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MARIO BONK
JUHA HEINONEN
PEKKA KOSKELA

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UNIFORMIZING GROMOV HYPERBOLIC SPACES

Mario Bonk
Juha Heinonen
Pekka Koskela

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M. Bonk
Institut für Analysis, Technische Universität Braunschweig, D-38106 Braunschweig, Germany.
Current address: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.
E-mail : M.Bonk@tu-bs.de

J. Heinonen
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

P. Koskela
Department of Mathematics, University of Jyväskylä, P.O. Box 35,
FIN-40351 Jyväskylä, Finland.
E-mail : pkoskela@math.jyu.fi

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Dedicated to Jussi Väisälä

UNIFORMIZING GROMOV HYPERBOLIC SPACES

Mario Bonk, Juha Heinonen, Pekka Koskela

Abstract. — The unit disk in the complex plane has two conformally related lives: one as an incomplete space with the metric inherited from $\mathbb{R}^2$, the other as a complete Riemannian 2-manifold of constant negative curvature. Consequently, problems in conformal analysis can often be formulated in two equivalent ways depending on which metric one chooses to use. The purpose of this volume is to show that a similar choice is available in much more generality. We shall replace the incomplete disk by a uniform metric space (defined as a generalization of a uniform domain in $\mathbb{R}^n$) and the space of constant negative curvature by a general Gromov hyperbolic space. We then prove that there is a one-to-one correspondence between quasiisometry classes of (proper, geodesic, and roughly starlike) Gromov hyperbolic spaces and the quasisimilarity classes of bounded locally compact uniform spaces. We study Euclidean domains that are Gromov hyperbolic with respect to the quasihyperbolic metric and the Martin boundaries of such domains. A characterization of planar Gromov hyperbolic domains is given. We also study quasiconformal homeomorphisms of Gromov hyperbolic spaces of bounded geometry; under mild conditions on the spaces we prove that such maps are rough quasiisometries. We employ a version of the classical Gehring-Hayman theorem, and methods from analysis on metric spaces such as modulus estimates on Loewner spaces.

Résumé **(Uniformisation des espaces hyperboliques de Gromov)**

On peut considérer le disque unité dans le plan complexe de deux façons différentes : comme un espace incomplet si on le munit de la métrique euclidienne de $\mathbb{R}^2$, et comme un espace complet s'il est équipé d'une métrique de courbure négative constante. Par conséquent, on peut souvent formuler des problèmes d'analyse conforme de deux manières différentes, suivant la métrique que l'on choisit d'utiliser. L'objet de ce volume est de montrer qu'un choix semblable est possible de manière beaucoup plus générale. On remplace le disque incomplet par un espace uniforme (défini comme une généralisation d'un domaine uniforme dans $\mathbb{R}^n$) et l'espace de courbure négative constante par un espace hyperbolique au sens de Gromov. On montre ensuite qu'il y a une correspondance univoque entre les classes de quasi-isométrie des espaces hyperboliques (qui sont de plus propres, géodésiques et grossièrement étoilés) et les classes de quasi-similitudes des espaces uniformes qui sont bornés et localement compacts. Nous étudions les domaines euclidiens munis de la métrique quasi-hyperbolique qui sont hyperboliques au sens de Gromov, et les frontières de Martin de ces domaines. On donne une caractérisation de domaines hyperboliques dans le plan. Nous étudions aussi les homéomorphismes quasi-conformes entre des espaces hyperboliques qui satisfont à une condition de géométrie bornée ; sous des hypothèses modérées, on démontre que les applications comme ci-dessus sont des quasi-isométries au sens large. Nous utilisons une version du théorème classique de Gehring-Hayman, et des méthodes d'analyse sur les espaces métriques comme des estimations de module dans les espaces de Loewner.

CONTENTS

CHAPTER 1

INTRODUCTION

The unit disk in the complex plane has two conformally related lives: one as an incomplete space with the metric inherited from $\mathbb{R}^2$, the other as a complete Riemannian 2-manifold of constant negative curvature. Consequently, problems in conformal analysis often can be formulated in two equivalent ways, depending on which metric one chooses to use. The purpose of this paper is to show that a similar choice is available in much more generality. We shall replace the incomplete disk by a uniform metric space (to be defined below as a generalization of a uniform domain in $\mathbb{R}^n$) and the space of constant negative curvature by a general Gromov hyperbolic metric space. We then have the following theorem:

Theorem 1.1. — *There is a one-to-one (conformal) correspondence between the quasiisometry classes of proper geodesic roughly starlike Gromov hyperbolic spaces and the quasisimilarity classes of bounded locally compact uniform spaces.*

The terminology of this introduction will be explained in the course of the paper. The proof of Theorem 1.1 is given in Chapter 4. It relies heavily on the ideas around the Gehring-Hayman theorem, which is discussed in Chapter 5. (We believe that our study of the Gehring-Hayman theorem in Chapter 5 has independent merit.) In subsequent chapters, we shall give several applications, old and new, of Theorem 1.1. Besides the Gehring-Hayman theorem, the paper contains other related studies of independent interest, as will be discussed later in this introduction.

A metric space is called *proper* if its closed balls are compact; it is called *geodesic* if every pair of points in it can be joined by a *geodesic*, that is, by a curve whose length equals the distance between the points. A geodesic metric space is *Gromov hyperbolic* if every geodesic triangle in it is *δ-thin* for some fixed $\delta \geqslant 0$. We also use the term *δ-hyperbolic* in this case. See **[Gr2]**, **[GhHa**, pages 16, 41, and 60], and Chapter 3 below for precise definitions.

Gromov hyperbolic spaces form a large and much studied class of metric spaces. They include all complete simply-connected Riemannian manifolds whose sectional curvature is everywhere less than a negative constant. On the other hand, an arbitrary complete manifold of constant negative curvature need not be Gromov hyperbolic; certain planar Riemann surfaces serve as easy examples to this effect. (Compare Theorems 1.12 and 1.13 below.)

The starting point for this paper was our desire to understand what Euclidean domains are Gromov hyperbolic when equipped with the quasihyperbolic metric. The *quasihyperbolic metric* k_Ω in a domain Ω of $\mathbb{R}^n$ is obtained by using the continuous density

$$\operatorname{dist}(x, \partial\Omega)^{-1}. \tag{1.2}$$

Thus,

$$k_\Omega(a,b) = \inf \int_\gamma \frac{|dx|}{\operatorname{dist}(x,\partial\Omega)}, \tag{1.3}$$

where the infimum is over all rectifiable curves γ joining the points a and b in Ω. The resulting metric space (Ω, k_Ω) is complete, proper, and geodesic **[GePa]**, **[GO]**.

For the record, in this paper a *domain* in $\mathbb{R}^n$ means an open connected set with nonempty complement in $\mathbb{R}^n$. We always assume $n \geqslant 2$.

If Ω is a plane domain whose complement in $\mathbb{R}^2$ has at least two points, then Ω admits a metric of constant negative curvature. This hyperbolic metric is obtained by a conformal change by using a density $\rho_\Omega(z)$ which in many cases, although not always, is comparable to the density (1.2), that is,

$$\rho_\Omega(z) \simeq \operatorname{dist}(z, \partial\Omega)^{-1}. \tag{1.4}$$

In fact, (1.4) holds precisely when the boundary $\partial\Omega$ is *uniformly perfect*. See **[BePo]**.

Recall that a domain Ω in $\mathbb{R}^n$ is said to be *uniform*, or *A-uniform*, where $A \geqslant 1$, if every pair of points $a, b \in \Omega$ can be joined by an arc γ so that

$$\operatorname{length}\gamma \leqslant A|a-b| \tag{1.5}$$

and, for each $x \in \gamma$,

$$\min\{\operatorname{length}\gamma(a,x),\ \operatorname{length}\gamma(x,b)\} \leqslant A \operatorname{dist}(x,\partial\Omega), \tag{1.6}$$

where $\gamma(z,w)$ denotes the subarc of γ between z and w.

Uniform domains were introduced independently by Martio and Sarvas **[MS]** and Jones **[J]**, and they have become the "nice domains" of analysis appearing in surprisingly many contexts **[Ge]**. In this paper, we are trying to explain uniformity through negative curvature.

Let (Ω, d) be a locally compact, *rectifiably connected* noncomplete metric space. The second assumption means that every pair of points in Ω can be joined by a rectifiable curve, where *curve* means a continuous map $\gamma\colon I \to \Omega$ from an interval

$I \subset \mathbb{R}$ to Ω. The *length* $\ell(\gamma)$ of a curve γ is defined in an obvious way, and the line integral

$$\int_\gamma \rho\, ds$$

is well defined over every locally rectifiable curve γ, and for every nonnegative Borel function ρ. Here the parameter interval I is allowed to be open or half-open, and a curve is said to be locally rectifiable if all its closed subcurves have finite length. (See the Appendix for a more detailed discussion.)

Given a real number $A \geqslant 1$, a curve $\gamma\colon [0,1] \to \Omega$ is called an *A-uniform curve* if

$$\ell(\gamma) \leqslant A\, d(\gamma(0), \gamma(1)) \tag{1.7}$$

and

$$\min\{\ell(\gamma|[0,t]), \ell(\gamma|[t,1])\} \leqslant A \operatorname{dist}(\gamma(t), \partial\Omega) \tag{1.8}$$

for all $t \in [0,1]$. Here the boundary $\partial\Omega$ is by definition the set

$$\partial\Omega = \overline{\Omega} - \Omega,$$

where $\overline{\Omega}$ denotes the completion Ω; by assumption the boundary $\partial\Omega$ is nonempty. The definition for a uniform curve extends, in an obvious way, to a situation where the parameter interval is open or half open. If γ is an embedding of the parameter interval, it is also called a *uniform arc.*

Definition 1.9. — A locally compact, rectifiably connected noncomplete metric space is called *A-uniform* if every pair of points in it can be joined by an *A-uniform* curve.

We also use the term uniform space, uniform curve, etc. if the parameter A need not be emphasized.

If Ω is a domain in $\mathbb{R}^n$ with the Euclidean metric, then the definition given in 1.9 agrees with the one given above for uniform domains; if instead we choose the metric

$$\ell_\Omega(a,b) = \inf \operatorname{length} \gamma, \tag{1.10}$$

where γ is a curve joining a and b in Ω, and length means Euclidean length, then we call Ω an *inner A-uniform* domain.

Clearly, every uniform domain is inner uniform. If Ω is the open unit disk in the plane minus the line segment [0,1], then Ω is inner uniform but not uniform.

Inner uniform domains in the plane were studied by Balogh and Volberg [**BV1**], [**BV2**] in connection with complex iteration. They called these domains *uniformly John.* (Inner uniform domains form a strict subclass of John domains.) For a thorough discussion of inner uniformity and related concepts, see a recent paper by Väisälä [**V3**].

We shall use the term *Gromov hyperbolic domain* for those domains in $\mathbb{R}^n$, or in $\overline{\mathbb{R}}^n$, that are Gromov hyperbolic in the quasihyperbolic metric. Throughout this paper, the metric notions for domains in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 2$, are understood in terms of the spherical metric (see Chapter 7). It turns out that (inner) uniform domains are Gromov hyperbolic, and that uniform domains can be characterized in terms of Gromov hyperbolicity:

Theorem 1.11. — *Inner uniform domains in $\mathbb{R}^n$ are Gromov hyperbolic. Moreover, a bounded domain in $\mathbb{R}^n$ is uniform if and only if it is both Gromov hyperbolic and its Euclidean boundary is naturally quasisymmetrically equivalent to the Gromov boundary.*

In the second assertion of Theorem 1.11, the assumption that the domain be bounded is simply for convenience. For unbounded domains one could either define the length in (1.10) by using the spherical metric in $\overline{\mathbb{R}}^n$, or replace the quasisymmetric equivalence by quasimöbius equivalence **[V1]**. (Note that the Gromov boundary of a proper and geodesic space is always a compact.) The first assertion and the necessity part of the second assertion in Theorem 1.11 are proved in Chapter 3; the sufficiency part is proved in Chapter 7, where we consider, more generally, domains in $\overline{\mathbb{R}}^n$.

We do not know if there is a result similar to the second assertion in Theorem 1.11 for inner uniform domains.

Quasiconformal mappings between Euclidean domains are rough quasiisometries in the quasihyperbolic metrics **[GO]**. Thus each quasiconformal image of a Gromov hyperbolic domain is again Gromov hyperbolic **[GhHa]**. In particular, there are Gromov hyperbolic domains that are not inner uniform, e.g. some simply connected proper subdomains of the plane.

Theorem 1.1 asserts that every (proper, geodesic, roughly starlike) Gromov hyperbolic space arises as a conformal image of a (bounded) uniform space. This uniformizing procedure turns the unbounded Gromov hyperbolic space into a bounded space with nice internal geometry. We hope that this point of view will be useful in understanding the large scale geometry of Gromov hyperbolic spaces, in particular, the structure of the boundary of a given Gromov hyperbolic space. The idea of a conformal change of metric in this context is of course not new. In recent times, the idea has been exploited by Floyd **[Fl]**, Gromov **[Gr2]**, **[Gr3]**, Gromov and Pansu **[GrPa]**, and others.

If one starts with a Gromov hyperbolic domain in $\mathbb{R}^n$, it is reasonable to ask if our uniformizing procedure produces a uniform domain in $\mathbb{R}^n$, perhaps up to an additional quasiconformal change in the metric. (Note that there are no other conformal maps in $\mathbb{R}^n$ for $n \geqslant 3$ than the Möbius transformations.) This is an interesting question, which we have not been able to answer, except in dimension $n = 2$:

Theorem 1.12. — *Gromov hyperbolic domains on the 2-sphere are precisely the conformal images of inner uniform slit domains.*

Recall that a subdomain of the Riemann sphere is called a *slit domain* if it contains ∞ and the components of its complement are either points or compact line segments parallel to a given direction. It would be interesting to know if there is a characterization of Gromov hyperbolic domains in $\mathbb{R}^n$, $n \geqslant 3$, akin to Theorem 1.12.

We conjecture that Theorem 1.12 can be improved as follows: *Gromov hyperbolic domains on the 2-sphere are precisely the conformal images of uniform circle domains.*

Recall that a domain on the Riemann sphere is called a *circle domain* if its boundary components are all round circles or points. Koebe's *Kreisnormierungsproblem* predicts that every domain in the extended plane is conformally equivalent to a circle domain. Thus, the above conjecture is just a special case of Koebe's conjecture, but in the stronger form where the circle domain is asked to be uniform.

Koebe's Kreisnormierungsproblem was solved for "collared" uniform domains by Herron [**Her**], and for arbitrary uniform domains by Herron and Koskela [**HerK**]. He and Schramm [**HS1**] established the conjecture for domains with countably many boundary components. For yet further results, see [**HS2**], [**Sc**]. It follows from Proposition 7.12 below that the above conjecture is true if Koebe's problem has an affirmative answer. In particular, in view of the He-Schramm theorem, the conjecture is true for domains with countably many boundary components.

We can verify our conjecture in one more special case, and in a slightly stronger form:

Theorem 1.13. — *A domain in $\overline{\mathbb{R}}^n$ with totally disconnected complement is Gromov hyperbolic if and only if it is uniform.*

MacManus [**M**] proved recently that the complement of a compact totally disconnected set K on the 2-sphere is uniform if and only if K lies on a quasicircle and is *uniformly disconnected.* This result together with Theorem 1.13 gives a simple geometric criterion for Gromov hyperbolicity of domains on the 2-sphere with totally disconnected complement. We do not know if there is a simple geometric characterization of Gromov hyperbolic domains in general. (See however a conjecture at the end of Chapter 7.)

Theorems 1.1, 1.11, 1.12, and 1.13 will be studied and reformulated more quantitatively in the ensuing sections.

Our proof for Theorem 1.12 in Chapter 6 is surprisingly indirect, using among other things the theory of modulus and Loewner spaces as developed recently in [**HeiK**], plus techniques from harmonic analysis. We do not know of an elementary proof for Theorem 1.12.

We note that Väisälä (partially in joint work with Alestalo) has extensively studied uniform domains in infinite-dimensional Banach spaces. See [**V4**] and the references there. Although the quasihyperbolic metric and Gromov hyperbolicity make sense in these spaces, our methods largely will not work due to the lack of local compactness. It would be interesting to know if there is a general relationship between uniformity and Gromov hyperbolicity that would cover infinite-dimensional situations as well.

In Chapter 6, en route to Theorem 1.12, we obtain results of independent interest. For example, we prove that uniform subdomains of an Ahlfors Q-regular Q-Loewner space are themselves Q-Loewner. (We refer to Chapter 6 for the terminology.) As a special case of this theorem we obtain that uniform domains in so-called *Carnot groups* are Q-Loewner spaces, where Q is the homogeneous dimension of the group. Such results have previously been proved only by relatively complicated Sobolev extension arguments. (In this connection, note that it is a nontrivial issue to prove that there exist uniform domains at all in Carnot groups; see **[CG]**, **[VG]**. For Sobolev extension results, see **[GN]**, **[N]**.) Even in $\mathbb{R}^n$, we do not know of a previous proof for the fact that uniform domains are Loewner spaces that would not use the Jones extension theorem; see **[J]**, **[GM]**. In any case, we obtain that inner uniform domains in $\mathbb{R}^n$ are n-Loewner in the inner metric of the domain; this result cannot be proved by extension methods.

Finally, we give applications of our studies first to potential theory on Euclidean domains and then to local-to-global phenomena for quasiconformal maps.

Theorem 1.14. — *If Ω is a Gromov hyperbolic domain in $\mathbb{R}^n$ and if the complement $\mathbb{R}^n \setminus \Omega$ satisfies a capacity density condition, then the Gromov and Martin boundaries of Ω are homeomorphic.*

The definition for the Martin boundary of a domain Ω is recalled below in Chapter 8, where Theorem 1.14 is proved (under slightly weaker hypotheses). For the capacity density condition, see (8.12).

Theorem 1.14, when coupled with the results on uniform spaces and hyperbolicity (Theorem 1.11 in particular), appears to give stronger conclusions than what can be found in the literature. Jerison and Kenig proved in **[JK]** that the Martin boundary of an *NTA-domain* is homeomorphic to the Euclidean boundary of the domain. Now an *NTA*-domain is a bounded uniform domain whose complement satisfies the capacity density condition, whence the Jerison-Kenig result follows from Theorems 1.11 and 1.14. What is new here is that the Martin boundary can be identified with the Euclidean inner boundary in case of an inner uniform domain (with complement satisfying a capacity density condition).

It follows, too, that a quasiconformal mapping of a plane Gromov domain (with complement satisfying a capacity density condition) onto another plane domain induces a homeomorphism between the respective Martin boundaries. This need not be true in general (see **[ST]**, **[Seg]**). A version of this latter observation is valid in higher dimensions as well, only the capacity density condition on the complement is not a quasiconformal invariant (it is for $n = 2$, see **[Po**, p. 302**]**).

Theorem 1.14 for hyperbolic graphs was proved by Ancona **[A3]** and we follow his ideas. It is through Theorem 1.11 that we obtain new results. It is possible that at least some of the consequences, e.g. those for John disks **[NV]**, were known to the experts. In lack of a precise reference, we include a rather detailed proof of

Theorem 1.14, to the extent it deviates from the arguments given by Ancona in [**A2**], [**A3**]. Finally, we point out that Theorem 1.14 contains as a special case a result of Arai [**Ar**] about the Martin boundary of strictly pseudoconvex domains in $\mathbb{C}^n$ (see Chapter 8).

Next we state an application to quasiconformal mapping theory.

Theorem 1.15. — *Let $f\colon X \to Y$ be a quasiconformal homeomorphism between two n-dimensional Hadamard manifolds, $n \geqslant 2$, that have Ricci bounded geometry. If X and Y are both Gromov hyperbolic, then $f\colon X \to Y$ is a rough quasiisometry and hence extends to a quasisymmetric homeomorphism $\partial_G X \to \partial_G Y$.*

We shall prove Theorem 1.15 in Chapter 9 below, where also the terminology is explained. Theorem 1.15 contains as a special case a result of Pansu [**Pa**, Corollary 4] which asserts a similar conclusion for quasiconformal diffeomorphisms between two Hadamard manifolds of pinched negative sectional curvature. On the other hand, Pansu derives his result as a corollary to a theorem of Gromov that uses an isoperimetric inequality argument; the relationship between Gromov's result and Theorem 1.15 is less clear.

In Chapter 9, we formulate and prove a yet more general result from which Theorem 1.15 follows. In particular, X and Y need not be manifolds there.

To finish this introduction, we define some classes of maps that will be important to our study. A homeomorphism $f\colon (X,d) \to (X',d')$ between two metric spaces is a *quasiisometry*, or an *L-quasiisometry*, if $L \geqslant 1$ and if

$$\frac{1}{L}\, d(x,y) \leqslant d'(f(x),f(y)) \leqslant L\, d(x,y) \tag{1.16}$$

for all $x, y \in X$. Thus our quasiisometries are the same as bi-Lipschitz homeomorphisms, and we warn the reader that many texts, e.g. [**GhHa**], use different terminology.

We call an arbitrary, not necessarily continuous, map $f\colon (X,d) \to (X',d')$ a *rough quasiisometry*, or an *(L,M)-rough quasiisometry*, if $L \geqslant 1$, $M \geqslant 0$, if

$$\frac{1}{L}\, d(x,y) - M \leqslant d'(f(x),f(y)) \leqslant L\, d(x,y) + M \tag{1.17}$$

for all $x, y \in X$, and if every point in X' lies within distance M from $f(X)$.

A homeomorphism $f\colon (M,d) \to (M',d')$ between two metric spaces is *quasisymmetric*, or *η-quasisymmetric*, if $\eta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism such that

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right) \tag{1.18}$$

for all triples of distinct points x, y, z in M. For example, a homeomorphism of the standard n-sphere, $n \geqslant 2$, to itself is quasisymmetric if and only if it is quasiconformal. Observe that the inverse of an η-quasisymmetric map is η'-quasisymmetric with

$\eta'(t) = 1/\eta^{-1}(1/t)$. See **[TV]** for the basic theory of quasisymmetric maps between metric spaces.

A homeomorphism $f\colon (\Omega, d) \to (\Omega', d')$ between two noncomplete metric spaces is a *quasisimilarity*, with *data* (η, L, λ), where $L \geqslant 1$ and $0 < \lambda \leqslant 1$, if

$$f \text{ is } \eta\text{-quasisymmetric} \tag{1.19}$$

and if for each $x \in \Omega$ there is $c_x > 0$ such that

$$\frac{c_x}{L}\, d(z,y) \leqslant d'\left(f(z), f(y)\right) \leqslant L c_x d(z,y) \tag{1.20}$$

whenever $z, y \in B(x, \lambda\ d_\Omega(x))$.

Here $d_\Omega(x)$ is the distance from x to the boundary of Ω (*cf.* next chapter), and $B(x, \lambda d_\Omega(x))$ is the open metric ball centered at x with radius $\lambda d_\Omega(x)$. Thus f is a quasisimilarity if it is both quasisymmetric and the restriction $f|B(x, \lambda d_\Omega(x))$ is uniformly a quasiisometry up to scaling. Again, this terminology is not standard; some authors have called a homeomorphism f between Euclidean domains quasisimilar if only the second condition (1.20) is satisfied.

A homeomorphism $f\colon (M, d) \to (M', d')$ between two metric spaces is *quasiconformal*, or *H-quasiconformal*, $H \geqslant 1$, if

$$H(x) := \limsup_{r \to 0} \frac{\sup\{d'\big(f(x), f(y)\big) : d(x,y) \leqslant r\}}{\inf\{d'\big(f(x), f(y)\big) : d(x,y) \geqslant r\}} \leqslant H \tag{1.21}$$

for all $x \in M$. Quasisymmetric maps are always quasiconformal and the (local) converse is often (but not always) true. For example, on domains in $\mathbb{R}^n$, $n \geqslant 2$, or more generally on Riemannian manifolds of dimension at least two, quasiconformal maps are locally quasisymmetric. See **[HeiK]** for a theory of quasiconformal maps between metric spaces.

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CHAPTER 2

THE QUASIHYPERBOLIC METRIC AND UNIFORM SPACES

Assume that (Ω, d) is a locally compact, rectifiably connected noncomplete metric space, and denote by $\overline{\Omega}$ its metric completion. Then the *boundary* $\partial\Omega = \overline{\Omega} - \Omega$ is nonempty, and we define the *quasihyperbolic metric* k_Ω in Ω by introducing the length element

$$\frac{|dz|}{d_\Omega(z)}, \qquad d_\Omega(z) = \operatorname{dist}(z, \partial\Omega) \text{ for } z \in \Omega.$$

Thus, $k_\Omega(x, y)$ is defined to be the infimum over the quantities

$$\int_\gamma \frac{|dz|}{d_\Omega(z)},$$

where γ runs over all rectifiable curves in Ω joining the points x and y. The inequality

$$|d_\Omega(z) - d_\Omega(w)| \leqslant d(z, w), \quad z, w \in \Omega, \tag{2.1}$$

implies that the density d_Ω is a continuous function on Ω, and because Ω is locally compact, d_Ω is positive. It follows easily that k_Ω is a metric.

There is a third natural metric in Ω, denoted by $\ell_\Omega(x, y)$ and defined as the infimum of the lengths (in the original d-metric) of all curves joining the points x and y in Ω (see the Appendix).

When the metric space Ω is understood as fixed, the abbreviations

$$k_\Omega(x, y) = k(x, y), \quad d_\Omega(x) = d(x), \quad \ell_\Omega(x, y) = \ell(x, y) \tag{2.2}$$

are commonly used.

Next, we record the following two elementary inequalities, valid for all $x, y \in \Omega$:

$$\left|\log \frac{d(x)}{d(y)}\right| \leqslant k(x, y), \tag{2.3}$$

$$\log\left(1 + \frac{d(x, y)}{d(x) \wedge d(y)}\right) \leqslant \log\left(1 + \frac{\ell(x, y)}{d(x) \wedge d(y)}\right) \leqslant k(x, y), \tag{2.4}$$

where $a \wedge b = \min\{a, b\}$.

If several metrics are simultaneously in use when length, diameter, ... are under consideration, the notation ℓ_e, diam_e, ... will be used, where e denotes the metric in which length, diameter, ... are measured.

Recall that a metric space is a *length space* if the distance between every pair of points in it is the infimum of the lengths of all curves joining the points. It is not hard to see that a complete locally compact length space is proper and geodesic [**Gr1**, p. 9].

If (Ω, d) is an arbitrary rectifiably connected metric space, any continuous density $\rho\colon \Omega \to (0, \infty)$ determines a metric space (Ω, d_ρ), where the new metric d_ρ is given by

$$d_\rho(x, y) = \inf \int_\gamma \rho\, ds; \tag{2.5}$$

the infimum is taken over all rectifiable curves γ joining the points x and y. It is easy to see that indeed d_ρ is a metric, and we call the space (Ω, d_ρ) a *conformal deformation* of (Ω, d) by the *conformal factor* ρ.

For completeness, a proof for the following lemma is provided in the appendix.

Lemma 2.6. — *Suppose that the identity map $(\Omega, d) \to (\Omega, \ell)$ is a homeomorphism. Then for each rectifiable curve $\gamma\colon I \to (\Omega, d)$ the length $\ell_\rho(\gamma)$ of γ in the metric space (Ω, d_ρ) is given by the line integral*

$$\ell_\rho(\gamma) = \int_\gamma \rho\, ds. \tag{2.7}$$

Proposition 2.8. — *If the identity map $(\Omega, d) \to (\Omega, \ell)$ is a homeomorphism, then it is a homeomorphism $(\Omega, d) \to (\Omega, k)$ and (Ω, k) is complete; in particular, (Ω, k) is proper and geodesic.*

Proof. — It is easy to see that the identity map $(\Omega, d) \to (\Omega, k)$ is a homeomorphism (compare Lemma A.4). Therefore, by Lemma 2.6 and by the remarks preceding it, it suffices to show that (Ω, k) is complete.

To this end, let (x_n) be a k-Cauchy sequence in Ω. Then it follows from (2.3) that

$$0 < m = \inf_n d(x_n) \leqslant \sup_n d(x_n) = M < \infty. \tag{2.9}$$

In particular, by (2.4)

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n)(\exp\{k(x_n, x_m)\} - 1) \\ &\leqslant M(\exp\{k(x_n, x_m)\} - 1) \end{aligned}$$

so that (x_n) is a d-Cauchy sequence as well. It follows from (2.9) and from the continuity of $x \mapsto d(x)$ that the limit $y \in \overline{\Omega}$ of this d-Cauchy sequence has to lie in Ω. Because $\mathrm{id}\colon (\Omega, d) \to (\Omega, k)$ is continuous, (x_n) has a limit in (Ω, k), as was to be proved. The proposition follows. □

Assume now that Ω is an A-uniform space as defined in the introduction. Then we have

$$d \leqslant \ell \leqslant Ad$$

so that id: $(\Omega, d) \to (\Omega, \ell)$ is an A-quasiisometry; in particular, Proposition 2.8 implies that (Ω, k) is complete, proper, and geodesic. The geodesic arcs in (Ω, k) will henceforth be called *quasihyperbolic geodesics.*

We shall next show that in a uniform space the quasihyperbolic geodesics are uniform curves.

Theorem 2.10. — *A quasihyperbolic geodesic in an A-uniform space is a B-uniform curve with B depending only on A.*

The proof of Theorem 2.10 is essentially the same given by Gehring and Osgood **[GO]** for Euclidean domains, only the generality of the situation necessitates some changes. We have decided to write down rather explicit constants in our estimates whenever this causes no extra trouble. For this purpose, we record the following two elementary inequalities:

$$\log\left(\frac{1}{1-x}\right) \leqslant 2A\log(1+2x), \quad A \geqslant 1,\ 0 \leqslant x \leqslant \frac{A}{A+1}; \tag{2.11}$$

$$\log(1+Ax) \leqslant A\log(1+x), \quad A \geqslant 1,\ x \geqslant 0. \tag{2.12}$$

Lemma 2.13. — *Let (Ω, d) be a locally compact, rectifiably connected noncomplete metric space. If $\gamma\colon [0,1] \to \Omega$ is a curve that satisfies, for all $t \in [0,1]$,*

$$\ell_d(\gamma|[0,t]) \wedge \ell_d(\gamma|[t,1]) \leqslant A\ d(\gamma(t)) \tag{2.14}$$

with end points $x = \gamma(0)$ and $y = \gamma(1)$, then

$$\begin{aligned} \ell_k(\gamma) &\leqslant 2A\log\left(1+\frac{\ell_d(\gamma)}{d(x)}\right)\left(1+\frac{\ell_d(\gamma)}{d(y)}\right) \\ &\leqslant 4A\log\left(1+\frac{\ell_d(\gamma)}{d(x)\wedge d(y)}\right). \end{aligned} \tag{2.15}$$

If Ω is an A-uniform space, then

$$\begin{aligned} k(x,y) &\leqslant 2A^2\log\left(1+\frac{d(x,y)}{d(x)}\right)\left(1+\frac{d(x,y)}{d(y)}\right) \\ &\leqslant 4A^2\log\left(1+\frac{d(x,y)}{d(x)\wedge d(y)}\right) \end{aligned} \tag{2.16}$$

for all $x, y \in \Omega$.

