}) \qquad (***)$$

From now on we most often consider equation (***) instead of equation (**) (if we can solve (***), then we can solve (**)).

5^0. Let the reducible group $\mathbf{M}$ be in block upper-triangular form (same as the reducibility type). Decompose any matrix $A\in \mathrm{gl}(n,\mathbf{C})$ in blocks, the decomposition being induced by the sizes of the small blocks of the reducibility type. Then for the following ordering of the blocks operator $\mathcal{L}'_p$, see (***), is block upper-triangular: if the blocks are denoted by Q_{ks}, k (s) being the number of row (of column) of blocks, then $Q_{k_1s_1}$ precedes $Q_{k_2s_2}$ if and only if $k_1-s_1<k_2-s_2$ or $k_1-s_1=k_2-s_2$ and $k_1<k_2$. Hence, it suffices to consider the action of $\mathcal{L}'_p$ upon matrices $Y_j,V_j,j=1,\dots,p+1$ whose elements outside a fixed block are equal to 0.

Lemma 3.5. *Denote by Q_{ijk} the restriction of M_k for $k=1,\dots,p$ or M_{p+1}^{-1} for $k=p+1$ to the block Q_{ij}. Then equation*

$$\sum_{k=1}^{p+1}Q_{iik}Z_k-Z_kQ_{jjk}=A$$

has a solution for any matrix $A\in gl(n,\mathbf{C})$ if and only if the two following conditions don't hold simultaneously:

1) Q_{ii} and Q_{jj} are of the same size (denoted by l);

2) there exists a matrix $B\in GL(l,\mathbf{C})$ such that $B^{-1}Q_{iik}B=Q_{jjk}$ for $k=1,\dots,p$ (hence, for $k=p+1$ as well).

Note that for $i > j$ the left hand-side of the equation gives the restriction of the image of operator $\mathcal{L}'_p$ to the block Q_{ij} (κ_p has no influence upon blocks under the diagonal). Conditions 1) and 2) together are a private case of the non-smoothnes condition from 3) of the theorem.

The block upper-triangular form of operator $\mathcal{L}'_p$ implies that one could try to solve equation (***) successively for each block, in the opposite order of the blocks. If we fail at one of them, then, probably, we can't solve equation (**) ('probably' means that solving equation (***) is not equivalent to solving equation (**)). If conditions 1) and 2) from the lemma hold simultaneously, then it doesn't follow from Lemma 3.5. that we can solve (***). If they don't, but the non-smoothness condition from 3) of the theorem holds, then we can solve equation (***) restricted to all blocks Q_{ij} under the diagonal, i.e. with $i > j$. For $i = j$ operator Ψ_p can give a solution only for matrices L' with $\mathrm{tr}L'|_{Q_{ii}} = 0$. Hence, operator κ_p must be used to make the trace of $L'|_{Q_{ii}}$ equal to 0 and he'll fail to do it for all small blocks simultaneously exactly if the variations of the traces of the small blocks are linearly dependent.

7^0. We proved in 6^0 that for strata verifying the non-smoothness condition from 3) of the theorem equation (**) possibly can't be solved and, hence, the variety $\mathcal{U}$, see 1^0, might not be transversal to the graph of (*). We prove now that non-transversality would imply local non-smoothness of their intersection. Introduce in $GL(n, \mathbf{C})$ local coordinates $q = (q_1, \ldots, q_{n^2})$ such that at the intersection point of $\mathcal{U}$ and the graph of (*) the projection of $\mathcal{U}$ into $\mathrm{GL}(n, \mathbf{C})$, i.e. $\mathcal{S}$, should be given by equations $q_1 = \ldots = q_s = 0$, $s = \dim\mathcal{S}$. Hence, the points of non-transversal intersection of the graph of (*) and $\mathcal{U}$ are the ones where the differential of (*) degenerates and we have $q_1 = \ldots = q_s = 0$. These are the singular points of the intersection of the image of (*) with $\{q_1 = \ldots = q_s = 0\}$, i.e. with $\mathcal{U}$.

If a superstratum of $(\mathrm{GL}(n, \mathbf{C}))^p$ contains among the Jordan normal forms of M_j, $j = 1, \ldots, p+1$ one with distinct eigenvalues, then their variations are independent and we never have conditions 1) and 2) of Lemma 3.5. fulfilled together. On the contrary, if there is at least a pair of equal eigenvalues in every M_j, $j = 1, \ldots, p+1$, then it is always possible to find a reducibility type such that conditions 1) and 2) of Lemma 3.5. will be fulfilled together.

Irreducible strata and superstrata in which at least one of the operators M_j is with distinct eigenvalues are connected. Really, let this be M_{p+1}. The image of (*) is a semi-algebraic subset of $GL(n, \mathbf{C})$ for every fixed set of Jordan normal forms of $M_1, \ldots, M_p$. The differential of (*) is non-degenerate, hence, the image is an open subset of $GL(n, \mathbf{C})$. Hence, if the Jordan normal forms of $M_1, \ldots, M_p$ are fixed, then the points of the graph of (*) for which M_{p+1} is not with distinct eigenvalues is locally a proper subvariety of the graph. This completes the proof of 2) and 3) of the theorem.

8^0. **Lemma 3.6.** *Let the group* M *be reducible. Then in its neighbourhood* $U \in (GL(n, \mathbf{C}))^p$ *there exists a holomorphic and holomorphically invertible matrix* C *such that it conjugates every group of the intersection of* U *with the stratum to which* M *belongs to one blocked as the reducibility type.*

The proof of 4) is similar to the one of 1). We consider the mapping (*) defined for $M_1, \ldots, M_p$ blocked as the reducibility type, making use of the lemma. Denote the set of these matrices by Σ. Consider the following subset of $(\Sigma)^p$: the multiplicities of the eigenvalues of every operator $M_1, \ldots, M_p$ and their distribution among the small blocks are fixed and the Jordan normal forms of the small blocks as well. Denote this set by $\mathcal{T}'$ and consider (*) as a mapping $(*){:}\mathcal{T}' \mapsto \Sigma$. For $M_{p+1}^{-1} \in \Sigma$ fix the multiplicities of its eigenvalues, their distribution among the small blocks and the Jordan normal forms of the small blocks. This defines a subset $\mathcal{S}' \subset \Sigma$.

