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A rigidity theorem for transverse dynamics of real analytic foliations of codimension one

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The purpose of this paper is to prove

Theorem 1. *Let $(M_i^n, \mathcal{F}_i), i = 1, 2$, be real analytic and orientable foliations of n-manifolds of codimension 1 and $h : (M_1^n, \mathcal{F}_1) \to (M_2^n, \mathcal{F}_2)$ a foliation preserving homeomorphism. Assume that all leaves of $\mathcal{F}_1$ are dense and there exists a leaf of $\mathcal{F}_1$ with holonomy group $\neq 1, \mathbb{Z}$. Then h is transversely real analytic.*

This applies to prove the following topological rigidity of the Godbillon-Vey class of real analytic foliations of codimension one.

Corollary 2. *Let $(M_i, \mathcal{F}_i), h$ be as in Theorem 1. Then $h^*(\mathrm{GV}(\mathcal{F}_2) = \mathrm{GV}(\mathcal{F}_1)$ holds.*

Here $\mathrm{GV}(\mathcal{F}_i) \in H^3(M, \mathbb{R})$ denotes the Godbillon-Vey class of $\mathcal{F}_i$, which is represented by the 3-form $\alpha \wedge d\alpha$ with a C^∞ -1-form α on M such that $d\theta = \theta \wedge \alpha$ holds with a C^∞ -1-form θ defining $\mathcal{F}$. It is easy to see that the Godbillon-Vey class is invariant under C^2-diffeomorphisms. Ghys, Tsuboi [9] and Raby [18] proved the invariance under C^1-diffeomorphisms, while the invariance is known to fail in some C^0-cases (see [5,9,11]). (Corollary 2 seems to admit the various generalisations allowing the existence of compact leaves. But we will not touch on those generalisations. See also the papers [5,7].)

The proof of the C^1-invariance due to Ghys and Tsuboi is based on a certain rigidity for C^1-conjugacies of transverse dynamics of foliations along compact leaves as well as minimal exceptional leaves cutting Cantor sets on transverse sections. The proof of Theorem 1 is based on the topological rigidity theorem for pseudogroups of diffeomorphisms of $\mathbb{R}$ (Theorem 3(1)).

To state Theorem 3 we prepare some notions. Let Γ_+^ω be the pseudogroup of real analytic and orientation preserving diffeomorphisms of open neighbourhoods of the line $\mathbb{R}$ respecting 0. We call a mapping $\phi : G \to \Gamma_+^\omega$ of a group G to the pseudogroup Γ_+^ω a *morphism* if the set $\phi(G)_0$ of germs of $\phi(f), f \in G$ form a group and ϕ induces a group homomorphism of G to $\phi(G)_0$. Therefore $\phi(f) : U_{\phi(f)}, 0 \to \phi(f)(U_{\phi(f)}), 0$ is a real analytic diffeomorphism of open neighbourhoods of $0 \in \mathbb{R}$ for $f \in G$ representing the germ of $\phi(f)$. We call $\phi(G)_0$ the germ of $\phi(G)$ and say ϕ is *solvable* (respectively *commutative, etc*) if $\phi(G)_0$ is so. The *orbit* $\mathcal{O}(x)$ of an $x \in \mathbb{R}$ is the set of those x_l joined by a sequence $(x_0, x_1, ..., x_l)$ with $x = x_0, x_{i+1} = \phi(f_i)(x_i), x_i \in U_{\phi(f_i)}, i = 0, ..., l-1$ for arbitrary $l \geq 0$. The *basin* $B_{\phi(G)}$ of 0 is the set of those x for which the closure of the orbit $\mathcal{O}(x)$ contains 0. If $\phi(G)$ is non trivial, i.e. $\phi(f) \neq \mathrm{id}$ for an $f \in G$, $B_{\phi(G)}$ is an open neighbourhood of 0 [17]. Morphisms $\phi, \psi : G \to \Gamma_+^\omega$ are *topologically* (resp. C^r-) *conjugate* if there exists a homeomorphism (resp. C^r-diffeomorphism) $h : U, 0 \to h(U), 0$ of open neighbourhoods of 0 such that $U_{\phi(f)}, \phi(f)(U_{\phi(f)}) \subset U, U_{\psi(f)}, \psi(f)(U_{\psi(f)}) \subset h(U)$ and $h \circ \phi(f) = \psi(f) \circ h$ holds on $U_{\phi(f)}$ for all $f \in G$. We call h a *linking homeomorphism* (resp. *linking diffeomorphism*) and we denote $h : \phi \to \psi$.

Theorem 3 (The rigidity theorem for pseudogroups). *Let $\phi, \psi : G \to \Gamma_+^\omega$ be morphisms which are topologically conjugate with each other and $h : \phi \to \psi$ a linking homeomorphism.*

(1) If $\phi(G)_0, \psi(G)_0$ are not isomorphic to $\mathbb{Z}$ and non trivial, the restriction $h : B_{\phi(G)} - 0 \to B_{\psi(G)} - 0$ is a real analytic diffeomorphism.