Proof. — Denote $L = \ell_d(\gamma)$ and let z be the point on γ which cuts γ into two subcurves of equal length. By symmetry, it suffices to show that

$$\ell_k(\gamma_1) \leqslant 2A\log\left(1+\frac{L}{d(x)}\right),$$

where γ_1 is the piece of γ from z to x. (This of course needs to be understood through a parameterization; γ need not be an arc.) If

$$L \leqslant \frac{2A}{A+1}\, d(x),$$

then, *cf.* (2.7),

$$\begin{aligned}
\ell_k(\gamma_1) &= \int_{\gamma_1} \frac{|du|}{d(u)} \leqslant \int_0^{L/2} \frac{ds}{d(x)-s} \\
&= \log\left(\frac{d(x)}{d(x)-L/2}\right) = \log\left(\frac{1}{1-\frac{L}{2d(x)}}\right) \\
&\leqslant 2A\log\left(1+\frac{L}{d(x)}\right),
\end{aligned}$$

where (2.11) was used in the last step. If in turn

$$L > \frac{2A}{A+1}\, d(x),$$

then we use (2.14) and find that

$$\begin{aligned}
\ell_k(\gamma_1) &\leqslant \int_0^{\frac{Ad(x)}{A+1}} \frac{ds}{d(x)-s} + A\int_{\frac{Ad(x)}{A+1}}^{L/2} \frac{ds}{s} \\
&= \log\left(\frac{1}{1-\frac{A}{A+1}}\right) + A\log\left(\frac{(A+1)L}{2Ad(x)}\right) \\
&\leqslant \log(1+A) + A\log\left(\frac{L}{d(x)}\right) \\
&\leqslant A\log 2 + A\log\left(1+\frac{L}{d(x)}\right) \\
&\leqslant 2A\log\left(1+\frac{L}{d(x)}\right),
\end{aligned}$$

where (2.12) was used.

Inequality (2.16) follows from (2.15) when applied to an A-uniform curve γ, for then $\ell_d(\gamma) \leqslant Ad(x,y)$. The proof of Lemma 2.13 is complete. □

Proof of Theorem 2.10. — Let γ be a quasihyperbolic geodesic in an A-uniform space Ω with end points y_1 and y_2. Put $D = \max_{x\in\gamma} d(x)$. For $i = 1,2$ let N_i be the unique nonnegative integer such that

$$\frac{D}{2^{N_i+1}} < d(y_i) \leqslant \frac{D}{2^{N_i}}.$$

For $k = 0,\dots,N_1$, let x_k^1 be the first point on γ with

$$d(x_k^1) = \frac{D}{2^k}$$

when traveling from y_1 towards y_2. Then define similarly x_k^2 for $k = 0, \dots, N_2$ with travel direction from y_2 to y_1. Use points x_k^1 and x_k^2 together with the end points y_1 and y_2 to divide γ into $(N_1 + N_2 + 3)$ nonoverlapping (modulo end points) subcurves γ_ν, $\nu \in [-N_1 - 1,\ N_2 + 1]$. (A curve containing an end point of γ, as well as the middle subcurve between x_0^1 and x_0^2, may be degenerate.) All subcurves γ_ν are quasihyperbolic geodesics between their respective end points, and

$$d(z) \leqslant \frac{D}{2^{|\nu|-1}} \quad \text{if } z \in \gamma_\nu,$$

$$d(z) \geqslant \frac{D}{2^{|\nu|}} \quad \text{if } z \text{ is an end point of } \gamma_\nu. \tag{2.17}$$

It thus follows from Lemma 2.13, formula (2.16), that

$$\frac{2^{|\nu|-1}}{D}\ell_d(\gamma_\nu) \leqslant \ell_k(\gamma_\nu) \leqslant 4A^2 \log\left(1 + \frac{2^{|\nu|}}{D}\ell_d(\gamma_\nu)\right). \tag{2.18}$$

An elementary computation, using formula (2.18) and the inequality $\log(1+x) \leqslant \sqrt{x}$ for $x \geqslant 0$ implies

$$\frac{2^{|\nu|}}{D}\ell_d(\gamma_\nu) \leqslant 64A^4$$

and thus

$$\ell_k(\gamma_\nu) \leqslant 32A^4.$$

By formulas (2.3) and (2.17) we hence obtain that

$$d(z) \geqslant \frac{D}{2^{|\nu|}} \exp\{-32A^4\} \qquad \text{for } z \in \gamma_\nu.$$

Therefore, for $z = \gamma(t) \in \gamma_\nu$,

$$\begin{aligned} \ell_d(\gamma|[0,t]) \wedge \ell_d(\gamma|[t,1]) &\leqslant \sum_{j \geqslant |\nu|} 64A^4 D 2^{-j} \\ &\leqslant 128A^4 D 2^{-|\nu|} \\ &\leqslant B_1(A) d(z), \end{aligned}$$

where $B_1(A) = 128A^4 \exp\{32A^4\}$. This completes the proof for the second requirement (1.8) placed on uniform curves.

It remains to show that γ also satisfies (1.7), that is

$$\ell_d(\gamma) \leqslant B(A) d(y_1, y_2). \tag{2.19}$$

The argument uses estimates (2.4) and (2.16). First choose points y_1' and y_2' from γ such that

$$\ell_d(\gamma_1') = \ell_d(\gamma_2') = \frac{1}{2}\, d(y_1, y_2),$$

where γ_i' is the subcurve from y_i to y_i', $i = 1, 2$, and that

$$d' = d(y_1') \wedge d(y_2') \geqslant \frac{d(y_1, y_2)}{2B_1},$$

where $B_1 = B_1(A)$ is the constant from the first part of the proof. Observe that

$$d(y_1', y_2') \leqslant 2d(y_1, y_2).$$

Estimates (2.4) and (2.16) now imply

$$\begin{aligned}\log\left(1+\frac{\ell_d(\gamma)-d(y_1,y_2)}{d'}\right) &\leqslant k(y_1',y_2')\\ &\leqslant 4A^2\log\left(1+\frac{2d(y_1,y_2)}{d'}\right)\\ &\leqslant 4A^2\log(1+4B_1),\end{aligned}$$

from which one infers that

$$\ell_d(\gamma) \leqslant d(y_1, y_2) + d'B_2(A).$$

If $d' \leqslant (4A^2+1)d(y_1,y_2)$, the assertion follows from this last estimate. Thus assume otherwise. Then it follows from (2.1) that

$$(4A^2+1)d(y_1,y_2) \leqslant d' \leqslant d(y_1)\wedge d(y_2) + \frac{1}{2}d(y_1,y_2)$$

and hence that

$$4A^2 d(y_1,y_2) \leqslant d(y_1)\wedge d(y_2).$$

Estimates (2.4) and (2.16) now give

$$\log\left(1+\frac{\ell_d(\gamma)}{d(y_1)\wedge d(y_2)}\right) \leqslant 4A^2\log\left(1+\frac{d(y_1,y_2)}{d(y_1)\wedge d(y_2)}\right) \leqslant 1.$$

Consequently, by (2.4) and (2.16),

$$\begin{aligned}\frac{\ell_d(\gamma)}{d(y_1)\wedge d(y_2)} &\leqslant 3\log\left(1+\frac{\ell_d(\gamma)}{d(y_1)\wedge d(y_2)}\right)\\ &\leqslant 3k(y_1,y_2)\\ &\leqslant 12A^2\log\left(1+\frac{d(y_1,y_2)}{d(y_1)\wedge d(y_2)}\right)\\ &\leqslant 12A^2\frac{d(y_1,y_2)}{d(y_1)\wedge d(y_2)}.\end{aligned}$$

Thus the desired bound (2.19) for $\ell_d(\gamma)$ can always be found, and the proof for Theorem 2.10 is complete. One can compute an estimate $B \leqslant \exp\{1000A^6\}$. □

To finish the chapter, we prove the following proposition.

Proposition 2.20. — *The completion of a uniform space is proper. In particular, if (Ω, d) is a bounded uniform space, then $(\overline{\Omega}, d)$ is compact.*

Proof. — We show that every bounded sequence (x_n) in a uniform space (Ω, d) contains a Cauchy subsequence. To this end, it suffices to show that for each $\varepsilon > 0$ there exists a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, x_{n_\ell}) \leqslant \varepsilon$ for all k, ℓ.

Thus, fix $\varepsilon > 0$ and fix a base point $w \in \Omega$; we may assume that $d(x_n, w) > \varepsilon/2$ for all n. Assuming that (Ω, d) is A-uniform, choose for each n an A-uniform curve γ_n from w to x_n. There exists a point $y_n \in \gamma_n$ such that $d(y_n) \geqslant \varepsilon/(4A)$ and $d(x_n, y_n) \leqslant \varepsilon/4$. In particular, (y_n) is a bounded sequence as well, and it follows from formula (2.16) that

$$k(w, y_n) \leqslant 4A^2 \log\left(1 + \frac{d(w, y_n)}{d(w) \wedge d(y_n)}\right) \leqslant R,$$

where R is independent of n. Since (Ω, k) is proper, (y_n) contains a convergent subsequence (y_{n_k}); and since (Ω, d) and (Ω, k) are homeomorphic, the sequence (y_{n_k}) is also convergent in (Ω, d) (see Proposition 2.8). By passing to a further subsequence if necessary, and by picking the corresponding points x_{n_k} we find a desired subsequence (x_{n_k}). The proof of Proposition 2.20 is complete. □

CHAPTER 3

UNIFORM SPACES ARE NEGATIVELY CURVED IN THE QUASIHYPERBOLIC METRIC

This chapter deals with the issue indicated in its title. We prove a general result (Theorem 3.6) which in particular implies the "only if" part of Theorem 1.11.

First we recall some basic facts about Gromov hyperbolic spaces and set up notation. For the proofs, see **[GhHa]**. Denote by $[x, y]$ any geodesic curve joining two points x and y in a metric space; that is, $[x, y]$ is a curve (an arc in fact) whose length is precisely the distance between x and y.

A geodesic metric space X is called *δ-hyperbolic*, $\delta \geqslant 0$, if for all triples of geodesics $[x, y], [y, z], [z, x]$ in X every point in $[x, y]$ is within distance δ from $[y, z] \cup [z, x]$. The property is often expressed by saying that geodesic triangles in X are *δ-thin*. In general, we say that a space is *Gromov hyperbolic* if it is δ-hyperbolic for some δ. All Gromov hyperbolic spaces in this paper are assumed unbounded.

From now on, we generically use the distance notation $|x - y|$ in any Gromov hyperbolic space, unless there is a specific need to do otherwise.

The *Gromov boundary* $\partial_G X$ of a proper geodesic Gromov hyperbolic space X is defined to be the set of equivalence classes of geodesic rays. A *geodesic ray* in X is an isometric image in X of the interval $[0, \infty)$, and two rays are equivalent if their Hausdorff distance in X is finite. We say that a geodesic ray γ *ends at* $a \in \partial_G X$ if γ represents the point a. The Gromov boundary $\partial_G X$ is always nonempty.

Alternatively, the Gromov boundary can be defined as the set of equivalence classes of sequences $(x_n) \subset X$ which *tend to infinity* in the sense that

$$\lim_{n,m \to \infty} (x_n | x_m)_w = \infty \ ,$$

where

$$(x|y)_w = \frac{1}{2}\{|x - w| + |y - w| - |x - y|\} \tag{3.1}$$

is the *Gromov product* between points $x, y \in X$ with respect to a base point $w \in X$. Two sequences $(x_n), (y_n)$ in X, tending to infinity, are equivalent if

$$\lim_{n \to \infty} (x_n | y_n)_w = \infty.$$

There is, for each $x \in X$ and $a \in \partial_G X$, a geodesic ray $[x, a]$ issuing from x and ending at a. Similarly, for every pair of distinct points $a, b \in \partial_G X$ there is a *geodesic line* $[a, b]$ from a to b which is an isometric image of $(-\infty, \infty)$ ending at a and b in the obvious sense.

The Gromov product extends to points $a, b \in \partial_G X$ and the geometric content of (3.1) is the following: if X is proper, geodesic and δ-hyperbolic, then

$$|(a|b)_w - \operatorname{dist}(w, [a, b])| \leqslant 8\delta \tag{3.2}$$

for any pair of points $a, b \in X \cup \partial_G X$ and any geodesic $[a, b]$ between the points.

The Gromov boundary does not, in general, possess a preferred metric, but there is a canonically defined quasisymmetric gauge on $\partial_G X$. (Recall the definition of quasisymmetry from (1.18).)

A *quasisymmetric gauge* on a set is a maximal collection of metrics on the set so that any two metrics in the collection are quasisymmetrically related by the identity map. Two gauges are *equivalent* if there is a bijection between the two underlying sets that is quasisymmetric with respect to some (hence each) pair of metrics from the gauges.

On the boundary $\partial_G X$ of a (proper, geodesic) δ-hyperbolic space X there is a canonically defined quasisymmetric gauge generated by the distance functions

$$\rho_{w,\varepsilon}(a, b) = \exp\{-\varepsilon(a|b)_w\}, \quad a, b \in \partial_G X, \tag{3.3}$$

where $\varepsilon > 0$ and $w \in X$ is a base point. Although, in general, expression (3.3) does not define a metric, there is $\varepsilon(\delta) > 0$ such that for $0 < \varepsilon < \varepsilon(\delta)$ one finds a metric $d_{w,\varepsilon}$ on $\partial_G X$ satisfying

$$\frac{1}{2}\rho_{w,\varepsilon}(a, b) \leqslant d_{w,\varepsilon}(a, b) \leqslant \rho_{w,\varepsilon}(a, b) \tag{3.4}$$

for $a, b \in \partial_G X$. All these changes in the distance functions including a base point change are quasisymmetric, so the gauge is well defined, depending only on X. We shall call this quasisymmetric gauge the *canonical (quasisymmetric) gauge* on $\partial_G X$. The Gromov boundary equipped with any metric from the canonical gauge is compact.

By combining (3.2), (3.3) and (3.4), we obtain

$$\frac{1}{C(\delta)} \exp\{-\varepsilon \operatorname{dist}(w, [a, b])\} \leqslant d_{w,\varepsilon}(a, b) \leqslant C(\delta) \exp\{-\varepsilon \operatorname{dist}(w, [a, b])\} \tag{3.5}$$

whenever $0 < \varepsilon < \varepsilon(\delta)$ and $a, b \in \partial_G X$.

A metric tree is δ-hyperbolic with $\delta = 0$; it can have arbitrarily long finite branches that are irrelevant to the construction of the (Gromov) boundary. We define a proper, geodesic Gromov hyperbolic space X to be *K-roughly starlike*, $K > 0$, with respect to a base point $w \in X$, if for every point $x \in X$ there exists some geodesic ray emanating from w whose distance to x is at most K. We call the space K-roughly starlike if it is K-roughly starlike with respect to some base point. Note that if a proper, geodesic δ-hyperbolic space is K-roughly starlike with respect to $w \in X$, then it is K'-roughly starlike with respect to any $w' \in X$, where $K' = K'(|w - w'|, \delta, K)$. Roughly starlike spaces are called visual in [**BoSc**]. All Cartan-Hadamard manifolds are roughly starlike. Gromov hyperbolic domains also have this property, as we shall see in Lemma 7.8.

In the following theorem, as everywhere in this paper, we let (Ω, d) denote a locally compact, rectifiably connected noncomplete metric space as in Chapter 2.

Theorem 3.6. — *If (Ω, d) is a uniform space, then (Ω, k) is a proper and geodesic Gromov hyperbolic space; moreover, if Ω is bounded, then (Ω, k) is roughly starlike and the quasisymmetric gauge determined by d on $\partial\Omega$ is naturally equivalent to the canonical gauge on $\partial_G\Omega$. The statement is quantitative in that the constants of the conclusion only depend on the uniformity constant of Ω.*

The gauge equivalence in Theorem 3.6 is induced by a natural bijection between the two boundaries, and is explained below in Proposition 3.12.

Thus, Theorem 3.6 implies the "only if" part of Theorem 1.11.

Because the Gromov boundary $\partial_G X$ is always compact and because quasisymmetric maps map bounded spaces to bounded spaces, the assumption Ω be bounded in the second assertion of Theorem 3.6 is necessary. If Ω is not bounded, the one point compactification $\partial^*\Omega$ of $\partial\Omega$ is still homeomorphic to $\partial_G\Omega$, and an appropriate *quasimöbius equivalence* [**V1**] can be established. However, we shall not discuss quasimöbius maps in this paper.

We now prove the first assertion in Theorem 3.6. Thus assume Ω is A-uniform. By Proposition 2.8 (*cf.* the remark after its proof), (Ω, k) is proper and geodesic. Next, let $x, y, z \in \Omega$ and let $[x, y], [y, z], [x, z]$ be quasihyperbolic geodesics. We have to show that, for each given point $u \in [x, y]$,

$$k(u,\ [x, z] \cup [y, z]) \leqslant \delta(A).$$

We assume, without loss of generality, that $\ell([x, u]) \leqslant \ell([u, y])$, and use Theorem 2.10 which asserts that quasihyperbolic geodesics are B-uniform curves, $B = B(A) \geqslant 1$, to obtain

$$d(u) \geqslant \frac{1}{B}\ell([x, u]). \tag{3.7}$$

and

$$\frac{1}{B}2\ell([x, u]) \leqslant \frac{1}{B}\ell([x, y]) \leqslant d(x, y) \leqslant \ell([x, z]) + \ell([z, y]).$$

If

$$\ell([x, z]) < \frac{1}{B}\ell([x, u]),$$

then we can find a point $v \in [z, y]$ such that

$$\ell([z, v]) = \frac{1}{2B}\ell([x, u]) \leqslant \ell([v, y]);$$

moreover,

$$
\begin{aligned}
d(u,v) &\leqslant \ell([x,u]) + \ell([x,z]) + \ell([v,z]) \\
&\leqslant \left(1 + \frac{1}{B} + \frac{1}{2B}\right)\ell([x,u]) \\
&= \frac{(2B+3)}{2B}\ell([x,u]),
\end{aligned}
\tag{3.8}
$$

while, by the uniformity of geodesic arcs (Theorem 2.10),

$$
d(v) \geqslant \frac{1}{B}\ell([z,v]) = \frac{1}{2B^2}\ell([x,u]). \tag{3.9}
$$

If, on the other hand,

$$
\ell([x,z]) \geqslant \frac{1}{B}\ell([x,u]),
$$

then we can find a point $v \in [x,z]$ such that

$$
\ell([x,v]) = \frac{1}{2B}\ell([x,u]) \leqslant \ell([v,z]);
$$

moreover,

$$
\begin{aligned}
d(u,v) &\leqslant \ell([x,u]) + \ell([x,v]) \\
&\leqslant \frac{(2B+1)}{2B}\ell([x,u]),
\end{aligned}
\tag{3.10}
$$

while

$$
d(v) \geqslant \frac{1}{B}\ell([x,v]) = \frac{1}{2B^2}\ell([x,u]). \tag{3.11}
$$

By combining formulas (3.7)–(3.10) and estimate (2.16), we arrive at

$$
\begin{aligned}
k(u,\ [x,z]\cup[y,z]) &\leqslant k(u,v) \\
&\leqslant 4A^2\log\left(1 + \frac{d(u,v)}{d(u)\wedge d(v)}\right) \\
&\leqslant \delta(A),
\end{aligned}
$$

where

$$
\delta(A) = 4A^2\log(1 + B(2B+3)).
$$

This proves the first assertion in Theorem 3.6. A computation shows that one can choose $\delta(A) = 10000A^8$.

Before we take up the second assertion in Theorem 3.6, we establish some results on (quasihyperbolic) geodesic rays and lines.

Proposition 3.12. — *Suppose that (Ω, d) is an A-uniform space and let $\gamma\colon [0, \ell_d(\gamma)) \to \Omega$ be a (quasihyperbolic) geodesic ray parameterized by arc length with respect to d.*

(a) If $\ell_d(\gamma) = \infty$, then $d(\gamma(s)) \geqslant s/B$ for $s \in [0,\infty)$, where $B = B(A) \geqslant 1$ is the constant in Theorem 2.10, and $d(\gamma(s), x) \to \infty$ as $s \to \infty$ for each $x \in \overline{\Omega}$. In particular, γ eventually leaves every bounded subset of Ω. Moreover, if α is another

geodesic ray, α has finite quasihyperbolic Hausdorff distance to γ if and only if $\ell_d(\alpha) = \infty$.

(b) If $\ell_d(\gamma) < \infty$, then there is a unique point $a \in \partial\Omega$ such that $d(\gamma(s), a) \to 0$ as $s \to \ell_d(\gamma)$ and γ is a B-uniform arc (with one end point in $\partial\Omega$). Moreover, if α is another geodesic ray, then α has finite quasihyperbolic Hausdorff distance to γ if and only if $\ell_d(\alpha) < \infty$ and $\alpha(s) \to a$ as $s \to \ell_d(\alpha)$.

(c) The map from the Gromov boundary $\partial_G\Omega$ to the set $\partial^*\Omega = \partial\Omega \cup \{\infty\}$ that assigns to each geodesic ray its end point in $\partial\Omega$ (case (b)) or ∞ (case (a)) is injective. If there are no rays as in case (a), which happens if and only if (Ω, d) is bounded, we set $\partial^*\Omega = \partial\Omega$.

(d) The mapping $\partial_G\Omega \to \partial^*\Omega$ is a bijection, and for every pair of distinct points $a, b \in \partial^*\Omega - \{\infty\}$ every geodesic line $[a, b]$ is a B-uniform curve, where $B = B(A) \geqslant 1$ is the constant in Theorem 2.10.

Proof. — Part (a) follows readily from Theorem 2.10: for the sufficiency part of the last assertion, use formula (2.16) to obtain

$$\begin{aligned} k(\gamma(s), \alpha(s)) &\leqslant 4A^2 \log\left(1 + \frac{2s + d(\gamma(0), \alpha(0))}{s/B}\right) \\ &\leqslant 4A^2 \log(1 + 3B) \end{aligned}$$

for $s \geqslant d(\gamma(0), \alpha(0))$; necessity follows from (2.3).

Part (b) is likewise easy to derive from Theorem 2.10 and formulas (2.3) and (2.16).

Part (c) follows from (a) and (b), except the necessity part of the last assertion. For this, assume that (Ω, d) is unbounded. Then we can choose a base point w and a sequence $(x_n) \subset \Omega$ such that $d(w, x_n) \to \infty$ as $n \to \infty$. Let γ_n be a quasihyperbolic geodesic from w to x_n. Because (Ω, k) is proper, standard arguments using the Arzela-Ascoli theorem show that the sequence (γ_n) subconverges uniformly on compacta to a geodesic ray $\gamma\colon [0, \infty) \to (\Omega, k)$ that leaves every bounded set in (Ω, d). Thus (c) is proved.

It remains to prove part (d). Fix a point $a \in \partial^*\Omega$. If $a = \infty$, so that (Ω, d) is unbounded, the existence of an unbounded ray was already discussed above. Thus assume $a \in \partial\Omega$. Pick a sequence $(x_n) \subset \Omega$ with $x_n \to a$ as $n \to \infty$, and choose quasihyperbolic geodesics γ_n from a base point w to x_n. Then $d(x_n) \to 0$, so that $k(w, x_n) \to \infty$ by the basic estimate (2.3). Hence we find, by using the Arzela-Ascoli theorem and the properness of (Ω, k), that a subsequence of (γ_n), still denoted the same, converges uniformly on compacta to a geodesic ray $\gamma\colon [0, \infty) \to (\Omega, k)$. It remains to show that $\operatorname{dist}(\gamma, a) = 0$.

To this end, fix $\varepsilon > 0$ (small compared to $d(a, w)$), and choose for each large enough n a point $y_n \in \gamma_n$ such that $d(y_n, x_n) = \varepsilon$; then the uniformity of γ_n together with standard estimates (2.16) imply that $k(w, y_n) \leqslant C$, where C is independent of n. By the properness of (Ω, k), a subsequence (y_{n_k}) converges to a point $y \in \gamma$ with

$k(w,y) \leqslant C$. By Proposition 2.8, (y_{n_k}) converges in (Ω, d) as well, so it follows that $d(a,y) \leqslant \varepsilon$.

Finally, if $a, b \in \partial^*\Omega - \{\infty\}$, then there is a geodesic line $[a,b]$ by the general discussion in the beginning of this chapter. This line is clearly a B-uniform curve by Theorem 2.10.

This completes the proof of Proposition 3.12. □

We now return to the proof of Theorem 3.6 and prove that (Ω, k) is roughly starlike if Ω is bounded. Choose $w \in \Omega$ such that

$$d(w) = \max_{x\in\Omega} d(x). \tag{3.13}$$

(The existence of such a point follows from Proposition 2.20.) Fix $x \in \Omega$ and let $x' \in \partial\Omega$ be a closest boundary point to x, so that $d(x) = d(x, x')$. By Proposition 3.12, a geodesic ray $[w, x']$ is a B-uniform arc with $B = B(A) \geqslant 1$ if (Ω, d) is A-uniform. Because $d(x) \leqslant d(w) \leqslant \ell_d([w, x'])$, we can select a point $y \in [w, x']$ such that $\ell_d([y, x']) = \frac{1}{2}d(x)$. Thus

$$d(x,y) \leqslant d(x,x') + d(x',y) \leqslant \tfrac{3}{2}d(x).$$

Because also

$$d(y) \geqslant \frac{1}{2B}d(x),$$

estimate (2.16) implies

$$k(x,y) \leqslant 4A^2 \log(1+3B)$$

as desired. A computation shows that (Ω, k) is roughly K-starlike with $K(A) \leqslant 5000A^8$.

Before we prove the last assertion of Theorem 3.6, a lemma is required.

Lemma 3.14. — *Let (Ω, d) be a bounded A-uniform space, let $w \in \Omega$ be a point as in (3.13), and let $\gamma = [a, w]$ be a quasihyperbolic ray from w to a point $a \in \overline{\Omega}$, parameterized by arc length with respect to d such that $\gamma(0) = a$. Let $b \in \overline{\Omega}$ be a point distinct from a, and define*

$$y = \begin{cases} \gamma(d(a,b)) & \text{if } d(a,b) \leqslant \frac{1}{2}d(a,w) \\ w & \text{if } d(a,b) > \frac{1}{2}d(a,w). \end{cases}$$

Then, for each quasihyperbolic geodesic $[a,b]$, we have that

(a) $k(y, [a,b]) \leqslant C_1$, *and*

(b) $k(w, [a,b]) - C_2 \leqslant k(y,w) \leqslant k(w,[a,b]) + C_2$,

where the constants $C_1, C_2 > 0$ depend only on A.

Proof. — Assume first that $d(a,b) > \frac{1}{2}d(a,w)$, so that $y = w$. Let $x \in [a,b]$ be the point such that the subarc of $[a,b]$ between a and x has length $\frac{1}{4}d(a,w)$; because

$\frac{1}{4}d(a,w) < \frac{1}{2}d(a,b) \leqslant \frac{1}{2}\ell_d([a,b])$ and $[a,b]$ is a $B(A)$-uniform arc, we have that

$$d(x) \geqslant \frac{1}{4B}d(a,w).$$

On the other hand,

$$d(w,x) \leqslant d(w,a) + d(a,x) \leqslant \frac{5}{4}d(w,a).$$

Since $d(x) \leqslant d(w)$, we get from (2.16)

$$k(w,[a,b]) \leqslant k(w,x) \leqslant 4A^2 \log\left(1 + \frac{d(w,x)}{d(x)}\right) \leqslant 4A^2 \log(1+5B).$$

So (a) and (b) follow in this case.

Next assume that $d(a,b) \leqslant \frac{1}{2}d(a,w)$. Let $x \in [a,b]$ be the point that divides $[a,b]$ into two subarcs of equal length. Since also γ is a $B(A)$-uniform arc, we have that

$$d(x) \wedge d(y) \geqslant \frac{1}{2B}d(a,b).$$

Because

$$\begin{aligned} d(x,y) &\leqslant d(x,a) + d(a,y) \\ &\leqslant \frac{B}{2}d(a,b) + d(a,b), \end{aligned}$$

we thus obtain from (2.16) that

$$k(x,y) \leqslant 4A^2 \log(1 + B(B+2)).$$

This implies (a) and the first inequality of (b). To prove the second inequality of (b), let $u \in [a,b]$ be a point with $k(w,u) = k(w,[a,b])$. If

$$\ell_d([a,u]) \wedge \ell_d([u,b]) \geqslant \frac{1}{2}d(a,b), \tag{3.15}$$

then the B-uniformity of $[a,b]$ implies

$$d(u) \geqslant \frac{1}{2B}\, d(a,b)$$

and we obtain as above that $k(u,y) \leqslant C(A)$. Hence we may assume that (3.15) does not hold. In this case, we first estimate

$$\begin{aligned} \ell_d([u,w]) &\geqslant d(w,u) \geqslant d(w,a) - d(a,b) - d(b,u) \wedge d(a,u) \\ &\geqslant \frac{1}{2}\, d(a,b). \end{aligned}$$

Let $u' \in [u,w]$ be the point so that $\ell_d([u,u']) = \frac{1}{4}\, d(a,b)$; because $[u,w]$ is a B-uniform arc, we have

$$d(u') \geqslant \frac{1}{4B}d(a,b).$$

Because also

$$\begin{aligned} d(u', y) &\leqslant d(u', u) + \ell_d([a, b]) + d(a, b) \\ &\leqslant \frac{9}{4} B \, d(a, b) \end{aligned}$$

by the definition of y and by the B-uniformity of $[a, b]$, formula (2.16) implies

$$k(u', y) \leqslant 4A^2 \log(1 + 9B^2) = C(A).$$

Since $u' \in [u, w]$, we thus obtain

$$\begin{aligned} k(w, [a, b]) = k(w, u) &\geqslant k(w, u') \\ &\geqslant k(w, y) - k(u', y) \\ &\geqslant k(w, y) - C(A). \end{aligned}$$

This completes the proof of Lemma 3.14. □

Remark 3.16. — The proof of Lemma 3.14 shows that for a general base point $w \in \Omega$ the constants C_1 and C_2 in (a) and (b) depend besides on A also on the ratio $\operatorname{diam} \Omega / d(w)$.

We shall now prove the last assertion of Theorem 3.6. Assume that (Ω, d) is a bounded A-uniform space. By Proposition 3.12, we have a bijection

$$\varphi \colon \partial_G \Omega \to \partial \Omega$$

that assigns to each geodesic ray $[w, a]$ its end point $a \in \partial\Omega$; here $w \in \Omega$ is a point that satisfies (3.13). Note that $\operatorname{diam} \overline{\Omega} \leqslant 2Ad(w)$. Fix a metric $d_{w,\varepsilon}$ from the canonical quasisymmetric gauge of $\partial_G X$ satisfying (3.5). It suffices to show that the inverse of φ is an η-quasisymmetric map $(\partial\Omega, d) \to (\partial_G \Omega, d_{w,\varepsilon})$ with $\eta = \eta(A)$.