Lemma 3.7. *$\mathcal{T}'$ and $\mathcal{S}'$ are smooth analytic subvarieties.*

The intersection of the graph of (*) with $\mathcal{S}' \times \mathcal{T}'$ (denote it by $\mathcal{R}$) consists of a finite number of strata of $(GL(n, \mathbf{C}))^p$ (not necessarily of a single one). Really, though the eigenvalues of M_j and the Jordan normal forms of their small blocks are fixed, the Jordan structures of M_j depend on the elements in the blocks above the diagonal as well.

Example : Consider the matrix $\begin{pmatrix} \lambda & 1 & a & b \\ 0 & \lambda & c & d \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$. For $c \neq 0$ it is conjugate to one 4×4 Jordan block. For $c = 0$, $a + d \neq 0$ it is conjugate to a matrix with one 3×3- and one 1×1-block, for $c = a + d = 0$ it is cojugate to a matrix with two 2×2-blocks.

9^0. In the case of non-special strata we have

Lemma 3.8. *Every reducible non-special stratum consists of a finite number of smooth analytic varieties.*

10^0. We prove 4) of the theorem for reducibility types with one big block only; for such with several big blocks the proof is similar. The proof is carried out by induction with respect of the number k of small blocks. Let $k = 2$. Set $M_j = \begin{pmatrix} P_j & Q_j \\ 0 & R_j \end{pmatrix}$. Let the stratum be special, i.e. the Jordan normal forms of P_j and R_j be the same for $j = 1, \ldots, p+1$ (for non-special strata the answer is given by Lemma 3.8.). By Lemma 3.5., equation (***) in which all matrices are block upper triangular (as M_j) can't be solved only if the sizes of P_j and R_j are equal and we have $B^{-1}P_jB = R_j$, $j = 1, \ldots, p+1$ for some matrix B.

Really, we first solve equation (***) for the diagonal blocks as in the irreducible case and then for the block B.

Let $P_j = R_j$, $j = 1, \ldots, p+1$. Denote by $\mathcal{C}$ the space of matrices blocked as $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Then we have

Lemma 3.9. *The image of operator $\mathcal{L}'_p$ restricted to $\mathcal{C}$ is either $\mathcal{C}$ or $\mathcal{C} \cap \{N \in \mathcal{C} | \mathrm{tr}(N|_B) = 0\}$. The second case occurs only if $Q_j = [D, P_j]$ for some matrix D, $j = 1, \ldots, p+1$.*

Lemma 3.10. *Let $M_j = \begin{pmatrix} P_j & Q_j \\ 0 & P_j \end{pmatrix}$ be conjugate to $\begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. Then $Q_j = [D_j, P_j]$ for some square matrices D_j. The opposite implication is also true.*

Hence, for $k = 2$ the only case in which equation (***) can't be solved is the one when M_j can be simultaneously conjugated to the form $\begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. But in this case the reducibility type has two big blocks.

11^0. Let the reducibility type contain $k \geq 3$ small blocks. Let in the block decomposition induced by the sizes of the small blocks $C \cup D$ be the set of blocks in the first row, $B \cup D$ – the one of blocks in the last column and A – the set of all other blocks on and above the diagonal, see Fig. 2. Item 3) from the definition of the stratification of $(GL(n, \mathbf{C}))^p$ implies that once the stratification of the restriction of the group $\mathbf{M}$ to A is defined, its definition for $\mathbf{M}|_{A \cup B}$, $\mathbf{M}|_{A \cup C}$, $\mathbf{M}|_{A \cup B \cup C \cup D}$ does not change the one of $\mathbf{M}|_A$.

Fix a stratum of $\mathbf{M}|_{A \cup B}$ and a stratum of $\mathbf{M}|_{A \cup C}$ such that their restrictions to A coincide. The stratification of $\mathbf{M}|_{A \cup B \cup C \cup D}$ imposes (for every stratum) analytic conditions on the block D. The restrictions $M_j|_D$, $j = 1, \ldots, p$, consist of a finite number of smooth analytic varieties for every stratum of Σ, see Lemma 2.1.. In (*) $M_{p+1}|_D$ is presented as an analytic function of $M_1, \ldots, M_p$ restricted to the corresponding strata. To prove the local smoothness of $\mathbf{M}|_D$ it suffices to prove that the graph of $M_{p+1}|_D$ is transversal to the level sets $M_{p+1}|_D$ where M_{p+1} is restricted to some stratum of Σ, as in the proof of 1).

12^0. Introduce the following notation for the blocks of M_j:

$$\begin{pmatrix} P & Q_1 & Q_2 & \ldots & Q_{k-2} & D \\ 0 & T & U_2 & \ldots & U_{k-2} & S_1 \\ 0 & 0 & V & \ldots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & W & S_{k-2} \\ 0 & 0 & 0 & \ldots & 0 & R \end{pmatrix}$$

The tangent space to the graph of $M_{p+1}|_D$ contains the space

$$\mathcal{Y} = \{Y|Y = (M_1 \dots M_p)|_P \sum_{j=1}^{p}((N_j|_P)X_j^0 - X_j^0(N_j|_R)\}$$

X_j^0 is of the size of D; see the definition of $\mathcal{L}_p$. Really, to see this it suffices to restrict in Ψ_p and $\mathcal{L}_p$ the matrices Z_j to D to obtain X_j^0. The space $\mathcal{Y}$ is not the whole space $\mathcal{D}$ (the space of matrices of the size of D) if and only if P and R are of the same size and we have $N_j|_P = B^{-1}(N_j|_R)B$, $j = 1, \dots, p$, see Lemma 3.5.. This implies that $M_j|_P = B^{-1}(M_j|_R)B$. Without loss of generality assume that $M_j|_P = M_j|_R$, $j = 1 \dots, p+1$.