(2) If $\phi(G)_0, \psi(G)_0$ are non commutative, h is unique and there exist even positive integers i, j such that $|h(\epsilon x^i)|^{1/j} : \tilde{B}^\epsilon_{\phi(G)} \to \tilde{B}^\epsilon_{\psi(G)}$ is a real analytic diffeomorphism for $\epsilon = \pm 1$. Here $\tilde{B}^\epsilon_{\phi(G)}$ is the set of those x such that $\epsilon x^i \in B_{\phi(G)}$ and $\tilde{B}^\epsilon_{\psi(G)}$ is the set of those x such that $x^j (resp. -x^j) \in B_{\psi(G)}$ if h maps $\mathbb{R}^\epsilon$ to $\mathbb{R}^+$ (resp. $\mathbb{R}^-$).

Now we apply the above rigidity theorem to the analytic action of the surface group on the circle S^1. Let Σ_g be the oriented closed surface of genus g and $\Gamma^g = \pi_1(\Sigma_g)$. For $r = 1, \dots, \infty$ and ω, $\mathrm{Diff}^r_+(S^1)$ denotes the group of orientation preserving C^r-diffeomorphisms of the circle. The ***suspension*** M of a homomorphism $\phi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ is the quotient of $S^1 \times D^2$ by the product $\phi \times \Gamma$ with a discrete cocompact subgroup $\Gamma^g \simeq \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acting freely on the interior of the Poincaré disc D^2. The second projection of $S^1 \times D^2$ induces the submersion of M onto $\Sigma_g = D^2/\Gamma$ with the fiber S^1. Since the action $\phi \times \Gamma$ respects the foliation of $S^1 \times D^2$ by the discs $x \times D^2, x \in S^1$, the suspension M is a foliated S^1-bundle of which the fibres are the quotients of the discs. In this way the topology of foliated S^1-bundles interchanges with that of the actions of Γ^g on S^1. The Euler number $\mathrm{eu}(\phi)$ of a homomorphism $\phi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ is defined to be that of the S^1-bundle associated to ϕ. The Milnor-Wood inequality [15,22] asserts

$$|eu(\phi)| \leq |\chi(\Sigma_g)| = 2g - 2.$$

The Euler number enjoys the following relations with the orbit structure:

(1) $\mathrm{eu}(\phi) = 0$ if there exists a finite orbit,

(2) If $\mathrm{eu}(\phi) \neq 0$, there exist a minimal set $\mathcal{M} \subset S^1$ of ϕ, an $x \in \mathcal{M}$ and an $f \in \mathrm{stab}(x)$ such that $\phi(f)|_{\mathcal{M}} \neq id$ [13], and if $r = \omega$ all orbits are dense [6] (see also [16]),

(3) If $|\mathrm{eu}(\phi)| = |\chi(\Sigma_g)|$ and $r \geq 2$, all orbits are dense [6],

where $\mathrm{stab}(x)$ denotes the stabiliser of x consisting of $f \in \Gamma^g$ with $\phi(f)(x) = x$. Homomorphisms $\phi, \psi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ are ***C^s-conjugate*** if there exists a C^s-diffeomorphism h of S^1 such that $\psi(f) \circ h = h \circ \phi(f)$ holds for $f \in \Gamma^g$. We say ϕ, ψ are ***topologically conjugate*** if $s = 0$, ***semi conjugate*** if h is monotone map of degree one (possibly discontinuous). We call h a ***linking homeomorphism*** and denote $h : \phi \to \psi$. It is known that the Euler number (and the bounded Euler class) concentrate the homotopic property of the action, namely

Theorem(Ghys [3]). *ϕ, ψ are semi conjugate if and only if $\phi^*(\chi_{\mathbb{Z}}) = \psi^*(\chi_{\mathbb{Z}})$*

in the bounded cohomology group $H_b^2(\Gamma^g : \mathbb{Z})$, *where* $\chi_{\mathbb{Z}} \in H_b^2(\mathrm{Diff}_+^0(S^1) : \mathbb{Z}) = \mathbb{Z}$ *is the generator, the bounded Euler class.*

Theorem (Matsumoto [13]). *If* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi) = \pm\chi(\Sigma_g)$, ϕ, ψ *are semi conjugate, and if* $2 \le r$, *they are topologically conjugate with each other, and in particular, conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *naturally acting on* S^1 *the boundary of the Poincaré disc.*

Theorem Ghys [8]. *If a homomorphism* $\phi : \Gamma^g \to \mathrm{Diff}_+^r(S^1)$ *attains the maximum of* $|\mathrm{eu}(\phi)|$ *and* $3 \le r$, ϕ *is* C^r*-smoothly conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

In contrast to the above results, the properties of homomorphisms with $|\mathrm{eu}(\phi)| \lneqq |\chi(\Sigma_g)|$ are less known (see [16]). Applying Theorem 3 to the action of the stabiliser subgroup $\mathrm{stab}(x)$ on (S^1, x) for an $x \in S^1$, we obtain

Corollary 4. *Let* $\phi, \psi : \Gamma_g \to \mathrm{Diff}_+^\omega(S^1)$ *be homomorphisms with* $|\mathrm{eu}(\phi)|$, $|\mathrm{eu}(\psi)| \ne 0, |\chi(\Sigma_g)|$, *which are topologically conjugate, and* $h : \phi \to \psi$ *a linking homeomorphism. Assume that for an* $x \in S^1$, *the stabiliser subgroup* $\mathrm{stab}(x) \subset \Gamma_g$ *of* x *is not isomorphic to* $\mathbb{Z}$ *and non trivial. Then* h *is a real analytic diffeomorphism and orientation preserving or reversing respectively whether* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi)$ *or* $\mathrm{eu}(\phi) = -\mathrm{eu}(\psi)$.