To this end, let $a, b, c \in \partial\Omega$ be three distinct points. Fix a geodesic ray $[w, a]$, and let y_b and y_c in $[w, a]$ be the points defined in Lemma 3.14, corresponding to b and c respectively. By the B-uniformity of $[w, a]$,

$$d(y_b) \geqslant \frac{1}{B}\big(d(w) \wedge d(a, b)\big)$$

and

$$d(y_c) \geqslant \frac{1}{B}\big(d(w) \wedge d(a, c)\big).$$

Assume first that

$$\frac{d(a, b)}{d(a, c)} = t \geqslant 1. \tag{3.17}$$

Then

$$\begin{aligned} d(y_b, y_c) &\leqslant d(y_b, a) + d(a, y_c) \\ &\leqslant 2d(a, b) + 2d(a, c) \\ &\leqslant 4d(a, b). \end{aligned}$$

It follows from (2.16), (2.12), and the above inequalities that

$$\begin{aligned} k(y_b, y_c) &\leqslant 4A^2 \log\left(1 + \frac{4Bd(a,b)}{d(w) \wedge d(a,c)}\right) \\ &\leqslant 32A^3 B \log(1+t), \end{aligned}$$

where in the last inequality we also used the fact that $d(a,c) \leqslant 2Ad(w)$.

Now we use estimate (3.5) and Lemma 3.14:

$$\begin{aligned} \frac{d_{w,\varepsilon}(a,b)}{d_{w,\varepsilon}(a,c)} &\leqslant C(A) \exp\{-\varepsilon\left(k(w,[a,b]) - k(w,[a,c])\right)\} \\ &\leqslant C(A) \exp\{-\varepsilon(k(y_b,w) - k(y_c,w))\} \\ &\leqslant C(A) \exp\{\varepsilon k(y_b,y_c)\} \\ &\leqslant C(A) \exp\{\varepsilon 32A^3 B \log(1+t)\}. \end{aligned}$$

This proves the quasisymmetry condition if (3.17) holds.

Next assume that

$$\frac{d(a,b)}{d(a,c)} = t < 1.$$

The definition for the points y_b and y_c then implies that we first hit y_b and then y_c when traveling from a to w along $[a,w]$; it may happen of course that $y_b = y_c = w$. Therefore,

$$k(w,y_b) - k(w,y_c) = k(y_c,y_b). \tag{3.18}$$

We also have

$$d(y_b) \leqslant d(a,y_b) \leqslant 2d(a,b),$$

and so

$$\begin{aligned} k(y_c,y_b) &\geqslant \log\frac{d(y_c)}{d(y_b)} \geqslant \log\frac{d(w) \wedge d(a,c)}{2Bd(a,b)} \\ &= \log\frac{d(w) \wedge d(a,c)}{d(a,b)} - \log 2B. \end{aligned}$$

Because $d(a,c) \leqslant 2Ad(w)$, we obtain

$$k(y_c,y_b) \geqslant \log\frac{1}{t} - \log 2A - \log 2B.$$

Finally, use again estimate (3.5) and Lemma 3.14, which together with (3.18) imply

$$\begin{aligned} \frac{d_{w,\varepsilon}(a,b)}{d_{w,\varepsilon}(a,c)} &\leqslant C(A) \exp\{-\varepsilon\left(k(y_b,w) - k(y_c,w)\right)\} \\ &= C(A) \exp\{-\varepsilon k(y_c,y_b)\} \\ &\leqslant C(A) t^{\varepsilon}. \end{aligned}$$

This completes the proof of Theorem 3.6. □

The above proof shows that the function η that controls the quasisymmetry of the map $(\partial\Omega, d) \to (\partial_G\Omega, d_{w,\varepsilon})$ can be chosen to be of the form

$$\eta(t) = \begin{cases} C(A)t^{\varepsilon}, & \text{if } \ 0 < t \leqslant 1, \\ C(A)t^{D(A)\varepsilon}, & \text{if } 1 \leqslant t < \infty. \end{cases} \tag{3.19}$$

Recall that $\varepsilon < \varepsilon(A)$ so that $d_{w,\varepsilon}$ is a metric. The particular choice of w satisfying (3.13) guarantees that η depends only on the uniformity constant A. A change in the base point would not change the quasiisometry gauge determined by $d_{w,\varepsilon}$. Alternatively, for an arbitrary base point w the constant $C(A)$ in (3.19) will also depend on the ratio $\operatorname{diam}\Omega/d(w)$. Note, however, that $D(A)$ depends only on A in all cases.

CHAPTER 4

UNIFORMIZATION

This chapter is devoted to the discussion and proof of Theorem 1.1.

Suppose X is a rectifiably connected metric space. We use the notation $|x-y|$ for the distance of points in X. We fix a base point w, and consider the family of conformal deformations of X by the densities

$$\rho_\varepsilon(x) = \exp\{-\varepsilon|x-w|\}, \quad \varepsilon > 0. \tag{4.1}$$

We denote the resulting metric spaces by $X_\varepsilon = (X, d_\varepsilon)$. Thus d_ε is a metric on X defined by

$$d_\varepsilon(a,b) = \inf \int_\gamma \rho_\varepsilon \, ds,$$

where the infimum is taken over all rectifiable curves in $(X, |x-y|)$ joining the points a and b (see the Appendix).

Recall from Lemma 2.6 that for any rectifiable curve γ in X we have

$$\ell_\varepsilon(\gamma) = \int_\gamma \rho_\varepsilon \, ds, \tag{4.2}$$

(with the obvious notation ℓ_ε) provided the identity map $(X, |x-y|) \to (X, \ell)$ is a homeomorphism. If X is geodesic, X_ε is always bounded, for given $x \in X$ and a geodesic segment $[w,x]$, we have that

$$d_\varepsilon(w,x) \leqslant \int_{[w,x]} \rho_\varepsilon \, ds \leqslant \int_0^\infty e^{-\varepsilon t} \, dt = \frac{1}{\varepsilon}, \tag{4.3}$$

and hence

$$\operatorname{diam}_\varepsilon X_\varepsilon \leqslant \frac{2}{\varepsilon}.$$

The triangle inequality implies that

$$\exp\{-\varepsilon|x-y|\} \leqslant \frac{\rho_\varepsilon(x)}{\rho_\varepsilon(y)} \leqslant \exp\{\varepsilon|x-y|\} \tag{4.4}$$

for $x, y \in X$ and $\varepsilon > 0$. This is a *Harnack type inequality*: the density ρ is roughly constant on balls of fixed radius. If X is geodesic, it follows from (4.4) that the identity map $X_\varepsilon \to X$ is a local quasiisometry; in particular, then, X_ε is rectifiably connected.

We shall show in this chapter that there is associated with each proper, geodesic and roughly starlike δ-hyperbolic space X a whole family of bounded uniform spaces X_ε for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$; the members in this family are quasisimilar to each other. This *uniformization* is obtained by a conformal deformation, where the density ρ_ε decays exponentially towards the Gromov boundary as in (4.1). Let us denote the association $X \to X_\varepsilon$ by $\mathcal{D}$ for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$, where $\mathcal{D}$ stands for *dampening*.

Similarly, we can *quasihyperbolize* each bounded uniform space Ω by considering the quasihyperbolic metric k in it; the resulting space (Ω, k) is a proper, geodesic and roughly starlike Gromov hyperbolic space (Theorem 3.6). Let us denote the association $\Omega \to (\Omega, k)$ by $\mathcal{Q}$.

The one-to-one correspondence between the two types of isomorphism classes of metric spaces, stated in Theorem 1.1, is given by the above associations $\mathcal{D}$ and $\mathcal{Q}$. We shall show that the composition $\mathcal{Q} \circ \mathcal{D}$ takes a given δ-hyperbolic space back to its quasiisometry class, and that the composition $\mathcal{D} \circ \mathcal{Q}$ preserves the quasisimilarity type of a given bounded A-uniform space Ω for $0 < \varepsilon \leqslant \varepsilon(A)$.

Note that, strictly speaking, the association $\mathcal{D}$ depends both on ε and the chosen base point w. We have chosen to suppress this dependence from the notation. The dependence on the base point is rather innocent: a change there results in a quasi-isometric change in the metric. On the other hand, two deformed spaces X_ε and $X_{\varepsilon'}$ are always quasisimilar; see Proposition 4.15.

We begin our study by showing that the deformations X_ε are uniform spaces. Here X need not be roughly starlike. Long "finite arms" in X will simply disappear in the uniformization.

Proposition 4.5. — *The conformal deformations $X_\varepsilon = (X, d_\varepsilon)$ of a proper, geodesic δ-hyperbolic space X are bounded $A(\delta)$-uniform spaces for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$.*

Proof. — By the discussion in the beginning of this chapter, X_ε is bounded, rectifiably connected and locally compact. It is also noncomplete, because any sequence (x_n) tending to infinity in X along a geodesic ray is a d_ε-Cauchy sequence, *cf.* (4.3). (Recall our standing assumption that all Gromov hyperbolic spaces are unbounded.) We denote

$$\partial_\varepsilon X = \partial X_\varepsilon = \overline{X}_\varepsilon - X_\varepsilon.$$

A simple estimate using the Harnack inequality (4.4) implies, for $|y - x| > 1/\varepsilon$, that

$$d_\varepsilon(y, x) \geqslant \frac{1}{\varepsilon e}\rho_\varepsilon(x).$$

In particular, we find that

$$d_\varepsilon(x) = d_\varepsilon(x, \partial_\varepsilon X) \geqslant \frac{1}{\varepsilon e}\rho_\varepsilon(x) \tag{4.6}$$

for $x \in X$.

By Theorem 5.1 in the next chapter, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that the *Gehring-Hayman theorem* is true for X_ε if $0 < \varepsilon < \varepsilon_0$. That is, for each pair of points $x, y \in X$ and for each curve γ in X with end points x and y, it holds that

$$\ell_\varepsilon([x,y]) \leqslant 20\ell_\varepsilon(\gamma), \tag{4.7}$$

where $[x, y]$ is a geodesic segment in the δ-hyperbolic space X. We shall show that for this choice of ε_0, the space X_ε is A-uniform for $A = \max\{20, \exp\{8\delta\varepsilon_0 + 1\}\}$ and for each $0 < \varepsilon \leqslant \varepsilon_0$. (Note in particular that A is independent of ε.) Indeed, we shall show that a geodesic segment $[x, y]$ joining two points $x, y \in X$ is an A-uniform curve.

The first condition for uniformity (1.7) is immediate from (4.7). To prove condition (1.8), pick $z \in [x, y]$. Consider a geodesic triangle

$$\Delta = [w, x] \cup [x, y] \cup [w, y].$$

There is a tripod map $f\colon \Delta \to T$, where T is a tripod consisting of three line segments satisfying the following two properties: f is an isometry on each of the sides of Δ and $f(u) = f(v)$ implies $|u - v| \leqslant 4\delta$. See [**GhHa**, p. 41] or [**Bo**, Proposition 3.1]. Let $x_1 \in [x, y]$ denote the unique point whose image under f is the origin of T. By symmetry, we may assume that $z \in [x, x_1]$.

By using the tripod map, one computes

$$|w - u| \geqslant |w - z| + |z - u| - 8\delta \tag{4.8}$$

for $u \in [x, z]$. Thus

$$\begin{aligned}
\ell_\varepsilon([x,z]) &= \int_{[x,z]} \rho_\varepsilon(u)\,|du| \\
&= \int_{[x,z]} \exp\{-\varepsilon|w-u|\}\,|du| \\
&\leqslant e^{8\delta\varepsilon} \exp\{-\varepsilon|w-z|\} \int_{[x,z]} \exp\{-\varepsilon|z-u|\}\,|du| \\
&\leqslant e^{8\delta\varepsilon} \rho_\varepsilon(z) \int_0^\infty e^{-\varepsilon t}\,dt = \frac{e^{8\delta\varepsilon}}{\varepsilon}\,\rho_\varepsilon(z) \\
&\leqslant \exp\{8\delta\varepsilon + 1\} d_\varepsilon(z),
\end{aligned}$$

where (4.6) was used in the last step. This completes the proof of Proposition 4.5. □

Remarks 4.9

(a) The restriction $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ in Proposition 4.5 is needed only so that the first condition (1.7) of uniformity is guaranteed (by the Gehring-Hayman theorem). The second uniformity condition (1.8) is true in X_ε for each $\varepsilon > 0$ with $A = \exp\{8\delta\varepsilon + 1\}$.

(b) The proof of Proposition 4.5 shows that geodesic segments, or more generally infinite geodesic rays or lines in X, are uniform curves in the deformed space X_ε.

We next establish a useful lemma:

Lemma 4.10. — *Let X be a proper, geodesic δ-hyperbolic space, and let $X_\varepsilon = (X, d_\varepsilon)$ be its uniformization for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ as in Proposition 4.5. Then there is a constant $C = C(\delta) \geqslant 1$ such that*

$$\begin{aligned}\frac{1}{C}\, d_\varepsilon(x,y) &\leqslant \frac{\exp\{-\varepsilon(x|y)_w\}}{\varepsilon}(1 \wedge \varepsilon|x-y|) \\ &\leqslant C\, d_\varepsilon(x,y)\end{aligned} \tag{4.11}$$

whenever $x, y \in X$.

Proof. — Fix $x, y \in X$ and $0 < \varepsilon \leqslant \varepsilon_0(\delta)$. Let $[x,y]$ be a geodesic between x and y, and let $x_1 \in [x,y]$ be the unique point that corresponds to the origin of the tripod under a tripod map of the geodesic triangle $[w,x] \cup [w,y] \cup [x,y]$. (Compare the proof of Proposition 4.5.) It then follows that $\big|\, |x_1 - w| - (x|y)_w \big| \leqslant 4\delta$. In view of this, it suffices to show that

$$d_\varepsilon(x,y) \simeq \frac{\rho_\varepsilon(x_1)}{\varepsilon}(1 \wedge \varepsilon|x-y|) \tag{4.12}$$

with constants depending only on δ. If $\varepsilon|x-y| \leqslant 1$, then (4.12) is clear by the Harnack inequality (4.4). Thus we may assume that $\varepsilon|x-y| > 1$.

By invoking the tripod map again, *cf.* formula (4.8), we have that

$$\exp\{-\varepsilon|w-u|\} \leqslant e^{8\delta\varepsilon}\rho_\varepsilon(x_1)\exp\{-\varepsilon|x_1-u|\}$$

for $u \in [x,y]$. Therefore

$$\begin{aligned}d_\varepsilon(x,y) &\leqslant e^{8\delta\varepsilon}\rho_\varepsilon(x_1)\int_{[x,y]} \exp\{-\varepsilon|x_1-u|\}\,|du| \\ &\leqslant 2e^{8\delta\varepsilon}\rho_\varepsilon(x_1)\int_0^\infty e^{-\varepsilon t}\,dt \\ &\leqslant 2e^{8\delta\varepsilon}\frac{\rho_\varepsilon(x_1)}{\varepsilon}.\end{aligned}$$

On the other hand,

$$\exp\{-\varepsilon|w-u|\} \geqslant \rho_\varepsilon(x_1)\exp\{-\varepsilon|x_1-u|\}$$

for $u \in [x,y]$. So the Gehring-Hayman theorem 5.1 gives

$$\begin{aligned}d_\varepsilon(x,y) &\geqslant \frac{1}{20}\ell_\varepsilon([x,y]) \\ &\geqslant \frac{1}{20}\rho_\varepsilon(x_1)\int_0^{\frac{1}{2}|x-y|} e^{-\varepsilon t}\,dt \\ &= \frac{1}{20}\,\frac{\rho_\varepsilon(x_1)}{\varepsilon}\left(1 - e^{-\frac{\varepsilon}{2}|x-y|}\right) \\ &\geqslant \frac{1}{20}\,\frac{\rho_\varepsilon(x_1)}{\varepsilon}\left(1 - e^{-\frac{1}{2}}\right).\end{aligned}$$

This proves Lemma 4.10. □

Proposition 4.13. — *Let X be a proper, geodesic δ-hyperbolic space, and let X_ε be its uniformization for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ as in Proposition 4.5. Then there is a natural quasisymmetric identification $\partial_G X \to \partial_\varepsilon X$.*

Proof. — It follows from (4.11) that each sequence in X that tends to infinity (as defined in the beginning of Chapter 3) converges in $\overline{X}_\varepsilon$ to a point in $\partial_\varepsilon X$. Moreover, two sequences tending to infinity are equivalent if and only if they determine the same limit point in $\partial_\varepsilon X$. We thus obtain an injective map $\partial_G X \to \partial_\varepsilon X$. By using (4.11) again, we easily see that this map is surjective as well, leading to a natural identification of the two boundaries. Finally, it follows from (4.11), (3.3), and (3.4) that this identification is quasisymmetric. (Indeed, the map $(\partial_G X, d_{w,\varepsilon}) \to \partial_\varepsilon X$ is a quasiisometry, where $d_{w,\varepsilon}$ is as in (3.4).) The proposition follows. □

Remarks 4.14

(a) The identification map $\partial_G X \to \partial_\varepsilon X$ in Proposition 4.13 can also be described as follows. For every equivalence class of geodesic rays there is a unique point $a \in \partial_\varepsilon X$ such that $\gamma(t) \to a$ in $\overline{X}_\varepsilon$ when $t \to \infty$ for each geodesic ray $\gamma\colon [0,\infty) \to X$ in the equivalence class. (Compare 3.12 (d).)

(b) By Proposition 2.20 we know that $\overline{X}_\varepsilon$ is compact for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$. Thus the spaces $\overline{X}_\varepsilon$ can be considered as compactifications of X obtained by adding the boundary $\partial_\varepsilon X$; these compactifications are all homeomorphic. A neighborhood basis for a point $a \in \partial_G X$ in this compactification is provided by the sets

$$N_{a,\lambda} = \{x \in X \cup \partial_G X : (x|a)_w \geqslant \lambda\}$$

for $\lambda \geqslant 0$.

The next proposition shows that the quasisimilarity class of X_ε is determined by the quasiisometry class of X. It follows, in particular, that X_ε and $X_{\varepsilon'}$ are quasisimilar if X is δ-hyperbolic and $0 < \varepsilon, \varepsilon' \leqslant \varepsilon_0(\delta)$.

Proposition 4.15. — *$\mathcal{D}$ maps mutually quasiisometric proper geodesic roughly starlike Gromov hyperbolic spaces to mutually quasisimilar spaces.*

We begin with a lemma which shows that inequality (4.6) is essentially sharp under the starlikeness assumption.

Lemma 4.16. — *Let X be proper and geodesic δ-hyperbolic space that is K-roughly starlike with respect to w. Then the density $\rho_\varepsilon(x) = \exp\{-\varepsilon|w - x|\}$ satisfies*

$$\frac{1}{\varepsilon e}\rho_\varepsilon(x) \leqslant d_\varepsilon(x) \leqslant \frac{(2e^{\varepsilon K} - 1)}{\varepsilon}\rho_\varepsilon(x) \tag{4.17}$$

for each $\varepsilon > 0$ and $x \in X$.

Proof. — The left inequality in (4.17) was already proved in (4.6). To prove the right inequality, fix $x \in X$. Let $[w,a]$ be an infinite ray from the base point w to a point $a \in \partial_G X$ so that

$$|x-y| \leqslant K$$

for some $y \in [w,a]$.

We then deduce from the Harnack inequality (4.4) that

$$\begin{aligned} d_\varepsilon(x) &\leqslant d_\varepsilon(x,y) + d_\varepsilon(y,a) \\ &\leqslant \int_{[x,y]} \rho_\varepsilon(z)\,|dz| + \int_{[y,a]} \rho_\varepsilon(z)\,|dz| \\ &\leqslant \rho_\varepsilon(x) \left[\int_0^K e^{\varepsilon t}\,dt + e^{\varepsilon K} \int_0^\infty e^{-\varepsilon t}\,dt \right] \\ &= \rho_\varepsilon(x) \frac{(2e^{\varepsilon K} - 1)}{\varepsilon} \end{aligned}$$

as desired. This proves Lemma 4.16. □

Considering long finite branches in a tree, one sees that estimate (4.17) is of the correct order. When $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ as in Proposition 4.5, formula (4.17) becomes

$$\frac{1}{e}\,\frac{\rho_\varepsilon(x)}{\varepsilon} \leqslant d_\varepsilon(x) \leqslant A\frac{\rho_\varepsilon(x)}{\varepsilon}\ , \tag{4.18}$$

where the constant $A \geqslant 1$ depends only on K and δ.

Proof of Proposition 4.15. — Suppose that $f\colon X \to X'$ is a quasiisometry between two proper, geodesic and roughly starlike Gromov hyperbolic spaces. Assuming that X is δ-hyperbolic and X' is δ'-hyperbolic, we consider the spaces X_ε and $X'_{\varepsilon'}$ for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ and $0 < \varepsilon' \leqslant \varepsilon_0(\delta')$. We are free to choose the respective base points $w \in X$ and $w' \in X'$ such that $f(w) = w'$, because a change in the base point only causes a quasiisometric change in the metric. We shall show that $f\colon X_\varepsilon \to X'_{\varepsilon'}$ is a quasisimilarity.

To this end, we first infer from (4.11), (3.1), and (4.18) that

$$\begin{aligned} 1 \wedge \varepsilon|x-y| &\leqslant C\,\varepsilon\,\frac{d_\varepsilon(x,y)}{\rho_\varepsilon(x)} \\ &\leqslant C\,\frac{d_\varepsilon(x,y)}{d_\varepsilon(x)}\ , \end{aligned} \tag{4.19}$$

where $C \geqslant 1$ depends only on the data associated with X. Now fix $z \in X$, $0 < \lambda < 1$, and $x, y \in B_\varepsilon(z, \lambda d_\varepsilon(z))$. Then

$$d_\varepsilon(x,y) \leqslant \frac{2\lambda}{1-\lambda}\,d_\varepsilon(x),$$

which together with (4.19) implies that

$$\varepsilon|x-y| \leqslant C\lambda, \tag{4.20}$$

and hence that

$$\varepsilon'|x'-y'| \leqslant C\,\frac{\varepsilon'}{\varepsilon}\lambda, \tag{4.21}$$

if $x, y \in B_\varepsilon(z, \lambda d_\varepsilon(z))$ and if λ is sufficiently small. Here, and throughout the rest of the proof, $C \geqslant 1$ is a constant that depends only on ε, ε', and the data associated with X and f. We also use the notation $f(a) = a'$ for $a \in X$.

Thus, if we pick λ small, but depending only on the data, we conclude, by (4.11), (4.20), (3.1), (4.4), and (4.18), that

$$\frac{1}{C}\,d_\varepsilon(x,y) \leqslant d_\varepsilon(z)\varepsilon|x-y| \leqslant C\,d_\varepsilon(x,y) \tag{4.22}$$

if $x, y \in B_\varepsilon(z, \lambda d_\varepsilon(z))$. On the other hand, if we choose λ yet smaller, now depending in addition on ε'/ε, we similarly have, by (4.21), that

$$\frac{1}{C}\,d_{\varepsilon'}(x',y') \leqslant d_{\varepsilon'}(z')\varepsilon'|x'-y'| \leqslant C\,d_{\varepsilon'}(x',y'). \tag{4.23}$$

Inequalities (4.22) and (4.23) show that the second requirement (1.20) for quasisimilarity holds.

To show that f is quasisymmetric, we shall prove that for each triple of distinct points $x, y, z \in X$ the following statement holds:

$$\frac{d_\varepsilon(x,y)}{d_\varepsilon(x,z)} \leqslant 1 \quad \text{implies} \quad \frac{d_{\varepsilon'}(x',y')}{d_{\varepsilon'}(x',z')} \leqslant C. \tag{4.24}$$

Indeed, because X_ε and $X_{\varepsilon'}$ are quasiconvex metric spaces, the required η-quasisymmetry of f follows from (4.24) by [**V4**, Theorem 6.6].

We shall consider cases corresponding to possible locations of the three points. Condition $d_\varepsilon(x,y) \leqslant d_\varepsilon(x,z)$ implies, by way of (4.11), that

$$\exp\{\varepsilon(x|z)_w - \varepsilon(x|y)_w\}\,\frac{1 \wedge \varepsilon|x-y|}{1 \wedge \varepsilon|x-z|} \leqslant C. \tag{4.25}$$

If $\varepsilon|x-z|$ is small, depending only on the data, we have as in (4.22) that

$$\frac{d_\varepsilon(x,y)}{d_\varepsilon(x)} \leqslant \frac{d_\varepsilon(x,z)}{d_\varepsilon(x)} \leqslant C\,\varepsilon|x-z|\,;$$

thus we are in the situation where (1.20) holds and (4.24) is clear. It follows that we only need to consider the case where

$$C\,\varepsilon|x-z| \geqslant 1. \tag{4.26}$$

Assume first that $\varepsilon|x-y| < 1$. It follows that

$$C \leqslant \varepsilon'|x'-z'| \quad \text{and} \quad \varepsilon'|x'-y'| \leqslant C. \tag{4.27}$$

We obtain from (4.11), (4.27), and (3.1) that

$$\begin{aligned}\frac{d_{\varepsilon'}(x',y')}{d_{\varepsilon'}(x',z')} &\leqslant C\,\frac{\exp\{-\varepsilon'(x'|y')_{w'}\}}{\exp\{-\varepsilon'(x'|z')_{w'}\}}\,\frac{1\wedge\varepsilon'|x'-y'|}{1\wedge\varepsilon'|x'-z'|}\\ &\leqslant C\exp\{\frac{\varepsilon'}{2}(|z'-w'|-|x'-z'|-|y'-w'|+|x'-y'|)\}\\ &\leqslant C\exp\{\varepsilon'|x'-y'|\}\leqslant C\end{aligned}$$

as desired.

If $\varepsilon|x-y|\geqslant 1$, then (4.25) and (4.26) show that $(x|z)_w \leqslant (x|y)_w + C$.

We now have

$$C\leqslant \varepsilon'|x'-z'| \quad\text{and}\quad (x'|z')_{w'}\leqslant (x'|y')_{w'}+C.$$

(For the second inequality, see [**BoSc**, Section 5].) Therefore,

$$\frac{d_{\varepsilon'}(x',y')}{d_{\varepsilon'}(x',z')}\leqslant C\,\frac{1\wedge\varepsilon'|x'-y'|}{1\wedge\varepsilon'|x'-z'|}\leqslant C,$$

so that (4.24) holds in this case too.

The proof of Proposition 4.15 is complete. □

We next study what happens to an A-uniform space Ω under the associations

$$\Omega \underset{\mathcal{Q}}{\longrightarrow} (\Omega,k) \underset{\mathcal{D}}{\longrightarrow} \Omega_\varepsilon.$$

Here we denote by Ω_ε the conformal deformation of the space (Ω,k) by the density $\rho_\varepsilon(x)=\exp\{-\varepsilon k(w,x)\}$, where $0<\varepsilon\leqslant\varepsilon_0(A)$ and $w\in\Omega$ is a base point satisfying (3.13).

Proposition 4.28. — *The identity map $\Omega\to\Omega_\varepsilon$, $0<\varepsilon\leqslant\varepsilon_0(A)$, is quasisimilar with constant depending only on A.*

Proof. — We first prove that there is $\lambda=\lambda(A)\in(0,1)$ such that

$$k(y,z)\leqslant\frac{1}{\varepsilon_0} \tag{4.29}$$

if $y,z\in B_\varepsilon(x,\lambda d(x))$, where $\varepsilon_0>0$ is a constant such that the conclusion of Proposition 4.5 holds for $0<\varepsilon\leqslant\varepsilon_0$. We can take $\varepsilon_0=\varepsilon_0(A)$ by Theorem 3.6. Indeed, (4.29) follows upon observing that $y,z\in B(x,\lambda d(x))$ implies both $d(y,z)\leqslant 2\lambda d(x)$ and $d(y)\wedge d(z)\geqslant(1-\lambda)d(x)$, whence

$$k(y,z)\leqslant 4A^2\log\left(1+\frac{2\lambda}{1-\lambda}\right)$$

by (2.16). Fix $\lambda=\lambda(A)\in(0,1)$ such that (4.29) holds.

It follows from (4.11) (compare the proof of (4.22)) that

$$\frac{1}{C}\rho_\varepsilon(x)k(y,z)\leqslant d_\varepsilon(y,z)\leqslant C\,\rho_\varepsilon(x)k(y,z) \tag{4.30}$$

if $y, z \in B(x, \lambda d(x))$, where $C = C(A) \geqslant 1$.

On the other hand, if $y, z \in B(x, \lambda d(x))$, then

$$k(y,z) \leqslant 4A^2 \log\left(1 + \frac{d(y,z)}{(1-\lambda)d(x)}\right) \leqslant \frac{4A^2 d(y,z)}{(1-\lambda)d(x)}$$

by (2.16), while

$$k(y,z) \geqslant \log\left(1 + \frac{d(y,z)}{(1+\lambda)d(x)}\right) \geqslant \frac{\log(1+a)}{a}\frac{d(y,z)}{(1+\lambda)d(x)}$$

by (2.4) because

$$\frac{d(y,z)}{(1+\lambda)d(x)} \leqslant \frac{2\lambda}{(1+\lambda)} =: a = a(A).$$

The inequality

$$a\log(1+t) \geqslant t\log(1+a) \tag{4.31}$$

for $0 \leqslant t \leqslant a$ was used here.

By combining these last estimates with (4.30), we obtain that the second requirement of quasisimilarity (1.20) is satisfied.

It remains to show that the identity map $\Omega \to \Omega_\varepsilon$ is quasisymmetric. Fix three distinct points $x, y, z \in \Omega$. As in the proof of Proposition 4.15, we use [**V4**, Theorem 6.6] and conclude that it suffices to show that there is $C = C(A) \geqslant 1$ such that

$$\frac{d(x,y)}{d(x,z)} \leqslant 1 \quad \text{implies} \quad \frac{d_\varepsilon(x,y)}{d_\varepsilon(x,z)} \leqslant C. \tag{4.32}$$

In view of Lemma 4.10, the second inequality in (4.32) is equivalent to

$$\frac{\exp\{-\varepsilon(x|y)_w\}}{\exp\{-\varepsilon(x|z)_w\}}\frac{(1\wedge \varepsilon k(x,y))}{(1\wedge \varepsilon k(x,z))} \leqslant C. \tag{4.33}$$

Let y_y and y_z denote the point $y \in [x,w]$ corresponding to the points y and z, respectively, as in Lemma 3.14. (Excuse the bad notation.) It then follows from Lemma 3.14 (b), (3.2) and (4.33) that we need to bound the expression

$$\frac{\exp\{-\varepsilon k(y_y,w)\}}{\exp\{-\varepsilon k(y_z,w)\}}\frac{(1\wedge \varepsilon k(x,y))}{(1\wedge \varepsilon k(x,z))}. \tag{4.34}$$

Because we are assuming that $d(x,y) \leqslant d(x,z)$, by definition we encounter y_y before y_z when traveling from x to w along $[x,w]$. Thus (4.34) becomes

$$\exp\{-\varepsilon k(y_y,y_z)\}\frac{(1\wedge \varepsilon k(x,y))}{(1\wedge \varepsilon k(x,z))} \leqslant \frac{1\wedge \varepsilon k(x,y)}{1\wedge \varepsilon k(x,z)}. \tag{4.35}$$

From (2.4) we have

$$d(x,y) \leqslant d(x,z) \leqslant d(x)\big(\exp\{k(x,z)\} - 1\big).$$

Hence there exists a constant $C_0 = C_0(A) > 0$ such that $k(x,z) \leqslant C_0$ implies $z, y \in B(x, \lambda d(x))$; this is as in (4.30) and the desired bound (4.33) clearly follows. On the other hand, if $k(x,z) > C_0$, then

$$\frac{1 \wedge \varepsilon k(x,y)}{1 \wedge \varepsilon k(x,z)} \leqslant C$$

and (4.33) again follows by way of (4.35).