13^0. The tangent space to the graph of $(*)|_D$ contains as well the space

$$\mathcal{Y}^1 = \{Y|Y = (M_1 \dots M_p)|_P \sum_{j=1}^{p}((N_j|_P)X_j^0 - X_j^0(M_j|_R))+$$

$$(M_1 \dots M_p)|_{Q_1} \sum_{j=1}^{p}(N_j|_{Q_1})X_j^1\}, \mathcal{Y} \subset \mathcal{Y}^1$$

where Y, (X_j^0, X_j^1) is of the size of D, (of D, of S_1) and X_j^1 belong to the subspace

$$\mathcal{Y}' = \{\sum_{j=1}^{p}((Mj|_T)X_j^1 - X_j^1(M_j|_R)) = 0\}$$

As $M_j|_P = M_j|_R$ and $\mathrm{M}|_P(\equiv \mathrm{M}|_R)$ is irreducible, then we have either $\mathcal{Y}^1 = \mathcal{D}$ or $\mathcal{Y}^1 = \mathcal{D} \cap \{\mathrm{tr}Y = 0\}$ ($\mathcal{D}$ is the space of matrices of the size of D); the second case occurs only if

$$\mathrm{tr}\sum_{j=1}^{p}(M_j|_{Q_1})X_j^1 = 0 \quad \forall (X_1^1, \dots, X_p^1) \in \mathcal{Y}' \qquad (****)$$

(the proof of this fact is similar to the one of Lemma 3.9.). Then condition (****) must be a corollary of equation

$$\sum_{j=1}^{p}((M_j|_T)X_j^1 - X_j^1(M_j|_R)) = 0$$

As in the proof of Proposition 3.17., we prove that the group M is conjugate to one blocked as M, with $M_j|_{Q_1} = 0$, $j = 1, \dots, p$. In this case note that the stratifications of the sets of monodromy groups blocked as

$$\begin{pmatrix} P & 0 & Q_2 & \dots & Q_{k-2} & D \\ 0 & T & U_2 & \dots & U_{k-2} & S_1 \\ 0 & 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & 0 & \dots & 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T & 0 & U_2 & \dots & U_{k-2} & S_1 \\ 0 & P & Q_2 & \dots & Q_{k-2} & D \\ 0 & 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & 0 & \dots & 0 & R \end{pmatrix}$$

are isomorphic, the isomorphism being generated by the conjugation with the permutation matrix which permutes the lines of blocks in which P and T are. Hence, the smoothness of the stratification of the block D for fixed strata of the other blocks follows from the inductive assumption applied to the set of monodromy groups blocked as

$$\begin{pmatrix} P & Q_2 & \dots & Q_{k-2} & D \\ 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & \dots & 0 & W \end{pmatrix}$$

This proves 4) of the theorem.

14^0. Prove 5). If in (**) all matrices except L are upper-triangular, the diagonal entries of each M_j being equal, then the image of $\mathcal{L}_p$ belongs to the subspace of matrices whose left lowest element is equal to 0. As in 7^0, this implies non-smoothness of the corresponding superstratum at $\mathbf{M}$.

Proof of Lemma 3.3.: 1^0. Making use of Lemma 3.1., we prove that every matrix $L \in gl(n, \mathrm{C})$, $\mathrm{tr}L = 0$ can be presented as $L = \Psi_p(\mathrm{M};\mathrm{X})$ (it would be, of course, more precise to write $\Psi_p(\mathrm{N},\mathrm{Z})$; we hope that the reader will not ge mixed up).

Proposition 3.11. *Let J be a Jordan matrix. Then any matrix $A \in gl(n, \mathrm{C})$ can be presented in a unique way as $A = [J,X]+Y$, where $[Y\ ,\ {}^tJ] = 0$; ${}^tJ$ denotes the transposed of J.*

Proof: Let the matrix J have one eigenvalue only. Then the matrices commuting with ${}^tJ$ are shown on Fig. 3. The numbers on one and the same interval are the same, all other numbers are equal to zero, see [2]. The intervals are parallel to the diagonal and they begin and end at the bords of the blocks. The block decomposition is in accordance with the Jordan structure. The number of intervals of a diagonal (of an off-diagonal) block is equal to the size (to the least of the sizes) of the block.

If on Fig. 3. we assume that the sum of all the elements lying on one and the same interval is equal to 0 (for every interval) and that no other conditions are imposed on the matrix, then we obtain the definition of an arbitrary matrix presentable as $[J, X]$ for some $X \in gl(n, \mathrm{C})$ (the reader will check this easily, because one needs to consider the action of the operator $[J, .]$ on every block on Fig. 3. separately).

If J has several eigenvalues, then one must take a direct sum of figures like Fig. 3. corresponding to the different eigenvalues. The proposition is proved.