The statement remains valid for morphisms of groups G into $\mathrm{Diff}_+^\omega(S^1)$ replacing the condition on the Euler number by the existence of a dense orbit.

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2. Sequence geometry

In this paper $f^{(n)}$ denotes the n-fold iteration $f \circ \cdots \circ f$ of $f : U_f \to f(U_f)$ in Γ_+^ω. Let $\mathcal{X} = \{x_i\}, \mathcal{Y} = \{y_i\}, i = 1.2. \dots$ be monotone sequences of positive numbers decreasing to 0. Define the *address function* $\mathrm{add}_{\mathcal{Y}}(x)$ of an $x > 0$ relative to $\mathcal{Y}$ to be the smallest integer i such that $y_i \le x$. It is easy to see that $\mathrm{add}_{\mathcal{Y}}(x)$ is a decreasing function of x and $y_{\mathrm{add}_{\mathcal{Y}}(x)-1} > x \ge y_{\mathrm{add}_{\mathcal{Y}}(x)}$.

Define the *address function* $\mathrm{add}_{\mathcal{X},\mathcal{Y}}$ by

$$\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i) = \mathrm{add}_{\mathcal{Y}}(x_i)$$

for $i = 1, 2, \ldots$. The address function enjoys the following inequality for a triple of sequences $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \{z_i\}$.

Proposition 6. *Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \{z_i\}$ be sequences of positive numbers decreasing to 0. Then*

$$\mathrm{add}_{\mathcal{Y},\mathcal{Z}}(\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i) - 1) \leq \mathrm{add}_{\mathcal{X},\mathcal{Z}}(i) \leq \mathrm{add}_{\mathcal{Y},\mathcal{Z}}(\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i))$$

for $x_i - 1 < y_0$.

We say two functions $P, Q : \mathrm{N} \cup 0 \to \mathrm{N} \cup 0$ are *equivalent* if there exist integers $c_1, \ldots, c_4$ such that

$$Q(i + c_1) + c_2 \leq P(i) \leq Q(i + c_3) + c_4$$

holds for all sufficiently large i.

Now let $\phi : G \to \Gamma_+^\omega$ be a morphism, and let $x_0 \in U_{\phi(g)}, y_0 \in U_{\phi(f)}$ be positive and sufficiently small and assume that $x_i = \phi(g)^{(i)}(x_0), y_i = \phi(f)^{(i)}(y_0)$ are decreasing to 0 as $i \to \infty$, replacing f, g by their inverses if necessary, and denote $\mathcal{X} = \{x_i\}, \mathcal{Y} = \{y_i\}$.

Proposition 7. *The equivalence class of the address function $\mathrm{add}_{\mathcal{X},\mathcal{Y}}$ is independent of the choice of the initial values x_0, y_0.*

proof. To prove the statement let $x_0 \neq x_0' > 0, y_0 \neq y_0' > 0$ and define the sequences $\mathcal{X}', \mathcal{Y}'$ similarly with x_0', y_0'. It is easy to see

$$\mathrm{add}_{\mathcal{X}',\mathcal{X}}(i) = i + c$$

for sufficiently large i, where

$$c = \begin{cases} \mathrm{add}_{\mathcal{X}}(x_0'), & \text{if } \ x_0 \geq x_0' \\ 1 - \mathrm{add}_{\mathcal{X}'}(x_0) & \text{if } \ x_0' > x_0,\ x_0 \neq x_j',\ j = 0, 1, \ldots \\ -\mathrm{add}_{\mathcal{X}'}(x_0) & \text{if } \ x_0' > x_0,\ x_0 \in \mathcal{X}' \end{cases}$$

From Proposition 6 we obtain

$$\text{add}_{\mathcal{X},\mathcal{Y}}(i+c-1) \le \text{add}_{\mathcal{X}',\mathcal{Y}}(i) \le \text{add}_{\mathcal{X},\mathcal{Y}}(i+c) \tag{1}$$

for sufficiently large i. Similarly we obtain

$$\begin{aligned} \text{add}_{\mathcal{X}',\mathcal{Y}} + c' - 1 &= (\text{add}_{\mathcal{Y},\mathcal{Y}'}(\text{add}_{\mathcal{X}',\mathcal{Y}} - 1) \\ &\le \text{add}_{\mathcal{X}',\mathcal{Y}'} \\ &\le \text{add}_{\mathcal{Y},\mathcal{Y}'}(\text{add}_{\mathcal{X}',\mathcal{Y}}) \\ &= \text{add}_{\mathcal{X}',\mathcal{Y}} + c' \end{aligned}$$

with

$$c' = \begin{cases} \text{add}_{\mathcal{Y}'}(y_0), & \text{if} \quad y'_0 \ge y_0 \\ 1 - \text{add}_{\mathcal{Y}}(y'_0), & \text{if} \quad y_0 > y'_0,\ y'_0 \ne y_j,\ j = 0, 1, \dots \\ -\text{add}_{\mathcal{Y}}(y'_0), & \text{if} \quad y_0 > y'_0,\ y'_0 \in \mathcal{Z} \end{cases}$$

and by (1),

$$\text{add}_{\mathcal{X},\mathcal{Y}}(i+c-1)c' - 1 \le \text{add}_{\mathcal{X}',\mathcal{Y}'}(i) \le \text{add}_{\mathcal{X},\mathcal{Y}}(i+c) + c'$$

for sufficiently large i. This completes the proof.