The proof of Proposition 4.28 is complete. □

To complete the proof of Theorem 1.1, we still have to show that under the association $\mathcal{Q}$, quasisimilar uniform spaces land in the same quasiisometry class, and that after the two deformations

$$(X, |x-y|) \underset{\mathcal{D}}{\longrightarrow} (X_\varepsilon, d_\varepsilon) \underset{\mathcal{Q}}{\longrightarrow} (X_\varepsilon, k_\varepsilon)$$

we land in the same quasiisometry class. Here X is a proper, geodesic and roughly starlike δ-hyperbolic space, and $0 < \varepsilon < \varepsilon(\delta)$ as in Proposition 4.5. These two facts are established in Propositions 4.36 and 4.37.

Proposition 4.36. — *A quasisimilarity between two uniform spaces is a quasiisometry in the quasihyperbolic metrics, quantitatively.*

Proof. — Let $f\colon (\Omega, d) \to (\Omega', d')$ be a quasisimilar map with data (η, L, λ). Because the corresponding spaces (Ω, k) and (Ω', k') are geodesic, and because of symmetry, it suffices to show that $f\colon (\Omega, k) \to (\Omega', k')$ is locally Lipschitz with constant depending only on the data. Fix $x, y \in \Omega$ such that

$$d(x,y) < \lambda(d(x) \wedge d(y)).$$

Then one easily computes from the quasisimilarity conditions (1.19) and (1.20) that

$$d'(f(x), f(y)) \leqslant C \frac{d'(f(z))}{d(z)}\, d(x,y)$$

for $z = x$ or $z = y$, where $C \geqslant 1$ depends only on the data. Thus

$$\begin{aligned} k'(f(x), f(y)) &\leqslant 4A^2 \log\left(1 + \frac{Cd(x,y)}{d(x) \wedge d(y)}\right) \\ &\leqslant 4A^2 C \log\left(1 + \frac{d(x,y)}{d(x) \wedge d(y)}\right) \\ &\leqslant 4A^2 C\, k(x,y) \end{aligned}$$

by (2.16), (2.12) and (2.4). This proves Proposition 4.36. □

Proposition 4.37. — *If X is K-roughly starlike, proper and geodesic δ-hyperbolic space, then for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ the identity map*

$$(X, \varepsilon|x-y|) \longrightarrow (X_\varepsilon, k_\varepsilon)$$

is a quasiisometry. Indeed, for all $x, y \in X$,

$$c\varepsilon|x-y| \leqslant k_\varepsilon(x,y) \leqslant e\varepsilon|x-y|,$$

where $c \in (0,1)$ depends only on K and δ.

Proof. — Let $x, y \in X$. Then

$$\begin{aligned} k_\varepsilon(x,y) &\leqslant \int_\gamma \frac{|d_\varepsilon z|}{d_\varepsilon(z)} = \int_0^{|x-y|} \frac{dL_\varepsilon(t)}{d_\varepsilon(\gamma(t))} \\ &= \int_0^{|x-y|} \frac{\rho_\varepsilon(\gamma(t))}{d_\varepsilon(\gamma(t))} \leqslant \varepsilon e|x-y| \end{aligned}$$

by (4.18), where

$$L_\varepsilon(t) = \int_0^t \rho_\varepsilon(\gamma(s))\, ds$$

and γ is a geodesic segment $[x, y]$ parameterized by arc length. Thus the identity map $X \to (X_\varepsilon, k_\varepsilon)$ is εe-Lipschitz. Note that this estimate holds for all $\varepsilon > 0$.

In the other direction, by (2.4), by the Gehring-Hayman theorem (4.7), and by (4.18),

$$\begin{aligned} k_\varepsilon(x,y) &\geqslant \log\left(1 + \frac{d_\varepsilon(x,y)}{d_\varepsilon(x) \wedge d_\varepsilon(y)}\right) \\ &\geqslant \log\left(1 + \frac{\ell_\varepsilon([x,y])}{20(d_\varepsilon(x) \wedge d_\varepsilon(y))}\right) \\ &\geqslant \log\left(1 + \frac{\varepsilon \ell_\varepsilon([x,y])}{C(K,\delta)(\rho_\varepsilon(x) \wedge \rho_\varepsilon(y))}\right). \end{aligned} \tag{4.38}$$

Assume first that $\varepsilon|x-y| \leqslant 2\log 2$. Then, by the Harnack inequality (4.4),

$$\rho_\varepsilon(z) \geqslant \frac{\rho_\varepsilon(x)}{4}$$

for all $z \in [x, y]$. Hence

$$\ell_\varepsilon([x,y]) \geqslant \frac{\rho_\varepsilon(x)}{4}\, |x-y|,$$

and we obtain from (4.38) and (4.31) that

$$\begin{aligned} k_\varepsilon(x,y) &\geqslant \log(1 + a\, \varepsilon|x-y|) \\ &\geqslant b\, \varepsilon|x-y|, \end{aligned}$$

for some constants $a, b \in (0,1)$ depending only on K and δ.

We are thus left with the case $\varepsilon|x-y| > 2\log 2$. Let $[x,y]$ be a geodesic segment and let $x_1 \in [x,y]$. By the Harnack inequality (4.4),

$$\begin{aligned} \ell_\varepsilon([x,y]) &\geqslant \rho_\varepsilon(x_1)\left(\int_0^{|x-x_1|} e^{-\varepsilon t}\,dt + \int_0^{|y-x_1|} e^{-\varepsilon t}\,dt\right) \\ &= \frac{\rho_\varepsilon(x_1)}{\varepsilon}\left(2 - e^{-\varepsilon|x-x_1|} - e^{-\varepsilon|y-x_1|}\right) \qquad (4.39) \\ &\geqslant \frac{\rho_\varepsilon(x_1)}{2\varepsilon}. \end{aligned}$$

We use this estimate for the point $x_1 \in [x,y]$ that corresponds to the origin under the tripod map

$$f\colon [w,x]\cup[x,y]\cup[w,y] \longrightarrow T.$$

(See the proof of Proposition 4.5.) In particular, with this choice of x_1,

$$\begin{aligned} |w-x| \vee |w-y| &\geqslant |w-x_1| + |x_1-x| \vee |x_1-y| - 4\delta \\ &\geqslant |w-x_1| + \frac{1}{2}\,|x-y| - 4\delta. \end{aligned}$$

This implies

$$\begin{aligned} \rho_\varepsilon(x)\wedge\rho_\varepsilon(y) &= \exp\{-\varepsilon(|w-x|\vee|w-y|)\} \\ &\leqslant e^{4\delta\varepsilon}\exp\{-\varepsilon|w-x_1| - \frac{\varepsilon}{2}|x-y|\} \qquad (4.40) \\ &= e^{4\delta\varepsilon}\rho_\varepsilon(x_1)\exp\{-\frac{\varepsilon}{2}|x-y|\}. \end{aligned}$$

Now use (4.40) together with (4.38), (4.39), and (2.12) to obtain

$$\begin{aligned} k_\varepsilon(x,y) &\geqslant \log\left(1 + a\,\frac{\rho_\varepsilon(x_1)}{(\rho_\varepsilon(x)\wedge\rho_\varepsilon(y))}\right) \\ &\geqslant \log\left(1 + b\exp\{\frac{\varepsilon}{2}|x-y|\}\right) \\ &\geqslant b\log\left(1 + \exp\{\frac{\varepsilon}{2}|x-y|\}\right) \\ &\geqslant b\,\frac{\varepsilon}{2}|x-y|, \end{aligned}$$

where $a, b \in (0,1)$ only depend on K and δ. This completes the proof of Proposition 4.37. □

The proof of Theorem 1.1 is now complete.

CHAPTER 5

THE GEHRING-HAYMAN THEOREM FOR GROMOV HYPERBOLIC SPACES

The Gehring-Hayman theorem in complex analysis states that the hyperbolic geodesic in a simply connected (hyperbolic) plane domain minimizes the Euclidean length among all the curves in the domain with the same end points, up to a universal multiplicative constant [**GeHa**]. This theorem has recently been generalized to quasihyperbolic geodesics in domains in $\mathbb{R}^n$ that are quasiconformally equivalent to uniform domains [**HN**], [**HR**]. A yet different generalization of the Gehring-Hayman theorem was established, and crucially applied, in the theory developed in [**BKR**]. In this chapter, we establish a similar theorem for general δ-hyperbolic, geodesic metric spaces, and for their conformal deformations. Essentially the same theorem appeared without proof in [**Gr2**, p. 191]. A complete proof of a similar result was given in [**CDP**, Prop. 1.6, p. 129], but there it was implicitly assumed that the space with the conformally deformed metric is geodesic. Since this assumption will not be true in general in the setting of the previous chapter, we shall provide a detailed proof of the general assertion.

Theorem 5.1 was used already in the Chapter 4. The present chapter can be read independently of the rest of this paper.

Theorem 5.1. — *Let X be a geodesic, δ-hyperbolic metric space, and let $\rho\colon X \to (0,\infty)$ be a continuous function that satisfies*

$$\frac{1}{C}\exp\{-\varepsilon|x-y|\} \leqslant \frac{\rho(x)}{\rho(y)} \leqslant C\exp\{\varepsilon|x-y|\} \tag{5.2}$$

for all $x, y \in X$ and for some fixed $C \geqslant 1$ and $\varepsilon > 0$. There exist $\varepsilon_0 = \varepsilon_0(\delta, C) > 0$ and $M = M(C) \geqslant 1$ such that, if $\varepsilon \leqslant \varepsilon_0$ in (5.2), then

$$\ell_\rho([x,y]) \leqslant M\ell_\rho(\gamma) \tag{5.3}$$

for each geodesic segment $[x, y]$ in X and for each curve γ joining x and y in X.

In (5.3), ℓ_ρ denotes the length in the deformed metric d_ρ; see (2.7). The proof will show that one can take $M = 20C^2$.

The proof of Theorem 5.1 is accomplished through a series of lemmas. We assume that X is a δ-hyperbolic, geodesic space with distance $|x-y|$, and that ρ satisfies (5.2); in particular, the constants ε and C are assumed fixed. The length of a curve γ in X is denoted by $\ell(\gamma)$.

Lemma 5.4. — *Let γ be a curve in X with end points x, y satisfying*

$$|x-y| \leqslant L = \frac{1}{12\varepsilon C^2} \tag{5.5}$$

and

$$\ell(\gamma) \geqslant 3C^2|x-y|. \tag{5.6}$$

Then

$$\ell_\rho([x,y]) \leqslant \ell_\rho(\gamma)$$

for each geodesic $[x,y]$.

Proof. — For a geodesic $[x,y]$, we compute

$$\begin{aligned}\ell_\rho([x,y]) &= \int_{[x,y]} \rho\, ds \leqslant C\rho(x) \int_0^{|x-y|} e^{\varepsilon t}\, dt \\ &= C\rho(x)\, \frac{1}{\varepsilon}\, \left(e^{\varepsilon|x-y|} - 1\right) \\ &\leqslant C\rho(x)\, \frac{3}{2}\, |x-y|.\end{aligned}$$

In the last step, we used (5.5) and the fact that $e^x - 1 \leqslant \frac{3}{2}x$ if $0 \leqslant x \leqslant 1/4$. On the other hand, by (5.6),

$$\begin{aligned}\ell_\rho(\gamma) &\geqslant \frac{\rho(x)}{C} \int_0^{3C^2|x-y|} e^{-\varepsilon t}\, dt \\ &= \frac{\rho(x)}{C}\frac{1}{\varepsilon}\left(1 - e^{-\varepsilon 3C^2|x-y|}\right) \\ &\geqslant 2C\rho(x)|x-y|.\end{aligned}$$

In the last step (5.5) was used again together with the fact that $1 - e^{-x} \geqslant \frac{2}{3}x$ if $0 \leqslant x \leqslant 1/4$. This proves the lemma. □

Lemma 5.7. — *Let $\widetilde{\gamma}\colon [a,b] \to X$ be a curve in X with finite length. Then there is a curve $\gamma\colon [a,b] \to X$ with the same end points as $\widetilde{\gamma}$ with $\ell_\rho(\gamma) \leqslant \ell_\rho(\widetilde{\gamma})$, and with*

$$\ell(\gamma(s,t)) \leqslant 3C^2|\gamma(s) - \gamma(t)| + 1 \tag{5.8}$$

whenever $a \leqslant s \leqslant t \leqslant b$ are such that $|\gamma(s) - \gamma(t)| \leqslant 1/(12\varepsilon C^2)$.

Proof. — Let Γ be the family of all curves in X with the same end points as $\widetilde{\gamma}$ and with ρ-length not exceeding $\ell_\rho(\widetilde{\gamma})$; because $\widetilde{\gamma} \in \Gamma$,

$$S = \inf_{\alpha \in \Gamma} \ell(\alpha) < \infty.$$

Pick $\gamma \in \Gamma$ such that $\ell(\gamma) \leqslant S + 1/2$. We claim that γ has the desired properties. We assume that γ is parameterized by the interval $[a,b]$, and so it suffices to show that (5.8) holds. To this end, suppose there exists an interval $[s,t] \subset [a,b]$ such that

$$|\gamma(s) - \gamma(t)| \leqslant \frac{1}{12\varepsilon C^2}$$

but

$$\ell(\gamma(s,t)) > 3C^2|\gamma(s) - \gamma(t)| + 1.$$

Lemma 5.4 then implies

$$\ell_\rho([\gamma(s), \gamma(t)]) \leqslant \ell_\rho(\gamma(s,t))$$

for a geodesic $[\gamma(s), \gamma(t)]$. By replacing $\gamma(s,t)$ with a geodesic $[\gamma(s), \gamma(t)]$, we obtain a new curve $\gamma' \in \Gamma$ with

$$\begin{aligned} S \leqslant \ell(\gamma') &= \ell(\gamma) - \ell(\gamma(s,t)) + |\gamma(s) - \gamma(t)| \\ &\leqslant S + \frac{1}{2} - 3C^2|\gamma(s) - \gamma(t)| - 1 + |\gamma(s) - \gamma(t)| \\ &\leqslant S - \frac{1}{2}, \end{aligned}$$

a contradiction. The lemma follows. □

Lemma 5.9. — *Let λ, μ, and L be positive numbers, and let $\gamma\colon [a,b] \to X$ be a rectifiable curve satisfying*

$$\ell(\gamma(s,t)) \leqslant \lambda|\gamma(s) - \gamma(t)| + \mu \tag{5.10}$$

whenever $[s,t] \subset [a,b]$ and $|\gamma(s) - \gamma(t)| \leqslant L$. Let $u = \gamma(a)$ and $v = \gamma(b)$ be the end points of γ. If for some geodesic segment $g = [x,y]$ in X it holds that

$$\operatorname{dist}(\gamma, g) = \operatorname{dist}(u, g) = \operatorname{dist}(v, g) \geqslant R := 1 + 4\delta + 4\delta\lambda, \tag{5.11}$$

then either

$$|u - v| > L \tag{5.12}$$

or

$$|u - v| \leqslant (1 + 8\delta\lambda)(2 + 16\delta + 8\delta\lambda) + 8\delta\mu. \tag{5.13}$$

Proof. — Let $u_0, v_0 \in [x,y]$ be points such that

$$|u_0 - u| = |v_0 - v| = \operatorname{dist}(\gamma, [x,y]).$$

By (5.11), we can find points $u_1 \in [u, u_0]$ and $v_1 \in [v, v_0]$ such that

$$|u - u_1| = |v - v_1| = R.$$

Consider a geodesic $[u_1, v_1]$; we claim that

$$\operatorname{dist}(\gamma, [u_1, v_1]) \geqslant r := R - 2\delta = 1 + 2\delta + 4\delta\lambda. \tag{5.14}$$

Indeed, assume for some points $z \in \gamma$ and $z_1 \in [u_1, v_1]$ we have that $|z - z_1| < r$. By δ-hyperbolicity, when applied to the geodesic rectangle $[u_0, u_1] \cup [u_1, v_1] \cup [v_0, v_1] \cup [u_0, v_0]$, there is a point $z_0 \in [u_0, u_1] \cup [v_0, v_1] \cup [u_0, v_0]$ such that $|z_1 - z_0| \leqslant 2\delta$. If $z_0 \in [u_0, v_0]$, then

$$R \leqslant |z_0 - z| \leqslant |z_0 - z_1| + |z_1 - z| < r + 2\delta = R$$

which is a contradiction; if $z_0 \in [u_0, u_1] \cup [v_0, v_1]$, we assume without loss of generality that $z_0 \in [u_0, u_1]$, and deduce

$$\begin{aligned} |u - u_0| &= \operatorname{dist}(\gamma, [x, y]) \leqslant \operatorname{dist}(z, [x, y]) \\ &\leqslant |z - z_1| + |z_1 - z_0| + |z_0 - u_0| \\ &< r + 2\delta + |u_1 - u_0| = R + |u_1 - u_0| = |u - u_0|, \end{aligned}$$

another contradiction. Thus (5.14) holds.

Next, we split $[a, b]$ into $n \geqslant 1$ intervals $[t_0, t_1], \ldots, [t_{n-1}, t_n]$, where $a = t_0 < t_1 < \cdots < t_n = b$ are such that

$$\begin{aligned} &\ell(\gamma(t_k, t_{k+1})) = 1 + 8\delta\lambda, \quad \text{for} \quad k = 0, \ldots, n-2, \\ &\ell(\gamma(t_{n-1}, t_n)) \leqslant 1 + 8\delta\lambda. \end{aligned}$$

Denote $x_k = \gamma(t_k)$ and choose $y_k \in [u_1, v_1]$ such that $|x_k - y_k| = \operatorname{dist}(x_k, [u_1, v_1])$. Note that

$$|x_k - x_{k+1}| \leqslant 1 + 8\delta\lambda < 2r - 4\delta.$$

The projection lemma [**Bo**, Lemma 3.2] thus implies that

$$|y_k - y_{k+1}| \leqslant 8\delta$$

for $k = 0, \ldots, n-1$. We also note that

$$|x_0 - y_0| = \operatorname{dist}(u, [u_1, v_1]) \leqslant |u - u_1| = R,$$

and similarly $|x_n - y_n| \leqslant R$.

Now assume that (5.12) does not hold, that is, $|u - v| \leqslant L$; then, by assumption (5.10), we have

$$\ell(\gamma) \leqslant \lambda |u - v| + \mu,$$

and so the number n of intervals $[t_k, t_{k+1}]$ satisfies

$$n \leqslant \frac{\ell(\gamma)}{1 + 8\delta\lambda} + 1 \leqslant \frac{\lambda |u - v| + \mu}{1 + 8\delta\lambda} + 1.$$

It follows from all the above that

$$\begin{aligned}|u-v| = |x_0 - x_n| &\leqslant |x_0 - y_0| + \sum_{k=0}^{n-1} |y_k - y_{k+1}| + |y_n - x_n| \\ &\leqslant 2R + n8\delta \\ &\leqslant 2R + 8\delta + \frac{8\delta\lambda}{1+8\delta\lambda}|u-v| + \frac{8\delta\mu}{1+8\delta\lambda},\end{aligned}$$

whence

$$|u-v| \leqslant (1+8\delta\lambda)(2R+8\delta) + 8\delta\mu$$

which is (5.13). The lemma follows. □

Lemma 5.15. — *Let $f\colon [a,b] \to [0,\infty)$ be a continuous function satisfying the following three properties:*

$$f(a) = f(b) = 0; \tag{5.16}$$

$$|f(s) - f(t)| \leqslant |s-t| \quad \text{for} \quad s,t \in [a,b]; \tag{5.17}$$

(5.18) there is $c > 0$ such that $(s,t) \in M_c$ implies $t - s < c$ or $t - s > 2c$, where

$$M_c = \{(s,t) \in [a,b] \times [a,b] : s \leqslant t \text{ and } c \leqslant f(s) = f(t) \leqslant f(x) \text{ for each } s \leqslant x \leqslant t\}.$$

Then

$$\max_{x\in[a,b]} f(x) < \frac{3}{2}\, c. \tag{5.19}$$

Proof. — Consider $N_c = \{(s,t) \in M_c : t - s \geqslant c\}$. We claim that N_c is empty. It is easy to see that both M_c and N_c are compact. If $N_c \neq \varnothing$, then we can choose $(u,v) \in N_c$ so that

$$v - u = \min\{t - s : (s,t) \in N_c\}. \tag{5.20}$$

By assumption (5.18) and by definition of N_c, we have that $v - u > 2c$. If there was a point $w \in (u,v)$ with $f(w) = f(u) = f(v)$, then $w - u \geqslant c$ or $v - w \geqslant c$, contradicting (5.20). Therefore $f(x) > f(u) = f(v) \geqslant c$ for $u < x < v$. Let

$$m = \min\big\{f(x) : x \in \big[u + \tfrac{1}{2}\, c,\ v - \tfrac{1}{2}\, c\big]\big\},$$

and let $u' \in (u, u + \frac{1}{2}c]$ be the largest number where f assumes the value m; similarly, let $v' \in [v - \frac{1}{2}c, v)$ be the smallest number where f assumes the value m. Obviously $(u',v') \in M_c$, and because $v' - u' \geqslant (v - \frac{1}{2}c) - (u + \frac{1}{2}c) = v - u - c > c$, we have in fact that $(u',v') \in N_c$. But $v' - u' < v - u$, contradicting (5.20). Thus $N_c = \varnothing$.

Now let $x \in [a,b]$ be arbitrary; we may assume that $f(x) > c$. Let $u \in (a,x]$ be the largest number where f assumes the value c; similarly, let $v \in [x,b)$ be the smallest such number. Then $(u,v) \in M_c$, and by what was proved above, we have $v - u < c$. Thus either $x - u < \frac{1}{2}c$ or $v - x < \frac{1}{2}c$. In any case, (5.17) implies $f(x) < \frac{3}{2}c$, and the lemma follows. □

Lemma 5.21. — *Let $\lambda \geqslant 1$ and $\mu > 0$ be positive numbers, and let $\gamma\colon [a,b] \to X$ be a rectifiable curve satisfying*

$$\ell(\gamma(s,t)) \leqslant \lambda|\gamma(s) - \gamma(t)| + \mu \tag{5.22}$$

whenever $[s,t] \subset [a,b]$ and $|\gamma(s) - \gamma(t)| \leqslant L$, where

$$L = 2\lambda(1 + 8\delta\lambda)(2 + 16\delta + 8\delta\lambda) + 16\delta\mu\lambda + 2\mu + 1. \tag{5.23}$$

Then γ belongs to the L-neighborhood of each geodesic $[\gamma(a), \gamma(b)]$.

Proof. — We may assume that $[a,b] = [0, \ell(\gamma)]$ and that γ is parameterized by arc length. Let $g = [\gamma(a), \gamma(b)]$ be a geodesic with the same end points as γ, and define

$$f(r) = \operatorname{dist}(\gamma(r),\ g), \qquad 0 \leqslant r \leqslant \ell(\gamma).$$

Then f satisfies conditions (5.16) and (5.17) in Lemma 5.15. We claim it also satisfies (5.18) with $c = L/2$. To see this, let $(s,t) \in M_c$. Then, for each $r \in [s,t]$,

$$\operatorname{dist}(\gamma(r),\ g) \geqslant \operatorname{dist}(\gamma(s),\ g) = \operatorname{dist}(\gamma(t),\ g) \geqslant \frac{L}{2} \geqslant R,$$

where R is given in (5.11). Thus Lemma 5.9 implies that either (5.12) or (5.13) holds, where $u = \gamma(s)$ and $v = \gamma(t)$. If (5.12) holds, then

$$2c = L < |\gamma(s) - \gamma(t)| \leqslant \ell(\gamma(s,t)) = t - s\ ;$$

if (5.13) holds, then

$$|\gamma(s) - \gamma(t)| \leqslant (1 + 8\delta\lambda)(2 + 16\delta + 8\delta\lambda) + 8\delta\mu \leqslant L,$$

which implies by way of (5.22) that

$$\begin{aligned} t - s = \ell(\gamma(s,t)) &\leqslant \lambda(1 + 8\delta\lambda)(2 + 16\delta + 8\delta\lambda) + 8\delta\mu\lambda + \mu \\ &= \frac{1}{2}(L - 1) < \frac{L}{2} = c. \end{aligned}$$

The assumptions of Lemma 5.15 are thus satisfied, and by (5.19), $f(r) \leqslant \frac{3}{4}\,L < L$ for $0 \leqslant r \leqslant \ell(\gamma)$. This proves Lemma 5.21. □

Proof of Theorem 5.1. — Choose $\lambda = 3C^2$ and $\mu = 1$ in Lemma 5.21, where $C \geqslant 1$ is as in (5.2), and let $L = L(\delta, C)$ be the constant given in (5.23). Define

$$\varepsilon_0 = \varepsilon_0(\delta, C) = \frac{1}{14LC^2}. \tag{5.24}$$

We claim that if $0 < \varepsilon \leqslant \varepsilon_0$ in (5.2), there exist $M = M(C)$ such that (5.3) holds for a given geodesic $g = [x,y]$ and for a given curve $\gamma\colon [a,b] \to X$ with $\gamma(a) = x$ and $\gamma(b) = y$. By Lemma 5.7, and by the choice of ε_0, we may assume that γ satisfies the assumptions of Lemma 5.21 with $\lambda = 3C^2$ and $\mu = 1$. Therefore, γ lies in the L-neighborhood $N_L(g)$ of the geodesic g. This implies (*cf.* [**Bo**, Corollary 2.7]) that g lies in the $2L$-neighborhood of γ,

$$g \subset N_{2L}(\gamma). \tag{5.25}$$

We consider two cases; in the first case, assume that the end points x and y satisfy

$$|x-y| \leqslant 7L. \tag{5.26}$$

Since g is a geodesic, there is a subcurve γ' of γ, starting from x, such that $\ell(\gamma') = \ell(g) = |x-y|$. It follows that

$$\ell_\rho(\gamma) \geqslant \ell_\rho(\gamma') \geqslant \frac{1}{C}\rho(x)e^{-\varepsilon_0 7L}\ell(g)$$

while

$$\ell_\rho(g) \leqslant C\rho(x)e^{\varepsilon_0 7L}\ell(g)$$

so that (5.3) holds with $M = C^2e^{2\varepsilon_0 7L} \leqslant 3C^2$, by (5.24).

Finally, assume that (5.26) does not hold. Then we can choose points $x_1 = x$, $x_2, \ldots, x_{n+1} = y$, $n \geqslant 2$, successively on g such that $|x_k - x_{k+1}| = 7L$ for $k = 1, \ldots, n-1$, and $|x_n - y| < 7L$. Let $g_k = [x_k, x_{k+1}] \subset g$. By (5.25), γ meets the balls $B(x_k, 2L)$ for all k, and γ cannot stay inside $B(x_k, 3L)$ because $n \geqslant 2$ and the balls $B(x_k, 3L)$ are pairwise disjoint for $k = 1, \ldots, n$. It follows that for $k = 1, \ldots, n$ there is a subcurve γ_k of γ inside $\overline{B}(x_k, 3L) \smallsetminus B(x_k, 2L)$ connecting $B(x_k, 2L)$ to $X \smallsetminus B(x_k, 3L)$. The curves γ_k are disjoint, and

$$\ell_\rho(\gamma) \geqslant \sum_{k=1}^{n} \ell_\rho(\gamma_k).$$

On the other hand,

$$\ell_\rho(\gamma_k) \geqslant \frac{1}{C}\rho(x_k)e^{-\varepsilon_0 3L}\ L$$

and

$$\ell_\rho(g_k) \leqslant C\rho(x_k)e^{\varepsilon_0 7L}\ 7L$$

for $k = 1, \ldots, n$. By summing these two inequalities over k , we obtain

$$\ell_\rho(g) \leqslant 7C^2e^{10\varepsilon_0 L}\ell_\rho(\gamma) \leqslant 20C^2\ell_\rho(\gamma)$$

by (5.24). This completes the proof of Theorem 5.1. □

CHAPTER 6

UNIFORM LOCALLY LOEWNER SPACES ARE LOEWNER

This chapter can be read independently of the rest of the paper, although some results and terminology from Chapter 2 are used.

The concept of a Loewner space was introduced in [**HeiK**]. Let M be a rectifiably connected metric space, and let μ be a Borel measure on M. Then M is called a *Q-Loewner space*, $Q > 1$, if the function

$$\varphi(t) = \inf\{\mathrm{mod}_Q(E, F; M) : \Delta(E, F) \leqslant t\} \tag{6.1}$$

is positive for each $t > 0$, where E and F designate nondegenerate disjoint continua in M with

$$\Delta(E, F) = \frac{\mathrm{dist}\ (E, F)}{\mathrm{diam}\, E \wedge\ \mathrm{diam}\, F} \tag{6.2}$$

their relative distance. Here and in the following $(E, F; U)$ denotes the family of all curves joining the sets $E, F \subset U \subset M$. The *Q-modulus* of a family Γ of curves in M is the number

$$\mathrm{mod}_Q\, \Gamma = \inf \int_M \rho^Q \, d\mu, \tag{6.3}$$

where the infimum is taken over all Borel functions $\rho\colon M \to [0, \infty]$ such that

$$\int_\gamma \rho \, ds \geqslant 1$$

for each $\gamma \in \Gamma$.

The Loewner condition quantifies the idea that a space has lots of rectifiable curves. It is, in essence, a conformally invariant condition, and plays an important role in the study of quasiconformal mappings in general spaces. Besides the Euclidean space $\mathbb{R}^Q$, quite a few examples of Loewner spaces are known, including those with Q not an integer. See [**HeiK**], [**BoPa**], [**La**] and the references there.

In this chapter, we shall prove the following result:

Theorem 6.4. — *A uniform and locally Q-Loewner space is a Q-Loewner space, quantitatively.*

We believe that Theorem 6.4 is interesting in its own right, in exhibiting new examples of Loewner spaces. Applications to our main theme are given in the next chapter.

We next define what we mean by a locally Loewner space. Let (Ω, d) be a locally compact, rectifiably connected noncomplete metric space as in Chapter 2, and let μ be a Borel measure on Ω. The metric measure space Ω is called a *locally Q-Loewner space*, or *locally Q-Loewner*, $Q > 1$, if there exist numbers $\kappa \geqslant 1$, $\varepsilon_0 \in (0, \kappa^{-1}]$, and a decreasing function $\psi\colon (0,\infty) \to (0,\infty)$ with the following property: if $x \in \Omega$, $0 < \varepsilon < \varepsilon_0$, and $E, F \subset B(x, \varepsilon d(x))$ are two disjoint nondegenerate continua with $\Delta(E,F) \leqslant t$, then

$$\operatorname{mod}_Q(E, F; B(x, \varepsilon\kappa d(x)) \geqslant \psi(t). \tag{6.5}$$

Recall that $d(x) = d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Thus, a locally Loewner space comes with data $v = (Q, \kappa, \varepsilon_0, \psi)$. Theorem 6.4 asserts that if Ω is an A-uniform space that is also locally Q-Loewner, then Ω is in fact Q-Loewner with the Loewner function φ, defined in (6.1), depending only on A and the data v.