Corollary 3.12. *Let $M_j = Q_j^{-1} J_j Q_j$ where J_j is the Jordan normal form of M_j. Then for a fixed Q_j (Q_j is not unique) any matrix $A \in gl(n, \mathrm{C})$ can be presented as $A = [M_j, X] + Y$, $[Y, Q_j^{-1}({}^tJ_j)Q_j] = 0$.*

2^0. **Proposition 3.13.** **(Schur's Lemma)** *If there exists a non-scalar matrix $S \in gl(n, \mathrm{C})$ such that $[S, M_j] = 0$, $j = 1, \dots, p$, then the group $\{M_1, \dots, M_p\}$ is reducible.*

Proof: Without loss of generality one can assume that S is in Jordan normal form. If it has at least two different eigenvalues, then M_j must be all block diagonal and the proposition is proved. If not, then Fig. 4. (it is the transposed of Fig. 3.) shows one possible way how to choose the invariant subspace in the case when there are Jordan blocks of different sizes; for the case of Jordan blocks of the same size see Fig. 5. The vectors to the right describe the invariant subspaces. Asteriscs denote elements which can be arbitrary.

3^0. **Proposition 3.14.** *For every group $\mathrm{M} = \{M_1, \dots, M_p\}$ there exists a group $\mathrm{K} = \{K_1, \dots, K_p\}$ such that*

i) M and K are simultaneously (ir)reducible;

ii) the images of the mappings $\Psi_p : (X_1, \dots, X_p) \mapsto [N_1, X_1] + \dots + [N_p, X_p]$ for $N_j = M_j$ and for $N_j = K_j$ are the same;

iii) the matrices K_j have one and the same Jordan normal form.

For the rest of the proof of the lemma we consider the Jordan normal forms of $M_1, \dots, M_p$ to be the same, making use of the proposition.

Proof: We have $[M_1, X_1] + [M_2, X_2] = [M_1 + \alpha M_2, X_1] + [M_2, X_2 - \alpha X_1]$. Hence, the change $(M_1, M_2) \mapsto (M_1 + \alpha M_2, M_2)$ preserves the image of Ψ_p and the (ir)reductibility of the group. If $M_1 + \alpha M_2$ fails to be non-degenerate, then we replace it by $M_1 + \alpha M_2 + \beta I$ for a suitable β; the image and the (ir)reductibility are preserved again. If M_1 and M_2 have different Jordan normal forms, then either

i) one of them (say, M_1) belongs to a stratum S_1 of $GL(n, \mathrm{C})$ from the closure of the stratum S_2 to which belongs the other or

ii) this is not the case.

In case *i*) either

i1) the whole line $M_1 + \alpha M_2$, $\alpha \in \mathbf{C}$ belongs to the closure of S_2 or

i2) for almost all α the matrix $M_1 + \alpha M_2$ belongs to a stratum S_3 of a higher dimension.

In case *i1*) we choose $\alpha \in \mathbf{C}$ such that $M_1+\alpha M_2 \in S_2$ (or $M_1+\alpha M_2+\beta I \in S_2$). In case *i2*) we choose $\alpha \in \mathbf{C}$ such that $M_1+\alpha M_2 \in S_3$ (or $M_1+\alpha M_2+\beta I \in S_3$). Case *ii*), in fact, coincides with *i2*). Hence, after a finite number of transformations all conditions of the proposition will be fulfilled, due to the finite number of strata of $GL(n, \mathbf{C})$.

Proposition 3.15. *Let all the matrices $M_1, \ldots, M_p$ have the same Jordan normal form, i.e. let $M_j = Q_j^{-1} J_j Q_j$, where $Q_j \in GL(n, \mathbf{C})$ and the Jordan matrices J_j belong to the same stratum of $GL(n, \mathbf{C})$, i.e. they have the same Jordan normal form. Set $J_1 = J$. Then the (ir)reducibility of the group and the image of the mapping Ψ_p (with $N_j = M_j$, see Proposition 3.14.) are preserved if the group $\{M_1, \ldots, M_p\}$ is replaced by the group $\{Q_j^{-1} J Q_j\}, j = 1, \ldots, p$.*

In accordance with this proposition, during the rest of the proof of the lemma we consider $M_1, \ldots, M_p$ to be from one and the same orbit. To prove the proposition it suffices to notice that the image of Ψ_p does not depend on the eigenvalues of the matrices J_j if the eigenvalues vary so that the Jordan normal forms of M_j are preserved and the matrices Q_j are fixed. The set of invariant subspaces is also preserved under such a change of the eigenvalues. The details are left for the reader.

4^0. **Proposition 3.16.** *Let $M_j = Q_j^{-1} J Q_j$, see Proposition 3.15., and let the image of the mapping Ψ_p with $N_j = M_j$, see Proposition 3.14., be not the whole of $sl(n, \mathbf{C})$. Then there exists a non-scalar matrix V such that $[{}^tV, Q_j^{-1} J Q_j] = 0, j = 1, \ldots, p$.*

Proof: Regard $\mathrm{gl}(n, \mathbf{C})$ as a vector space of dimension n^2. Denote its coordinates by x_{ks}, $1 \leq k, s \leq n$. Let $\mathcal{S}_j$ be the set of elements on one interval on Fig. 3., see the proof of Proposition 3.11. Define the linear forms φ_j on $\mathrm{gl}(n, \mathbf{C})$ as $\varphi_j = \sum_{\mathcal{S}_j} x_{ks}$. Set $d = \sharp \varphi_j$. Suppose (which is not restrictive) that $Q_1 = I$, i.e. $M_1 = J$ is in Jordan normal form. Then the image of the mapping $X_1 \mapsto [M_1, X_1]$ is given by $\varphi_j = 0, j = 1, \ldots, d$. Hence, the image of the mapping Ψ_p with $N_j = M_j$ is a subspace of $\mathrm{sl}(n, \mathbf{C})$ described by a system of equations of the kind $\varphi \equiv \sum_{j=1}^{d} s_j \varphi_j = 0$, $s_j \in \mathbf{C}$. Fix one such equation. It denotes the set of zeros of a linear form (of x_{ks}) on $\mathrm{gl}(n, \mathbf{C})$. The coefficients of the form are the coordinates of an $n \times n$-matrix V commuting with ${}^tJ$, see Proposition 3.11.