3. Formal invariants for non solvable pseudogroups

It is shown in the paper [17] that the non solvable group $\phi(G)$ contains diffeomorphisms $\phi(f), f \in G$ with Taylor expansion at $x = 0$

$$\phi(f)(x) = x - \frac{K}{i}(x^{i+1} + \cdots),$$

$K \ne 0$ with i greater than an arbitrary large integer. So let

$$\phi(g)(x) = x - \frac{L}{j}(x^{j+1} + \cdots),$$

$L \ne 0, i < j$ for a $g \in G$. We call the i, j the *orders of the flatness* for $\phi(f), \phi(g)$ respectively. By Proposition 6 the equivalence class of the address function

$\text{add}_{\mathcal{X},\mathcal{Y}}$ is independent of the choice of x_0, y_0. We denote the equivalence class by $\text{add}_{\phi(g),\phi(f)}$.

First we consider the orbit $\mathcal{Y}$ of y_0 under $\phi(f)$. It is known ([20]) that with a suitable analytic coordinate we may assume $\phi(f)$ has the Taylor expantion

$$\phi(f)(x) = x - \frac{K}{i}(x^{i+1} + (-A + \frac{i+1}{2})x^{2i+1} + \cdots),$$

which is formally conjugate with

$$\phi'(f)(x) = \exp - \frac{K}{i}(\frac{x^{i+1}}{1 + Ax^i})\partial/\partial x.$$

The $-iA/K$ is known as the *residue* of f. By a result due to Takens [20] there exists a C^∞ diffeomorphism $\lambda : \mathbb{R}, 0 \to \mathbb{R}, 0$ i-flat at 0 such that $\lambda \circ \phi(f) = \phi'(f) \circ \lambda$ holds on $U_{\phi(f)}$ shrinking $U_{\phi(f)}$. Introducing the coordinate $\tilde{x} = \xi_{i,A}(x) = x^{-i} + A \log x^{-i}$ for $x > 0$, $\phi'(f)$ induces the translation $\tilde{\phi}(f) = \exp K\partial/\partial\tilde{x}$ on the $\tilde{x}$-line at ∞. Let $y'_n = \lambda(y_n)$ and $\tilde{y}_n = \xi_{i,A}(y'_n)$ for $n = 0, 1, \ldots$. Then

$$\text{(a)} \qquad \tilde{y}_n = \tilde{\phi}(f)^{(n)}(\tilde{y}_0) = \tilde{y}_0 + nK.$$

(The existence of the coordinate $\tilde{x}$ with Property (a) is proved by the sectorial normalisation theorem [12,21] as well as the existence of the solution of Abel's equation by Szekeres [19]. Those results imply the existence of the nomalising diffeomorphism λ real analyticity off 0. But the differentiability at 0 is not an obvious consequence. The analyticity of the conjugacy h off 0 in Theorem 3(1) follows from that of λ. In this paper the smoothness of h (Proposition 9) is first proved and analyticity is proved by the uniqueness (Proposition 10) and the convergence of the formal conjugacy due to Cerveau and Moussu [2].)

We apply the same argument to the slow dynamics $\phi(g)$. Let $\mu : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a C^∞ diffeomorphism j-flat at 0 such that $\mu \circ \phi(g) = \phi'(g) \circ \mu$ holds on $U_{\phi(g)}$, where $\phi'(g)(x) = \exp - \frac{L}{j}(\frac{x^{j+1}}{1+Bx^j})\partial/\partial x$ with a constant B. Let

$\tilde{\tilde{x}} = \xi_{j,B}(x) = x^{-j} + B\log x^{-j}$ for $x > 0$. On the $\tilde{\tilde{x}}$-line, $\phi'(g)$ lifts to the translation $\tilde{\tilde{\phi}}(g) = \exp L\, \partial/\partial\tilde{\tilde{x}}$ at ∞.

Let $x'_n = \mu(x_n)$ and $\tilde{\tilde{x}}_n = \xi_{j,B}(x'_n)$ for $n = 0, 1, \ldots$. Then $\tilde{\tilde{x}}_n = \tilde{\tilde{x}}_0 + nL$, from which we obtain the estimate for the $\phi(g)$-orbit $\mathcal{X}$, $x_n = (nL)^{-1/j} + o(n^{-1/j})$ for $n = 0, 1, \ldots$. To compare $\mathcal{X}$ to $\mathcal{Y}$, let

$$\text{(b)} \qquad \tilde{x}_n = x_n^{-i} + A\,\log x_n^{-i} = (nL)^{i/j} + o(n^{i/j}).$$

From (a) and (b) we obtain

$$\text{(c)} \qquad \mathrm{add}_{\phi(g),\phi(f)}(n) = \frac{L^{i/j}}{K}\, n^{\frac{i}{j}} + o(n^{\frac{i}{j}}).$$

Proposition 8. *$L^{\frac{i}{j}}/K$ and $\frac{i}{j}$ are topological invariants for the pseudogroup generated by $\phi(f)$ and $\phi(g)$.*

Proof. Assume h is orientation preserving. The linking homeomorphism h sends the pairs of the orbits of x_0 under $\phi(f), \phi(g)$ to that of $h(x_0)$ under $\psi(f), \psi(g)$, and those pairs have the same topological structure and define the same address function up to the equivalence relation. By (c) the i/j is the exponent of the address function and $L^{\frac{i}{j}}/K$ is its coefficient, which are clearly invariant under the equivalence relation. If h is orientation reversing, an alternative argument goes through.