Remarks 6.6

(a) A ball in $\mathbb{R}^n$ is an n-Loewner space with $\varphi = \varphi_n$. Thus Theorem 6.4 implies that every uniform domain in $\mathbb{R}^n$ is a Loewner space with Loewner function depending only on n and on the constant associated with uniformity. This result was first proved by Gehring and Martio [**GM**]. Their proof was based on the Sobolev extension theorem of Jones [**J**], which is not available in our setting.

(b) Let Ω be a domain in $\mathbb{R}^n$ and let m be a metric in Ω such that $|x-y| \leqslant m(x,y) \leqslant \ell_\Omega(x,y)$ for $x, y \in \Omega$, where ℓ_Ω is defined in (1.10). If (Ω, m) is uniform, it follows from Theorem 6.4 that it is an n-Loewner space. In particular, inner uniform domains are n-Loewner spaces. Note that this result cannot be proved by using Jones's extension theorem, and is new as far as we can tell. (For earlier results in this direction, see [**V2**], [**Hei**].) For a generalization, see Theorem 6.47 below.

(c) In (a) and (b), we can replace Ω by a domain in $\overline{\mathbb{R}}^n$ equipped with the spherical metric and measure (see Chapter 7). Then similar remarks hold.

A metric (Borel) measure space (M, μ) is called (*Ahlfors*) *Q-regular*, $Q > 0$, if the measure μ satisfies

$$\frac{1}{C} R^Q \leqslant \mu(B_R) \leqslant C R^Q \tag{6.7}$$

for each metric ball B_R of radius $0 < R \leqslant \operatorname{diam} M$ and for some constant $C \geqslant 1$ independent of the ball. It was proved in [**HeiK**, Section 4] that a quasiconformal homeomorphism between Q-regular Q-Loewner spaces is quasisymmetric, provided it maps bounded sets to bounded sets. Here quasiconformality is defined by the infinitesimal condition (1.21). (For the issue of quantitativity of this assertion, see [**HeiK**, 4.25].) In view of 6.6 (b) and (c) above, this leads to some (possibly) new observations about quasiconformal maps between (inner) uniform domains.

We shall show later in this chapter, after the proof of Theorem 6.4, that uniform domains in locally compact Q-regular Q-Loewner spaces are Q-Loewner. This does not follow directly from Theorem 6.4; the localization issue is slightly complicated.

We now begin the proof of Theorem 6.4. First we record the following estimate:

$$\frac{1}{8A^2}\, d(x)k(x,y) \leqslant d(x,y) \leqslant \ell_d(\gamma) \leqslant 2d(x)k(x,y), \tag{6.8}$$

whenever (Ω, d) is an A-uniform space, k is the quasihyperbolic metric in Ω, and $\gamma = [x,y]$ is a quasihyperbolic geodesic joining two points x, y in Ω with $k(x,y) \leqslant 1/4$. (See Chapter 2.) The first inequality follows from (2.3) and (2.16). The second inequality is trivial, and the last follows from the estimate

$$\log\left(1 + \frac{\ell_d(\gamma)}{d(x)}\right) = \int_0^{\ell_d(\gamma)} \frac{ds}{d(x)+s} \leqslant \ell_k(\gamma) = k(x,y)$$

together with the assumption $k(x,y) \leqslant 1/4$.

Lemma 6.9. — *Let (Ω, d) be an A-uniform space, and let $\gamma = [x,y]$ be a quasihyperbolic geodesic in Ω joining two points x and y. If $k(x,y) \geqslant \varepsilon_1$, where $0 < \varepsilon_1 \leqslant 1/8$, then there exist points $z_0 = x,\ z_1, \ldots, z_N = y$ on γ such that*

$$\varepsilon_1 \leqslant k(z_{\nu-1}, z_\nu) \leqslant 2\varepsilon_1, \quad \nu = 1, \ldots, N, \tag{6.10}$$

and that

$$\sum_{\nu=0}^{N} d(z_\nu)^\alpha \leqslant C(A, \alpha, \varepsilon_1)\ell_d(\gamma)^\alpha \tag{6.11}$$

for each $\alpha > 0$.

Proof. — Since $k(x,y) \geqslant \varepsilon_1$, it is clear that we can choose points $z_0, z_1, \ldots, z_N$ as in (6.10), by simply choosing them successively when we traverse γ from x to y, such that $k(z_{\nu-1}, z_\nu) = \varepsilon_1$ except possibly for $\nu = N$.

Next, let $L = \ell_d(\gamma)$ and let $w \in \gamma$ be the unique point with $L/2 = \ell_d([x,w]) = \ell_d([w,y])$, where $[x,w]$ and $[w,y]$ denote the subgeodesics of $\gamma = [x,y]$ from x to w and w to y, respectively. We claim that

$$d(w) \leqslant \frac{8A^2}{\varepsilon_1}\, L. \tag{6.12}$$

Indeed, we may assume both that $d(w) \geqslant L$ and that $k(x,w) \geqslant \varepsilon_1/2$, the latter by symmetry; thus

$$d(x,w) \leqslant \frac{L}{2} \leqslant \frac{d(w)}{2}$$

which implies

$$d(x) \geqslant d(w) - d(x,w) \geqslant \frac{d(w)}{2},$$

and therefore, by (2.16),

$$\frac{\varepsilon_1}{2} \leqslant k(x,w) \leqslant 4A^2 \log\left(1 + \frac{L/2}{d(w)/2}\right)$$
$$\leqslant 4A^2 \frac{L}{d(w)}.$$

This is (6.12).

Now relabel the points $z_0, z_1, \ldots, z_N$ so that $y_0, \ldots, y_{N_1}$ are the points that we pass if we travel from w to y along γ, in this order, and $y'_0, \ldots, y'_{N_2}$ are the points that we pass if we travel from w to x along γ, in this order. Thus $x = y'_{N_2}, y = y_{N_1}$, and it may happen that $y_0 = y'_0 = w$. We have that $k(w, y_0) \leqslant 2\varepsilon_1 \leqslant 1/4$, and so

$$d(y_0) \leqslant \exp\{k(w,y_0)\}d(w) \leqslant \frac{16A^2}{\varepsilon_1}L \tag{6.13}$$

by (2.3) and (6.12).

By Theorem 2.10, γ is a B-uniform arc for some $B = B(A)$. This implies that

$$d(y_n) \geqslant \frac{1}{B}\ell_d\left(\gamma(y_n, y)\right)$$
$$= \frac{1}{B}\sum_{\nu=n}^{N_1-1} \ell_d\left(\gamma(y_\nu, y_{\nu+1})\right)$$

for all $n = 0, \ldots, N_1 - 1$. Because $\varepsilon_1 \leqslant k(y_\nu, y_{\nu+1}) \leqslant 2\varepsilon_1 \leqslant 1/4$, estimate (6.8) gives

$$d(y_{\nu+1}) \leqslant \frac{8A^2}{\varepsilon_1}\ell_d\left(\gamma(y_\nu, y_{\nu+1})\right),$$

whence

$$d(y_n) \geqslant \frac{\varepsilon_1}{8A^2B}\sum_{\nu=n+1}^{N_1} d(y_\nu)$$

for all $n = 0, \ldots, N_1 - 1$. A simple lemma on geometric series together with (6.13) thus implies

$$d(y_{\nu+1}) \leqslant K\left(\frac{K}{K+1}\right)^\nu d(y_0) \leqslant \frac{K16A^2}{\varepsilon_1}\left(\frac{K}{K+1}\right)^\nu L$$

for $\nu = 0, \ldots, N_1 - 1$, where $K = 8A^2B/\varepsilon_1$. A similar reasoning applies to the points $y'_0, \ldots, y'_{N_2}$, and we conclude that (6.11) holds. The proof of Lemma 6.9 is complete. □

Proof of Theorem 6.4. — Let (Ω, d, μ) be a locally Q-Loewner space with data $v = (Q, \kappa, \varepsilon_0, \psi)$; we also assume that Ω is A-uniform. Fix two disjoint nondegenerate continua $E, F \subset \Omega$, and let $\rho\colon \Omega \to [0,\infty]$ be a Borel function such that

$$\int_\gamma \rho\, ds \geqslant 1$$

for every rectifiable curve γ joining E and F in Ω. We have to show that there is a positive lower bound

$$\int_\Omega \rho^Q\, d\mu \geqslant C_0 > 0, \tag{6.14}$$

where C_0 depends only on the data v and A, and on the number

$$t = \Delta(E,F) = \frac{\operatorname{dist}\,(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F}. \tag{6.15}$$

It is not hard to see that we may assume that $t \geqslant 1$ and that

$$\operatorname{diam} E \leqslant \operatorname{diam} F \leqslant 2\ \operatorname{diam} E.$$

(See the reasoning in [**HeiK**, proof of Theorem 3.6, p. 19].) Moreover, if there are points $a \in E$, $b \in F$ such that $k(a,b) < \varepsilon_0/12$, then by estimate (6.8) we have

$$\begin{aligned}\operatorname{diam} E \vee \operatorname{diam} F &\leqslant 2\operatorname{dist}(E,F)\\ &\leqslant 2d(a,b) \leqslant 4d(a)k(a,b)\\ &< \frac{\varepsilon_0}{3} d(a).\end{aligned}$$

It follows that E and F both lie in $B(a, \varepsilon_0 d(a))$, and we have by the local Loewner property that

$$\int_\Omega \rho^Q\, d\mu \geqslant \int_{B(a,\kappa\varepsilon_0 d(a))} \rho^Q\, d\mu \geqslant \psi(t) > 0$$

as required. Therefore, we may assume that

$$k(a,b) \geqslant \frac{\varepsilon_0}{12} \tag{6.16}$$

whenever $a \in E,\ b \in F$.

Next, denote $D = \operatorname{dist}(E,F)$, and for $\lambda > 0$ define

$$\Lambda_\lambda = \left\{ a \in \Omega : \int_{B(a,\kappa\varepsilon_0 d(a))} \rho^Q\, d\mu \geqslant \lambda\, \frac{d(a)}{D} \right\}. \tag{6.17}$$

Lemma 6.18. — *There is $\lambda_0 = \lambda_0(v, A, t) > 0$ such that the set*

$$H_{\lambda_0} = \bigcup_{a \in \Lambda_{\lambda_0}} B(a, 2Bd(a)) \tag{6.19}$$

contains either E or F, where $B = B(A)$ is the constant in Theorem 2.10.

Suppose that the lemma has been proved, and suppose that $E \subset H_{\lambda_0}$. Because E is compact, it belongs to a finite number of balls from the collection that comprises

H_{λ_0}, and by standard covering arguments [**Fe**, 2.8] we find points $a_1, \ldots, a_p \in \Lambda_{\lambda_0}$ such that the balls $B(a_i, 2Bd(a_i))$, $i = 1, \ldots, p$, are pairwise disjoint, and that

$$E \subset \bigcup_{i=1}^{p} B\left(a_i, 10Bd(a_i)\right).$$

We have $2B \geqslant 1 \geqslant \kappa\varepsilon_0$, so that

$$\begin{aligned}
\frac{1}{t} = \frac{\operatorname{diam} E \wedge \operatorname{diam} F}{\operatorname{dist}(E,F)} &= \frac{\operatorname{diam} E}{D} \\
&\leqslant \frac{1}{D}\sum_{i=1}^{p} \operatorname{diam} B(a_i, 10Bd(a_i)) \\
&\leqslant \frac{20B}{D}\sum_{i=1}^{p} d(a_i) \leqslant \frac{20B}{\lambda_0}\sum_{i=1}^{p}\int_{B(a_i,2Bd(a_i))} \rho^Q \, d\mu \\
&\leqslant \frac{20B}{\lambda_0}\int_\Omega \rho^Q \, d\mu.
\end{aligned}$$

This gives the desired inequality (6.14).

It therefore remains to prove Lemma 6.18. To this end, fix $\lambda > 0$ and assume that neither E nor F belongs to the set

$$H_\lambda = \bigcup_{a \in \Lambda_\lambda} B(a,\ 2Bd(a)).$$

We have to produce a positive lower bound for λ, depending only on t, A, and v. By assumption there are points $x \in E$ and $y \in F$ such that $x, y \notin H_\lambda$. Let $\gamma = [x,y]$ be a quasihyperbolic geodesic joining x and y in Ω. By Theorem 2.10, γ is a B-uniform curve; in particular,

$$d(x,z) \wedge d(z,y) \leqslant Bd(z)$$

for each $z \in \gamma$, which implies that γ does not meet Λ_λ. Therefore, by the definition of Λ_λ in (6.17),

$$\int_{B(z,\kappa\varepsilon_0 d(z))} \rho^Q \, d\mu < \frac{\lambda d(z)}{D} \tag{6.20}$$

for each $z \in \gamma$.

Next, let $L = \ell_d(\gamma)$. Then $d(x,y) \leqslant L \leqslant Bd(x,y)$. On the other hand,

$$D = \operatorname{dist}(E,F) \leqslant d(x,y) \leqslant \operatorname{dist}(E,F) + \operatorname{diam} E + \operatorname{diam} F \leqslant 4D$$

by the assumptions made in the beginning of the proof. It follows that

$$D \leqslant L = \ell_d(\gamma) \leqslant 4BD. \tag{6.21}$$

Now let $\varepsilon_1 = \varepsilon_0/12$. Then $k(x,y) \geqslant \varepsilon_1$ by (6.16). According to Lemma 6.9, there exist points $z_0 = x,\ z_1, \ldots, z_N = y$ on γ such that

$$\frac{\varepsilon_0}{12} \leqslant k(z_{\nu-1}, z_\nu) \leqslant \frac{\varepsilon_0}{6}, \qquad \nu = 1, \ldots, N, \tag{6.22}$$

and that

$$\sum_{\nu=0}^{N} d(z_\nu)^{1/Q} \leqslant C_1 L^{1/Q}, \tag{6.23}$$

where $C_1 = C_1(A, Q, \varepsilon_0)$. Estimates (6.8) and (6.22) now give

$$\frac{\varepsilon_0}{96A^2}\, d(z_\nu) \leqslant \left\{ \begin{matrix} d(z_\nu, z_{\nu+1}) \\ d(z_\nu, z_{\nu-1}) \end{matrix} \right\} \leqslant \frac{\varepsilon_0}{3}\, d(z_\nu),\ \nu \in \left\{ \begin{matrix} 0,\dots,N-1 \\ 1,\dots,N \end{matrix} \right\}. \tag{6.24}$$

The second inequality in (6.24) implies

$$\frac{1}{2} d(z_\nu) \leqslant d(z_{\nu-1}) \leqslant 2d(z_\nu), \qquad \nu = 1, \dots, N. \tag{6.25}$$

Next, we choose for $\nu = 1, \dots, N-1$ a subarc γ_ν of γ that joins z_ν inside $\overline{B}(z_\nu, \frac{\varepsilon_0 d(z_\nu)}{1000\, A^2})$ to the boundary of that ball; then

$$\frac{\varepsilon_0 d(z_\nu)}{1000A^2} \leqslant \operatorname{diam} \gamma_\nu \leqslant \frac{\varepsilon_0 d(z_\nu)}{500A^2}, \qquad \nu = 1, \dots, N-1. \tag{6.26}$$

The subarcs γ_ν can be found because of (6.24).

We choose subcontinua γ_0 of E and γ_N of F such that $x \in \gamma_0$, $y \in \gamma_N$, and

$$\frac{D}{2t} \wedge \frac{\varepsilon_0 d(z_\nu)}{1000\, A^2} \leqslant \operatorname{diam} \gamma_\nu \leqslant 2\left(\frac{D}{2t} \wedge \frac{\varepsilon_0 d(z_\nu)}{1000A^2}\right), \quad \nu = 0, N. \tag{6.27}$$

The subcontinua γ_0, γ_N can be found by the choice of t and by [**HY**, 2.16]. By (6.21), (6.23), and (6.27),

$$c_2\, d(z_\nu) \leqslant \operatorname{diam} \gamma_\nu \leqslant \frac{\varepsilon_0}{500A^2}\, d(z_\nu), \qquad \nu = 0, N, \tag{6.28}$$

where $c_2 = c_2(A, Q, \varepsilon_0, t) > 0$. It follows from (6.24), (6.25), (6.26) and (6.28) that

$$\frac{\varepsilon_0}{250\, A^2} d(z_\nu) \leqslant \operatorname{dist}(\gamma_\nu, \gamma_{\nu+1}) \leqslant d(z_\nu, z_{\nu+1}) \leqslant \frac{\varepsilon_0}{3} d(z_\nu) \tag{6.29}$$

and that

$$\operatorname{diam} \gamma_\nu \wedge \operatorname{diam} \gamma_{\nu+1} \geqslant c_3\, d(z_\nu) \tag{6.30}$$

for $\nu = 0, \dots, N-1$, where $c_3 = c_3(A, Q, \varepsilon_0, t) > 0$. Moreover,

$$\gamma_\nu \cup \gamma_{\nu+1} \subset B(z_\nu, \varepsilon_0 d(z_\nu)), \qquad \nu = 0, \dots, N-1. \tag{6.31}$$

By using (6.29), (6.30), (6.31), and the local Loewner property of Ω, we deduce that

$$\operatorname{mod}_Q(\gamma_\nu, \gamma_{\nu+1}\, ; B(z_\nu, \kappa\varepsilon_0 d(z_\nu)) \geqslant c_4 > 0$$

for $\nu = 0, \dots, N-1$, where $c_4 = c_4(A, Q, \psi, \varepsilon_0, t)$. Therefore, if

$$\int_\sigma \rho\, ds \geqslant \left(\frac{\lambda d(z_\nu)}{c_4 D}\right)^{1/Q}$$

for all paths σ joining γ_ν and $\gamma_{\nu+1}$ in $B(z_\nu, \kappa\varepsilon_0 d(z_\nu))$, it would follow from (6.20) that

$$c_4 \leqslant \left(\frac{c_4 D}{\lambda d(z_\nu)}\right) \int_{B(z_\nu, \kappa\varepsilon_0 d(z_\nu))} \rho^Q\, d\mu < c_4$$

which is absurd. We thus find, for each $\nu = 0, \ldots, N-1$, a curve α_ν joining γ_ν and $\gamma_{\nu+1}$ in $B(z_\nu, \kappa\varepsilon_0 d(z_\nu)) \subset \Omega$ with

$$\int_{\alpha_\nu} \rho\, ds \leqslant \left(\frac{\lambda d(z_\nu)}{c_4 D}\right)^{1/Q}. \tag{6.32}$$

Next we analyze the relative of position of the curves α_ν. By (6.25) and (6.29),

$$\operatorname{diam}\alpha_{\nu-1} \wedge \operatorname{diam}\alpha_\nu \geqslant \frac{\varepsilon_0}{500\, A^2}\, d(z_\nu), \qquad \nu = 1, \ldots, N-1.$$

We can therefore select subcurves α'_ν of $\alpha_{\nu-1}$ and α''_ν of α_ν such that both α'_ν and α''_ν have one end point on γ_ν, and that

$$\begin{aligned} \frac{\varepsilon_0}{1000\, A^2}\, d(z_\nu) &\leqslant \operatorname{diam}\alpha'_\nu \wedge \operatorname{diam}\alpha''_\nu \\ &\leqslant \operatorname{diam}\alpha'_\nu \vee \operatorname{diam}\alpha''_\nu \leqslant \frac{\varepsilon_0}{500\, A^2}\, d(z_\nu) \end{aligned} \tag{6.33}$$

for $\nu = 1, \ldots, N-1$. Because

$$\operatorname{dist}(\alpha'_\nu, \alpha''_\nu) \leqslant \operatorname{diam}\gamma_\nu \leqslant \frac{\varepsilon_0}{500 A^2}\, d(z_\nu) \tag{6.34}$$

by (6.26), we have that

$$\alpha'_\nu \cup \alpha''_\nu \subset B(z_\nu, \varepsilon_0 d(z_\nu)), \qquad \nu = 1, \ldots, N-1. \tag{6.35}$$

As before, by using (6.33), (6.34), and (6.35), and the local Loewner property of Ω, we deduce that

$$\operatorname{mod}_Q(\alpha'_\nu, \alpha''_\nu; B(z_\nu, \kappa\varepsilon_0 d(z_\nu))) \geqslant \psi(2) = c_5 > 0 \tag{6.36}$$

for $\nu = 1, \ldots, N-1$. Moreover, as before, we find a curve β_ν joining α'_ν and α''_ν in $B(z_\nu, \kappa\varepsilon_0 d(z_\nu)) \subset \Omega$ with

$$\int_{\beta_\nu} \rho\, ds \leqslant \left(\frac{\lambda d(z_\nu)}{c_5 D}\right)^{1/Q}, \qquad \nu = 1, \ldots, N-1. \tag{6.37}$$

Note in particular that β_ν joins $\alpha_{\nu-1}$ and α_ν, for $\nu = 1, \ldots, N-1$. It follows that the set

$$\alpha_0 \cup \beta_1 \cup \alpha_1 \cup \cdots \cup \beta_{N-1} \cup \alpha_{N-1}$$

is connected and joins E to F in Ω. We choose a curve σ from this union such that σ joins E to F; then

$$\begin{aligned} \int_\sigma \rho\, ds &\leqslant \sum_{\nu=0}^{N-1} \int_{\alpha_\nu} \rho\, ds + \sum_{\nu=1}^{N-1} \int_{\beta_\nu} \rho\, ds \\ &\leqslant \sum_{\nu=0}^{N-1} \left(\frac{\lambda d(z_\nu)}{c_4\, D}\right)^{1/Q} + \sum_{\nu=1}^{N-1} \left(\frac{\lambda d(z_\nu)}{c_5\, D}\right)^{1/Q} \\ &\leqslant c_6 \left(\frac{\lambda}{D}\right)^{1/Q} \sum_{\nu=0}^{N} d(z_\nu)^{1/Q} \leqslant c_7 \lambda^{1/Q} \end{aligned}$$

by (6.32), (6.37), (6.23), and (6.21), where $c_7 = c_7(A, Q, \varepsilon_0, \psi, t)$. On the other hand, since ρ is an admissible test function for the curve family $(E, F; \Omega)$, we have

$$1 \leqslant \int_\sigma \rho\, ds \leqslant c_7 \lambda^{1/Q}.$$

This gives the required lower bound for λ, and Lemma 6.18 follows.

Theorem 6.4 is thereby completely proved. □

Remark 6.38. — The argument in the proof of Theorem 6.4 can be used to prove a modulus estimate where the sets E and F are allowed to lie in $\overline{\Omega}$, i.e., they may contain part of the boundary of Ω. To be specific, similarly as above let $(E, F; \Omega)$ denote the family of all curves that connect E and F and lie in Ω with the possible exception of their endpoints. Then, if Ω is a uniform and locally Q-Loewner space, there is a positive lower bound for $\operatorname{mod}_Q(E, F; \Omega)$ only depending on $t > 0$ whenever $E, F \subset \overline{\Omega}$ are continua with $\Delta(E, F) \leqslant t$. To see this, the following modifications for the proof of Theorem 6.4 are necessary. First note that there is a version of Lemma 6.9 where the geodesic $[x, y]$ is allowed to be a geodesic ray or an infinite geodesic. In this case the points z_ν will be defined for all $\nu \in \mathbb{N}$, or all $\nu \in \mathbb{Z}$, respectively. In the considerations of the proof of Theorem 6.4 it may happen that the points x and y lie in $\partial\Omega$. To deal with this situation one has to appeal to the modified version of Lemma 6.9. We leave the details to the reader.

We shall next examine when the local Loewner property is stable under conformal deformations.

Let (Ω, d) be a locally compact, rectifiably connected metric space, and let μ be a Borel measure on Ω. Given a Borel function $\rho\colon \Omega \to (0, \infty)$, denote by

$$\lambda_\rho(\gamma) = \int_\gamma \rho\, ds$$

the ρ-length of a rectifiable curve γ, and by $d_\rho(x, y)$ the corresponding distance function,

$$d_\rho(x, y) = \inf \lambda_\rho(\gamma),$$

where the infimum is taken over all rectifiable curves γ in Ω joining x and y. We assume in this chapter that ρ is continuous, and that the identity map $(\Omega, d) \to (\Omega, d_\rho)$ is a homeomorphism; the latter happens if, for example, (Ω, d) is quasiconvex. Moreover, in this case, a curve γ in Ω has finite length $\ell_d(\gamma)$ in the metric d if and only if it has finite length $\lambda_\rho(\gamma)$ in the metric d_ρ. (See Appendix for these facts.)

Recall that a metric space is called *A-quasiconvex*, $A \geqslant 1$, if every pair of points in the space can be joined by a curve whose length is no more than A times the distance between the points.

Next, fix $Q > 1$. Let μ_ρ be a Borel measure in Ω defined by $d\mu_\rho = \rho^Q d\mu$. Then the Q-modulus of a family of curves in Ω is the same in the two metric measure spaces

(Ω, d, μ) and $(\Omega, d_\rho, \mu_\rho)$. This conformal invariance of modulus follows directly from the definition (6.3).

Theorem 6.39. — *Let (Ω, d, μ) be a locally compact, noncomplete and A-quasiconvex locally Q-Loewner space. Assume that $\rho\colon \Omega \to (0, \infty)$ is a continuous function such that (Ω, d_ρ) is incomplete and such that the following two conditions hold for some $C \geqslant 1$:*

$$\frac{1}{C} \leqslant \frac{\rho(x)}{\rho(y)} \leqslant C \tag{6.40}$$

whenever $x, y \in B(z, d(z)/2)$ and $z \in \Omega$, and

$$\frac{1}{C}\, \rho(x)d(x) \leqslant d_\rho(x) \leqslant C\rho(x)d(x) \tag{6.41}$$

for each $x \in \Omega$. Then $(\Omega, d_\rho, \mu_\rho)$ is a locally Q-Loewner space, quantitatively.

Proof. — First we prove that

$$\frac{1}{C}\, \rho(z)d(x,y) \leqslant d_\rho(x,y) \leqslant AC\rho(z)d(x,y) \tag{6.42}$$

whenever $x, y \in B(z, d(z)/(8A))$ and $z \in \Omega$. To see the first inequality, let γ be a rectifiable curve joining x and y in Ω. If $\gamma \subset B(z, d(z)/2)$, then

$$\lambda_\rho(\gamma) \geqslant \frac{1}{C}\, \rho(z)\ell_d(\gamma) \geqslant \frac{1}{C}\, \rho(z)d(x,y);$$

if γ leaves $B(z, d(z)/2)$, then it has a subcurve γ_0 in $B(z, d(z)/2)$ with length $\ell_d(\gamma_0) \geqslant \frac{3}{8}d(z) \geqslant d(x,y)$. Thus the first inequality in (6.42) follows. To prove the second inequality, choose a curve γ that joins x and y such that $\ell_d(\gamma) \leqslant Ad(x,y) \leqslant \frac{1}{4}\, d(z)$; thus $\gamma \subset B(z, d(z)/2)$ and hence

$$\begin{aligned} d_\rho(x,y) &\leqslant \lambda_\rho(\gamma) \leqslant C\rho(z)\ell_d(\gamma) \\ &\leqslant AC\rho(z)d(x,y) \end{aligned}$$

as required.

Next, it is easy to see that

$$d_\rho(x,y) \geqslant \frac{1}{2C}\rho(x)d(x) \tag{6.43}$$

whenever $d(x,y) \geqslant d(x)/2$. It follows that

$$B_\rho(x, \varepsilon d_\rho(x)) \subset B(x, \varepsilon C^2 d(x)) \tag{6.44}$$

whenever $x \in \Omega$ and $0 < \varepsilon \leqslant 1/(2C^2)$, where B_ρ denotes a ball in the metric d_ρ. Indeed, if $y \in B_\rho(x, \varepsilon d_\rho(x)) \cap B(x, d(x)/2)$, then

$$\begin{aligned} \frac{1}{C}\, \rho(x)d(x,y) &\leqslant d_\rho(x,y) < \varepsilon d_\rho(x) \\ &\leqslant \varepsilon\, C\rho(x)d(x), \end{aligned}$$

by (6.43) and (6.41), while if $y \in B_\rho(x, \varepsilon d_\rho(x)) \setminus B(x, d(x)/2)$, then

$$\begin{aligned}\frac{1}{2C}\, \rho(x)d(x) &\leqslant d_\rho(x,y) < \varepsilon d_\rho(x) \leqslant \varepsilon\, C\rho(x)d(x) \\ &\leqslant \frac{1}{2C}\rho(x)d(x)\end{aligned}$$

by (6.43), which is impossible.

Similarly one shows that

$$B(x, \varepsilon d(x)) \subset B_\rho(x, \varepsilon AC^2 d_\rho(x)) \tag{6.45}$$

whenever $x \in \Omega$ and $0 < \varepsilon \leqslant 1/(8A)$.

By assumption, (Ω, d, μ) is a locally Q-Loewner space with data $(\kappa, \varepsilon_0, \psi)$, say. We claim that $(\Omega, d_\rho, \mu_\rho)$ is a locally Q-Loewner space with data $(\kappa', \varepsilon_0', \psi')$, where

$$\varepsilon_0' = \frac{\varepsilon_0}{8\kappa AC^4}\ , \ \kappa' = \kappa AC^4, \quad \psi'(t) = \psi(AC^2 t).$$

(Note that $\varepsilon_0'\kappa' \leqslant 1$.) To prove this claim, let E and F be two disjoint continua in $B_\rho(z, \varepsilon d_\rho(z))$, where $z \in \Omega$ and $0 < \varepsilon \leqslant \varepsilon_0'$. Let

$$t = \frac{\operatorname{dist}_\rho(E,F)}{\operatorname{diam}_\rho\, E \wedge \operatorname{diam}_\rho\, F}.$$

Then

$$\begin{aligned}B_\rho(z, \varepsilon d_\rho(z)) &\subset B(z, \varepsilon C^2 d(z)) \subset B(z, \varepsilon\kappa C^2 d(z)) \\ &\subset B_\rho(z, \varepsilon\kappa AC^4 d_\rho(z))\end{aligned} \tag{6.46}$$

by (6.44) and (6.45). In particular, E and F lie in $B(z, d(z)/(8A))$. It follows from (6.42) that

$$\begin{aligned}\frac{\operatorname{dist}(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F} &\leqslant \frac{C}{\rho(z)}\ \frac{\operatorname{dist}_\rho(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F} \\ &\leqslant AC^2\ \frac{\operatorname{dist}_\rho(E,F)}{\operatorname{diam}_\rho\, E \wedge \operatorname{diam}_\rho\, F} = AC^2 t.\end{aligned}$$

Therefore, by (6.46) and by the local Loewner property of (Ω, d, μ), we have

$$\operatorname{mod}_Q\big(E, F; B_\rho(z, \varepsilon\kappa AC^4\, d_\rho(z))\big) \geqslant \operatorname{mod}_Q\big(E, F; B(z, \varepsilon\kappa C^2\, d(z))\big) \geqslant \psi(AC^2 t).$$

The conformal invariance of modulus was also used here. This completes the proof of Theorem 6.39. □

We call an open subset Ω of a locally compact metric space (M, d) a *uniform subdomain* of M if (Ω, d) is a uniform space in the sense of Definition 1.9. We also say that Ω is a *uniform domain* in M.

The remainder of this chapter is devoted to a proof of the following result, and to a discussion of some of its corollaries.

Theorem 6.47. — *An open connected subset of a locally compact Q-regular Q-Loewner space is locally Q-Loewner. In particular, uniform subdomains of such spaces are Q-Loewner. The statement is quantitative in the usual sense.*

The second assertion of Theorem 6.47 follows from the first in view of Theorem 6.4. (Recall the definition for Q-regularity from (6.7).)