The conjugation $X \mapsto Q_j^{-1} J Q_j$ induces an automorphism of $\mathrm{gl}(n, \mathbf{C})$: $Y = AX$ where A is $n^2 \times n^2$ and X is considered as a vector column (the $(k+1)$-st column of the matrix X follows the k-th one). The matrix A is explicitely described below. The form φ can be presented as $\varphi = \tilde{\varphi}X$ where $\tilde{\varphi}$ is a vector-line of size n^2; if V is considered as a vector-column, then we have ${}^t\tilde{\varphi} = V$. After the conjugation the form $\tilde{\varphi}$ changes to $\tilde{\varphi}A^{-1}$. Really, if $Y = AX$ are the new coordinates, then $\tilde{\varphi}X = \tilde{\varphi}A^{-1}Y$.

We have $\tilde{\varphi}A^{-1}Y = {}^tY^t(A^{-1})^t\tilde{\varphi} \equiv {}^t(\tilde{\varphi}A^{-1}Y)$. The n^2-vector-column ${}^t(A^{-1})^t\tilde{\varphi}$ is (in the sense above) equal to an $n \times n$-matrix. This is the matrix $({}^tQ_j)V({}^tQ_j)^{-1}$. Really, for $n = 3$ let $Q_j = \begin{pmatrix} m & n & p \\ q & r & s \\ t & u & v \end{pmatrix}$. Then we have $A = LR = RL$ where $L = \begin{pmatrix} Q_j^{-1} & 0 & 0 \\ 0 & Q_j^{-1} & 0 \\ 0 & 0 & Q_j^{-1} \end{pmatrix}$ corresponds to multiplying by Q_j^{-1} to the left and $R = \begin{pmatrix} mI & qI & tI \\ nI & rI & uI \\ pI & sI & vI \end{pmatrix}$ corresponds to multiplying by Q_j to the right. It is clear how to construct L and R for arbitrary n. Hence, ${}^tA^{-1} = {}^t(L^{-1})^tR^{-1}$, i.e. ${}^tA^{-1}$ is the matrix of the transformation $X \mapsto ({}^tQ_j)X({}^tQ_j)^{-1}$. We have $[({}^tQ_j)V({}^tQ_j)^{-1}, ({}^tQ_j)({}^tJ)({}^tQ_j)^{-1}] = 0$.

The linear form φ describes a subspace of $\mathrm{sl}(n, \mathbf{C})$ to which the image of every mapping $X \mapsto [Q_j^{-1}JQ_j, X]$ belongs. Hence, the matrix V corresponding to φ plays the role of the matrix ${}^tQ_jV^tQ_j^{-1}$ above as well, i.e. it commutes with ${}^tJ$ and ${}^t(Q_j)^tJ^t(Q_j^{-1}), j = 2, \ldots, p$, i.e. $[{}^tV, J] = [{}^tV, Q_j^{-1}JQ_j] = 0$. The proposition is proved.

5^0. Sum up the proof of the lemma. Suppose that the image of Ψ_p is not the whole of $\mathrm{sl}(n, \mathbf{C})$. Then one can replace the group $\mathbf{M}$ by another group which satisfies the conclusion of Proposition 3.15. (namely, that all generators belong to one and the same orbit), without changing the image of Ψ_p. This leads to the existence of the nonscalar matrix ${}^tV$ commuting with the generators of the new (irreducible!) group which is a contradiction with Proposition 3.13. The lemma is proved.

Proof of Lemma 3.4.: It suffices to vary any eigenvalue of any of the matrices $M_1, \ldots, M_p$.

Proof of Lemma 3.5.: The proof resembles the one of Lemma 3.3. Consider the matrices $P_k = \begin{pmatrix} Q_{iik} & 0 \\ 0 & Q_{jjk} \end{pmatrix}$. The left hand-side of the equation in Lemma 3.5. is the restriction of the action of Ψ_p with $N_j = P_j$ to the left lower block; Ψ_p is defined in Lemma 3.1.

Like in Proposition 3.16. we show that if the equation from Lemma 3.5. has no solution for some choice of the right hand-side, then there exists a nonscalar matrix V whose right upper and diagonal blocks are equal to 0 such that $[{}^tV, P_k] = 0, k = 1, \dots, p$. Conjugating V and $P_1, \dots, P_p$ by one and the same block-diagonal matrix, we can achieve the following form of ${}^tV$: zeros in the left lower and in the diagonal blocks, the right upper block being of the form $V' = \begin{pmatrix} I & 0 \\ S & T \end{pmatrix}$, where the blocks S and T are equal to 0; we assume that Q_{iik} is of size $n_1 \leq n_2$ where n_2 is the size of Q_{jjk}. The size r of I is equal to the rank of ${}^tV$, S and/or T can be empty.

The condition $[{}^tV, P_k] = 0$ implies that for $r < n_1 \leq n_2$ last $n_1 - r$ columns of $Q_{iik}V'$ and last $n_2 - r$ rows of $V'Q_{jjk}$ are equal to 0. Hence, we must have that the elements in last $n_1 - r$ rows and first $n_1 - r$ columns of Q_{iik} must be zeros for $k = 1, \dots, p$, i.e. the group $\{Q_{iik}\}, k = 1, \dots, p$ is reducible which is a contradiction. If $r = n_1 < n_2$, then the elements of last $n_2 - r$ columns and first $n_2 - r$ rows of Q_{jjk} must be zeros (for $k = 1, \dots, p$), i.e. the group $\{Q_{jjk}\}, k = 1, \dots, p$ is reducible which again is a contradiction. Finally, if $r = n_1 = n_2$, we have $Q_{iik} = Q_{jjk}, k = 1, \dots, p$ which gives the result claimed by the lemma.