4. Proof of The Theorem 3 for non solvable pseudogroups

First we prove Theorem 3(1) for non solvable pseudogroups. If the linking homeomorphism h is orientation reversing, the homeomorphism $-h$ is orientation preserving and links ϕ to the reversed pseudogroup ψ' consinting of the orientation preserving diffeomorphisms $\psi'(f) : -U_f \to -f(U_f), f \in G$ defined by $\psi'(f)(x) = -\psi(f)(-x)$. So we assume that h is orientation preserving throughout this section. Let $\psi(f)(x) = x - \frac{K'}{i'}(x^{i'+1} + \ldots)$ and $\psi(g)(x) = x - \frac{L'}{j'}(x^{j'+1} + \ldots)$. First assume $(i, j) = (i', j')$ and h is orientation preserving for simplicity. By a linear coordinate transformation we may assume $K = K'$ and then it follows $L = L'$ from Proposition 8. By an analytic coordinate transformation we may assume

$$\psi(f)(x) = x - \frac{K}{i}(x^{i+1} + (-A' + \frac{i+1}{2})x^{2i+1} + \cdots).$$

Let $\lambda' : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a C^∞-diffeomorphism j-flat at 0 such that $\lambda' \circ \psi(f) = \psi'(f) \circ \lambda'$ holds on $U_{\psi(f)}$, where

$$\psi'(f) = \exp\ -\frac{K}{i}\frac{x^{i+1}}{1 + A'x^i}\ \partial/\partial x.$$

Let $\tilde{y} = \xi_{i,A'}(x) = x^{-i} + A' \log x^{-i}$. Since $\phi(f)^{(n)}(x_0) \to 0$, we see $K > 0$.

On the $\tilde{x}$-line the diffeomorphism $\phi(g)$ induces the "non-linear translation"

$$\tilde{\phi}(g)(\tilde{x}) = \tilde{x} + \frac{i}{j}L\ \tilde{x}^{\frac{i-j}{i}} + o(\tilde{x}^{\frac{i-j}{i}})$$

from which

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g) \circ \tilde{\phi}(f)^{(n)}(\tilde{x}) = \tilde{x} + \frac{i}{j}L(nK)^{\frac{i-j}{i}} + o(n^{\frac{i-j}{i}})$$

from which

$$\lim_{n\to\infty} n^{\frac{j-i}{i}}(\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g) \circ \tilde{\phi}(f)^{(n)}\ -\ \mathrm{id})\ \partial/\partial\tilde{x} = \frac{iL}{j}K^{\frac{i-j}{i}}\partial/\partial\tilde{x}$$

holds at the end of the $\tilde{x}$-line. The flow of the above limit vector field is approximated arbitrarily closely by the discrete dynamical system of type

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g)^{(m)} \circ \tilde{\phi}(f)^{(n)}, \qquad m = 0, 1, \ldots$$

with a sufficiently large $n > 0$ ([17]).

Similarly the $\tilde{\psi}(f), \tilde{\psi}(g)$ define the vector field $\frac{iL}{j}K^{\frac{i-j}{i}}\partial/\partial\tilde{y}$ on the $\tilde{y}$-line. The lift $\tilde{h}_+ : \tilde{x} - \text{line}, \infty \to \tilde{y} - \text{line}, \infty$ of the restiction h_+ of h to $\mathbb{R}^+$ sends the orbit of

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g)^{(m)} \circ \tilde{\phi}(f)^{(n)}$$

to that of

$$\tilde{\psi}(f)^{(-n)} \circ \tilde{\psi}(g)^{(m)} \circ \tilde{\psi}(f)^{(n)}.$$

Therefore $\tilde{h}_+$ is compatible with the above flows respecting time hence it is a translation by a constant α_+ (see [17] for a detailed argument) and

$$h_+(x) = \lambda'^{(-1)} \circ \xi_{i,A'}^{-1}(\xi_{i,A} \circ \lambda(x) + \alpha_+),$$

which is i-flat at 0. Similarly we can show that the restriction h_- of h to $\mathbb{R}^-$ is of the form

$$h_-(x) = \lambda'^{(-1)} \circ \xi_{i,A'}^{-1}(\xi_{i,A} \circ \lambda(x) + \alpha_-),$$

with a constant α_-, which is i-flat at 0. With both the above smoothness of h_+ and h_-, we see that the linking homeomorphism h is a C^i-smooth diffeomorphism on a neighbourhood of 0 and i-flat at 0.

Proposition 9. *The linking homeomorphism h is C^∞-smooth on a neighbourhood of 0.*

Proof. Since $\phi(G)_0$ is non solvable, the i can be chosen arbitrary large. Therefore h is C^∞-smooth at 0. The smoothness off 0 is clear by the form of $h_\pm$ above presented.