The proof of Theorem 6.47 requires two propositions (with independent merit).

Proposition 6.48. — *Let (X, d, μ) be a Q-regular Q-Loewner space. Then there exist functions $\psi\colon (0,\infty) \to (0,\infty)$ and $\kappa\colon (0,\infty) \to [1,\infty)$ with the following property: if E and F are two disjoint nondegenerate continua in a ball $B(x,r)$ in X, then*

$$\operatorname{mod}_Q(E, F; B(x, \kappa(t)r)) \geqslant \psi(t)$$

whenever

$$\Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F} \leqslant t.$$

The statement is quantitative in the usual sense.

Proof. — By assumption, there exists a decreasing function $\varphi\colon (0,\infty) \to (0,\infty)$ such that

$$\operatorname{mod}_Q \Gamma \geqslant \varphi(t),$$

where $\Gamma = (E, F; X)$ is the family of curves that join E and F in X. Fix a constant $\kappa > 2$, and let Γ_1 be the subfamily of Γ consisting of curves that lie in $B(x, \kappa r)$, and let $\Gamma_2 = \Gamma \setminus \Gamma_1$. Then

$$\begin{aligned} \operatorname{mod}_Q \Gamma &\leqslant \operatorname{mod}_Q \Gamma_1 + \operatorname{mod}_Q \Gamma_2 \\ &\leqslant \operatorname{mod}_Q \Gamma_1 + C(\log \kappa)^{1-Q}, \end{aligned}$$

where the second inequality follows from **[HeiK**, 3.14**]**. (The Q-regularity of X is used here.) We can choose $\kappa = \kappa(Q, \varphi, t)$ so large that

$$\operatorname{mod}_Q \Gamma_1 \geqslant \frac{1}{2}\,\varphi(t).$$

This proves the proposition. □

Our aim is to show that the function κ in Proposition 6.48 can be chosen to be constant, i.e., independent of the parameter t. The following is a handy sufficient criterion for the local Loewner property; for a global version of this result, see Remarks 6.67 (a) below.

Proposition 6.49. — *Let (Ω, d, μ) be a locally compact noncomplete and A-quasiconvex metric measure space, and let $Q > 1$. Assume that there exist numbers $\kappa \geqslant 1$, $\varepsilon_0 \in (0, \kappa^{-1}]$, and $\zeta > 0$ with the following property: if E and F are disjoint nondegenerate continua in a ball $B(x, \varepsilon d(x))$ in Ω, $0 < \varepsilon \leqslant \varepsilon_0$, and if*

$$\Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\operatorname{diam} E \wedge \operatorname{diam} F} \leqslant 16,$$

then

$$\operatorname{mod}_Q\big(E,F;B(x,\varepsilon\kappa d(x))\big) \geqslant \zeta.$$

Then (Ω, d, μ) *is a locally Q-Loewner space, quantitatively.*

Proof. — Define

$$\varepsilon_0' = \frac{1}{2+4A+6\kappa} \wedge \varepsilon_0, \quad \kappa' = 1+2A+6\kappa. \tag{6.50}$$

(Note that $\varepsilon_0'\kappa' \leqslant 1$.) Let E and F be two disjoint nondegenerate continua in $B(a,\varepsilon d(a))$, where $a \in \Omega$ and $0 < \varepsilon \leqslant \varepsilon_0'$. Fix a Borel function $\rho\colon B(a,\varepsilon\kappa' d(a)) \to [0,\infty]$ such that

$$\int_\gamma \rho\, ds \geqslant 1$$

for each curve γ joining E and F in $B(a,\varepsilon\kappa' d(a))$. We shall show that

$$\int_{B(a,\varepsilon\kappa' d(a))} \rho^Q\, d\mu \geqslant \zeta_1 > 0, \tag{6.51}$$

where ζ_1 depends only on the data associated with Ω, and on a number $t \geqslant \Delta(E,F)$ that is fixed from now on.

To this end, fix $x \in E$ and $y \in F$ such that

$$d(x,y) = \operatorname{dist}(E,F) \leqslant 2\varepsilon d(a), \tag{6.52}$$

and then choose a curve σ joining x and y in Ω such that

$$\ell(\sigma) \leqslant A d(x,y). \tag{6.53}$$

We claim that

$$B(z, s\kappa d(z)) \subset B(a,\varepsilon\kappa' d(a)) \tag{6.54}$$

for each $z \in \sigma$ whenever $s \leqslant \varepsilon$. Indeed, if $z \in \sigma$, then

$$\begin{aligned} d(a,z) &\leqslant d(a,x) + d(x,z) \\ &\leqslant \varepsilon d(a) + A d(x,y) \leqslant (1+2A)\varepsilon d(a) \\ &\leqslant \frac{1}{2}\, d(a) \end{aligned}$$

by (6.52), (6.53), and the choice of ε_0' in (6.50); thus

$$\frac{1}{2}\, d(a) \leqslant d(z) \leqslant \frac{3}{2}\, d(a), \quad z \in \sigma. \tag{6.55}$$

Hence, if $w \in B(z, s\kappa d(z))$, we have

$$\begin{aligned} d(a,w) &\leqslant d(a,z) + d(z,w) \\ &\leqslant (1+2A)\varepsilon d(a) + s\kappa \frac{3}{2}\, d(a) \\ &\leqslant \left(1+2A+\frac{3\kappa}{2}\right)\varepsilon d(a) < \kappa'\varepsilon d(a), \end{aligned}$$

from which (6.54) follows. Now set

$$s = \min\left\{\varepsilon, \frac{\operatorname{diam} E}{d(a)}, \frac{\operatorname{diam} F}{d(a)}\right\}.$$

Then we can find subcontinua $E' \subset E$ and $F' \subset F$ such that $x \in E'$ and $y \in F'$, and that

$$\begin{aligned} \operatorname{diam} E' &\geqslant \frac{s}{64}\, d(a), \qquad E' \subset B\left(x, \frac{s}{32}\, d(a)\right), \\ \operatorname{diam} F' &\geqslant \frac{s}{64}\, d(a), \qquad F' \subset B\left(y, \frac{s}{32}\, d(a)\right). \end{aligned} \tag{6.56}$$

(This choice is possible by [**HY**, 2.16].)

We can now consider two cases depending on whether $d(x,y) \leqslant \frac{s}{8}\, d(a)$ or not.

Assume first that $d(x,y) \leqslant \frac{s}{8}\, d(a)$. Then

$$E' \cup F' \subset B\left(x, \frac{5s}{32}\, d(a)\right) \subset B(x, sd(x)) \tag{6.57}$$

and

$$\Delta(E', F') \leqslant 8 \tag{6.58}$$

by our assumption and (6.55). Because ρ is an admissible test function for the curve family $\big(E, F; B(a, \varepsilon\kappa' d(a))\big)$, it is also admissible for $\big(E', F'; B(x, \varepsilon\kappa d(x))\big)$ by (6.54). It thus follows from (6.57), (6.58), (6.54), and from the assumed local Loewner property of Ω, that the integral in (6.51) is bounded below by a number ζ_1 depending only on the data associated with Ω.

It remains to examine the case where $d(x,y) > \frac{s}{8}\, d(a)$. By traveling along σ from x to y, we can pick points $z_0 = x, z_1, \ldots, z_N = y$ on σ in successive order such that

$$\frac{s}{8}\, d(a) \leqslant d(z_\nu, z_{\nu+1}) \leqslant \frac{s}{4}\, d(a) \tag{6.59}$$

for $\nu = 0, 1, \ldots, N-1$. By (6.53) and (6.52),

$$\frac{sN}{8}\, d(a) \leqslant \ell(\sigma) \leqslant Ad(x,y) \leqslant 2\varepsilon A d(a).$$

This together with the definition of s implies

$$\begin{aligned} 1 \leqslant N &\leqslant \max\{16A,\ 8A\Delta(E,F)\} \\ &\leqslant 16A \max\{1, t\}. \end{aligned} \tag{6.60}$$

We claim that the inequality

$$\int_{B\left(z_\nu, \kappa s d(z_\nu)\right)} \rho^Q \, d\mu \ < \ \frac{\zeta}{(2N-1)^Q} \tag{6.61}$$

cannot be true for all $\nu = 0, 1, \ldots, N$. This claim and (6.60) imply (6.51), because

$$B\big(z, s\kappa d(z)\big) \subset B\big(a, \varepsilon\kappa' d(a)\big)$$

for $z \in \sigma$ by (6.54).

To complete the proof of the proposition, it therefore suffices to prove the above claim. Assume on the contrary that (6.61) holds for each $\nu = 0, 1, \ldots, N$. For $\nu = 1, \ldots, N-1$ we can choose subcurves σ_ν of σ such that $z_\nu \in \sigma_\nu$, that $\operatorname{diam} \sigma_\nu \geqslant \frac{s}{64}\, d(a)$ and that $\sigma_\nu \subset B\left(z_\nu,\ \frac{s}{32}\, d(a)\right)$; moreover, let $\sigma_0 = E'$ and $\sigma_N = F'$. Then

$$\frac{s}{16}\, d(a) \leqslant \operatorname{dist}(\sigma_\nu, \sigma_{\nu+1}) \leqslant \frac{s}{4}\, d(a) \tag{6.62}$$

by (6.56) and (6.59), and hence also

$$\Delta(\sigma_\nu, \sigma_{\nu+1}) \leqslant 16$$

for $\nu = 0, \ldots, N-1$; moreover

$$\sigma_\nu \cup \sigma_{\nu+1} \subset B\left(z_\nu\ ,\ \frac{10s}{32}\, d(a)\right) \subset B\left(z_\nu\ ,\ sd(z_\nu)\right) \tag{6.63}$$

for $\nu = 0, \ldots, N-1$, where the second inclusion follows from (6.55). Therefore, because $0 < s \leqslant \varepsilon \leqslant \varepsilon_0$, the assumption gives

$$\operatorname{mod}_Q\left(\sigma_\nu, \sigma_{\nu+1}; B\big(z_\nu,\ \kappa sd(z_\nu)\big)\right) \geqslant \zeta. \tag{6.64}$$

In view of this and our contrapositive assumption, the density

$$(2N-1)\rho$$

cannot be admissible for the curve family $\left(\sigma_\nu, \sigma_{\nu+1}; B\big(z_\nu,\ \kappa sd(z_\nu)\big)\right)$ for each $\nu = 0, \ldots, N-1$; that is, for each such ν we can find a rectifiable curve α_ν joining σ_ν and $\sigma_{\nu+1}$ in $B\big(z_\nu,\ \kappa sd(z_\nu)\big)$ such that

$$\int_{\alpha_\nu} \rho\, ds < \frac{1}{2N-1}. \tag{6.65}$$

Note that $\operatorname{diam} \alpha_\nu \geqslant \frac{s}{16}\, d(a)$ by (6.62). This means that we can find subcurves α'_ν of $\alpha_{\nu-1}$ and α''_ν of α_ν, $\nu = 1, \ldots, N-1$, that both have an end point on σ_ν such that

$$\begin{aligned}\frac{s}{32}\, d(a) &\leqslant \operatorname{diam} \alpha'_\nu \wedge \operatorname{diam} \alpha''_\nu \\ &\leqslant \operatorname{diam} \alpha'_\nu \vee \operatorname{diam} \alpha''_\nu \leqslant \frac{s}{16}\, d(a).\end{aligned}$$

Because $\sigma_\nu \subset B(z_\nu,\ \frac{s}{32} d(a))$, we thus have

$$\alpha'_\nu \cup \alpha''_\nu \subset B\left(z_\nu,\ \frac{4s}{32} d(a)\right) \subset B(z_\nu,\ sd(z_\nu));$$

moreover

$$\Delta(\alpha'_\nu, \alpha''_\nu) \leqslant 4.$$

It follows from the assumption that

$$\operatorname{mod}_Q\left(\alpha'_\nu, \alpha''_\nu\ ; B\left(z_\nu,\ \kappa sd(z_\nu)\right)\right) \geqslant \zeta$$

for each $\nu = 1, \ldots, N-1$, and as above we conclude that there exist curves β_ν joining α'_ν and α''_ν, hence $\alpha_{\nu-1}$ and α_ν, in the ball $B(z_\nu,\ \kappa s d(z_\nu))$ such that

$$\int_{\beta_\nu} \rho\, ds < \frac{1}{2N-1}, \tag{6.66}$$

for $\nu = 1, \ldots, N-1$. By construction, in the union

$$\alpha_0 \cup \beta_1 \cup \alpha_1 \cup \beta_2 \cup \cdots \cup \beta_{N-1} \cup \alpha_{N-1}$$

there is a curve γ that joins E and F inside

$$\bigcup_{\nu=0}^{N} B\left(z_\nu,\ s\kappa d(z_\nu)\right) \subset B(a, \varepsilon\kappa' d(a));$$

for the inclusion, see (6.54). The line integral of ρ over γ satisfies, by (6.65) and (6.66),

$$\int_\gamma \rho\, ds < \frac{N}{2N-1} + \frac{N-1}{2N-1} = 1,$$

which contradicts the assumption of admissibility of the density ρ.

The proof of Proposition 6.49 is complete. □

Remarks 6.67

(a) The proof of Proposition 6.49 above gives another (quantitative) conclusion which we next describe, and which is crucial to the proof of Theorem 6.47.

Let M be a quasiconvex metric measure space with the following property: if E and F are disjoint nondegenerate continua in a ball $B(x,r)$ in M such that $\Delta(E,F) \leqslant 16$, then $\operatorname{mod}_Q(E,F;B(x,\kappa r)) \geqslant \zeta$ for some constants $Q > 1$, $\kappa \geqslant 1$, and $\zeta > 0$, independent of E, F, x, and r. Then there is a constant $\kappa' \geqslant 1$ and a function $\varphi\colon (0,\infty) \to (0,\infty)$ such that

$$\operatorname{mod}_Q\left(E,F;B(x,\kappa' r)\right) \geqslant \varphi(t)$$

whenever E and F are disjoint nondegenerate continua in $B(x,r)$ with $\Delta(E,F) \leqslant t$ and $t > 0$. In particular, M is a Q-Loewner space. Note that M need not be locally compact here.

(b) The hypothesis that (Ω, d, μ) be quasiconvex in Proposition 6.49 can be replaced by the assumption that Ω is a domain in a locally compact quasiconvex space with nonempty complement. The proof works the same.

It was shown in [**HeiK**, Section 3] that Q-regular Q-Loewner spaces are quasiconvex, quantitatively. Thus the remarks made in 6.67 (a) above yield the following corollary:

Corollary 6.68. — *In Proposition 6.48, the function κ can be chosen to be constant.*

Proof of Theorem 6.47. — Let (M,d) be a locally compact Q-regular Q-Loewner space, and let Ω be a domain in M. We may naturally assume that $\Omega \neq M$. A Loewner space is by definition (pathwise) connected, so that (Ω, d) is noncomplete.

As usual, we let $d(x)$ denote the distance from a point x in Ω to the boundary of Ω obtained by completing the space (Ω, d). (Note that this completion boundary of Ω could be different from the boundary of Ω as a subspace of M.) Let κ be the constant in Corollary 6.68, and let $\varepsilon_0 = \kappa^{-1}$. Then, if E and F are disjoint nondegenerate continua in a ball $B(x, \varepsilon d(x))$ for $x \in \Omega$ and $0 < \varepsilon \leqslant \varepsilon_0$, we have by Proposition 6.48 and Corollary 6.68 that

$$\operatorname{mod}_Q (E, F;\ B(x, \kappa\varepsilon d(x)) \geqslant \psi(t)$$

for some function $\psi\colon (0,\infty) \to (0,\infty)$, whenever $\Delta(E,F) \leqslant t$. It follows that (Ω, d) is a local Q-Loewner space, and the proof of Theorem 6.47 is complete. □

CHAPTER 7

GROMOV HYPERBOLIC SPHERICAL DOMAINS

In this chapter, we study Gromov hyperbolicity (in the quasihyperbolic metric) of domains Ω in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 2$. In particular, we shall prove Theorems 1.12, 1.13, and the sufficiency part of Theorem 1.11.

The spherical metric σ on $\overline{\mathbb{R}}^n$ is determined by the length element

$$|dz|_\sigma = \frac{2|dz|}{1+|z|^2}\,,$$

where $|dz|$ is the Euclidean length element and $|z|$ is the Euclidean norm of a point $z \in \mathbb{R}^n$. All metric notions in this chapter refer to the spherical metric σ, unless otherwise stated. Note that $\operatorname{diam} \overline{\mathbb{R}}^n = \pi$.

A *domain* is an open and connected subset of $\overline{\mathbb{R}}^n$. We assume that each domain $\Omega \subset \overline{\mathbb{R}}^n$ has nonempty boundary, so that the quasihyperbolic metric $k = k_\Omega$ can be defined as in Chapter 2. If $\overline{\mathbb{R}}^n \setminus \Omega$ contains a neighborhood of the point at infinity (that is, if Ω is a bounded domain in $\mathbb{R}^n$), then, in the ensuing discussion, the Euclidean metric could be used instead of the spherical metric; this only requires notational changes.

We call a domain Ω in $\overline{\mathbb{R}}^n$ (δ-)*hyperbolic*, or *Gromov hyperbolic*, if (Ω, k) is (δ-)-hyperbolic as defined in Chapter 3.

We begin our study of Gromov hyperbolic domains by dividing points in an arbitrary domain into two classes as follows. For $0 < \lambda \leqslant 1/2$, a point x in a domain Ω is called a λ-*annulus point* if there is a point $\overline{x} \in \partial\Omega$,

$$\sigma(\overline{x}, x) = d(x) = \operatorname{dist}(x, \partial\Omega),$$

such that $\partial\Omega$ does not meet the annulus

$$A(\overline{x}, \lambda) = \left\{ y \in \overline{\mathbb{R}}^n : \lambda d(x) < \sigma(\overline{x}, y) < \frac{d(x)}{\lambda} \right\}. \tag{7.1}$$

If x is not a λ-annulus point, it is called a λ-*arc point*. The following lemma explains the terminology.

Lemma 7.2. — *If $x \in \Omega$ is a λ-arc point, then it lies on an $A(\lambda)$-uniform arc γ in Ω joining two boundary points. Moreover, the arc γ satisfies*

$$\ell_\sigma(\gamma) \leqslant A(\lambda)\ d(x) \tag{7.3}$$

and

$$\ell_k(\gamma(z,w)) \leqslant B(\lambda)k(z,w) \tag{7.4}$$

for each pair of points $z, w \in \gamma$. Here $A(\lambda)$ and $B(\lambda)$ are positive constants that depend only on λ.

Condition (7.4) means that the arc γ is a *quasigeodesic* in the metric space (Ω, k). It is an important (and characteristic [**Bo**]) property of Gromov hyperbolic spaces that quasigeodesics are never far from honest geodesics [**GhHa**, Théorème 11, p. 87]. Thus we have the following corollary to Lemma 7.2.

Corollary 7.5. — *Suppose that Ω is δ-hyperbolic and that $x \in \Omega$ is a λ-arc point. Then there are $C = C(\delta, \lambda) > 0$ and a geodesic line $[a, b]$ joining two boundary points $a, b \in \partial_G\Omega$ such that*

$$k(x, [a, b]) \leqslant C. \tag{7.6}$$

Proof of Lemma 7.2. — Let $x \in \Omega$ be a λ-arc point and let $\overline{x} \in \partial\Omega$ be a point such that $d = d(x) = \sigma(\overline{x}, x)$. Upon performing a spherical isometry, we may assume that $\overline{x} = 0$. By assumption, there is a point $y \in \partial\Omega$ such that

$$\lambda d < \sigma(0, y) = r < \frac{d}{\lambda}.$$

In particular, $x \neq \infty$. Choose a (Euclidean) ray L_y from 0 to ∞ that passes through y; this ray is unique unless $y = \infty$. Let α be a shortest arc on the sphere

$$S_d = \left\{ z \in \overline{\mathbb{R}}^n : \sigma(0, z) = d \right\}$$

from x to $L_y \cap S_d = \{x_0\}$; note that α may be degenerate, which happens if $x_0 = x$. We have that

$$\operatorname{dist}(\alpha, y) = |r - d| < \left(\frac{1}{\lambda} - 1 \right) d.$$

Now slide a closed ball of radius

$$r_1 = \frac{1}{2}\ (1 - \lambda)d$$

first along the arc α from x to x_0 and then along the line segment $[x_0, y]$ so that the center of the ball always stays on the arc $\beta = \alpha \cup [x_0, y]$. There is a first point x_1 on β such that the closed ball $\overline{B}(x_1, r_1)$ meets $\partial\Omega$ at a point y_1. Then $\sigma(x_1, y_1) = r_1$ and

$$\lambda d < \sigma(0, y_1) < \frac{d}{\lambda}.$$

Note that $x_1 \neq x$ and that in general $y_1 \neq y$.

We claim that

$$\gamma = [0,x]_\sigma \cup \beta(x,x_1) \cup [x_1,y_1]_\sigma$$

is the desired arc, where $\beta(x,x_1)$ denotes the subarc of β between x and x_1, and where $[0,x]_\sigma$ and $[x_1,y_1]_\sigma$ denote the (unique) spherical geodesics between $0,x$ and x_1,y_1, respectively.

A simple computation shows that the (spherical) length of γ satisfies

$$\ell_\sigma(\gamma) \leqslant d + \pi d + \left(\frac{1}{\lambda} - 1\right) d = \left(\pi + \frac{1}{\lambda}\right) d,$$

so that the first condition (1.7) of uniformity holds with $A = \frac{(\pi+1/\lambda)}{\lambda}$; this inequality also proves (7.3). To verify the second condition (1.8), observe that for points in $[0,x]_\sigma$ or in $[x_1,y_1]_\sigma$ we can choose $A = 1$ in (1.8), while for $z \in \beta(x,x_1)$ we have

$$d(z) \geqslant \frac{1}{2}(1-\lambda)d \geqslant \frac{1}{2}(1-\lambda)\frac{1}{(\pi+1/\lambda)}\,\ell_\sigma(\gamma).$$

Thus for (1.8) we can choose $A = 4(\pi + 1/\lambda)$.

It remains to verify (7.4). It is easy to see from the construction that γ satisfies a *chord arc* condition in the spherical metric:

$$\ell_\sigma\big(\gamma(z,w)\big) \leqslant C(\lambda)\sigma(z,w) \tag{7.7}$$

for each pair of points $z,w \in \gamma$, where $C(\lambda) \geqslant 1$ depends only on λ. The desired estimate (7.4) now follows from (2.15), (7.7), (2.12), and (2.4). One can choose $B(\lambda) = 4A(\lambda)C(\lambda)$. The proof of the lemma is complete. □

Lemma 7.8. — *If Ω is δ-hyperbolic, then it is $K(\delta)$-roughly starlike with respect to a base point $w \in \Omega$ that satisfies (3.13) (with $d = \sigma$).*

Proof. — Let $x \in \Omega$. If x is a $\frac{1}{2}$-arc point, then it lies within a bounded distance from a geodesic line by Corollary 7.5, and hence within a distance $K(\delta)$ from a geodesic ray emanating from w. Thus we may assume that x is a $\frac{1}{2}$-annulus point. Let $\overline{x} \in \partial\Omega$ be a point such that $\sigma(x,\overline{x}) = d(x)$ and that there are no boundary points $y \in \partial\Omega$ satisfying

$$\frac{1}{2}\,d(x) < \sigma(\overline{x},y) < 2d(x). \tag{7.9}$$

It is not hard to see that there is a quasihyperbolic line $[w,\overline{x}]$ emanating from w and ending at $\overline{x}$; indeed, the line $[x,\overline{x}]$ is a quasihyperbolic ray, and there must be an equivalent ray $[w,\overline{x}]$ starting at w (see Chapter 3). Then, because $d(w) \geqslant d(x)$, there is a point $z \in [w,\overline{x}]$ such that $\sigma(\overline{x},z) = d(x)$, and we easily compute using (7.9) that $k(x,z) \leqslant 4\pi$. This proves the lemma. □

We do not know an example of a domain that is not roughly starlike.

A domain Ω in $\overline{\mathbb{R}}^n$ is locally compact, rectifiably connected and noncomplete in any metric m that satisfies

$$\sigma \leqslant m \leqslant \ell_\sigma \ , \tag{7.10}$$

where $\ell_\sigma(z, w)$ is the infimum of the (spherical) lengths of all curves in Ω joining the points z and w. The quasihyperbolic metric $k = k_\Omega$ in (Ω, m) is independent of the chosen metric m satisfying (7.10), and so is the length of any curve in Ω. In Chapter 3, we showed that if (Ω, m) is a uniform space, then (Ω, k) is a proper, geodesic, and roughly starlike Gromov hyperbolic space, and the boundary $\partial_m\Omega$ of (Ω, m) is (naturally) quasisymmetrically (gauge) equivalent to the Gromov boundary of $\partial_G\Omega$ of (Ω, k) (Theorem 3.6). (We need to observe here that if (Ω, m) is a uniform space, then it is bounded. Without the uniformity assumption, this clearly need not be the case for a metric m as in (7.10).)

If Ω is a Gromov hyperbolic domain and if m is a metric in Ω satisfying (7.10), we say that the Gromov boundary $\partial_G\Omega$ is *naturally* homeomorphic to $\partial_m\Omega$ if all equivalent quasihyperbolic rays in (Ω, k) end at a unique point in $\partial_m\Omega$, and if this correspondence is a homeomorphism. Similarly, we say that the canonical gauge on $\partial_G\Omega$ is *naturally equivalent* to the quasisymmetric gauge determined by $\partial_m\Omega$ if the above correspondence is a quasisymmetric homeomorphism. The proof of Theorem 3.6 shows that the quasisymmetric gauge equivalence of $\partial_G\Omega$ and $\partial_m\Omega$ is natural, if (Ω, m) is uniform.

Next we shall prove a converse assertion, which in particular establishes the sufficiency part of Theorem 1.11. (Recall that for bounded domains in $\mathbb{R}^n$, we can replace the spherical metric by the Euclidean metric throughout this chapter.)

Theorem 7.11. — *Let Ω be a δ-hyperbolic domain in $\overline{\mathbb{R}}^n$. If the canonical gauge on $\partial_G\Omega$ is naturally equivalent to the quasisymmetric gauge determined by σ on $\partial_\sigma\Omega$, then (Ω, σ) is a uniform space.*

The statement in Theorem 7.11 is quantitative in the following sense: if $d_{w,\varepsilon}$ is a metric in the canonical gauge of $\partial_G\Omega$ as in (3.4) and if the natural map between $(\partial_G\Omega, d_{w,\varepsilon})$ and $(\partial_\sigma\Omega, \sigma)$ is η-quasisymmetric, then (Ω, σ) is A-uniform with $A = A(\delta, \eta)$.

We do not know whether Theorem 7.11 is true when σ is replaced with ℓ_σ (or more generally with a metric m as in (7.10)). The answer for $m = \ell_\sigma$ would be *yes* if Proposition 7.13 below is true for ℓ_σ.

Here, and throughout this chapter, $w \in \Omega$ is a fixed base point satisfying (3.13); recall that the distance from a point $x \in \Omega$ to the boundary of Ω is independent of a metric m satisfying (7.10). We also fix a metric $d_\varepsilon = d_{w,\varepsilon}$ once and for all.

The proof of Theorem 7.11 consists of a reduction to the following proposition which is interesting in its own right.

Proposition 7.12. — *Let Ω be a δ-hyperbolic domain in $\overline{\mathbb{R}}^n$, and let m be either σ or ℓ_σ. If (Ω, m) is linearly locally connected with constant c, then (Ω, m) is a uniform space with constant A depending only on δ and c. Moreover, if (Ω, σ) only satisfies the second requirement of linear local connectivity, (LLC_2) below, then (Ω, ℓ_σ) is a uniform space.*

A metric space (M, d) is said to be *linearly locally connected (with constant $c \geqslant 1$)* if the following two conditions are satisfied:

(LLC_1): every pair of points a, b in a ball $B(x, r)$ in M can be joined by a continuum in $B(x, cr)$.

(LLC_2): every pair of points a, b in $M \setminus \overline{B}(x, r)$ can be joined by a continuum in $M \setminus \overline{B}(x, r/c)$.

Here $B(x, r)$ and $\overline{B}(x, r)$ denote open and closed balls, respectively.

Proposition 7.12 can be viewed as an extension of the following well known principle in the theory of quasiconformal mapping: *a linearly locally connected domain in $\mathbb{R}^n$ that is quasiconformally equivalent to a uniform domain is itself uniform.* This principle was first recorded by Gehring and Martio **[GM]**. The metric used in **[GM]** is the Euclidean metric; for generalizations to inner metrics, see **[V2]**, **[Hei]**. Our proof of Proposition 7.12 follows the reasoning of these papers. However, to make the arguments work, we need both the uniformization theory as developed in Chapter 4 and the fact, proved below by using the results of Chapter 6, that the uniformized hyperbolic domains are Loewner spaces.

Note that the necessity part of Theorem 1.13 in the introduction directly follows from Proposition 7.12, for domains with totally disconnected complement are linearly locally connected (in the spherical metric). Similarly, circle domains on the Riemann sphere are linearly locally connected in the spherical metric, and hence the conjecture that we made in the introduction is valid within the class of domains that are conformally equivalent to circle domains. (By Koebe's Kreisnormierung prediction, this class contains all domains.)

Proposition 7.13. — *Under the assumptions of Theorem 7.11, (Ω, σ) is linearly locally connected.*

The statement in 7.13 is quantitative in the same sense as Theorem 7.11 is. We conclude that to prove Theorem 7.11, it suffices to prove Propositions 7.12 and 7.13.

We do not know whether Proposition 7.13 is true for ℓ_σ in place of σ. (Compare the remark after Theorem 7.11.)

To prove propositions 7.12 and 7.13, we first state the following proposition, interesting in its own right.

Proposition 7.14. — *If Ω is a δ-hyperbolic domain in $\overline{\mathbb{R}}^n$, then its uniformization $(\Omega, d_\varepsilon, \mu_\varepsilon)$ is a uniform n-Loewner space for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$, with the Loewner function φ depending only on δ and n.*

Here (Ω, d_ε) for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ means a conformal deformation of (Ω, k) as in Proposition 4.5. The measure μ_ε is defined by

$$\mu_\varepsilon(E) = \int_E \frac{\rho_\varepsilon(x)^n}{d(x)^n} d\sigma_n(x), \tag{7.15}$$

where $\rho_\varepsilon(x) = \exp\{-\varepsilon k(w, x)\}$ as in (4.1) and σ_n is the spherical n-measure on $\overline{\mathbb{R}}^n$. Recall the definition for a Loewner space from Chapter 6.