Proof of Lemma 3.6.: We prove the lemma in the case of one big and two small blocks. In the general case the proof consists in repeating the same construction the necessary number of times. There exists a holomorphic and holomorphically invertible matrix $C'(\varepsilon)$ conjugating $M_1(\varepsilon)$ with its Jordan normal form; ε denotes the local coordinates in the neighbourhood of M. The invariant subspaces of $M_1|_{\varepsilon=0}$ are described on Fig. 4 and Fig. 5. One of them must be invariant for $M_2|_{\varepsilon=0}, \dots, M_{p+1}|_{\varepsilon=0}$. But then at least one such subspace must be invariant for $M_2, \dots, M_{p+1}$ for all $\varepsilon \in U$ as well – the number of invariant subspaces is finite, the set of values of ε for which an invariant subspace of $C'^{-1}M_1C'$ is such for $C'^{-1}M_jC'$, $j = 2, \dots, p+1$ as well is closed. Hence, there exists a cojugation with a permutation matrix such that all the matrices M_j will be blocked as follows: $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. The superposition of the two conjugations gives the necessary matrix C.

Proof of Lemma 3.7.: We have $\mathcal{T}' = \mathcal{T}^* \times \mathcal{D}$ where $\mathcal{D}$ is the subspace of $(\mathrm{gl}(n, \mathrm{C}))^p$ consisting of the p-tuples of matrices with the same reducibility type as the ones of Σ, the elements of whose small blocks and of the blocks outside the big blocks are 0 and the elements of whose superdiagonal blocks in the big blocks are arbitrary. $\mathcal{T}^*$ is the set of matrices the elements of whose off-diagonal blocks are 0 and whose small blocks are same as the ones of $\mathcal{T}'$.

Consider a fixed small block. A matrix of the size of the small block – $T = D^{-1}JD$, J being its Jordan normal form which is fixed – can be locally parametrised by the matrix D and by the eigenvalues of J; for D one can fix a subspace of minimal dimension (D is not defined uniquely but modulo multiplication by matrices commuting with J). Hence, for every small block one can introduce local coordinates (d,g) where d are the coordinates of D and g are the ones of the eigenvalues of J. The matrices with a fixed Jordan normal form depend analytically on (d,g) and their locally smooth analytic variety is an analytic fibration over the base g. One can locally parametrise $\mathcal{T}^*$ as follows: for every M_j parametrise its every small block independently as above; denote the coordinates by $(d_1,g_1),\ldots,(d_s,g_s)$. If the second small block has eigenvalues equal to such of the first one, then we set $g_{2j}=g_{1k}$ for the corresponding eigenvalues and obtaining a subset g' of g. In the analytic fibration $(d_1,g_1)\times(d_2,g_2)\mapsto(d_1,g_1)$ consider the subset $\{g_{2j}=g_{1k}\}$ – this is an analytic subvariety of the initial one. We obtain in the same way the subspaces $g'_3,\ldots,g'_s$ (considering the fibrations $(d_1,g_1)\times(d_3,g_3)\mapsto(d_1,g_1)$ etc.). The variety obtained in this way (setting $g_{3j}=g_{1k}$) is a smooth analytic subvariety of the initial one. Replacing in the reasoning above g_1 and g_2 by $g_1\cup g'_2$ and g_3, we obtain the subspaces $g''_3,\ldots,g''_s$ etc. Finally, we obtain the subspace $g_1\cup g'_2\cup g''_3\cup\ldots\cup g_s^{(s-1)}\subset g_1\cup\ldots\cup g_s$ and the necessary smooth analytic subvariety, i.e. $\mathcal{T}^*$ which has local coordinates $(d_1,\ldots,d_s,g_1,g'_2,\ldots,g_s^{(s-1)})$ (constructed for every M_j separately).

For $\mathcal{S}'$ the proof is similar to the one for $\mathcal{T}'$.

Proof of Lemma 3.8.: 1^0. Consider the case when the reducibility type is $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, i.e. it consists of one big and two small blocks. Consider M_1. Suppose that its blocks A and C are in Jordan normal form for all values of the coordinates upon which the elements of $\mathcal{R}$ (see 8^0 of the proof of the theorem) depend. A holomorphic conjugation to such a form exists (locally). Let λ_1 be an eigenvalue of M_1. Then condition $\text{rk}(M_1-\lambda_1 I)\leq a_1$, $a_1\in \mathbf{N}$ ('rk'='rank') defines a finite number of smooth subvarieties $\mathcal{T}_k^{(1)}$ of $\mathcal{T}'$. The same is true for $M_2,\ldots,M_p$. If we consider conditions $\text{rk}(M_s-\lambda_s I)\leq a_s$, $s=1,\ldots,p$ simultaneously, then this defines a finite number of smooth analytic subvarieties $\mathcal{T}''_k$. Applied to M_{p+1}, (i.e. for $s=p+1$), these conditions define a finite number of smooth analytic subvarieties $\mathcal{S}''_j$ of $\mathcal{S}'$. The graph of (*) restricted to $\mathcal{T}''_k$ intersects $\mathcal{S}''_j\times\mathcal{T}''_k$ and the intersection (for each (k,j)) is a smooth analytic subvariety. This is proved in the same way as of 1) of the theorem, considering the restriction of equation (***) to each block on or above the diagonal (in the big blocks), in the opposite to the order of the blocks as described in 5^0 of the proof of the theorem. This is possible because the strata are non-special and we don't have problems with Lemma 3.5.