By the proposition $\phi(f)$ and $\psi(f)$ are C^∞-conjugate. Since the residues A, A' are invariant under formal conjugacy relation of germs of analytic diffeomorphisms, we obtain$A = A'$ hence $\tilde{\phi}(f) = \tilde{\psi}(f)$ and

$$\begin{cases} \lambda' \circ h_+ \circ \lambda^{(-1)} = \exp \frac{-\alpha_+}{i}\chi & \text{on} \quad \mathbb{R}^+ \\ \lambda' \circ h_- \circ \lambda^{(-1)} = \exp \frac{-\alpha_-}{i}\chi & \text{on} \quad \mathbb{R}^-, \end{cases}$$

where χ denotes $\frac{x^{i+1}}{1+Ax^i}\partial/\partial x$.

Proposition 10. *$\alpha_+ = \alpha_-$ and the germ of h at 0 is unique.*

Proof. Since $h_+^{(-1)} \circ \phi(g) \circ h_+ = \psi(g)$ and $h_-^{(-1)} \circ \phi(g) \circ h_- = \psi(g)$ hold on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively at 0, we obtain the formal equalities

$$\lambda^{(-1)} \circ \exp \frac{\alpha_+}{i}\chi \circ \lambda' \circ \phi(g) \circ \lambda'^{(-1)} \circ \exp \frac{-\alpha_+}{i}\chi \circ \lambda = \phi(f)$$

and

$$\lambda^{(-1)} \circ \exp \frac{\alpha_-}{i}\chi \circ \lambda' \circ \phi(g) \circ \lambda'^{(-1)} \circ \exp \frac{-\alpha_-}{i}\chi \circ \lambda = \phi(f).$$

This shows that $\lambda'^{(-1)} \circ \exp \frac{\alpha_+ - \alpha_+}{i}\chi \circ \lambda$ commutes with $\phi(g)$, and by formal calculation, it follows $\alpha_+ = \alpha_- = \alpha$ (since $i \neq j$). Therefore $h = \lambda^{(-1)} \circ \exp \frac{\alpha}{i}\chi \circ \lambda'$.

Next assume $h' = \lambda^{(-1)} \circ \exp\ -\frac{\beta}{i}\chi \circ \lambda'$ satisfies $h'^{(-1)} \circ \phi(g) \circ h' = \psi(g)$. Then it follows $\alpha = \beta$ from a similar argument. This shows the uiqueness of h.

By a result due to Cerveau and Moussu [2], a formal conjugacy is convergent to give a real analytic conjugacy for non solvable groups of germs of diffeomorphisms. Therefore the Taylor series of h at 0 is convergent to an analytic diffeomorphism $\tilde{h}$ linking $\phi(G)_0$ to $\psi(G)_0$. Then the uniqueness of the linking homeomorphism (Proposition 10) asserts that the germ of h is nothing but the $\tilde{h}$ real analytic on a neighbourhood of 0. The analyticity propagates to whole $B_{\phi(G)}$ by the same argument in the proof of Theorem 1 in §6. This completes the proof of Theorem 3 for the case $(i, j) = (i', j')$ and h is orientation preserving.

Now we prove the theorem for general non solvable pseudogroups. Assume that $\phi(f), \phi(g)$ and $\psi(f), \psi(g)$ have the orders of flatness i, j and i', j' respectively. By Proposition 7, we may write $i'/i = j'/j = p/q$ with even positive integers p, q. Define the lift $\phi_p^\epsilon : G \to \Gamma_+^\omega$ by $\phi_p^\epsilon(f) : U_{\phi_p^\epsilon(f)} \to \phi_p^\epsilon(f)(U_{\phi_p^\epsilon(f)})$, $\phi_p^\epsilon(f)(x) = (\epsilon\phi(f)(\epsilon x^p))^{1/p}$ for $\epsilon = \pm 1$, where $U_{\phi\epsilon_p(f)}$ is the preimage of $U_{\phi(f)}$ by $x :\longrightarrow \epsilon x^p$. Define the lift $\psi_q^\epsilon : G \to \Gamma_+^\omega$ similarly. Then $\phi_p^\epsilon(f), \phi_p^\epsilon(g)$ have the orders of flatness pi, pj respectively. The linking homeomorphism h lifts to the orientation preserving homeomorphism $K^\epsilon = (\epsilon\ h(\epsilon x^p))^{1/q}$ of $U_p^\epsilon = \{x|\ \epsilon x^p \in U\}$ to $U_q^\epsilon = \{y|\ \epsilon y^q \in h(U)\}$, which is linking ϕ_p^ϵ to ψ_q^ϵ for $\epsilon = \pm 1$.

Proposition 11. *(1) ϕ is solvable if and only if ϕ_p^1 is solvable if and only if ϕ_p^{-1} is solvable.*

(2) $B_{\phi_p^\epsilon} = \{x|\ \epsilon x^p \in B_\phi\}$ *for* $\epsilon = \pm 1$.

Proof. The homomorphism of pseudogroups which asigns $\phi_p^\epsilon(f)$ to $\phi(f)$ for $f \in G$ induces a group isomorphism of the germs $\phi(G)_0$ to $\phi_p^\epsilon(G)_0$ for $\epsilon = \pm 1$. So Statement (1) is clear. Statement (2) for the basin follows from the definition.

By the result obtained previously in this section, the lift K^ϵ is a unique real analytic diffeomorphism. In particular h is unique and the restriction $h : B_\phi(G) - 0 \to B_\psi(G) - 0$ is a real analytic diffeomorphism. This completes the proof of Theorem 3 for non solvable pseudogroups.