Proof of Proposition 7.14. — The metric measure space $(\Omega, \ell_\sigma, \sigma_n)$ is a locally compact, noncomplete and quasiconvex locally n-Loewner space. This is obvious when one remembers that Euclidean balls are n-Loewner with function $\varphi = \varphi_n$. The uniform space (Ω, d_ε) is obtained from (Ω, ℓ_σ) by using the conformal deformation $\rho \colon \Omega \to (0, \infty)$,

$$\rho(x) = \frac{\rho_\varepsilon(x)}{d(x)}.$$

We understand here that $0 < \varepsilon \leqslant \varepsilon_0(\delta)$ is fixed, where $\varepsilon_0(\delta)$ is as in Proposition 4.5. We now apply Theorem 6.39, with $d = \ell_\sigma$ and $\mu = \sigma_n$. First, by the triangle inequality and (4.4) it is easy to see that condition (6.40) is satisfied by our density ρ. Second, by Lemma 7.8 we have that (Ω, k) is roughly starlike, so that $\rho_\varepsilon(x) = \rho(x)d(x)$ satisfies

$$\frac{1}{\varepsilon e}\rho_\varepsilon(x) \leqslant d_\varepsilon(x) \leqslant \frac{C(\delta)}{\varepsilon}\rho_\varepsilon(x) \tag{7.16}$$

by Lemma 4.16; this gives condition (6.41). Theorem 6.39 thus implies that $(\Omega, d_\varepsilon, \mu_\varepsilon)$ is a locally n-Loewner space. Note that the constants in this assertion only depend on δ and not on ε, because of the scaling in (7.16).

Finally, because (Ω, d_ε) is also a uniform space (by Proposition 4.5), it is an n-Loewner space by the main result (Theorem 6.4) of Chapter 6. This proves Proposition 7.14. □

Proof of Proposition 7.13. — We first prove that (Ω, σ) satisfies condition (LLC_1). Let a and b be two points in a ball $B_\sigma(x, r)$ in Ω. Suppose that a and b cannot be joined in Ω within a spherical ball $B_\sigma(x, cr)$ for some $c \geqslant 4$. Let γ be an arc in Ω joining a and b; then γ necessarily leaves the ball $B_\sigma(x, cr)$. Let a_1 and b_1 be the first and the last point, respectively, on γ such that $\sigma(a_1, x) = r = \sigma(b_1, x)$ when traveling from a to b along γ. Define points a_2 and b_2 on γ in a similar fashion by requiring that $\sigma(a_2, x) = \sqrt{c}r = \sigma(b_2, x)$.

Next, choose two arcs, γ_1 on $\partial B_\sigma(x, r)$ and γ_2 on $\partial B_\sigma(x, \sqrt{c}r)$, that connect a_1 to b_1 and a_2 to b_2, respectively. The arcs γ_1 and γ_2 must meet $\partial\Omega$. Hence there exists a first point $\overline{a}_1 \in \gamma_1 \cap \partial\Omega$ and a last point $\overline{b}_1 \in \gamma_1 \cap \partial\Omega$ when traveling from a_1 to b_1 along γ_1. Define points $\overline{a}_2 \in \gamma_2 \cap \partial\Omega$ and $\overline{b}_2 \in \gamma_2 \cap \partial\Omega$ in a similar fashion. It may happen that $\overline{a}_1 = \overline{b}_1$ or that $\overline{a}_2 = \overline{b}_2$, but in any case $\{\overline{a}_1, \overline{b}_1\} \cap \{\overline{a}_2, \overline{b}_2\} = \varnothing$. Now let α be the curve which is the union of the half open subarc of γ_1 in Ω connecting a_1 and $\overline{a}_1$, the subarc $\gamma(a_1, a_2)$ of γ, and the half open subarc of γ_2 in Ω connecting a_2 and $\overline{a}_2$. Similarly, define a curve β that joins $\overline{b}_1$ to $\overline{b}_2$ in Ω.

Note that α and β lie in the closed ball $\overline{B}_\sigma(x, \sqrt{c}\, r)$. By assumption, the open curves α and β cannot be joined in Ω within $B_\sigma(x, cr)$, and standard modulus estimates thus imply

$$\operatorname{mod}_n(\alpha, \beta; \Omega) \leqslant C(n) \left(\log \sqrt{c}\right)^{1-n}.$$

By the conformal invariance of modulus, we have that

$$\operatorname{mod}_n(\alpha, \beta; \Omega_\varepsilon) \leqslant C(n) \left(\log \sqrt{c}\right)^{1-n} \tag{7.17}$$

as well, where $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ is as in Proposition 7.14. We shall now show, by using the fact that Ω_ε is a Loewner space (Proposition 7.14), that the left hand side of (7.17) has a lower bound depending only on the data in the assumptions.

By the η-quasisymmetry of the (natural) boundary map, we have that

$$\frac{d_\varepsilon(\overline{a}_1, \overline{b}_1)}{d_\varepsilon(\overline{a}_1, \overline{a}_2)} \leqslant \eta\left(\frac{\sigma(\overline{a}_1, \overline{b}_1)}{\sigma(\overline{a}_1, \overline{a}_2)}\right) \leqslant \eta\left(\frac{2}{\sqrt{c}-1}\right) \leqslant \eta(2). \tag{7.18}$$

Similarly, we obtain

$$\frac{d_\varepsilon(\overline{a}_1, \overline{b}_1)}{d_\varepsilon(\overline{b}_1, \overline{b}_2)} \leqslant \eta(2). \tag{7.19}$$

We have

$$\operatorname{dist}_\varepsilon(\alpha, \beta) \leqslant d_\varepsilon(\overline{a}_1, \overline{b}_1)$$

while, by (7.18) and (7.19),

$$\begin{aligned}\min\{\operatorname{diam}_\varepsilon \alpha,\ \operatorname{diam}_\varepsilon \beta\} &\geqslant \min\{d_\varepsilon(\overline{a}_1, \overline{a}_2), d_\varepsilon(\overline{b}_1, \overline{b}_2)\}\\ &\geqslant \frac{1}{\eta(2)}\, d_\varepsilon(\overline{a}_1, \overline{b}_1).\end{aligned}$$

It follows that

$$\frac{\operatorname{dist}_\varepsilon(\alpha, \beta)}{\min\{\operatorname{diam}_\varepsilon \alpha,\ \operatorname{diam}_\varepsilon \beta\}} \leqslant \eta(2),$$

which contradicts (7.17) for c too large in view of the Loewner property of Ω_ε. This proves that (Ω, σ) has property (LLC_1), quantitatively.

The proof of property (LLC_2) runs along similar lines. Assuming that $a, b \in \Omega \setminus \overline{B}_\sigma(x, r)$ are points that cannot be joined by a curve in $\Omega \setminus \overline{B}_\sigma(x, r/c)$, where $c \geqslant 4$, we can find open curves α, β in $\Omega \setminus B_\sigma(x, r/\sqrt{c})$ such that α has end points $\overline{a}_1 \in \partial B_\sigma(x, r) \cap \partial\Omega$, $\overline{a}_2 \in \partial B_\sigma(x, r/\sqrt{c}) \cap \partial\Omega$, β has end points $\overline{b}_1 \in \partial B_\sigma(x, r) \cap \partial\Omega$, $\overline{b}_2 \in \partial B_\sigma(x, r/\sqrt{c}) \cap \partial\Omega$, and that there is no curve joining α and β in $\Omega \setminus \overline{B}_\sigma(x, r/c)$. By using a modulus estimate and conformal invariance as in (7.17) together with the quasisymmetry of the natural boundary map as in (7.18) and (7.19), we easily obtain an upper bound for c, depending only on the data.

This completes the proof of Proposition 7.13. □

Proof of Proposition 7.12. — Let Ω be a δ-hyperbolic domain in $\overline{\mathbb{R}}^n$, and fix $\varepsilon = \varepsilon_0(\delta) > 0$ as in Proposition 4.5. First assume that $m = \ell_\sigma$ or $m = \sigma$, and that (Ω, m) is linearly locally connected. By Proposition 7.14, $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ is a uniform n-Loewner space; recall that the measure μ_ε has density $d\mu_\varepsilon(x) = \rho(x)^n d\sigma_n(x)$, where $\rho(x) = \rho_\varepsilon(x)/d(x)$ for $x \in \Omega$, and σ_n is the spherical n-measure. The density ρ has the following two properties:

$$\frac{1}{A} \leqslant \frac{\rho(x)}{\rho(y)} \leqslant A, \tag{7.20}$$

if $x, y \in B\left(z, d(z)/2\right)$ for $z \in \Omega$, and

$$\frac{1}{A}\rho_\varepsilon(x) \leqslant d_\varepsilon(x) \leqslant A\rho_\varepsilon(x) \tag{7.21}$$

for $x \in \Omega$; the latter follows from Lemma 4.16 and Lemma 7.8. Here and throughout the proof, we let $A \geqslant 1$ denote any positive constant that depends only on the data, which consists of δ and the constant $c \geqslant 1$ in the assumed linear local connectedness of (Ω, m).

Because quasihyperbolic geodesics in (Ω, k) are uniform curves in Ω_ε, we have that

$$\ell_\varepsilon([a,b]) \leqslant A d_\varepsilon(a,b) \tag{7.22}$$

and

$$\ell_\varepsilon([a,x]) \wedge \ell_\varepsilon([x,b]) \leqslant A d_\varepsilon(x) \tag{7.23}$$

whenever $[a, b]$ is such a geodesic and $x \in [a, b]$.

We claim that every quasihyperbolic geodesic $[a, b]$ in (Ω, k) satisfies

$$\operatorname{diam}_m [a,x] \wedge \operatorname{diam}_m [x,b] \leqslant A d(x) \tag{7.24}$$

for $x \in [a, b]$. To prove the claim, assume that $x \in [a, b]$ is such that neither $[a, x]$ nor $[x, b]$ belong to $\overline{B}_m(x, Bd(x))$ for some $B > c$. There exist $a_1 \in [a, x]$ and $b_1 \in [x, b]$, both outside the ball $\overline{B}_m(x, Bd(x))$; because (Ω, m) is linearly locally connected with constant c, there exists a curve γ joining a_1 to b_1 in $\Omega \setminus \overline{B}_m(x, Bd(x)/c)$. Then

$$\operatorname{diam}_\varepsilon \overline{B}_m\left(x, d(x)/4\right) \geqslant \frac{1}{A} d_\varepsilon(x) \tag{7.25}$$

and

$$\begin{aligned} \operatorname{diam}_\varepsilon \gamma \geqslant d_\varepsilon(a_1, b_1) &\geqslant \frac{1}{A}\ell_\varepsilon([a_1, b_1]) \\ &\geqslant \frac{1}{A}\rho(x)\ell_\sigma\left([a_1, b_1] \cap \overline{B}_m\left(x, d(x)/2\right)\right) \\ &\geqslant \frac{1}{A}\rho(x)d(x) \geqslant \frac{1}{A}\, d_\varepsilon(x). \end{aligned} \tag{7.26}$$

by (7.20)–(7.22). Moreover,

$$\operatorname{dist}_\varepsilon\left(\overline{B}_m\left(x, d(x)/4\right), \gamma\right) \leqslant \ell_\varepsilon([a_1, x]) \wedge \ell_\varepsilon([b_1, x]) \leqslant A d_\varepsilon(x) \tag{7.27}$$

by (7.23). It therefore follows from the Loewner property of Ω_ε, that

$$\operatorname{mod}_n \Gamma \geqslant \frac{1}{A} > 0, \tag{7.28}$$

where Γ is the family of all curves in Ω joining $\overline{B}_m(x, d(x)/4)$ and γ. By the conformal invariance, we can assume that the modulus in (7.28) is measured either in the space $(\Omega, d_\varepsilon, \mu_\varepsilon)$, or in (Ω, m, σ_n). In any case, because γ lies outside $\overline{B}_m(x, Bd(x)/c)$, we have that (*cf.* **[HeiK**, 3.14])

$$\frac{1}{A} \leqslant \operatorname{mod}_n \Gamma \leqslant A\left(\log \frac{4B}{c}\right)^{1-n},$$

which gives an upper bound for B, thereby proving (7.24).

It follows from (7.24) and Lemma 7.33 below that the quasihyperbolic geodesic $[a,b]$ in fact satisfies the following stronger property:

$$\ell_\sigma([a,x]) \wedge \ell_\sigma([x,b]) \leqslant Ad(x) \tag{7.29}$$

for all $x \in [a,b]$. It therefore remains to show that

$$\ell_\sigma([a,b]) \leqslant Am(a,b) \tag{7.30}$$

if $[a,b]$ is a quasihyperbolic geodesic in (Ω, k). To this end, it suffices to prove the following:

$$\ell_\sigma([a,b]) \leqslant A \ \operatorname{diam}_\sigma \gamma \tag{7.31}$$

for each curve γ joining a and b in Ω. Indeed, if $m = \ell_\sigma$, then (7.31) clearly implies (7.30); if $m = \sigma$, then the assumed linear local connectivity condition implies that there is a curve γ joining a to b in Ω such that $\operatorname{diam}_\sigma \gamma \leqslant 2c\sigma(a,b)$, which by way of (7.31) similarly implies (7.30).

We now prove (7.31). Fix a quasihyperbolic geodesic $[a,b]$ and choose a point $x_0 \in [a,b]$ such that $\ell_\sigma([a,x_0]) = \ell_\sigma([x_0,b])$. By (7.29),

$$\ell_\sigma([a,b]) \leqslant Ad(x_0). \tag{7.32}$$

If both a and b belong to the ball $B_m(x_0, d(x_0)/20)$, one easily estimates that the geodesic $[a,b]$ stays inside $B_m(x_0, d(x_0)/2)$ and that $\ell_\sigma([a,b]) \leqslant A\sigma(a,b)$. Thus, by symmetry, assume that $a \notin B_m(x_0, d(x_0)/20)$. Let γ be a curve joining a to b in Ω. If γ meets the ball $B_m(x_0,\ d(x_0)/40)$, then

$$\operatorname{diam}_\sigma \gamma \geqslant \frac{d(x_0)}{40} \geqslant \frac{1}{A}\, \ell_\sigma([a,b]),$$

by (7.32) as desired. Thus we may assume that γ lies outside $B_m(x_0,\ d(x_0)/40)$. By computing as in (7.25), (7.26), and (7.27), and observing the Loewner property of Ω_ε, we obtain that the n-modulus of the curve family Γ, consisting of curves joining γ and $\overline{B}_m(x_0,\ d(x_0)/80)$ has a lower bound depending only on the data,

$$\operatorname{mod}_n \Gamma \geqslant \frac{1}{A} > 0.$$

On the other hand, either $d(x_0) \leqslant 100\ \mathrm{diam}_\sigma \gamma$ or

$$\mathrm{mod}_n \Gamma \leqslant A \left(\log \frac{200 d(x_0)}{\mathrm{diam}_\sigma \gamma} \right)^{1-n},$$

as shown by simple geometric considerations. This and (7.32) imply that (7.31) holds in all cases, and the proof of the first assertion in Proposition 7.12 is complete.

Next, if (Ω, σ) only satisfies the second requirement of linear local connectivity, (LLC_2), then as above we conclude that (7.29) holds for all hyperbolic geodesics $[a, b]$ in Ω. Again, (7.31) is true, and we obtain (7.30) for $m = \ell_\sigma$. This proves that (Ω, ℓ_σ) is a uniform space, and the proof of Proposition 7.12 is complete. □

In consequence, Theorem 7.11 is completely established. In the proof of Proposition 7.12 above we needed the following lemma.

Lemma 7.33. — *Let Ω be a domain in $\overline{\mathbb{R}}^n$ and let $\gamma = [a, b]$ be a quasihyperbolic geodesic in Ω. If there is a constant $A \geqslant 1$ such that*

$$\sigma(a, x) \wedge \sigma(x, b) \leqslant A d(x)$$

for each $x \in \gamma$, then there is a constant $B = B(A) \geqslant 1$ such that

$$\ell_\sigma([a, x]) \wedge \ell_\sigma([x, b]) \leqslant B d(x)$$

for each $x \in \gamma$.

Results like Lemma 7.33 appear in the literature in the case of the Euclidean metric for domains Ω in $\mathbb{R}^n$; see e.g. [**MS**, p. 385–386] or [**NV**, p. 7–9]. Only trivial modifications are needed to transfer these proofs to the spherical case, and we leave them to the tenacious reader.

Proof of Theorem 1.12. — We have already proven that inner uniform domains, and hence their conformal images are hyperbolic. (See Chapters 3 and 4).

To prove the other direction, assume that Ω is a hyperbolic domain on the two-sphere. Because hyperbolicity is preserved under conformal maps, without loss of generality, we can assume that Ω is a *slit domain*; that is, $\infty \in \Omega$ and the complementary components of Ω consists of line segments in $\mathbb{R}^2 \subset \overline{\mathbb{R}}^2$, all parallel to a fixed axis, and points in $\mathbb{R}^2$. (See e.g. [**Ts**, Theorem IX.22, p. 400].) It is easy to see that such a domain satisfies condition (LLC_2) with respect to the spherical metric. The claim therefore follows from Proposition 7.12. Theorem 1.12 follows. □

Remark 7.34. — In a recent paper [**BuSt**], Buckley and Stanoyevitch have shown that a bounded product domain in $\mathbb{R}^m \times \mathbb{R}^n$ is quasiconformally equivalent to an inner uniform domain in $\mathbb{R}^{m+n}$ only if it is itself inner uniform. (Here by a *bounded product domain* in $\mathbb{R}^m \times \mathbb{R}^n$ we mean a domain Ω that is of the form $\Omega = \Omega_1 \times \Omega_2$ for some bounded domains $\Omega_1 \subset \mathbb{R}^m$ and for $\Omega_2 \subset \mathbb{R}^n$, where $m, n \geqslant 1$.) This result also follows from Proposition 7.12, because (this is not hard to see) bounded product domains always satisfy condition (LLC_2), and because quasiconformal images of hyperbolic domains are hyperbolic.

It would be interesting to characterize Gromov hyperbolic domains in $\overline{\mathbb{R}}^n$ by geometric properties that are more easily verifiable than the original definition in terms of thin geodesic triangles. It can be shown that the following properties are necessary for the Gromov hyperbolicity:

(i) *Gehring-Hayman property*: There exists a constant $C \geqslant 1$ such that for every geodesic $[x, y]$ in (Ω, k) and for every curve γ in Ω with end points x and y we have that
$$\ell_\sigma([x, y]) \leqslant C\ell_\sigma(\gamma).$$

(ii) *Separation property*: There exists a constant $C \geqslant 1$ such that whenever $[x, y]$ is a geodesic in (Ω, k), $z \in [x, y]$, and γ is a curve in Ω connecting $[x, z)$ and $(z, y]$, then
$$B_{\ell_\sigma}\big(z, Cd(z)\big) \cap \gamma \neq \varnothing.$$

The Gehring-Hayman property of a Gromov hyperbolic domain can be established by using the ideas in [**HR**], taking Proposition 7.14 into account. See also [**BKR**]. The separation property was implicitly established in the proof of Proposition 7.12.

We conjecture that above properties (i) and (ii) are also sufficient for the Gromov hyperbolicity of (Ω, k).

CHAPTER 8

THE MARTIN BOUNDARY OF A GROMOV HYPERBOLIC DOMAIN

In this chapter, we prove Theorem 1.14 following the ideas of Ancona [**A2**], [**A3**]; indeed, we prove the more general result Theorem 8.15. Throughout, we conform with the notation of Chapter 3. In particular, we let X denote a geodesic δ-hyperbolic space with metric written as $|x-y|$.

We begin with a discussion of some geometric facts that will be useful later on in connection with the Martin boundary of a Gromov hyperbolic domain.

Lemma 8.1. — *Let $I=(0,a)$ be an open interval, $0<a\leqslant\infty$, and let $\gamma\colon \overline{I}\to X$ be an isometric embedding. For $t\in I$ define*

$$U_t=\Big\{x\in X : \operatorname{dist}\Big(x,\gamma([t,a))\Big)<\operatorname{dist}\Big(x,\gamma((0,t])\Big)\Big\},$$

and denote its closure in X by $\overline{U}_t$. Then, for all $t\in I$,

(a) $\gamma((0,t))\cap\overline{U}_t=\varnothing$,
(b) $\gamma((t,a))\subset U_t$, *and*
(c) $\gamma(t)\in\partial\overline{U}_t\subset\partial U_t$.

The proof of Lemma 8.1 is left to the reader.

Lemma 8.2. — *Let $\gamma\colon\overline{I}\to X$ and U_t, $t\in I$, be as in Lemma 8.1. For $0<t<s<a$ set $x=\gamma(t)$, $y=\gamma(s)$, and $z=\gamma((s+t)/2)$. Then*

(a) $U_t\supset U_s$;

moreover, for each $u\in X\smallsetminus U_t$ and $v\in\overline{U}_s$,

(b) *if* $|x-y|>8\delta$, *then* $|u-v|\geqslant|u-x|+|v-y|-6\delta$ *and* $\operatorname{dist}(z,[u,v])\leqslant 2\delta$ *for each geodesic* $[u,v]$,
(c) *if* $|x-y|>8\delta$, *then* $|u-v|\geqslant\frac{1}{2}|x-y|-3\delta>0$, *and*
(d) $|u-z|\wedge|v-z|\geqslant\frac{1}{4}\,|x-y|$.

In particular, $U_t\supset\overline{U}_s$ if $|x-y|>8\delta$ by (c).

Proof. — The proof of (a) is clear. To prove (b), pick $t' \in [0,t]$ and $s' \in [s,a]$ such that

$$\operatorname{dist}(u,\gamma) = |u-\gamma(t')|, \qquad \operatorname{dist}(v,\gamma) = |v-\gamma(s')|.$$

Denote $x' = \gamma(t')$, $y' = \gamma(s')$, and let $[x,y] \subset [x',y']$ be geodesic subarcs of γ. By the δ-hyperbolicity of X, we find a point $z' \in [x',u] \cup [u,v] \cup [v,y']$ such that $|z'-z| \leqslant 2\delta$. We claim that $z' \in [u,v]$. If this is not the case, we assume without loss of generality that $z' \in [u,x']$. Then

$$\begin{aligned} |z'-x'| &\geqslant |x'-z| - 2\delta \geqslant |x-z| - 2\delta \\ &= \frac{1}{2}\,|x-y| - 2\delta > 2\delta \end{aligned}$$

by assumption, and hence

$$\begin{aligned} |u-x'| = \operatorname{dist}(u,\gamma) &\leqslant |u-z| \leqslant |u-z'| + 2\delta \\ &= |u-x'| - |z'-x'| + 2\delta < |u-x'|, \end{aligned}$$

which is a contradiction. Thus $z' \in [u,v]$, and it follows that $\operatorname{dist}(z,[u,v]) \leqslant 2\delta$. The δ-thinness of the triangle $[x',z] \cup [z,u] \cup [u,x']$ implies that there is a point $x'' \in [u,x'] \cup [z,u]$ such that $|x''-x| \leqslant \delta$. We then have that

$$\begin{aligned} |x-u| &\leqslant |x''-u| + \delta \leqslant |u-x'| \vee |u-z| + \delta \\ &= |u-z| + \delta. \end{aligned}$$

Similarly, $|y-v| \leqslant |v-z| + \delta$, and therefore

$$\begin{aligned} |u-v| &= |u-z'| + |z'-v| \\ &\geqslant |u-z| - |z-z'| + |v-z| - |z-z'| \\ &\geqslant |x-u| + |y-v| - 6\delta. \end{aligned}$$

Thus (b) follows.

Next, (b) gives

$$\begin{aligned} |x-y| &\leqslant |x-u| + |u-v| + |v-y| \\ &\leqslant |u-v| + |u-v| + 6\delta, \end{aligned}$$

and thus (c) follows.

To prove (d), we compute

$$|x-z| \leqslant |x'-z| \leqslant |z-u| + |u-x'| \leqslant 2\,|z-u|,$$

which implies $|z-u| \geqslant \frac{1}{4}\,|x-y|$, and similarly $|z-v| \geqslant \frac{1}{4}\,|x-y|$. Thus (d) follows, and Lemma 8.2 is thereby established. □

Assume now that X is proper. Then the conformal deformations $X_\varepsilon = (X, d_\varepsilon)$, $0 < \varepsilon \leqslant \varepsilon_0(\delta)$, are bounded uniform spaces by the uniformization theory in Chapter 4. In particular, the completion $\overline{X}_\varepsilon$ of (X, d_ε) is compact by Proposition 2.20. The compact spaces $\overline{X}_\varepsilon$ are all homeomorphic to each other by Remark 4.14 (b), for $0 < \varepsilon \leqslant \varepsilon_0(\delta)$, and the Gromov boundary $\partial_G X$ is homeomorphic to ∂X_ε by Proposition 4.13. We can compactify X by adding $\partial_G X$ to it; then $X \cup \partial_G X$ is homeomorphic to $\overline{X}_\varepsilon$. In the following, we consider the topological structure of this space. The notation is as in Lemma 8.1.

Lemma 8.3. — *Let $\gamma\colon [0,\infty] \to X$ be a geodesic ray in a proper and geodesic δ-hyperbolic space, and let $w = \gamma(0)$ be the fixed base point in X. The sets*

$$V_t = U_t \cup \{b \in \partial_G X : b \text{ can be represented by a sequence in } U_t\}, \qquad t > 0,$$

form a neighborhood base of the point in $\partial_G X$ that is represented by the ray γ.

Proof. — This lemma likely appears somewhere in the literature. Rather than searching for the references, we shall sketch a proof. (For the tacitly used facts about Gromov hyperbolic geometry, see [**GhHa**, Ch. 8].) Let $c_1, c_2, \ldots$ denote positive universal constants; the notation $A = B \pm c_i\delta$ means $|A - B| \leqslant c_i\delta$.

If $x, y \in X$ and $u \in [w, y]$ is a point such that $|u - x| = \operatorname{dist}(x, [w, y])$, then the Gromov product satisfies (use the tripod map [**GhHa**, p. 41])

$$(x|y)_w = |u - w| \pm c_1\delta. \tag{8.4}$$

Thus, if $x \in X$ and $x' \in \gamma$ is a point such that $|x - x'| = \operatorname{dist}(x, \gamma)$, (8.4) implies that

$$(x|\gamma(s))_w = |x' - w| \pm c_1\delta$$

for all sufficiently large s. Hence

$$(x|a)_w = |x' - w| \pm c_2\delta, \tag{8.5}$$

where $a \in \partial_G X$ is the boundary point determined by γ. This shows that if $x \in U_t$, then $(x|a)_w \geqslant t - c_2\delta$. If now $y \in V_t \cap \partial_G X$, it can be represented by a sequence (y_n) in U_t, so that

$$\begin{aligned}(y|a)_w &\geqslant \liminf_{n\to\infty} (y_n|a)_w - c_3\delta \\ &\geqslant t - c_4\delta.\end{aligned}$$

If follows that

$$(x|a)_w \geqslant t - c_5\delta \tag{8.6}$$

if $x \in V_t$. Because the sets $\{x \in X \cup \partial_G X : (x|a)_w > M\}$, $M > 0$, form a neighborhood basis of a, we conclude that every neighborhood of a contains a set V_t for t sufficiently large.

It remains to show that every set V_t, $t > 0$, is a neighborhood of a. If $x \in X$ and $(x|a)_w > t + c_2\delta$, then (8.5) shows that $|x' - w| > t$ for every point $x' \in \gamma$ such that $|x' - x| = \operatorname{dist}(x, \gamma)$; in particular, we have that

$$\operatorname{dist}\Big(x, \gamma([t,\infty))\Big) < \operatorname{dist}\Big(x, \gamma((0,t])\Big),$$

which implies $x \in U_t$. If $x \in \partial_G X$ and (x_n) is a sequence representing x, then

$$\liminf_{n\to\infty}(x_n|a)_w \geqslant (x|a)_w - c_6\delta;$$

in particular, if $(x|a)_w > t + (c_6 + c_2)\delta$, we can pass to a subsequence and assume, initially, that $(x_n|a)_w > t + c_2\delta$, which implies that $x \in V_t$. In conclusion, there is a constant $c_7 > 0$ such that $x \in X \cup \partial_G X$ and $(x|a)_w > t + c_7\delta$ imply $x \in V_t$. Thus V_t is a neighborhood of a, and the lemma follows. □

The following definition is due to Ancona [**A2**, p. 514], [**A3**, p. 5].

Definition 8.7. — Given a positive increasing function $\Phi\colon [0,\infty) \to (0,\infty)$ with $\Phi(t) \to \infty$ as $t \to \infty$, a Φ-*chain* in a metric space (M,d) is a decreasing sequence $\Gamma_1 \supset \cdots \supset \Gamma_m$ of open sets in M together with a sequence of points $x_1,\ldots,x_m$ in M such that $x_i \in \partial\overline{\Gamma}_i$ for each $i = 1,\ldots,m$ and that

(a) $0 < \Phi(0) = c_0 \leqslant d(x_i,x_{i+1}) \leqslant 1/c_0$ for $i = 1,\ldots,m-1$,

(b) $\operatorname{dist}(x,\Gamma_{i+1}) \geqslant \Phi(d(x,x_i))$ for all $x \in \partial\Gamma_i$ and $i = 1,\ldots,m-1$.

Here $\overline{\Gamma}_i$ is the closure of Γ_i in M.

We also call a sequence of points $x_1,\ldots,x_m$ in M a Φ-*chain* if they are associated with sets $\Gamma_1,\ldots,\Gamma_m$ as above.

In a δ-hyperbolic space X we set

$$\Phi_\delta(t) = \begin{cases} c_0 = \min\{\delta, \frac{1}{22\delta}\}\ , & 0 \leqslant t < 7\delta, \\ t - 6\delta, & 7\delta \leqslant t. \end{cases} \tag{8.8}$$

We assume here, as we may, that $\delta > 0$.

The next lemma appears in [**A3**, Theorem 6.9] for graphs. For completeness we include a similar proof.

Lemma 8.9. — *Let $[x,y]$ be a geodesic segment in a δ-hyperbolic space X, $\delta > 0$, such that $|x-y| > 44\delta$. Choose points $x = x_0, x_1, \ldots, x_{m+1} = y$ successively from $[x,y]$ when traveling from x to y such that $|x_i - x_{i+1}| = 22\delta$ for $i = 0,\ldots,m-1$, and that $|x_m - x_{m+1}| \leqslant 22\delta$. Then the points $x_1,\ldots,x_m$ form a Φ_δ-chain in X, where Φ_δ is given in (8.8).*

Proof. — First note that $m \geqslant 2$ because we assume $|x-y| > 44\delta$. Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = |x-y|$ be the preimages of the points $x_0,\ldots,x_{m+1}$ under a fixed isometry $\gamma\colon [0,|x-y|] \to [x,y]$. Define open sets U_t for $0 < t < |x-y|$ as in Lemma 8.1, and let $\Gamma_i = U_{t_i}$ for $i = 1,\ldots,m$. Then $x_i \in \partial\overline{\Gamma}_i$ for $i = 1,\ldots,m$ by Lemma 8.1 (c). Since $|x_i - x_{i+1}| = 22\delta$ for $1 \leqslant i \leqslant m-1$, we have

$$c_0 = \min\left\{\delta,\ \frac{1}{22\delta}\right\} \leqslant |x_i - x_{i+1}| \leqslant \frac{1}{c_0}$$

for $i = 1, \dots, m-1$, verifying condition (a) in Definition 8.7. Note also that $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_m$ by Lemma 8.2 (a). Next, if $u \in \partial\Gamma_i$ and $v \in \Gamma_{i+1}$, we have by Lemma 8.2 (b) and (c) that

$$\begin{aligned}|u-v| &\geqslant \max\left\{|u-x_i| - 6\delta,\ \frac{1}{2}\ |x_i - x_{i+1}| - 3\delta\right\}\\ &\geqslant \max\{|u-x_i| - 6\delta, 8\delta\}\\ &\geqslant \Phi_\delta(|u-x_i|).\end{aligned}$$

Hence $\operatorname{dist}(u, \Gamma_{i+1}) \geqslant \Phi_\delta(|u-x_i|)$, verifying condition (b) in Definition 8.7. The lemma follows. □

Proposition 8.10. — *Let X be a proper and geodesic δ-hyperbolic space, $\delta > 0$. Each point $\zeta \in \partial_G X$ admits a neighborhood basis $\{V_1', V_2', \dots\}$ in $X \cup \partial_G X$ such that for the sets $U_k' = V_k' \cap X$ the following two assertions hold:*

(a) the sets U_k' form a decreasing sequence of nonempty open subsets of X with $\overline{U}'_{k+1} \subset U_k'$ and $\cap_{k=1}^{\infty} U_k' = \varnothing$;

(b) for each integer $k \geqslant 1$ there exists a point $p_k \in U_{2k}' \setminus \overline{U}'_{2k+1}$ such that

$$|u-v| \wedge |u-p_k| \wedge |v-p_k| \geqslant 75\delta$$

whenever $u \in \partial U_{2k}'$ and $v \in \partial U_{2k+1}'$.