On each $(\mathcal{S}''_j,\mathcal{T}''_k)$ consider conditions $\text{rk}(M_s-\lambda_s I)^2\leq b_s$, $s=1,\ldots,p+1$. They define smooth analytic subvarieties; the graph of (*) restricted to $\mathcal{T}''_k$ intersects $\mathcal{S}''_i\times\mathcal{T}''_k$ and the intersection is a smooth analytic variety. Then we consider conditions $\mathrm{rk}(M_s-\lambda_s I)^3 \leq c_s$ etc. These conditions define the closures of the strata; they are connected with the finding of the Jordan normal form of M_s (more precisely – how many blocks and of what size correspond to the given eigenvalue). When $\mathrm{rk}(M_s-\lambda_s I)$ is maximal possible, then this doesn't define a subvariety, but the compliment to the analytic varieties from which the other strata are composed. In the case of one big and two small blocks the proof of the smoothness is easy because the B-blocks of the matrices $(M_s-\lambda_s I)^k$ are equal to $\sum_{j=0}^{k} P_s^j Q_s R_s^{k-j}$ where $P_s = M_s|_A$, $Q_s = M_s|_B$, $R_s = M_s|_C$, i.e. they depend linearly on Q_s.

2^0. Let the reducibility type consist of one big and r small blocks:

$$\begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1,r-1} & Q_{1r} \\ 0 & Q_{22} & \dots & Q_{2,r-1} & Q_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q_{r-1,r-1} & Q_{r-1,r} \\ 0 & 0 & \dots & 0 & Q_{rr} \end{pmatrix}$$

In this case the proof is carried out by induction with respect to r, in the same way as in 1^0; the roles of A, B and C are played respectively by $Q' = \begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1,r-1} \\ 0 & Q_{22} & \dots & Q_{2,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{r-1,r-1} \end{pmatrix}$, $\begin{pmatrix} Q_{1r} \\ \vdots \\ Q_{r-1,r} \end{pmatrix}$ and Q_{rr}. It is assumed that the Q'- and Q_{rr}-blocks of $M_1, \dots, M_{p+1}$ are restricted to irreducible components of given strata. Smoothness of strata is proved as it is explained in 1^0. For the case of many big blocks the lemma is proved in the same way.

Proof of Lemma 3.9.: Set in operator $\mathcal{L}'_p$, see (***), $Y_j = \begin{pmatrix} Y'_j & Y''_j \\ 0 & Y'''_j \end{pmatrix}$. Find first Y'_j, Y'''_j and V_j, $j = 1, \dots, p+1$ as in the irreducible case; they solve the restriction of equation (***) to the A- and C-blocks. After this the equation can be solved for the B-block as well if its trace after the fixing of Y'_j, Y'''_j is 0. If not, then we can try to make the substitution $Y'_j \mapsto Y'_j + U_j$ (or $Y'''_j \mapsto Y'''_j + U_j$) where U_j are matrices of the size of P_j such that $\sum_{j=1}^{p}[U_j, P_j] = 0$. This will not change the A- and C-blocks. It will fail to change the trace of the B- block if and only if $\mathrm{tr}\sum_{j=1}^{p} U_j Q_j (\equiv \mathrm{tr}\sum_{j=1}^{p} Q_j U_j) = 0$ for every set of U_j such that $\sum_{j=1}^{p}[U_j, P_j] = 0$.

Proposition 3.17. *Let* $tr\sum_{j=1}^{p} U_j Q_j (\equiv tr\sum_{j=1}^{p} Q_j U_j) = 0$ *for every set of* U_j *such that* $\sum_{j=1}^{p}[U_j, P_j] = 0$. *Then* $Q_j = [P_j, D^0]$ *for some matrix* D^0 *of the size of* P_j, $j = 1, \dots, p+1$.

The lemma follows from the proposition, setting $D = -D^0$.

Proof : Condition $\mathrm{tr}\sum_{j=1}^{p} U_jQ_j = 0$ must be a corollary from condition $\sum_{j=1}^{p}[U_j, P_j] = 0$. Every such corollary is of the form $\mathrm{tr}\sum_{j=1}^{p}[U_j, P_j]D^0 = 0$. We have $\mathrm{tr}\sum_{j=1}^{p}[U_j, P_j]D^0 = \mathrm{tr}\sum_{j=1}^{p}(P_jD^0U_j - D^0P_jU_j)$. Hence, we must have $Q_j = [P_j, D^0]$.

Proof of Lemma 3.10.: For every $\varepsilon \in \mathbf{C}$ the matrix $\begin{pmatrix} P_j & \varepsilon Q_j \\ 0 & P_j \end{pmatrix}$ is conjugate to $M_j^0 = \begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. Hence, the matrix $\begin{pmatrix} 0 & Q_j \\ 0 & 0 \end{pmatrix}$ belongs to the tangent space to the orbit of M_j^0 for every ε, i.e. $Q_j = [P_j, D_j]$ for some D_j. The opposite implication follows from $M_j = S^{-1}M_j^0S$ with $S = \begin{pmatrix} I & -D_j \\ 0 & I \end{pmatrix}$.

Proof of Lemma 2.1.: We combine the ideas used in the proofs of Lemmas 3.7. and 3.8. For every stratum of Σ the Jordan normal forms of its small blocks define smooth analytic varieties in Σ. This is proved as the smoothness of $\mathcal{T}^*$, see the proof of Lemma 3.7. Let λ be an eigenvalue of $M \in \Sigma$. Then conditions $\mathrm{rk}(M - \lambda I)^i \leq a_i$, $i = 1, 2, \ldots$, $a_i \in \mathbf{N}$ define a finite number of smooth analytic varieties, see the proof of Lemma 3.8., which are the closures of a finite number of strata. The lemma is proved.