5. Proof of Theorem 3 for solvable pseudogroups

Theorem 12 ([17]). *A solvable subgroup H of the group of germs of analytic diffeomorphisms of* $\mathbb{R}$ *respecting 0 is* C^ω*-conjugate with one of the following:*

(1) H consists of linear functions ax *with the coefficients* a *in a subgroup* L *of* $\mathbb{R}^*$.

(2) H consists of $f^{(\alpha)} = x + \alpha K x^{i+1} + \cdots, \alpha \neq 0$ *with* α *in a subgroup* $\Lambda \subset \mathbb{R}$, $1 \in \Lambda$. *Here* $f \in H, f(x) = x + Kx^{i+1} + \cdots$ *and* $f^{(\alpha)}$ *is the unique real analytic diffeomorphism with the Taylor expantion* $f^{(\alpha)}(x) = x + \alpha K x^{i+1} + \cdots$ *such that* $f^{(\alpha)} \circ f = f \circ f^{(\alpha)} = f^{(\alpha+1)}$. *If* Λ *is dense in* $\mathbb{R}$, *those* $f^{(\alpha)}$ *are written as exp* $\alpha\chi$ *with an i-flat real analytic vector field* χ *on* $\mathbb{R}$. *(for the definition of the α-times iteration* $f^{(\alpha)}$ *see the papers [17.19].)*

(3) H consists of those $f^{(\alpha)}$ *and* $-f^{(\alpha+\beta)}$ *with* $\alpha \in \Lambda \subset \mathbb{R}$ *and a* $\beta, 2\beta \in \Lambda$ *and* f *satisfies the relation* $f(-x) = -f(x)$.

(4) H consists of those f^α *in (2) and* $af^{(\alpha+\beta(a))}$ *with* a *in a subgroup* $L \subset \mathbb{R}^*, a^i \neq 1$. *Here* f *satisfies the relation* $a^{-1}f(ax) = f^{(a^i)}$ *for* $a \in L$ *and* $\beta : L \to \mathbb{R}$ *is a function and* $\mathrm{res}(f) = 0$. *i.e.* f *is formally and* C^∞*-conjugate with exp* $Kx^{i+1}\partial/\partial x, K \neq 0$.

In Cases (1),(2) and (3), the H is commutative. and in Case (4), H is non commutative but solvable.

Since the members of our pseudogroups $\phi(G), \psi(G)$ are all orientation preserving, the germs $\phi(G)_0, \psi(G)_0$ are C^ω-conjugate to one of the H in Cases (1),(2) and (4). In the following we assume the germs are of the form in those cases and prove the the analyticity of the restrictions h_+, h_- of the linking homeomorphism h to $\mathbb{R}^+, \mathbb{R}^-$ on a neighbourhood of 0. The differentiability propagates to whole $B_{\phi(G)} - 0$ by the same argument as in the proof of theorem 1 in §6.

Case (1). Assume $\phi(G)_0 \neq \mathbb{Z}$. This assumption is equivalent to that the linear term group L_ϕ of $\phi(G)_0$ is a dense subgroup of $\mathbb{R}^*$, in other words, all orbits are dense nearby 0. Let $\log L_\phi$ denote the subgroup of $\mathbb{R}$ consisting of the logarithms of the linear terms of $\phi(f), f \in G$. Since h sends the $\phi(G)$-orbit of an x to the $\psi(G)$-orbit of $h(x)$, h induces a homomorphism $\tilde{h}$ of the subgroups $\log L_\phi$ to $\log L_\psi$, which extends to a linear function kx. By this form we see $\log \circ h \circ \exp(x)$ is an affine transformation $kx + l$, from which $h(x) = (\exp l)x^k$ for $x > 0$. A similar argument shows the analyticity of h_-.

Case (2). In this case the germs of $\phi(f)^{(\alpha)}$ are of the form $\exp \alpha\chi$ with a flat analytic vector field χ and α in a subgroup $\Lambda \subset \mathbb{R}$. The hypothesis that $\phi(G)_0$ is not isomorphic to $\mathbb{Z}$ implies that Λ is a dense subgroup. Let $\Lambda' \subset \mathbb{R}$ be the group associated to $\psi(G)$. The correspondence of $\phi(G)$-orbits and $\psi(G)$-orbits in $\mathbb{R}^+$ by h induces a linear transformation of Λ to Λ', which describes the h conversely. Therefore the h_+ is real analytic off 0, and similarly it is shown that h_- is analytic off 0.

Case (4). Let $\phi(G)_0^0 \subset \phi(G)_0$ denote the subgroup consisting of the i-flat germs of diffeomorphisms $\phi(f)^{(\alpha)}, \alpha \in \Lambda \subset \mathbb{R}$ of $\phi(G)$, and $\psi(G)_0^0 \subset \psi(G)_0$ the subgroup consisting of j-flat germs of diffeomorphisms $\psi(f)^{(\alpha)}, \alpha \in \Lambda \subset \mathbb{R}$. It suffices here to prove the analyticity of h for the case $i = j$.