Moreover, there exists a Φ_δ-chain $x_1, \dots, x_m$, $m \geqslant 3$, such that

$$|u-x_1| \vee |v-x_m| \vee |p_k - x_i| \leqslant 24\delta$$

for some $i = 2, \dots, m-1$, where Φ_δ is given in (8.8).

Proof. — Fix a base point $w \in X$. For $\zeta \in \partial_G X$, let $\gamma\colon [0,\infty) \to X$ be a geodesic ray that represents ζ with $\gamma(0) = w$. For $t > 0$ define U_t as in Lemma 8.1, and define V_t as in Lemma 8.3. For $k \geqslant 1$ set

$$\begin{aligned} V_k' &= V_{300k\delta}\,, & U_k' &= U_{300k\delta}\,,\\ x_k &= \gamma(300\cdot 2k\delta), & y_k &= \gamma\left(300(2k+1)\delta\right),\\ p_k &= \gamma\left(300(2k+1/2)\delta\right). \end{aligned}$$

Lemmas 8.2 and 8.3 give that (V_k') is a neighborhood basis of ζ and that (a) is true. Note that $x_k \in \partial\overline{U}'_{2k}$, $y_k \in \partial\overline{U}'_{2k+1}$, $p_k \in U_{2k}' \setminus \overline{U}'_{2k+1}$, and that

$$|x_k - p_k| = |y_k - p_k| = \frac{1}{2}|x_k - y_k| = 150\delta.$$

Next, if $u \in \partial U_{2k}'$ and $v \in \partial U_{2k+1}'$, we obtain from Lemma 8.2 (c) and (d) that

$$|u-v| \geqslant \frac{1}{2}\ |x_k - y_k| - 3\delta = 147\delta$$

and that

$$|u-p_k| \wedge |v-p_k| \geqslant \frac{1}{4}\ |x_k - y_k| \geqslant 75\delta,$$

and hence the first part of (b) follows.

Finally, let $x_1, \dots, x_m$ be the Φ_δ-chain on $[u,v]$ constructed in Lemma 8.9, where $[u,v]$ is a geodesic from u to v. Then $|u-x_1| = |x_i - x_{i+1}| = 22\delta$ for $i = 1, \dots, m-1$, and $|v - x_m| \leqslant 22\delta$. Notice that $\operatorname{dist}(p_k, [u,v]) \leqslant 2\delta$ by Lemma 8.2 (b). It therefore follows that $|p_k - x_i| \leqslant 24\delta$ for some $i = 1, \dots, m$. But we must have $i \neq 1, m$, for otherwise

$$75\delta \leqslant |u - p_k| \wedge |v - p_k| \leqslant 46\delta$$

which is impossible. (Recall that we assume $\delta > 0$.) This proves Proposition 8.10. □

We shall next apply Proposition 8.10 similarly to Ancona [**A2**, Theorems 7 and 8, p. 517–518].

Let Ω be a domain in $\mathbb{R}^n$. Recall that $d(x)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ (which we always assume nonempty). We consider three conditions that Ω may or may not satisfy.

(i) *Strong barrier*: there exists $\varepsilon > 0$ and a C^2-smooth positive superharmonic function s in Ω such that

$$\Delta s(x) + \varepsilon d(x)^{-2}\, s(x) \leqslant 0 \tag{8.11}$$

for $x \in \Omega$, where Δ is the Laplacian.

(ii) *Capacity density condition*: there is a constant $c > 0$ such that

$$\operatorname{cap}\Big((\mathbb{R}^n \setminus \Omega) \cap \overline{B}(x,r)\Big) \geqslant \begin{cases} cr^{n-2}, & n \geqslant 3, \\ cr, & n = 2, \end{cases} \tag{8.12}$$

for each $x \in \partial\Omega$ and $r > 0$, where cap denotes the Newtonian capacity if $n \geqslant 3$, and the logarithmic capacity if $n = 2$.

(iii) *Corkscrew condition*: there exists a constant $c > 0$ such that for each $x \in \partial\Omega$ and $r > 0$ there is a point $y \in B(x,r)$ with

$$B(y, cr) \cap \Omega = \varnothing. \tag{8.13}$$

Then we have the following (quantitative) implications

$$\text{(iii)} \implies \text{(ii)} \implies \text{(i)}.$$

In dimension $n = 2$, conditions (ii) and (i) are equivalent but not so for $n \geqslant 3$. See [**A1**], [**A2**], [**Le**], [**Po**].

The domain Ω can be turned into a complete Riemannian manifold (Ω, g) that is moreover quasiisometric (in the sense of (1.16)) to (Ω, k_Ω) in a standard way by mollifying the distance function $d(x)$ appropriately; this gives a C^∞-function $\overline{d}(x)$ in Ω such that $\overline{d}(x) \approx d(x)$ and the Riemannian metric g is obtained by using the length element $ds^2 = \overline{d}(x)^{-2}|dx|^2$. (See [**A2**, p. 521], [**St**, p. 171].)

Now let Δ be the Euclidean Laplacian. Condition (8.11) is equivalent to condition

$$\Delta s(x) + \varepsilon \overline{d}(x)^{-2}\, s(x) \leqslant 0, \tag{8.14}$$

upon a change in ε; in the language of Ancona, (8.14) means that the operator $\overline{d}(x)^2\,\Delta$ is *weakly coercive* on the Riemannian manifold (Ω, g); see [**A2**, p. 521].

We want to determine the Martin boundary $\partial_M\Omega$ of Ω with respect to the Laplacian Δ. Let us recall the definition of $\partial_M\Omega$. Fix a base point $w \in \Omega$; then call a sequence $\zeta = (x_k)$ in Ω *admissible* if it has no limit points in Ω and if there is a function H_ζ in Ω such that

$$\frac{G(x, x_k)}{G(w, x_k)} \longrightarrow H_\zeta(x), \qquad k \to \infty,$$

locally uniformly on Ω, where $G(x, y)$ denotes the Green function in Ω (we assume G exists). Two admissible sequences ζ_1 and ζ_2 are defined to be equivalent if $H_{\zeta_1} = H_{\zeta_2}$, and the *Martin boundary* $\partial_M\Omega$ is the set of equivalence classes of admissible sequences. The set $\Omega_M = \Omega \cup \partial_M\Omega$ is a compact topological space when equipped with the *Martin topology*, and is a compactification of Ω. In this topology, $H_{\zeta_n}(x) \longrightarrow H_\zeta(x)$ if $\zeta_n \longrightarrow \zeta$ and $x \in \Omega$ (See [**Hel**, Ch. 12] for the description of the Martin topology.)

Theorem 8.15. — *Let Ω be a Gromov hyperbolic domain in $\mathbb{R}^n$. If Ω admits a strong barrier (8.11), then the Martin boundary $\partial_M\Omega$ and the Gromov boundary $\partial_G\Omega$ are homeomorphic. More precisely, the identity map extends to a homeomorphism*

$$\Omega \cup \partial_G\Omega \longrightarrow \Omega \cup \partial_M\Omega.$$

Proof. — Because the operator $\overline{d}(x)^2\,\Delta$ is weakly coercive as explained above, Theorem 8 in [**A2**, p. 518] applies. Because the Riemannian manifold (Ω, g) is quasiisometric to the space (Ω, k), we only need to check that Ancona's conditions [**A2**, (G.A.)] are satisfied on (Ω, g). This is essentially Proposition 8.10 when applied to the Gromov hyperbolic space (Ω, g). Note that Gromov hyperbolicity is preserved under quasiisometries, and that (Ω, g) is proper and geodesic as a complete Riemannian manifold. It is also clear that Ancona's requirement of bounded geometry is satisfied by the space (Ω, g).

There is one minor point that we have to address here, and this concerns the proof of Theorem 7 in [**A2**, p. 517]. (Ancona derives his Theorem 8 from Theorem 7.) Namely, our Φ_δ-chain, given in Proposition 8.10, does not necessarily pass through the points u, v, p_k; this was required in [**A2**, Theorem 7]. However, we can get the crucial estimate [**A2**, (6.4)],

$$G(u, v) \leqslant c\, G(p_k, u) G(p_k, v) \tag{8.16}$$

for $u \in \partial U'_{2k}$, $v \in \partial U'_{2k+1}$, by applying Proposition 8.10 (b); indeed, by [**A2**, Theorem 5, p. 515] we have that

$$G(x_1, x_m) \leqslant c\, G(x_1, x_i) G(x_i, x_m), \tag{8.17}$$

and because the points u, v, p_k are uniformly separated by 8.10 (b), (8.16) follows from (8.17) by a repeated use of Harnack's inequality. The proof of Theorem 7 now continues as in [**A2**]. This completes the proof of Theorem 8.15. □

Remarks 8.18

(a) Obviously, either of the stronger assumptions (ii) or (iii) can be used in place of (i) in Theorem 8.15. We stated Theorem 1.14 by using (ii). Jerison and Kenig proved in **[JK]** that the Martin boundary $\partial_M \Omega$ of a uniform domain Ω identifies with its Euclidean boundary $\partial\Omega$ if, in addition, (iii) is satisfied.

(b) We could have considered more general elliptic operators and their associated potential theories in Theorem 8.15, *cf.* **[A2]**. A particularly interesting case is the Laplace-Beltrami operator corresponding to the Bergman metric on a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with sufficiently smooth boundary. By an explicit construction of Φ-chains, Arai **[Ar]** proved that the Martin boundary $\partial_M \Omega$ can be identified with the Euclidean boundary $\partial\Omega$. A recent result of Balogh and the first author **[BB1]**, **[BB2]** shows that a strictly pseudoconvex domain Ω is Gromov hyperbolic in the Bergman metric; moreover, the Gromov boundary $\partial_G \Omega$ coincides with the Euclidean boundary as a set. On the other hand, it is easy to see that an analog of Theorem 8.15 holds in this setting. Hence we have the identification of the three boundaries: $\partial\Omega = \partial_M \Omega = \partial_G \Omega$.

CHAPTER 9

QUASICONFORMAL MAPS BETWEEN GROMOV HYPERBOLIC SPACES OF BOUNDED GEOMETRY

Recall definition (1.21) for quasiconformal maps between metric spaces. It is a well-known fact that every quasiconformal self-homeomorphism $f\colon \mathbb{B}^n \to \mathbb{B}^n$ of the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, $n \geqslant 2$, extends to a quasiconformal homeomorphism of the boundary sphere. However, this is not an easy result to prove starting from definition (1.21). First proofs are due to Callender, Gehring, and Väisälä around 1960. One can prove, in fact, that $f\colon \mathbb{B}^n \to \mathbb{B}^n$ is a rough quasiisometry (definition (1.17)) in the hyperbolic metric on $\mathbb{B}^n$. Thus the homeomorphic, and in fact quasisymmetric, extension to $\partial\mathbb{B}^n$ is provided by a general theorem of Efremovich and Tihomirova **[ET]**.

In this chapter, we shall demonstrate that the above phenomena remain valid under quite general circumstances. In particular, we shall prove Theorem 1.15. Given the results in Chapter 6 of this paper, and the theory developed in **[HeiK]**, our arguments follow the standard lines.

First we introduce metric spaces of bounded geometry; the concept may have some independent interest.

Let $X = (X, d, \mu)$ be a pathwise connected, proper metric (Borel) measure space, and let $Q \geqslant 1$. We say that X is of *Q-bounded geometry*, or a *Q-BG space*, if the following two conditions are satisfied:

(BG_1): there exist $R_0 > 0$ and $C_0 \geqslant 1$ such that

$$\frac{1}{C_0} R^Q \leqslant \mu(B_R) \leqslant C_0 R^Q \tag{9.1}$$

for all open balls B_R in X of radius $0 < R < R_0$;

(BG_2): there exist $C_1 \geqslant 1$ and $\tau \geqslant 1$ such that

$$\frac{1}{\mu(B_R)} \int_{B_R} |u - u_{B_R}|\, d\mu \leqslant C_1 R \left(\frac{1}{\mu(B_{\tau R})} \int_{B_{\tau R}} \rho^Q \, d\mu \right)^{1/Q} \tag{9.2}$$

for all open balls B_R in X of radius $0 < R < R_0/\tau$, for all bounded measurable functions u in $B_{\tau R}$, and for all Borel functions $\rho\colon B_{\tau R} \to [0,\infty]$ that satisfy

$$|u(\gamma(a)) - u(\gamma(b))| \leqslant \int_\gamma \rho\, ds \tag{9.3}$$

whenever $\gamma\colon [a,b] \to B_{\tau R}$ is a rectifiable curve. In (9.2), we denote by $B_{\tau R}$ the open ball with the same center as B_R and radius τR and by u_{B_R} the mean value of u over B_R.

Thus, X is of Q-bounded geometry if it is uniformly locally Ahlfors Q-regular (9.1) and supports uniformly locally a Poincaré inequality (9.2). Poincaré inequalities based on Borel functions satisfying (9.3), or *upper gradients*, were introduced in **[HeiK]**. Indeed, by localizing the arguments in [**HeiK**, Section 5], one obtains the following characterization of metric spaces of bounded geometry:

Proposition 9.4. — *A metric measure space (X, d, μ) as above is of Q-bounded geometry, $Q > 1$, if and only if the following two conditions are satisfied:*

(BG_1'): there exist $R_0' > 0$ and $C_0' \geqslant 1$ such that

$$\mu(B_R) \leqslant C_0' R^Q \tag{9.5}$$

for all open balls B_R in X of radius $0 < R < R_0'$;

(BG_2'): there exists $\kappa \geqslant 1$ and a decreasing function $\phi\colon (0,\infty) \to (0,\infty)$ such that

$$\operatorname{mod}_Q(E, F; B_{\kappa R}) \geqslant \phi(t), \tag{9.6}$$

whenever E and F are two disjoint nondegenerate continua in a ball B_R, $0 < R < R_0'/\kappa$, with $\operatorname{dist}(E,F) \leqslant t(\operatorname{diam} E \wedge \operatorname{diam} F)$.

The statement is quantitative in the usual sense.

Thus, X is of Q-bounded geometry if the measure μ satisfies a uniform local volume growth condition (9.5) and if X is a locally Q-Loewner space in the sense of (BG_2'). (Recall the definition for Q-modulus mod_Q from (6.3). Also compare (BG_2') and the definition for a locally Loewner space in Chapter 6.)

Examples 9.7

(a) If X^n is a complete, connected Riemannian n-manifold with positive injectivity radius $\operatorname{inj} X$ and with a lower bound for the Ricci curvature, $\operatorname{Ric} \geqslant -(n-1)k^2$, $k \geqslant 0$, then X is of n-bounded geometry as defined above. Indeed, we can choose $R_0 = \frac{1}{2} \operatorname{inj} X$. The second inequality in (9.1) follows from the classical comparison theorems, while the first one is due to Croke **[C]**. The local Poincaré inequality follows from work of Buser **[Bu]** (*cf.* **[SC]**). In the literature (*cf.* **[Ho]**), such a manifold X is often termed a manifold of *(Ricci) bounded geometry.*

(b) The examples in **[BoPa]** and in **[La]** show that there are Q-BG metric spaces for each real number $Q > 1$. For more examples, see **[HeiK]**, **[Sem]**, and the references given there.

Now we are ready to state our main result of this chapter.

Theorem 9.8. — *Let X and Y be geodesic metric spaces of Q-bounded geometry, $Q > 1$. Assume in addition that Y is a roughly starlike Gromov hyperbolic space whose Gromov boundary $\partial_G Y$ is a nondegenerate continuum. Then every quasiconformal homeomorphism $f\colon X \to Y$ is a rough Lipschitz map, quantitatively.*

In particular, if X is also a roughly starlike Gromov hyperbolic space whose Gromov boundary $\partial_G X$ is a nondegenerate continuum, then every quasiconformal homeomorphism $f\colon X \to Y$ is a rough quasiisometry, and hence extends to a quasisymmetric homeomorphism between the Gromov boundaries.

A *rough Lipschitz* map between metric spaces is a map that satisfies the second inequality in (1.17). For the terms related to Gromov hyperbolicity, see Chapter 3.

A *Hadamard manifold* X is a complete simply connected Riemannian manifold of nonpositive sectional curvature. If X is Gromov hyperbolic, then $\partial_G X$ is homeomorphic to a $(n-1)$-dimensional sphere, where $n = \dim X$, and hence connected when $n \geqslant 2$. A Hadamard manifold X is 0-roughly starlike with respect to each basepoint $w \in X$. In view of this and the discussion in 9.7 (a), Theorem 1.15 follows from Theorem 9.8. One should observe that by the arguments in [**HeiK**, Section 4] the inverse of a quasiconformal mapping is quasiconformal under the assumptions of Theorem 9.8.

Proof of Theorem 9.8. — Let $f\colon X \to Y$ be as in the hypotheses. We need to show that there exists $L \geqslant 1$ and $M \geqslant 0$, depending only on the data associated with f, X, and Y, such that

$$|f(x) - f(y)| \leqslant L|x - y| + M \tag{9.9}$$

for each pair of points $x, y \in X$. (We use the generic distance notation $|a - b|$ here.) Thus, fix $x, y \in X$.

Assume first that $y \in B(x, \lambda R_0)$ for some $0 < \lambda < 1$, where R_0 is a constant R_0' such that (BG_1') and (BG_2') are satisfied in X. Denote by R_1 a similar constant for Y. Suppose that

$$|f(x) - f(y)| \geqslant R_1, \tag{9.10}$$

and let α be a geodesic joining x and y in X. We claim that

$$\operatorname{mod}_Q \Gamma \geqslant C > 0, \tag{9.11}$$

where C is independent of x, y, and α, and where Γ is the family of all curves γ in Y joining $f(\alpha)$ to $\partial_G Y$; that is, γ meets $f(\alpha)$ and is not contained in any compact set in Y. To this end, we invoke both the uniformization theory of Chapter 4 and the local to global Loewner theory of Chapter 6. Fix $\varepsilon > 0$ so that the uniformized space $Y_\varepsilon = (Y, d_\varepsilon)$ is a bounded uniform space, as in Theorem 4.5. It follows from the definition of the metric d_ε and from (4.4) that

$$d_\varepsilon\big(f(x), f(y)\big) \geqslant \rho_\varepsilon(f(x)) \int_0^{R_1} e^{-\varepsilon t}\, dt \geqslant C\rho_\varepsilon(f(x)).$$

From now on, C denotes any positive constant that depends only on the data associated with f, X, and Y.

Thus, because Y is roughly starlike, Lemma 4.16 implies that

$$d_\varepsilon\big(f(x), f(y)\big) \geqslant Cd_\varepsilon(f(x)).$$

It follows that

$$\operatorname{dist}_\varepsilon\big(f(\alpha), \partial Y_\varepsilon\big) \leqslant C\big(\operatorname{diam}_\varepsilon f(\alpha) \wedge \operatorname{diam}_\varepsilon \partial Y_\varepsilon\big),$$

whence the general boundary version of Theorem 6.4 (see Remark 6.38) gives the claim in (9.11), provided $(Y_\varepsilon, d_\varepsilon, \nu_\varepsilon)$ is a locally Q-Loewner space. (Here $d\nu_\varepsilon = \rho_\varepsilon^Q\, d\nu$, where ν is the underlying measure in Y.) But this latter property of Y_ε is easy to verify by using condition (BG_2') and the fact, easily derived from (4.18) and the definition of d_ε, that

$$B(z, \lambda_1 R_1) \subset B_\varepsilon(z, \lambda_2 d_\varepsilon(z)) \subset B(z, R_1),$$

where $z \in Y$, the numbers $0 < \lambda_1, \lambda_2 < 1$ depend only on the data, B_ε is a ball in Y_ε, and B is a ball in Y. Note that both the conformal invariance of the modulus and the rough starlikeness of Y are used here. Thus (9.11) follows.

On the other hand, the volume growth condition (9.5) implies by [**HeiK**, Lemma 3.14] that

$$\operatorname{mod}_Q f^{-1}\Gamma \leqslant C\big(\log(1/\lambda)\big)^{1-Q}. \tag{9.12}$$

We claim that this in turn implies a definite positive lower bound for λ. Indeed, under the assumptions of Q-bounded geometry, the arguments in [**HeiK**, Section 4] imply that f is locally uniformly quasisymmetric in the sense that every point in X has a neighborhood where f is η-quasisymmetric as defined in (1.18) with η depending only on the data. Then we can invoke the work of Tyson [**Ty**] which implies that

$$\frac{1}{C}\operatorname{mod}_Q \Gamma' \leqslant \operatorname{mod}_Q f\Gamma' \leqslant C\operatorname{mod}_Q \Gamma' \tag{9.13}$$

for all curve families Γ' in X. The combination of (9.13), (9.12), and (9.11) gives that $\lambda \geqslant C > 0$.

What we have proved at this point is that there exists a constant $\lambda \in (0,1)$, depending only on the data, such that $|x-y| \leqslant \lambda R_0$ for $x, y \in X$ implies $|f(x)-f(y)| \leqslant R_1$.

Next, assume $|x - y| > \lambda R_0$. Pick a geodesic γ from x to y, and let $x = x_0, x_1, \ldots, x_{N+1} = y$ be points on γ so that

$$|x_i - x_{i+1}| = \lambda R_0, \quad i = 0, \ldots, N-1$$

and

$$|x_N - x_{N+1}| \leqslant \lambda R_0.$$

In particular, $\lambda R_0 N \leqslant |x-y| \leqslant \lambda R_0(N+1)$. Hence we have, by what was proved above, that

$$\begin{aligned} |f(x)-f(y)| &\leqslant \sum_{i=0}^{N} |f(x_i)-f(x_{i+1})| \\ &\leqslant R_1(N+1) \leqslant \frac{R_1}{\lambda R_0}|x-y| + R_1. \end{aligned}$$

We conclude that f is a rough Lipschitz map. This finishes the proof of Theorem 9.8. □

Remarks 9.14

(a) The condition on rough starlikeness is necessary for the conclusion of Theorem 9.8. Indeed, by conformally changing the metric in a sequence of small disjoint disks one can pull out towers from the hyperbolic plane so that each tower has finite length but the lengths have no fixed bound. The matters can easily be arranged so that the resulting space Y is proper, geodesic and Gromov hyperbolic, but the identity map $\mathbb{H}^2 \to Y$ (which is conformal) is not a rough Lipschitz map.

(b) The conclusion of Theorem 9.8 remains valid if we only assume that f is a quasiconformal homeomorphism of X onto an open subset of Y. (We thank Bruce Kleiner for pointing this fact out to us.) The proof is essentially the same, because the key fact (9.11) continues to hold; now Γ is the family of curves joining $f(\alpha)$ to the boundary of $f(X)$ in the extended sense.

APPENDIX: LENGTHS AND METRICS

In this appendix, we gather some basic facts about curves and line integration in metric spaces. We prove a result (Proposition A.7) which in particular implies Lemma 2.6. Undoubtedly, Proposition A.7 can be found somewhere in the literature, but in lack of a precise reference, we provide a proof.

Let (M, d) be a metric space, and let γ be a compact curve in M, that is, $\gamma\colon [a, b] \to M$ is a continuous map of a compact interval into M. The *length* $\ell_d(\gamma)$ of γ with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup \sum_{k=0}^{n-1} d\left(\gamma(t_k), \gamma(t_{k+1})\right), \tag{A.1}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. If $\ell_d(\gamma) < \infty$, then γ is said to be *rectifiable.* For a rectifiable curve γ we can define the *arc length* $s\colon [a, b] \to [0, \infty)$ along γ by

$$s(t) = \ell_d(\gamma|[a, t]).$$

The function s is a function of bounded variation on $[a, b]$.

Next, let $\rho\colon M \to [0, \infty]$ be a Borel function. For each rectifiable curve $\gamma\colon [a, b] \to M$ we can define its *ρ-length* by

$$\lambda_\rho(\gamma) = \int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t))\, ds(t). \tag{A.2}$$

If M is rectifiably connected (that is, if every pair of points in M can be joined by a rectifiable curve), then ρ determines a distance function

$$d_\rho(x, y) = \inf \lambda_\rho(\gamma), \tag{A.3}$$

where the infimum is taken over all rectifiable curves γ joining x and y in M. In general, the distance function d_ρ need not be a metric; it is a metric if ρ is positive and continuous. If $\rho \equiv 1$, then $\lambda_\rho(\gamma) = \ell_d(\gamma)$ is the length of γ with respect to the metric d; in this case, we denote the corresponding metric in (A.3) by ℓ_d.

Assume from now on that (M,d) is rectifiably connected and that $\rho\colon M \to (0,\infty)$ is continuous. For any two metrics d_1 and d_2 on M we write $d_1 \sim d_2$ if they induce the same topology on M, i.e. if the identity map $(M,d_1) \to (M,d_2)$ is a homeomorphism.

The following lemma is easy to prove; we omit the proof.

Lemma A.4. — *We always have $d_\rho \sim \ell_d$. In particular, $d \sim \ell_d$ if and only if $d \sim d_\rho$.*

Note that the identity map $(M,\ell_d) \to (M,d)$ is always 1-Lipschitz and hence continuous, but it need not be a homeomorphism. In the following, we make the additional assumption that $d \sim \ell_d$, which then implies that $d \sim \ell_d \sim d_\rho$ by Lemma A.4.

Next, denote by $\ell_\rho(\gamma) = \ell_{d_\rho}(\gamma)$ the length of a curve γ with respect to the metric d_ρ. That is, $\ell_\rho(\gamma)$ is given by the expression in (A.1) if we replace the metric d by the metric d_ρ.

The following two lemmas are easy to establish; we leave the proofs to the reader.

Lemma A.5. — *For a curve γ we have that $\ell_d(\gamma) < \infty$ if and only if $\ell_\rho(\gamma) < \infty$. That is, γ is rectifiable with respect to d if and only if it is rectifiable with respect to d_ρ.*

Lemma A.6. — *If γ is rectifiable, then $\ell_\rho(\gamma) \leqslant \lambda_\rho(\gamma)$.*

In view of Lemma A.5, the rectifiability of a curve is independent of the chosen metric d_ρ. We shall henceforth speak of rectifiable curves without specifying the metric. Recall that we have the standing assumption that $d \sim \ell_d$ and that ρ is positive and continuous.

We can now prove the following result:

Proposition A.7. — *Let (M,d) be a rectifiably connected metric space such that $d \sim \ell_d$, and let $\rho\colon (M,d) \to (0,\infty)$ be continuous. Then*

$$\ell_\rho(\gamma) = \lambda_\rho(\gamma)$$

for each rectifiable curve $\gamma\colon [a,b] \to (M,d)$.

Proof. — Let $\gamma\colon [a,b] \to (M,d)$ be rectifiable. By Lemma A.6, it suffices to show that $\lambda_\rho(\gamma) \leqslant \ell_\rho(\gamma)$. (Note that $\ell_\rho(\gamma)$ is finite by Lemma A.5). Since $d \sim d_\rho$ by Lemma A.4, the curve γ is compact in both (X,d) and (X,d_ρ). Thus, for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{(A.8)} \qquad x \in \gamma,\ y \in X \ \text{ and } \ d(x,y) \wedge d_\rho(x,y) < \delta \ \text{ imply } \ |\rho(x) - \rho(y)| < \varepsilon.$$

Now fix $\varepsilon > 0$, and let $0 < \delta < 1$ be as in (A.8). Choose a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that

$$\text{(A.9)} \qquad d(x_k,\gamma(t)) \wedge d_\rho(x_k,\gamma(t)) < \delta/2$$

for $t_{k-1} \leqslant t \leqslant t_k$ and $k = 1,\ldots,n$, where $x_k = \gamma(t_k)$. Next, for each k choose $t_k^0 = t_{k-1} < t_k^1 < \cdots < t_k^{\nu_k} = t_k$ such that, with $x_k^i = \gamma(t_k^i)$, and $\gamma_k = \gamma|[t_{k-1},t_k]$, we have

$$\ell_d(\gamma_k) \leqslant \sum_{i=0}^{\nu_k-1} d\left(x_k^i, x_k^{i+1}\right) + \frac{\varepsilon}{n}. \tag{A.10}$$

Then

$$\begin{aligned}\int_{\gamma_k} \rho\, ds &\leqslant (\rho(x_k)+\varepsilon)\, \ell_d(\gamma_k) \\ &\leqslant (\rho(x_k)+\varepsilon) \left(\sum_{i=0}^{\nu_k-1} d\left(x_k^i, x_k^{i+1}\right) + \frac{\varepsilon}{n} \right),\end{aligned} \tag{A.11}$$

by (A.8), (A.9) and (A.10). If, on the other hand, α is an arbitrary curve joining x_k^i and x_k^{i+1}, and $\alpha' = \alpha \cap B_d(x_k^i, d(x_k^i, x_k^{i+1}))$, one computes

$$\begin{aligned}\int_{\alpha} \rho\, ds &\geqslant \int_{\alpha'} \rho\, ds \\ &\geqslant \left(\rho(x_k^i) - \varepsilon\right) d\left(x_k^i, x_k^{i+1}\right) \\ &\geqslant \left(\rho(x_k) - 2\varepsilon\right) d\left(x_k^i, x_k^{i+1}\right),\end{aligned}$$

which implies that

$$\rho(x_k) d(x_k^i, x_k^{i+1}) \leqslant d_\rho(x_k^i, x_k^{i+1}) + 2\varepsilon d(x_k^i, x_k^{i+1}). \tag{A.12}$$

Next combine (A.11) and (A.12) to obtain

$$\int_{\gamma_k} \rho\, ds \leqslant \ell_\rho(\gamma_k) + 3\varepsilon \ell_d(\gamma_k) + \frac{M\varepsilon}{n} + \frac{\varepsilon^2}{n}, \qquad M = \max_{z\in\gamma} \rho(z).$$

The claim follows from this by summing over all subcurves γ_k, and by letting $\varepsilon \to 0$. The proof of Proposition A.7 is complete. □

Remarks A.13

(a) The assumption $d \sim \ell_d$ in Proposition A.7 is satisfied, for example, if (X, d) is locally quasiconvex.

(b) If the hypotheses of Proposition A.7 are satisfied, then (M, d_ρ) is a *length space*; that is, $d_\rho(x, y)$ can always be given as the infimum of the lengths of the curves joining x and y.

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