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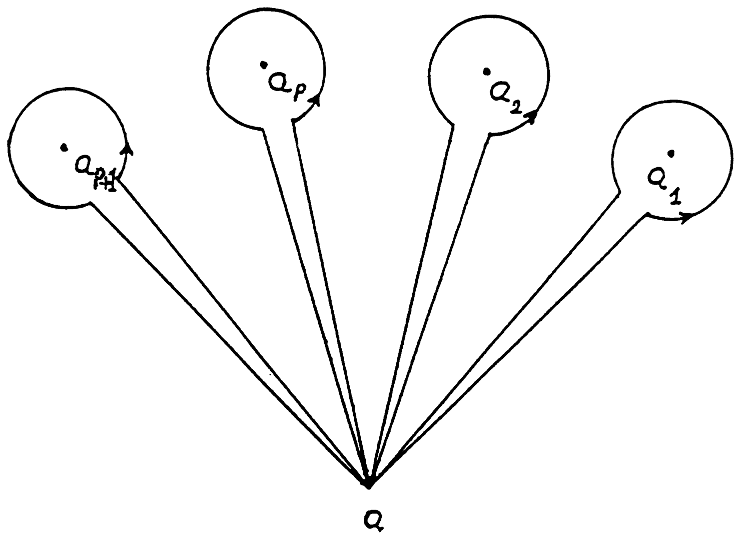

Fig. 1

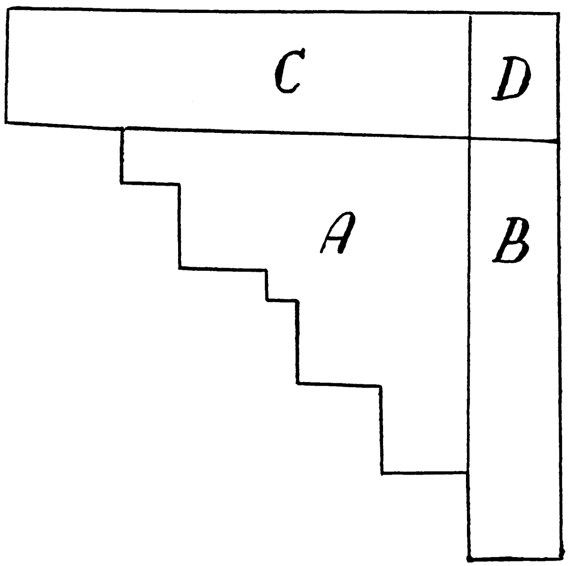

Fig. 2.

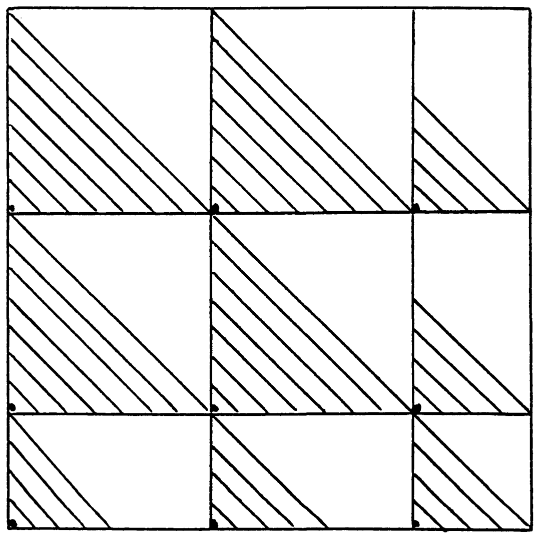

Fig. 3.

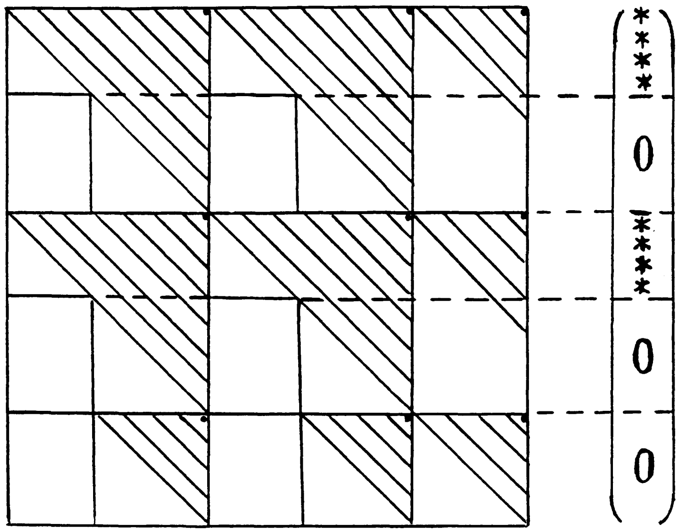

Fig. 4.

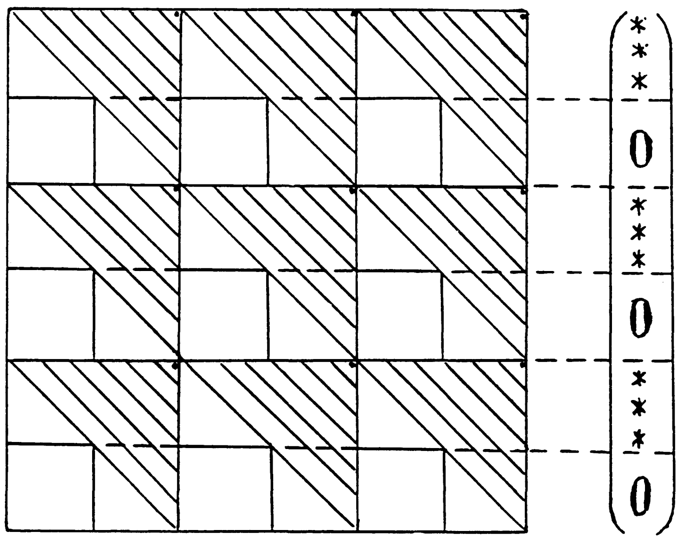

Fig. 5.