Lemma 13. *Let $\phi(f), \psi(f) : \mathbb{R}, 0 \to \mathbb{R}, 0$ be germs of analytic diffeomorphisms with the linear term x and the order of flatness $i \geq 1$, and let $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a germ of homeomorphism such that $h \circ \phi(f) = \psi(f) \circ h$. Then h is differentiable at 0.*

Proof. By C^∞- coordinate change we may assume $\phi(f) = \exp - \frac{K}{i}\frac{x^{i+1}}{1+Ax^i}\partial/\partial x$ and $\psi(f) = \exp - \frac{L}{i}\frac{x^{i+1}}{1+Bx^i}\partial/\partial x$, and by a linear coordinate transformation, $K = L > 0$. These diffeomorphisms lift to the translations by K respectively on the $\tilde{x}$-line, $\tilde{x} = \xi_{i,A}(x) = x^{-i} + A\log x^{-i} (x > 0)$, and the $\tilde{y}$-line, $\tilde{y} = \xi_{i,B}(y)$. And these translations are conjugate by the lift $\tilde{h} : \tilde{x} - line \to \tilde{y} - line$ of h . So we obtain an extimate $|\tilde{h}(\tilde{x}) - \tilde{x} - T| \le K$, with a constant T, from which

$$\xi_{i,B}^{-1}(\xi_{i,A}(x) + T + K) \le h(x) \le \xi_{i,B}^{-1}(\xi_{i,A}(x) + T - K)$$

This implies the differentiability of h at 0.

Next let $\phi(g)(x) = ax + \cdots$, $a \ne 0, 1$ be a diffeomorphism non commutative with $\phi(f)$ and $\psi(g)(x) = a'x + \cdots$ $a' \ne 0, 1$. By assumption $\psi(g) \circ h = h \circ \phi(g)$ holds, and by the differentiability of h at 0, we obtain $a = a'$.

Lemma 14. *Let $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be the germ of a mapping commutating with a linear function ax. If h is differentiable at 0, h is linear.*

Proof. By the commutativity, $h(a^i x)/a^i x = h(x)/x$ for all x and $i = 0, 1, \ldots$. By the differentiability, $h(x)/x$ is a constant independent of x.

By the Poincaré linearization theorem $\phi(g), \psi(g)$ are analytically conjugate with ax. Here Lemma 14 applies to say that the germ of h at 0 is linear. In this situation the relation $h \circ \phi(f) = \psi(f) \circ h$ admits the unique linear map h. This completes the proof of Theorem 3.

6. Proof of Theorem 1and Corollaries 2,4

Proof of Theorem 1. Let L be a leaf of $\mathcal{F}_1$ with holonomy group $\ne 0, \mathbb{Z}$. Then the image $h(L)$ has holonomy isomorphic to that of L and, by Theorem 4, h is transversely analytic on a deleted neighbourhood $U - L$ of an $x \in L$. Let $x' \in M_1$ be an arbitrary point. The leaf $L_{x'}$ of $\mathcal{F}_1$ containing x' is dense by assumption, hence a point $x'' \in L_{x'}$ is contained in $U - L$. Clearly the translation $T_{x',x''}$ along a path in $L_{x'}$ sending the transverse section at x' to that of x'' is analytic, and the germs of h at x', x'' link the $T_{x',x''}$ to the transverse dynamics $T_{h(x'),h(x'')}$ along $h(L_{x'}) = L_{h(x')}$. Therefore the transverse analyticity of h at x'' induces the transverse analyticity on a neighbourhood of x'. This completes the proof of Theorem 1.

Proof of Corollary 2. The Godbillon-Vey class $\mathrm{GV}(\mathcal{F})$ of $\mathcal{F}$ may be defined by the pull back $\rho(\mathcal{F})^*c$ of a cocycle $c \in H^3(B\Gamma_{\mathbb{R}}^{\infty}, \mathbb{R})$ of the classifying space $B\Gamma_{\mathbb{R}}^{\infty}$ of the pseudogroup $\Gamma_{\mathbb{R}}^{\infty}$ of orientation preserving C^{∞}-diffeomrphisms of open subsets of $\mathbb{R}$ by the classifying map $\rho(\mathcal{F}) : M \longrightarrow B\Gamma_{\mathbb{R}}^{\infty}$ ([1]). Since $h(\mathcal{F}) = \mathcal{F}'$ and h is transversely real analytic, if follows $\rho(\mathcal{F}') \circ h = \rho(\mathcal{F}')$, from which $\mathrm{GV}(\mathcal{F}) = h^*\mathrm{GV}(\mathcal{F}')$. This completes the proof of Corollary 2.

Proof of Corollary 4. Let $\phi, \psi : \Gamma^g \longrightarrow \mathrm{Diff}_+^{\omega}(S^1)$ be homomorphisms and $h : \phi \longrightarrow \psi$ a linking homeomorhism. Let $\mathrm{stab}(x_0) \subset \Gamma^g$ be the stabiliser of an $x_0 \in S^1$. Then h links the restriction of ϕ to $\mathrm{stab}(x_0)$ to that of ψ. Assume that $\phi(\mathrm{stab}(x_0))$ is not isomorphic to $\mathbb{Z}$ and non trivial. Then by the rigidity theorem (Theorem 3), h is a real analytic diffeomorphism on a deleted neighbourhood $U - x_0$ of x_0 in S^1. By a result due to Ghys [6], if $|\mathrm{eu}(\phi)| \neq 0$, all orbits are dense in S^1. So, for any $y \in S^1$, there is a $g \in G$ such that $\phi(g)(y) \in U - x_0$. Then the equality $h \circ \phi(g) = \psi(g) \circ h$ implies that h is a real analytic diffeomorphism at y. This completes the proof of Corollary 4.

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